An N-Body gravitational simulation model

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Abstract

This report presents an N-body gravitational simulation model designed to simulate and analyse the dynamics of celestial bodies under gravitational interactions, using Newton's laws of motion and gravitation as well as the Velocity Verlet integration algorithm. Key outputs of the model include a 2D animation of the orbital motions, the calculation of orbital eccentricities and periods, and the analysis of the system's energies as well as the errors. These are showcased on the 3-Body Earth, Sun and Moon system.

1 Introduction

An N-Body simulation involves numerically modeling the motion of a system of N-interacting bodies over time by accounting for the forces acting between them.

While most commonly used in Astrophysics to study the motions of gravitationally interacting celestial bodies [1], it also finds a range of other applications in the sciences and engineering, such as in molecular dynamics. For instance, the N-body approach has been used to model the interatomic potential for a metal by computing the potential energies of N interacting atoms [2], as well as to simulate heavy ion collisions, where atomic nuclei interact during high-energy events [3].

The model described in this report uses the N-body approach to simulate and analyse gravitational dynamics, with the main focus being on the collective motions of the Earth, the Moon, and the Sun.

Each body in the system moves according to Newton's equations of motion:

$$\frac{\delta \mathbf{r}_i}{\delta t} = \mathbf{v}_i \tag{1}$$

$$m\frac{\delta \mathbf{v}_i}{\delta t} = \mathbf{F}_i, \tag{2}$$

The force in this case is the net gravitational force acting on the body, which is the vector sum of the gravitational forces exerted by all the other bodies in the system.

The gravitational force between two masses can be calculated using Newton's law of gravitation:

$$\mathbf{F}_{12}(t) = G\frac{m_1 m_2}{|\mathbf{r}_{12}(t)|^2}\hat{\mathbf{r}}_{12}(t) \tag{3}$$

where G is the gravitational constant, $\hat{\mathbf{r}}_{12}$ is the unit vector pointing from mass m_1 to mass m_2 (indicating the direction of the force), and $\mathbf{r}_{12}$ is the distance vector between the two masses. The magnitude of the force is thus proportional to the product of the two masses and decreases with the square of the distance between the two bodies.

From this net force, the acceleration vector of the body ($\frac{\delta^2 \mathbf{x}}{\delta t^2}$) can then be calculated using Newton's 2nd law (equation 2).

For a 3-body system such as the Earth, Moon and the Sun), the net acceleration on the first body due to the forces from the second and third bodies would thus be:

$$\mathbf{a}_1(t) = \frac{\mathbf{F}_{12}(t)}{m_1} + \frac{\mathbf{F}_{13}(t)}{m_1} = G\frac{m_2}{|\mathbf{r}_{12}(t)|^2}\hat{\mathbf{r}}_{12}(t) + G\frac{m_3}{|\mathbf{r}_{13}(t)|^2}\hat{\mathbf{r}}_{13}(t) \tag{4}$$

and equivalently for the other two bodies.

To find the position of each body over time, a numerical integration method needs to be used that computes the positions and velocities at discrete intervals, as an analytical solution is not possible.

Generally, this is done by calculating the force (and thereby the acceleration) based on the current position and then using this acceleration to update the velocity and position at the next time step.

In an N-body system, the total energy is conserved - this provides a reliable measure of the accuracy of the simulation. In a gravitational system, the total energy is composed of the kinetic energy and the gravitational potential energy of the system.

The total kinetic energy is the sum of the kinetic energies of each of the bodies:

$$KE = \sum_{i=1}^{N} \frac{1}{2} m_i |\mathbf{v}_i|^2 \tag{5}$$

The gravitational potential energy is computed by summing the potential energies arising between each unique pair of bodies in the system:

$$PE = \sum_{i=1}^{N} \sum_{j=i+1}^{N} -G\frac{m_i m_j}{|\mathbf{r}_{ij}|} \tag{6}$$

whereby m_i and m_j are the masses of a pair of bodies separated by a distance $|r_{ij}|$).

As the total energy is conserved, the kinetic and potential energies follow an inverse relationship - as a body moves further away from another body, its kinetic energy decreases (due to a reduction in velocity) and its gravitational potential energy increases.

Gravitational forces cause planets to move in elliptical orbits around their host stars. This is due to it acting as a centripetal force which is balanced with the inertial force generated from the planet's velocity, preventing the planet from being pulled into the star.

These orbits are not perfectly circular, however, but possess a certain amount of eccentricity (e), which measures the deviation of the orbit from circularity:

$$e = \frac{r_a - r_p}{r_a + r_p}. \tag{7}$$

where r_a is the aphelion (the furthest distance of the planet from the host star) and r_b is the perihelion (the closest distance) - with a value of 0 representing perfect circularity and 1 a very elliptical orbit.

The orbital period of a planet can be distinguished into a sidereal amd synodic period. The sidereal period hereby denotes the planet's true orbit - the time taken for it to complete one full cycle around its host star, (relative to a background of distant, fixed stars). The synodic period, on the other hand, is the time taken for a planet to return to the same position relative to other planets. For the Moon, for example, this means the time taken for it to be in the same relative position with respect to the SUn and the Earth again (i.e., the time taken for it to be in same position in night sky again).

2 Analysis and Discussion

This section will describe what results the gravitational model is capable of outputting. The model's functionality is showcased via results from the 3-Body Earth, Sun and Moon test case.

The model is generalised to work for any given number N of bodies. All that is required for it to run is the specification of a set of attributes for each body: its mass, initial position, and initial velocity. These parameters are then stored in that system's designated dictionary. Additionally, the duration of the simulation as well as the size of the time-step used in the integration algorithm need to be specified (whereby the default is 3 years with a timestep of an hour).

With these initial parameters, the Velocity Verlet integration algorithm is used to compute the positions, velocities, and accelerations of all bodies at each time step, storing their evolution over time for later analysis.

The code for the integration is hereby written in C++, as it can be expected to be a lot faster and efficieny than Python, especially with many time steps (and if many bodies want to be simulated).

Unlike the more well-known Leapfrog integration method, where position and velocity are updated at different time steps (with velocity calculated at half-integer steps and position at integer time steps), the Velocity Verlet method computes both position and velocity at the same time step. It works in the following way:

1. First, the position is updated:
$$\mathbf{r}(t + \Delta t) = \mathbf{r}(t) + \mathbf{v}(t)\Delta t + \frac{1}{2}\mathbf{a}(t)\Delta t^2$$
where $\mathbf{r}(t)$ is the position, $\mathbf{v}(t)$ is the velocity, and $\mathbf{a}(t)$ is the acceleration at time t.
2. Then the new acceleration $\mathbf{a}(t + \Delta t)$ is calculated using the updated position $\mathbf{r}(t + \Delta t)$.
3. Finally, the velocities are updated using the current velocity, the current acceleration, and the updated acceleration:
$$\mathbf{v}(t + \Delta t) = \mathbf{v}(t) + \frac{\mathbf{a}(t) + \mathbf{a}(t + \Delta t)}{2}\Delta t$$

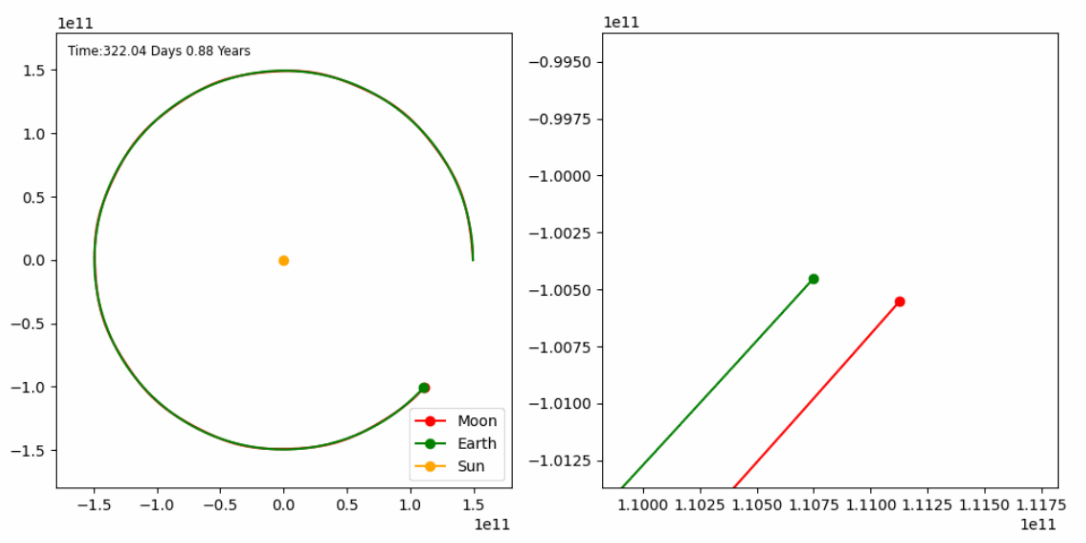

Figure 1: A snapshot of the gravitational ·-body animation of the Sun, Earth and Moon system, with the right box showing a zoomed-in view of the Moon orbiting the Earth.

2.1 Animation of the orbits

The main output of this model is a 2-dimensional animation of the positions of each of the bodies over time (obtained through the integration method). This is also generalised for any number of bodies, and includes a zoomed-in animation of the motion of two chosen bodies.

This is especially useful for the simulation of the Sun, Earth and Moon system, as the elliptical orbit of the Moon around the Earth can additionally be seen as the Earth performs a larger elliptical orbit around the Sun. A snapshot of the animation is depicted in figure 1.

2.2 Energies and error analysis

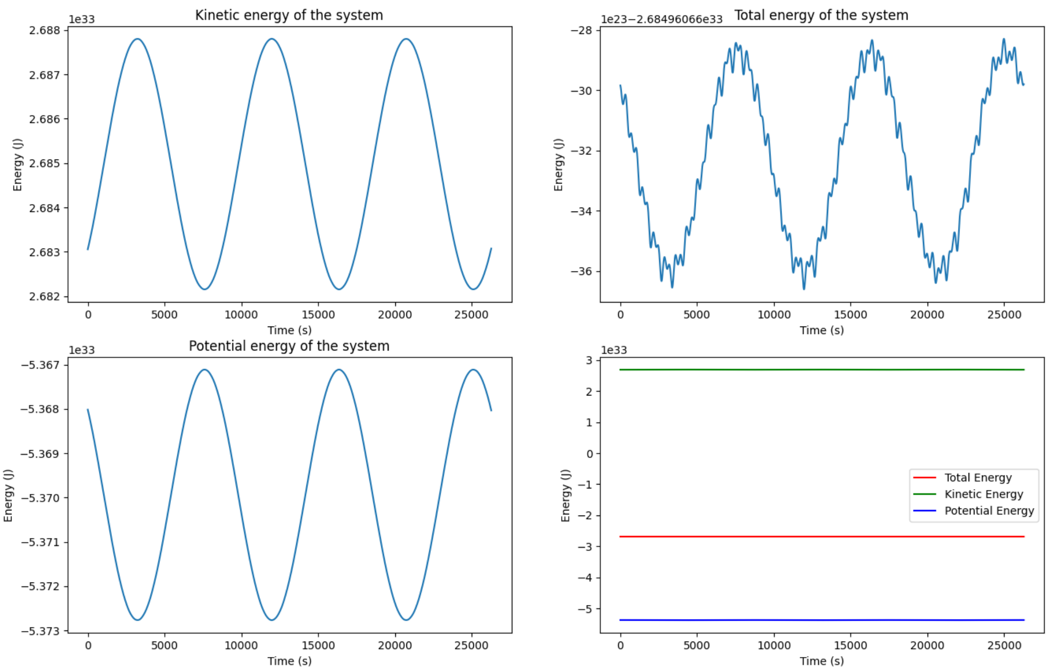

Figure 2: The system's kinetic, potential, and total energy.

As a next step, the model uses the stored positions and velocities to calculate and plot the kinetic, potential and total energy of the system, as well as the relative error of the total energy. As seen in figure 2, the kinetic and potential

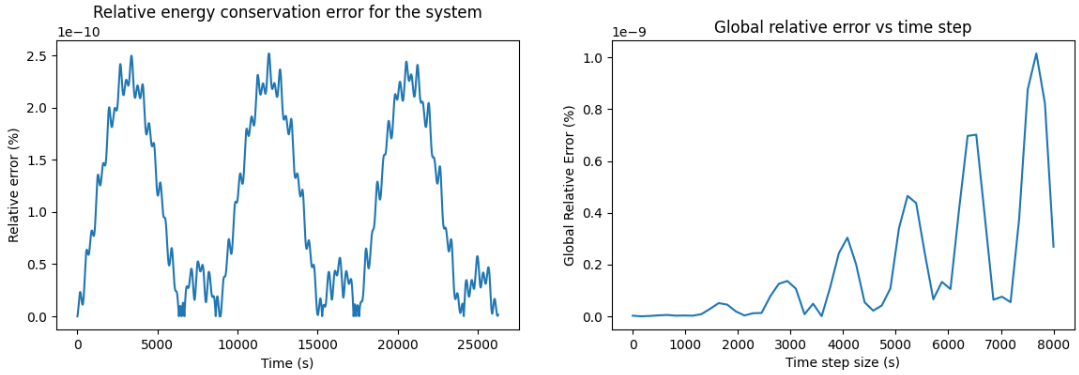

Figure 3: The relative conservation error over time (left) and the global relative error versus size of the time step.

energies of the 3-body solar system exhibit the expected inverse relationship. Additionally, the total energy is negative, as is the case for bound systems such as this one, where gravity holds the bodies together in stable, inescapable orbits.

The plot however also shows slight oscillations in the total energy, instead of a straight line, which would indicate its perfect conservation over time - due to numerical errors in the integration algorithm.

However, plotting the local relative error in the total energy in the left box of figure 3 shows how these deviations are neglibile, with the errors being in the order of magnitude of about 10^-10.

Running the simulation with increasing step-sizes leads to a oscillatory but steady increase in the global relative error of the total energy (i.e, the relative error of the last time step). Even at double the size of the default step-size (so 7200 seconds), the global error remains in the order of magnitude of 10^-9, indicating a highly accurate simulation.

These results are a crucial aspect of the model, as it is important to know exactly how reliable the results of the simulation are.

2.3 Eccentricities and orbital periods

The model is also able to calculate the eccentricity of a selected body's orbit, by using the stored positions of that body and the one it is orbiting.

The eccentricity of Earth's orbit around the Sun is hereby calculated as being 0.0005567787. However this value significantly deviates from the value found in literature (0.01671). This is most likely due to the chosen initial conditions, which the resulting orbits are highly dependent on. This simple 3-body model also does not take into account perturbations caused by the other planets in the solar system.

Finally, the model also performs an analysis of the planet's orbital periods. The sidereal period of a selected body is calculated by tracking the orbiting body's angular position over time and identifying when full cycles occur. Computing the sidereal periods of the Earth and the Moon give values of 364.8125 and 27.5021 days, respectively, which are very close to the values found in literature (365.2564 and 27.3217).

Additionally, in the specific solar system 3-body case, the Moon's synodic period can be retrieved by calculating the differences between the Moon and the Sun distances and the Moon and the Earth distances over time, giving a measure of their relative alignment. By tracking the periodic peaks in this difference, one can thus calculate how long it takes for a given alignment to repeat (ie, find the synodic period). This gives an orbital synodic period of 29.7350 days for the Moon, which is very close to the actual value (29.5306).

Lastly, and specific for this solar-system case, the eccentricity of Earth's orbit is plotted against an increasing solar mass, and the Moon's synodic period is plotted against an increasing lunar mass (figure 4).

This shows an increased eccentricity for solar masses below the actual mass, and for masses above the actual mass. The moon's synodic period decreases steadily with an increasing lunar mass.

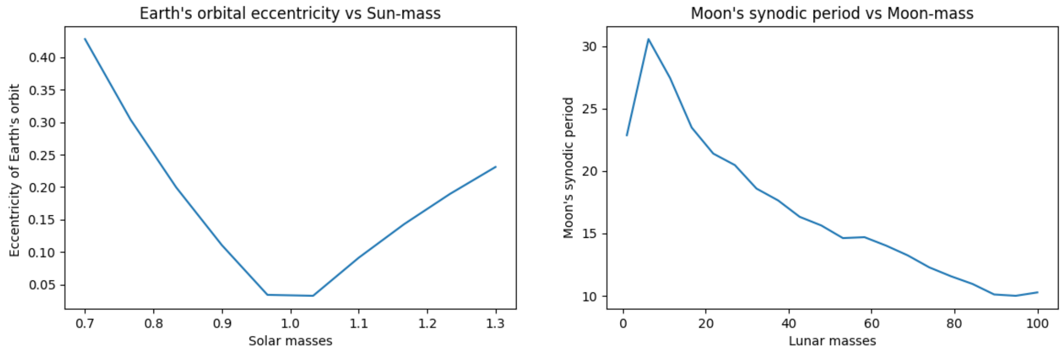

Figure 4

3 Conclusions

In conclusion, this report describes how to make use of a developed N-body gravitational model to simulate and explain the dynamics of celestial bodies. Specifically, its functionality was showcased on the 3-body Earth, Sun and Moon system, succesfully capturing the gravitational interactions between them. These are displayed via a 2D animation of their orbital motions, an analysis of the system's energies and the simulation's errors, which are negligible, as well as the computation of the orbits' eccentricities and periods. The general setup of this model allows these results to be obtained for any astronomical system, given accurate initial parameters, and can thus also be used to study exoplanets.

References

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