Advanced topics in Machine Learning
Assignment 0

Mariuta Nicolae (rqt629) September 6, 2015

Question 1

1. Sample space S = {(red, red), (red, orange) ,(red, blue) (orange, red), (orange,orange), (orange,blue), (blue, red), (blue, orange)}

2. P{(red, red)} = $\frac{5}{9} * \frac{4}{8} = \frac{5}{18}$

P{(red,orange) } = $\frac{5}{9} * \frac{3}{8} = \frac{5}{24}$

P{(red,blue)}= $\frac{5}{9} * \frac{1}{8} = \frac{5}{72}$

P{(orange,red)}=$\frac{3}{9} * \frac{5}{8} = \frac{5}{24}$

P{(orange,orange)}=$\frac{3}{9} * \frac{2}{8} = \frac{1}{12}$

P{(orange,blue)}=$\frac{3}{9} * \frac{1}{8} = \frac{1}{24}$

P{(blue,red)}=$\frac{1}{9} * \frac{5}{8} = \frac{5}{72}$

P{(blue,orange)}= $\frac{1}{9} * \frac{3}{8} = \frac{3}{72}$

3. Possible values for X are: 0,1,2

4. P{X=0} = $\frac{5}{18} + \frac{5}{72} + \frac{5}{72} = \frac{15}{36}$ I summed the probabilities of all draws that do not contain orange ball.

5. P{X=1} = $\frac{5}{24} + \frac{5}{24} + \frac{1}{24} + \frac{3}{72} = \frac{37}{72}$

P{X=2} = $\frac{1}{12}$

$$E[X] = 0 * \frac{15}{36} + 1 * \frac{37}{72} + 2 * \frac{1}{12} = \frac{49}{72}$$

Question 2

1. If the expectation of a random variable is $\sum_x xP(X = x)$ I can then expand the expectation of sums to:

$$E[X + Y] = \sum_{x,y} (x + y)P(X = x \; AND \; Y = y) = \sum_{x,y} xP(X = x \; AND \; Y = y) + \sum_{x,y} yP(X = x \; AND \; Y = y)$$

If I consider only the first element of the sum, then:

$$\sum_{x,y} xP(X = x \; AND \; Y = y) = \sum_x x \sum_y P(X = x \; AND \; Y = y) = \sum_x xP(X = x) \; = \mathrm{E[X]}$$

So the inner sum here is $xP(X = x)$ (the event: "X=x" is the same as the event "X=x and Y takes any value ").

In the same way we obtain that

$$\sum_{x,y} yP(X = x \; AND \; Y = y) = E[Y]$$

By combining these results, we obtain that $E[X + Y] = E[X] + E[Y]$ as required.

2.

$$E[XY] = \sum_{x,y} xyp_{xy}(x, y)$$

Because X and Y are independent, then:

$$\sum_{x,y} xyp_{xy}(x, y) \; = \sum_{x,y} xyp_x(x)p_y(y) = \sum_x xp_x(x) \sum_y yp_y(y) = \; E[X]E[Y]$$

3. I take the case of tossing two fair dices. X is the number of dices greater or equal with 3 and Y is the maximum number being shown. For these random variables, I have the following tables with all possible values for X and Y for each roll dice:

X	1	2	3	4	5	6
1	0	0	1	1	1	1
2	0	0	1	1	1	1
3	1	1	2	2	2	2
4	1	1	2	2	2	2
5	1	1	2	2	2	2
6	1	1	2	2	2	2

Y	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	4	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	6

The joint probability distribution table is the following:

X/Y	1	2	3	4	5	6	sum
0	1/36	3/36	0	0	0	0	4/36
1	0	0	4/36	4/36	4/36	4/36	16/36
2	0	0	1/36	3/36	5/36	7/36	16/36
sum	1/36	3/36	5/36	7/36	9/36	11/36	1

The expectations for each variable:

$$E[X] = 0 * \frac{4}{36} + 1 * \frac{16}{36} + 2 * \frac{16}{36} = \frac{48}{36} = 1.33$$

$$E[Y] = 1 * \frac{1}{36} + 2 * \frac{3}{36} + 3 * \frac{5}{36} + 4 * \frac{7}{36} + 5 * \frac{4}{36} + 6 * \frac{11}{36} = \frac{135}{36} = 3.75$$

$$E[X]E[Y] = \frac{48}{36} * \frac{136}{36} = 5$$

$$E[XY] = 1 * 3 * \frac{4}{36} + 1 * 4 * \frac{4}{36} + 3 * \frac{5}{36} + 5 * \frac{4}{36} + 6 * \frac{4}{36} + \frac{2}{36} + 2 * \frac{2}{36} + 2 * \frac{5}{36} + 2 * \frac{7}{36} = \frac{104}{36} = 2.88$$

Since $E[XY] \neq E[X]E[Y]$ then, the two random variables are dependent.

4. $E\big[E[X]\big] = E[X]$

In this case, it is like the E[X] in interior of the first expectation is a function (we can say f(x)).

In this case, I can write like the expectation of a function:

$$E[f(X)] = \sum f(x)p_X(x)$$

But in our case, f(x) is expectation of X so we can expand and write like:

$$E\big[E[X]\big] = \sum xp_X(x) \sum p_X(x)$$

And the sum of all probabilities for a random variable is 1 so we can conclude that:

$$E\big[E[X]\big] = \sum xp_X(x) \sum p_X(x) = \sum xp_X(x) * 1 = E[X]$$

It is certainly not the optimal way to write that proof but I hope I got the right idea.

5.

$V[X] = E[(X - E[X])^2]$

I will start by just opening the parenthesis:

$$V[X] = E[(X - E[X])^2] = E[X^2 - 2XE[X] + E[X]^2] = \sum (x^2 - 2xE[X] + E[X]^2)\ p(x)$$
$$= \sum x^2 p(x)$$
$$- 2E[X] \sum x + E[X]^2 == E[x^2] - 2E[X]E[X] + +E[x]^2 = E[X^2] - E[X]^2$$

I have written the proof as for the discrete case but I should have written for the continuous case by using integrals instead of sums.

Question 3

1. I will define the random variable S as suggested in the hint:

$$S = \sum_{i=1}^{10} X_i$$

By doing this, we can bound the probability by using the Markov's inequality by using the formula:

$$P(S \geq 9) \leq \frac{E[S]}{9}$$

Since there is equal chance for each X_i to take value 0 or 1, then the expectation of X_i is 0.5 and then $E[S] = 0.5 * 10 = 5$. As a conclusion, we can bound the inequality:

$$P(S \geq 9) \leq \frac{5}{9} = 0.55$$

2. I am not sure if I understood correctly the Hoeffding's inequality but since we are interested in the upper limit of the sum, I only have to calculate the upper bound of the equality:

$$P[S - E[S] > \varepsilon] \leq \exp(-\frac{2\varepsilon^2}{\sum_{i=1}^{10}(1-0)^2})$$

Since we are interested in the probability that S to be smaller or equal than 9 and I have previously found that E[X] = 5, then ε I think it should be 4.

$$P[S - 5 \geq 4] \leq \exp(-\frac{2 * 4^2}{\sum_{i=1}^{10}(1-0)^2})$$

$$P[S \geq 9] \leq \exp\left(-\frac{32}{10}\right) = 0.04076$$

3. Since there are 10 possible combinations of X1,X2...X10 that have sum 9 and 1 combination that has sum 10 out of 2^{10} combinations in total, then the probability of the event is:

$$P[S \geq 9] = \frac{11}{2^{10}} = 0.0107$$

4. By looking at the results obtained previously, I can say that Hoeffding's inequality provides a more accurate bounding of the probabilities.

Question 4

1. I assign random variable $X_i = 1$ if the passenger shows up and 0 otherwise. In this case, we have the probabilities:

$P[X_i = 1\,] = 0.95$ and $P[X_i = 0\,] = 0.05$

I will again assign random variable S for the total number of passengers and the expectation of S for 100 passengers is:

$$S = 0.95 * 100 = 95$$

By using the Markov's inequality, I obtain the bound:

$$P(S \geq 99) \leq \frac{95}{99} = \ 0.9595$$

By using the Hoeffding's inequality, I obtain:

$$P[S - 95 \geq 4] \leq \exp\left(-\frac{2 * 4^2}{\sum_{i=1}^{100}(1-0)^2}\right)$$

$$P[S \geq 99] \leq \exp\left(-\frac{32}{100}\right) = 0.726$$