Vision and Image Processing: Linear Algrebra

François Lauze
Department of Computer Science
University of Copenhagen

Plan for today

- Vectors
- Matrices
- Traces and determinants
- Linear mappings

Outline

1. Vectors
2. Matrices
3. Square Matrices, Trace, Determinant
4. Linear Mappings

Vectors and Matrices

Ordered collections of real numbers that represent some quantities

- Position in plane, space, velocity, some geometric transformations, images...
- Series of basic (and less basic operations) defined on them.

Vectors

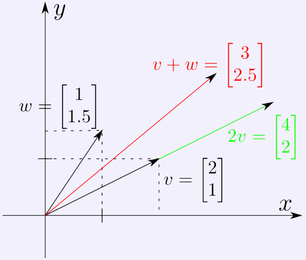
- A *n*-vector is a *n*-uple of real values:

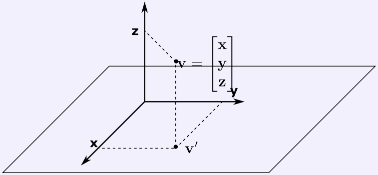
$$v = \begin{bmatrix} x_1, & \dots, & x_n \end{bmatrix} \text{ (row vector) }, \quad v = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} \text{ (column vector, preferred)}$$

- Addition: same length vectors $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$
- Multiplication by a scalar $\lambda \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{bmatrix}$
- Transposition $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^\top = \begin{bmatrix} x_1, & \dots, & x_n \end{bmatrix}, \begin{bmatrix} x_1, & \dots, & x_n \end{bmatrix}^\top = \begin{bmatrix} x_1, \\ \vdots \\ x_n \end{bmatrix}$
- To save space, I often write a column vector as a transpose of a line vector:

$$\mathbf{x} = \begin{bmatrix} x_1, & \dots, & x_n \end{bmatrix}^\top$$

Vectors, coordinates, operations – Highschool stuffs!





- Vector space $\mathbb{R}^n$, set of vectors of length n.
- n is the dimension of the vector space.
- Vector subspace: lines (going through origin), planes (going through origin), etc...
- Line: dimension 1, plan dimension 2, etc.

Inner Product, Orthogonality, Norm, Distance

- Two vector $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n = \sum_{i=1}^n x_i y_i$

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- Distance between $\mathbf{y}$ and $\mathbf{y}$: $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.

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- Distance between $\mathbf{y}$ and $\mathbf{y}$: $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.
- Exercise: develop the expression $\|\mathbf{x} - \mathbf{y}\|^2$.

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Matrices

- A $n \times m$ matrix is an array of numbers with n rows and m columns
- A $2{\times}3$ matrix F

$$F = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

- 2 matrices of the same size can be added together: just add the entries:

$$\begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -2 & 1 \\ 7 & -3 & 0 \end{bmatrix} = \ ?$$

- a matrix can be multiplied by a scalar: just multiply all entries

$$4 \begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \ ?$$

- Null matrix: matrix with all entries = 0: $\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$

- Transposition of a Matrix (Matrix transpose): $(n \times m) \to (m \times n)$

$$A = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nm} \end{bmatrix}, \quad A^\top = \begin{bmatrix} a_{11} & \dots & a_{n1} \\ \vdots & \vdots & \vdots \\ a_{1m} & \dots & a_{nm} \end{bmatrix}$$

- Example

$$A = \begin{bmatrix} 2 & 3 & 0 & 1 \\ 1 & 8 & 5 & 7 \end{bmatrix}, \quad A^\top = \begin{bmatrix} 2 & 1 \\ 3 & 8 \\ 0 & 5 \\ 1 & 7 \end{bmatrix}$$

- a square matrix A is symmetric if $A = A^T$

$$\underbrace{A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}}_{\text{symmetric}}, \quad \underbrace{B = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}}_{\text{not symmetric}}$$

Product of a Matrix and a Vector

- A matrix of size $m \times n$ and a vector of length m can be multiplied to form a vector of length n.
- Formal rule:

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}, v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$Av = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \dots a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \dots a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{n2}v_2 + \dots a_{mn}v_n \end{bmatrix}$$

- Each line of A is multiplied in “inner product way” with v.

Product of Matrices

- Dimension rule: Dimension of A and B must be compatible

$$(m,p).(q,n) \implies \begin{cases} p \neq q : \text{impossible} \\ (m,p).(p,n) \to (m,\cancel{p}).(\cancel{p},n) \to (m,n) \end{cases}$$

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- Algebraic rule: a_{ij} entry (i,j) of A, b_{jk} entry (j,k) of B

$$A = (a_{ij})_{\substack{i=1\dots m,\\ j=1\dots p}} \quad B = (b_{jk})_{\substack{j=1\dots p\\ k=1\dots n}}$$

Denote entry (i,k) of product $C = AB$ by c_{ik}:

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- Example

$$\begin{bmatrix} 2 & 2 \\ 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 6 & 2 \\ -5 & 3 & 3 \\ 3 & 3 & -1 \end{bmatrix}$$

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- What does matrix multiplication means? Later!

Special Products

- Row vector $\mathbf{x} = [x_1, \dots, x_n]$: matrix of size $1 \times n$. Column vector $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$: matrix of size n. The products $\mathbf{x}\,\mathbf{y}$ and $\mathbf{y}\,\mathbf{x}$ well defined.
- $\mathbf{x}\,\mathbf{y}$: dimensions rule says $(1, n)(n, 1) \rightarrow (1, 1)$. A $(1, 1)$ dimension matrix? a single number!

$$\mathbf{x}\,\mathbf{y} = [x_1, \dots, x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

- $\mathbf{y}\,\mathbf{x}$. What does dimension rule says: $(n, 1).(1, n) \rightarrow (n, n)$: A square matrix.

$$\mathbf{y}\,\mathbf{x} = \begin{bmatrix} y_1 x_1 & y_1 x_2 & \dots & y_1 x_n \\ y_2 x_1 & y_2 x_2 & \dots & y_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ y_n x_1 & y_n x_2 & \dots & y_n x_n \end{bmatrix}$$

- $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$. $\mathbf{x}^T\mathbf{y}$ satisfies the dimensions rule, result is scalar.

$$\mathbf{x}^T\mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i.$$

  This is the inner product!
- $\mathbf{x}\,\mathbf{y}^\top$ satisfies the dimensions rule

$$\mathbf{x}\,\mathbf{y}^\top = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \dots & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \dots & x_2 y_n \\ \vdots & \vdots & \vdots & \vdots \\ x_n y_1 & x_n y_2 & \dots & x_n y_n \end{bmatrix}$$

  Outer product. Outer product works in fact for column vectors of different dimensions. Not the case for inner product.

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Square Matrices

- The product of two $n \times n$ square matrices has the same size.
$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}, \quad AB = \begin{bmatrix} -2 & 5 \\ -24 & 5 \end{bmatrix}$
- Beware that $AB \neq BA$ in general! $BA = \begin{bmatrix} 1 & 12 \\ -9 & 2 \end{bmatrix}$
- I can have $A \neq 0$, $B \neq 0$, $AB = 0$!
- $AA = A^2$: powers of a matrix. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
- Identity matrix: 1 on the diagonal, 0 elsewhere:

$$I_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Identity because $AI = IA = A$

Trace of a Square Matrix

- $\mathrm{Tr}(A)$: Trace of A = sum of the diagonal elements of A:

$$\mathrm{Tr}\left(\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 1 \\ -1 & 4 & 5 \end{bmatrix}\right) = 1 + 3 + 5 = 9.$$

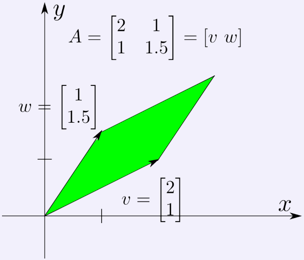
- Invariant to a lot of transformations, used massively in linear algebra.
- Linear:

$$\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B) \quad \mathrm{Tr}(\lambda A) = \lambda\,\mathrm{Tr}(A).$$

- Product: $\mathrm{Tr}(AB) = \mathrm{Tr}(BA) \neq \mathrm{Tr}(A)\,\mathrm{Tr}(B)$.

Determinant of a square matrix

- $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$



- Area of the green parallelogram spanned by v and w

$$\det A = 2 \times 1.5 - 1 \times 1 = 2$$

- Order of vectors matters

$$det \begin{bmatrix} b & a \\ d & c \end{bmatrix} = -\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- $\det[w\ v] = -\det[v\ w]$. Reversing the orientation changes the sign
- if $v = \lambda w$: parallelogram is flat, area is 0.

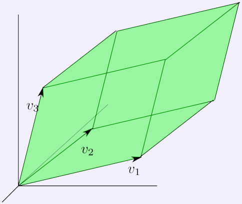
$$\det \begin{bmatrix} a & \lambda a \\ b & \lambda b \end{bmatrix} = \lambda ab - \lambda ab = 0.$$

- A matrix with null determinant is singular.
- Important rule

$$\boxed{det(AB) = \det(BA) = \det(A)\det(B)}$$

Note that
$\det(A+B) \neq \det(A) + \det(B)$

In 3D



$$\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} =$$

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_2b_1c_3 - a_1b_3c_2 - a_3b_2c_1$$

- $v_1 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, v_2 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, v_3 = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$
- $\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \det[v_1, v_2, v_3]$
- Volume of the parallelepiped spanned by v_1, v_2 and v_3.
- Order of the vectors counts!
- If one vector is combination of the others: parallepiped flat, volume is 0.
- Matlab `det` command! - Python Numpy has a similar one (in linalg module). Not limited to 3x3 matrices.

Inverse Matrices

- $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I, \quad BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

  A and B are inverse of each other: $A = B^{-1}$, $B = A^{-1}$.
- A is invertible iff $\det(A) \neq 0$.
- Example: system of equations

$$\begin{cases} 3x + 2y & = 5 \\ 2x + y & = -1 \end{cases} \text{ in matrix form: } \underbrace{\overbrace{\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}}^{C}}_{\det = 3-4=-1\neq 0} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

- Solution

$$C^{-1} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}, \quad C^{-1}C \begin{bmatrix} x \\ y \end{bmatrix} = I \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = C^{-1} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

- Matlab and Python-numpy have functions to invert matrices and solve linear systems.

For Non-Square-Matrices

- Notions of right-inverse or left inverses.
- General construction of the Moore-Penrose Pseudo-Inverse. Works both with matrices with more lines than columns – overdetermined linear systems and the opposite: less lines than columns – underdetermined linear systems.
- `pinv` function in Matlab, `pinv` function in `numpy.linalg` python package.
- Intimately connected to linear least-squares problems and the Singular Value Decomposition.
- In turn intimately connected to eigenvalues and eigenvectors problems (spectral theory) for square matrices.

Outline

1. Vectors
2. Matrices
3. Square Matrices, Trace, Determinant
4. Linear Mappings

Linear Mapping

- Mapping between vectors with only addition of coordinates, multiplications by scalar and no constant terms.
- Example
$$f\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}x+3y\\z-2x\end{bmatrix}$$
- Non linear example
$$g\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}x^2+3yz\\z-2x^2+1\end{bmatrix}$$
There are powers and constant terms.

Linearity

- This means $f(v + \lambda v') = f(v) + \lambda f(v')$

$$
\begin{aligned}
f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \lambda \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}\right) &= f\begin{bmatrix} x + \lambda x' \\ y + \lambda y' \\ z + \lambda z' \end{bmatrix} \\
&= \begin{bmatrix} x + \lambda x' + 3(y + \lambda y') \\ z + \lambda z' - 2(x + \lambda x') \end{bmatrix} \\
&= \begin{bmatrix} x + 3y \\ z - 2x \end{bmatrix} + \lambda \begin{bmatrix} x' + 3y' \\ z' - 2x' \end{bmatrix} \\
&= f\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \lambda f\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}
\end{aligned}
$$

- f is linear.

- Example: Compute the product of

$$A = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix} \text{ and } v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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- We find precisely the value of

$$f\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 3y \\ -2x + z \end{bmatrix} = \begin{bmatrix} x + 3y \\ z - 2x \end{bmatrix}$$

- Each linear mapping can be written that way. Often use the same notation for the matrix and the linear mapping.

Matrices /linear mappings as geometric transformations

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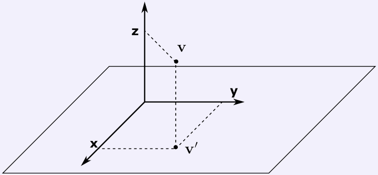
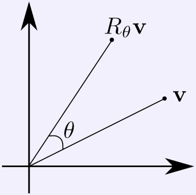
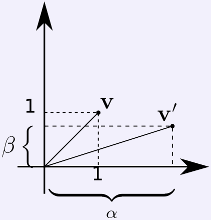
Projection on $x - y$ plane

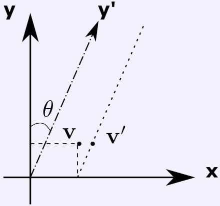




Rotation of angle θ



anisotropic scaling



shear

- projection $\mathbb{R}^3 \to \mathbb{R}^2$

$$F \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Rotation of angle θ from $\mathbb{R}^2 \to \mathbb{R}^2$:

$$R_\theta \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}, \quad R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- Scaling by a factor α in x and β in y:

$$S \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha x \\ \beta y \end{bmatrix}, \quad S = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

- Shear of the y-axis with angle θ:

$$\mathcal{S} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + \sin\theta y \\ y \end{bmatrix}, \quad \mathcal{S} = \begin{bmatrix} 1 & \sin\theta \\ 0 & 1 \end{bmatrix}$$

Eigenvalues/vectors of a Square Matrix

- $A = \begin{bmatrix} 11 & 27 \\ -4 & -10 \end{bmatrix}$. $\quad A\begin{bmatrix} 9 \\ -4 \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \end{bmatrix}$. $\quad A\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$

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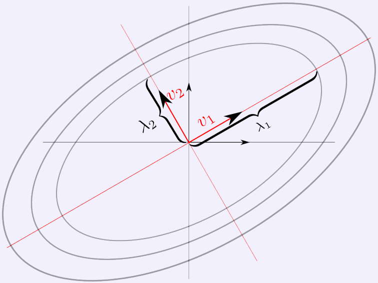
- Any vector of $\mathbb{R}^2$ can be written as $\alpha[9, -4]^\top + \beta[-3, 1]^\top$. Action of A:

$$A\left(\alpha[9, -4]^\top + \beta[-3, 1]^\top\right) = -\alpha[9, -4]^\top + 2\beta[-3, 1]^\top.$$

Case of Symmetric Matrices

- Eigenvectors for different eigenvalues are orthogonal. Can be chosen with norm 1.
- Eigenvectors + eigenvalues: Linear“elliptic–like” scaling.



- Can be interpreted as “Rotate, scale in each axis direction, then rotate back”.

Matrix Rank

- Linear mapping $f : \mathbb{R}^n \to \mathbb{R}^m$, A matrix associated. Rank of A = dimension of the image of f, i.e. dimension of the set made of the $f(x_1, \dots, x_n)$.
- Rank of projection F above: 2: all vectors of $\mathbb{R}^3$ are projected on the plan of vectors $[x, y, 0]^T$.
- Take $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. A has rank 1.

$$f(x, y) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 2x + 4y \end{bmatrix}$$

  All the values of f belong to the line $y = 2x$, dimension 1 subspace of $\mathbb{R}^2$.
- The square matrix A, says size $n \times n$, is invertible is its rank is n.
- For a square matrix, its rank is the number of non-zeros eigenvalues.

Meaning of the Product

- M and N the linear mappings $\begin{bmatrix} 2 & 2 \\ 1 & 3 \\ 1 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix}$
- Apply N to $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and M to the result:

$$N\mathbf{v} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 3y \\ -2x + z \end{bmatrix}$$

- and

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$$\begin{bmatrix} 2 & 2 \\ 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x + 3y \\ -2x + z \end{bmatrix} = \begin{bmatrix} -2x + 6y + 2z \\ -5x + 3y + 3z \\ 3x + 3y - z \end{bmatrix} =$$

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Matrix Product as Chain Application of Linear Mappings

- We found that

$$M(N\mathbf{v}) = \underbrace{MN}_{\text{Matrix product}} \mathbf{v}$$

- Very Important Property: Matrix product corresponds to chain application (composition) of linear mappings!

So far

- We talked of vectors, vector spaces, dimension, inner products
- Matrices, operations on them, transposition, product of matrices,
- Square matrices and their algebra, symmetric matrices, Traces, determinants.
- linear mappings, ranks, eigenvalues/vectors.
- matrix products and composition of linear mappings.

Read the Linear Algebra Tutorial and Reference on Absalon!

This will also be useful for other courses!