Shape and geometry
Signal and Image Processing 2014

1. **Experiments on translations**

   1.1. Let $I(i,j)$ be a 2D image. Consider the image $\tilde{I}$ obtained by translation of I by one pixel to the right. Express what is $\tilde{I}(i,j)$ with respect to I. Write the matrix filter corresponding to this transformation.

   1.2. Create a zero-valued image of odd size in x and y containing a centered white square. Write a code to translate this image from any specified number of pixels in both directions. Discuss on the different boundary conditions that you can put and what they do.

   1.3. Remind what is the Fourier transform of a translated image with respect to the Fourier transform of the original image. Write a second MATLAB code that performs translation based on Fourier transform and compare outputs with the previous approach.

   1.4. Can you do non-integer translations with the Fourier method ? Comment on what happens for the previous white square image. Apply it also on other images.

2. **Procrustes transformations**

   2.1. Consider two sets of points $(x_i, y_i)_{i=1,..,N}$ and $(x'_i, y'_i)_{i=1,..,N}$ belonging to two different images. We assume that for all i, these points are located on corresponding objects and intend to find the 2D Procrustes transformation (translation, rotation and scaling) that maps the source points on the target ones. How many free parameters are there in this problem and what should be the minimum value of N that one would need to determine these parameters ?

   2.2. Load the two images *westconcordaerial.png*, *westconcordorthophoto.png* and pick up some corresponding points between the two of them (*getpts* function may be useful). Using MATLAB *cp2tform* function, write what transformations parameters for the Procrustes transformation you find. Then apply this transformation to the first image (use *imtransform* function) and compare the result with the target image.

3. **Bonus questions : optimal rotation for Procrustes alignment**
   The following questions aim at proving the results of (7.23) for the rotation solution in Procrustes alignment (the translation and scaling derivation being quite straightforward).

   3.1. Denoting X and Y the distribution matrices, we wish to minimize the functional $R \mapsto \mathrm{Tr}((RX-Y)^T(RX-Y))$ over the set of all $N \times N$

rotation matrices (in practice $N = 2$ or $N = 3$). Using the cyclic property of the trace and its invariance to transposition, show that this is equivalent to maximize $\mathrm{Tr}(RXY^T)$.

3.2. Based on the SVD decomposition, we write $XY^T = USV^T$ where U and V are both orthonormal $N \times N$ matrices and S is a diagonal matrix with strictly positive diagonal coefficients. Show that the quantity to maximize becomes $\mathrm{Tr}(HS)$ where $H \doteq V^T RU$. What can you say about the matrix H ?

3.3. Develop explicitely $\mathrm{Tr}(HS)$ with respect to the coefficients of H and S and show that the maximum is reached only for $H = \mathrm{I}_N$. Conclude.

3.4. What is the implicit assumption on the rank of XY^T to actually have all non-zero diagonal coefficients in S ? What does it imply on the number of points N for instance ? What happens if this assumption is no more satisfied ?

4. **Affine and projective alignments**

4.1. Explain briefly and in your own words the interest of homogeneous coordinates for general Procrustes, affine or projective alignment.

4.2. Consider a first distribution X given by the 4 points $\{(0,0),(1,0),(1,1),(0,1)\}$ in the 2D plane and a second one Y given by the corresponding 4 points $\{(0.6,0.6),(1.3,0.8),(1,1.6),(0.4,1)\}$. Show these two quadrilaterals on a common figure. Is there a Procrustes transformation mapping exactly the points of X on the points of Y and why ? Show what the procrustes alignment function gives for this example.

4.3. Same question for affine transformations.

4.4. Consider now a general 2D projective transform as presented on page 181. How many degrees of freedom does it have and is it thus possible to find one that matches points of X to the ones of Y ? Display the solution provided by MATLAB for this projective mapping.