A collection of Calculus of Variations exercises

June 16, 2024

Chapter 1

Scalar Case

Exercise 1

Let

$$I[u] = \int_a^b 2u'(x)^3 dx.$$

Find a certain function $u : [a, b] \to \infty$ that is a minimum of the given integral functional.

Solution

$$F(x, u(x), u'(x)) = F(u'(x)) = 2u'(x)^3.$$

We can rewrite the integrand as

$$F(u'(x)) = F(p) = 2p^3$$

From the Euler-Lagrange equations:

$$\frac{d}{dx}\frac{\partial F}{\partial u'}(u'(x)) = \frac{\partial F}{\partial p}(p) = \frac{\partial F}{\partial u}(p)$$

we have

$$\frac{d}{dx}\frac{\partial F}{\partial p}(p) = 0.$$

Since

$$\frac{\partial F}{\partial p}(p) = \frac{\partial(2p^3)}{\partial p} = 6p^2 = 6u'(x)^2$$

then

$$\frac{d}{dx}\frac{\partial F}{\partial u'}(u'(x)) = 12u'(x)u''(x).$$

We obtain an ordinary differential equation of the second order

$$12u'(x)u''(x) = 0$$

$$u'(x) = C_1$$

$$u(x) = C_2 x.$$

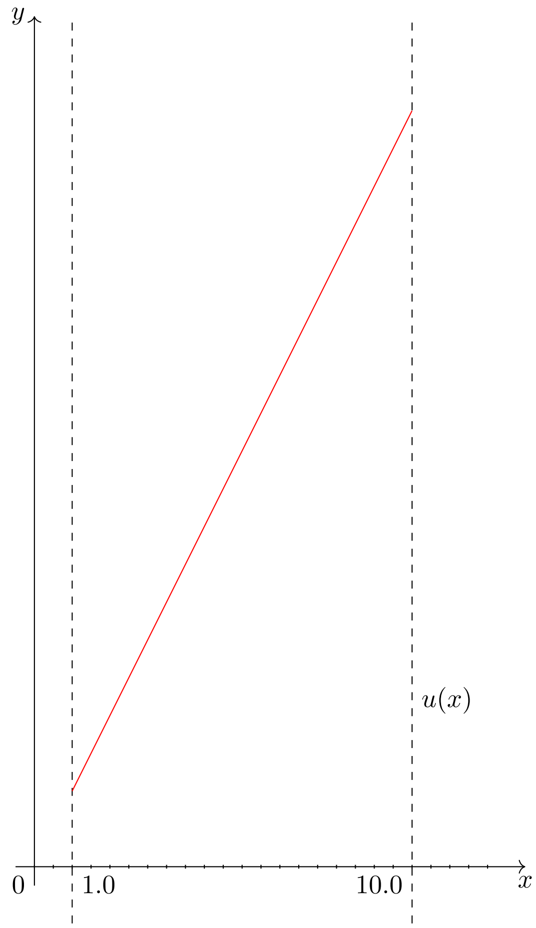
We now assume $a = 1$ and $b = 10$.

$$\begin{cases} u(x) = C_2 x \\ u'(x) = C_1 \\ u(1) = 2 \\ u'(10) = 4 \end{cases}$$

$$C_1 = 4$$

$$C_2 = 2$$

$$u(x) = 2x$$

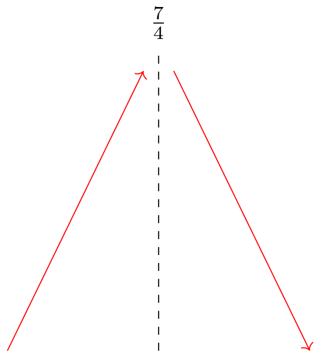

y
u(x)
0
1.0
10.0
x

Exercise 2

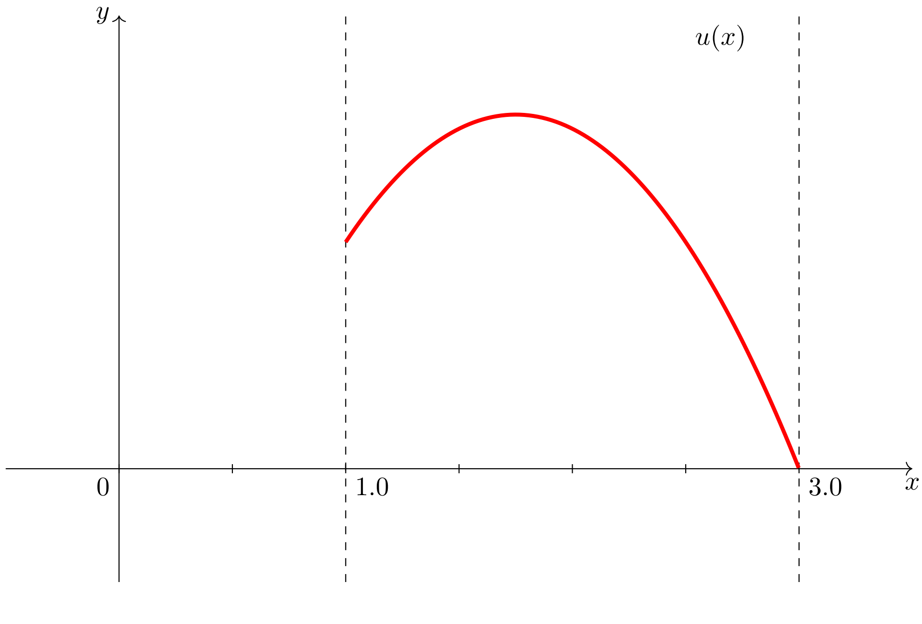
Let
$$I[u] = \int_a^b 4x(u'(x)) + u'(x)^2 dx.$$
Find a certain function $u : [a, b] \to \infty$ that is a minimum of the given integral functional.

Solution

We have
$$F(x, u(x), u'(x)) = 4x(u'(x)) + u'(x)^2$$
that can be rewritten as
$$F(x, y, z) = 4xz + z^2.$$
Now we can apply the Euler-Lagrange equations:
$$\frac{d}{dx}\frac{\partial F}{\partial z}(z) = \frac{\partial F}{\partial y}(z).$$
Since $\frac{\partial F}{\partial y}(z) = 0$, we obtaint the following equation:
$$\frac{d}{dx}\frac{\partial F}{\partial z}(x, z) = 0.$$
Since
$$\frac{\partial F}{\partial z} = 4x + 2z,$$
then
$$\frac{d}{dx}\frac{\partial F}{\partial z} = 4 + 2u''(x).$$
Finally we obtain the following ODE:
$$4 + 2u''(x) = 0$$
$$u''(x) = -2$$
integrating
$$u'(x) = -2x + C_1$$
$$u(x) = -x^2 + C_1 x + C_2.$$
If we assume $a = 1$, $b = 3$ and the boundary conditions
$$u(a) = u(1) = 1,$$
$$u(b) = u(3) = 0,$$
we obtain
$$C_1 = \frac{7}{2}$$
$$C_2 = \frac{-3}{2}.$$
$$u(x) = -x^2 + \frac{7}{2}x - \frac{3}{2}$$
$$u(1) = 1$$
$$u(3) = 0.$$
$$\frac{d}{dx}u(x) = -2x + \frac{7}{2}$$
$$\frac{d}{dx}u(x) \geq 0$$
for
$$x \leq \frac{7}{4}.$$



$\frac{7}{4}$



y
$u(x)$
0
1.0
3.0
x

Exercise 3

Let $v \in C_c^\infty([-1,1],\mathbb{R})$ given by

$$v(x) = e^{-\frac{1}{(1-x^2)}}.$$

Determine the value of

$$\int_a^b u(x) \cdot v'(x)dx$$

in the case

1. $u(x) = x$;
2. $u(x) = 1$;
3. $u(x) = 10$.

Solution

Case 1

$$\int_{-1}^1 u(x)v'(x)dx = \int_{-1}^1 xv'(x)dx =?$$

$$v'(x) = [e^{-\frac{1}{(1-x^2)}}]' = e^{-\frac{1}{(1-x^2}} \cdot (\frac{-2x}{(1-x^2)^2}) = -\frac{2x}{(1-x^2)^2}e^{-\frac{1}{(1-x^2)}}.$$

Then

$$\int_{-1}^1 u(x)v'(x)dx = \int_{-1}^1 x(-\frac{2x}{(1-x^2)^2}e^{-\frac{1}{(1-x^2)}}dx = \int_{-1}^1 -\frac{2x^2}{(1-x^2)^2}e^{-\frac{1}{(1-x^2)}}dx = \int_{-1}^1 x[e^{-\frac{1}{(1-x^2)}}]'dx = \star$$

we need an integration by parts

$$\int fg' = fg - \int gf'$$

$$\star = [xe^{-\frac{1}{(1-x^2)}}]_{-1}^1 - \int_{-1}^1 e^{-\frac{1}{(1-x^2)}}dx = \star$$

the integral is equal to zero, then

$$\star = (e^0 - e^0) = 2 \neq 0.$$

Case 2

$$\int_{-1}^1 u(x)v'(x)dx = \int_{-1}^1 1 \cdot (-\frac{2x}{(1-x^2)^2}e^{-\frac{1}{(1-x^2)}})dx = \int_{-1}^1 [-\frac{2x}{(1-x^2)^2}e^{-\frac{1}{(1-x^2)}}dx$$

$$= \int_{-1}^1 [e^{-\frac{1}{(1-x^2)}}]'dx = [e^{-\frac{1}{(1-x^2)}}]_{-1}^1 = (e^0 - e^0) = 1 - 1 = 0.$$

Case 3

$$\int_{-1}^1 u(x)v'(x)dx = \int_{-1}^1 10(-\frac{2x}{(1-x^2}e^{-\frac{1}{(1-x^2)}})dx = \int_{-1}^1 10[e^{-\frac{1}{(1-x^2)}}]'dx = 10[e^{-\frac{1}{(1-x^2)}}]_{-1}^1 = 10(e^0 - e^0) = 0.$$

Exercise 4

Find, if it exists, the minimum of the following functional:

$$I[u] = \int_{-1}^{1} x^4 u'(x)^2 dx,$$

such that $u(-1) = -1$, $u(1) = 1$, where $u : [-1, 1] \to \mathbb{R}$.

Solution

We have that

$$\int_{-1}^{1} x^4 u'(x)^2 dx \geq 0$$

since

$$x^4 u'(x)^2 \geq 0 \qquad \text{for} \qquad x \in [-1, 1].$$

Then

$$I[u] \geq 0$$

that is the functional is lower bounded by zero. We can rewrite

$$F(x, u(x), u'(x)) = F(x, y, z) = x^4 z^2.$$

Now we apply the Euler-Lagrange equations:

$$\frac{d}{dx}\frac{\partial F}{\partial z}(x, z) = \frac{\partial F}{\partial y}(y).$$

Since $\frac{\partial F}{\partial y}(y) = 0$, we obtain

$$\frac{\partial F}{\partial z}(x, z) = x^4 2z$$

$$\frac{d}{dx}(x^4 2z) = 0$$

$$4x^3 2u''(x) = 0.$$

Integrating we obtain

$$u(x) = -\frac{A}{x^3} + B,$$

and considering the boundary conditions:

$$u(-1) = -1 \qquad u(1) = 1,$$

$$\begin{cases} A + B = -1 \\ B - A = 1 \end{cases}$$

we obtain

$$A = -1 \qquad B = 0.$$

Then

$$u(x) = \frac{1}{x^3}.$$

Now let's do a function study.

$$u(x) = \frac{1}{x^3} \qquad x \in [-1, 1].$$

horizontal asymptotes

$$\lim_{x \to 1} u(x) = 1 \qquad \lim_{x \to -1} u(x) = -1,$$

there are not horizontal asymptotes.

vertical asymptotes

$$\lim_{x\to 0^{\pm}} u(x) = \lim_{x\to\pm} \frac{1}{x^3} = \pm\infty,$$

then $x = 0$ is a vertical asymptote.

first derivative

$$u'(x) = [x^{-3}]' = -3x^{-3} = -\frac{3}{x^4}$$

$$u'(x) \geq 0 \qquad -\frac{3}{x^4} \geq 0 \qquad \nexists x \in [-1,1] \qquad \text{s.t.} \qquad u'(x) \geq 0$$

$$\Longrightarrow u'(x) < 0 \qquad \forall x \in [-1,1].$$

Then the function is strictly decreasing.

second derivative

$$u''(x) = [-3x^{-4}]' = 12x^{-5} = \frac{12}{x^5}$$

$$u''(x) \geq 0 \qquad \frac{12}{x^5} \geq 0 \qquad \text{for} \qquad x > 0.$$

Now we know that $u \notin C^1([-1,1])$ since it diverges around the origin.

We now can show if there exists a minimizing sequence. We consider $0z\epsilon < 1$ and a function

$$v_\epsilon(x) = \begin{cases} -1 & \text{if} \quad x \leq -\epsilon \\ \frac{x}{\epsilon} & \text{if} \quad -\epsilon < x < \epsilon \\ 1 & \text{if} \quad x \geq \epsilon \end{cases}$$

The functional becomes

$$I[v_\epsilon] = \int_{-1}^{1} x^4 v_\epsilon'^2(x)dx = \int_{-1}^{-\epsilon} x^4(-1)^2 dx = \int_{-\epsilon}^{\epsilon} x^4 (\frac{1}{\epsilon})^2 dx + \int_{\epsilon}^{1} x^4 dx$$

$$[\frac{x^5}{5}]_{-1}^{-\epsilon} + \frac{1}{\epsilon^2}[\frac{x^5}{5}]_{-\epsilon}^{\epsilon} + [\frac{x^5}{5}]_{\epsilon}^{1} = \frac{2}{5}\epsilon^3.$$

Then

$$I[v_\epsilon] = \frac{2}{5}\epsilon^3.$$

For $\epsilon \to 0^{\pm}$, we obtain the function

$$v_0(x) = \begin{cases} -1 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x > 0. \end{cases}$$

This function is not continuous and not even of class $C^1([-1,1])$. The fact that the functional is lower bounded does not insure the existence of the minimum.