

A collection of Real Analysis exercises

June 6, 2024

Chapter 1

Metric Spaces

Exercise 1

Let the distances in \mathbb{R}^2 :

1. $d_1(X, Y) = \sum_{i=1}^2 |y_i - x_i| = |y_1 - x_1| + |y_2 - x_2|$;
2. $d_2(X, Y) = \sqrt{\sum_{i=1}^2 |y_i - x_i|^2} = \sqrt{|y_1 - x_1|^2 + |y_2 - x_2|^2}$;
3. $d_\infty(X, Y) = \max_{i=1,2} |y_i - x_i| = \max\{|y_1 - x_1|, |y_2 - x_2|\}$.

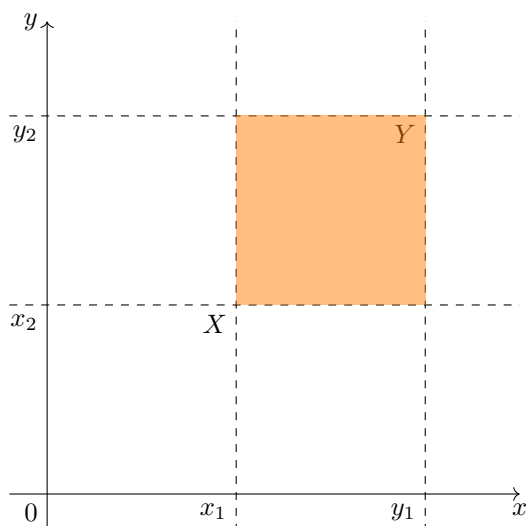
Construct the open balls related to these distances.

Solution

Point 1

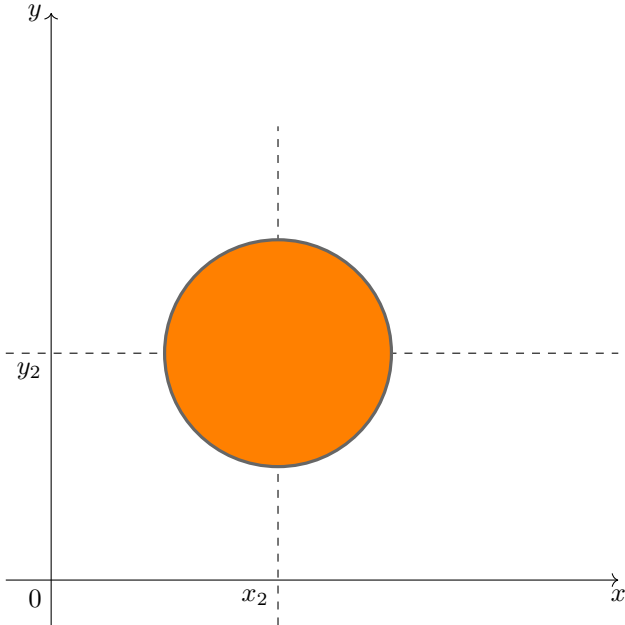
$$\begin{aligned} d_1(X, Y) &< r \\ \text{iff} \\ \sum_{i=1}^2 |y_i - x_i| &< r \\ \text{iff} \\ |y_1 - x_1| + |y_2 - x_2| &< r \end{aligned}$$

Point 2

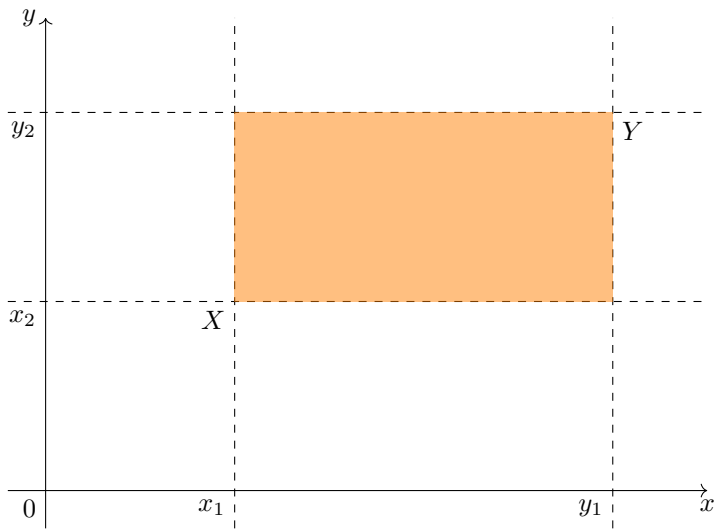


$$\begin{aligned} d_2(X;Y) &< r \\ \text{iff} \\ \sqrt{|y_1 - x_1|^2 + |y_2 - x_2|^2} &< r \\ \text{iff} \\ |y_1 - x_1|^2 + |y_2 - x_2|^2 &< r^2 \end{aligned}$$

Point 3



$$\begin{aligned} d_\infty(X,Y) &< r \\ \text{iff} \\ \max\{|y_1 - x_1|, |y_2 - x_2|\} &< r \end{aligned}$$



Exercise 2

Let (X, d) a metric space, $A \subset X$ not empty. Show if the following statements are true or false:

1. A open $\implies \overset{\circ}{A} \cap \partial A = \emptyset$;
2. if $\overset{\circ}{A} \cap \partial A = \emptyset \implies A$ closed;

Solution

Point 1

The first statement is always true. It is also true if A is closed or if A is neither open or closed.

$$\begin{aligned} \partial A &= \{x \in X \text{ s.t. } x \notin \overset{\circ}{A} \text{ and } x \in \overset{\circ}{X \setminus A}\} \\ &\implies \overset{\circ}{A} \cap \partial A = \emptyset \text{ is true.} \end{aligned}$$

Point 2

The second statement is false. Counterexample:

$$A =]0, 1[$$

is an open set with

$$\partial A = \{0, 1\}$$

and we have that $\overset{\circ}{A} \cap \partial A = \emptyset \implies A$ is closed is false because A is open.

Exercise 3

Let (X, d) a metric space $A \subset X$ closed and $A \neq \emptyset$. Furthermore let

$$f : X \rightarrow \mathbb{R}$$

with

$$f(x) = \inf_{a \in A \setminus \{x\}} d(x, a).$$

Tell whether the following statements are true or false.

1. $x \in A \implies f(x) = 0$;
2. $f(x) = 0 \implies x \in A$;

Solution

Point 1

This statement is not true $\forall x$. Counterexample:

$$A = [0, 1] \cup \{2\}$$

we have that $2 \in A$, but $f(2) \neq 0$ because $f(2) = \inf d(2, a)$ with $a \in [0, 1]$. If $a \in [0, 1]$ the distance of 2 from a is greater (or equal) than the distance of 2 from 1.

$$\text{If } a \in [0, 1]$$

$$\text{then } d(2, a) \geq d(2, 1) = 1$$

so that

$$\text{if } d(2, a) \geq 1 \implies \forall a \in [0, 1] \quad d(2, a) \geq 1.$$

Then

$$\inf d(2, a) \geq 1 \quad a \in [0, 1]$$

and it can't be equal to zero.

Point 2

Remember that x is an accumulation point for A iff $\inf_{a \in A \setminus \{x\}} d(x, a) = 0$.

$$f(x) = 0 \implies x \text{ is an accumulation point for } A$$

$$\text{then } x \in \overline{A} = A \quad \text{since } A \text{ is closed}$$

then the second statement is true.

Exercise 4

Let $X = C^0([0, 1])$, $d_\infty(f, g) = \sup_{t \in [0, 1]} |f(t) - g(t)|$ and $d_2(f, g) = \sqrt{\int_0^1 |f(t) - g(t)|^2 dt}$. Show that d_2 and d_∞ are not equivalent.

Solution

If we choose $f_n(t) = t^n \quad \forall n \in \mathbb{N}$, we have that:

$$d_\infty(f_n, 0) = \sup_{t \in [0, 1]} |f_n(t)| = \sup_{t \in [0, 1]} |t^n| = 1.$$

It is a maximum.

$$(d_2(f_n, 0))^2 = \int_0^1 |f_n(t)|^2 dt = \int_0^1 (t)^{2n} dt = \left[\frac{t^{2n+1}}{2n+1} \right]_0^1 = \frac{1}{2n+1}$$

then

$$d_2(f_n, 0) = \frac{1}{\sqrt{2n+1}}$$

for $n \rightarrow \infty$, $d_2 \rightarrow 0$. So that

$$\forall c \in \mathbb{R} \exists n \in \mathbb{N} \quad \text{s.t.} \quad d_\infty(f_n, 0) > c d_2(f_n, 0)$$

then d_2 and d_∞ are not equivalent.

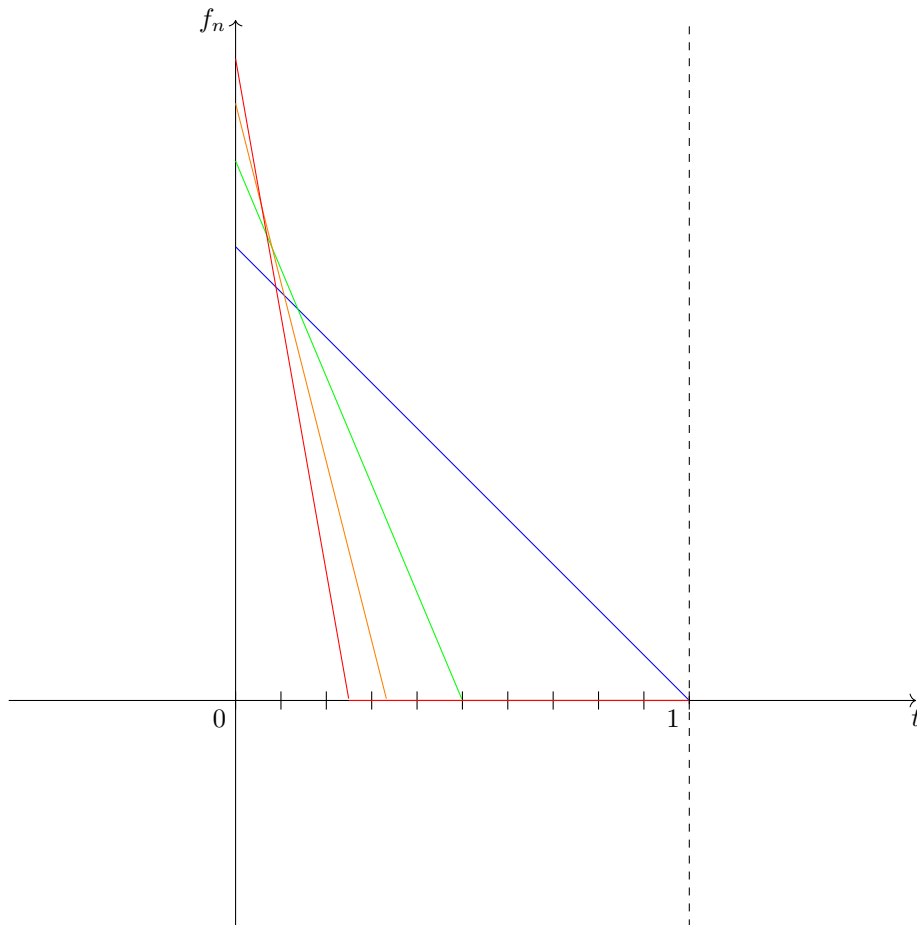
Exercise 5

Let $X = C^0([0, 1])$. Show that the open ball $B_{d_2}(0, 1)$ is unbounded with respect to d_∞ .

Solution

Consider the following sequence of functions:

$$f_n(t) = \begin{cases} -n^{\frac{5}{4}}(t - \frac{1}{n}) & \text{for } t \in [0, \frac{1}{n}] \\ 0 & \text{for } t \in [\frac{1}{n}, 1] \end{cases}$$



$$d_2(f, g) = \sqrt{\int_0^1 |g(t) - f(t)|^2 dt}$$

$$d_\infty(f_n, 0) = \sup_{t \in [0, 1]} |f_n(t)| = f_n(0) = \sqrt[4]{n} \rightarrow \infty$$

$$(d_2(f_n, 0))^2 = \int_0^{\frac{1}{n}} -n^{\frac{5}{2}}(t - \frac{1}{n}) dt = [-\frac{n^{\frac{5}{2}}}{3}(t - \frac{1}{n})^3]_0^{\frac{1}{n}} = \frac{n^{\frac{5}{2}}}{3} \frac{1}{n^3} = \frac{1}{3\sqrt{n}}$$

then

$$d_s(f_n, 0) = \frac{1}{\sqrt{3}\sqrt{n}}.$$

With respect to d_∞ the function is unbounded because contains a sequence the goes to infinity.

Exercise 6

Say if $[0, +\infty[$ is bounded in (\mathbb{R}, d_0) and in (\mathbb{R}, d) , with

- d_0 the discrete metric;
- d the Euclidean metric;

Solution

The discrete metric is characterized by the fact that the distance between two points is equal to zero or one.

$$d_0(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

so that

$$\text{diam}([0, +\infty[) = \sup_{x, y \in [0, +\infty[} d_0(x, y) \leq 1,$$

then $[0, +\infty[$ is bounded in (\mathbb{R}, d_0) .

If we consider the Euclidean distance

$$\text{diam}([0, +\infty[) = \sup_{x, y \in [0, +\infty[} d(x, y) = \sup_{x, y \in [0, +\infty[} |y - x| \geq n \quad \forall n \in \mathbb{N},$$

then $\text{diam}([0, +\infty[) = +\infty$, so that $[0, +\infty[$ is unbounded with d .

Exercise 7

Let (X, d) a metric space, $f : X \rightarrow \mathbb{R}$ a continuous function and $A \subset X$ bounded. Say if the following statements are true or false.

1. $f(A)$ is connected;
2. $f(A)$ is compact;
3. $f(A)$ is open;

Solution

Point 1

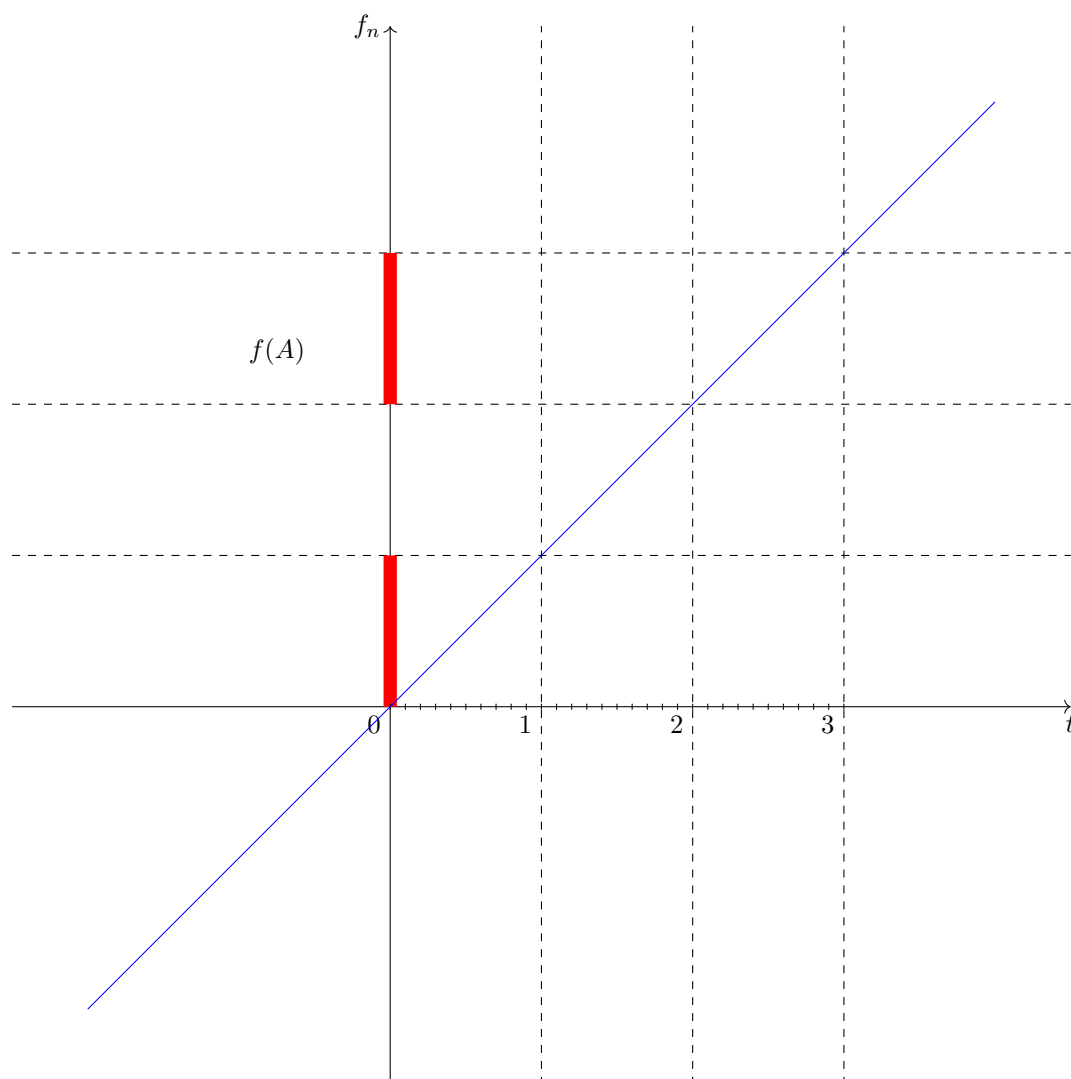
Consider $f : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x) = x$$

and consider $A = [0, 1] \cup [2, 3]$, there is no request on A , so that

$$f(A) = [0, 1] \cup [2, 3]$$

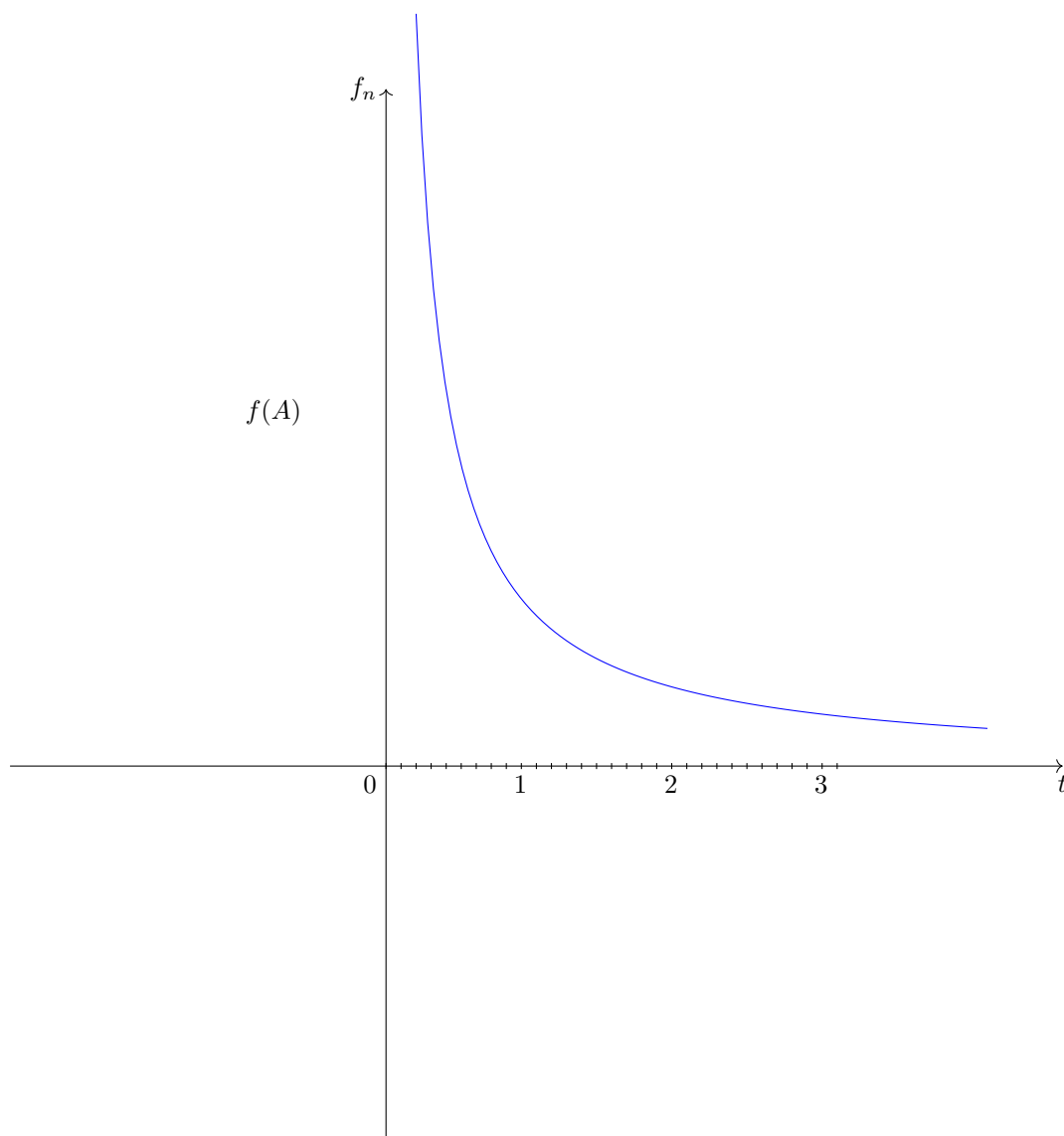
is not connected. **Point 2**



Consider $f :]0, +\infty[\rightarrow \mathbb{R}$, given by

$$f(x) = \frac{1}{x}$$

If we choose $A =]0, 1]$ we have $f(A) = [1, +\infty[$ that is not compact because it is not bounded.



Point 3

If we consider $f : \mathbb{R} \rightarrow \mathbb{R}$ with $f(x) = 1$,

$$A = [0, 1]$$

$$f(A) = \{1\} \quad \text{that is closed.}$$

Then all the three statements are false.

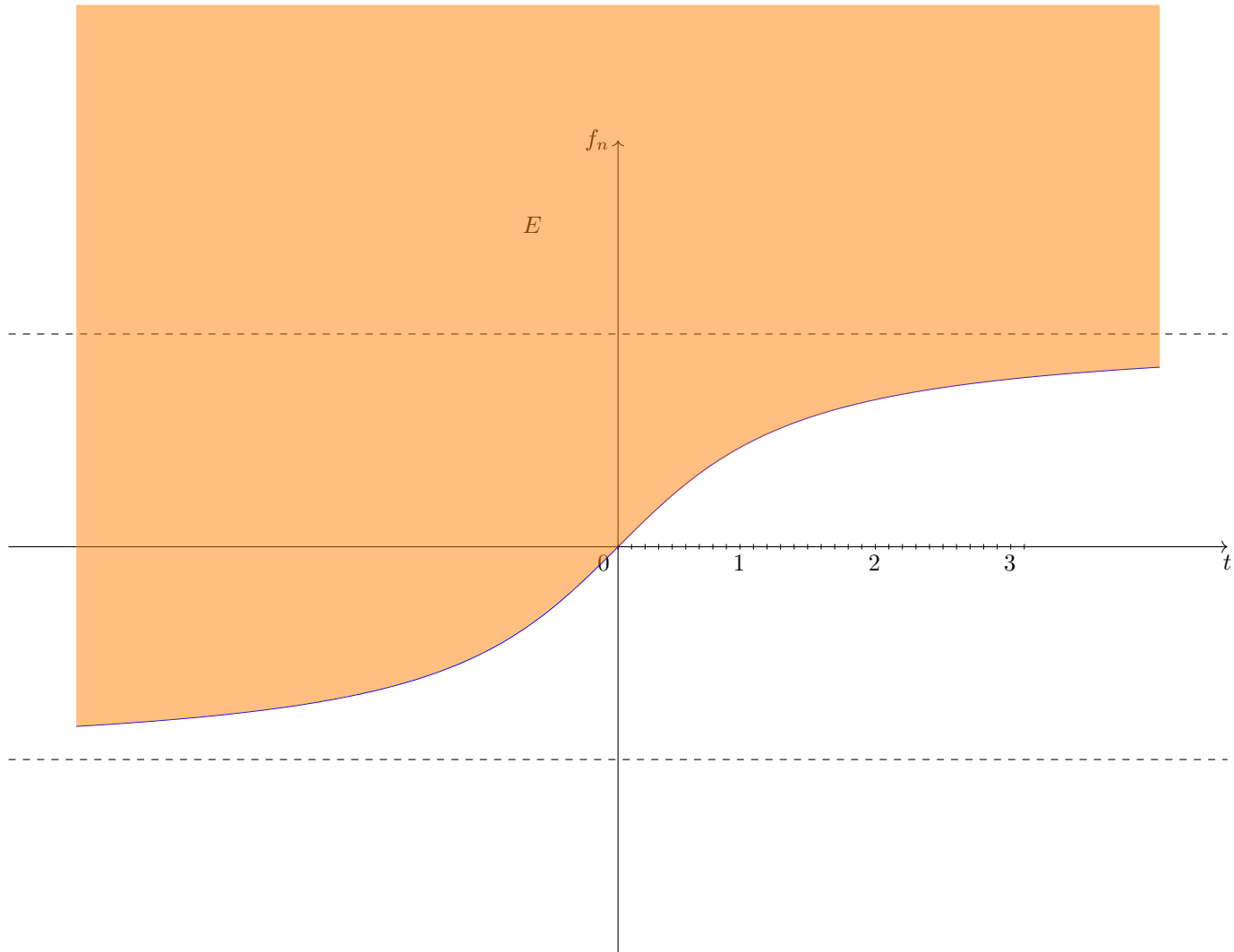
Exercise 8

Let $X = \mathbb{R}^2$ with the Euclidean distance. Say if the set

$$E = \{(x, y) \in \mathbb{R}^2 \quad \text{s.t.} \quad y \geq \arctan x\}$$

is complete and if it is compact.

Solution



We can see that E is not bounded so it is not compact. Now we see if it is complete. Consider

$$f(x, y) = y - \arctan(x),$$

we have

$$E = f^{-1}([0, +\infty[).$$

It is a contrainage of a closed set, then E is closed. We know that a closed subset of a complete metric space is complete. Since \mathbb{R}^2 is complete, then E is complete.

Exercise 9

Let $X = \mathbb{R}^2$ with the Euclidean distance and let

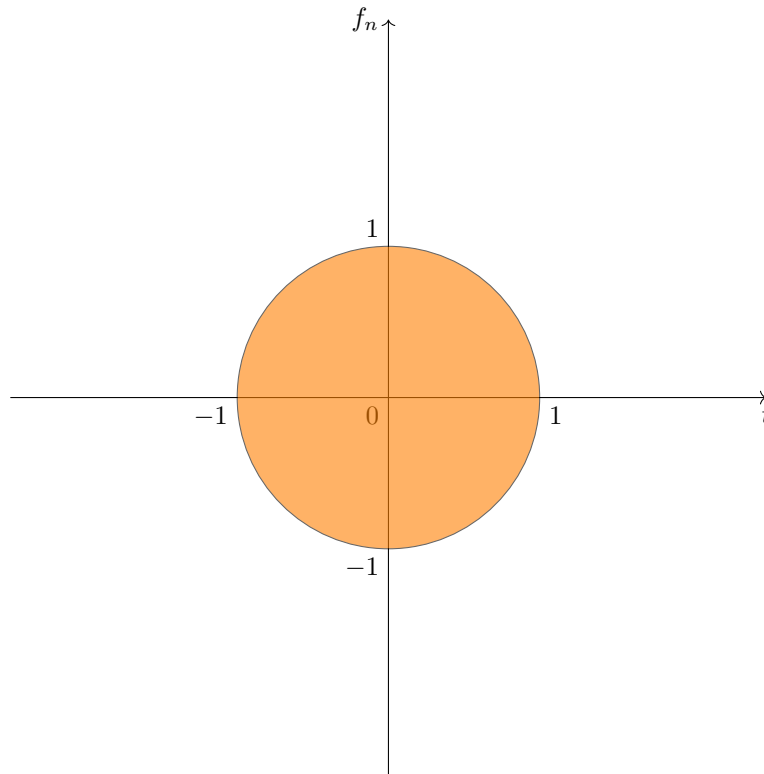
$$A = [0, 1] \times [0, +\infty[$$

$$B = \{(x, y) \in \mathbb{R}^2 \quad \text{with} \quad x^2 + y^2 < 1\}.$$

Say if A and B with the Euclidean distance are complete.

Solution

A is closed and $A \subset \mathbb{R}^2$ that is complete with the Euclidean distance. Then A is complete. B is open,



we need to find a Cauchy sequence in B that doesn't converge in B . Consider

$$x_n = \left(1 - \frac{1}{n}, 0\right) \quad x_n \in B \quad \forall n.$$

We have that

$$x_n \rightarrow (1, 0) \quad \text{in} \quad \mathbb{R}^2,$$

but $(1, 0) \notin B$. We have a sequence that converge in the space, that is a Cauchy sequence, but that doesn't converge in B . Then B is not complete.

Exercise 10

Let (X, d) a metric space and x_n a sequence of elements of X . Say if the following statements are true or false.

1. $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0 \implies x_n$ bounded;
2. x_n convergent $\implies \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$;
3. x_n Cauchy $\implies \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$.

Solution

Point 1

The fact that $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$ doesn't imply that x_n is bounded. Counterexample:

$$x_n = \log n$$

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = \lim_{n \rightarrow \infty} |\log n + 1 - \log n| = \lim_{n \rightarrow \infty} \left| \log \frac{n+1}{n} \right| = 0.$$

The distances between two consecutive elements become shorter but x_n is not bounded.

$$\sup_{n \in \mathbb{N}} \{\log n, \quad n \in \mathbb{N}\} = +\infty.$$

Then the first statement is false.

Point 2

If x_n is convergent then there exists $x_\infty \in X$ s.t. :

$$\lim_{n \rightarrow \infty} x_n = x_\infty.$$

Then

$$0 \leq d(x_n, x_{n+1}) \leq d(x_n, x_\infty) + d(x_\infty, x_{n+1}),$$

since $d(x_n, x_\infty) \rightarrow 0$, $d(x_\infty, x_{n+1}) \rightarrow 0$, then by "the two carabinieri theorem" we have that $d(x_n, x_{n+1}) \rightarrow 0$.

Point 3

x_n is a Cauchy sequence iff

$$\forall \epsilon > 0 \exists \nu \in \mathbb{N} \text{ s.t. } \forall n, m \geq \nu \in \mathbb{N} \implies d(x_n, x_m) < \epsilon.$$

If we fix n , then m can be very far, so is stronger the Cauchy condition with respect to (x_n, x_{n+1}) , so that:

$$\forall \epsilon > 0 \exists \nu \in \mathbb{N} \text{ s.t. } \forall n > \nu d(x_n, x_{n+1}) < \epsilon \implies \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0.$$

Exercise 11

Determine as α varies the limit in (\mathbb{R}^2, d_2) of the sequence:

$$x_n = \left(\frac{1}{n}, (-1)^n \frac{n^\alpha - 1}{n^2}\right).$$

If the limit doesn't exist find eventual convergent subsequences.

Solution

A sequence in \mathbb{R}^2 is like having two sequences in \mathbb{R} :

$$a_n = \frac{1}{n} \rightarrow 0$$

$$b_n = (-1)^n \frac{n^\alpha - 1}{n^2} = (-1)^n n^{\alpha-2} + \frac{(-1)^n}{n^2} \rightarrow 0.$$

- If $\alpha - 2 < 0$ then $b_n \rightarrow 0$;
- if $\alpha = 2$ then $b_n \rightarrow \nexists$;
- if $\alpha - 2 > 0$ then $b_n \rightarrow \nexists$.

Then if $\alpha < 2$ then

$$\lim_{n \rightarrow \infty} x_n = (0, 0)$$

if $\alpha \geq 2$

$$\lim_{n \rightarrow \infty} x_n = \nexists.$$

If we consider $\alpha > 2$ since $|b_n| \rightarrow +\infty$ there aren't convergent subsequences. If $\alpha = 2$ we have the subsequence of even indices

$$x_n = \left(\frac{1}{n}; \frac{n^2 + 1}{n^2}\right) \rightarrow (0, 1),$$

the subsequence of odd indices:

$$x_n = \left(\frac{1}{n}; -\frac{n^2 + 1}{n^2}\right) \Rightarrow (0, -1).$$

Exercise 12

Let (X, d) a metric space, $A \subseteq X$, $A \neq \emptyset$, x_n a sequence in A that converges to $x_\infty \in X$. Say if the following statements are false or true.

1. x_∞ is an accumulation point for A ;
2. $x_\infty \in \overline{A}$;
3. $x_\infty \in \overset{\circ}{A}$;
4. $x_\infty \in \partial A$.

Solution

Point 1

The first statement is false. Counterexample:

$$A = [0, 1] \cup \{2\}$$

$$x_n = 2 \quad \text{constant}$$

$$\lim_{n \rightarrow \infty} x_n = x_\infty = 2$$

that is not an accumulation point since 2 is an isolated point for A .

Point 2

If x_∞ is an isolated point the statement follows by the point 1, if x_∞ is not an isolated point it will be an accumulation point then the statement trivially follows. So the statement 2 is true.

Point 3

$x_\infty \in \overset{\circ}{A}$ is false. Counterexample:

$$A =]0, 1]$$

$$x_n = \frac{1}{n} \implies x_\infty \rightarrow 0 \notin \overset{\circ}{A}.$$

Point 4

$x_\infty \in \partial A$ is false. Counterexample:

$$A =]0, 1[$$

$$x_n = \frac{1}{2} + \frac{1}{n}$$

$$x_n \rightarrow \frac{1}{2}$$

that is not in ∂A .

Exercise 13

Show that the metric space $(C^0([0, 1]), d_\infty)$ is not compact.

Solution

Consider

$$f_n(t) = t^n \quad t \in [0, 1]$$

suppose that there is a subsequence that converges to $f \in C^0([0, 1])$:

$$f_{n_k} \rightarrow f,$$

that is

$$d_\infty(f_{n_k}, f) \rightarrow 0, \quad \forall t \in [0, 1].$$

$$|f_{n_k}(t) - f(t)| \leq d_\infty(f_{n_k}, f) \rightarrow 0.$$

If this holds then

$$f_{n_k}(t) \rightarrow f(t) \quad \forall t \in [0, 1]$$

$$f_n(t) \rightarrow g(t) = \begin{cases} 0 & \text{if } t \in [0, 1[\\ 1 & \text{if } t = 1 \end{cases}$$

If the sequence converge to $g(t)$ then also the subsequences tends to $g(t)$, but f_n can't have convergent subsequences because they must to converge to $g(t) \notin C^0([0, 1])$ because $g(t)$ is not continuous. Then $C^0([0, 1])$ is not compact.

Exercise 14

Consider the sequence

$$f_n(t) = \sqrt{\frac{1+n^2t^2}{n}} \quad \text{for } t \in [-1, 1].$$

Show that $f_n \rightarrow f$ with $f(t) = |t|$ with the distance d_∞ . Furthermore deduce that the space $C^1([-1, 1], d_\infty)$ is not complete.

Solution

$$|f_n(t) - f(t)| = \left| \frac{\sqrt{1+n^2t^2}}{n} - |t| \right| = \left| \frac{1}{n\sqrt{1+n^2t^2} + n|t|} \right| \leq \frac{1}{n}$$

Since the denominator is greater or equal to n we have that the fraction is lower or equal to $\frac{1}{n} \quad \forall t \in [-1, 1]$. Then

$$0 \leq d_\infty(f_n, f) \leq \frac{1}{n} \rightarrow 0.$$

Then $f_n \rightarrow f$ is a Cauchy sequence, but $f \notin C^1([-1, 1])$. Then the sequence doesn't converge in $(C^1[-1, 1], d_\infty)$ and so it is not complete.

Exercise 15

Let $X = C^0([0, 1])$ with the distance d_2 . Show that the sequence

$$f_n(t) = \begin{cases} 0 & \text{if } t \in [0, \frac{1}{2} - \frac{1}{n}[\\ \sqrt{2nt + 2 - n} & \text{if } t \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2}] \\ \sqrt{-2nt + 2 + n} & \text{if } t \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{n}[\\ 0 & \text{if } t \in [\frac{1}{2} + \frac{1}{n}, 1] \end{cases}$$

1. converges to the null function in $x = 2$:
2. does not converge to the null function in the metric space (X, d_∞) ;
3. admits limit in (X, d_∞) .

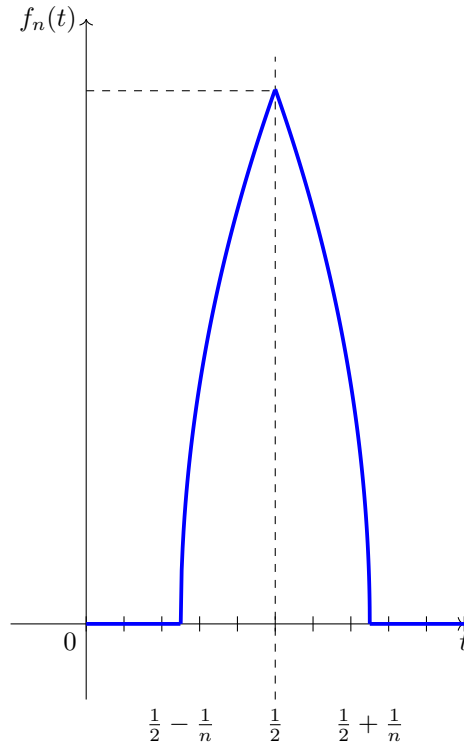
Solution

Point 1

We need to verify that $d_2(f_n, 0) \rightarrow 0$ where

$$d_2(f_n, 0) = \sqrt{\int_0^1 |f_n(t)|^2 dt}$$

We have that



$$(d_2(f_n, 0))^2 \leq \frac{2}{n} \rightarrow 0.$$

Point 2

$$d_\infty(f_n, 0) = \sup_{t \in [0, 1]} |f_n(t)| = f_n\left(\frac{1}{2}\right) = \sqrt{2}.$$

It doesn't tend to zero then it doesn't tend to the null function with the distance d_∞ . **Point 3**

We suppose that the sequence admits a limit and that

$$\exists g \in X \quad \text{s.t.} \quad d_{\infty}(f, g) \rightarrow 0.$$

Then

$$\forall t \in [0, 1] \quad 0 \leq |f_n(t) - g(t)| \leq d_{\infty}(f_n, g)$$

since the "due carabinieri" theorem

$$\forall t \in [0, 1] \quad f_n(t) \rightarrow g(t).$$

If $t \neq \frac{1}{2}$ we have that $f_n(t) \rightarrow 0 \implies g(t) = 0$.

If $t = \frac{1}{2}$ $f_n(t) = \sqrt{2} \implies g(\frac{1}{2}) = \sqrt{2} \implies g \notin X$. Then the limit with d_{∞} doesn't exist.