A collection of Game Theory exercises

June 10, 2024

Exercise 1

Two people are employed in a joint project. If every person $i = 1, 2$ spends an amount of resources x_i, where $0 \le x_i \le 1$, incurring a cost $k_i(x_i)$, the project will have a revenue of $f(x_1, x_2)$. The revenue is equally divided by the two persons, without considering the resources employed by each person.

- write the payoff function of each player;
- formulate this problem as a strategic game and determine the possible Nash equilibria in the cases:
  1. $f(x_1, x_2) = 3x_1x_2$ and $k_i(x_i) = x_i^1$ for $i = 1, 2$;
  2. $f(x_1, x_2) = 4x_1x_2$ and $k_1(x_1) = x_1$, $k_2(x_2) = \frac{2}{3}x_2$.

In the prevoius cases can we deduce a priori that a Nash equilibrium exists?

Solution

case a)
i) proceed:

$$f(x_1, x_2) = 3x_1x_2,$$

the proceed for every single person is:

$$f_i(x_1, x_2) = \frac{3}{2}x_1x_2.$$

The payoff functions are:

$$u_1(x_1, x_2) = \frac{3}{2}x_1x_2 - x_1^2$$

$$u_2(x_1, x_2) = \frac{3}{2}x_1x_2 - x_2^2.$$

The strategic game $(A_1 \times A_2, (u_1, u_2))$ is such that:

$$A_1 = A_2 = [0, 1]$$

$$u_1(x_1, x_2) = \frac{3}{2}x_1x_2 - x_1^2$$

$$u_2(x_1, x_2) = \frac{3}{2}x_1x_2 - x_2^2.$$

We now need to calculate the best reply:

$$BR_i := A_{-i} \to A_i \qquad \forall i = 1, 2, \cdots, n,$$

that is

$$BR_1 := A_2 \to A_1 \qquad \forall x_2 \in [0, 1],$$

given by

$$BR_1(x_2) = \arg\max_{x_1 \in [0,1]} u_1(x_1, x_2) = \arg\max_{x_1 \in [0,1]} \frac{3}{2}x_1x_2 - x_1^2 = \arg\max_{x_1 \in [0,1]} (\frac{3}{2}x_2 - x_1)x_1.$$

We have that

$$\frac{\partial u_1(x_1, x_2)}{\partial x_1} = \frac{3}{2}x_2 - 2x_1 = 0$$

for

$$x_1 = \frac{3}{4}x_2,$$

so that

$$BR_1(x_2) = \frac{3}{4}x_2.$$

The best reply for the player two is:

$$BR_2 := A_1 \to A_2 \qquad \forall x_1 \in [0, 1]$$

given by

$$BR_1(x_1) = \arg\max_{x_2 \in [0,1]} u_2(x_1, x_2) = \arg\max_{x_2 \in [0,1]} \frac{3}{2} x_1 x_2 - x_2^2$$

so that

$$\frac{\partial u_2(x_1, x_2)}{\partial x_2} = \frac{3}{2} x_1 - x_2 = 0$$

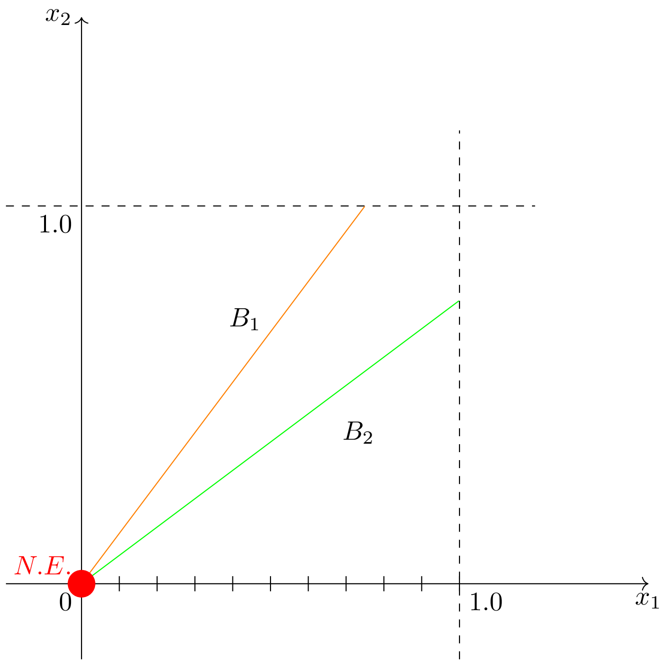
for

$$x_2 = \frac{3}{4} x_1,$$

so that

$$BR_2(x_1) = \frac{3}{4} x_1.$$

We can also solve the following system:



$$\begin{cases} x_1 = \frac{3}{4} x_2 \\ x_2 = \frac{3}{4} x_1 \end{cases} \implies \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$$

$(0,0)$ is the unique Nash Equilibrium.

case b)

i) Payoff functions:

$$f(x_1, x_2) = 4x_1 x_2,$$

then for each player the proceeds is

$$f_i(x_1, x_2) = 2x_1 x_2,$$

so that the payoff functions are:

$$u_1(x_1, x_2) = 2x_1 x_2 - x_1$$

$$u_2(x_1, x_2) = 2x_1 x_2 - \frac{2}{3} x_1.$$

ii) Now we consider the two players strategic game $(A_1 \times A_2, (u_1, u_2))$ such that

$$A_1 = A_2 = [0, 1]$$

$$u_1(x_1, x_2) = 2x_1 x_2 - x_1$$

$$u_2(x_1, x_2) = 2x_1x_2 - \frac{2}{3}x_2.$$

Now we can consider the Best Reply

$$BR_i := A_{-i} \to A_i \qquad \forall i = 1, 2, \cdots, n$$

that are

$$BR_1 := A_2 \to A_1$$

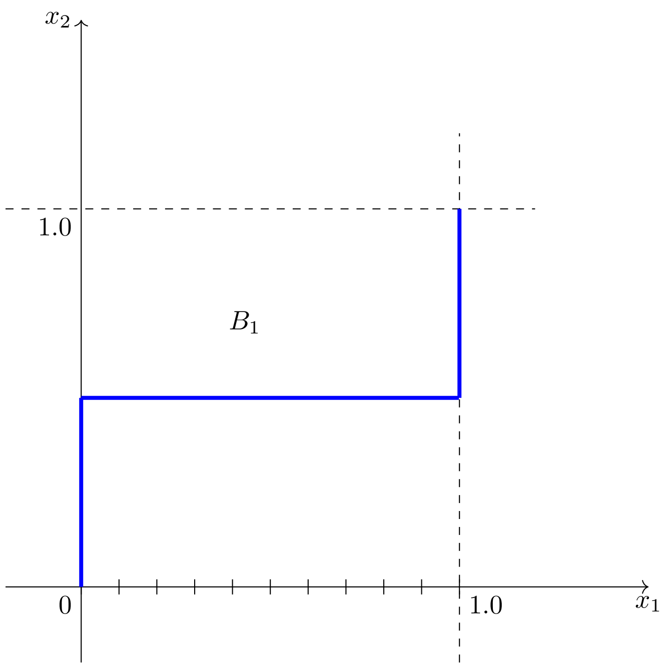
$$BR_2 := A_1 \to A_2.$$

From the definition we have

$$BR_i(a_{-i}) = \{a_i \in A_i \qquad \text{s.t.} \qquad u_i(a_{-i}, a_i) \geq u_i(a_{-i}, \hat{a}_i) \qquad \forall \hat{a}_i \in A_i\}$$

$$= \arg\max_{\hat{a}_i \in A_i} u_i(a_{-i}, \hat{a}_i)$$

$$BR_1(x_2) = \arg\max_{x_1 \in [0,1]} (2x_1 - 1)x_1 = \begin{cases} \{0\} & \text{if} \quad x_2 < \frac{1}{2} \\ [0,1] & \text{if} \quad x_2 = \frac{1}{2} \\ \{1\} & \text{if} \quad x_2 > \frac{1}{2} \end{cases}$$



$$BR_2 := A_1 \to A_2 \qquad \forall x_1 \in [0,1]$$

$$BR_2(x_1) = \arg\max_{x_2 \in [0,1]} (2x_1 - \frac{2}{3})x_2 = \begin{cases} \{0\} & \text{if} \quad x_1 < \frac{1}{3} \\ [0,1] & \text{if} \quad x_1 = \frac{1}{3} \\ \{1\} & \text{if} \quad x_1 > \frac{1}{3} \end{cases}$$

From the definition

$$a^\star \in A \qquad \text{is a Nash Equilibrium}$$

$$\text{iff}$$

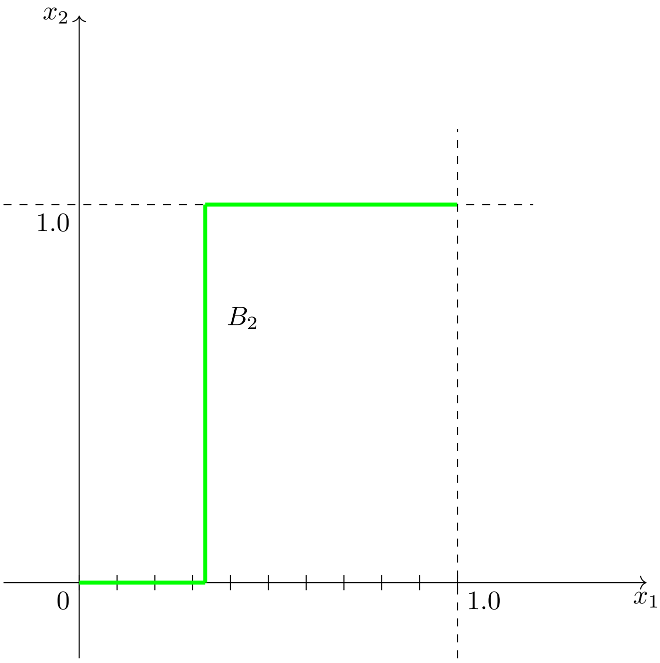
$$a_i^\star \in BR_i(a_{-i}^\star \qquad \forall i$$

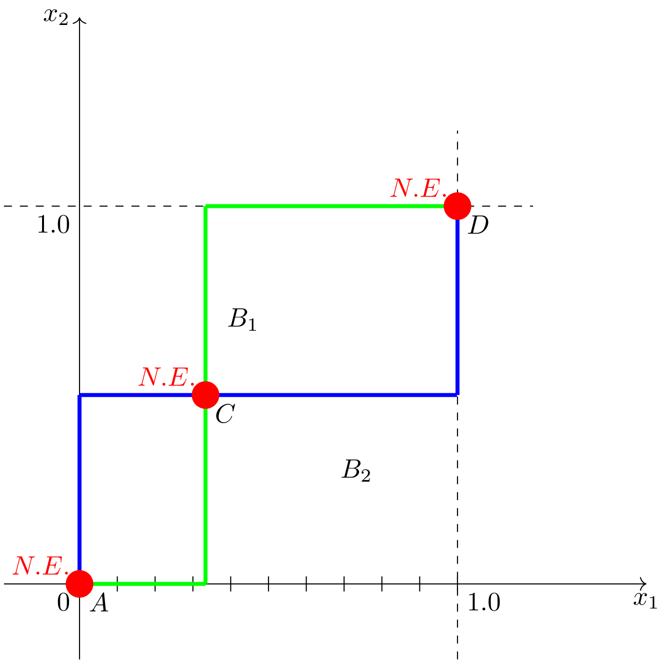
The Nash Equilibrium are

$$B_1 \cap B_2 = \{A, C, D\}$$

where

$$A = (0, 0)$$





$$C = (1./3, 1./2)$$

$$D = (1., 1.)$$

The functions $u_i : A_1 \times A_2 \to \mathbb{R}$ are continuous. The map $x_1 \mapsto u_1(x_1, x_2)$ with $x_2 \in A$ fixed is linear in x_1 and so it is concave. The same for the map $x_2 \mapsto u_2(x_1, x_2)$ with $x_1 \in A$ fixed. Then the Nash Theorem guarantees the existence of at least a Nash Equilibrium.