Chapter 1

Functional Spaces

Exercise 1

Compute

$$\lim_{n\to+\infty}\int_1^\infty f_n(x)dx$$

where

$$f_n(x)=\frac{\sin(nx)}{x^3}e^{-n\sqrt{x}}$$

for all $x\geq 1$ and for all $n\in\mathbb{N}$.

Solution

This exercise is trivial using the Dominated Convergence Theorem.

First we calculate the **punctual convergence**.

$$\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}\frac{\sin(nx)}{x^3}e^{-n\sqrt{x}}=0$$

for all $x\geq 1$ since $\lim_{n\to\infty}\frac{\sin(nx)}{x^3}=0$ and $e^{-n\sqrt{x}}$ is bounded.

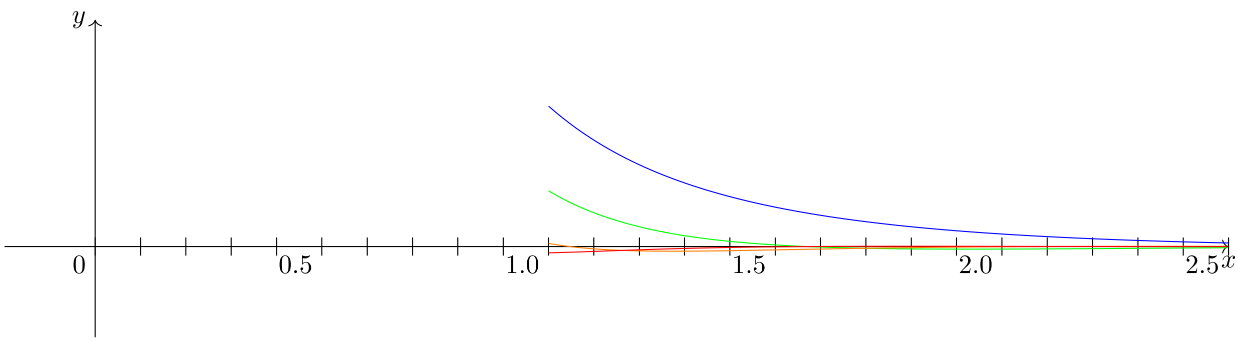


Figure 1.1: The sequence of functions $f_n(x)$.

$f(x)=0 \qquad \forall x\geq 1$ is the punctal limit. Now we search a dominant function.

$$|f_n(x)|=|\frac{\sin(nx)}{x^3}e^{-n\sqrt{x}}|\leq\star,$$

since the sine and $e^{-n\sqrt{x}}$ are bounded functions:

$$-1\leq\sin(nx)\leq 1 \qquad \forall n\in\mathbb{N} \qquad \forall x\in\mathbb{R}$$

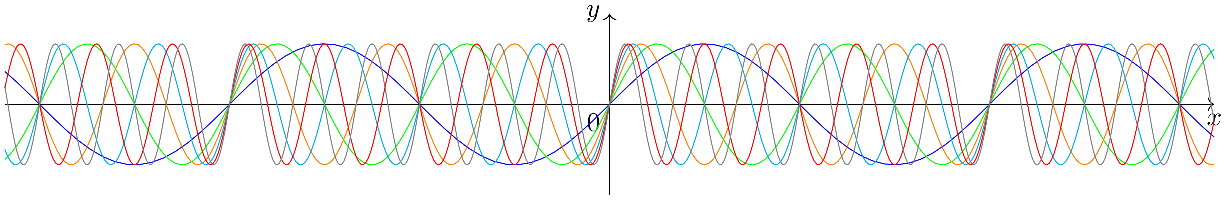


Figure 1.2: The sine function.

$$e^{-n\sqrt{x}} \leq 1 \qquad \forall n \in \mathbb{N} \qquad x \geq 1$$

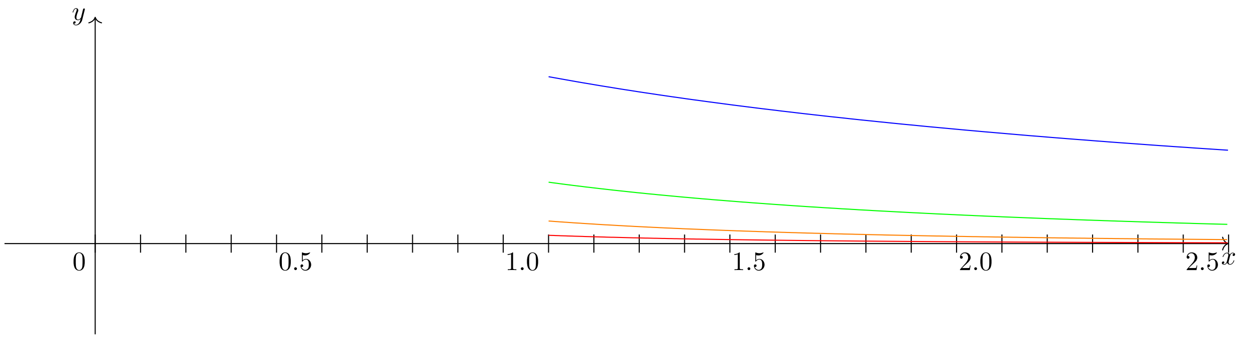


Figure 1.3: The sequence of functions $e^{-n\sqrt{x}}$.

$$\star \leq |\frac{1}{x^3}| = \frac{1}{x^3} = g(x) \qquad \forall n \in \mathbb{N}$$

since $x \in [0, +\infty)$. Now we need to verify if $g \in L^1([0, +\infty))$.

$$\int_1^{+\infty} |g(x)| dx = \int_1^{+\infty} |\frac{1}{x^3}| dx = \int_1^{+\infty} \frac{1}{x^3} dx < +\infty$$

since the summability criteria:

$$\int_1^{+\infty} \frac{1}{x^\alpha} dx = \begin{cases} \frac{1}{\alpha - 1} & if \quad \alpha > 1 \\ +\infty & if \quad \alpha \leq 1 \end{cases}$$

Now we can apply the Dominated Convergence Theorem (or Lebesgue Theorem):

$$\lim_{n \to +\infty} f_n(x) dx = \int_1^{+\infty} \lim_{n \to +\infty} f_n(x) dx = \int_1^{+\infty} \lim_{n \to +\infty} \frac{\sin(nx)}{x^3} e^{-n\sqrt{x}} dx = \int_1^{+\infty} 0 dx = 0.$$

Then the solution is

$$\lim_{n \to +\infty} \int_1^{+\infty} f_n(x) dx = \lim_{n \to +\infty} \frac{\sin(nx)}{x^3} e^{-n\sqrt{x}} dx = 0 \qquad \forall x \leq 1.$$

Chapter 2

L^p Spaces

Exercise 1

Analyze the convergence in $L^p([0,1])$ with $1 \leq p < \infty$ of

$$f_n(x) = \frac{\cos(nx)e^{-nx}}{\sqrt[4]{x}} \qquad for \quad x \in [0,1] \qquad \forall n \in \mathbb{N}.$$

For which L^p the sequence converge to a certain function?

Solution

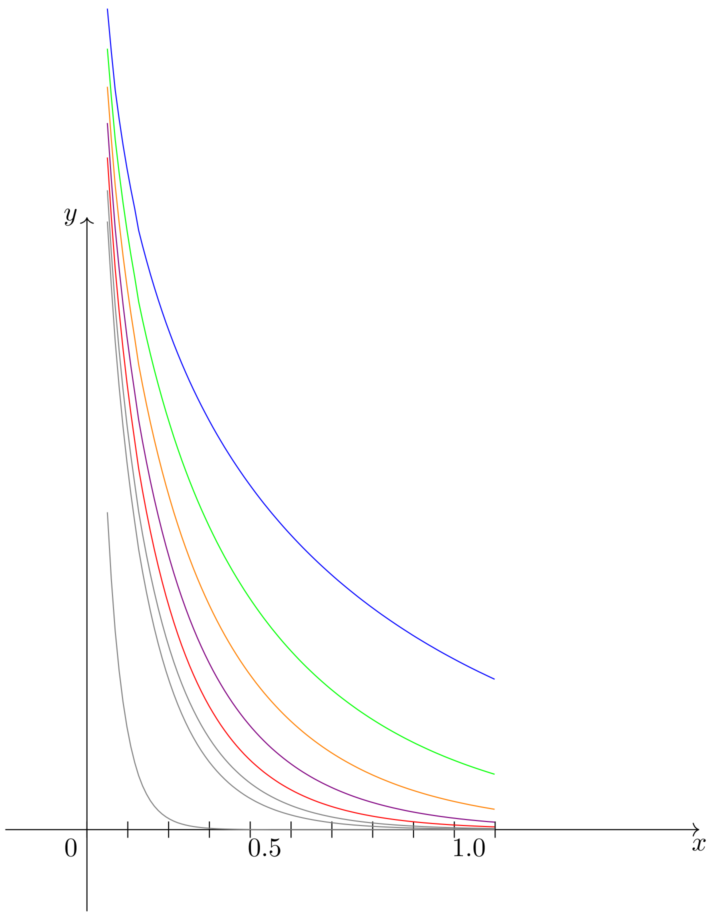


Figure 2.1: The sequence of functions $f_n(x)$.

First of all we search for which p this sequence belongs to some L^p, applying the Dominated Convergence Theorem.

$$|f_n(x)| = |\frac{\cos(nx)}{\sqrt[4]{x}}e^{-nx}| \le \frac{1}{\sqrt[4]{x}} = g(x) \qquad x \in [0,1]$$

We know that $f_n(x)$ belongs to some L^p if and only if

$$\int_0^1 |f_n(x)|^p dx < +\infty.$$

The exponents p that satisfy this relations are the candidates.

$$\int_0^1 |f_n(x)|^p dx = \int_0^1 |\frac{\cos(nx)}{\sqrt[4]{x}}e^{-nx}|^p dx \le \int_0^1 |\frac{1}{\sqrt[4]{x}}|^p dx = \int_0^1 \frac{1}{x^{\frac{p}{4}}}dx.$$

From the summability criteria:

$$\int_1^{+\infty} \frac{1}{x^\alpha}dx = \begin{cases} \frac{1}{\alpha-1} & if \quad \alpha > 1 \\ +\infty & if \quad \alpha \le 1 \end{cases} \quad ,$$

since

$$|\frac{\cos(nx)}{\sqrt[4]{x}}e^{-nx}|^p \le (\frac{1}{\sqrt[4]{x}})^p$$

we have that for $p \in [1,4) \qquad f_n(x) \in L^p([0,1]) \qquad \forall n \in \mathbb{N}$.

- $f_n \in L^1([0,1])$;
- $f_n \in L^2([0,1])$;
- $f_n \in L^3([0,1])$.

Punctual Convergence:

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} \frac{\cos(nx)}{\sqrt[4]{x}}e^{-nx} = \lim_{n\to+\infty} \frac{\cos(nx)}{\sqrt[4]{x}e^{nx}} \to 0$$

so

$$f_n \to 0 \qquad \text{pointwise} \qquad \forall x \in [0,1],$$

we can apply the comparison criterium.

$$\lim_{x\to0^+} f_n(x)\sqrt[4]{x} = \lim_{x\to0^+} \frac{\cos(nx)}{\sqrt[4]{x}}e^{-nx}\sqrt[4]{x} = 1$$

$$f_n(x) \sim \frac{1}{\sqrt[4]{x}} \qquad \text{for} \qquad x \to 0^+.$$

Now we can analyze the convergence in $L^p([0,1])$

$$\|f_n - f\|_{L^p}$$

$$\|f_n(x) - f(x)\|^p_{L^p([0,1])} = \|f_n(x)\|^p_{L^p([0,1])} = \left\|\frac{\cos(nx)}{\sqrt[4]{x}e^{nx}}\right\|^p_{L^p([0,1])} = \int_0^1 |\frac{\cos(nx)}{\sqrt[4]{x}e^{-nx}}{}_{L^p([0,1])}|^p dx = \star$$

since

$$|\frac{\cos(nx)}{\sqrt[4]{x}e^{nx}}{}_{L^p([0,1])}|^p \le g(x) = \frac{1}{x^{\frac{p}{4}}}$$

where

$$g \in L^p([0,1]) \qquad \text{for} \qquad 1 \le p < 4,$$

we can apply the Dominated Convergence Theorem

$$\lim_{n\to+\infty} \|f_n(x) - f(x)\|^p_{L^p([0,1])} = \lim_{n\to+\infty} \int_0^1 |\frac{\cos(nx)}{\sqrt[4]{x}}e^{-nx}|^p dx = \int_0^1 \lim_{n\to+\infty} |\frac{\cos(nx)}{\sqrt[4]{x}}e^{-nx}|^p dx = 0.$$

So

$$\lim_{n\to+infty} \|f_n(x)-f(x)\|_{L^p([0,1])} \to 0$$

$$f_n(x)\to 0 \qquad in \qquad L^p([0,1]) \qquad \forall p\in[1;4).$$

Since

$$\lim_{x\to 0^+} \frac{|f_n(x)|}{g(x)} = 1 \qquad \forall n\in\mathbb{N}$$

we have

$$f_n\in L^p([0,1]) \leftrightarrow g\in L^p([0,1])$$

so that

$$f_n \notin L^p([0,1]) \qquad \text{if} p\geq 4.$$

The sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ can't converge in $L^p([0,1])$ spaces if $p\geq 4$.

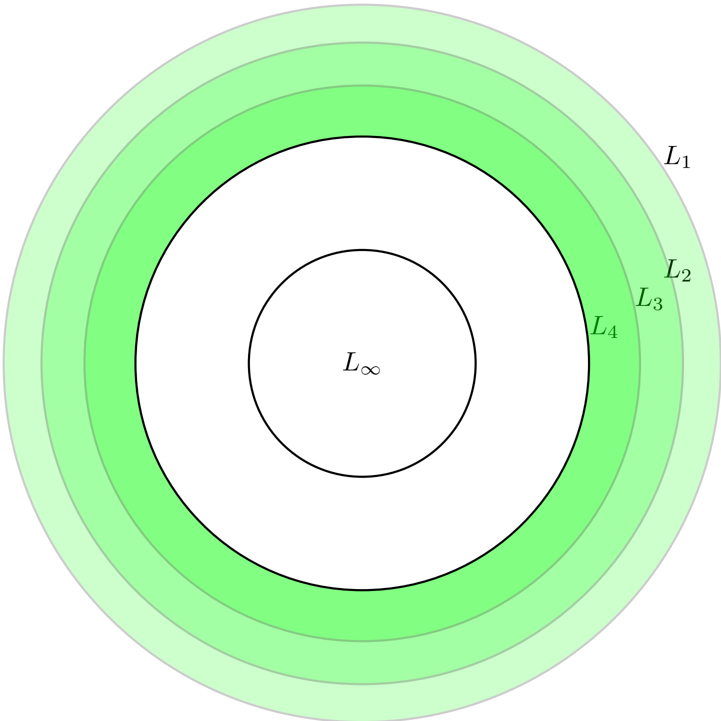
In the case $p=+\infty$, we have

$$\|f_n(x)\|_\infty = \operatorname*{ess\,sup}_{x\in(0,1)} |\frac{\cos(nx)}{\sqrt[4]{x}}e^{-nx}| \leq \sup_{x\in(0,1)} |\frac{1}{\sqrt[4]{x}}| \to +\infty,$$

so

$$f_n \nrightarrow 0 \qquad \text{in} \qquad L^\infty((0,1)).$$

Since $[0,1]$ is a bounded set we have the following embeddings: The sequence $f_n(x)$ lives in "green"



spaces.

Chapter 3

Hilbert Spaces

Exercise 1