

A collection of Real Analysis exercises

June 14, 2024

Chapter 1

Metric Spaces

Exercise 1

Let the distances in \mathbb{R}^2 :

1. $d_1(X, Y) = \sum_{i=1}^2 |y_i - x_i| = |y_1 - x_1| + |y_2 - x_2|$;
2. $d_2(X, Y) = \sqrt{\sum_{i=1}^2 |y_i - x_i|^2} = \sqrt{|y_1 - x_1|^2 + |y_2 - x_2|^2}$;
3. $d_\infty(X, Y) = \max_{i=1,2} |y_i - x_i| = \max\{|y_1 - x_1|, |y_2 - x_2|\}$.

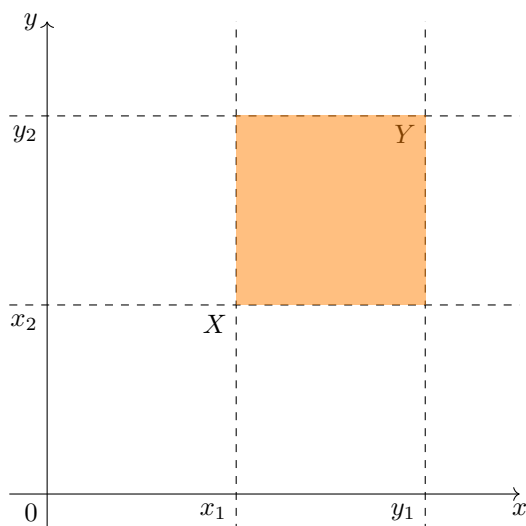
Construct the open balls related to these distances.

Solution

Point 1

$$\begin{aligned}
 d_1(X, Y) &< r \\
 \text{iff} \\
 \sum_{i=1}^2 |y_i - x_i| &< r \\
 \text{iff} \\
 |y_1 - x_1| + |y_2 - x_2| &< r
 \end{aligned}$$

Point 2



$$d_2(X;Y) < r$$

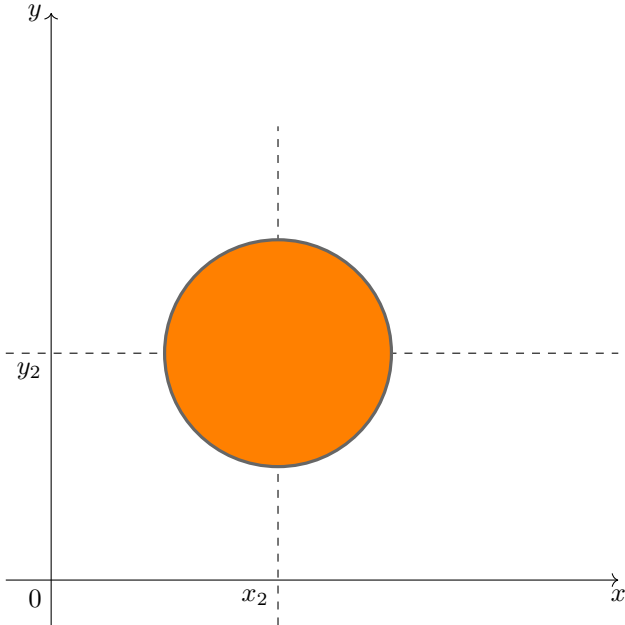
iff

$$\sqrt{|y_1 - x_1|^2 + |y_2 - x_2|^2} < r$$

iff

$$|y_1 - x_1|^2 + |y_2 - x_2|^2 < r^2$$

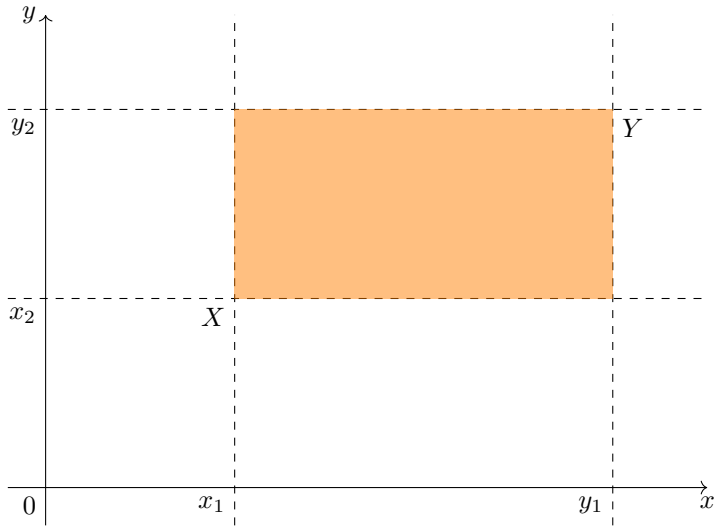
Point 3



$$d_\infty(X,Y) < r$$

iff

$$\max\{|y_1 - x_1|, |y_2 - x_2|\}$$



Exercise 2

Let (X, d) a metric space, $A \subset X$ not empty. Show if the following statements are true or false:

1. A open $\implies \overset{\circ}{A} \cap \partial A = \emptyset$;
2. if $\overset{\circ}{A} \cap \partial A = \emptyset \implies A$ closed;

Solution

Point 1

The first statement is always true. It is also true if A is closed or if A is neither open or closed.

$$\begin{aligned} \partial A &= \{x \in X \text{ s.t. } x \notin \overset{\circ}{A} \text{ and } x \in \overset{\circ}{X \setminus A}\} \\ &\implies \overset{\circ}{A} \cap \partial A = \emptyset \text{ is true.} \end{aligned}$$

Point 2

The second statement is false. Counterexample:

$$A =]0, 1[$$

is an open set with

$$\partial A = \{0, 1\}$$

and we have that $\overset{\circ}{A} \cap \partial A = \emptyset \implies A$ is closed is false because A is open.

Exercise 3

Let (X, d) a metric space $A \subset X$ closed and $A \neq \emptyset$. Furthermore let

$$f : X \rightarrow \mathbb{R}$$

with

$$f(x) = \inf_{a \in A \setminus \{x\}} d(x, a).$$

Tell whether the following statements are true or false.

1. $x \in A \implies f(x) = 0$;
2. $f(x) = 0 \implies x \in A$;

Solution

Point 1

This statement is not true $\forall x$. Counterexample:

$$A = [0, 1] \cup \{2\}$$

we have that $2 \in A$, but $f(2) \neq 0$ because $f(2) = \inf d(2, a)$ with $a \in [0, 1]$. If $a \in [0, 1]$ the distance of 2 from a is greater (or equal) than the distance of 2 from 1.

$$\text{If } a \in [0, 1]$$

$$\text{then } d(2, a) \geq d(2, 1) = 1$$

so that

$$\text{if } d(2, a) \geq 1 \implies \forall a \in [0, 1] \quad d(2, a) \geq 1.$$

Then

$$\inf d(2, a) \geq 1 \quad a \in [0, 1]$$

and it can't be equal to zero.

Point 2

Remember that x is an accumulation point for A iff $\inf_{a \in A \setminus \{x\}} d(x, a) = 0$.

$$f(x) = 0 \implies x \text{ is an accumulation point for } A$$

$$\text{then } x \in \overline{A} = A \quad \text{since } A \text{ is closed}$$

then the second statement is true.

Exercise 4

Let $X = C^0([0, 1])$, $d_\infty(f, g) = \sup_{t \in [0, 1]} |f(t) - g(t)|$ and $d_2(f, g) = \sqrt{\int_0^1 |f(t) - g(t)|^2 dt}$. Show that d_2 and d_∞ are not equivalent.

Solution

If we choose $f_n(t) = t^n \quad \forall n \in \mathbb{N}$, we have that:

$$d_\infty(f_n, 0) = \sup_{t \in [0, 1]} |f_n(t)| = \sup_{t \in [0, 1]} |t^n| = 1.$$

It is a maximum.

$$(d_2(f_n, 0))^2 = \int_0^1 |f_n(t)|^2 dt = \int_0^1 (t)^{2n} dt = \left[\frac{t^{2n+1}}{2n+1} \right]_0^1 = \frac{1}{2n+1}$$

then

$$d_2(f_n, 0) = \frac{1}{\sqrt{2n+1}}$$

for $n \rightarrow \infty$, $d_2 \rightarrow 0$. So that

$$\forall c \in \mathbb{R} \exists n \in \mathbb{N} \quad \text{s.t.} \quad d_\infty(f_n, 0) > c d_2(f_n, 0)$$

then d_2 and d_∞ are not equivalent.

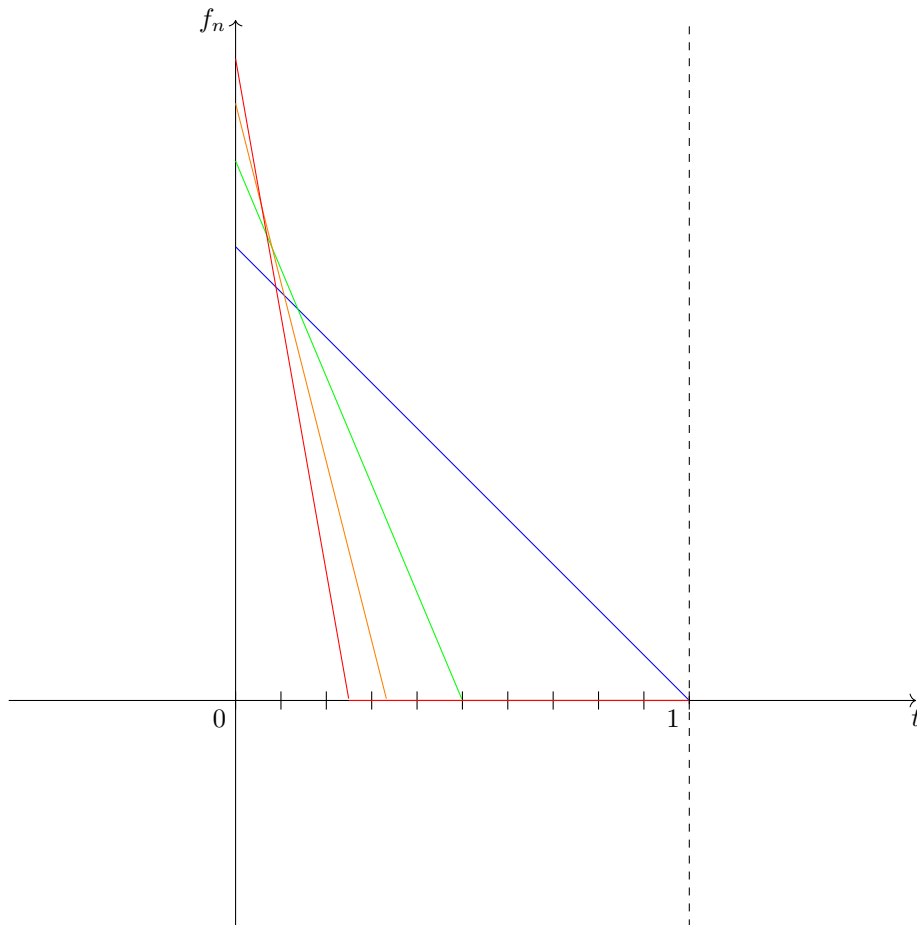
Exercise 5

Let $X = C^0([0, 1])$. Show that the open ball $B_{d_2}(0, 1)$ is unbounded with respect to d_∞ .

Solution

Consider the following sequence of functions:

$$f_n(t) = \begin{cases} -n^{\frac{5}{4}}(t - \frac{1}{n}) & \text{for } t \in [0, \frac{1}{n}] \\ 0 & \text{for } t \in [\frac{1}{n}, 1] \end{cases}$$



$$d_2(f, g) = \sqrt{\int_0^1 |g(t) - f(t)|^2 dt}$$

$$d_\infty(f_n, 0) = \sup_{t \in [0, 1]} |f_n(t)| = f_n(0) = \sqrt[4]{n} \rightarrow \infty$$

$$(d_2(f_n, 0))^2 = \int_0^{\frac{1}{n}} -n^{\frac{5}{2}}(t - \frac{1}{n}) dt = [-\frac{n^{\frac{5}{2}}}{3}(t - \frac{1}{n})^3]_0^{\frac{1}{n}} = \frac{n^{\frac{5}{2}}}{3} \frac{1}{n^3} = \frac{1}{3\sqrt{n}}$$

then

$$d_s(f_n, 0) = \frac{1}{\sqrt{3}\sqrt{n}}.$$

With respect to d_∞ the function is unbounded because contains a sequence the goes to infinity.

Exercise 6

Say if $[0, +\infty[$ is bounded in (\mathbb{R}, d_0) and in (\mathbb{R}, d) , with

- d_0 the discrete metric;
- d the Euclidean metric;

Solution

The discrete metric is characterized by the fact that the distance between two points is equal to zero or one.

$$d_0(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

so that

$$\text{diam}([0, +\infty[) = \sup_{x, y \in [0, +\infty[} d_0(x, y) \leq 1,$$

then $[0, +\infty[$ is bounded in (\mathbb{R}, d_0) .

If we consider the Euclidean distance

$$\text{diam}([0, +\infty[) = \sup_{x, y \in [0, +\infty[} d(x, y) = \sup_{x, y \in [0, +\infty[} |y - x| \geq n \quad \forall n \in \mathbb{N},$$

then $\text{diam}([0, +\infty[) = +\infty$, so that $[0, +\infty[$ is unbounded with d .

Exercise 7

Let (X, d) a metric space, $f : X \rightarrow \mathbb{R}$ a continuous function and $A \subset X$ bounded. Say if the following statements are true or false.

1. $f(A)$ is connected;
2. $f(A)$ is compact;
3. $f(A)$ is open;

Solution

Point 1

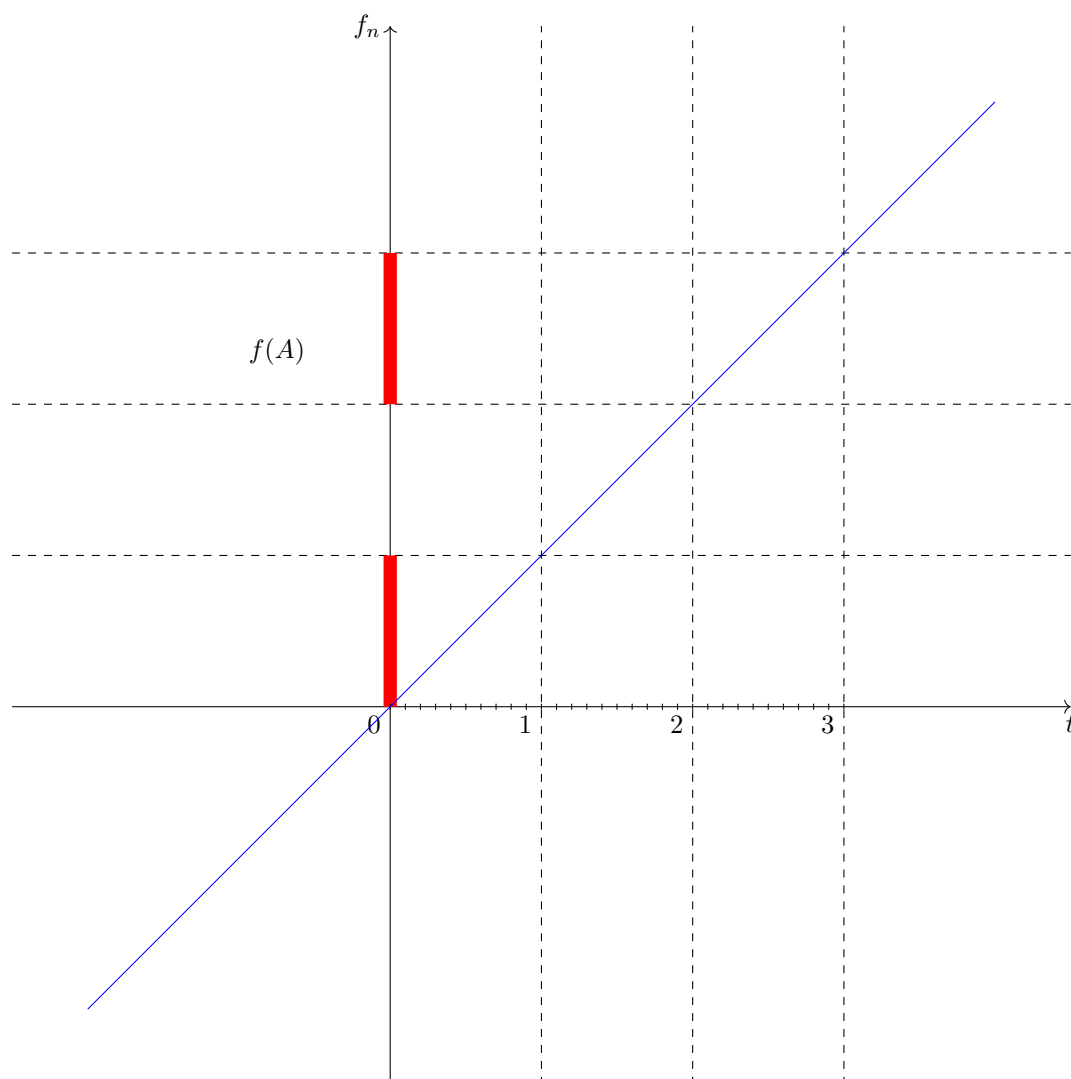
Consider $f : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x) = x$$

and consider $A = [0, 1] \cup [2, 3]$, there is no request on A , so that

$$f(A) = [0, 1] \cup [2, 3]$$

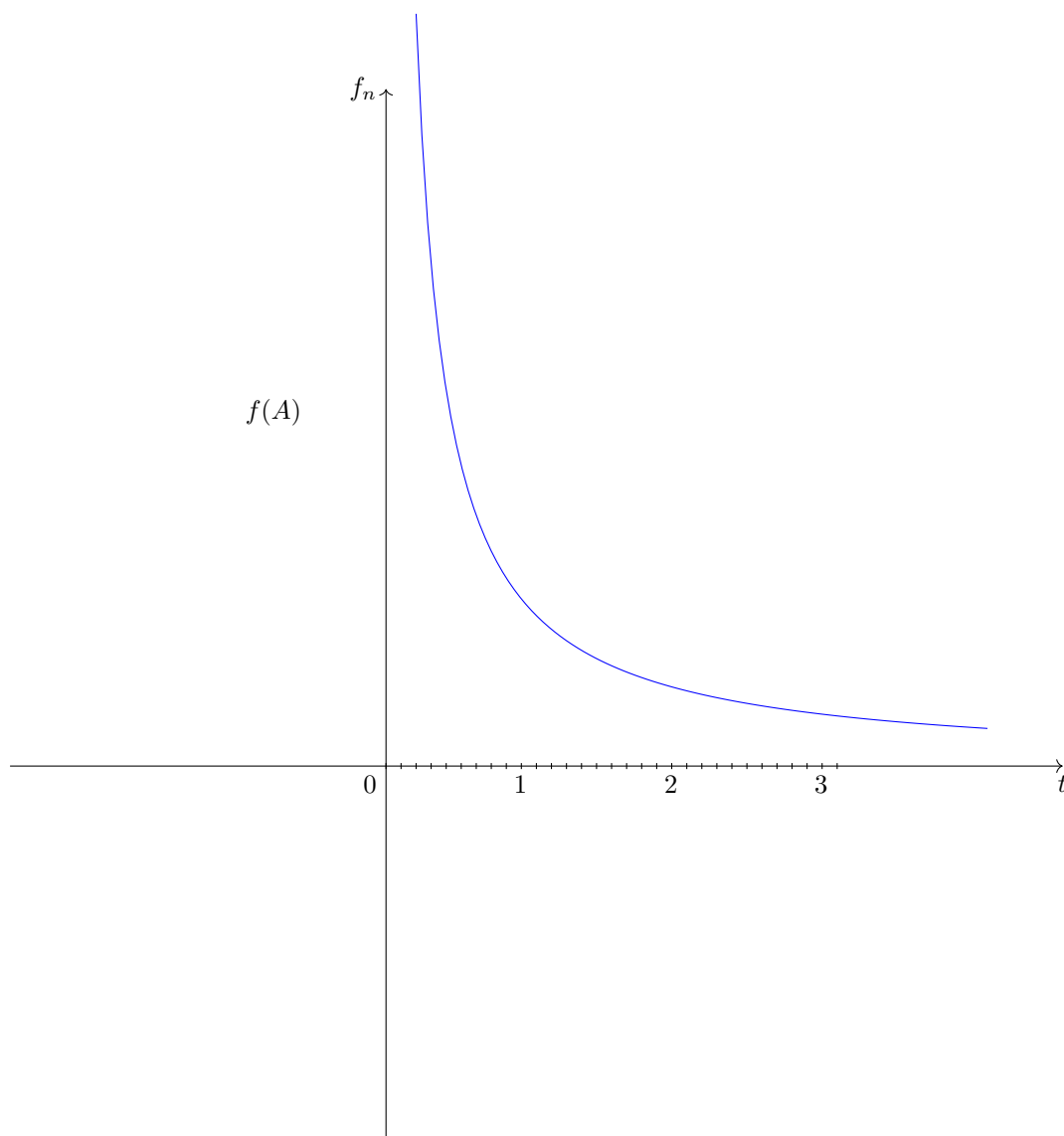
is not connected. **Point 2**



Consider $f :]0, +\infty[\rightarrow \mathbb{R}$, given by

$$f(x) = \frac{1}{x}$$

If we choose $A =]0, 1]$ we have $f(A) = [1, +\infty[$ that is not compact because it is not bounded.



Point 3

If we consider $f : \mathbb{R} \rightarrow \mathbb{R}$ with $f(x) = 1$,

$$A = [0, 1]$$

$$f(A) = \{1\} \quad \text{that is closed.}$$

Then all the three statements are false.

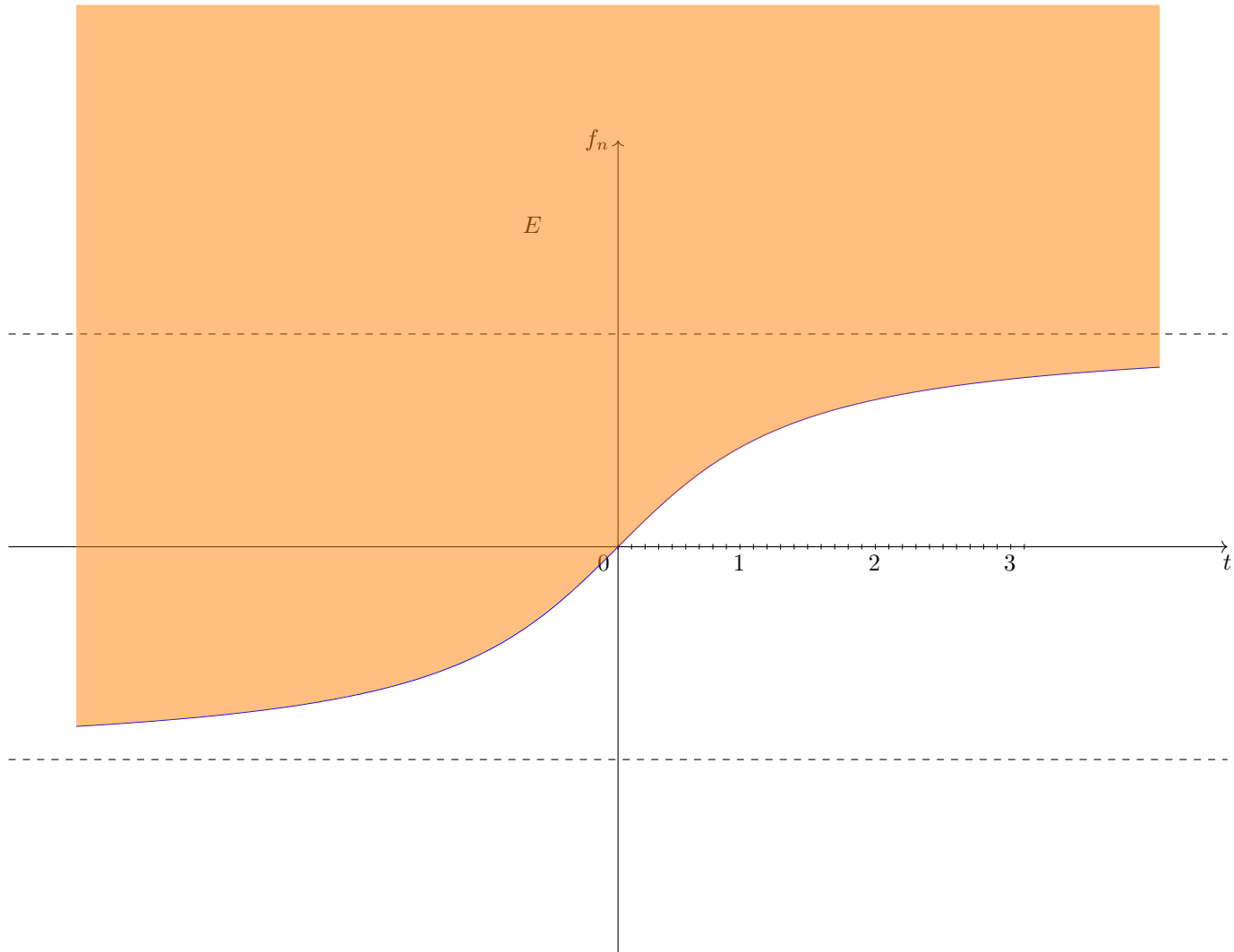
Exercise 8

Let $X = \mathbb{R}^2$ with the Euclidean distance. Say if the set

$$E = \{(x, y) \in \mathbb{R}^2 \quad \text{s.t.} \quad y \geq \arctan x\}$$

is complete and if it is compact.

Solution



We can see that E is not bounded so it is not compact. Now we see if it is complete. Consider

$$f(x, y) = y - \arctan(x),$$

we have

$$E = f^{-1}([0, +\infty[).$$

It is a contrainage of a closed set, then E is closed. We know that a closed subset of a complete metric space is complete. Since \mathbb{R}^2 is complete, then E is complete.

Exercise 9

Let $X = \mathbb{R}^2$ with the Euclidean distance and let

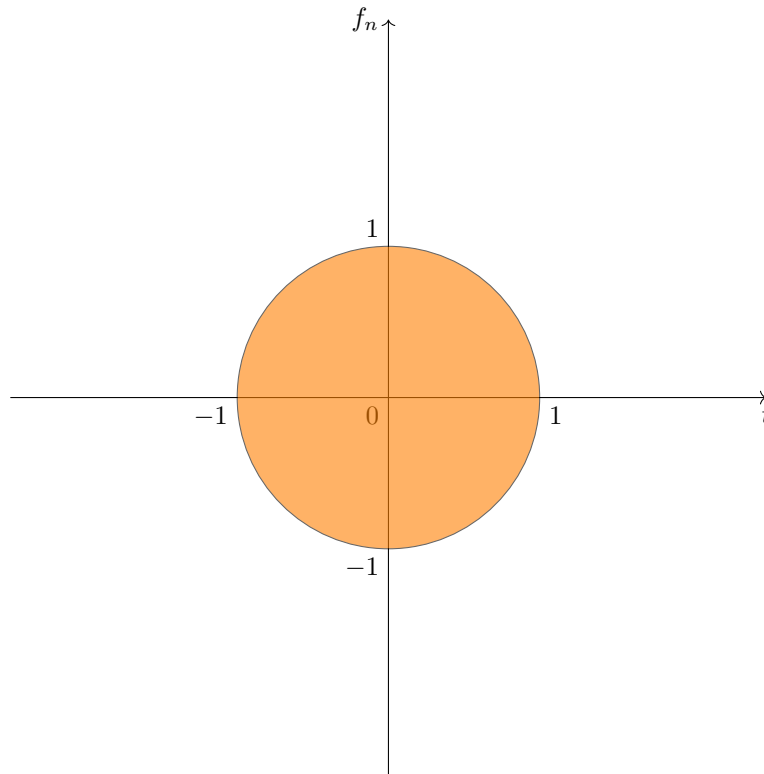
$$A = [0, 1] \times [0, +\infty[$$

$$B = \{(x, y) \in \mathbb{R}^2 \quad \text{with} \quad x^2 + y^2 < 1\}.$$

Say if A and B with the Euclidean distance are complete.

Solution

A is closed and $A \subset \mathbb{R}^2$ that is complete with the Euclidean distance. Then A is complete. B is open,



we need to find a Cauchy sequence in B that doesn't converge in B . Consider

$$x_n = \left(1 - \frac{1}{n}, 0\right) \quad x_n \in B \quad \forall n.$$

We have that

$$x_n \rightarrow (1, 0) \quad \text{in} \quad \mathbb{R}^2,$$

but $(1, 0) \notin B$. We have a sequence that converge in the space, that is a Cauchy sequence, but that doesn't converge in B . Then B is not complete.

Exercise 10

Let (X, d) a metric space and x_n a sequence of elements of X . Say if the following statements are true or false.

1. $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0 \implies x_n$ bounded;
2. x_n convergent $\implies \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$;
3. x_n Cauchy $\implies \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$.

Solution

Point 1

The fact that $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$ doesn't imply that x_n is bounded. Counterexample:

$$x_n = \log n$$

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = \lim_{n \rightarrow \infty} |\log n + 1 - \log n| = \lim_{n \rightarrow \infty} \left| \log \frac{n+1}{n} \right| = 0.$$

The distances between two consecutive elements become shorter but x_n is not bounded.

$$\sup_{n \in \mathbb{N}} \{\log n, \quad n \in \mathbb{N}\} = +\infty.$$

Then the first statement is false.

Point 2

If x_n is convergent then there exists $x_\infty \in X$ s.t. :

$$\lim_{n \rightarrow \infty} x_n = x_\infty.$$

Then

$$0 \leq d(x_n, x_{n+1}) \leq d(x_n, x_\infty) + d(x_\infty, x_{n+1}),$$

since $d(x_n, x_\infty) \rightarrow 0$, $d(x_\infty, x_{n+1}) \rightarrow 0$, then by "the two carabinieri theorem" we have that $d(x_n, x_{n+1}) \rightarrow 0$.

Point 3

x_n is a Cauchy sequence iff

$$\forall \epsilon > 0 \exists \nu \in \mathbb{N} \text{ s.t. } \forall n, m \geq \nu \in \mathbb{N} \implies d(x_n, x_m) < \epsilon.$$

If we fix n , then m can be very far, so is stronger the Cauchy condition with respect to (x_n, x_{n+1}) , so that:

$$\forall \epsilon > 0 \exists \nu \in \mathbb{N} \text{ s.t. } \forall n > \nu d(x_n, x_{n+1}) < \epsilon \implies \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0.$$

Exercise 11

Determine as α varies the limit in (\mathbb{R}^2, d_2) of the sequence:

$$x_n = \left(\frac{1}{n}, (-1)^n \frac{n^\alpha - 1}{n^2}\right).$$

If the limit doesn't exist find eventual convergent subsequences.

Solution

A sequence in \mathbb{R}^2 is like having two sequences in \mathbb{R} :

$$a_n = \frac{1}{n} \rightarrow 0$$

$$b_n = (-1)^n \frac{n^\alpha - 1}{n^2} = (-1)^n n^{\alpha-2} + \frac{(-1)^n}{n^2} \rightarrow 0.$$

- If $\alpha - 2 < 0$ then $b_n \rightarrow 0$;
- if $\alpha = 2$ then $b_n \rightarrow \nexists$;
- if $\alpha - 2 > 0$ then $b_n \rightarrow \nexists$.

Then if $\alpha < 2$ then

$$\lim_{n \rightarrow \infty} x_n = (0, 0)$$

if $\alpha \geq 2$

$$\lim_{n \rightarrow \infty} x_n = \nexists.$$

If we consider $\alpha > 2$ since $|b_n| \rightarrow +\infty$ there aren't convergent subsequences. If $\alpha = 2$ we have the subsequence of even indices

$$x_n = \left(\frac{1}{n}; \frac{n^2 + 1}{n^2}\right) \rightarrow (0, 1),$$

the subsequence of odd indices:

$$x_n = \left(\frac{1}{n}; -\frac{n^2 + 1}{n^2}\right) \Rightarrow (0, -1).$$

Exercise 12

Let (X, d) a metric space, $A \subseteq X$, $A \neq \emptyset$, x_n a sequence in A that converges to $x_\infty \in X$. Say if the following statements are false or true.

1. x_∞ is an accumulation point for A ;
2. $x_\infty \in \overline{A}$;
3. $x_\infty \in \overset{\circ}{A}$;
4. $x_\infty \in \partial A$.

Solution

Point 1

The first statement is false. Counterexample:

$$A = [0, 1] \cup \{2\}$$

$$x_n = 2 \quad \text{constant}$$

$$\lim_{n \rightarrow \infty} x_n = x_\infty = 2$$

that is not an accumulation point since 2 is an isolated point for A .

Point 2

If x_∞ is an isolated point the statement follows by the point 1, if x_∞ is not an isolated point it will be an accumulation point then the statement trivially follows. So the statement 2 is true.

Point 3

$x_\infty \in \overset{\circ}{A}$ is false. Counterexample:

$$A =]0, 1]$$

$$x_n = \frac{1}{n} \implies x_\infty \rightarrow 0 \notin \overset{\circ}{A}.$$

Point 4

$x_\infty \in \partial A$ is false. Counterexample:

$$A =]0, 1[$$

$$x_n = \frac{1}{2} + \frac{1}{n}$$

$$x_n \rightarrow \frac{1}{2}$$

that is not in ∂A .

Exercise 13

Show that the metric space $(C^0([0, 1]), d_\infty)$ is not compact.

Solution

Consider

$$f_n(t) = t^n \quad t \in [0, 1]$$

suppose that there is a subsequence that converges to $f \in C^0([0, 1])$:

$$f_{n_k} \rightarrow f,$$

that is

$$d_\infty(f_{n_k}, f) \rightarrow 0, \quad \forall t \in [0, 1].$$

$$|f_{n_k}(t) - f(t)| \leq d_\infty(f_{n_k}, f) \rightarrow 0.$$

If this holds then

$$f_{n_k}(t) \rightarrow f(t) \quad \forall t \in [0, 1]$$

$$f_n(t) \rightarrow g(t) = \begin{cases} 0 & \text{if } t \in [0, 1[\\ 1 & \text{if } t = 1 \end{cases}$$

If the sequence converge to $g(t)$ then also the subsequences tends to $g(t)$, but f_n can't have convergent subsequences because they must to converge to $g(t) \notin C^0([0, 1])$ because $g(t)$ is not continuous. Then $C^0([0, 1])$ is not compact.

Exercise 14

Consider the sequence

$$f_n(t) = \sqrt{\frac{1+n^2t^2}{n}} \quad \text{for } t \in [-1, 1].$$

Show that $f_n \rightarrow f$ with $f(t) = |t|$ with the distance d_∞ . Furthermore deduce that the space $C^1([-1, 1], d_\infty)$ is not complete.

Solution

$$|f_n(t) - f(t)| = \left| \frac{\sqrt{1+n^2t^2}}{n} - |t| \right| = \left| \frac{1}{n\sqrt{1+n^2t^2} + n|t|} \right| \leq \frac{1}{n}$$

Since the denominator is greater or equal to n we have that the fraction is lower or equal to $\frac{1}{n} \quad \forall t \in [-1, 1]$. Then

$$0 \leq d_\infty(f_n, f) \leq \frac{1}{n} \rightarrow 0.$$

Then $f_n \rightarrow f$ is a Cauchy sequence, but $f \notin C^1([-1, 1])$. Then the sequence doesn't converge in $(C^1[-1, 1], d_\infty)$ and so it is not complete.

Exercise 15

Let $X = C^0([0, 1])$ with the distance d_2 . Show that the sequence

$$f_n(t) = \begin{cases} 0 & \text{if } t \in [0, \frac{1}{2} - \frac{1}{n}[\\ \sqrt{2nt + 2 - n} & \text{if } t \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2}] \\ \sqrt{-2nt + 2 + n} & \text{if } t \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{n}[\\ 0 & \text{if } t \in [\frac{1}{2} + \frac{1}{n}, 1] \end{cases}$$

1. converges to the null function in $x = 2$:
2. does not converge to the null function in the metric space (X, d_∞) ;
3. admits limit in (X, d_∞) .

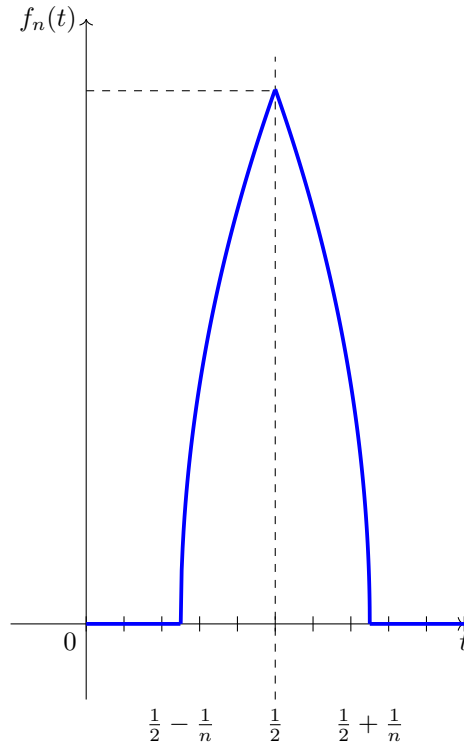
Solution

Point 1

We need to verify that $d_2(f_n, 0) \rightarrow 0$ where

$$d_2(f_n, 0) = \sqrt{\int_0^1 |f_n(t)|^2 dt}$$

We have that



$$(d_2(f_n, 0))^2 \leq \frac{2}{n} \rightarrow 0.$$

Point 2

$$d_\infty(f_n, 0) = \sup_{t \in [0, 1]} |f_n(t)| = f_n\left(\frac{1}{2}\right) = \sqrt{2}.$$

It doesn't tend to zero then it doesn't tend to the null function with the distance d_∞ . **Point 3**

We suppose that the sequence admits a limit and that

$$\exists g \in X \quad \text{s.t.} \quad d_\infty(f, g) \rightarrow 0.$$

Then

$$\forall t \in [0, 1] \quad 0 \leq |f_n(t) - g(t)| \leq d_\infty(f_n, g)$$

since the "due carabinieri" theorem

$$\forall t \in [0, 1] \quad f_n(t) \rightarrow g(t).$$

If $t \neq \frac{1}{2}$ we have that $f_n(t) \rightarrow 0 \implies g(t) = 0$.

If $t = \frac{1}{2}$ $f_n(t) = \sqrt{2} \implies g(\frac{1}{2}) = \sqrt{2} \implies g \notin X$. Then the limit with d_∞ doesn't exist.

Exercise 16

Let $f \in C^1(\mathbb{R}, \mathbb{R})$, 2π -periodic, such that its Fourier series is of the form

$$\sum_{n=3}^{\infty} \alpha_n \sin(nx)$$

. Let the Fourier series associated to f^3 of the form

$$\sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

Which of the following statements is certainly true?

1. $b_n = \alpha_n^3 \quad \forall n$;
2. $a_n = 0 \quad \forall n$.

Solution

Trivially we can see that f is an odd function, then f^3 is also an odd function, so that $a_n = 0 \quad \forall n$ is certainly true. Then

$$\mathcal{F}_{f^3(x)} = \sum_{n=0}^{+\infty} b_n \sin(nx).$$

If we consider

$$\begin{aligned} \alpha_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f^3(x) \sin(nx) dx. \end{aligned}$$

Generally is not true that $b_n = \alpha_n^3$. Counterexample:

$$f(x) = \sin(x)$$

$$\alpha_3 = 1 \quad \text{and} \quad \alpha_n = 0 \quad \forall n \neq 3.$$

All the terms such as $\sin(4x), \sin(5x), \dots, \sin(1000x)$ have null coefficients.

$$f^3(x) = \sin^3(3x)$$

$$\alpha_3 = 1$$

$$b_3 = 1^{3??}$$

$$\begin{aligned} b_3 &= \int_{-\pi}^{\pi} f^3(x) \sin(3x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^4(3x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2(3x) (1 - \cos^2(3x)) dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2(3x) dx - \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2(3x) \cos^2(3x) dx = \star \end{aligned}$$

using the duplication and bisection formulas

$$\sin^2(3x) = \frac{1 - \cos(6x)}{2}$$

$$\sin^2(3x) \cos^2(3x) = \frac{\sin^2(6x)}{4} = \frac{1 - \cos(12x)}{8}$$

$$\star = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1 - \cos(6x)}{2} dx - \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1 - \cos(12x)}{8} dx$$

$$= \frac{1}{2\pi} [x]_{-\pi}^{\pi} - \frac{1}{12\pi} [\sin(6x)]_{-\pi}^{\pi} - \frac{1}{8\pi} [x]_{-\pi}^{\pi} + \frac{1}{96\pi} [\sin(12x)]_{-\pi}^{\pi} = \frac{3}{4}.$$

Then

$$\alpha_3 = 1 \quad b_3 = \frac{3}{4} \neq 1^3.$$

Exercise 17

Let $f \in C^1(\mathbb{R}^2, \mathbb{R})^2$ such that $f(3, 6) = (0, 0)$. In a neighborhood of $(3, 1)$, which of the following statements are certainly true?

1. $\exists \alpha \in \mathbb{R}$ such that $x \mapsto x + \alpha f(x)$ satisfies the hypotheses of the Implicit Function Theorem.
2. $\forall \alpha \in \mathbb{R}$ $x \mapsto x + \alpha f(x)$ satisfies the hypotheses of the Inverse Function Theorem.

Solution

Point 1

We need to show that the derivative of f in $(3, 1)$ is an invertible matrix. Counterexample: let $\alpha = 0$ and consider a function $g(x) = x \mapsto x$. Let

$$g(x_1, x_2) = (x_1, x_2)$$

$$Dg(x_1, x_2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The determinant $\det Dg = 1 \neq 0$. Then the gradient matrix is invertible and since it is constant it is also invertible in $(3, 1)$. So the first statement is true.

Point 2

Counterexample. We need to find a function f with two variables such that $Df = 0$, starting from a function g not invertible. Consider

$$g(x_1, x_2) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

with $\alpha = 1$, we have

$$Dg(x_1, x_2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The gradient matrix Dg is not invertible and it doesn't satisfy the Inverse Function Theorem hypotheses.

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} x + f_1(x) \\ y + f_2(y) \end{bmatrix}$$

$$f_1(x, y) = 3 - x$$

$$f_2(x, y) = 1 - y$$

$$f(3, 1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then the second statement is false.

Exercise 18

Let $f \in C^2(\mathbb{R}^2, \mathbb{R})$ such that $f(1, 2) = 0$ and consider a neighborhood of $(1, 2)$. Which of the following statements are certainly true?

1. $\forall \alpha \in \mathbb{R} \quad x + \alpha f(x, y) + y - 3 = 0$ satisfies the Implicit Function Theorem hypotheses;
2. $\exists \alpha \in \mathbb{R}$ such that $x + \alpha f(x, y) + y - 3 = 0$ satisfies the Implicit Function Theorem hypotheses.

Solution

$$g(x, y) = x + \alpha f(x, y) + y - 3$$

$$g(1, 2) = 0$$

$$f \in C^2 \implies g \in C^2$$

We need to verify if

$$\frac{g}{y}(1, 2) \neq 0?$$

$$\frac{g}{x}(1, 2) \neq 0?$$

Point 2

We take $\alpha = 0$ so that $g(x, y) = x + y - 3$. Then

$$\frac{\partial g}{\partial y}(1, 2) = 1 \neq 0.$$

Then the first statement is true.

Point 1

We need a counterexample, starting from a function $g(x, y) = x^2 + y^2$ we can consider:

$$g(x, y) = (x - 1)^2 + (y - 2)^2$$

so that

$$\nabla g(1, 2) = [0, 0].$$

We have

$$x + \alpha f(x, y) + y - 3 = (x - 1)^2 + (y - 2)^2.$$

We take $\alpha = 1$.

$$x + f(x, y) + y - 3 = (x - 1)^2 + (y - 2)^2$$

$$f(x, y) = (x - 1)^2 + (y - 2)^2 - x - y + 3$$

$$f(1, 2) = (0, 0)$$

and $f \in C^2$, but

$$\nabla f(1, 2) = [0, 0]$$

so that f is not invertible. Then the second statement is not true.

Exercise 19

Let $f_\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by:

$$f_\alpha(x, y) = \begin{cases} \frac{|x|^\alpha y^2}{\sqrt{4x^2 + 3y^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Studying with respect to $\alpha \in \mathbb{R}$ the differentiability and continuity of f_α .

Solution

First: continuity

We have problems in all the points of the y -axis. First we consider the points $(0, b)$ with $b \neq 0$ that is a generic point of the y axis.

$$\lim_{(x,y) \rightarrow (0,b)} \frac{|x|^\alpha y^2}{\sqrt{4x^2 + 3y^2}} = \frac{b^2}{\sqrt{3b^2}} \lim_{(x,y) \rightarrow (0,b)} |x|^\alpha = \begin{cases} +\infty & \text{if } \alpha < 0 \\ \frac{b^2}{\sqrt{3b^2}} & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha > 0. \end{cases}$$

Then f is continuous in $(0, b)$ with $b \neq 0 \quad \forall \alpha > 0$. Now we consider the point $(0, 0)$.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{|x|^\alpha y^2}{\sqrt{4x^2 + 3y^2}}$$

in polar coordinates

$$\lim_{\rho \rightarrow 0} \frac{\rho^\alpha |\cos \theta|^\alpha \rho^2 (\sin^2 \theta)}{\rho \sqrt{4 \cos^2 \theta + 3 \sin^2 \theta}}$$

making some additions

$$|\rho^{\alpha+1} \frac{|\cos \theta|^\alpha (\sin^2 \theta)}{\sqrt{4 \cos^2 \theta + 3 \sin^2 \theta}}| \leq \rho^{\alpha+1} \frac{|\cos \theta|^\alpha |\sin^2 \theta|}{\sqrt{3} |\sin \theta|} = \frac{\rho^{\alpha+1}}{\sqrt{3}} |\cos \theta|^\alpha |\sin \theta| \leq \frac{\rho^{\alpha+1}}{\sqrt{3}}$$

uniformly in θ .

If $\alpha > 0$ then $L = 0$ uniformly in θ .

If $\alpha \leq -1$ then $L = \nexists$.

Remains the case $-1 < \alpha < 0$.

$$\lim_{\rho \rightarrow 0} \frac{\rho^{\alpha+1} |\cos \theta|^\alpha \sin^2 \theta}{\sqrt{4 \cos^2 \theta + 3 \sin^2 \theta}} = 0$$

this limit is equivalent to

$$\lim_{\rho \rightarrow 0} \rho^{\alpha+1} h(z) = 0 \quad \forall \theta \neq \frac{\pi}{2} \quad \frac{3\pi}{2}$$

but the limit is not uniform since it is not possible to increase the function. Then for $-1 < \alpha < 0$ the limit goes to $+\infty$. The local boundedness theorem is violated.

Then f is continuous for $\alpha \geq 0$, f is continuous in $(0, b)$ for $\alpha \geq 0$.

Second: differentiability

We consider the points $(0, b)$ with $b \neq 0$ and with $\alpha > 0$.

$$\frac{\partial f}{\partial x}(0, b) = \lim_{t \rightarrow 0} \frac{f(t, b) - f(0, b)}{t} = \lim_{t \rightarrow 0} \frac{|t|^\alpha b^2}{\sqrt{4t^2 + 3b^2}} \frac{1}{t} = \begin{cases} \nexists & \text{if } \alpha < 1 \\ \nexists & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha > 1 \end{cases}$$

$$\frac{\partial f}{\partial y}(0, b) = \lim_{t \rightarrow 0} \frac{f(0, b+t) - f(0, b)}{t} = \lim_{t \rightarrow 0} \frac{0}{t} = 0.$$

Now we consider the case $\alpha > 1$ and show if f is differentiable. For $\alpha > 1$, $\nabla f(0, 0) = [0, 0]$.

$$\lim_{(h,k) \rightarrow (0,0)} \frac{f(h, b+k) - f(0, b) - \nabla f(0, b) \begin{bmatrix} h \\ k \end{bmatrix}}{\sqrt{h^2 + k^2}} = \lim_{(h,k) \rightarrow (0,0)} \frac{|h|^\alpha (b+k)^2}{\sqrt{4h^2 + 3(b+k)^2} \sqrt{h^2 + k^2}}$$

$$= \lim_{\rho \rightarrow 0} \frac{\rho^{\alpha-1} |\cos \theta|^\alpha (b + \rho \sin \theta)^2}{\sqrt{4\rho^2 \cos^2 \theta + 3(b + \rho \sin \theta)^2}} = 0 \quad \forall \theta.$$

Now we show if it is uniformly.

$$\left| \frac{\rho^{\alpha-1} |\cos \theta|^\alpha (b + \rho \sin \theta)^2}{\sqrt{4\rho^2 \cos^2 \theta + 3(b + \rho \sin \theta)^2}} \right| \leq \rho^{\alpha-1} \frac{|b + \rho \sin \theta|}{\sqrt{3}} \leq \rho^{\alpha-1} \frac{|b| + \rho}{\sqrt{3}}.$$

The right term doesn't depend on θ so the limit is uniform in θ . If $\alpha > 1$ f is differentiable in $(0, b)$ with $b \neq 0$. Now we consider the case $\alpha \geq 0$ in $(0, 0)$.

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{t \rightarrow 0} \frac{f(t, 0) - f(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{0}{t} = 0$$

$$\frac{\partial f}{\partial y}(0, 0) = \lim_{t \rightarrow 0} \frac{f(0, t) - f(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{0}{t} = 0$$

$$\lim_{(h,k) \rightarrow (0,0)} \frac{f(h, k) - f(0, 0) - \nabla f(0, 0) \begin{bmatrix} h \\ k \end{bmatrix}}{\sqrt{h^2 + k^2}} = \lim_{(h,k) \rightarrow (0,0)} \frac{f(h, k)}{\sqrt{h^2 + k^2}} = \lim_{(h,k) \rightarrow (0,0)} \frac{|h|^\alpha k^2}{\sqrt{4h^2 + 3k^2} \sqrt{h^2 + k^2}}$$

if $\alpha = 0$ the limit is

$$\lim_{(h,k) \rightarrow (0,0)} \frac{k^2}{\sqrt{4h^2 + 3k^2} \sqrt{h^2 + k^2}} = \nexists.$$

If $\alpha > 0$ in polar coordinates:

$$\lim_{\rho \rightarrow 0} \frac{\rho^\alpha |\cos \theta|^\alpha \rho^2 (\sin^2 \theta)}{\rho \rho \sqrt{4 \cos^2 \theta + 3 \sin^2 \theta}} = 0 \quad \forall \theta.$$

Now we show if it is uniform in θ .

$$\left| \frac{\rho^\alpha |\cos \theta|^\alpha \sin^2 \theta}{\sqrt{4 \cos^2 \theta + 3 \sin^2 \theta}} \right| \leq \rho^\alpha \frac{|\cos \theta|^\alpha \sin^2 \theta}{\sqrt{3} \sin^2 \theta} = \frac{\rho^\alpha |\cos \theta|^\alpha \sin^2 \theta}{\sqrt{3} \sin^2 \theta} = \rho^2 \frac{|\cos \theta|^\alpha |\sin \theta|}{\sqrt{3}} \leq \frac{\rho^2}{\sqrt{3}}.$$

It is uniform in θ . Then $L = 0$ uniformly in θ .

Then f is differentiable in the origin for $\alpha > 0$.