A collection of Functional Analysis exercises

June 15, 2024

Chapter 1

Functional Spaces

Exercise 1

Compute

$$\lim_{n\to+\infty}\int_1^\infty f_n(x)dx$$

where

$$f_n(x) = \frac{\sin(nx)}{x^3}e^{-n\sqrt{x}}$$

for all $x \geq 1$ and for all $n \in \mathbb{N}$.

Solution

This exercise is trivial using the Dominated Convergence Theorem.

First we calculate the **pointwise convergence**.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{\sin(nx)}{x^3}e^{-n\sqrt{x}} = 0$$

for all $x \geq 1$ since $\lim_{n\to\infty}\frac{\sin(nx)}{x^3} = 0$ and $e^{-n\sqrt{x}}$ is bounded.

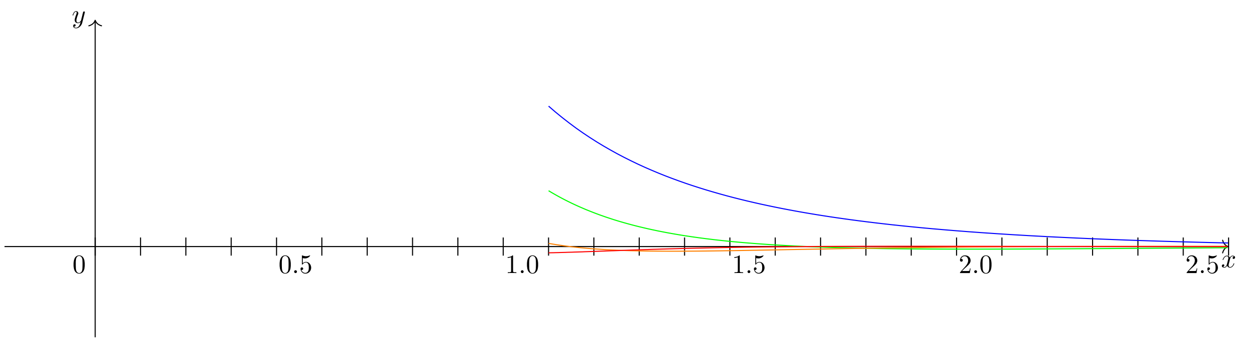


Figure 1.1: The sequence of functions $f_n(x)$.

$f(x) = 0 \qquad \forall x \geq 1$ is the punctal limit. Now we search a dominant function.

$$|f_n(x)| = |\frac{\sin(nx)}{x^3}e^{-n\sqrt{x}}| \leq \star,$$

since the sine and $e^{-n\sqrt{x}}$ are bounded functions:

$$-1 \leq \sin(nx) \leq 1 \qquad \forall n \in \mathbb{N} \qquad \forall x \in \mathbb{R}$$

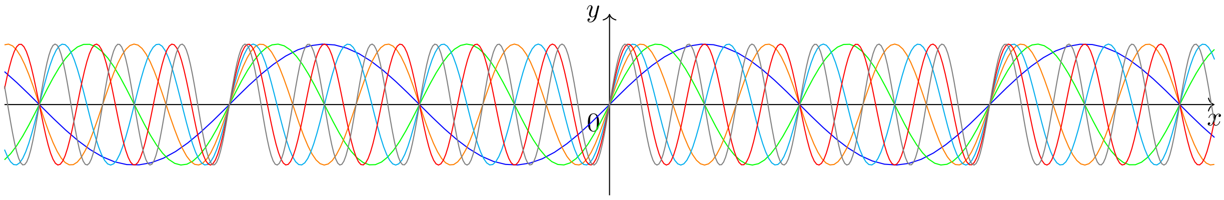


Figure 1.2: The sine function.

$$e^{-n\sqrt{x}} \leq 1 \qquad \forall n \in \mathbb{N} \qquad x \geq 1$$

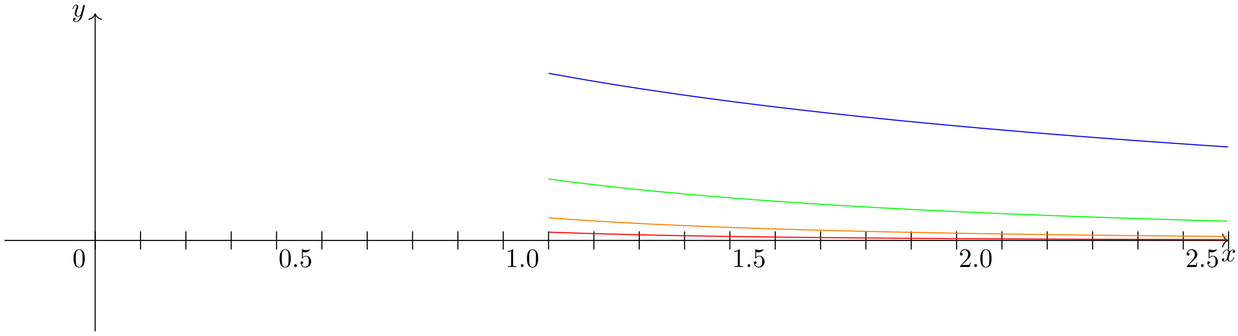


Figure 1.3: The sequence of functions $e^{-n\sqrt{x}}$.

$$\star \leq |\frac{1}{x^3}| = \frac{1}{x^3} = g(x) \qquad \forall n \in \mathbb{N}$$

since $x \in [0, +\infty)$. Now we need to verify if $g \in L^1([0, +\infty))$.

$$\int_1^{+\infty} |g(x)|dx = \int_1^{+\infty} |\frac{1}{x^3}|dx = \int_1^{+\infty} \frac{1}{x^3}dx < +\infty$$

since the summability criteria:

$$\int_1^{+\infty} \frac{1}{x^\alpha}dx = \begin{cases} \frac{1}{\alpha-1} & if \quad \alpha > 1 \\ +\infty & if \quad \alpha \leq 1 \end{cases}$$

Now we can apply the Dominated Convergence Theorem (or Lebesgue Theorem):

$$\lim_{n\to+\infty} f_n(x)dx = \int_1^{+\infty} \lim_{n\to+\infty} f_n(x)dx = \int_1^{+\infty} \lim_{n\to+\infty} \frac{\sin(nx)}{x^3}e^{-n\sqrt{x}}dx = \int_1^{+\infty} 0dx = 0.$$

Then the solution is

$$\lim_{n\to+\infty} \int_1^{+\infty} f_n(x)dx = \lim_{n\to+\infty} \frac{\sin(nx)}{x^3}e^{-n\sqrt{x}}dx = 0 \qquad \forall x \leq 1.$$

Exercise 2

Compute

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx$$

where

$$f_n(x)=\frac{x}{1+x^{2n}} \qquad \text{with} \qquad x\in(0,1).$$

Solution

We need to apply the Dominated Converge Theorem.

First of all we analyze the **pointwise convergence**.

$$\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}\frac{x}{1+x^{2n}},$$

if $x\in[0,1)$ we have

$$\lim_{n\to\infty}\frac{x}{1+x^{2n}}=x$$

since $x^{2n}\to 0$ for $x\in[0,1)$.

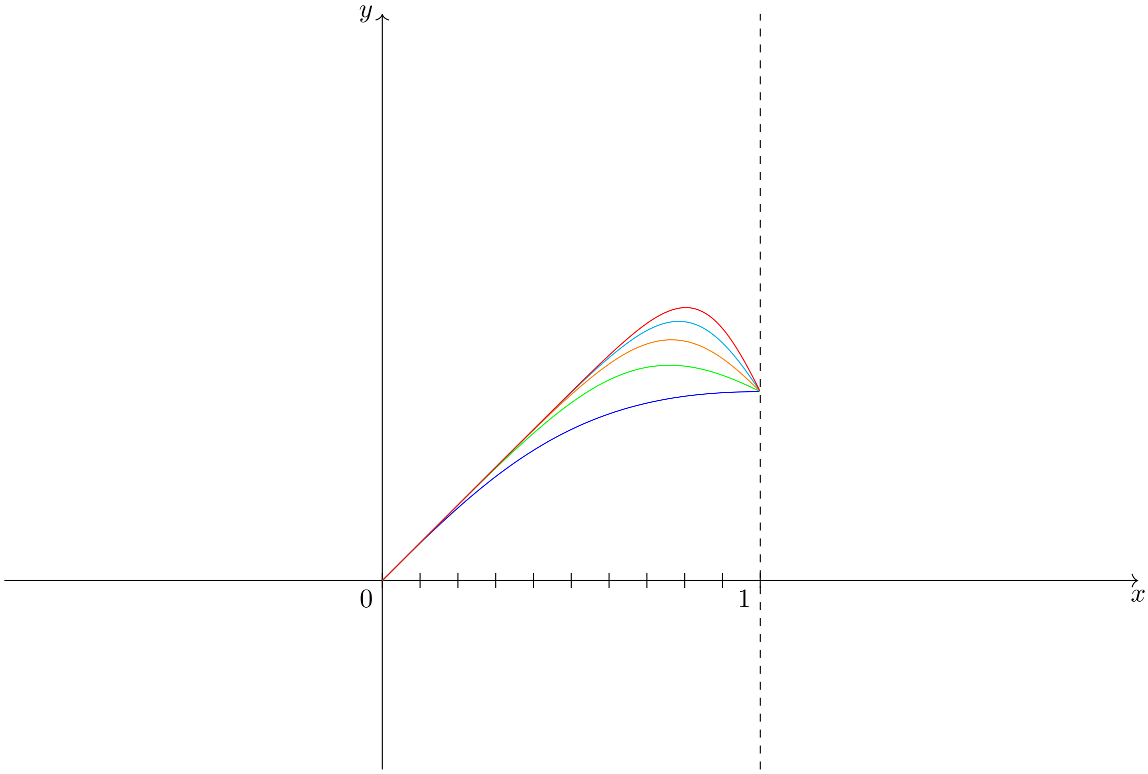
If $x=1$,

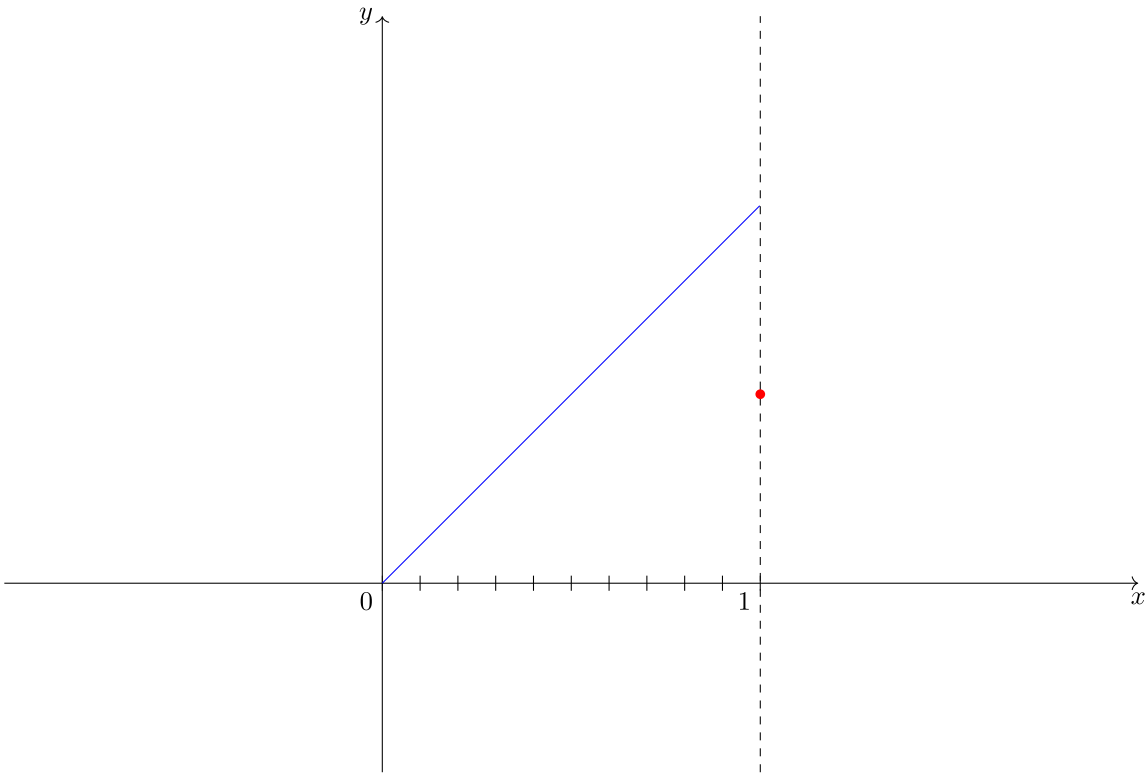
$$\lim_{n\to\infty}\frac{x}{1+x^{2n}}=\lim_{n\to\infty}\frac{1}{1+1^{2n}}=\frac{1}{2}.$$

The pointwise limit is

$$\lim_{n\to\infty} f_n(x)=\begin{cases} x & \text{if} \quad x\in[0,1) \\ \frac{x}{2} & \text{if} \quad x=1 \end{cases}$$

Now we find the dominant function.





$$\exists g \in L^1(0,1) \qquad \text{s. t.} \qquad |f_n(x)| \leq g?$$

$$|f_n(x)| = |\frac{x}{1+x^{2n}}| \leq \frac{x}{1+x^{2n}} \leq x = g(x) \qquad \forall n \in \mathbb{N}, \qquad \forall x \in (0,1).$$

$$\int_0^1 |g(x)| dx = \int_0^1 |x| dx = \int_0^1 x dx = [\frac{x^2}{2}]_0^1 = \frac{1}{2} < +\infty$$

so that

$$g \in L^1(0,1).$$

We can now apply the Dominated Convergence Theorem.

$$\lim_{n\to\infty} \int_0^1 \frac{x}{1+x^{2n}} dx = \int_0^1 \lim_{n\to\infty} \frac{x}{1+x^{2n}} dx = \int_0^1 x dx = \frac{1}{2}.$$

The solution is

$$\lim_{n\to\infty} \int_0^1 \frac{x}{1+x^{2n}} dx = \frac{1}{2}.$$

Exercise 3

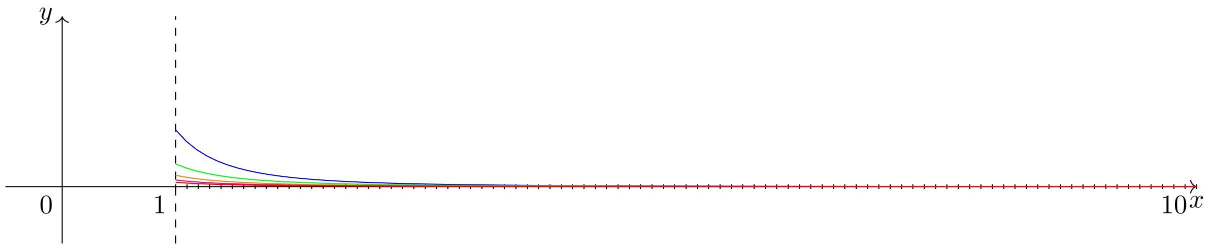
Studying convergence in $L^1([1,+\infty))$ of

$$f_n(x) = \frac{\cos(nx)}{n^2+x} \frac{1}{x^2} \qquad \forall x \geq 1 \qquad \forall n \in \mathbb{N}.$$

Solution

Convergence in $L^1([0,+\infty))$:

$$\|f_n - f\|_{L^1([1;+\infty))} = \int_1^{+\infty} |f_n - f| dx \to 0$$



POINTWISE CONVERGENCE

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\cos(nx)}{n^2+x}\frac{1}{x^2} = \star$$

$$-1 \le \cos(nx) \le 1$$

$$\star = 0,$$

ù so the pointwise limit is

$$f(x) = 0 \qquad \forall x \ge 1.$$

Now we need to find a dominant function:

$$\exists g \in L^1([1;+\infty)) \qquad \text{s.t.} \qquad |f_n(x)| \le g \qquad \forall x \ge 1 \qquad \forall n \in \mathbb{N}$$

$$|f_n(x)| = |\frac{\cos(nx)}{n^2+x}\frac{1}{x^2}| \le \frac{1}{n^2+x}\frac{1}{x^2} \le \frac{1}{x}\frac{1}{x^2} = \frac{1}{x^3} = g(x) \qquad \forall x \ge 1 \qquad \forall n \in \mathbb{N}$$

$$\int_1^{+\infty} |g(x)| dx = \int_1^{+\infty} |\frac{1}{x^3}| dx = \int_1^{+\infty} \frac{1}{x^3} dx < +\infty$$

summability criteria:

$$\int_1^{+\infty} \frac{1}{x^\alpha} dx = \begin{cases} \frac{1}{\alpha-1} & if \quad \alpha > 1 \\ +\infty & if \quad \alpha \le 1, \end{cases}$$

then

$$g \in L^1[1;+\infty).$$

$$\lim_{n\to\infty} \|f_n - f\|_{L^1([1;+\infty))} \to 0$$

$$\lim_{n\to\infty} |f_n - f| dx \to 0$$

$$iff$$

$$\lim_{n\to\infty} \int_1^{+\infty} |f_n| dx \to 0.$$

Now we can apply the Dominated Convergence Theorem:

$$\lim_{n\to+\infty} \int_1^{+\infty} |f_n| dx = \int_1^{+\infty} \lim_{n\to+\infty} |f_n| dx = \int_1^{+\infty} \lim_{n\to+\infty} |\frac{\cos(nx}{n^2+x}\frac{1}{x^2}| dx$$

$$= \int_1^{+\infty} \lim_{n\to+\infty} (\frac{\cos(nx)}{n^2+x}\frac{1}{x^2} dx = \int_1^{+\infty} 0 dx = 0,$$

so that

$$f_n \to 0 \qquad \text{in} \qquad L^1([1;+\infty)).$$

Exercise 4

Let $f_n(x) = \sum_{n=1}^{\infty} \frac{|\sin(nx)|}{2^n}$ $\qquad x \in [0, \pi]$. Compute

$$\int_0^\pi f(x)dx.$$

Solution

Remember that a series is the limit of the partial sums. If the partial sums are formed by positive terms, then the series are monotone. We consider

$$f_h(x) = \sum_{n=1}^{h} \frac{|\sin(nx)|}{2^n}.$$

We have truncated the series up to the h term. If we consider the truncated series we have that $f_h(x)$ is monotone, in fact

$$0 \leq f_h \leq f_{h+1}.$$

Furthermore

$$f_h(x) = \sum_{h=1}^{h} \frac{|\sin(nx)|}{2^n} \to f(x) \qquad \text{for} \qquad h \to \infty.$$

Then we can apply the Beppo Levi's Theorem:

$$\int_0^\pi f(x)dx = \int_0^\pi \lim_{h\to\infty} f_h(x)dx = \lim_{h\to\infty} \int_0^\pi f_h(x)dx = \lim_{h\to\infty} \int_0^\pi \sum_{n=1}^{h} \frac{|\sin(nx)|}{2^n} dx = \lim_{h\to\infty} \sum_{n=1}^{h} \frac{1}{2^n} \int_0^\pi |\sin(nx)|dx.$$

Now we have to compute the integral. We know that

$$\int_0^\infty |\sin(y)|dy = n \int_0^\pi \sin(y)dy$$

so that

$$\int_0^\pi |\sin(nx)|dx = \int_0^{n\pi} |siny| \frac{dy}{n} = \int_0^\pi \sin y dy = [-\cos y]_0^\pi = 2.$$

Then

$$\int_0^\pi f(x)dx = \sum_{k=1}^{\infty} (\frac{1}{2})^k 2 = \sum_{k=1}^{\infty} \frac{2}{2^k} = 2.$$

Then

$$\int_0^\pi f(x)dx = 2.$$

Exercise 5

Compute

$$\int_0^{+\infty} \frac{1}{\sqrt{x}} \sum_{k=0}^{\infty} e^{-(\alpha^n)\sqrt{x}} dx$$

as $\alpha \geq 0$ varies.

Solution

Consider

$$\sum_{k=0}^{\infty} e^{l(\alpha)^k \sqrt{x}},$$

it is a series with positive terms, then it converges or positively diverges, that is well defined. If we consider its truncated

$$\sum_{k=0}^{n} e^{-(\alpha^k)\sqrt{x}}$$

we can construct the functions

$$f_{\alpha,n}(x) = \sum_{k=0}^{n} e^{-(\alpha^k)\sqrt{x}}.$$

This is a sequence of functions $f_n \geq 0$ with

$$0 \leq f_n \leq f_{n+1} \qquad \forall n \in \mathbb{N},$$

then it is monotone. We can use the Beppo Levi Theorem:

$$\int_0^{+infty} \frac{1}{\sqrt{x}} \sum_{k=0}^{+\infty} e^{-(\alpha^k)\sqrt{x}} dx = \int_0^{+\infty} \frac{1}{\sqrt{x}} \lim_{n\to\infty} \sum_{k=0}^{n} e^{-(\alpha^k)\sqrt{x}} dx = \star$$

now we can apply Beppo Levi Theorem

$$\star = \lim_{n\to\infty} \sum_{k=0}^{n} \int_0^{+\infty} \frac{e^{-(\alpha^k)\sqrt{x}}}{\sqrt{x}} dx = \star$$

now we can make a change of variables,

$$y = \sqrt{x}$$
$$dy = \frac{1}{2}\frac{1}{y} dx$$
$$dx = 2y dy$$

$$\star = \lim_{n\to\infty} \sum_{k=0}^{n} \int_0^{\infty} \frac{e^{-(\alpha^k)y}}{y} 2y dy = \lim_{n\to\infty} \sum_{k=0}^{\infty} 2 \int_0^{\infty} e^{-(\alpha^k)y} dy$$

Now we have that

$$[e^{-(\alpha^k)y}]' = -(\alpha^k) e^{-(\alpha^k)y}$$

then

$$\lim_{n\to\infty} \sum_{k=0}^{n} 2 \frac{1}{(-\alpha^k)} \int_0^{\infty} -(\alpha^k) e^{-(\alpha^k)y} dy = \lim_{n\to\infty} \sum_{k=0}^{n} 2 \frac{1}{-(\alpha^k)} \int_0^{\infty} [e^{-(\alpha^k)y}]'$$

$$= \sum_{k=0}^{+\infty} \frac{2}{-\alpha^k} \left(\lim_{c\to\infty} [e^{-(\alpha^k)y}]_0^c\right) = \sum_{k=0}^{\infty} \frac{2}{\alpha^k} = 2 \sum_{k=0}^{\infty} \left(\frac{1}{\alpha}\right)^k = \star$$

this is a geometric series, then

$$\star = 2 \begin{cases} +\infty & \text{if} \quad \frac{1}{\alpha} \geq 1 \\ \frac{1}{1-\frac{1}{\alpha}} & \text{if} \quad |\frac{1}{\alpha}| < 1 \\ \text{indet.} & \text{if} \quad \frac{1}{\alpha} \leq -1. \end{cases}$$

But we have that $\alpha \geq 0$, then

$$\int_0^{\infty} \frac{1}{\sqrt{x}} \sum_{k=0}^{\infty} e^{-(\alpha^k)\sqrt{x}} dx = \begin{cases} \frac{2\alpha}{\alpha-1} & \text{if} \quad \frac{1}{\alpha} \in (0,1) \\ \infty & \text{if} \quad \frac{1}{\alpha} \geq 1 \end{cases} = \begin{cases} \frac{2\alpha}{\alpha-1} & \text{if} \quad \alpha > 1 \\ \infty & \text{if} \quad \alpha \in (0,1]. \end{cases}$$

Chapter 2

L^p Spaces

Exercise 1

Analyze the convergence in $L^p([0,1])$ with $1 \leq p < \infty$ of

$$f_n(x) = \frac{\cos(nx)e^{-nx}}{\sqrt[4]{x}} \qquad for \quad x \in [0,1] \qquad \forall n \in \mathbb{N}.$$

For which L^p the sequence converge to a certain function?

Solution

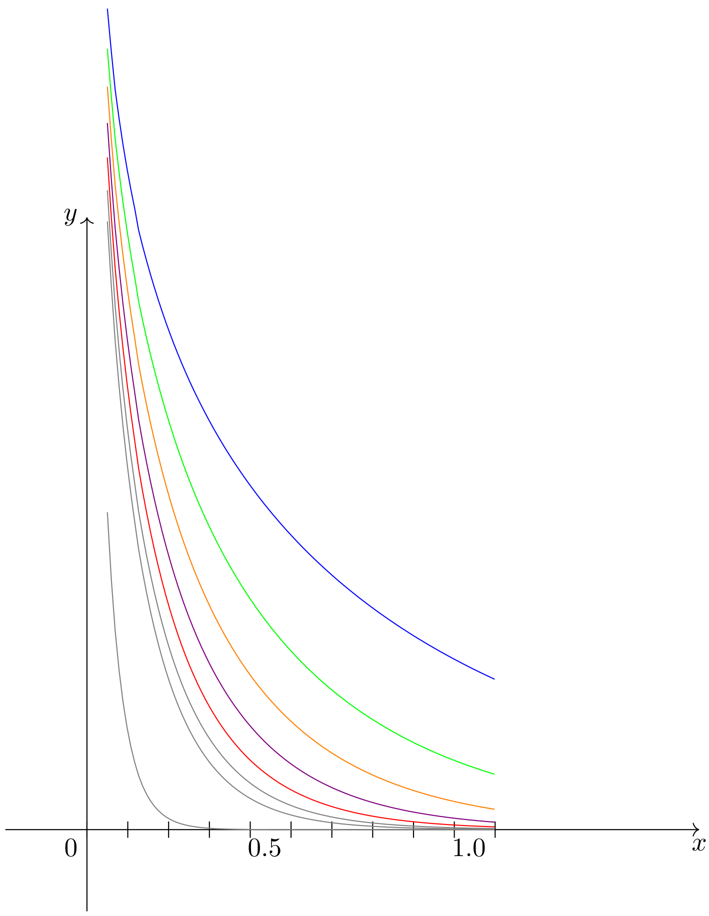


Figure 2.1: The sequence of functions $f_n(x)$.

First of all we search for which p this sequence belongs to some L^p, applying the Dominated Convergence Theorem.

$$|f_n(x)| = |\frac{\cos(nx)}{\sqrt[4]{x}}e^{-nx}| \leq \frac{1}{\sqrt[4]{x}} = g(x) \qquad x \in [0,1]$$

We know that $f_n(x)$ belongs to some L^p if and only if

$$\int_0^1 |f_n(x)|^p dx < +\infty.$$

The exponents p that satisfy this relations are the candidates.

$$\int_0^1 |f_n(x)|^p dx = \int_0^1 |\frac{\cos(nx)}{\sqrt[4]{x}}e^{-nx}|^p dx \leq \int_0^1 |\frac{1}{\sqrt[4]{x}}|^p dx = \int_0^1 \frac{1}{x^{\frac{p}{4}}} dx.$$

From the summability criteria:

$$\int_1^{+\infty} \frac{1}{x^\alpha} dx = \begin{cases} \frac{1}{\alpha-1} & if \quad \alpha > 1 \\ +\infty & if \quad \alpha \leq 1 \end{cases},$$

since

$$|\frac{\cos(nx)}{\sqrt[4]{x}}e^{-nx}|^p \leq (\frac{1}{\sqrt[4]{x}})^p$$

we have that for $p \in [1,4) \qquad f_n(x) \in L^p([0,1]) \qquad \forall n \in \mathbb{N}$.

- $f_n \in L^1([0,1])$;
- $f_n \in L^2([0,1])$;
- $f_n \in L^3([0,1])$.

Pointwise Convergence:

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} \frac{\cos(nx)}{\sqrt[4]{x}}e^{-nx} = \lim_{n\to+\infty} \frac{\cos(nx)}{\sqrt[4]{x}e^{nx}} \to 0$$

so

$$f_n \to 0 \qquad \text{pointwise} \qquad \forall x \in [0,1],$$

we can apply the comparison criterium.

$$\lim_{x\to0^+} f_n(x)\sqrt[4]{x} = \lim_{x\to0^+} \frac{\cos(nx)}{\sqrt[4]{x}}e^{-nx}\sqrt[4]{x} = 1$$

$$f_n(x) \sim \frac{1}{\sqrt[4]{x}} \qquad \text{for} \qquad x \to 0^+.$$

Now we can analyze the convergence in $L^p([0,1])$

$$\|f_n - f\|_{L^p}$$

$$\|f_n(x) - f(x)\|^p_{L^p([0,1])} = \|f_n(x)\|^p_{L^p([0,1])} = \left\|\frac{\cos(nx)}{\sqrt[4]{x}e^{nx}}\right\|^p_{L^p([0,1])} = \int_0^1 |\frac{\cos(nx)}{\sqrt[4]{x}e^{-nx}}_{L^p([0,1])}|^p dx = \star$$

since

$$|\frac{\cos(nx)}{\sqrt[4]{x}e^{nx}}_{L^p([0,1])}|^p \leq g(x) = \frac{1}{x^{\frac{p}{4}}}$$

where

$$g \in L^p([0,1]) \qquad \text{for} \qquad 1 \leq p < 4,$$

we can apply the Dominated Convergence Theorem

$$\lim_{n\to+\infty} \|f_n(x) - f(x)\|^p_{L^p([0,1])} = \lim_{n\to+\infty} \int_0^1 |\frac{\cos(nx)}{\sqrt[4]{x}}e^{-nx}|^p dx = \int_0^1 \lim_{n\to+\infty} |\frac{\cos(nx)}{\sqrt[4]{x}}e^{-nx}|^p dx = 0.$$

So

$$\lim_{n\to+infty} \|f_n(x) - f(x)\|_{L^p([0,1])} \to 0$$

$$f_n(x) \to 0 \qquad in \qquad L^p([0,1]) \qquad \forall p \in [1;4).$$

Since

$$\lim_{x\to 0^+} \frac{|f_n(x)|}{g(x)} = 1 \qquad \forall n \in \mathbb{N}$$

we have

$$f_n \in L^p([0,1]) \leftrightarrow g \in L^p([0,1])$$

so that

$$f_n \notin L^p([0,1]) \qquad \text{if} \qquad p \geq 4.$$

The sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ can't converge in $L^p([0,1])$ spaces if $p \geq 4$.

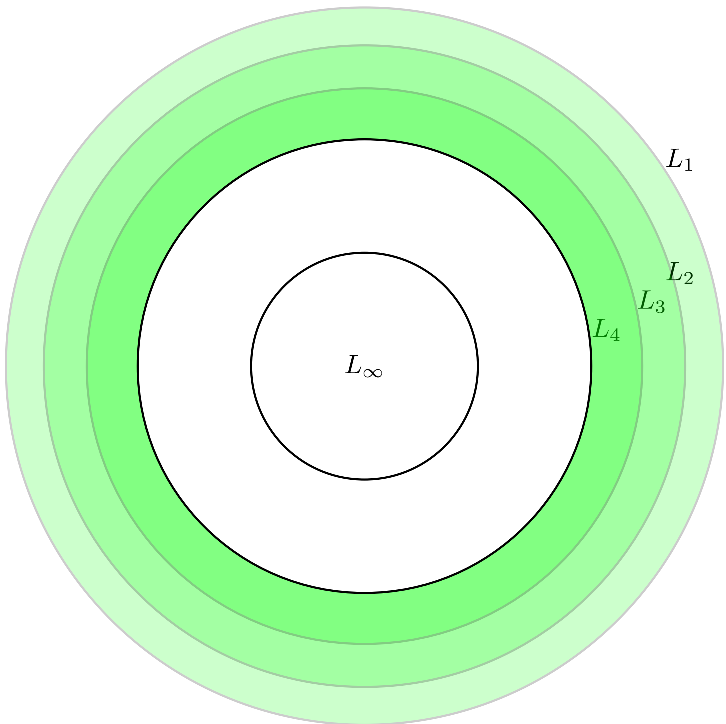
In the case $p = +\infty$, we have

$$\|f_n(x)\|_\infty = \operatorname*{ess\,sup}_{x\in(0,1)} |\frac{\cos(nx)}{\sqrt[4]{x}} e^{-nx}| \leq \sup_{x\in(0,1)} |\frac{1}{\sqrt[4]{x}}| \to +\infty,$$

so

$$f_n \nrightarrow 0 \qquad \text{in} \qquad L^\infty((0,1)).$$

Since $[0,1]$ is a bounded set we have the following embeddings: The sequence $f_n(x)$ lives in "green"



spaces.

Exercise 2

Let $f \in L^\infty([0,+\infty))$ and suppose it is a monotone non-increasing function (weakly decreasing). Let $f \geq 0$.Show that

$$x = f(x) \to 0 \qquad \text{for} \qquad x \to +\infty.$$

Solution

f is weakly decreasing or monotone non-increasing and it is positive. Then

$$\forall x_1 \leq x_2 \implies f(x_2) \leq f(x_1),$$

furthemore it is positive, then

$$\lim_{x\to+\infty} f(x) = l$$

that is

$$\lim_{x\to+\infty} f(x) = l = \inf\{f(x) \qquad \text{s.t.} \qquad x \in dom f, \qquad x > l\}$$

$$f \in L^1([0,+\infty)) \implies l = 0$$

$f \in L^1([0,+\infty))$ means that $\int_0^{+\infty} |f|dx < +\infty$. If it were $l \neq 0$ we would have

$$f \neq L^1([0,+\infty))$$

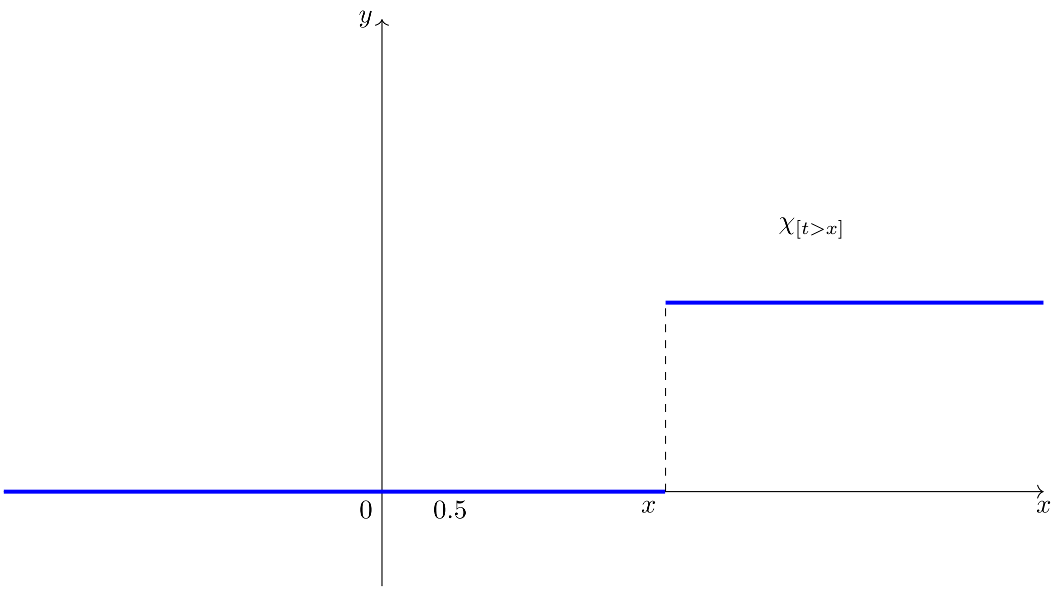
since

$$\int_0^{+\infty} |f|dx \to +\infty.$$

If $f \in L^1([0,+\infty))$ it must be $l = 0$. We now need to show that the product $x \cdot f(x) \to 0$ for $x \to +\infty$. We have

$$\int_0^{+\infty} f(t)dt = \int_0^{+\infty} f(t)\chi_{[t>x]}dt$$

We have that for $x \to +\infty \implies \chi_{[t>x]} \to 0$, in fact: for $x \to +\infty$ it becomes Now we search the

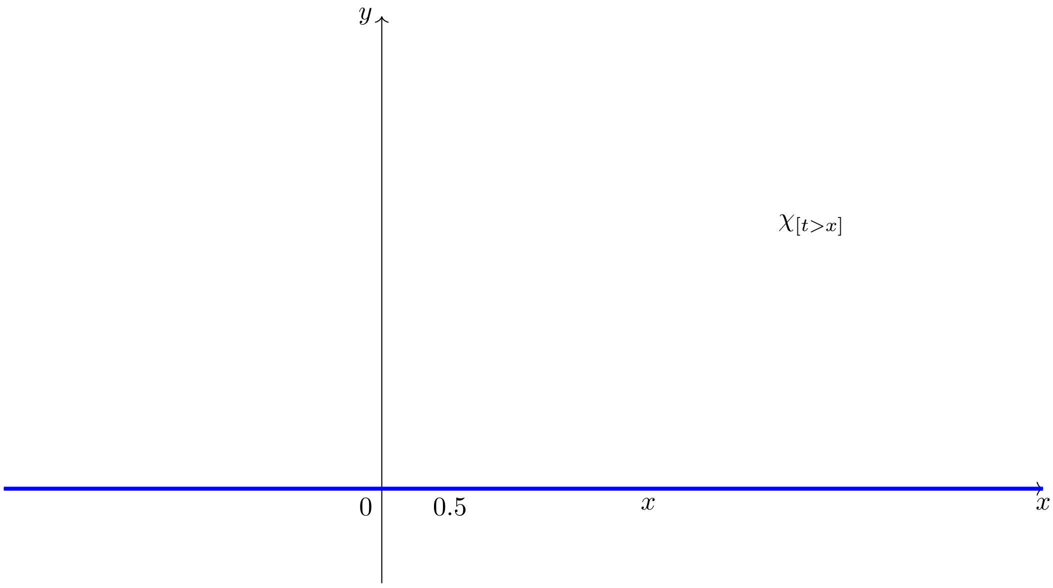


domination for $f(t)\chi_{[t>x]}(t)$, beacouse we want to apply the Dominated Convergence Theorem.

$$f(t)\chi_{[t>x]} \leq |f(t)| \qquad \text{with} \qquad f \in L^1([0,+\infty))$$

$$\text{for} \qquad x \to +\infty \qquad \text{we have}$$

$$f(t)\chi_{[t>x]} \to 0 \qquad \text{a.e.}\mu$$



This function is dominate by $|f(t)|$ that is a L^1 function. Then we can apply the Dominated Convergence Theorem.

$$\lim_{t\to+\infty}\int_0^{+\infty} f(t)dt = \int_0^{+\infty}\lim_{t\to+\infty} f(t)dt = 0$$

$$\forall\epsilon > 0 \qquad \exists x_\epsilon > 0 \qquad \text{s.t.} \qquad \int_{x_\epsilon}^{+\infty} f(x) < \epsilon.$$

Now we take $x \geq x_\epsilon$, we have

$$xf(x) = x_\epsilon f(x) + xf(x) - x_\epsilon f(x) = x_\epsilon f(x) + f(x)\int_{x_\epsilon}^{x} 1dt \leq \star$$

since it is weakly decreasing we can put f inside the integral

$$\star \leq x_\epsilon f(x) + \int_{x_\epsilon}^{x} f(t)dt \leq \star$$

since f is positive then we can integrate between x_ϵ and $+\infty$

$$\star \leq x_\epsilon f(x) + \int_{x_\epsilon}^{+\infty} f(t)dt.$$

Now we can pass to the limit for $x \to +\infty$ and we obtain

$$xf(x) \leq x_\epsilon f(x) + \int_{x_\epsilon}^{+\infty} f(t)dt \leq \epsilon$$

$$\text{for} \qquad x \to +\infty \qquad xf(x) \leq \epsilon \qquad \forall\epsilon > 0$$

then for $x \to +\infty$, $xf(x) \to 0$.

Exercise 3

Find $f \in (L^1(\mathbb{R}) \cap L^\infty_{loc}(\mathbb{R}) \setminus L^2(\mathbb{R})$ and $g \in (L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})) \setminus L^1(\mathbb{R})$.

Solution

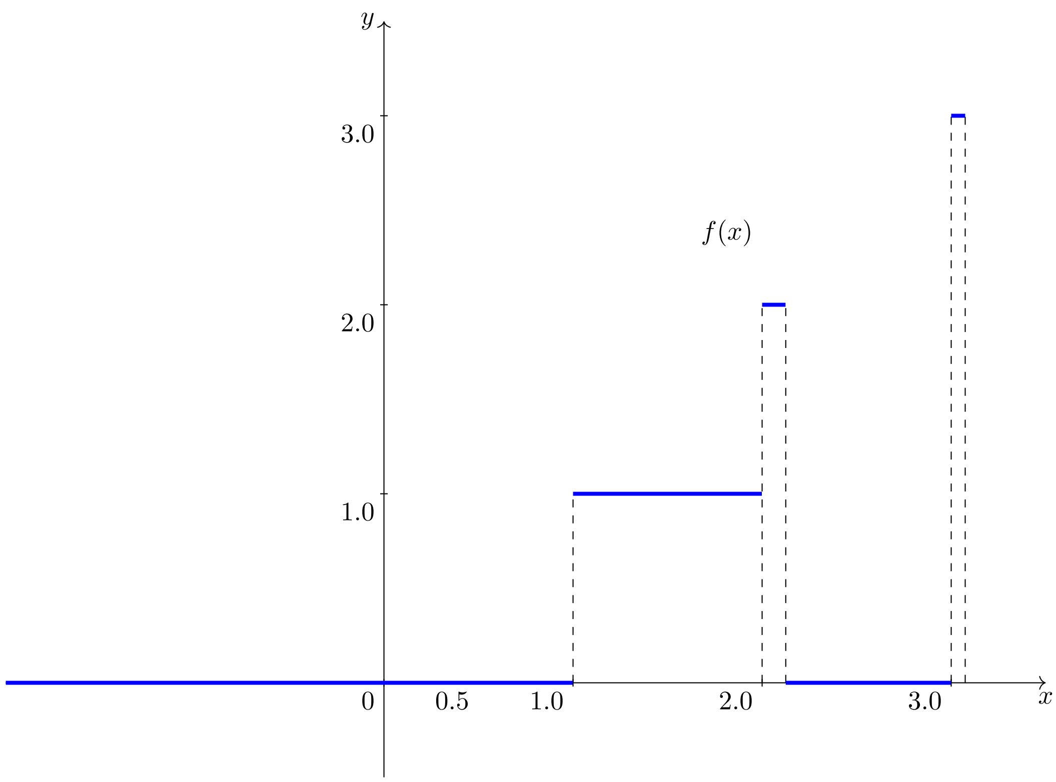
We want to find a function f that belongs to $(L^1(\mathbb{R}) \cap L^\infty_{loc}(\mathbb{R}) \setminus L^2(\mathbb{R})$.

First of all we can construct a function that is in $L^1(\mathbb{R})$ and that is locally bounded (essentially bounded), but that is not in $L^2(\mathbb{R})$.

Generally when think to the term "local" we mean "restricted to a compact set".

We can think to the following function:

$$f(x) = \sum_{k=1}^{\infty} k\chi_{[k,k+\frac{1}{k^3}]}(x) \qquad x \in \mathbb{R}$$



$$f(x) = \chi_{[1,2]} + 2\chi_{[2,\frac{17}{8}]} + 3\chi_{[3,\frac{82}{27}]} + \cdots$$

This function is surely in $L^\infty_{loc}(\mathbb{R})$. In fact $\forall a > 0 \exists k_a$ maximal such that $k_a \le a$. Then

$$f \in L^\infty_{loc}.$$

We know that

$$\sup_{a \in A} f = k_a,$$

since the rung at the k-th step is k high. The norm L^1 is given by the sum of the area of each rectangle. Then

$$\|f\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |f| dx < +\infty,$$

$$\frac{1}{k^3} \to 0 \qquad \text{for} \qquad k \to +\infty.$$

Now we want to show that $f \notin L^2(\mathbb{R})$. We consider the truncated

$$f_n = \sum_{k=1}^{n} k\chi_{k,k+\frac{1}{k^3}}(x),$$

and make all the calculus. Trivially $f_n \geq 0$, then we have a series with positive terms. Furthermore f_n is monotone since

$$0 \leq f_n \leq f_{n+1}.$$

We can apply the Beppo Levi Theorem (or the Monotone Convergence Theorem).

$$\|f\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |f|dx = \int_{\mathbb{R}} \lim_{n\to+\infty} \sum_{k=1}^{n} k\chi_{[k,k+\frac{1}{k^3}]}(x)dx = \star$$

we now apply the Beppo Levi Theorem

$$\star = \sum_{k=1}^{\infty} \int_{\mathbb{R}} k\chi_{[k,k+\frac{1}{k^3}]}(x)dx = \sum_{k=1}^{\infty} k \int_{\mathbb{R}} \chi_{[k,k+\frac{1}{k^3}]}(x)dx = \sum_{k=1}^{\infty} k[x]_k^{k+\frac{1}{k^3}} = \sum_{k=1}^{\infty} k(k+\frac{1}{k^3}-k) = \sum_{k=1}^{\infty} \frac{1}{k^2} < +\infty$$

this is a generalized harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n^\lambda} = \begin{cases} \text{divergent} & \text{if} \quad \lambda \leq 1 \\ \text{convergent} & \text{if} \quad \lambda > 1 \end{cases}$$

Then we have that

$$\|f\|_{L^1(\mathbb{R})} < +\infty \implies f \in L^1(\mathbb{R}).$$

This shows us that we can have a function that is locally bounded ($L^\infty_{loc}(\mathbb{R})$) and that is also summable on $\mathbb{R}$, ($L^1(\mathbb{R})$). Now we show that $f \notin L^2(\mathbb{R})$. We have that

$$f \in L^2(\mathbb{R}) \iff (\int_{\mathbb{R}} |f|^2 dx)^{\frac{1}{2}} = \|f\|_{L^2(\mathbb{R})} < +\infty.$$

We still apply the Beppo Levi Theorem.

$$f_n^2(x) = \sum_{k=1}^{n} k^2\chi_{[k,k+\frac{1}{k^3}]}(x) \qquad f_n^2 \geq 0 \qquad 0 \leq f_n^2 \leq f_{n+1}^2$$

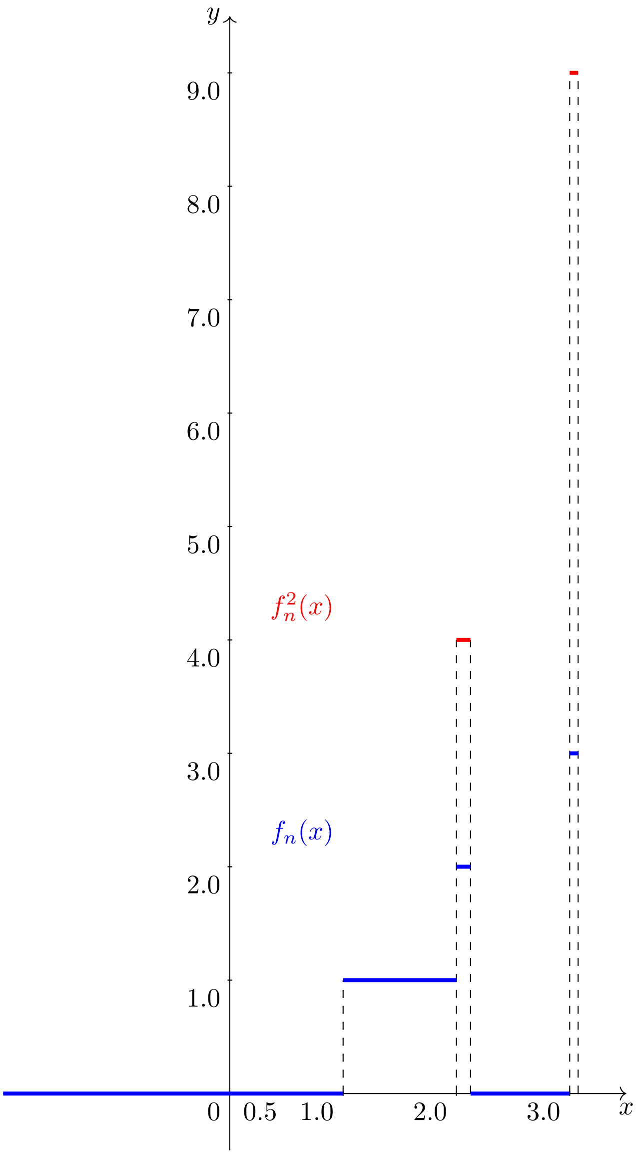
$$\|f_n\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |f(x)|^2 = \lim_{n\to+\infty} \int_{\mathbb{R}} \sum_{k=1}^{n} k^2\chi_{[k,k+\frac{1}{k^3}]}(x)dx = star$$

we apply the Beppo Levi Theorem

$$\star = \int_{\mathbb{R}} \sum_{k=1}^{\infty} k^2\chi[k,k+\frac{1}{k^3}](x)dx = \sum_{k=1}^{\infty} k^2 \int_{\mathbb{R}} \chi_{[k,k+\frac{1}{k^3}]}(x)dx = \sum_{k=1}^{\infty} k^2[x]_k^{k+\frac{1}{k^3}} = \sum_{k=1}^{\infty} \frac{1}{k} \to +\infty,$$

since it is a harmonic series:

$$\sum_{n=1}^{\infty} \frac{1}{n} \to +\infty$$



that diverges positively. Then

$$f \notin L^2(\mathbb{R}).$$

We have found a function

$$f(x) = \sum_{k=1}^{\infty} k\chi_{[k,k+\frac{1}{k^3}]}(x)$$

such that

$$f \in (L^1(\mathbb{R}) \cap L^{\infty}_{loc}(\mathbb{R})) \setminus L^2(\mathbb{R}).$$

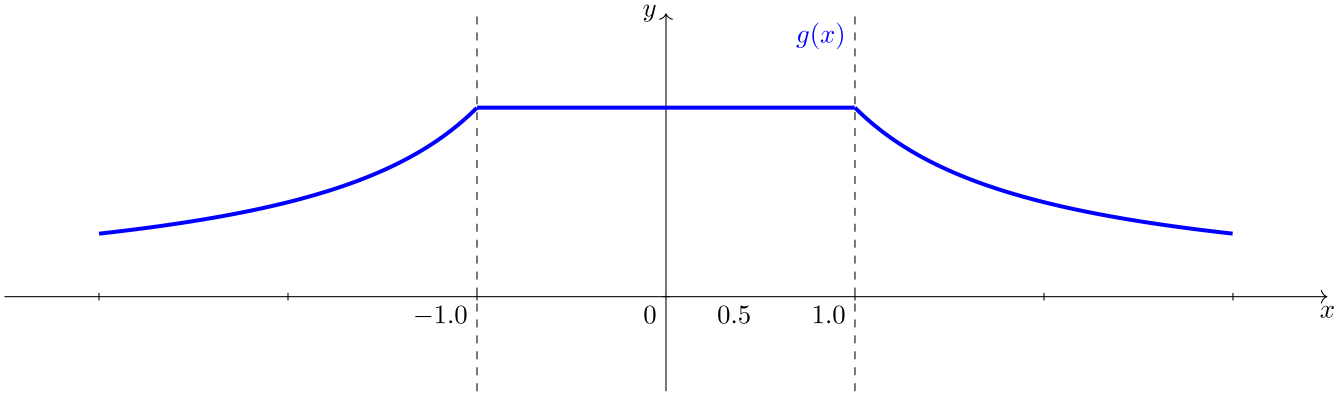
Now we want to find a function

$$g \in (L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})) \setminus L^1(\mathbb{R}).$$

We consider a function

$$g(x) = \begin{cases} 1 & \text{if} \quad |x| \leq 1 \\ \frac{1}{|x|} & \text{if} \quad |x| > 1 \end{cases}$$

g is certainly bounded since



$$\operatorname*{ess\,sup}_{\mathbb{R}} g(x) = \sup_{\mathbb{R}} g(x) = 1$$

then

$$g \in L^\infty(\mathbb{R}).$$

Now we show that $g \notin L^1(\mathbb{R})$, in fact

$$\|g\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |g(x)| dx = \int_{-\infty}^{+\infty} g(x) dx = \int_{-1}^{1} g(x) dx + \int_{|x|>1} \frac{1}{|x|} dx,$$

the first integral,

$$\int_{-1}^{1} 1 dx = [x]_{-1}^{1} = 2,$$

the second integral

$$\int_{-\infty}^{-1} -\frac{1}{x} dx + \int_{1}^{+\infty} \frac{1}{x} \to +\infty.$$

Then

$$g \notin L^1(\mathbb{R}).$$

Finally we show that $g \in L^2(\mathbb{R})$.

$$\|g(x)\|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{+\infty} g(x)^2 dx = \int_{-1}^{1} g(x)^2 dx + \int_{|x|>1} \frac{1}{|x|^2} dx = \int_{-1}^{1} 1 dx + \int_{-\infty}^{-1} \frac{1}{(-x)^2} dx + \int_{1}^{+\infty} \frac{1}{x^2} dx < +\infty,$$

since the generalized harmonic series. Then

$$g \in L^2(\mathbb{R}).$$

We have found a function $g \in (L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})) \setminus L^1(\mathbb{R})$.

Chapter 3

Hilbert Spaces

Exercise 1

Let $X = (C(0,1); \|\cdot\|_\infty)$ and consider

$$K = \{f \in X : \int_0^{\frac{1}{2}} f(t)dt - \int_{\frac{1}{2}}^1 f(t)dt = 1\}.$$

Show that K is closed and not empty, and determine the projection of 0 over the set K.

Solution

K is not empty. To show that we can take:

$$f(t) = \frac{\pi}{2}\sin(2\pi t).$$

Now we can consider:

$$u(t) = \chi_{(0,\frac{1}{2})}(t) - \chi_{(\frac{1}{2},1}(t)$$

and consider the following operator:

$$(Tf) = \int_0^1 fudt$$

where

$$T : C(0,1) \to \mathbb{R}$$
$$K = T^{-1}(\{1\})$$

since $\{1\}$ is a singleton then it is closed; the contraimage of a closed set must be closed, so K is closed.

$$|Tf| \leq C\,\|f\|_\infty$$

$$f \in C(0,1) \qquad \|f\|_\infty \leq 1$$

then

$$f \notin K,$$

that is the elements of K are of the form

$$\|\cdot\|_\infty > 1.$$

We have that

$$f(x) \leq \|f\|_\infty \leq 1 \qquad \forall x \in (0,1)$$

"by contradiction"

$$f \in K \qquad \int_0^1 fu = 1$$

$$1 = \int_0^1 fu \leq \int_0^1 |f||u| \leq \|f\|_\infty \int_0^1 dt = \|f\|_\infty \leq 1.$$

We have that

$$|fu| \leq 1 \qquad \int_0^1 |fu| = 1$$

so that

$$|fu| = 1 \quad a.e.$$

$$\int_0^1 (1 - |fu|) = 0 \qquad a.e.$$

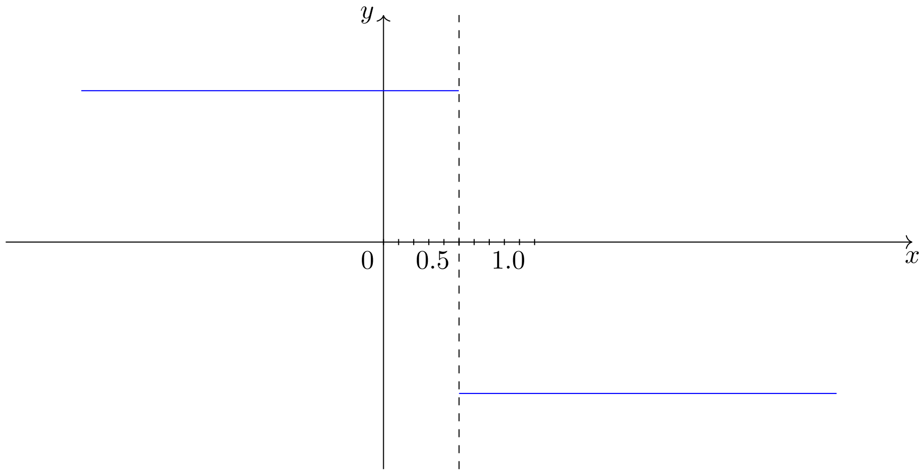
so that

$$fu = 1 \qquad a.e..$$

So we obtain this contradiction:

$$\begin{cases} f = 1 & if \quad x \in (0, \frac{1}{2}) \\ f = -1 & if \quad x \in (\frac{1}{2}, 1) \end{cases}$$

Now we consider:



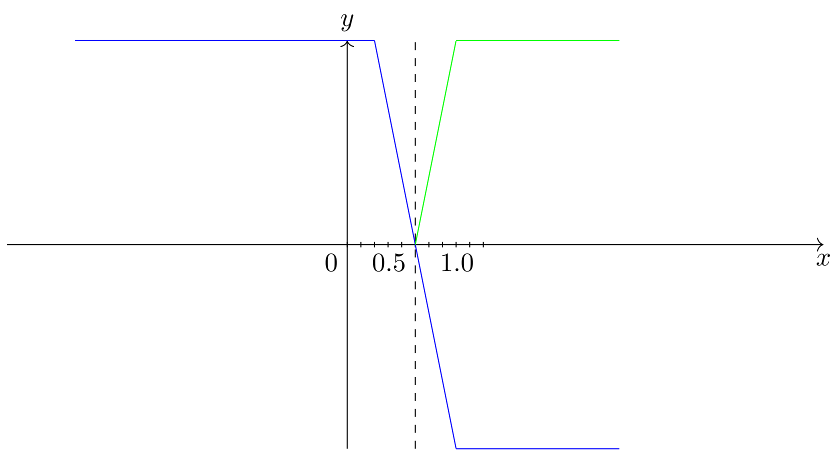
$$d = \inf\{\|f\|_\infty : f \in K\}$$

$$\|f\|_\infty \leq 1 \qquad \Longrightarrow \qquad f \notin K$$

$$d = \inf\{\|f\|_\infty : f \in K\} \geq 1,$$

we can take $1 < \alpha < 2$ and $\epsilon = \frac{\alpha - 1}{\alpha}$.

$$f_\alpha = -\frac{\alpha}{\epsilon}(x - \frac{1}{2})$$



Now we show that $f_\alpha = -\frac{\alpha}{\epsilon}(x - \frac{1}{2})$ belongs to K.

$$\int_0^{\frac{1}{2}} f_\alpha - \int_{\frac{1}{2}}^1 f_\alpha = 1 \qquad \forall \alpha \in (1,2)$$

$$\|f_\alpha\|_\infty = \alpha,$$

α is the supremum,

$$d = \inf\{\|f\|_\infty : f \in K\} \leq \|f_\alpha\|_\infty = \alpha$$

$$\alpha \in (1,2)$$

$$\begin{cases} d \leq 2 \\ \forall \alpha \in (1,2) \end{cases} \implies d \leq 1$$

so that

$$d = \inf\{\|f\|_\infty : f \in K\} = 1,$$

but this inf is not assumed, this is not a minimum, then

$$\nexists f \in K \qquad \text{s.t.} \qquad d = \|f\|_\infty = 1,$$

$$d = d(0,K)$$

$$0 \notin K.$$

Chapter 4

Operators

Exercise 1

Let

$$a(x) = \begin{cases} x & if \quad x \in (0, \frac{1}{2}] \\ 0 & if \quad x \in (\frac{1}{2}, 1] \end{cases}$$

and consider the operator

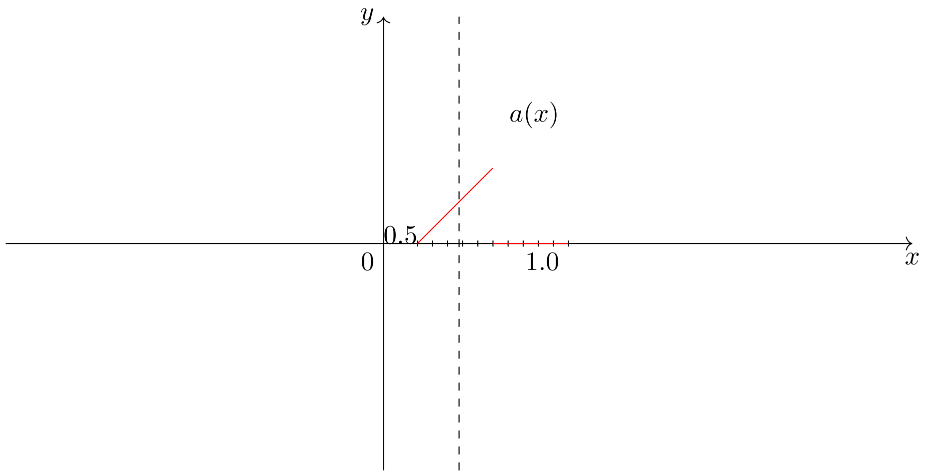
$$T : L^2(0,1) \to L^2(0,1)$$

given by

$$Tf(x) = a(x)f(x), \qquad x \in (0,1).$$

Show that $T \in \mathcal{L}(L^2[0,1])$ and compute $\|T\|$.

Solution



$$f \in L^2(0,1)$$

$$\|Tf\|^2_{L^2(0,1)} = \int_0^1 a(x)^2 f(x)^2 dx \leq \frac{1}{4} \int_0^1 f(x)^2 dx = \frac{1}{4} \|f\|^2_{L^2(0,1)}$$

so that

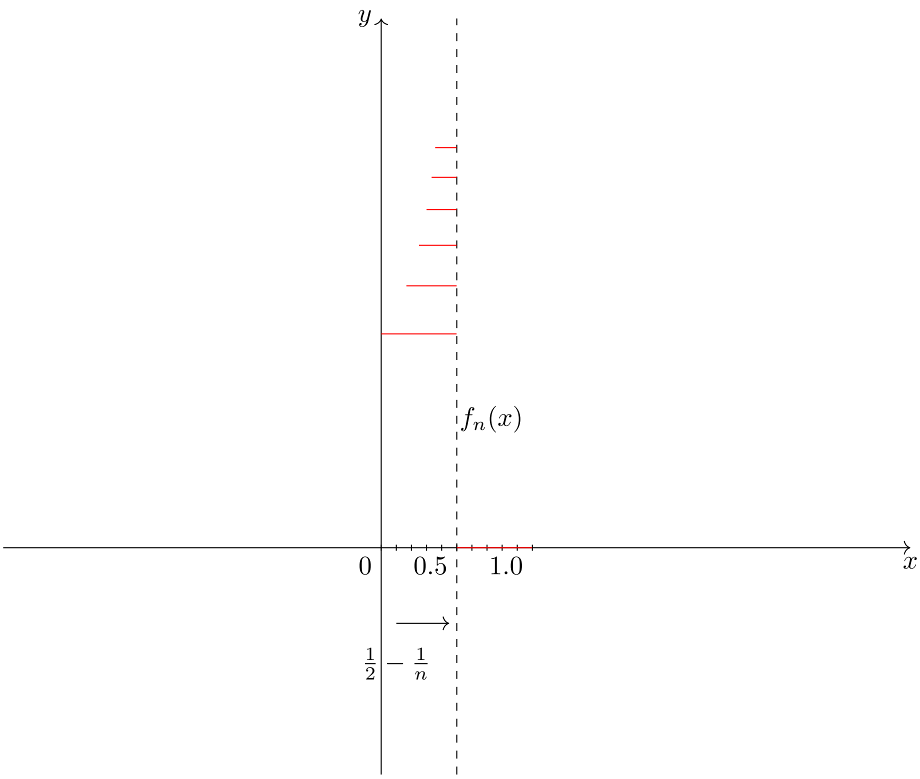
$$\|T\|_{\mathcal{L}(L^2(0,1))} \leq \frac{1}{2}.$$

Then we want to show that $\frac{1}{2}$ is the value of the norm, that is

$$\|T\| = \frac{1}{2}.$$

We define

$$f_n(x) = \begin{cases} \sqrt{n} & if \quad x \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2}], \qquad n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$



$$\|f_n(x)\|^2_{L^2(0,1)} = \int_0^1 f_n(x)^2 = 1$$

Now we compute the norm of the image:

$$\|Tf_n(x)\|^2_{L^2(0,1)} = \int_0^1 a(x)^2 f_n(x)^2 dx \geq \star$$

we can minor the integral with

$$\star \geq \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}} n(\frac{1}{2} - \frac{1}{n})^2 dx = (\frac{1}{2} - \frac{1}{n})^2 \to \frac{1}{4} \qquad \text{for} \quad n \to +\infty$$

so that

$$\|T\|_{\mathcal{L}(L^2(0,1))} = \frac{1}{2}.$$

We can characterize the norm in various ways.

$$\|T\|_{\mathcal{L}(L^2(0,1))} = \sup_{f \in L^2(0,1) \quad \|f\|_{L^2}=1} \frac{\|Tf\|_{L^2(0,1)}}{\|f\|_{L^2(0,1)}} \leq \frac{1}{2}.$$

We have shown that

$$\|Tf_n(x)\|^2_{L^2(0,1)} \geq \frac{1}{4},$$

that is

$$\|Tf_n(x)\|_{L_2(0,1)} \geq \frac{1}{2},$$

but at the same time we have

$$\|T\| \leq \frac{1}{2}$$

then

$$\|Tf_n(x)\| = \frac{1}{2}.$$

Exercise 2

Let $(f'_h)_{h\in\mathbb{N}} \in L^p$ for some p, with the following hypotheses:

- $(f'_h)_{h\in\mathbb{N}}$ is bounded in L^p for some p;
- $f_h(0)$ is bounded.

Show that $(f_h)_{h\in\mathbb{N}}$ is compact (relatively) in $(C([0,1]), \|\cdot\|_\infty$.

Solution

From the hypotheses we can suppose that

$$f_h(x) = f_h(0) + \int_0^x f'_h(y)dy \qquad x \in [0,1].$$

Since we have to show the compactness in the set of continuous fuunctions we need to utilize the Ascoli-Arzelà theorem.

Let $C > 0$ constant.

Equiboundedness

$$h \in \mathbb{N}, \qquad x \in [0,1]$$

$$|f_h(x)| \leq |f_h(0)| + \int_0^x |f'_h(y)|dy \leq \star$$

using the hypothesis 2 and the Hölder inequality

$$\star \leq C + \|f'_h\|_{L^p(0,1)} x^{\frac{1}{p'}} \leq M$$

so that

$$|f_h(x)| \leq M,$$

that is f_h is equibounded.

Equicontinuity

$$x, y \in [0,1], \qquad x < y$$

$$f_h(y) - f_h(x) \leq \int_x^y |f'_h(w)|dw \leq \star$$

using the Hölder inequality

$$\star \leq \|f'_h\|_{L^p(0,1)} |y-x|^{\frac{1}{p'}}.$$

This shows that the fuunctions f_h are equi-hölder with exponent $\frac{1}{p'}$, in particular they are eqicontinuous.

Then from the Ascoli-Arzelà theorem they are relatively compact.