A collection of Real Analysis exercises

June 14, 2024

Chapter 1

Metric Spaces

Exercise 1

Let the distances in $\mathbb{R}^2$:

1. $d_1(X,Y) = \sum_{i=1}^{2} |y_i - x_i| = |y_1 - x_1| + |y_2 - x_2|$;
2. $d_2(X,Y) = \sqrt{\sum_{i=1}^{2} |y_i - x_i|^2} = \sqrt{|y_1 - x_1|^2 + |y_2 - x_2|^2}$;
3. $d_\infty(X,Y) = \max_{i=1,2} |y_i - x_i| = \max\{|y_1 - x_1|, |y_2 - x_2|\}$.

Construct the open balls related to these distances.

Solution

Point 1

$$d_1(X,Y) < r$$

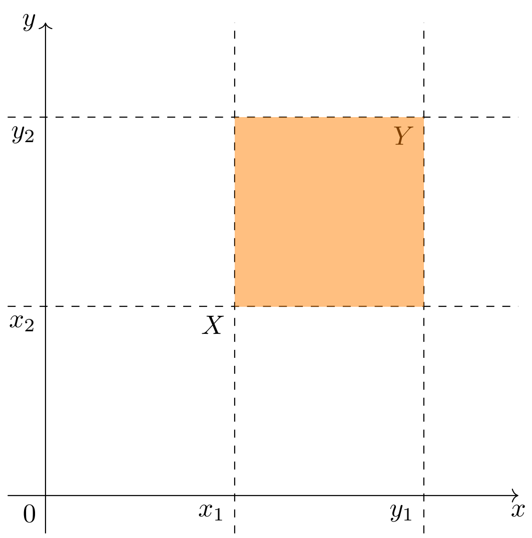
$$\text{iff}$$

$$\sum_{i=1}^{2} |y_i - x_i| < r$$

$$\text{iff}$$

$$|y_1 - x_1| + |y_2 - x_2| < r$$

Point 2



$$d_2(X;Y) < r$$

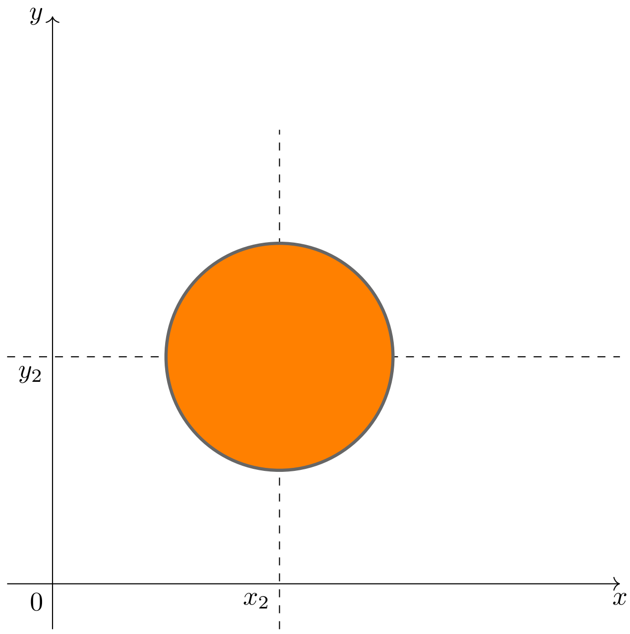
iff

$$\sqrt{|y_1 - x_1|^2 + |y_2 - x_2|^2} < r$$

iff

$$|y_1 - x_1|^2 + |y_2 - x_2|^2 < r^2$$

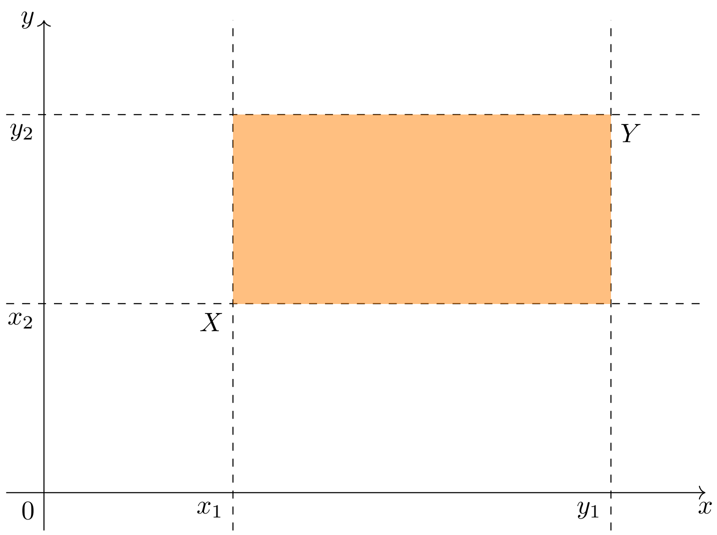
Point 3



$$d_\infty(X,Y) < r$$

iff

$$\max\{|y_1 - x_1|, |y_2 - x_2|\}$$



Exercise 2

Let (X, d) a metric space, $A \subset X$ not empty. Show if the following statements are true or false:

1. $A \quad \text{open} \quad \implies \mathring{A} \cap \partial A = \emptyset$;
2. if $\mathring{A} \cap \partial A = \emptyset \implies A \quad \text{closed}$;

Solution

Point 1
The first statement is always true. It is also true if A is closed or if A is neither open or closed.

$$\partial A = \{x \in X \text{s.t.} x \notin \mathring{A} \quad \text{and} \quad x \notin \overset{\circ}{\overbrace{X \setminus A}}\ \}$$

$$\implies \mathring{A} \cap \partial A = \emptyset \quad \text{is} \quad \text{true.}$$

Point 2
The second statement is false. Counterexample:

$$A =]0, 1[$$

is an open set with

$$\partial A = \{0, 1\}$$

and we have that $\mathring{A} \cap \partial A = \emptyset \implies A \quad \text{is} \quad \text{closed}$ is false because A is open.

Exercise 3

Let (X,d) a metric space $A \subset X$ closed and $A \neq \emptyset$. Furthermore let

$$f : X \to \mathbb{R}$$

with

$$f(x) = \inf_{a \in A \setminus \{x\}} d(x,a).$$

Tell wether the following statements are true or false.

1. $x \in A \implies f(x) = 0$;
2. $f(x) = 0 \implies x \in A$;

Solution

Point 1

This statement is not true $\forall x$. Counterexample:

$$A = [0,1] \cup \{2\}$$

we have that $2 \in A$, but $f(2) \neq 0$ because $f(2) = \inf d(2,a)$ with $a \in [0,1]$. If $a \in [0,1]$ the distance of 2 from a is greater(or equal) than the distance of 2 from 1.

$$\text{If} \qquad a \in [0,1]$$

$$\text{then} d(2,a) \geq d(2,1) = 1$$

so that

$$\text{if} d(2,a) \geq 1 \implies \forall a \in [0,1] \qquad d(2,a) \geq 1.$$

Then

$$\inf d(2,a) \geq 1 \qquad a \in [0,1]$$

and it can't be equal to zero.

Point 2

Remeber that x is an accumulation point for A iff $\inf_{a \in A \setminus \{x\}} d(x,a) = 0$.

$$f(x) = 0 \implies x \qquad \text{is an accumulation point for A}$$

$$\text{then} \qquad x \in \overline{A} = A \qquad \text{since A is closed}$$

then the second statement is true.

Exercise 4

Let $X = C^0([0,1])$, $d_\infty(f,g) = \sup_{t\in[0,1]} |f(t)-g(t)|$ and $d_2(f,g) = \sqrt{\int_0^1 |f(t)-g(t)|^2 dt}$. Show that d_2 and d_∞ are not equivalent.

Solution

If we choose $f_n(t) = t^n \qquad \forall n \in \mathbb{N}$, we have that:

$$d_\infty(f_n, 0) = \sup_{t\in[0,1]} |f_n(t)| = \sup_{t\in[0,1]} |t^n| = 1.$$

It is a maximum.

$$(d_2(f_n,0))^2 = \int_0^1 |f_n(t)|^2 dt = \int_0^1 (t)^{2n} dt = [\frac{t^{2n+1}}{2n+1}]_0^1 = \frac{1}{2n+1}$$

then

$$d_2(f_n,0) = \frac{1}{2n+1}$$

for $n \to \infty$, $d_2 \to 0$. So that

$$\forall c \in \mathbb{R} \exists n \in \mathbb{N} \qquad \text{s.t.} \qquad d_\infty(f_n,0) > c d_2(f_n,0)$$

then d_2 and d_∞ are not equivalent.

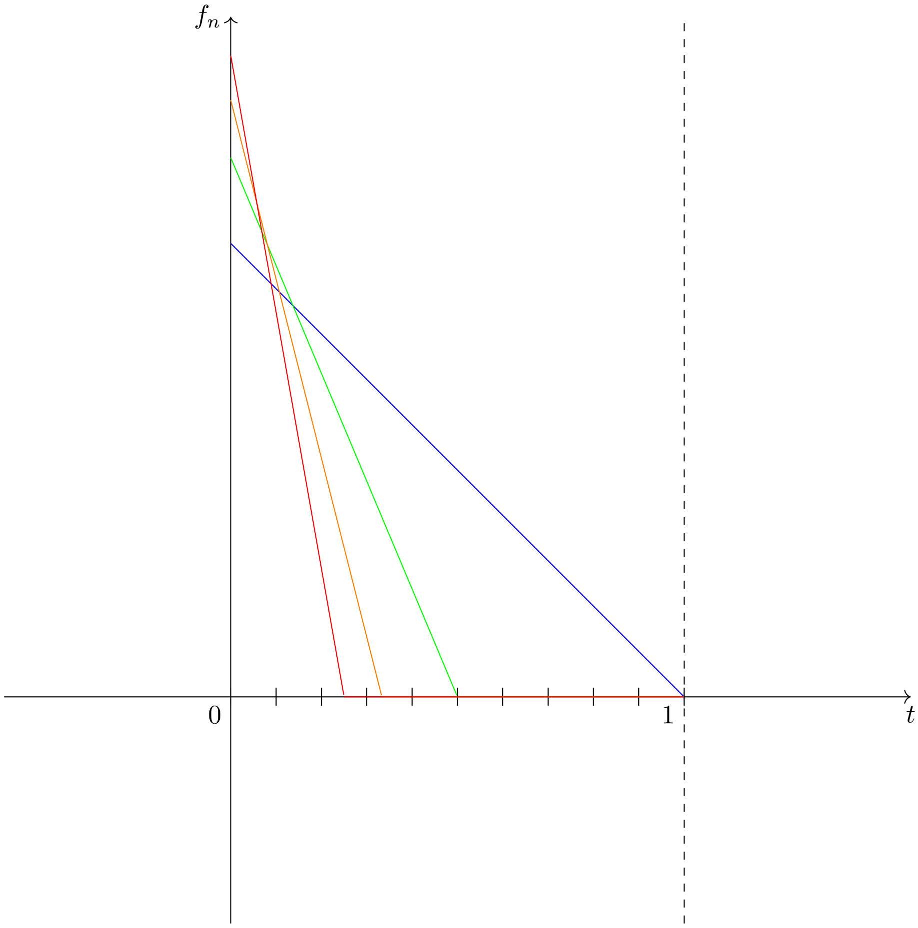
Exercise 5

Let $X = C^0([0,1])$. Show that the open ball $B_{d_2}(0,1)$ is unbounded with respect to d_∞.

Solution

Consider the following sequence of functions:

$$f_n(t) = \begin{cases} -n^{\frac{5}{4}}(t - \frac{1}{n}) & \text{for} \quad t \in [0, \frac{1}{n}] \\ 0 \quad \text{for} & t \in]\frac{1}{n}, 1] \end{cases}$$



$$d_2(f,g) = \sqrt{\int_0^1 |g(t) - f(t)|^2 dt}$$

$$d_\infty(f_n, 0) = \sup_{t \in [0,1]} |f_n(t)| = f_n(0) = \sqrt[4]{n} \to \infty$$

$$(d_2(f_n, 0))^2 = \int_0^{\frac{1}{n}} -n^{\frac{5}{2}}(t - \frac{1}{n})dt = [-\frac{n^{\frac{5}{2}}}{3}(t - \frac{1}{n})^3]_0^{\frac{1}{n}} = \frac{n^{\frac{5}{2}}}{3}\frac{1}{n^3} = \frac{1}{3\sqrt{n}}$$

then

$$d_s(f_n, 0) = \frac{1}{\sqrt{3\sqrt{n}}}.$$

With respect to d_∞ the function is unbounded because contains a sequence the goes to infinity.

Exercise 6

Say if $[0, +\infty[$ is bounded in $(\mathbb{R}, d_0)$ and in $(\mathbb{R}, d)$, with

- d_0 the discrete metric;
- d the Euclidean metric;

Solution

The discrete metric is characterized by the fact that the distance between two points is equal to zero or one.

$$d_0(x, y) = \begin{cases} 1 & \text{if} \quad x \neq y \\ 0 & \text{if} \quad x = y \end{cases}$$

so that

$$diam([0, +\infty[) = \sup_{x,y \in [0,+\infty[} d_0(x, y) \leq 1,$$

then $[0, +\infty[$ is bounded in $(\mathbb{R}, d_0)$.

If we consider the Euclidean distance

$$diam([0, +\infty[) = \sup_{x,y \in [0,+\infty[} d(x, y) = \sup_{x,y \in [0,+\infty[} |y - x| \geq n \qquad \forall n \in \mathbb{N},$$

then $diam([0, +\infty[) = +\infty$, so that $[0, +\infty[$ is unbounded with d.

Exercise 7

Let (X,d) a metric space, $f : X \to \mathbb{R}$ a continous function and $A \subset X$ bounded. Say if the following statements are true or false.

1. $f(A)$ is connected;
2. $f(A)$ is compact;
3. $f(A)$ is open;

Solution

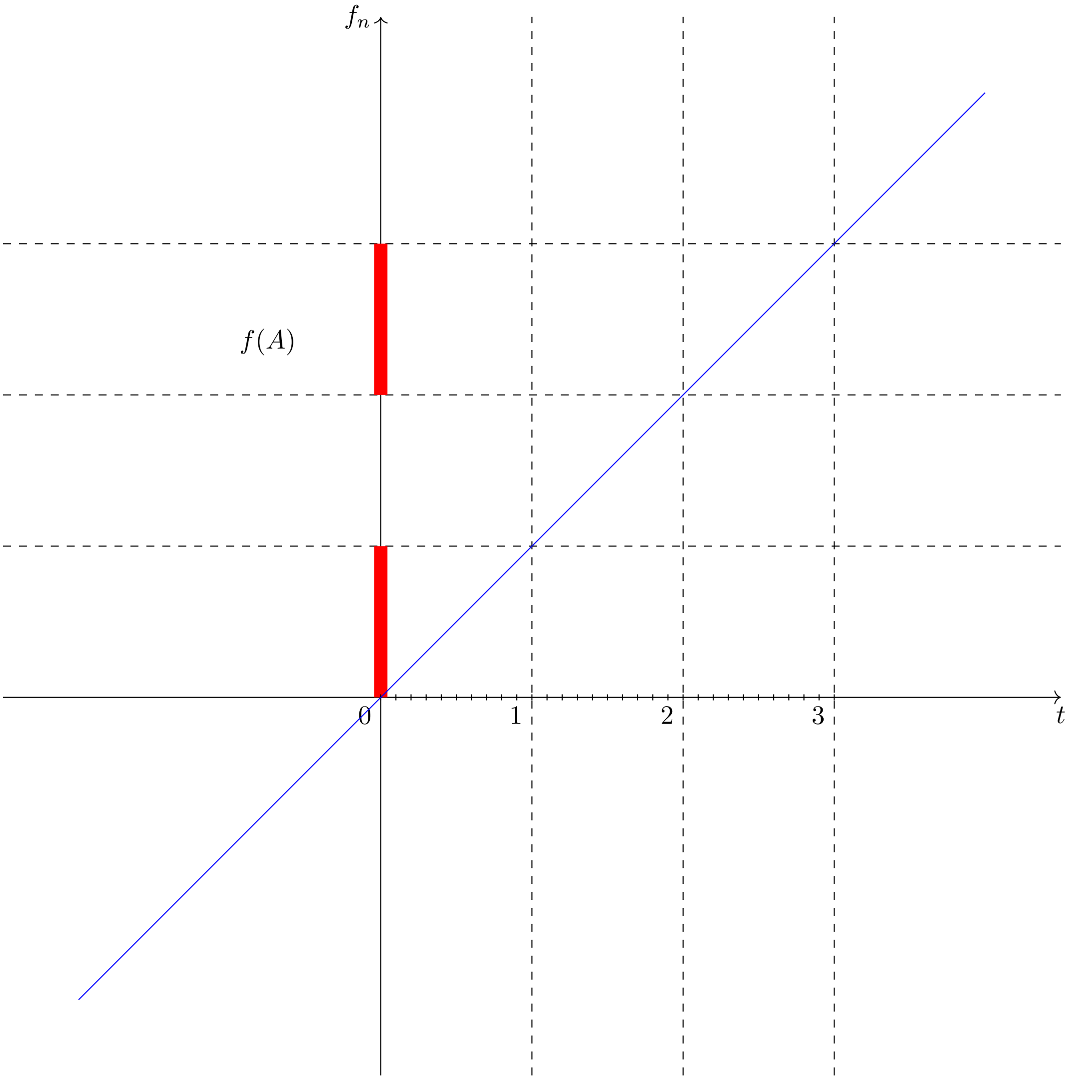
Point 1
Consider $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x$$

and consider $A = [0,1] \cup [2,3]$, there is no request on A, so that

$$f(A) = [0,1] \cup [2,3]$$

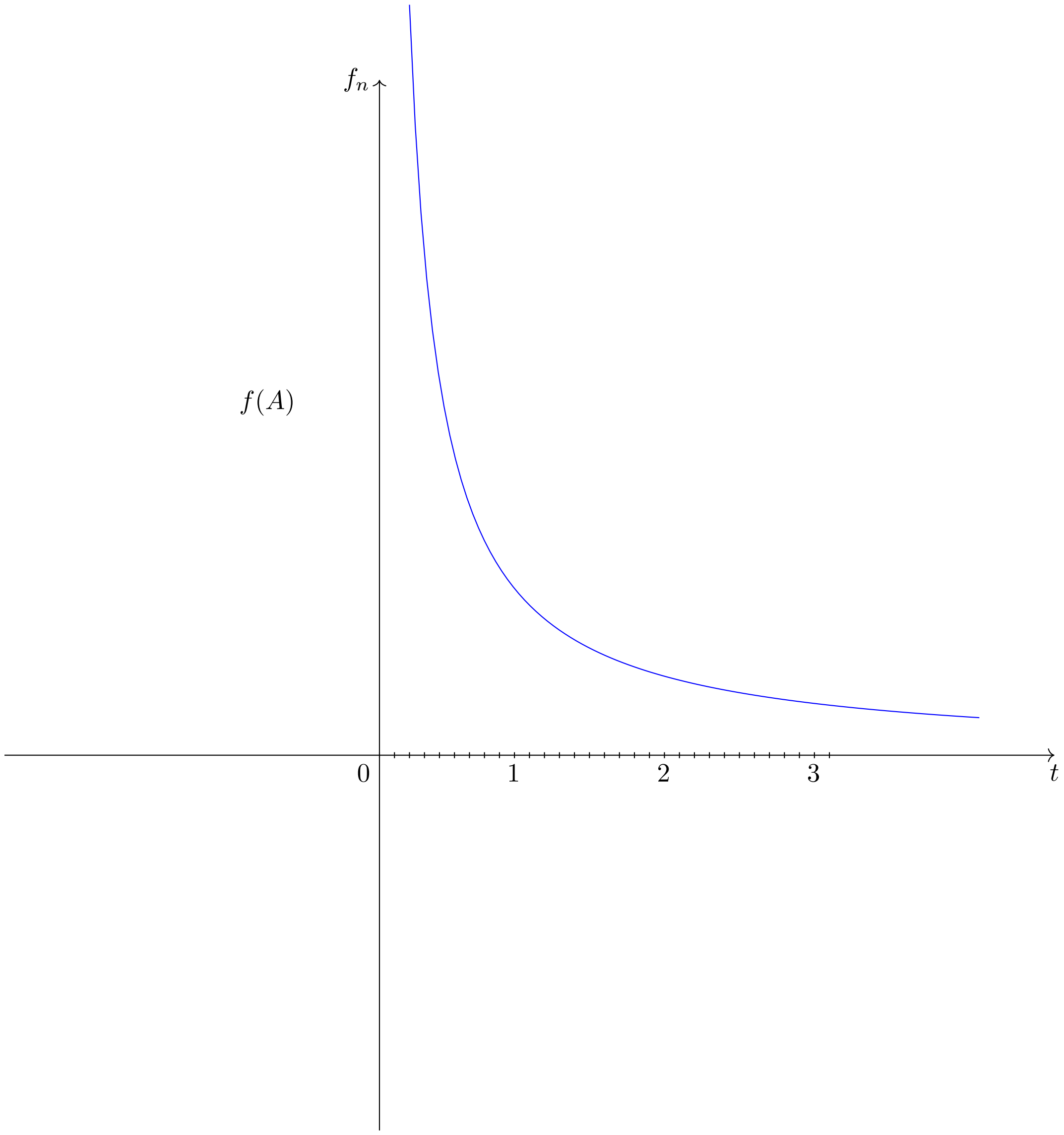
is not connected. **Point** 2



Consider $f :]0, +\infty[\to \mathbb{R}$, given by

$$f(x) = \frac{1}{x}$$

If we choose $A =]0, 1]$ we have $f(a) = [1, +\infty[$ that is not compact because it is not bounded.



Point 3
If we consider $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 1$,

$$A = [0, 1]$$

$$f(A) = \{1\} \qquad \text{that is closed.}$$

Then all the three statements are false.

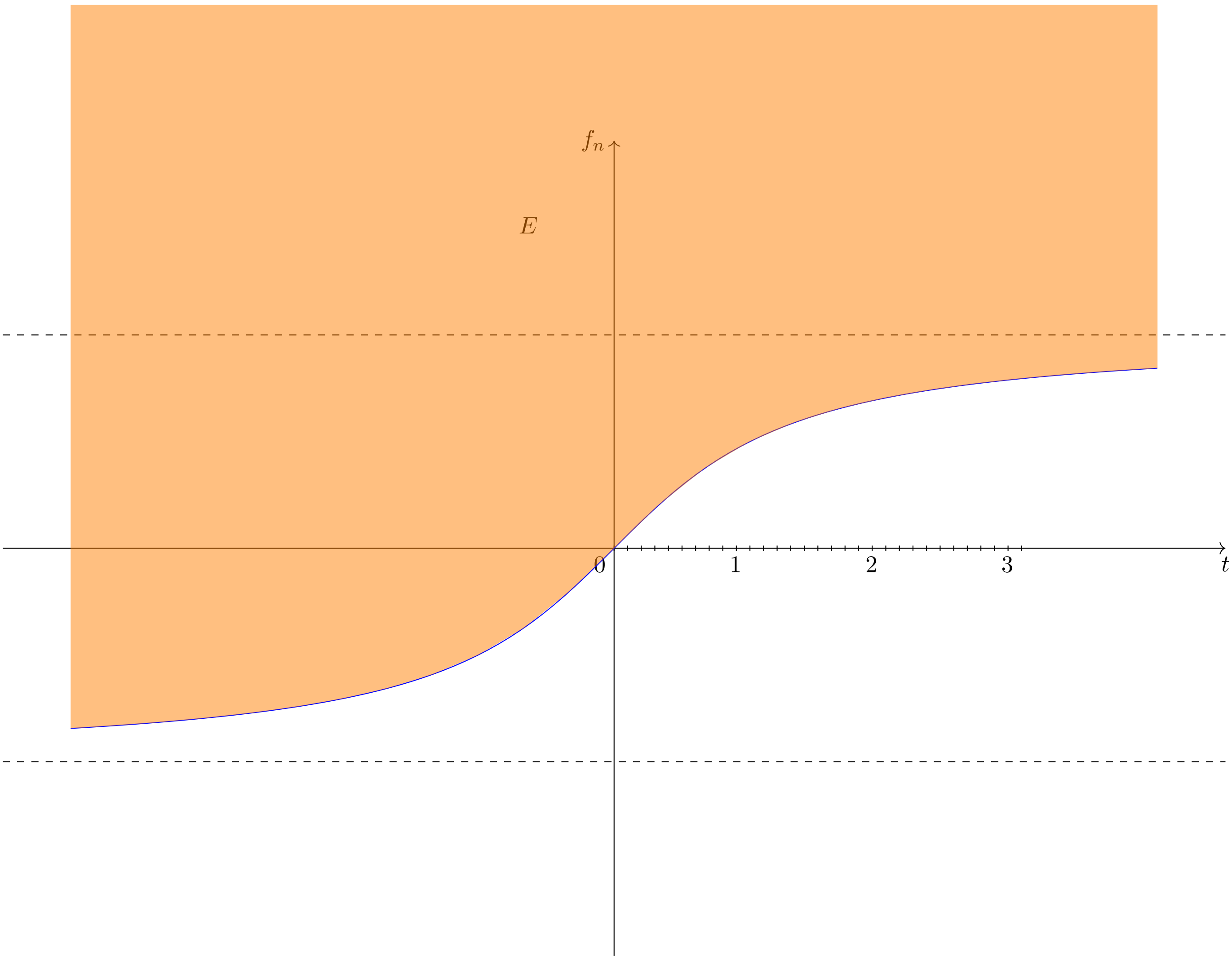
Exercise 8

Let $X = \mathbb{R}^2$ with the Euclidean distance. Say if the set

$$E = \{(x, y) \in \mathbb{R}^2 \qquad \text{s.t.} \qquad y \geq \arctan x\}$$

is complete and if it is compact.

Solution



We can see that E is not bounded so it is not compact. Now we see if it is complete. Consider

$$f(x, y) = y - \arctan(x),$$

we have

$$E = f^{-1}([0, +\infty[).$$

It is a contraimage of a closed set, then E is closed. We know that a closed subset of a complete metric space is complete. Since $\mathbb{R}^2$ is complete, then E is complete.

Exercise 9

Let $X = \mathbb{R}^2$ with the Euclidean distance and let

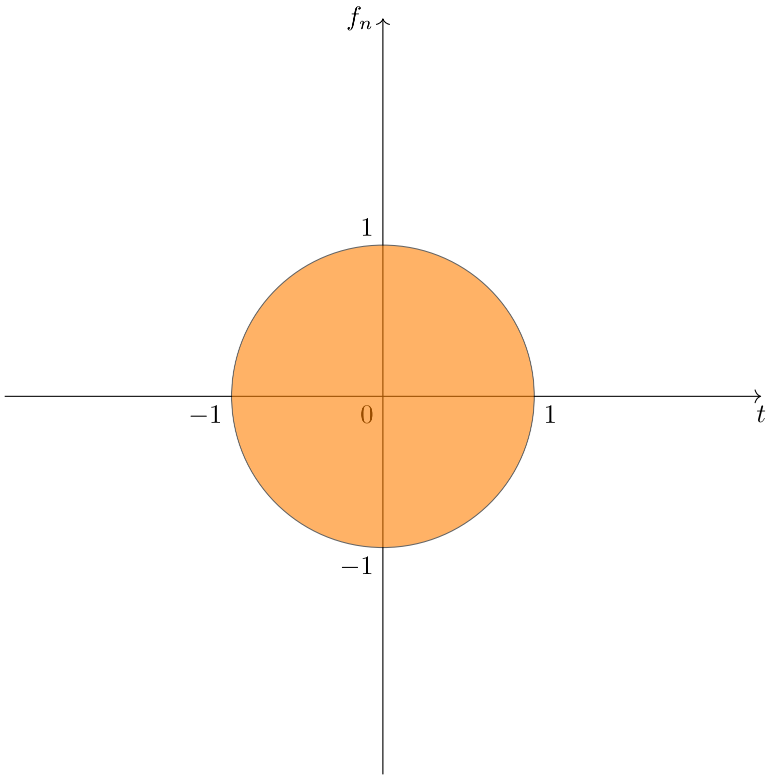
$$A = [0, 1] \times [0, +\infty[$$

$$B = \{(x, y) \in \mathbb{R}^2 \qquad \text{with} \qquad x^2 + y^2 < 1\}.$$

Say if A and B with the Euclidean distance are complete.

Solution

A is closed and $A \subset \mathbb{R}^2$ that is complete with the Euclidean diastance. Then A is complete. B is open,



we need to find a Cauchy sequence in B that doesn't converge in B. Consider

$$x_n = (1 - \frac{1}{n}, 0) \qquad x_n \in B \qquad \forall n.$$

We have that

$$x_n \to (1, 0) \qquad \text{in} \qquad \mathbb{R}^2,$$

but $(1, 0) \notin B$. We have a sequence that converge in the space, that is a Cauchy sequence, but that doesn't converge in B. Then B is not complete.

Exercise 10

Let (X, d) a metric space and x_n a sequence of elements of X. Say if the following statements are true or false.

1. $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \implies x_n \quad$ bounded;
2. $x_n \quad$ convergent $\implies \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$;
3. $x_n \quad$ Cauchy $\implies \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Solution

Point 1

The fact that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ doesn't imply that x_n is bounded. Counterexample:

$$x_n = \log n$$

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} |\log n + 1 - \log n| = \lim_{n\to\infty} |\log \frac{n+1}{n}| = 0.$$

The distances between two consecutive elements become shorter but x_n is not bounded.

$$\sup_{n\in\mathbb{N}}\{\log n, \qquad n \in \mathbb{N}\} = +\infty.$$

Then the first statement is false.

Point 2

If x_n is convergent then there exists $x_\infty \in X$ s.t. :

$$\lim_{n\to\infty} x_n = x_\infty.$$

Then

$$0 \leq d(x_n, x_{n+1}) \leq d(x_n, x_\infty) + d(x_\infty, x_{n+1}),$$

since $d(x_n, x_\infty) \to 0$, $d(x_\infty, x_{n+1}) \to 0$, then by "the two carabinieri theorem" we have that $d(x_n, x_{n+1}) \to 0$.

Point 3

x_n is a Cauchy sequence iff

$$\forall \epsilon > 0 \exists \nu \in \mathbb{N} \text{s.t.} \forall n, m \geq \nu \in \mathbb{N} \implies d(x_n, x_m) < \epsilon.$$

If we fix n, then m can be very far, so is stronger the Cauchy condition with respect to (x_n, x_{n+1}), so that:

$$\forall \epsilon > 0 \exists \nu \in \mathbb{N} \text{s.t.} \forall n > \nu d(x_n, x_{n+1}) < \epsilon \implies \lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Exercise 11

Determine as α varies the limit in $(\mathbb{R}^2, d_2)$ of the sequence:

$$x_n = (\frac{1}{n}, (-1)^n \frac{n^\alpha - 1}{n^2}.$$

If the limit doesn't exists find eventual convergent subsequences.

Solution

A sequence in $\mathbb{R}^2$ is like having two sequences in $\mathbb{R}$:

$$a_n = \frac{1}{n} \to 0$$

$$b_n = (-1)^n \frac{n^\alpha - 1}{n^2} = (-1)^n n^{\alpha-2} + \frac{(-1)^n}{n^2} \to 0.$$

- If $\alpha - 2 < 0$ then $b_n \to 0$;
- if $\alpha = 2$ then $b_n \to \nexists$;
- if $\alpha - 2 > 0$ then $b_n \to \nexists$.

Then if $\alpha < 2$ then

$$\lim_{n\to\infty} x_n = (0, 0)$$

if $\alpha \geq 2$

$$\lim_{n\to\infty} x_n = \nexists.$$

If we consider $\alpha > 2$ since $|b_n \to +\infty$ there aren't convergent subsequences. If $\alpha = 2$ we the subsequence of even indices

$$x_n = (\frac{1}{n}; \frac{n^2+1}{n^2}) \to (0, 1),$$

the subsequence of odd indices:

$$x_n = (\frac{1}{n}; -\frac{n^2+1}{n^2} = \to (0, -1).$$

Exercise 12

Let (X, d) a metric space, $A \subseteq X$, $A \neq \emptyset$, x_n a sequence in A that converges to $x_\infty \in X$. Say if the following statements are false or true.

1. x_∞ is an accumulation point for A;
2. $x_\infty \in \overline{A}$;
3. $x_\infty \in \mathring{A}$;
4. $x_\infty \in \partial A$.

Solution

Point 1

The first statement is false. Counterexample:

$$A = [0, 1] \cup \{2\}$$

$$x_n = 2 \qquad \text{constant}$$

$$\lim_{n\to\infty} x_n = x_\infty = 2$$

that is not an accumulation point since 2 is an isolated point for A.

Point 2

If x_∞ is an isolated point the statement follows by the point 1, if x_∞ is not an isolated point it will be an accumulation point then the statement trivially follows. So the statement 2 is true.

Point 3

$x_\infty \in \mathring{A}$ is false. Counterexample:

$$A =]0, 1]$$

$$x_n = \frac{1}{n} \implies x_\infty \to 0 \notin \mathring{A}.$$

Point 4

$x_\infty \in \partial A$ is false. Counterexample:

$$A =]0, 1[$$

$$x_n = \frac{1}{2} + \frac{1}{n}$$

$$x_n \to \frac{1}{2}$$

that is not in ∂A.

Exercise 13

Show that the metric space $(C^0([0,1]), d_\infty)$ is not compact.

Solution

Consider

$$f_n(t) = t^n \qquad t \in [0,1]$$

suppose that there is a subsequence that converges to $f \in C^0([0,1])$:

$$f_{nk} \to f,$$

that is

$$d_\infty(f_{nk}, f) \to 0, \qquad \forall t \in [0,1].$$

$$|f_{nk}(t) - f_n(t)| \leq d_\infty(f_{nk}, f) \to 0.$$

If this holds then

$$f_{nk}(t) \to f_n(t) \qquad \forall t \in [0,1]$$

$$f_n(t) \to g(t) = \begin{cases} 0 & \text{if} t \in [0,1[\\ 1 & \text{if} t = 1 \end{cases}$$

If the sequence converge to $g(t)$ then also the subsequences tends to $g(t)$, but f_n can't have convergent subsequences because they must to converge to $g(t) \notin C^0([0,1])$ because $g(t)$ is not continuous. Then $C^0([0,1])$ is not compact.

Exercise 14

Consider the sequence

$$f_n(t) = \sqrt{\frac{1+n^2t^2}{n}} \qquad \text{for} t \in [-1,1].$$

Show that $f_n \to f$ with $f(t) = |t|$ with the distance d_∞. Furthermore deduce that the space $C^1([-1,1], d_\infty)$ is not complete.

Solution

$$|f_n(t) - f(t)| = |\frac{\sqrt{1+n^2t^2}}{n} - |t|| = |\frac{1}{n\sqrt{1+n^2t^2} + n|t|}| \leq \frac{1}{n}$$

Since the denominator is greater or equal to n we have that the fraction is lower or equal to $\frac{1}{n} \quad \forall t \in [-1,1]$. Then

$$0 \leq d_\infty(f_n, f) \leq \frac{1}{n} \to 0.$$

Then $f_n \to f$ is a Cauchy sequence, but $f \notin C^1([-1,1])$. Then the squence doesn't converge in $(C^1[-1,1], d_\infty)$ and so it is not complete.

Exercise 15

Let $X = C^0([0,1])$ with the distance d_2. Show that the sequence

$$f_n(t) = \begin{cases} 0 & \text{if} \quad t \in [0, \frac{1}{2} - \frac{1}{n}[\\ \sqrt{2nt + 2 - n} & \text{if} \quad t \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2} \\ \sqrt{-2nt + 2 + n} & \text{if} \quad t \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{n}[\\ 0 & \text{if} \quad t \in [\frac{1}{2} + \frac{1}{n}, 1] \end{cases}$$

1. converges to the null function in $x = 2$:
2. does not converge to the null function in the metric space (X, d_∞);
3. admits limit in (X, d_∞).

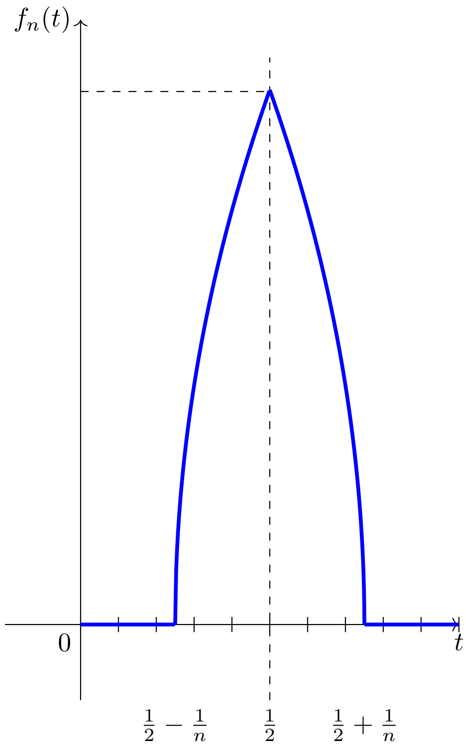
Solution

Point 1

We need to verify that $d_2(f_n, 0) \to 0$ where

$$d_2(f_n, 0) = \sqrt{\int_0^1 |f_n(t)|^2 dt}$$

We have that



$$(d_2(f_n, 0))^2 \leq \frac{2}{n} \to 0.$$

Point 2

$$d_\infty(f_n, 0) = \sup_{t \in [0,1]} |f_n(t)| = f_n(\frac{1}{2}) = \sqrt{2}.$$

It doesn't tend to zero then it doesn't tend to the null function with the distance d_∞. **Point** 3

We suppose that the sequence admits a limit and that

$$\exists g \in X \qquad \text{s.t.} \qquad d_\infty(f, g) \to 0.$$

Then

$$\forall t \in [0,1] \qquad 0 \leq |f_n(t) - g(t)| \leq d_\infty(f_n, g)$$

since the "due carabinieri" theorem

$$\forall t \in [0,1] \qquad f_n(t) \to g(t).$$

If $t \neq \frac{1}{2}$ we have that $f_n(t) \to 0 \implies g(t) = 0$.

If $t = \frac{1}{2} \qquad f_n(t) = \sqrt{2} \implies g(\frac{1}{2}) = \sqrt{2} \implies g \notin X$. Then the limit with d_∞ doesn't exist.

Exercise 16

Let $f \in C^1(\mathbb{R}, \mathbb{R})$, 2π-periodic, such that its Fourier series is of the form

$$\sum_{n=3}^{\infty} \alpha_n \sin(nx)$$

. Let the Fourier series associated to f^3 of the form

$$\sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

Which of the following statements is certainly true?

1. $b_n = \alpha_n^3 \qquad \forall n$;
2. $a_n = 0 \qquad \forall n$.

Solution

Trivially we can see that f is an odd function, then f^3 is also an odd function, so that $a_n = 0 \qquad \forall n$ is certainly true. Then

$$\mathcal{F}_{f^3(x)} = \sum_{n=0}^{+\infty} b_n \sin(nx).$$

If we consider

$$\alpha_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f^3(x) \sin(nx) dx.$$

Generally is not true that $b_n = \alpha_n^3$. Counterexample:

$$f(x) = \sin(x)$$

$$\alpha_3 = 1 \qquad \text{and} \qquad \alpha_n = 0 \qquad \forall n \neq 3.$$

All the terms such as $\sin(4x), \sin(5x), \cdots \sin(1000x)$ have null coefficients.

$$f^3(x) = \sin^3(3x)$$

$$\alpha_3 = 1$$

$$b_3 = 1^3??$$

$$b_3 = \int_{-\pi}^{\pi} f^3(x) \sin(3x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^4(3x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2(3x)(1 - \cos^2(3x) dx$$

$$= \frac{1}{\pi} \sin^2(3x) dx - \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2(3x) \cos^2(3x) dx = \star$$

using the duplication and bisection formulas

$$\sin^2(3x) = \frac{1 - \cos(6x)}{2}$$

$$\sin^2(3x) \cos^2(3x) = \frac{\sin^2(6x)}{4} = \frac{1 - \cos(12x)}{8}$$

$$\star = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1 - \cos(6x)}{2} dx - \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1 - \cos(12x)}{8} dx$$

$$= \frac{1}{2\pi} [x]_{-\pi}^{\pi} - \frac{1}{12\pi} [\sin(nx)]_{-\pi}^{\pi} - \frac{1}{8\pi} [x]_{-\pi}^{\pi} + \frac{1}{96\pi} [\sin(12x)]_{-\pi}^{\pi} = \frac{3}{4}.$$

Then

$$\alpha_3 = 1 \qquad b_3 = \frac{3}{4} \neq 1^3.$$

Exercise 17

Let $f \in C^1(\mathbb{R}^2, \mathbb{R})^2$ such that $f(3,6) = (0,0)$. In a neighborhood of $(3,1)$, which of the following statements are certainly true?

1. $\exists \alpha \in \mathbb{R}$ such that $x \mapsto x + \alpha f(x)$ satisfies the hypotheses of the Implicit Function Theorem.
2. $\forall \alpha \in \mathbb{R} \qquad x \mapsto x + \alpha f(x)$ satisfies the hypotheses of the Inverse Function Theorem.

Solution

Point 1

We need to show that the derivative of f in $(3,1)$ is an invertible matrix. Counterexample: let $\alpha = 0$ and consider a function $g(x) = x \mapsto x$. Let

$$g(x_1, x_2) = (x_1, x_2)$$

$$Dg(x_1, x_2) = \begin{bmatrix} 1 & 0 \\ 0 & \end{bmatrix}$$

The determinant $\det Dg = 1 \neq 0$. Then the gradient matrix is invertible and since it is constant it is also invertible in $(3,1)$. So the first statement is true.

Point 2

Counterexample. We need to find a function f with two variables such that $Df = 0$, starting from a function g not invertible. Consider

$$g(x_1, x_2) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

with $\alpha = 1$, we have

$$Dg(x_1, x_2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The gradient matrix Dg is not invertible and it doesn't satisfy the Inverse Function Theorem hypotheses.

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} x + f_1(x) \\ y + f_2(y) \end{bmatrix}$$

$$f_1(x, y) = 3 - x$$

$$f_2(x, y) = 1 - y$$

$$f(3, 1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then the second statement is false.

Exercise 18

Let $f \in C^2(\mathbb{R}^2, \mathbb{R})$ such that $f(1,2) = 0$ and consider a neighborhood of $(1,2)$. Which of the following statements are certainly true?

1. $\forall \alpha \in \mathbb{R} \qquad x + \alpha f(x,y) + y - 3 = 0$ satisfies the Implicit Function Theorem hypotheses;
2. $\exists \alpha \in \mathbb{R}$ such that $x + \alpha f(x,y) + y - 3 = 0$ satisfies the Implicit Function Theorem hypotheses.

Solution

$$g(x,y) = x + \alpha f(x,y) + y - 3$$
$$g(1,2) = 0$$
$$f \in C^2 \implies g \in C^2$$

We need to verify if

$$\frac{g}{y}(1,2) \neq 0?$$
$$\frac{g}{x}(1,2) \neq 0?$$

Point 2

We take $\alpha = 0$ so that $g(x,y) = x + y - 3$. Then

$$\frac{\partial g}{\partial y}(1,2) = 1 \neq 0.$$

Then the first statement is true.

Point 1

We need a counterexample, starting from a function $g(x,y) = x^2 + y^2$ we can consider:

$$g(x,y) = (x-1)^2 + (y-2)^2$$

so that

$$\nabla g(1,2) = [0,0].$$

We have

$$x + \alpha f(x,y) + y - 3 = (x-1)^2 + (y-2)^2.$$

We take $\alpha = 1$.

$$x + f(x,y) + y - 3 = (x-1)^2 + (y-2)^2$$
$$f(x,y) = (x-1)^2 + (y-2)^2 - x - y + 3$$
$$f(1,2) = (0,0)$$

and $f \in C^2$, but

$$\nabla f(1,2) = [0,0]$$

so that f is not invertible. Then the second statement is not true.

Exercise 19

Let $f_\alpha : \mathbb{R}^2 \to \mathbb{R}$ given by:

$$f_\alpha(x,y) = \begin{cases} \frac{|x|^\alpha y^2}{\sqrt{4x^2+3y^2}} & \text{if} \quad x \neq 0 \\ 0 \quad \text{if} & x = 0 \end{cases}$$

Studying with respect to $\alpha \in \mathbb{R}$ the differentiability and continuity of f_α.

Solution

First: continuity

We have problems in all the points of the y-axis. First we consider the points $(0,b)$ with $b \neq 0$ that is a generic point of the y axis.

$$\lim_{(x,y)\to(0,b)} \frac{|x|^\alpha y^2}{\sqrt{4x^2+3y^2}} = \frac{b^2}{\sqrt{3b^2}} \lim_{(x,y)\to(0,b)} |x|^\alpha = \begin{cases} +\infty & \text{if} \quad \alpha < 0 \\ \frac{b^2}{\sqrt{3b^2}} & \text{if} \quad \alpha = 0 \\ 0 \quad \text{if} & \alpha > 0. \end{cases}$$

Then f is continous in $(0,b)$ with $b \neq 0 \qquad \forall \alpha > 0$. Now we consider the point $(0,0)$.

$$\lim_{(x,y)\to(0,0)} \frac{|x|^\alpha y^2}{\sqrt{4x^2+3y^2}}$$

in polar coordinates

$$\lim_{\rho\to 0} \frac{\rho^\alpha |\cos\theta|^\alpha \rho^2 (\sin^2\theta)}{\rho\sqrt{4\cos^2\theta + 3\sin^2\theta}}$$

making some additions

$$|\rho^{\alpha+1} \frac{|\cos\theta|^\alpha (\sin^2\theta)}{\sqrt{4\cos^2\theta+3\sin^2\theta}}| \leq \rho^{\alpha+1} \frac{|\cos\theta|^\alpha |\sin^2\theta|}{\sqrt{3}|\sin\theta|} = \frac{\rho^{\alpha+1}}{\sqrt{3}} |\cos\theta|^\alpha |\sin\theta| \leq \frac{\rho^{\alpha+1}}{\sqrt{3}}$$

uniformly in θ.

If $\alpha > 0$ then $L = 0$ uniformly in θ.

If $\alpha \leq -1$ then $L = \nexists$.

Remains the case $-1 < \alpha < 0$.

$$\lim_{\rho\to 0} \frac{\rho^{\alpha+1} |\cos\theta|^\alpha \sin^2\theta}{\sqrt{4\cos^2\theta+3\sin^2\theta}} = 0$$

this limit is equivalent to

$$\lim_{\rho\to 0} \rho^{\alpha+1} h(z) = 0 \qquad \forall \theta \neq \frac{\pi}{2} \quad \frac{3\pi}{2}$$

but the limit is not uniform since it is not possible to increase the function. Then for $-1 < \alpha < 0$ the limit goes to $+\infty$. The local boundedness theorem is violated.

Then f is continuous for $\alpha \geq 0$, f is continuous in $(0,b)$ for $\alpha \geq 0$.

Second: differentiability

We consider the points $(0,b)$ with $b \neq 0$ and with $\alpha > 0$.

$$\frac{\partial f}{\partial x}(0,b) = \lim_{t\to 0} \frac{f(t,b)+f(0,b)}{t} = \lim_{t\to 0} \frac{|t|^\alpha b^2}{\sqrt{4t^2+3b^2}} \frac{1}{t} = \begin{cases} \nexists & \text{if} \quad \alpha < 1 \\ \nexists & \text{if} \quad \alpha = 1 \\ 0 & \text{if} \quad \alpha > 1 \end{cases}$$

$$\frac{\partial f}{\partial y}(0,b) = \lim_{t\to 0} \frac{f(0,b+t)-f(0,b)}{t} = \lim_{t\to 0} \frac{0}{t} = 0.$$

Now we consider the case $\alpha > 1$ and show if f is differentiable. For $\alpha > 1$, $\nabla f(0,0) = [0,0]$.

$$\lim_{(h,k)\to(0,0)} \frac{f(h,b+k) - f(0,b) - \nabla f(0,b) \begin{bmatrix} h \\ k \end{bmatrix}}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{|h|^\alpha (b+k)^2}{\sqrt{4h^2+3(b+k)^2}\sqrt{h^2+k^2}}$$

$$= \lim_{\rho\to 0} \frac{\rho^{\alpha-1}|\cos\theta|^\alpha (b+\rho\sin\theta)^2}{\sqrt{4\rho^2\cos^2\theta + 3(b+\rho\sin\theta)^2}} = 0 \qquad \forall\theta.$$

Now we show if it is uniformly.

$$|\frac{\rho^{\alpha-1}|\cos\theta|^\alpha (b+\rho\sin\theta)^2}{\sqrt{4\rho^2\cos^2\theta + 3(b+\rho\sin\theta)^2}}| \le \rho^{\alpha-1}\frac{|b+\rho\sin\theta|}{\sqrt{3}} \le \rho^{\alpha-1}\frac{|b|+\rho}{\sqrt{3}}.$$

The right term doesn't depend on *theta* so the limit is uniform in θ. If $\alpha > 1$ f is differentiable in $(0,b)$ with $b \neq 0$. Now we consider the case $\alpha \ge 0$ in $(0,0)$.

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0}\frac{0}{t} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0}\frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0}\frac{0}{t} = 0$$

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-\nabla f(0,0)\begin{bmatrix} h \\ k\end{bmatrix}}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)}\frac{f(h,k)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)}\frac{|h|^\alpha k^2}{\sqrt{4h^2+3k^2}\sqrt{h^2+k^2}}$$

if $\alpha = 0$ the limit is

$$\lim_{(h,k)\to(0,0)}\frac{k^2}{\sqrt{4h^2+3k^2}\sqrt{h^2+k^2}} = \nexists.$$

If $\alpha > 0$ in polar coordinates:

$$\lim_{\rho\to 0}\frac{\rho^\alpha|\cos\theta|^\alpha\rho^2(\sin^2\theta)}{\rho\rho\sqrt{4\cos^2\theta+3\sin^2\theta}} = 0 \qquad \forall\theta.$$

Now we show if it is uniform in θ.

$$|\frac{\rho^\alpha|\cos\theta|^\alpha\sin^2\theta}{\sqrt{4\cos^2\theta+3\sin^2\theta}}| \le \rho^\alpha\frac{|\cos\theta|^\alpha\sin^2\theta}{\sqrt{3\sin^2\theta}} = \frac{\rho^\alpha|\cos\theta|^\alpha|\sin^2\theta|}{\sqrt{3}\sin\theta} = \rho^2\frac{|\cos\theta|^\alpha|\sin\theta|}{\sqrt{3}} \le \frac{\rho^2}{\sqrt{3}}.$$

It is uniform in θ. Then $L = 0$ uniformly in θ.

Then f is differentiable in the origin for $\alpha > 0$.

Exercise 20

Let $A \subseteq \mathbb{R}^n$ open and $f : A \to \mathbb{R}^m$ differentiable on A. Show that:

1. if A is convex and $\|Df(x)\| \leq L \qquad \forall xinA$ then f is Lip;
2. if f is Lip with constant L then $\|Df(x)\| \leq L \qquad \forall x \in A$.

Solution

Point 1

We apply the Finite Accretion Theorem. If A is convex, then

$$\forall x'', x' \in A \qquad \|f(x'') - f(x')\| \leq \sup_{\xi \in A} \|Df(\xi)\| \, \|x'' - x'\| \leq \star$$

since $\|Df(x)\| \leq L$ then also the sup $\leq L$, then

$$\star \leq L \, \|x'' - x'\|$$

then f is Lip.

Point 2

Let $x \in A$, $v \in \mathbb{R}^m$, $t \in \mathbb{R}$.

$$\|f(x + tv) - f(x)\| \leq L \, \|tv\| = L|t| \, \|v\|$$

$$\frac{\|f(x + tv) - f(x)\|}{t} \leq L \, \|v\| \, .$$

Since f is differentiable and derivable, the limit exists, so passing to the limit for $t \to 0$ we obtain:

$$\|Df(x) \cdot v\| \leq L \, \|v\| \, .$$

Now we take all the vectors for norm equal to 1:

$$w \in \mathbb{R}^n \qquad \|w\| = 1$$

then

$$\|Df(x) \cdot w\| \leq L,$$

if it is valid for all the previous vectors we obtain

$$\sup_{w \in \mathbb{R}^n, \quad \|w\|=1} \|Df(x)w\| \leq L$$

then

$$\|Df(x)\| \leq L.$$