

# Chapter 1

## Functional Spaces

### Exercise 1

Compute

$$\lim_{n \rightarrow +\infty} \int_1^{\infty} f_n(x) dx$$

where

$$f_n(x) = \frac{\sin(nx)}{x^3} e^{-n\sqrt{x}}$$

for all  $x \geq 1$  and for all  $n \in \mathbb{N}$ .

### Solution

This exercise is trivial using the Dominated Convergence Theorem.

First we calculate the **punctual convergence**.

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{\sin(nx)}{x^3} e^{-n\sqrt{x}} = 0$$

for all  $x \geq 1$  since  $\lim_{n \rightarrow \infty} \frac{\sin(nx)}{x^3} = 0$  and  $e^{-n\sqrt{x}}$  is bounded.

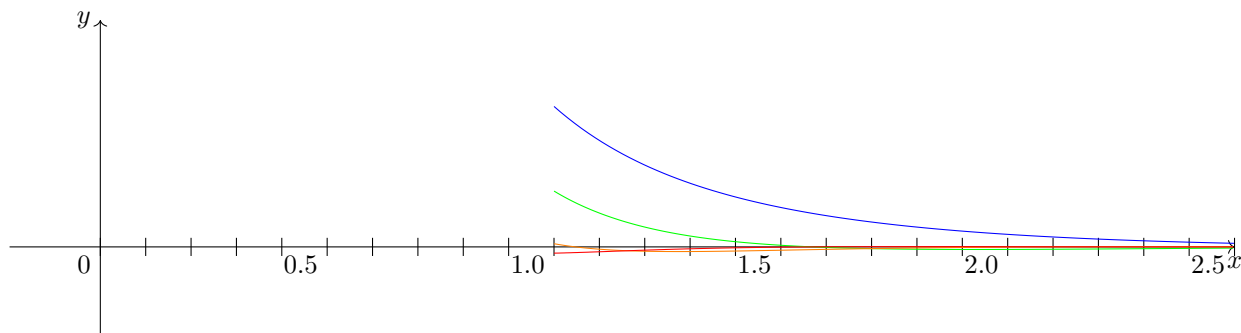


Figure 1.1: The sequence of functions  $f_n(x)$ .

$f(x) = 0 \quad \forall x \geq 1$  is the punctual limit. Now we search a dominant function.

$$|f_n(x)| = \left| \frac{\sin(nx)}{x^3} e^{-n\sqrt{x}} \right| \leq \star,$$

since the sine and  $e^{-n\sqrt{x}}$  are bounded functions:

$$-1 \leq \sin(nx) \leq 1 \quad \forall n \in \mathbb{N} \quad \forall x \in \mathbb{R}$$

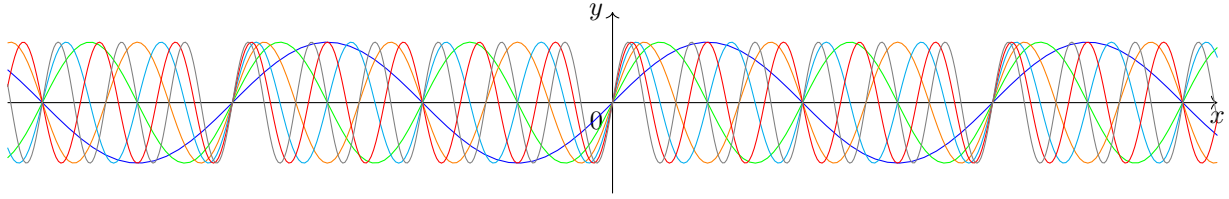
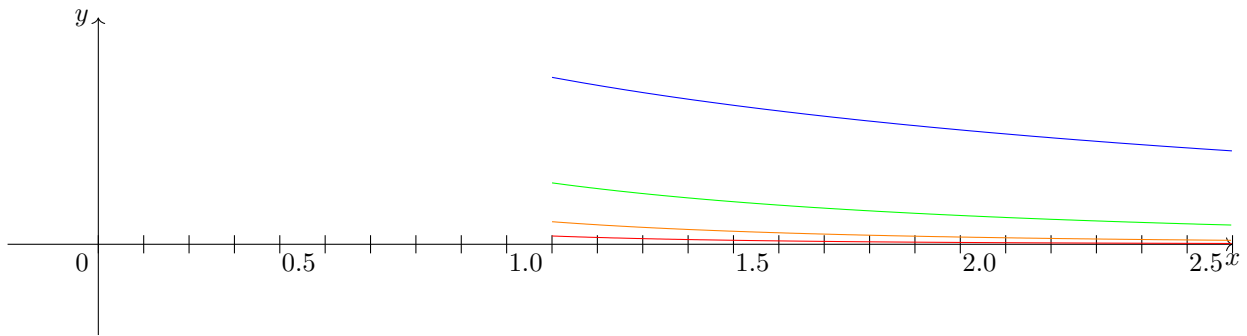


Figure 1.2: The sine function.

$$e^{-n\sqrt{x}} \leq 1 \quad \forall n \in \mathbb{N} \quad x \geq 1$$

Figure 1.3: The sequence of functions  $e^{-n\sqrt{x}}$ .

$$\star \leq \left| \frac{1}{x^3} \right| = \frac{1}{x^3} = g(x) \quad \forall n \in \mathbb{N}$$

since  $x \in [0, +\infty)$ . Now we need to verify if  $g \in L^1([0, +\infty))$ .

$$\int_1^{+\infty} |g(x)| dx = \int_1^{+\infty} \left| \frac{1}{x^3} \right| dx = \int_1^{+\infty} \frac{1}{x^3} dx < +\infty$$

since the summability criteria:

$$\int_1^{+\infty} \frac{1}{x^\alpha} dx = \begin{cases} \frac{1}{\alpha-1} & \text{if } \alpha > 1 \\ +\infty & \text{if } \alpha \leq 1 \end{cases}$$

Now we can apply the Dominated Convergence Theorem (or Lebesgue Theorem):

$$\lim_{n \rightarrow +\infty} \int_1^{+\infty} f_n(x) dx = \int_1^{+\infty} \lim_{n \rightarrow +\infty} f_n(x) dx = \int_1^{+\infty} \lim_{n \rightarrow +\infty} \frac{\sin(nx)}{x^3} e^{-n\sqrt{x}} dx = \int_1^{+\infty} 0 dx = 0.$$

Then the solution is

$$\lim_{n \rightarrow +\infty} \int_1^{+\infty} f_n(x) dx = \lim_{n \rightarrow +\infty} \int_1^{+\infty} \frac{\sin(nx)}{x^3} e^{-n\sqrt{x}} dx = 0 \quad \forall x \leq 1.$$

# Chapter 2

## $L^p$ Spaces

### Exercise 1

Analyze the convergence in  $L^p([0, 1])$  with  $1 \leq p < \infty$  of

$$f_n(x) = \frac{\cos(nx)e^{-nx}}{\sqrt[n]{x}} \quad \text{for } x \in [0, 1] \quad \forall n \in \mathbb{N}.$$

For which  $L^p$  the sequence converge to a certain function?

### Solution

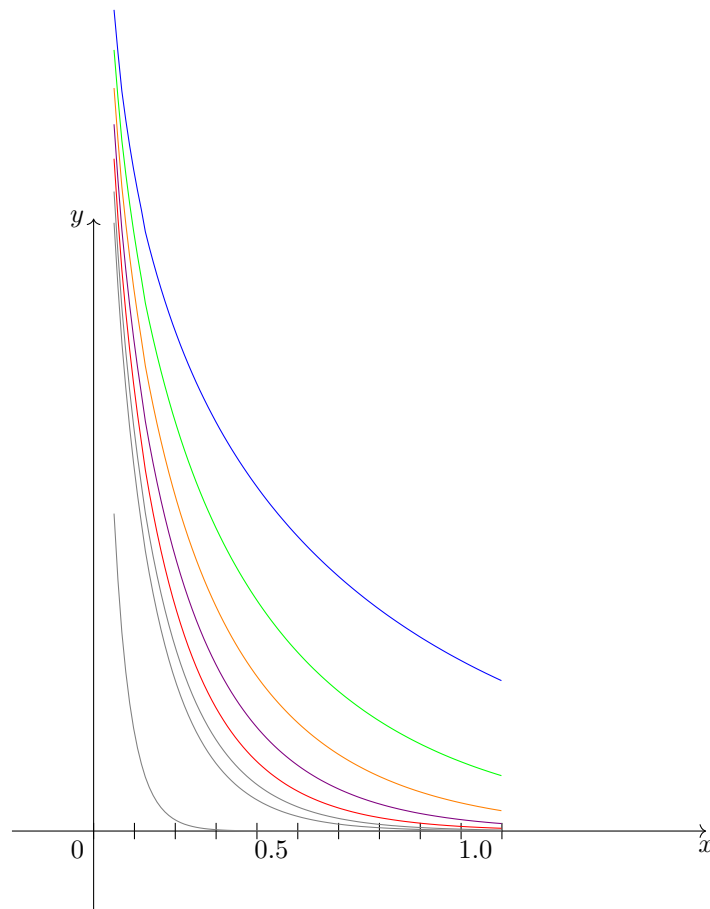


Figure 2.1: The sequence of functions  $f_n(x)$ .

First of all we search for which  $p$  this sequence belongs to some  $L^p$ , applying the Dominated Convergence Theorem.

$$|f_n(x)| = \left| \frac{\cos(nx)}{\sqrt[4]{x}} e^{-nx} \right| \leq \frac{1}{\sqrt[4]{x}} = g(x) \quad x \in [0, 1]$$

We know that  $f_n(x)$  belongs to some  $L^p$  if and only if

$$\int_0^1 |f_n(x)|^p dx < +\infty.$$

The exponents  $p$  that satisfy this relations are the candidates.

$$\int_0^1 |f_n(x)|^p dx = \int_0^1 \left| \frac{\cos(nx)}{\sqrt[4]{x}} e^{-nx} \right|^p dx \leq \int_0^1 \left| \frac{1}{\sqrt[4]{x}} \right|^p dx = \int_0^1 \frac{1}{x^{\frac{p}{4}}} dx.$$

From the summability criteria:

$$\int_1^{+\infty} \frac{1}{x^\alpha} dx = \begin{cases} \frac{1}{\alpha-1} & \text{if } \alpha > 1 \\ +\infty & \text{if } \alpha \leq 1 \end{cases},$$

since

$$\left| \frac{\cos(nx)}{\sqrt[4]{x}} e^{-nx} \right|^p \leq \left( \frac{1}{\sqrt[4]{x}} \right)^p$$

we have that for  $p \in [1, 4)$   $f_n(x) \in L^p([0, 1]) \quad \forall n \in \mathbb{N}$ .

- $f_n \in L^1([0, 1])$ ;
- $f_n \in L^2([0, 1])$ ;
- $f_n \in L^3([0, 1])$ .

**Punctual Convergence:**

$$\lim_{n \rightarrow +\infty} f_n(x) = \lim_{n \rightarrow +\infty} \frac{\cos(nx)}{\sqrt[4]{x}} e^{-nx} = \lim_{n \rightarrow +\infty} \frac{\cos(nx)}{\sqrt[4]{x} e^{nx}} \rightarrow 0$$

so

$$f_n \rightarrow 0 \quad \text{pointwise} \quad \forall x \in [0, 1],$$

we can apply the comparison criterium.

$$\lim_{x \rightarrow 0^+} f_n(x) \sqrt[4]{x} = \lim_{x \rightarrow 0^+} \frac{\cos(nx)}{\sqrt[4]{x}} e^{-nx} \sqrt[4]{x} = 1$$

$$f_n(x) \sim \frac{1}{\sqrt[4]{x}} \quad \text{for } x \rightarrow 0^+.$$

Now we can analyze the convergence in  $L^p([0, 1])$

$$\|f_n - f\|_{L^p}$$

$$\|f_n(x) - f(x)\|_{L^p([0,1])}^p = \|f_n(x)\|_{L^p([0,1])}^p = \left\| \frac{\cos(nx)}{\sqrt[4]{x} e^{nx}} \right\|_{L^p([0,1])}^p = \int_0^1 \left| \frac{\cos(nx)}{\sqrt[4]{x} e^{nx}} \right|_{L^p([0,1])}^p dx = \star$$

since

$$\left| \frac{\cos(nx)}{\sqrt[4]{x} e^{nx}} \right|_{L^p([0,1])}^p \leq g(x) = \frac{1}{x^{\frac{p}{4}}}$$

where

$$g \in L^p([0, 1]) \quad \text{for } 1 \leq p < 4,$$

we can apply the Dominated Convergence Theorem

$$\lim_{n \rightarrow +\infty} \|f_n(x) - f(x)\|_{L^p([0,1])}^p = \lim_{n \rightarrow +\infty} \int_0^1 \left| \frac{\cos(nx)}{\sqrt[4]{x}} e^{-nx} \right|^p dx = \int_0^1 \lim_{n \rightarrow +\infty} \left| \frac{\cos(nx)}{\sqrt[4]{x}} e^{-nx} \right|^p dx = 0.$$

So

$$\lim_{n \rightarrow +\infty} \|f_n(x) - f(x)\|_{L^p([0,1])} \rightarrow 0$$

$$f_n(x) \rightarrow 0 \quad \text{in} \quad L^p([0,1]) \quad \forall p \in [1; 4).$$

Since

$$\lim_{x \rightarrow 0^+} \frac{|f_n(x)|}{g(x)} = 1 \quad \forall n \in \mathbb{N}$$

we have

$$f_n \in L^p([0,1]) \leftrightarrow g \in L^p([0,1])$$

so that

$$f_n \notin L^p([0,1]) \quad \text{if } p \geq 4.$$

The sequence  $\{f_n(x)\}_{n \in \mathbb{N}}$  can't converge in  $L^p([0,1])$  spaces if  $p \geq 4$ .

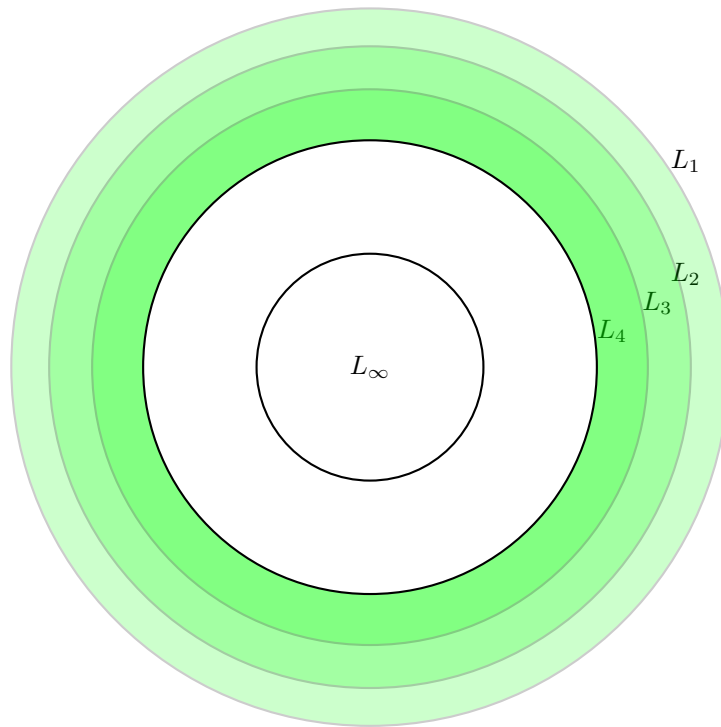
In the case  $p = +\infty$ , we have

$$\|f_n(x)\|_\infty = \operatorname{ess\,sup}_{x \in (0,1)} \left| \frac{\cos(nx)}{\sqrt[4]{x}} e^{-nx} \right| \leq \sup_{x \in (0,1)} \left| \frac{1}{\sqrt[4]{x}} \right| \rightarrow +\infty,$$

so

$$f_n \not\rightarrow 0 \quad \text{in} \quad L^\infty((0,1)).$$

Since  $[0,1]$  is a bounded set we have the following embeddings: The sequence  $f_n(x)$  lives in "green"



spaces.