A collection of Functional Analysis exercises

June 5, 2024

Chapter 1

Functional Spaces

Exercise 1

Compute

$$\lim_{n\to+\infty}\int_1^{\infty} f_n(x)dx$$

where

$$f_n(x) = \frac{\sin(nx)}{x^3}e^{-n\sqrt{x}}$$

for all $x \geq 1$ and for all $n \in \mathbb{N}$.

Solution

This exercise is trivial using the Dominated Convergence Theorem.

First we calculate the **pointwise convergence**.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{\sin(nx)}{x^3}e^{-n\sqrt{x}} = 0$$

for all $x \geq 1$ since $\lim_{n\to\infty}\frac{\sin(nx)}{x^3} = 0$ and $e^{-n\sqrt{x}}$ is bounded.

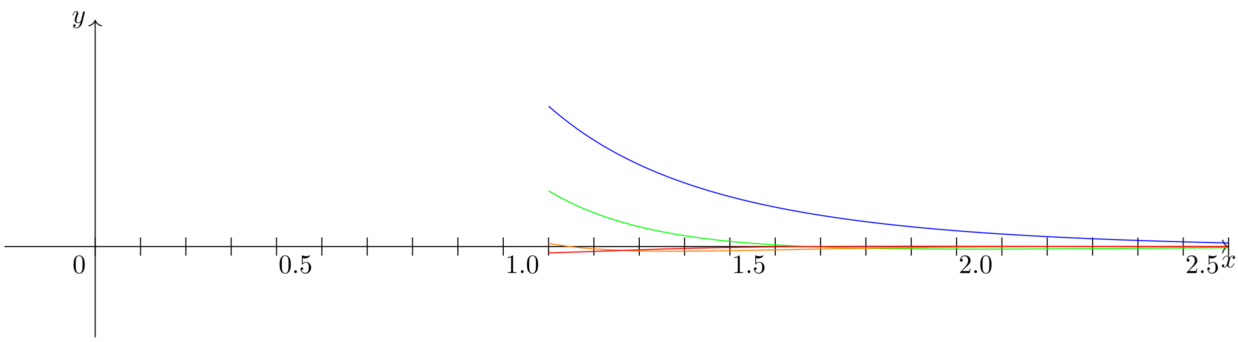


Figure 1.1: The sequence of functions $f_n(x)$.

$f(x) = 0 \qquad \forall x \geq 1$ is the punctal limit. Now we search a dominant function.

$$|f_n(x)| = |\frac{\sin(nx)}{x^3}e^{-n\sqrt{x}}| \leq \star,$$

since the sine and $e^{-n\sqrt{x}}$ are bounded functions:

$$-1 \leq \sin(nx) \leq 1 \qquad \forall n \in \mathbb{N} \qquad \forall x \in \mathbb{R}$$

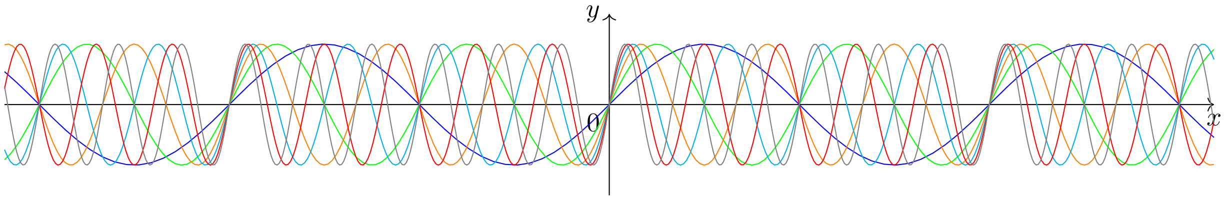


Figure 1.2: The sine function.

$$e^{-n\sqrt{x}} \leq 1 \qquad \forall n \in \mathbb{N} \qquad x \geq 1$$

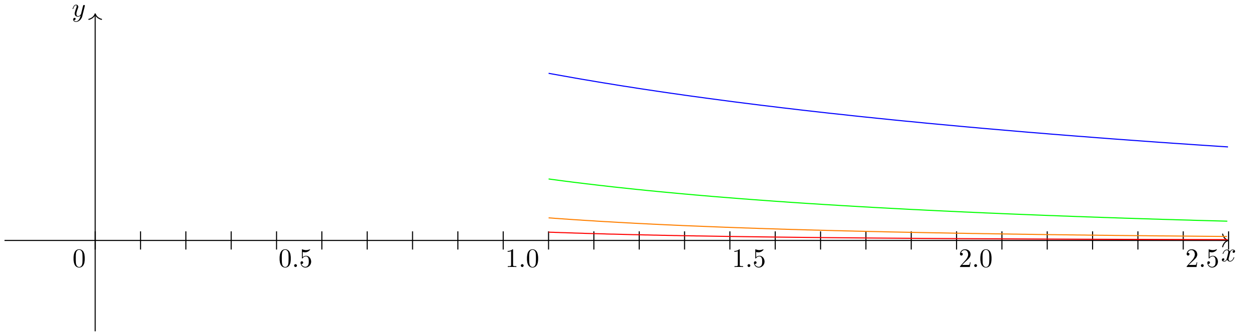


Figure 1.3: The sequence of functions $e^{-n\sqrt{x}}$.

$$\star \leq |\frac{1}{x^3}| = \frac{1}{x^3} = g(x) \qquad \forall n \in \mathbb{N}$$

since $x \in [0, +\infty)$. Now we need to verify if $g \in L^1([0, +\infty))$.

$$\int_1^{+\infty} |g(x)| dx = \int_1^{+\infty} |\frac{1}{x^3}| dx = \int_1^{+\infty} \frac{1}{x^3} dx < +\infty$$

since the summability criteria:

$$\int_1^{+\infty} \frac{1}{x^\alpha} dx = \begin{cases} \frac{1}{\alpha - 1} & if \quad \alpha > 1 \\ +\infty & if \quad \alpha \leq 1 \end{cases}$$

Now we can apply the Dominated Convergence Theorem (or Lebesgue Theorem):

$$\lim_{n \to +\infty} f_n(x) dx = \int_1^{+\infty} \lim_{n \to +\infty} f_n(x) dx = \int_1^{+\infty} \lim_{n \to +\infty} \frac{\sin(nx)}{x^3} e^{-n\sqrt{x}} dx = \int_1^{+\infty} 0 dx = 0.$$

Then the solution is

$$\lim_{n \to +\infty} \int_1^{+\infty} f_n(x) dx = \lim_{n \to +\infty} \frac{\sin(nx)}{x^3} e^{-n\sqrt{x}} dx = 0 \qquad \forall x \leq 1.$$

Exercise 2

Compute

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx$$

where

$$f_n(x)=\frac{x}{1+x^{2n}} \qquad \text{with} \qquad x\in(0,1).$$

Solution

We need to apply the Dominated Converge Theorem.

First of all we analyze the **pointwise convergence**.

$$\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}\frac{x}{1+x^{2n}},$$

if $x\in[0,1)$ we have

$$\lim_{n\to\infty}\frac{x}{1+x^{2n}}=x$$

since $x^{2n}\to 0$ for $x\in[0,1)$.

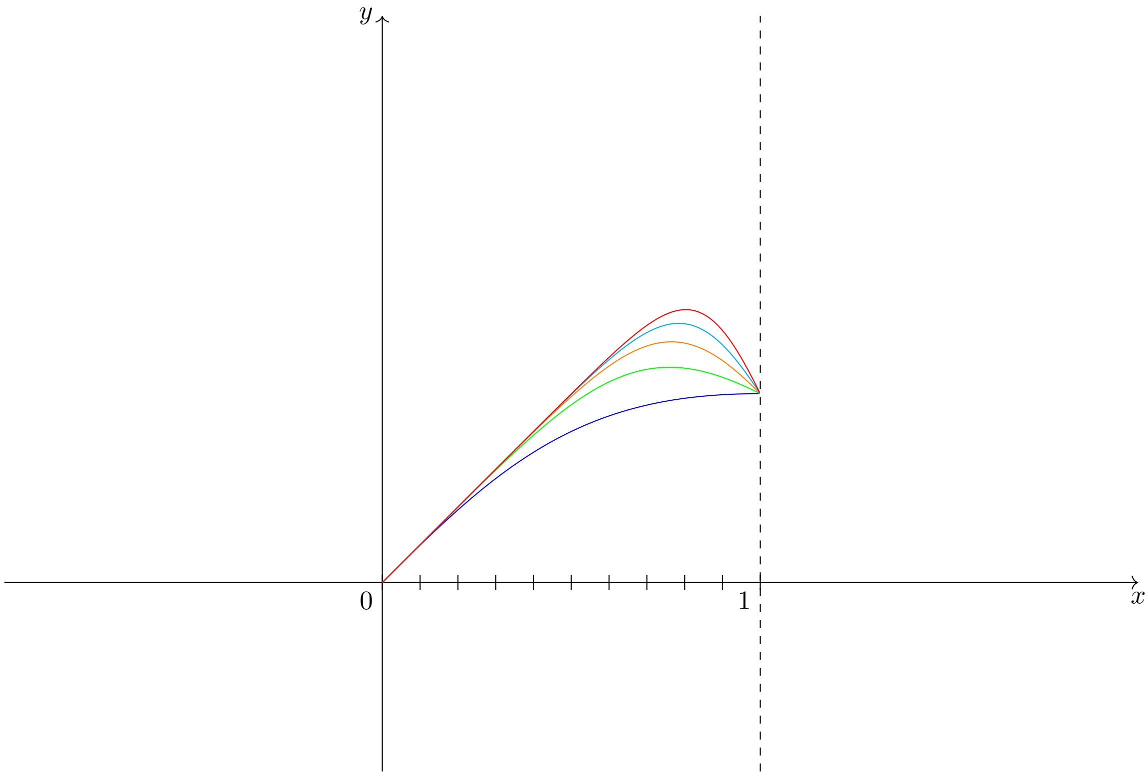
If $x=1$,

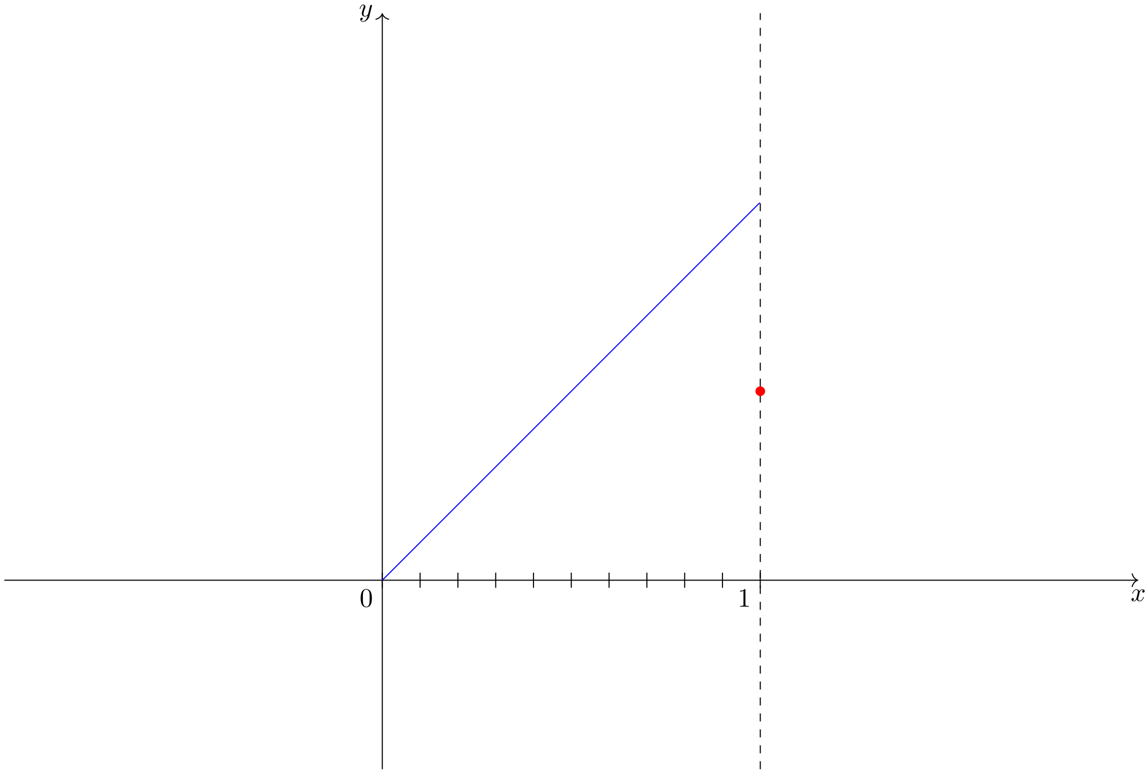
$$\lim_{n\to\infty}\frac{x}{1+x^{2n}}=\lim_{n\to\infty}\frac{1}{1+1^{2n}}=\frac{1}{2}.$$

The pointwise limit is

$$\lim_{n\to\infty} f_n(x)=\begin{cases} x & \text{if} \quad x\in[0,1) \\ \frac{x}{2} & \text{if} \quad x=1 \end{cases}$$

Now we find the dominant function.





$$\exists g \in L^1(0,1) \qquad \text{s. t.} \qquad |f_n(x)| \leq g?$$

$$|f_n(x)| = |\frac{x}{1+x^{2n}}| \leq \frac{x}{1+x^{2n}} \leq x = g(x) \qquad \forall n \in \mathbb{N}, \qquad \forall x \in (0,1).$$

$$\int_0^1 |g(x)|dx = \int_0^1 |x|dx = \int_0^1 xdx = [\frac{x^2}{2}]_0^1 = \frac{1}{2} < +\infty$$

so that

$$g \in L^1(0,1).$$

We can now apply the Dominated Convergence Theorem.

$$\lim_{n\to\infty} \int_0^1 \frac{x}{1+x^{2n}}dx = \int_0^1 \lim_{n\to\infty} \frac{x}{1+x^{2n}}dx = \int_0^1 xdx = \frac{1}{2}.$$

The solution is

$$\lim_{n\to\infty} \int_0^1 \frac{x}{1+x^{2n}}dx = \frac{1}{2}.$$

Exercise 3

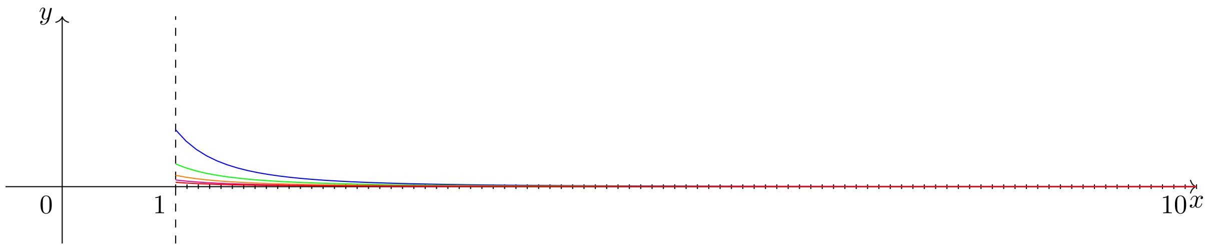
Studying convergence in $L^1([1,+\infty))$ of

$$f_n(x) = \frac{\cos(nx)}{n^2+x}\frac{1}{x^2} \qquad \forall x \geq 1 \qquad \forall n \in \mathbb{N}.$$

Solution

Convergence in $L^1([0,+\infty))$:

$$\|f_n - f\|_{L^1([1;+\infty))} = \int_1^{+\infty} |f_n - f|dx \to 0$$



POINTWISE CONVERGENCE

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\cos(nx)}{n^2+x}\frac{1}{x^2} = \star$$

$$-1 \leq \cos(nx) \leq 1$$

$$\star = 0,$$

ù so the pointwise limit is

$$f(x) = 0 \qquad \forall x \geq 1.$$

Now we need to find a dominant function:

$$\exists g \in L^1([1;+\infty)) \qquad \text{s.t.} \qquad |f_n(x)| \leq g \qquad \forall x \geq 1 \qquad \forall n \in \mathbb{N}$$

$$|f_n(x)| = |\frac{\cos(nx)}{n^2+x}\frac{1}{x^2}| \leq \frac{1}{n^2+x}\frac{1}{x^2} \leq \frac{1}{x}\frac{1}{x^2} = \frac{1}{x^3} = g(x) \qquad \forall x \geq 1 \qquad \forall n \in \mathbb{N}$$

$$\int_1^{+\infty} |g(x)|dx = \int_1^{+\infty} |\frac{1}{x^3}|dx = \int_1^{+\infty} \frac{1}{x^3}dx < +\infty$$

summability criteria:

$$\int_1^{+\infty} \frac{1}{x^\alpha}dx = \begin{cases} \frac{1}{\alpha-1} & if \quad \alpha > 1 \\ +\infty & if \quad \alpha \leq 1, \end{cases}$$

then

$$g \in L^1[1;+\infty).$$

$$\lim_{n\to\infty} \|f_n - f\|_{L^1([1;+\infty))} \to 0$$

$$\lim_{n\to\infty} |f_n - f|dx \to 0$$

$$iff$$

$$\lim_{n\to\infty} \int_1^{+\infty} |f_n|dx \to 0.$$

Now we can apply the Dominated Convergence Theorem:

$$\lim_{n\to+\infty} \int_1^{+\infty} |f_n|dx = \int_1^{+\infty} \lim_{n\to+\infty} |f_n|dx = \int_1^{+\infty} \lim_{n\to+\infty} |\frac{\cos(nx}{n^2+x}\frac{1}{x^2}|dx$$

$$= \int_1^{+\infty} \lim_{n\to+\infty} (\frac{\cos(nx)}{n^2+x}\frac{1}{x^2}dx = \int_1^{+\infty} 0dx = 0,$$

so that

$$f_n \to 0 \qquad \text{in} \qquad L^1([1;+\infty)).$$

Exercise 4

Let $f_n(x) = \sum_{n=1}^{\infty} \frac{|\sin(nx)|}{2^n}$ $\qquad x \in [0,\pi]$. Compute

$$\int_0^{\pi} f(x)dx.$$

Solution

Remember that a series is the limit of the partial sums. If the partial sums are formed by positive terms, then the series are monotone. We consider

$$f_h(x) = \sum_{n=1}^{h} \frac{|\sin(nx)|}{2^n}.$$

We have truncated the series up to the h term. If we consider the truncated series we have that $f_h(x)$ is monotone, in fact

$$0 \leq f_h \leq f_{h+1}.$$

Furthermore

$$f_h(x) = \sum_{h=1}^{h} \frac{|\sin(nx)|}{2^n} \to f(x) \qquad \text{for} \qquad h \to \infty.$$

Then we can apply the Beppo Levi's Theorem:

$$\int_0^{\pi} f(x)dx = \int_0^{\pi} \lim_{h\to\infty} f_h(x)dx = \lim_{h\to\infty} \int_0^{\pi} f_h(x)dx = \lim_{h\to\infty} \int_0^{\pi} \sum_{n=1}^{h} \frac{|\sin(nx)|}{2^n} dx = \lim_{h\to\infty} \sum_{n=1}^{h} \frac{1}{2^n} \int_0^{\pi} |\sin(nx)|dx.$$

Now we have to compute the integral. We know that

$$\int_0^{\infty} |\sin(y)|dy = n \int_0^{\pi} \sin(y)dy$$

so that

$$\int_0^{\pi} |\sin(nx)|dx = \int_0^{n\pi} |siny| \frac{dy}{n} = \int_0^{\pi} \sin y dy = [-\cos y]_0^{\pi} = 2.$$

Then

$$\int_0^{\pi} f(x)dx = \sum_{k=1}^{\infty} (\frac{1}{2})^k 2 = \sum_{k=1}^{\infty} \frac{2}{2^k} = 2.$$

Then

$$\int_0^{\pi} f(x)dx = 2.$$

Chapter 2

L^p Spaces

Exercise 1

Analyze the convergence in $L^p([0,1])$ with $1 \leq p < \infty$ of

$$f_n(x) = \frac{\cos(nx)e^{-nx}}{\sqrt[4]{x}} \qquad for \quad x \in [0,1] \qquad \forall n \in \mathbb{N}.$$

For which L^p the sequence converge to a certain function?

Solution

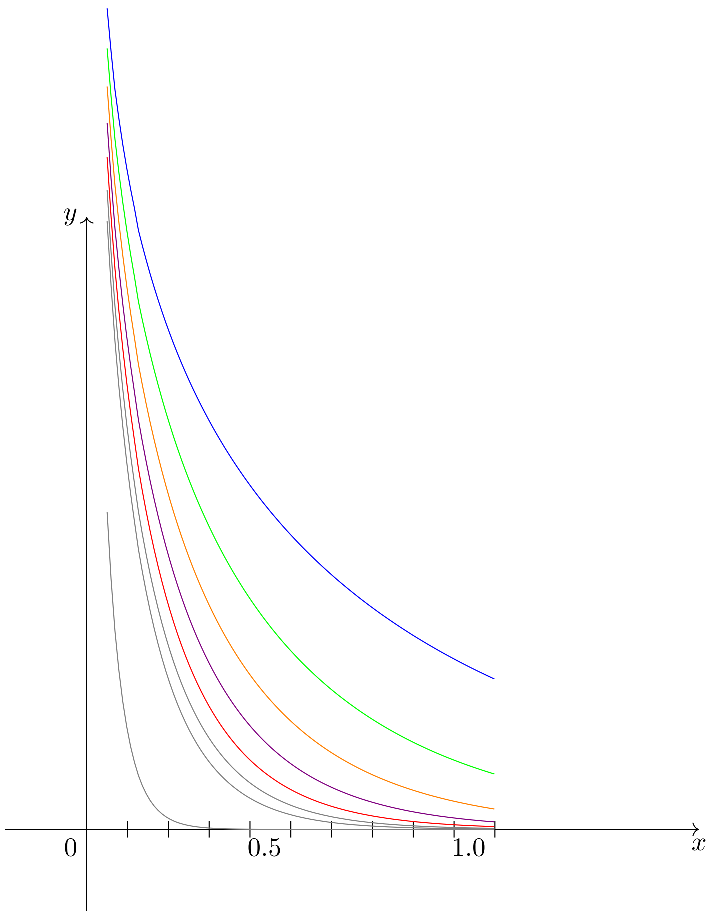


Figure 2.1: The sequence of functions $f_n(x)$.

First of all we search for which p this sequence belongs to some L^p, applying the Dominated Convergence Theorem.

$$|f_n(x)| = |\frac{\cos(nx)}{\sqrt[4]{x}} e^{-nx}| \leq \frac{1}{\sqrt[4]{x}} = g(x) \qquad x \in [0,1]$$

We know that $f_n(x)$ belongs to some L^p if and only if

$$\int_0^1 |f_n(x)|^p dx < +\infty.$$

The exponents p that satisfy this relations are the candidates.

$$\int_0^1 |f_n(x)|^p dx = \int_0^1 |\frac{\cos(nx)}{\sqrt[4]{x}} e^{-nx}|^p dx \leq \int_0^1 |\frac{1}{\sqrt[4]{x}}|^p dx = \int_0^1 \frac{1}{x^{\frac{p}{4}}} dx.$$

From the summability criteria:

$$\int_1^{+\infty} \frac{1}{x^\alpha} dx = \begin{cases} \frac{1}{\alpha-1} & if \quad \alpha > 1 \\ +\infty & if \quad \alpha \leq 1 \end{cases},$$

since

$$|\frac{\cos(nx)}{\sqrt[4]{x}} e^{-nx}|^p \leq (\frac{1}{\sqrt[4]{x}})^p$$

we have that for $p \in [1,4) \qquad f_n(x) \in L^p([0,1]) \qquad \forall n \in \mathbb{N}$.

- $f_n \in L^1([0,1])$;
- $f_n \in L^2([0,1])$;
- $f_n \in L^3([0,1])$.

Pointwise Convergence:

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} \frac{\cos(nx)}{\sqrt[4]{x}} e^{-nx} = \lim_{n\to+\infty} \frac{\cos(nx)}{\sqrt[4]{x} e^{nx}} \to 0$$

so

$$f_n \to 0 \qquad \text{pointwise} \qquad \forall x \in [0,1],$$

we can apply the comparison criterium.

$$\lim_{x\to 0^+} f_n(x) \sqrt[4]{x} = \lim_{x\to 0^+} \frac{\cos(nx)}{\sqrt[4]{x}} e^{-nx} \sqrt[4]{x} = 1$$

$$f_n(x) \sim \frac{1}{\sqrt[4]{x}} \qquad \text{for} \qquad x \to 0^+.$$

Now we can analyze the convergence in $L^p([0,1])$

$$\|f_n - f\|_{L^p}$$

$$\|f_n(x) - f(x)\|^p_{L^p([0,1])} = \|f_n(x)\|^p_{L^p([0,1])} = \left\| \frac{\cos(nx)}{\sqrt[4]{x} e^{nx}} \right\|^p_{L^p([0,1])} = \int_0^1 |\frac{\cos(nx)}{\sqrt[4]{x} e^{-nx}}_{L^p([0,1])}|^p dx = \star$$

since

$$|\frac{\cos(nx)}{\sqrt[4]{x} e^{nx}}_{L^p([0,1])}|^p \leq g(x) = \frac{1}{x^{\frac{p}{4}}}$$

where

$$g \in L^p([0,1]) \qquad \text{for} \qquad 1 \leq p < 4,$$

we can apply the Dominated Convergence Theorem

$$\lim_{n\to+\infty} \|f_n(x) - f(x)\|^p_{L^p([0,1])} = \lim_{n\to+\infty} \int_0^1 |\frac{\cos(nx)}{\sqrt[4]{x}} e^{-nx}|^p dx = \int_0^1 \lim_{n\to+\infty} |\frac{\cos(nx)}{\sqrt[4]{x}} e^{-nx}|^p dx = 0.$$

So

$$\lim_{n\to+infty}\|f_n(x)-f(x)\|_{L^p([0,1])}\to 0$$

$$f_n(x)\to 0 \qquad in \qquad L^p([0,1]) \qquad \forall p\in[1;4).$$

Since

$$\lim_{x\to 0^+}\frac{|f_n(x)|}{g(x)}=1 \qquad \forall n\in\mathbb{N}$$

we have

$$f_n\in L^p([0,1])\leftrightarrow g\in L^p([0,1])$$

so that

$$f_n\notin L^p([0,1]) \qquad \text{if} \qquad p\geq 4.$$

The sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ can't converge in $L^p([0,1])$ spaces if $p\geq 4$.

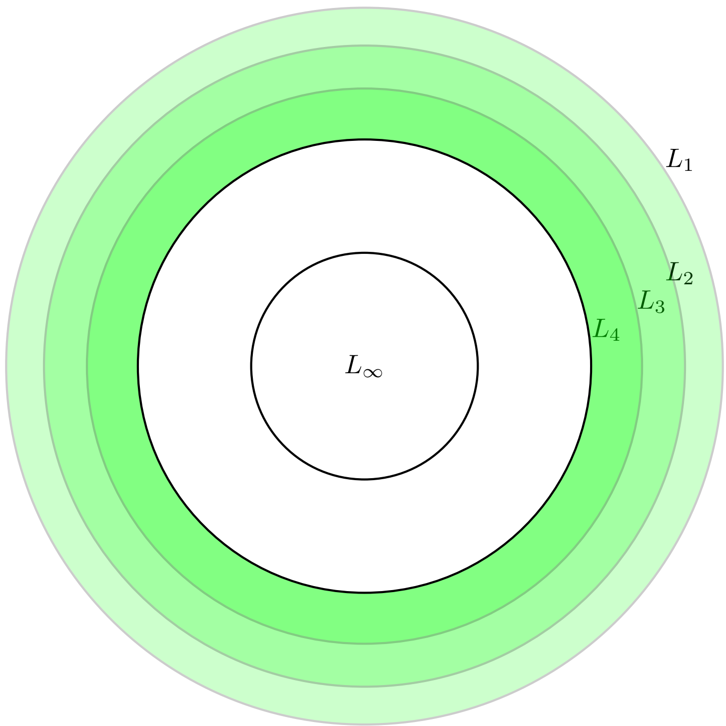
In the case $p=+\infty$, we have

$$\|f_n(x)\|_\infty=\operatorname*{ess\,sup}_{x\in(0,1)}|\frac{\cos(nx)}{\sqrt[4]{x}}e^{-nx}|\leq\sup_{x\in(0,1)}|\frac{1}{\sqrt[4]{x}}|\to+\infty,$$

so

$$f_n\nrightarrow 0 \qquad \text{in} \qquad L^\infty((0,1)).$$

Since $[0,1]$ is a bounded set we have the following embeddings: The sequence $f_n(x)$ lives in "green"



spaces.

Chapter 3

Hilbert Spaces

Exercise 1

Let $X = (C(0,1); \|\cdot\|_\infty)$ and consider

$$K = \{f \in X : \int_0^{\frac{1}{2}} f(t)dt - \int_{\frac{1}{2}}^1 f(t)dt = 1\}.$$

Show that K is closed and not empty, and determine the projection of 0 over the set K.

Solution

K is not empty. To show that we can take:

$$f(t) = \frac{\pi}{2}\sin(2\pi t).$$

Now we can consider:

$$u(t) = \chi_{(0,\frac{1}{2})}(t) - \chi_{(\frac{1}{2},1}(t)$$

and consider the following operator:

$$(Tf) = \int_0^1 fu dt$$

where

$$T : C(0,1) \to \mathbb{R}$$

$$K = T^{-1}(\{1\})$$

since $\{1\}$ is a singleton then it is closed; the contraimage of a closed set must be closed, so K is closed.

$$|Tf| \leq C\,\|f\|_\infty$$

$$f \in C(0,1) \qquad \|f\|_\infty \leq 1$$

then

$$f \notin K,$$

that is the elements of K are of the form

$$\|\cdot\|_\infty > 1.$$

We have that

$$f(x) \leq \|f\|_\infty \leq 1 \qquad \forall x \in (0,1)$$

"by contradiction"

$$f \in K \qquad \int_0^1 fu = 1$$

$$1 = \int_0^1 fu \leq \int_0^1 |f||u| \leq \|f\|_\infty \int_0^1 dt = \|f\|_\infty \leq 1.$$

We have that

$$|fu| \leq 1 \qquad \int_0^1 |fu| = 1$$

so that

$$|fu| = 1 \quad a.e.$$

$$\int_0^1 (1 - |fu|) = 0 \qquad a.e.$$

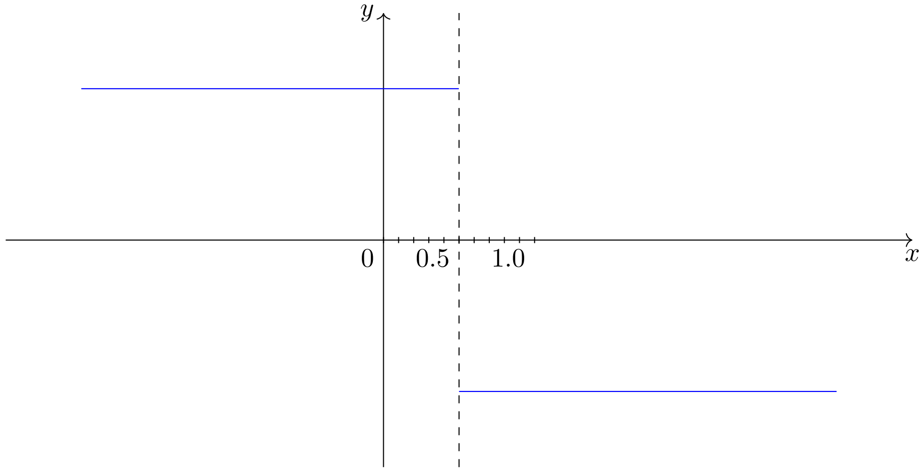
so that

$$fu = 1 \qquad a.e..$$

So we obtain this contradiction:

$$\begin{cases} f = 1 & if \quad x \in (0, \frac{1}{2}) \\ f = -1 & if \quad x \in (\frac{1}{2}, 1) \end{cases}$$

Now we consider:



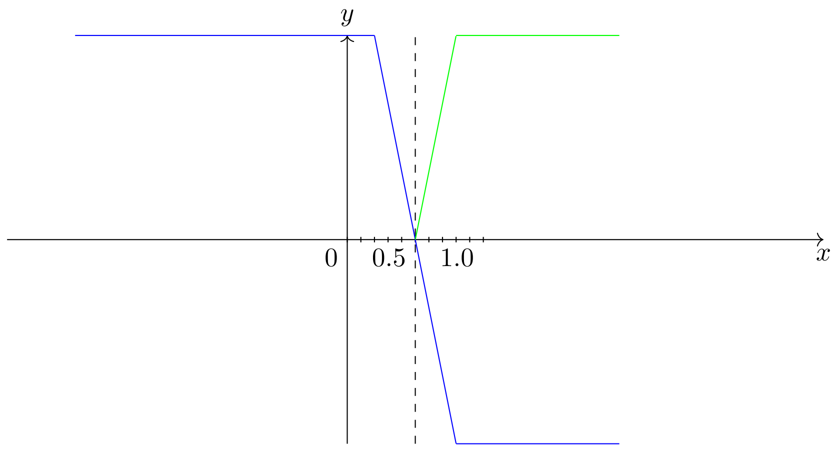
$$d = \inf\{\|f\|_\infty : f \in K\}$$

$$\|f\|_\infty \leq 1 \qquad \Longrightarrow \qquad f \notin K$$

$$d = \inf\{\|f\|_\infty : f \in K\} \geq 1,$$

we can take $1 < \alpha < 2$ and $\epsilon = \frac{\alpha-1}{\alpha}$.

$$f_\alpha = -\frac{\alpha}{\epsilon}(x - \frac{1}{2})$$



Now we show that $f_\alpha = -\frac{\alpha}{\epsilon}(x - \frac{1}{2})$ belongs to K.

$$\int_0^{\frac{1}{2}} f_\alpha - \int_{\frac{1}{2}}^1 f_\alpha = 1 \qquad \forall \alpha \in (1,2)$$

$$\|f_\alpha\|_\infty = \alpha,$$

α is the supremum,

$$d = \inf\{\|f\|_\infty : f \in K\} \leq \|f_\alpha\|_\infty = \alpha$$

$$\alpha \in (1,2)$$

$$\begin{cases} d \leq 2 \\ \forall \alpha \in (1,2) \end{cases} \implies d \leq 1$$

so that

$$d = \inf\{\|f\|_\infty : f \in K\} = 1,$$

but this inf is not assumed, this is not a minimum, then

$$\nexists f \in K \qquad \text{s.t.} \qquad d = \|f\|_\infty = 1,$$

$$d = d(0, K)$$

$$0 \notin K.$$

Chapter 4

Operators

Exercise 1

Let

$$a(x) = \begin{cases} x & if \quad x \in (0, \frac{1}{2}] \\ 0 & if \quad x \in (\frac{1}{2}, 1] \end{cases}$$

and consider the operator

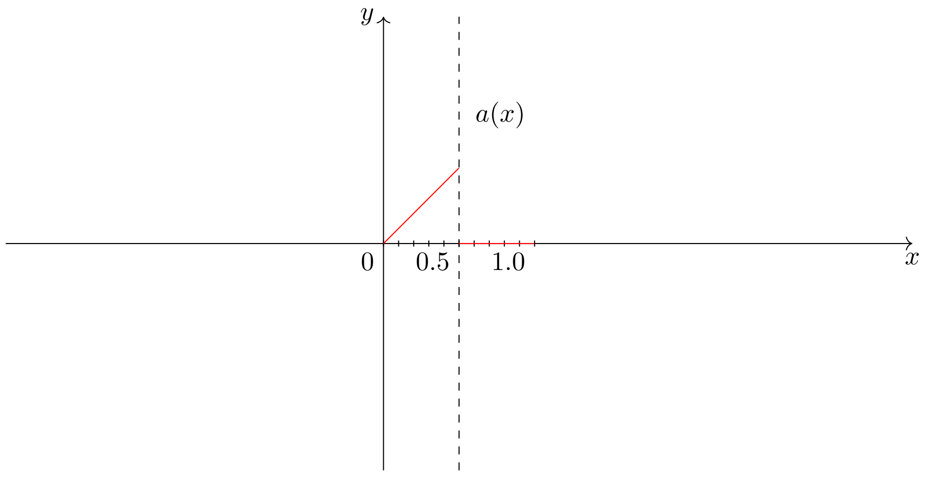
$$T : L^2(0,1) \to L^2(0,1)$$

given by

$$Tf(x) = a(x)f(x), \qquad x \in (0,1).$$

Show that $T \in \mathcal{L}(L^2[0,1])$ and compute $\|T\|$.

Solution



$$f \in L^2(0,1)$$

$$\|Tf\|^2_{L^2(0,1)} = \int_0^1 a(x)^2 f(x)^2 dx \leq \frac{1}{4} \int_0^1 f(x)^2 dx = \frac{1}{4} \|f\|^2_{L^2(0,1)}$$

so that

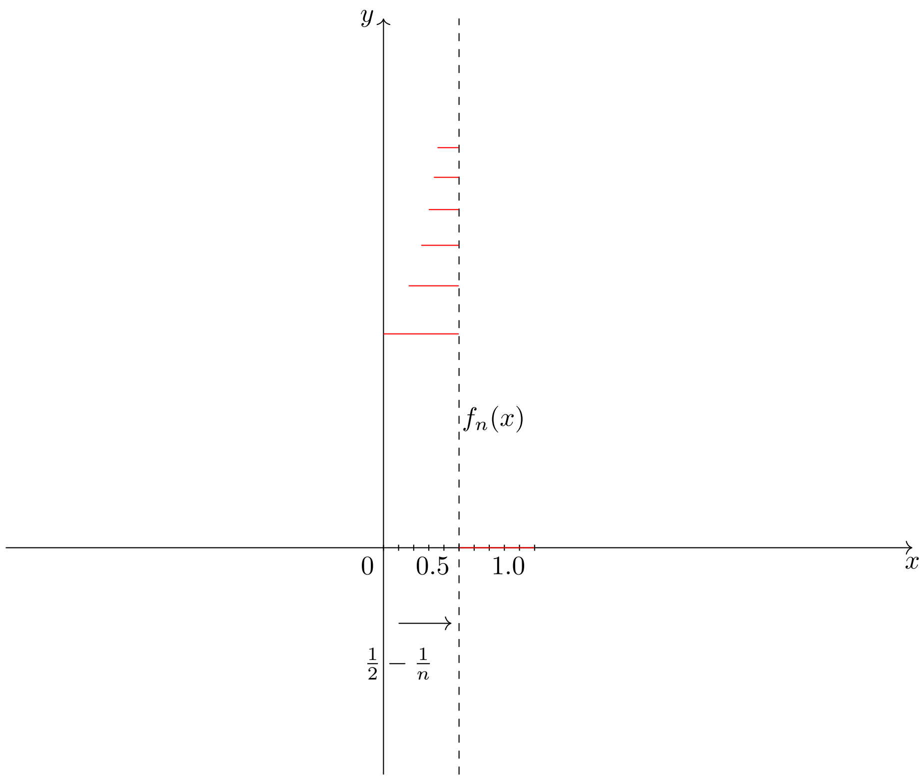
$$\|T\|_{\mathcal{L}(L^2(0,1))} \leq \frac{1}{2}.$$

Then we want to show that $\frac{1}{2}$ is the value of the norm, that is

$$\|T\| = \frac{1}{2}.$$

We define

$$f_n(x) = \begin{cases} \sqrt{n} & if \quad x \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2}], \qquad n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$



$$\|f_n(x)\|^2_{L^2(0,1)} = \int_0^1 f_n(x)^2 = 1$$

Now we compute the norm of the image:

$$\|Tf_n(x)\|^2_{L^2(0,1)} = \int_0^1 a(x)^2 f_n(x)^2 dx \geq \star$$

we can minor the integral with

$$\star \geq \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}} n(\frac{1}{2} - \frac{1}{n})^2 dx = (\frac{1}{2} - \frac{1}{n})^2 \to \frac{1}{4} \qquad \text{for} \quad n \to +\infty$$

so that

$$\|T\|_{\mathcal{L}(L^2(0,1))} = \frac{1}{2}.$$

We can characterize the norm in various ways.

$$\|T\|_{\mathcal{L}(L^2(0,1))} = \sup_{f \in L^2(0,1) \quad \|f\|_{L^2}=1} \frac{\|Tf\|_{L^2(0,1)}}{\|f\|_{L^2(0,1)}} \leq \frac{1}{2}.$$

We have shown that

$$\|Tf_n(x)\|^2_{L^2(0,1)} \geq \frac{1}{4},$$

that is

$$\|Tf_n(x)\|_{L_2(0,1)} \geq \frac{1}{2},$$

but at the same time we have

$$\|T\| \leq \frac{1}{2}$$

then

$$\|Tf_n(x)\| = \frac{1}{2}.$$

Exercise 2

Let $(f'_h)_{h\in\mathbb{N}} \in L^p$ for some p, with the following hypotheses:

- $(f'_h)_{h\in\mathbb{N}}$ is bounded in L^p for some p;
- $f_h(0)$ is bounded.

Show that $(f_h)_{h\in\mathbb{N}}$ is compact (relatively) in $(C([0,1]), \|\cdot\|_\infty$.

Solution

From the hypotheses we can suppose that

$$f_h(x) = f_h(0) + \int_0^x f'_h(y)dy \qquad x \in [0,1].$$

Since we have to show the compactness in the set of continuous fuunctions we need to utilize the Ascoli-Arzelà theorem.

Let $C > 0$ constant.

Equiboundedness

$$h \in \mathbb{N}, \qquad x \in [0,1]$$

$$|f_h(x)| \leq |f_h(0)| + \int_0^x |f'_h(y)|dy \leq \star$$

using the hypothesis 2 and the Hölder inequality

$$\star \leq C + \|f'_h\|_{L^p(0,1)}\, x^{\frac{1}{p'}} \leq M$$

so that

$$|f_h(x)| \leq M,$$

that is f_h is equibounded.

Equicontinuity

$$x, y \in [0,1], \qquad x < y$$

$$f_h(y) - f_h(x) \leq \int_x^y |f'_h(w)|dw \leq \star$$

using the Hölder inequality

$$\star \leq \|f'_h\|_{L^p(0,1)}\, |y-x|^{\frac{1}{p'}}.$$

This shows that the fuunctions f_h are equi-hölder with exponent $\frac{1}{p'}$, in particular they are eqicontinuous.

Then from the Ascoli-Arzelà theorem they are relatively compact.