A collection of Game Theory exercises

June 11, 2024

Exercise 1

Two people are employed in a joint project. If every person $i = 1, 2$ spends an amount of resources x_i, where $0 \leq x_i \leq 1$, incurring a cost $k_i(x_i)$, the project will have a revenue of $f(x_1, x_2)$. The revenue is equally divided by the two persons, without considering the resources employed by each person.

- write the payoff function of each player;
- formulate this problem as a strategic game and determine the possible Nash equilibria in the cases:
    1. $f(x_1, x_2) = 3x_1x_2$ and $k_i(x_i) = x_i^1$ for $i = 1, 2$;
    2. $f(x_1, x_2) = 4x_1x_2$ and $k_1(x_1) = x_1$, $k_2(x_2) = \frac{2}{3}x_2$.

In the prevoius cases can we deduce a priori that a Nash equilibrium exists?

Solution

case a)

i) proceed:

$$f(x_1, x_2) = 3x_1x_2,$$

the proceed for every single person is:

$$f_i(x_1, x_2) = \frac{3}{2}x_1x_2.$$

The payoff functions are:

$$u_1(x_1, x_2) = \frac{3}{2}x_1x_2 - x_1^2$$

$$u_2(x_1, x_2) = \frac{3}{2}x_1x_2 - x_2^2.$$

The strategic game $(A_1 \times A_2, (u_1, u_2))$ is such that:

$$A_1 = A_2 = [0, 1]$$

$$u_1(x_1, x_2) = \frac{3}{2}x_1x_2 - x_1^2$$

$$u_2(x_1, x_2) = \frac{3}{2}x_1x_2 - x_2^2.$$

We now need to calculate the best reply:

$$BR_i := A_{-i} \to A_i \qquad \forall i = 1, 2, \cdots, n,$$

that is

$$BR_1 := A_2 \to A_1 \qquad \forall x_2 \in [0, 1],$$

given by

$$BR_1(x_2) = \arg\max_{x_1 \in [0,1]} u_1(x_1, x_2) = \arg\max_{x_1 \in [0,1]} \frac{3}{2}x_1x_2 - x_1^2 = \arg\max_{x_1 \in [0,1]} (\frac{3}{2}x_2 - x_1)x_1.$$

We have that

$$\frac{\partial u_1(x_1, x_2)}{\partial x_1} = \frac{3}{2}x_2 - 2x_1 = 0$$

for

$$x_1 = \frac{3}{4}x_2,$$

so that

$$BR_1(x_2) = \frac{3}{4}x_2.$$

The best reply for the player two is:

$$BR_2 := A_1 \to A_2 \qquad \forall x_1 \in [0, 1]$$

given by

$$BR_1(x_1) = \arg\max_{x_2 \in [0,1]} u_2(x_1, x_2) = \arg\max_{x_2 \in [0,1]} \frac{3}{2} x_1 x_2 - x_2^2$$

so that

$$\frac{\partial u_2(x_1, x_2)}{\partial x_2} = \frac{3}{2} x_1 - x_2 = 0$$

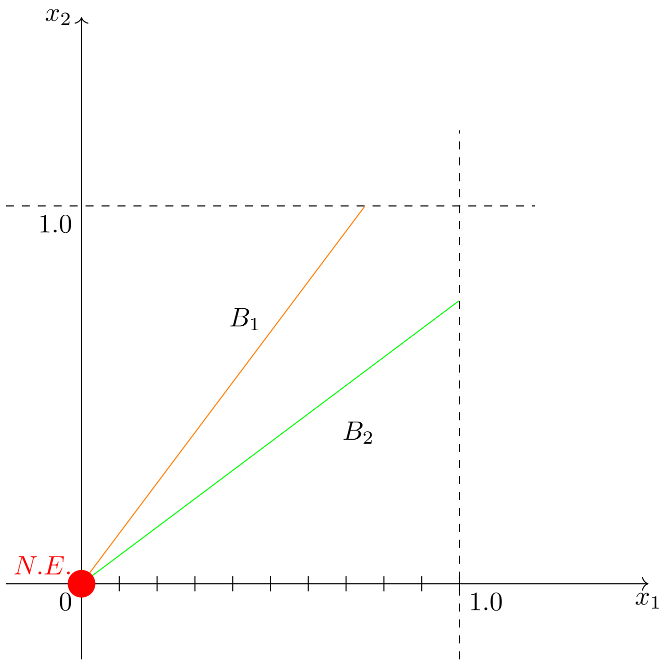
for

$$x_2 = \frac{3}{4} x_1,$$

so that

$$BR_2(x_1) = \frac{3}{4} x_1.$$

We can also solve the following system:



$$\begin{cases} x_1 = \frac{3}{4} x_2 \\ x_2 = \frac{3}{4} x_1 \end{cases} \implies \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$$

$(0, 0)$ is the unique Nash Equilibrium.

case b)

i) Payoff functions:

$$f(x_1, x_2) = 4 x_1 x_2,$$

then for each player the proceeds is

$$f_i(x_1, x_2) = 2 x_1 x_2,$$

so that the payoff functions are:

$$u_1(x_1, x_2) = 2 x_1 x_2 - x_1$$

$$u_2(x_1, x_2) = 2 x_1 x_2 - \frac{2}{3} x_1.$$

ii) Now we consider the two players strategic game $(A_1 \times A_2, (u_1, u_2))$ such that

$$A_1 = A_2 = [0, 1]$$

$$u_1(x_1, x_2) = 2 x_1 x_2 - x_1$$

$$u_2(x_1, x_2) = 2x_1x_2 - \frac{2}{3}x_2.$$

Now we can consider the Best Reply

$$BR_i := A_{-i} \to A_i \qquad \forall i = 1, 2, \cdots, n$$

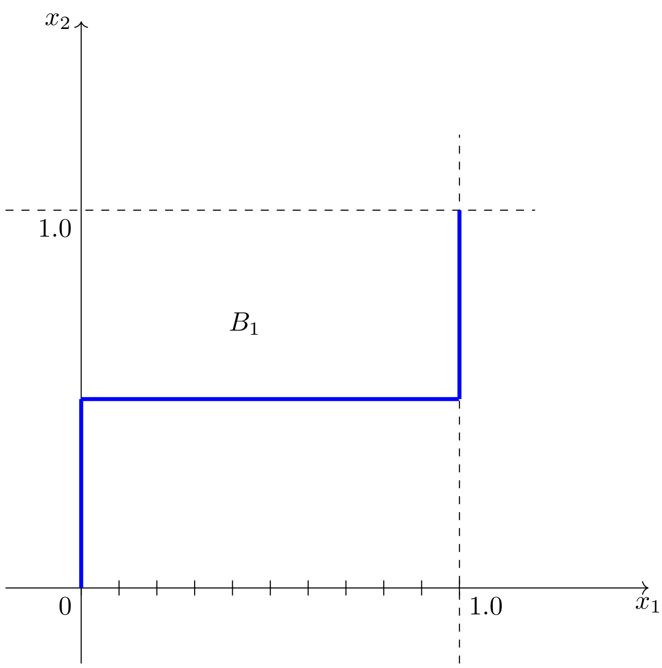
that are

$$BR_1 := A_2 \to A_1$$
$$BR_2 := A_1 \to A_2.$$

From the definition we have

$$BR_i(a_{-i}) = \{a_i \in A_i \qquad \text{s.t.} \qquad u_i(a_{-i}, a_i) \geq u_i(a_{-i}, \hat{a}_i) \qquad \forall \hat{a}_i \in A_i\}$$
$$= \arg\max_{\hat{a}_i \in A_i} u_i(a_{-i}, \hat{a}_i)$$

$$BR_1(x_2) = \arg\max_{x_1 \in [0,1]} (2x_1 - 1)x_1 = \begin{cases} \{0\} & \text{if} \quad x_2 < \frac{1}{2} \\ [0,1] & \text{if} \quad x_2 = \frac{1}{2} \\ \{1\} & \text{if} \quad x_2 > \frac{1}{2} \end{cases}$$



$$BR_2 := A_1 \to A_2 \qquad \forall x_1 \in [0,1]$$

$$BR_2(x_1) = \arg\max_{x_2 \in [0,1]} (2x_1 - \frac{2}{3})x_2 = \begin{cases} \{0\} & \text{if} \quad x_1 < \frac{1}{3} \\ [0,1] & \text{if} \quad x_1 = \frac{1}{3} \\ \{1\} & \text{if} \quad x_1 > \frac{1}{3} \end{cases}$$

From the definition

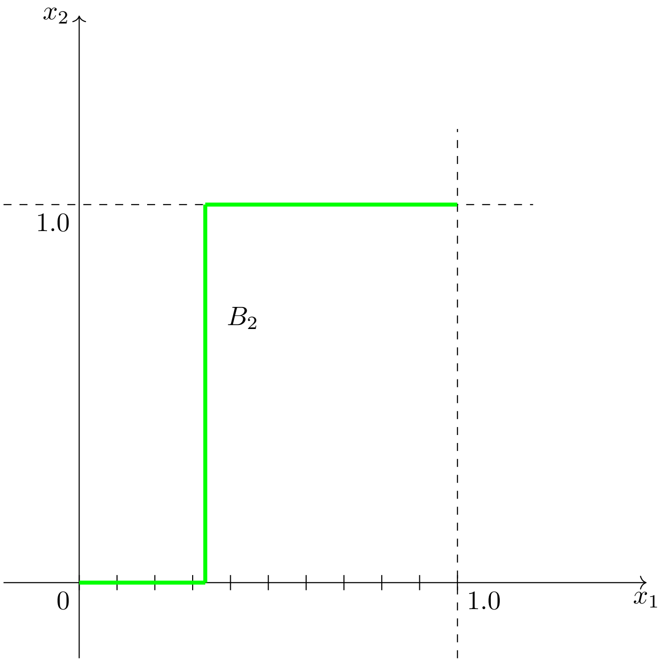
$$a^\star \in A \qquad \text{is a Nash Equilibrium}$$
$$\text{iff}$$
$$a_i^\star \in BR_i(a_{-i}^\star \qquad \forall i$$

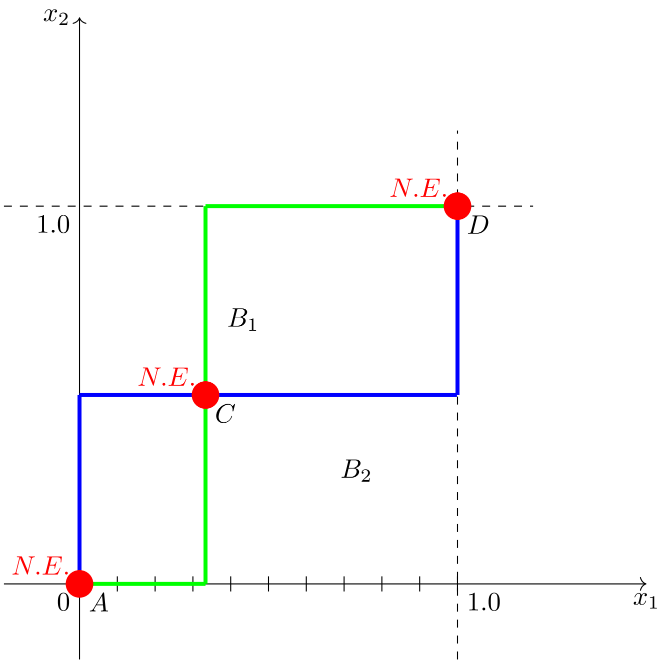
The Nash Equilibrium are

$$B_1 \cap B_2 = \{A, C, D\}$$

where

$$A = (0, 0)$$





$$C = (1./3, 1./2)$$

$$D = (1., 1.)$$

The functions $u_i : A_1 \times A_2 \to \mathbb{R}$ are continuous. The map $x_1 \mapsto u_1(x_1, x_2)$ with $x_2 \in A$ fixed is linear in x_1 and so it is concave. The same for the map $x_2 \mapsto u_2(x_1, x_2)$ with $x_1 \in A$ fixed. Then the Nash Theorem guarantees the existence of at least a Nash Equilibrium.

Exercise 2

Consider a two player non-cooperative game, in which every player controls a unique variable, which we indicate, respectively with x_1 for the first player and x_2 for the second player. The alternative set for the first player is:

$$X_1 = \{x_1 \qquad \text{such that} \qquad -6 \leq x_1 \leq 2\}$$

and for the second player is:

$$X_2 = \{x_2 \qquad \text{such that} \qquad -2 \leq x_2 \leq 41\}.$$

The payoff functions for the two players are:

$$C_1(x_1, x_2) = \frac{1}{2}x_1^2 - x_1(2x_2 - 4) + 7x_2$$
$$C_2(x_1, x_2) = (3 - x_2)(1 - x_1).$$

a. Can be stated "a priori" the existence of a Nash Equilibrium?

b. Identify for each player the Best Reply functions.

c. Identify the Nash Equilibrium of the game, if they exist.

Solution

Point a)

The two player strategic game is

$$(A_1 \times A_2, (u_1, u_2))$$

with the alternative sets:

$$A_1 = [-6, 2]$$
$$A_2 = [-2, 4],$$

and payoff functions

$$u_1(x_1, x_2) = \frac{1}{2}x_1^2 - x_1(2x_2 - 4) + 7x_2$$
$$u_2(x_1, x_2) = (3 - x_2)(1 - x_1),$$

where the two players must minimize. Now we verify the hypotheses of the Nash Theorem.

A_1 and A_2 are closed and bounded sets of $\mathbb{R}$ so that they are non empty, convex and compact subsets of $\mathbb{R}$. The functions

$$u_i : A_1 \times A_2 \to \mathbb{R}$$

are continuous. The map

$$x_1 \mapsto u_1(x_1, x_2)$$

with $x_2 \in A_2$ fixed non-linear with respect to the variable x_1, but since we are considering a minimizing problem the map $x_1 \mapsto u_1(x_1, x_2)$ is convex for every x_2 fixed and so also for the map $x_2 \mapsto u_2(x_1, x_2)$ for every x_1 fixed.

The hypotheses of the Nash Theorem are satisfied so we know that at least a Nash Equilibrium exists.

Point b)

$$BR_i : A_{-i} \to A_i$$

$$BR_1 : A_2 \rightrightarrows A_1 \qquad \forall x_2 \in [-2, 4]$$

is given by

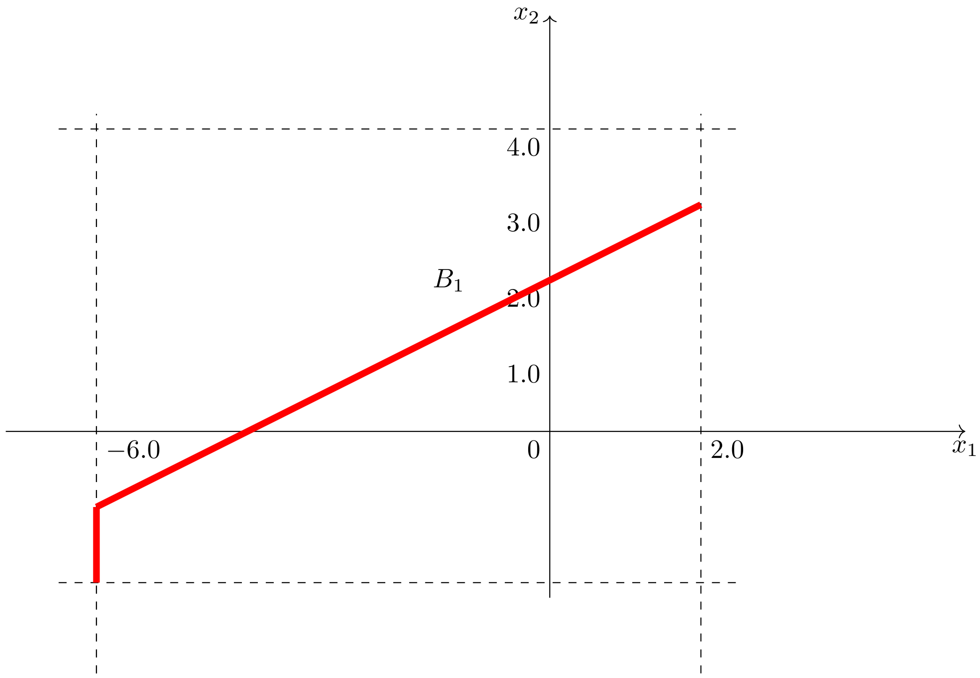
$$BR_1(x_2) = \underset{x_1 \in [-6,2]}{\arg\min}\, u_1(x_1, x_2) = \underset{x_1 \in [-6,2]}{\arg\min}\, \frac{1}{2}x_1^2 - x_1(2x_2 - 4) + 7x_2$$

so that

$$\frac{\partial u_1(x_1, x_2)}{\partial x_1} = x_1 - (2x_2 - 4) = 0$$

for

$$x_1 = 2x_2 - 4.$$



$$BR_1(x_2) = \underset{x_1 \in [-6,2]}{\arg\min}\, u_1(x_1, x_2) = \begin{cases} \{-6\} & \text{if} \quad x_2 < -11 \\ \{2x_2 - 4\} & \text{if} \quad -1 < x_2 < 3 \\ \{2\} & \text{if} \quad x_2 > 3 \end{cases}$$

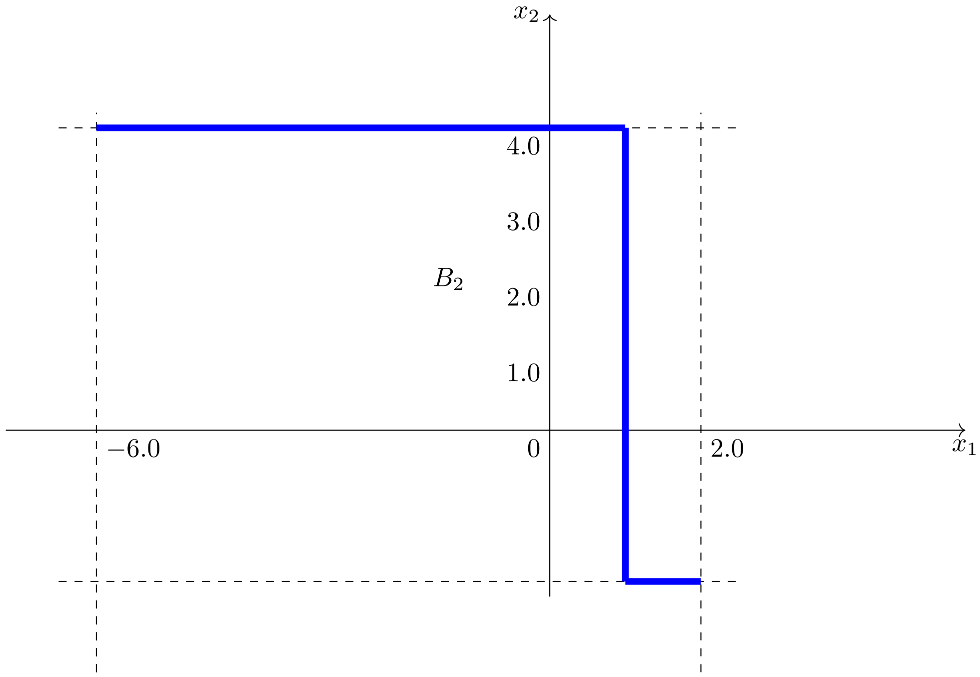
$$BR_2(x_1) = \underset{x_2 \in [-2,4]}{\arg\min}\, u_2(x_1, x_2) = \underset{x_2 \in [-2,4]}{\arg\min}\, (3 - x_2)(1 - x_1)$$

$$\frac{\partial u_2(x_1, x_2)}{\partial x_2} = x_1 - 1 = const. = \begin{cases} \{4\} & \text{if} \quad x_1 < 1 \\ \{-2, 4\} & \text{if} \quad x_1 = 1 \\ \{-2\} & \text{if} \quad x_1 > 1 \end{cases}$$

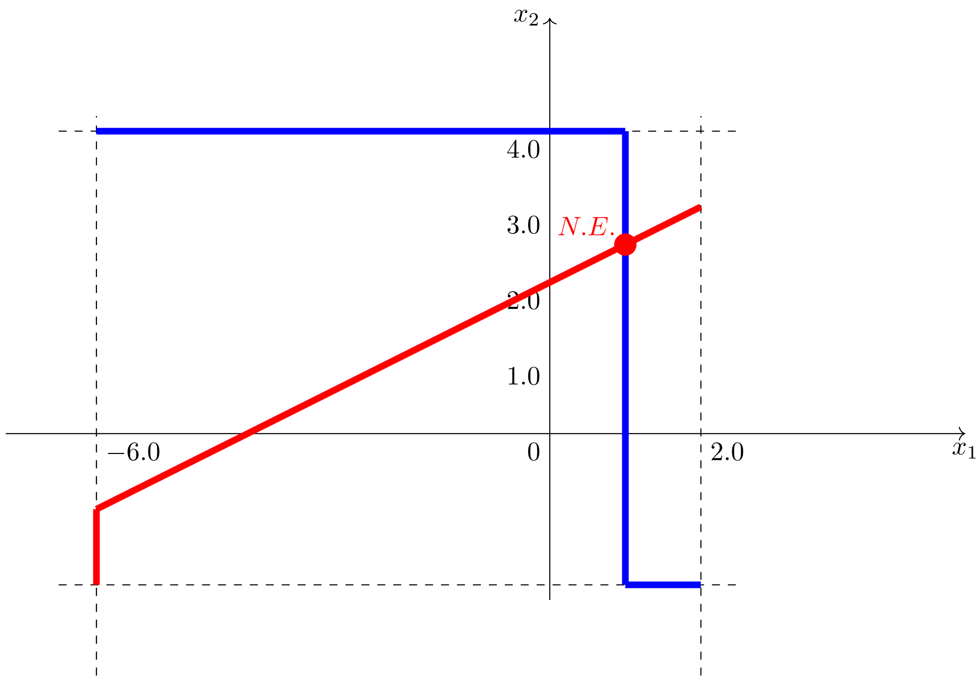
Now to determine the Nash Equilibrium we need to make the intersection $B_1 \cap B_2$. We can also solve the following system:

$$\begin{cases} x_1 = 2x_2 - 4 \\ x_1 = 1 \end{cases}$$

The the unique Nash Equilibrium is $(1; \frac{5}{2})$.



x_2
4.0
3.0
B_2
2.0
1.0
-6.0
0
2.0
x_1



x_2
4.0
3.0
$N.E.$
2.0
1.0
-6.0
0
2.0
x_1

Exercise 3

Consider the following game:

	Second Player					
First Player	D		E		F	
A	3	0	0	0	0	3
B	0	0	1	1	0	0
C	0	3	0	0	3	0

a. Determine the eventual Nash Equilibrium in pure strategy;

b. Show that there exists an Equilibrium in mixed strategies with the player 1 that plays (A, B, C) with probability $(\frac{1}{5}, \frac{3}{5}, \frac{1}{5})$ and the player 2 that plays (D, E, F) with probability $(\frac{1}{5}, \frac{3}{5}, \frac{1}{5})$.

Solution

Point a)

- if player 2 plays D player 1 to maximize chooses A;
- if player 2 plays E player 1 to maximize chooses B;
- if player 2 plays F player 1 to maximize chooses C;
- if player 1 plays A player 2 to maximize chooses F;
- if player 1 plays B player 2 to maximize chooses E;
- if player 1 plays C player 1 to maximize chooses D;

Then the unique Nash Equilibrium in pure strategies is (B, E). **Point b)**

$$A_1 = \{A, B, C\}$$
$$A_2 = \{D, E, F\}$$
$$S_i := \Delta A_i$$
$$S_1 = \Delta A_1 = x_1 A + x_2 B + (1 - x_1 - x_2)C$$

with $x_1 + x_2 = 1 \qquad x_i \geq 0$.

$$S_2 = \Delta A_2 = y_1 D + y_2 E + (1 - y_1 - y_2)F$$

with $y_1 + y_2 = 1 \qquad y_i \geq 0$. Now we can construct the payoff functions.

$$u_1(x_1, x_2, x_3, y_1, y_2, y_3) = [x_1, x_2, 1 - x_1 - x_2] \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ 1 - y_1 - y_2 \end{bmatrix}$$

$$= [3x_1, x_2, 3(1 - x_1 - x_2)] \begin{bmatrix} y_1 \\ y_2 \\ 1 - y_1 - y_2 \end{bmatrix} = 3x_1 y_1 + y_2 x_2 + 3(1 - x_1 - x_2)(1 - y_1 - y_2)$$

$$= 3x_1 y_1 + y_2 x_2 + 3(1 - y_1 - y_2 - x_1 + x_1 y_1 + x_2 y_2 - x_2 + x_2 y_1 + x_2 y_2)$$

$$= 6x_1 y_1 + 4x_2 y_2 + 3x_1 y_2 + 3x_2 y_1 - 3y_2 - 3x_1 - 3x_2 - 3y_1 + 3.$$

Thr payoff function for the second player:

$$u_2(x_1, x_2, x_3, y_1, y_2, y_3) = [x_1, x_2, 1 - x_1 - x_2] \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ 1 - y_1 - y_2 \end{bmatrix}$$

$$= [3(1-x_1-x_2), x_2, 3x_1]\begin{bmatrix} y_1 \\ y_2 \\ 1-y_1-y_2 \end{bmatrix} = 3y_1 - 6x_1y_1 + x_2y_2 - 3x_2y_1 - 3x_1y_2 + 3x_1$$

Summarize:

$$u_1(x_1,x_2,y_1,y_2) = 6x_1y_1 + 4x_2y_2 + 3x_1y_2 + 3x_2y_1 - 3y_2 - 3x_1 - 3x_2 - 3y_1 + 3$$

$$u_2(x_1,x_2,y_1,y_2) = -6x_1y_1 + x_2y_2 - 3x_2y_1 - 3x_1y_2 + 3y_1 + 3x_1$$

$$BR_i : A_{-i} \rightrightarrows A_i$$

$$BR_1 : A_2 \rightrightarrows A_1.$$

In terms of a mixed strategic game $E(G)$, we have

$$BR_i : S_{-i} \rightrightarrows S_i$$

$$BR_1 : S_2 \rightrightarrows S_1.$$

Now we fix $(y_1, y_2) \in [0,1] \times [0,1]$,

$$BR_1(y_1,y_2) = \underset{(x_1,x_2)\in[0,1]^2}{\arg\max}\ u_1(x_1,x_2,y_1,y_2) = \underset{(x_1,x_2)\in[0,1]^2}{\arg\max}\ 6x_1y_1 + 4x_2y_2 + 3x_1y_2 + 3x_2y_1 - 3y_2 - 3x_1$$

$$= \underset{(x_1,x_2)\in[0,1]^2}{\arg\max}\ x_1(6y_1 + 3y_2 - 3) + x_2(4y_2 + 3y_1 - 3).$$

$$BR_1(\frac{1}{5},\frac{3}{5},\frac{1}{5}) = \underset{(x_1,x_2)\in[0,1]^2}{\arg\max}\ x_1(\frac{6}{5}+\frac{9}{5}-\frac{15}{5}) + x_2(\frac{12}{5}+\frac{3}{5}-\frac{15}{5}) = \underset{(x_1,x_2)\in[0,1]^2}{\arg\max}\ 0 = [0,1]\times[0,1].$$

$$S^\star = ((\frac{1}{5},\frac{3}{5},\frac{1}{5}),(\frac{1}{5},\frac{3}{5},\frac{1}{5}))$$

$$S_1^\star = (\frac{1}{5},\frac{3}{5},\frac{1}{5})$$

$$BR_1^\star(S_2^\star) = x_1A + x_2D + (1-x_1-x_2)C$$

with $x_1, x_2 \in [0,1]\times[0,1]$. Considering

$$\frac{1}{5}A + \frac{3}{5}B + \frac{1}{5}C,$$

then we have verified that $S_1^\star \in BR_1(S_2^\star)$.

$$BR_2 : S_1 \rightrightarrows S_2$$

we now fix $x_1, x_2 \in [0,1]\times[0,1]$ so that

$$BR_2(x_1,x_2,1-x_1-x_2) = \underset{(y_1,y_2)\in[0,1]^2}{\arg\max}\ u_2(x_1,x_2,y_1,y_2) = \underset{(y_1,y_2)\in[0,1]^2}{\arg\max}\ -6x_1y_1 + x_2y_2 - 3x_2y_1 - 3x_1y_2 + 3y_1 + 3x_1$$

$$= \underset{(y_1,y_2)\in[0,1]^2}{\arg\max}\ y_1(-6x_1 - 3x_2 + 3) + y_2(x_2 - 3x_1)$$

$$BR_2(\frac{1}{5},\frac{3}{5},\frac{1}{5}) = \underset{(y_1,y_2)\in[0,1]^2}{\arg\max}\ y_1(-\frac{6}{5}-\frac{9}{5}+\frac{15}{5}) + y_2(\frac{3}{5}-\frac{3}{5}) = \underset{(y_1,y_2)\in[0,1]^2}{\arg\max}\ 0 = [0,1]\times[0,1]$$

$$BR_2(S_1^\star) = [0,1]\times[0,1]$$

so that

$$BR_2^\star(S_1^\star) = y_1D + y_2E + (1-y_1-y_2)F$$

with $(y_1,y_2)\in[0,1]^2$. Since

$$S_2^\star = (\frac{1}{5},\frac{3}{5},\frac{1}{5}),$$

we have

$$\frac{1}{5}D + \frac{3}{5}E + \frac{1}{5}F$$

then $S_2^\star \in BR_2(S_1^\star)$.

Since

$$S_1^\star \in BR_1(S_2^\star)$$

and

$$S_2^\star \in BR_2(S_1^\star)$$

there exists a Nash Equilibrium in mixed strategies with the player 1 that plyas

$$\frac{1}{5}A + \frac{3}{5}B + \frac{1}{5}C$$

and the player 2 that plays

$$\frac{1}{5}D + \frac{3}{5}E + \frac{1}{5}F.$$