A collection of Real Analysis exercises

December 26, 2024

Chapter 1

Metric Spaces

Exercise 1

Let the distances in $\mathbb{R}^2$:

1. $d_1(X,Y) = \sum_{i=1}^{2} |y_i - x_i| = |y_1 - x_1| + |y_2 - x_2|$;
2. $d_2(X,Y) = \sqrt{\sum_{i=1}^{2} |y_i - x_i|^2} = \sqrt{|y_1 - x_1|^2 + |y_2 - x_2|^2}$;
3. $d_\infty(X,Y) = \max_{i=1,2} |y_i - x_i| = \max\{|y_1 - x_1|, |y_2 - x_2|\}$.

Construct the open balls related to these distances.

Solution

Point 1

$$d_1(X,Y) < r$$

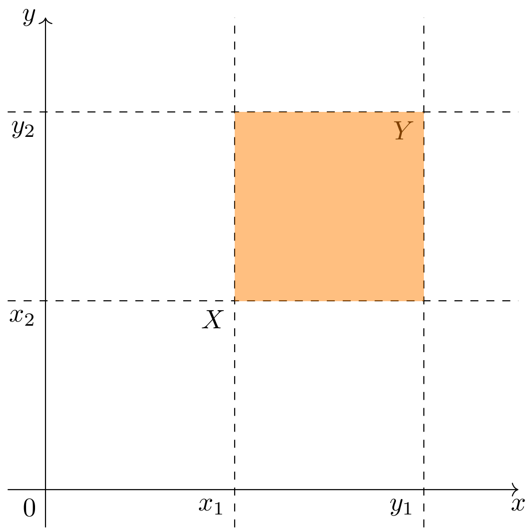
$$\text{iff}$$

$$\sum_{i=1}^{2} |y_i - x_i| < r$$

$$\text{iff}$$

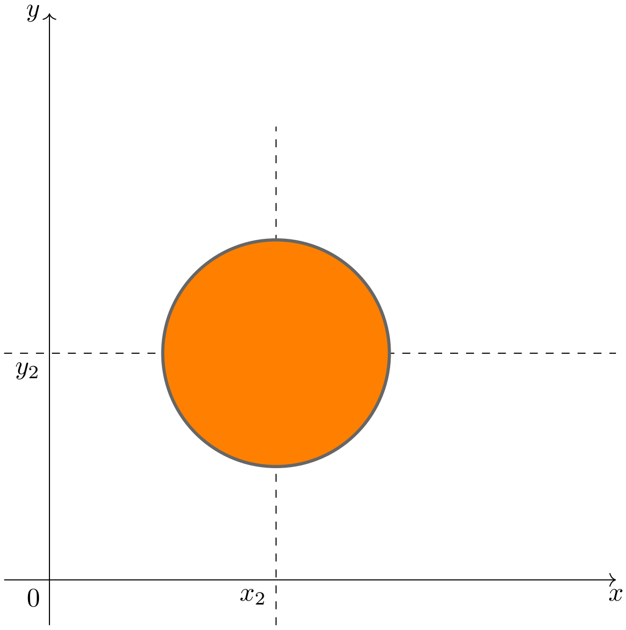
$$|y_1 - x_1| + |y_2 - x_2| < r$$

Point 2

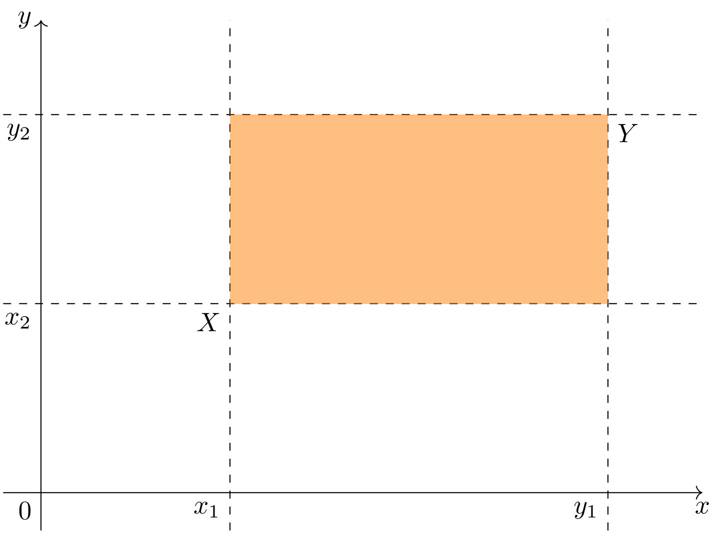


$$d_2(X;Y) < r$$
iff
$$\sqrt{|y_1 - x_1|^2 + |y_2 - x_2|^2} < r$$
iff
$$|y_1 - x_1|^2 + |y_2 - x_2|^2 < r^2$$

Point 3



$$d_\infty(X,Y) < r$$
iff
$$\max\{|y_1 - x_1|, |y_2 - x_2|\}$$



Exercise 2

Let (X, d) a metric space, $A \subset X$ not empty. Show if the following statements are true or false:

1. $A \quad \text{open} \quad \implies \mathring{A} \cap \partial A = \emptyset$;
2. if $\mathring{A} \cap \partial A = \emptyset \implies A \quad \text{closed}$;

Solution

Point 1
The first statement is always true. It is also true if A is closed or if A is neither open or closed.

$$\partial A = \{x \in X \text{s.t.} x \notin \mathring{A} \quad \text{and} \quad x \notin \overset{\circ}{\overbrace{X \setminus A}}\ \}$$

$$\implies \mathring{A} \cap \partial A = \emptyset \quad \text{is} \quad \text{true.}$$

Point 2
The second statement is false. Counterexample:

$$A =]0, 1[$$

is an open set with

$$\partial A = \{0, 1\}$$

and we have that $\mathring{A} \cap \partial A = \emptyset \implies A \quad \text{is} \quad$ closed is false because A is open.

Exercise 3

Let (X, d) a metric space $A \subset X$ closed and $A \neq \emptyset$. Furthermore let

$$f : X \to \mathbb{R}$$

with

$$f(x) = \inf_{a \in A \setminus \{x\}} d(x, a).$$

Tell wether the following statements are true or false.

1. $x \in A \implies f(x) = 0$;
2. $f(x) = 0 \implies x \in A$;

Solution

Point 1
This statement is not true $\forall x$. Counterexample:

$$A = [0, 1] \cup \{2\}$$

we have that $2 \in A$, but $f(2) \neq 0$ because $f(2) = \inf d(2, a)$ with $a \in [0, 1]$. If $a \in [0, 1]$ the distance of 2 from a is greater(or equal) than the distance of 2 from 1.

$$\text{If} \qquad a \in [0,1]$$

$$\text{then} d(2, a) \geq d(2, 1) = 1$$

so that

$$\text{if} d(2, a) \geq 1 \implies \forall a \in [0, 1] \qquad d(2, a) \geq 1.$$

Then

$$\inf d(2, a) \geq 1 \qquad a \in [0, 1]$$

and it can't be equal to zero.
Point 2
Remeber that x is an accumulation point for A iff $\inf_{a \in A \setminus \{x\}} d(x, a) = 0$.

$$f(x) = 0 \implies x \qquad \text{is an accumulation point for A}$$

$$\text{then} \qquad x \in \overline{A} = A \qquad \text{since A is closed}$$

then the second statement is true.

Exercise 4

Let $X = C^0([0,1])$, $d_\infty(f,g) = \sup_{t\in[0,1]} |f(t) - g(t)|$ and $d_2(f,g) = \sqrt{\int_0^1 |f(t) - g(t)|^2 dt}$. Show that d_2 and d_∞ are not equivalent.

Solution

If we choose $f_n(t) = t^n \qquad \forall n \in \mathbb{N}$, we have that:

$$d_\infty(f_n, 0) = \sup_{t\in[0,1]} |f_n(t)| = \sup_{t\in[0,1]} |t^n| = 1.$$

It is a maximum.

$$(d_2(f_n,0))^2 = \int_0^1 |f_n(t)|^2 dt = \int_0^1 (t)^{2n} dt = [\frac{t^{2n+1}}{2n+1}]_0^1 = \frac{1}{2n+1}$$

then

$$d_2(f_n,0) = \frac{1}{2n+1}$$

for $n \to \infty$, $d_2 \to 0$. So that

$$\forall c \in \mathbb{R} \exists n \in \mathbb{N} \qquad \text{s.t.} \qquad d_\infty(f_n,0) > cd_2(f_n,0)$$

then d_2 and d_∞ are not equivalent.

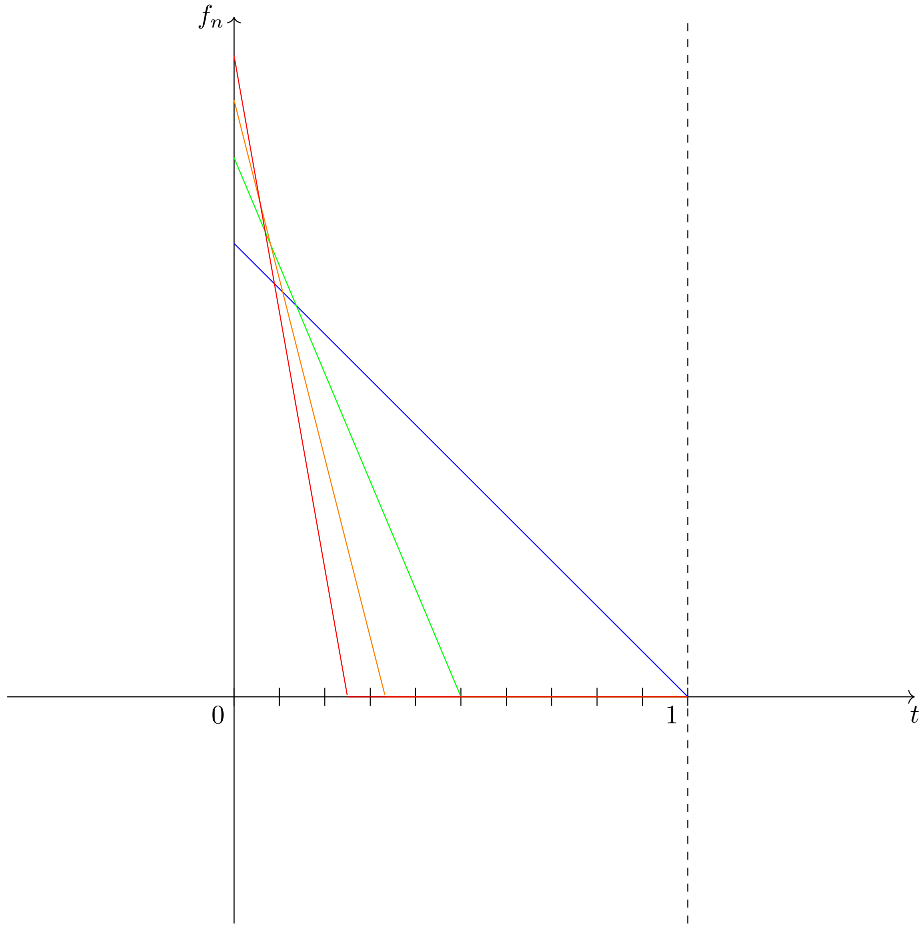
Exercise 5

Let $X = C^0([0,1])$. Show that the open ball $B_{d_2}(0,1)$ is unbounded with respect to d_∞.

Solution

Consider the following sequence of functions:

$$f_n(t) = \begin{cases} -n^{\frac{5}{4}}(t - \frac{1}{n}) & \text{for} \quad t \in [0, \frac{1}{n}] \\ 0 \quad \text{for} & t \in]\frac{1}{n}, 1] \end{cases}$$



$$d_2(f,g) = \sqrt{\int_0^1 |g(t) - f(t)|^2 dt}$$

$$d_\infty(f_n, 0) = \sup_{t \in [0,1]} |f_n(t)| = f_n(0) = \sqrt[4]{n} \to \infty$$

$$(d_2(f_n,0))^2 = \int_0^{\frac{1}{n}} -n^{\frac{5}{2}}(t - \frac{1}{n})dt = [-\frac{n^{\frac{5}{2}}}{3}(t - \frac{1}{n})^3]_0^{\frac{1}{n}} = \frac{n^{\frac{5}{2}}}{3}\frac{1}{n^3} = \frac{1}{3\sqrt{n}}$$

then

$$d_s(f_n, 0) = \frac{1}{\sqrt{3\sqrt{n}}}.$$

With respect to d_∞ the function is unbounded because contains a sequence the goes to infinity.

Exercise 6

Say if $[0,+\infty[$ is bounded in $(\mathbb{R}, d_0)$ and in $(\mathbb{R}, d)$, with

- d_0 the discrete metric;
- d the Euclidean metric;

Solution

The discrete metric is characterized by the fact that the distance between two points is equal to zero or one.

$$d_0(x,y) = \begin{cases} 1 & \text{if} \quad x \neq y \\ 0 & \text{if} \quad x = y \end{cases}$$

so that

$$diam([0,+\infty[) = \sup_{x,y\in[0,+\infty[} d_0(x,y) \leq 1,$$

then $[0,+\infty[$ is bounded in $(\mathbb{R}, d_0)$.

If we consider the Euclidean distance

$$diam([0,+\infty[) = \sup_{x,y\in[0,+\infty[} d(x,y) = \sup_{x,y\in[0,+\infty[} |y-x| \geq n \qquad \forall n \in \mathbb{N},$$

then $diam([0,+\infty[) = +\infty$, so that $[0,+\infty[$ is unbounded with d.

Exercise 7

Let (X, d) a metric space, $f : X \to \mathbb{R}$ a continous function and $A \subset X$ bounded. Say if the following statements are true or false.

1. $f(A)$ is connected;
2. $f(A)$ is compact;
3. $f(A)$ is open;

Solution

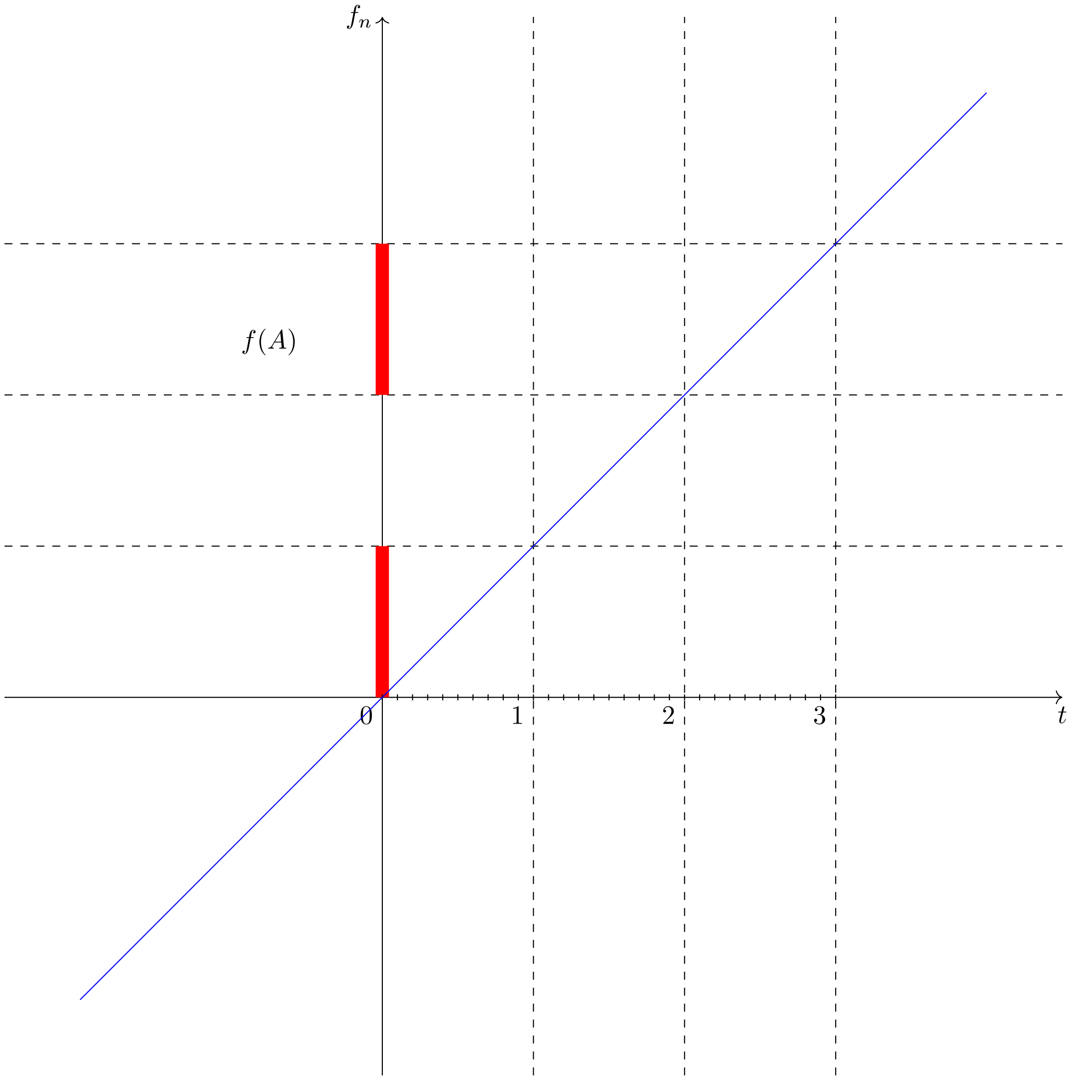
Point 1
Consider $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x$$

and consider $A = [0, 1] \cup [2, 3]$, there is no request on A, so that

$$f(A) = [0, 1] \cup [2, 3]$$

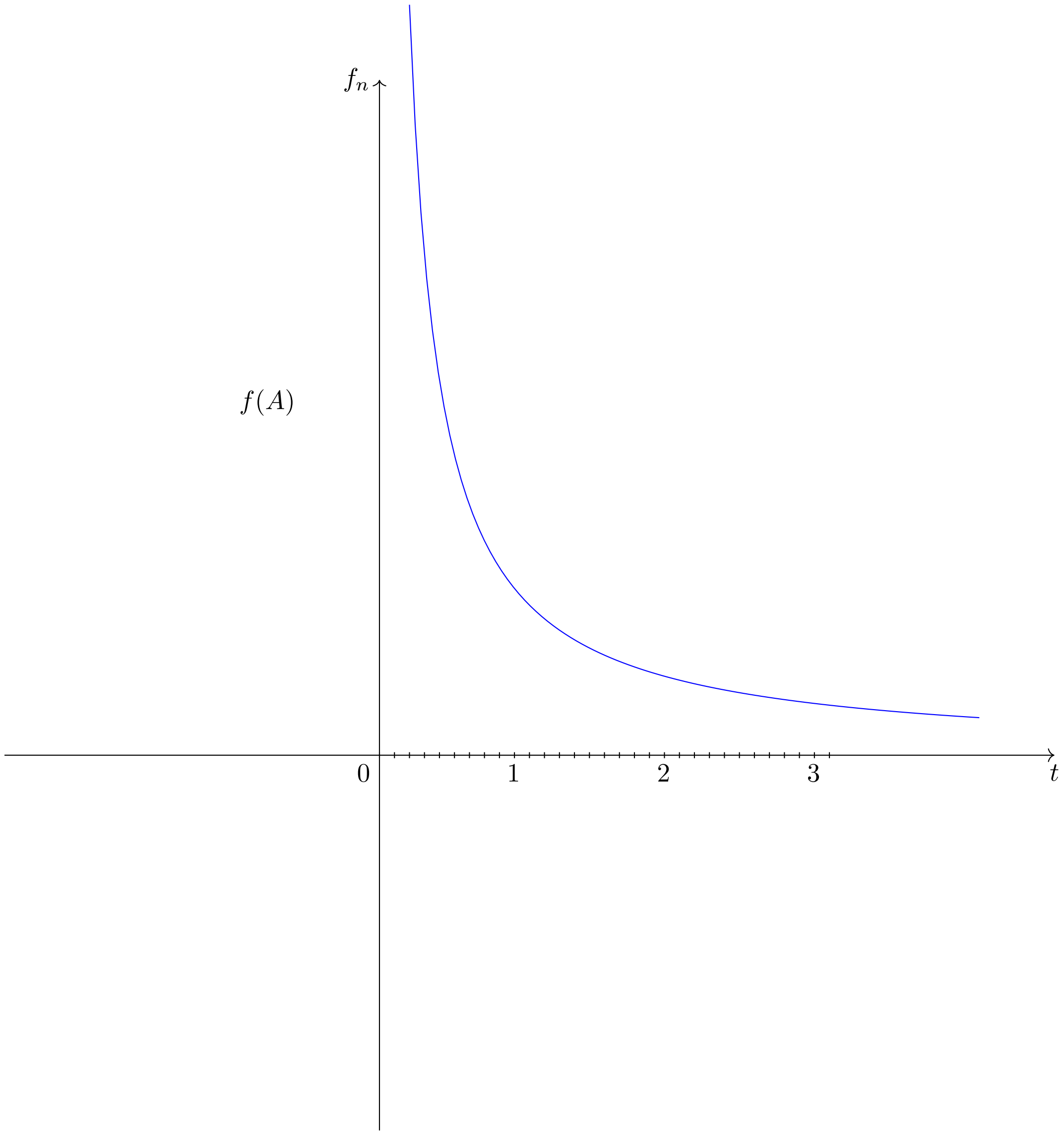
is not connected. **Point** 2



Consider $f :]0, +\infty[\to \mathbb{R}$, given by

$$f(x) = \frac{1}{x}$$

If we choose $A =]0, 1]$ we have $f(a) = [1, +\infty[$ that is not compact because it is not bounded.



Point 3
If we consider $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 1$,

$$A = [0, 1]$$

$$f(A) = \{1\} \qquad \text{that is closed.}$$

Then all the three statements are false.

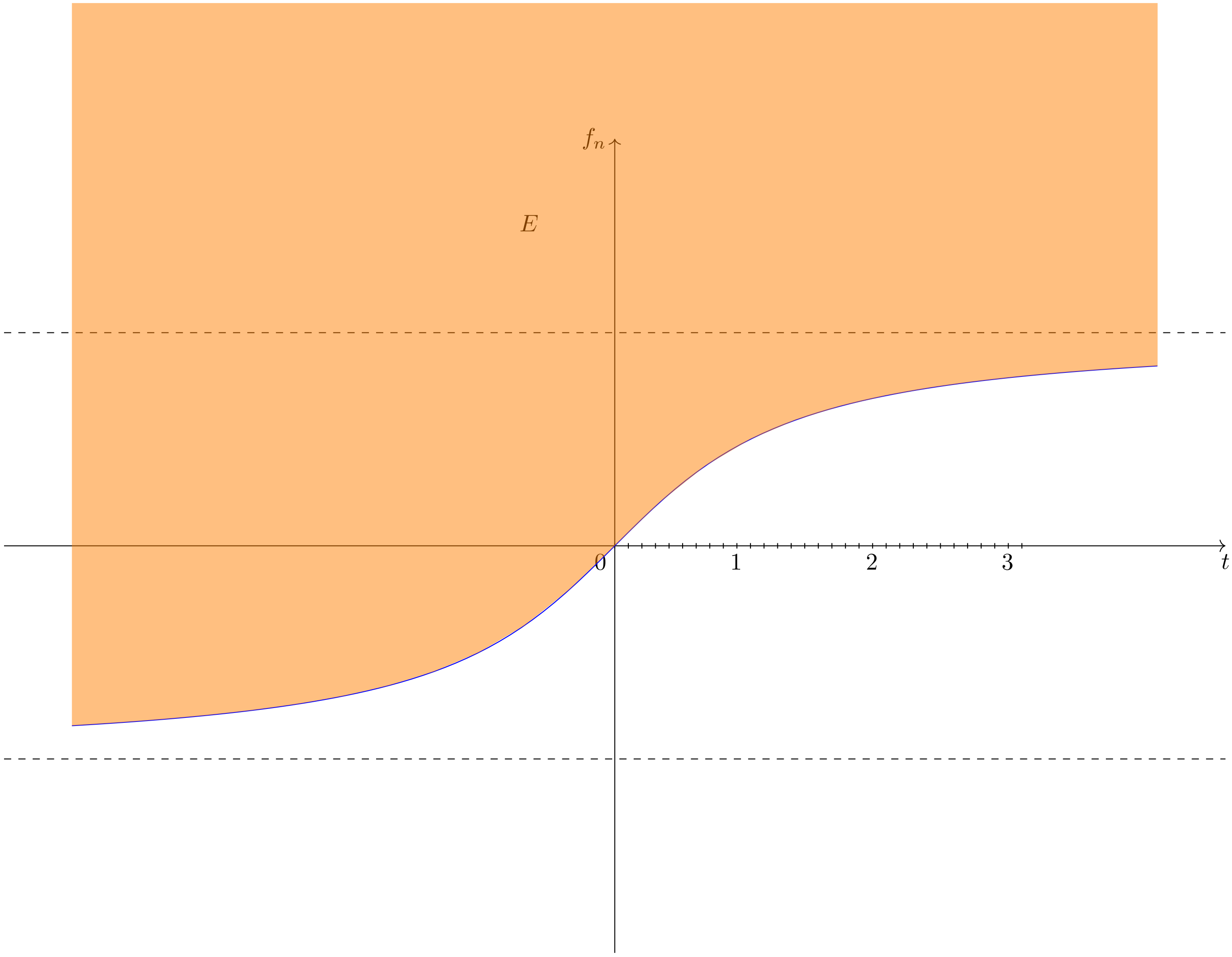
Exercise 8

Let $X = \mathbb{R}^2$ with the Euclidean distance. Say if the set

$$E = \{(x, y) \in \mathbb{R}^2 \qquad \text{s.t.} \qquad y \geq \arctan x\}$$

is complete and if it is compact.

Solution



We can see that E is not bounded so it is not compact. Now we see if it is complete. Consider

$$f(x, y) = y - \arctan(x),$$

we have

$$E = f^{-1}([0, +\infty[).$$

It is a contraimage of a closed set, then E is closed. We know that a closed subset of a complete metric space is complete. Since $\mathbb{R}^2$ is complete, then E is complete.

Exercise 9

Let $X = \mathbb{R}^2$ with the Euclidean distance and let

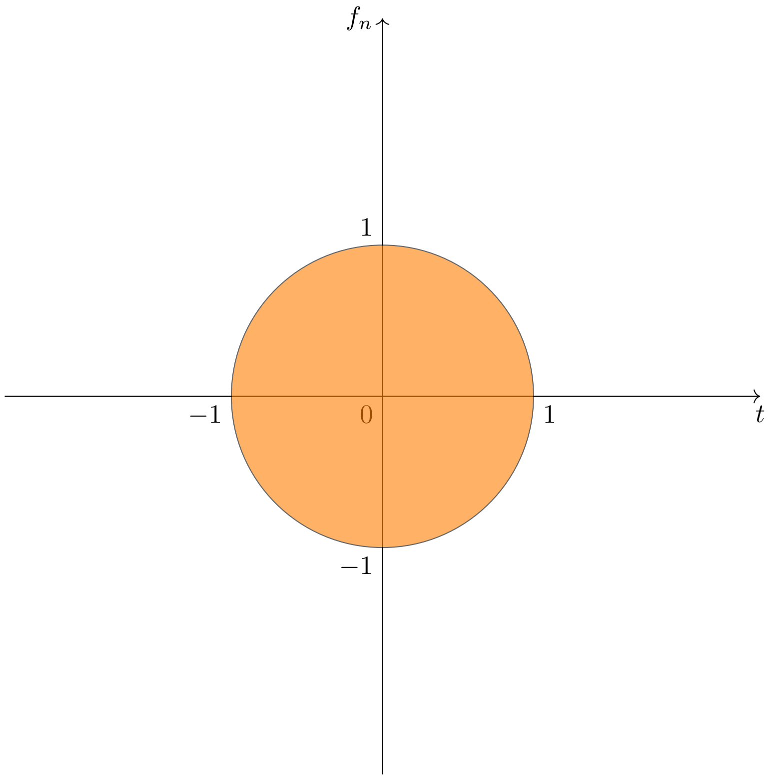
$$A = [0,1] \times [0, +\infty[$$

$$B = \{(x,y) \in \mathbb{R}^2 \qquad \text{with} \qquad x^2 + y^2 < 1\}.$$

Say if A and B with the Euclidean distance are complete.

Solution

A is closed and $A \subset \mathbb{R}^2$ that is complete with the Euclidean diastance. Then A is complete. B is open,



we need to find a Cauchy sequence in B that doesn't converge in B. Consider

$$x_n = (1 - \frac{1}{n}, 0) \qquad x_n \in B \qquad \forall n.$$

We have that

$$x_n \to (1,0) \qquad \text{in} \qquad \mathbb{R}^2,$$

but $(1,0) \notin B$. We have a sequence that converge in the space, that is a Cauchy sequence, but that doesn't converge in B. Then B is not complete.

Exercise 10

Let (X, d) a metric space and x_n a sequence of elements of X. Say if the following statements are true or false.

1. $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \implies x_n \quad \text{bounded}$;
2. $x_n \quad \text{convergent} \implies \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$;
3. $x_n \quad \text{Cauchy} \implies \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Solution

Point 1

The fact that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ doesn't imply that x_n is bounded. Counterexample:

$$x_n = \log n$$

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} |\log n + 1 - \log n| = \lim_{n\to\infty} |\log \frac{n+1}{n}| = 0.$$

The distances between two consecutive elements become shorter but x_n is not bounded.

$$\sup_{n\in\mathbb{N}}\{\log n, \qquad n \in \mathbb{N}\} = +\infty.$$

Then the first statement is false.

Point 2

If x_n is convergent then there exists $x_\infty \in X$ s.t. :

$$\lim_{n\to\infty} x_n = x_\infty.$$

Then

$$0 \le d(x_n, x_{n+1}) \le d(x_n, x_\infty) + d(x_\infty, x_{n+1}),$$

since $d(x_n, x_\infty) \to 0$, $d(x_\infty, x_{n+1}) \to 0$, then by "the two carabinieri theorem" we have that $d(x_n, x_{n+1}) \to 0$.

Point 3

x_n is a Cauchy sequence iff

$$\forall \epsilon > 0 \exists \nu \in \mathbb{N} \text{s.t.} \forall n, m \ge \nu \in \mathbb{N} \implies d(x_n, x_m) < \epsilon.$$

If we fix n, then m can be very far, so is stronger the Cauchy condition with respect to (x_n, x_{n+1}), so that:

$$\forall \epsilon > 0 \exists \nu \in \mathbb{N} \text{s.t.} \forall n > \nu d(x_n, x_{n+1}) < \epsilon \implies \lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Exercise 11

Determine as α varies the limit in $(\mathbb{R}^2, d_2)$ of the sequence:

$$x_n = (\frac{1}{n}, (-1)^n \frac{n^\alpha - 1}{n^2}.$$

If the limit doesn't exists find eventual convergent subsequences.

Solution

A sequence in $\mathbb{R}^2$ is like having two sequences in $\mathbb{R}$:

$$a_n = \frac{1}{n} \to 0$$

$$b_n = (-1)^n \frac{n^\alpha - 1}{n^2} = (-1)^n n^{\alpha-2} + \frac{(-1)^n}{n^2} \to 0.$$

- If $\alpha - 2 < 0$ then $b_n \to 0$;
- if $\alpha = 2$ then $b_n \to \nexists$;
- if $\alpha - 2 > 0$ then $b_n \to \nexists$.

Then if $\alpha < 2$ then

$$\lim_{n\to\infty} x_n = (0,0)$$

if $\alpha \geq 2$

$$\lim_{n\to\infty} x_n = \nexists.$$

If we consider $\alpha > 2$ since $|b_n \to +\infty$ there aren't convergent subsequences. If $\alpha = 2$ we the subsequence of even indices

$$x_n = (\frac{1}{n}; \frac{n^2+1}{n^2}) \to (0,1),$$

the subsequence of odd indices:

$$x_n = (\frac{1}{n}; -\frac{n^2+1}{n^2} =\to (0,-1).$$

Exercise 12

Let (X, d) a metric space, $A \subseteq X$, $A \neq \emptyset$, x_n a sequence in A that converges to $x_\infty \in X$. Say if the following statements are false or true.

1. x_∞ is an accumulation point for A;
2. $x_\infty \in \overline{A}$;
3. $x_\infty \in \mathring{A}$;
4. $x_\infty \in \partial A$.

Solution

Point 1

The first statement is false. Counterexample:

$$A = [0, 1] \cup \{2\}$$

$$x_n = 2 \qquad \text{constant}$$

$$\lim_{n\to\infty} x_n = x_\infty = 2$$

that is not an accumulation point since 2 is an isolated point for A.

Point 2

If x_∞ is an isolated point the statement follows by the point 1, if x_∞ is not an isolated point it will be an accumulation point then the statement trivially follows. So the statement 2 is true.

Point 3

$x_\infty \in \mathring{A}$ is false. Counterexample:

$$A =]0, 1]$$

$$x_n = \frac{1}{n} \implies x_\infty \to 0 \notin \mathring{A}.$$

Point 4

$x_\infty \in \partial A$ is false. Counterexample:

$$A =]0, 1[$$

$$x_n = \frac{1}{2} + \frac{1}{n}$$

$$x_n \to \frac{1}{2}$$

that is not in ∂A.

Exercise 13

Show that the metric space $(C^0([0,1]), d_\infty)$ is not compact.

Solution

Consider

$$f_n(t) = t^n \qquad t \in [0,1]$$

suppose that there is a subsequence that converges to $f \in C^0([0,1])$:

$$f_{nk} \to f,$$

that is

$$d_\infty(f_{nk}, f) \to 0, \qquad \forall t \in [0,1].$$

$$|f_{nk}(t) - f_n(t)| \leq d_\infty(f_{nk}, f) \to 0.$$

If this holds then

$$f_{nk}(t) \to f_n(t) \qquad \forall t \in [0,1]$$

$$f_n(t) \to g(t) = \begin{cases} 0 & \text{if} t \in [0,1[\\ 1 & \text{if} t = 1 \end{cases}$$

If the sequence converge to $g(t)$ then also the subsequences tends to $g(t)$, but f_n can't have convergent subsequences because they must to converge to $g(t) \notin C^0([0,1])$ because $g(t)$ is not continuous. Then $C^0([0,1])$ is not compact.

Exercise 14

Consider the sequence

$$f_n(t) = \sqrt{\frac{1+n^2t^2}{n}} \qquad \text{for} t \in [-1,1].$$

Show that $f_n \to f$ with $f(t) = |t|$ with the distance d_∞. Furthermore deduce that the space $C^1([-1,1], d_\infty)$ is not complete.

Solution

$$|f_n(t) - f(t)| = |\frac{\sqrt{1+n^2t^2}}{n} - |t|| = |\frac{1}{n\sqrt{1+n^2t^2}+n|t|}| \leq \frac{1}{n}$$

Since the denominator is greater or equal to n we have that the fraction is lower or equal to $\frac{1}{n} \quad \forall t \in [-1,1]$. Then

$$0 \leq d_\infty(f_n, f) \leq \frac{1}{n} \to 0.$$

Then $f_n \to f$ is a Cauchy sequence, but $f \notin C^1([-1,1])$. Then the squence doesn't converge in $(C^1[-1,1], d_\infty)$ and so it is not complete.

Exercise 15

Let $X = C^0([0,1])$ with the distance d_2. Show that the sequence

$$f_n(t) = \begin{cases} 0 & \text{if} \quad t \in [0, \frac{1}{2} - \frac{1}{n}[\\ \sqrt{2nt + 2 - n} & \text{if} \quad t \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2} \\ \sqrt{-2nt + 2 + n} & \text{if} \quad t \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{n}[\\ 0 & \text{if} \quad t \in [\frac{1}{2} + \frac{1}{n}, 1] \end{cases}$$

1. converges to the null function in $x = 2$:
2. does not converge to the null function in the metric space (X, d_∞);
3. admits limit in (X, d_∞).

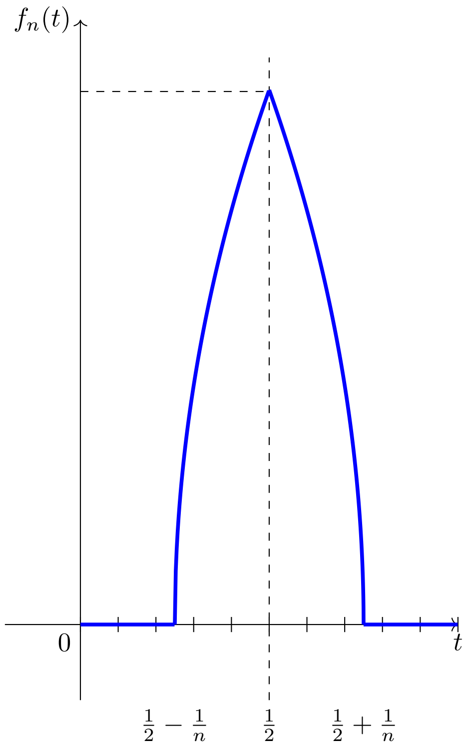
Solution

Point 1

We need to verify that $d_2(f_n, 0) \to 0$ where

$$d_2(f_n, 0) = \sqrt{\int_0^1 |f_n(t)|^2 dt}$$

We have that



$$(d_2(f_n, 0))^2 \leq \frac{2}{n} \to 0.$$

Point 2

$$d_\infty(f_n, 0) = \sup_{t \in [0,1]} |f_n(t)| = f_n(\frac{1}{2}) = \sqrt{2}.$$

It doesn't tend to zero then it doesn't tend to the null function with the distance d_∞. **Point** 3

We suppose that the sequence admits a limit and that

$$\exists g \in X \qquad \text{s.t.} \qquad d_\infty(f,g) \to 0.$$

Then

$$\forall t \in [0,1] \qquad 0 \leq |f_n(t) - g(t)| \leq d_\infty(f_n, g)$$

since the "due carabinieri" theorem

$$\forall t \in [0,1] \qquad f_n(t) \to g(t).$$

If $t \neq \frac{1}{2}$ we have that $f_n(t) \to 0 \implies g(t) = 0$.
If $t = \frac{1}{2}$ $\qquad f_n(t) = \sqrt{2} \implies g(\frac{1}{2}) = \sqrt{2} \implies g \notin X$. Then the limit with d_∞ doesn't exist.

Exercise 16

Let $f \in C^1(\mathbb{R}, \mathbb{R})$, 2π-periodic, such that its Fourier series is of the form

$$\sum_{n=3}^{\infty} \alpha_n \sin(nx)$$

. Let the Fourier series associated to f^3 of the form

$$\sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

Which of the following statements is certainly true?

1. $b_n = \alpha_n^3 \qquad \forall n;$
2. $a_n = 0 \qquad \forall n.$

Solution

Trivially we can see that f is an odd function, then f^3 is also an odd function, so that $a_n = 0 \qquad \forall n$ is certainly true. Then

$$\mathcal{F}_{f^3(x)} = \sum_{n=0}^{+\infty} b_n \sin(nx).$$

If we consider

$$\alpha_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f^3(x) \sin(nx) dx.$$

Generally is not true that $b_n = \alpha_n^3$. Counterexample:

$$f(x) = \sin(x)$$

$$\alpha_3 = 1 \qquad \text{and} \qquad \alpha_n = 0 \qquad \forall n \neq 3.$$

All the terms such as $\sin(4x), \sin(5x), \cdots \sin(1000x)$ have null coefficients.

$$f^3(x) = \sin^3(3x)$$

$$\alpha_3 = 1$$

$$b_3 = 1^3??$$

$$\begin{aligned} b_3 = & \int_{-\pi}^{\pi} f^3(x) \sin(3x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^4(3x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2(3x)(1 - \cos^2(3x) dx \\ & = \frac{1}{\pi} \sin^2(3x) dx - \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2(3x) \cos^2(3x) dx = \star \end{aligned}$$

using the duplication and bisection formulas

$$\sin^2(3x) = \frac{1 - \cos(6x)}{2}$$

$$\sin^2(3x)\cos^2(3x) = \frac{\sin^2(6x)}{4} = \frac{1 - \cos(12x)}{8}$$

$$\star = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1 - \cos(6x)}{2} dx - \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1 - \cos(12x)}{8} dx$$

$$= \frac{1}{2\pi} [x]_{-\pi}^{\pi} - \frac{1}{12\pi} [\sin(nx)]_{-\pi}^{\pi} - \frac{1}{8\pi} [x]_{-\pi}^{\pi} + \frac{1}{96\pi} [\sin(12x)]_{-\pi}^{\pi} = \frac{3}{4}.$$

Then

$$\alpha_3 = 1 \qquad b_3 = \frac{3}{4} \neq 1^3.$$

Exercise 17

Let $f \in C^1(\mathbb{R}^2, \mathbb{R})^2$ such that $f(3,6) = (0,0)$. In a neighborhood of $(3,1)$, which of the following statements are certainly true?

1. $\exists \alpha \in \mathbb{R}$ such that $x \mapsto x + \alpha f(x)$ satisfies the hypotheses of the Implicit Function Theorem.
2. $\forall \alpha \in \mathbb{R} \qquad x \mapsto x + \alpha f(x)$ satisfies the hypotheses of the Inverse Function Theorem.

Solution

Point 1

We need to show that the derivative of f in $(3,1)$ is an invertible matrix. Counterexample: let $\alpha = 0$ and consider a function $g(x) = x \mapsto x$. Let

$$g(x_1, x_2) = (x_1, x_2)$$

$$Dg(x_1, x_2) = \begin{bmatrix} 1 & 0 \\ 0 & \end{bmatrix}$$

The determinant $\det Dg = 1 \neq 0$. Then the gradient matrix is invertible and since it is constant it is also invertible in $(3,1)$. So the first statement is true.

Point 2

Counterexample. We need to find a function f with two variables such that $Df = 0$, starting from a function g not invertible. Consider

$$g(x_1, x_2) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

with $\alpha = 1$, we have

$$Dg(x_1, x_2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The gradient matrix Dg is not invertible and it doesn't satisfy the Inverse Function Theorem hypotheses.

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} x + f_1(x) \\ y + f_2(y) \end{bmatrix}$$

$$f_1(x, y) = 3 - x$$

$$f_2(x, y) = 1 - y$$

$$f(3,1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then the second statement is false.

Exercise 18

Let $f \in C^2(\mathbb{R}^2, \mathbb{R})$ such that $f(1,2) = 0$ and consider a neighborhood of $(1,2)$. Which of the following statements are certainly true?

1. $\forall \alpha \in \mathbb{R} \qquad x + \alpha f(x,y) + y - 3 = 0$ satisfies the Implicit Function Theorem hypotheses;
2. $\exists \alpha \in \mathbb{R}$ such that $x + \alpha f(x,y) + y - 3 = 0$ satisfies the Implicit Function Theorem hypotheses.

Solution

$$g(x,y) = x + \alpha f(x,y) + y - 3$$
$$g(1,2) = 0$$
$$f \in C^2 \implies g \in C^2$$

We need to verify if

$$\frac{g}{y}(1,2) \neq 0?$$
$$\frac{g}{x}(1,2) \neq 0?$$

Point 2

We take $\alpha = 0$ so that $g(x,y) = x + y - 3$. Then

$$\frac{\partial g}{\partial y}(1,2) = 1 \neq 0.$$

Then the first statement is true.

Point 1

We need a counterexample, starting from a function $g(x,y) = x^2 + y^2$ we can consider:

$$g(x,y) = (x-1)^2 + (y-2)^2$$

so that

$$\nabla g(1,2) = [0,0].$$

We have

$$x + \alpha f(x,y) + y - 3 = (x-1)^2 + (y-2)^2.$$

We take $\alpha = 1$.

$$x + f(x,y) + y - 3 = (x-1)^2 + (y-2)^2$$
$$f(x,y) = (x-1)^2 + (y-2)^2 - x - y + 3$$
$$f(1,2) = (0,0)$$

and $f \in C^2$, but

$$\nabla f(1,2) = [0,0]$$

so that f is not invertible. Then the second statement is not true.

Exercise 19

Let $f_\alpha : \mathbb{R}^2 \to \mathbb{R}$ given by:

$$f_\alpha(x,y) = \begin{cases} \frac{|x|^\alpha y^2}{\sqrt{4x^2+3y^2}} & \text{if} \quad x \neq 0 \\ 0 \quad \text{if} \quad x = 0 \end{cases}$$

Studying with respect to $\alpha \in \mathbb{R}$ the differentiability and continuity of f_α.

Solution

First: continuity

We have problems in all the points of the y-axis. First we consider the points $(0,b)$ with $b \neq 0$ that is a generic point of the y axis.

$$\lim_{(x,y)\to(0,b)} \frac{|x|^\alpha y^2}{\sqrt{4x^2+3y^2}} = \frac{b^2}{\sqrt{3b^2}} \lim_{(x,y)\to(0,b)} |x|^\alpha = \begin{cases} +\infty & \text{if} \quad \alpha < 0 \\ \frac{b^2}{\sqrt{3b^2}} & \text{if} \quad \alpha = 0 \\ 0 \quad \text{if} \quad \alpha > 0. \end{cases}$$

Then f is continous in $(0,b)$ with $b \neq 0 \qquad \forall \alpha > 0$. Now we consider the point $(0,0)$.

$$\lim_{(x,y)\to(0,0)} \frac{|x|^\alpha y^2}{\sqrt{4x^2+3y^2}}$$

in polar coordinates

$$\lim_{\rho\to 0} \frac{\rho^\alpha |\cos\theta|^\alpha \rho^2(\sin^2\theta)}{\rho\sqrt{4\cos^2\theta+3\sin^2\theta}}$$

making some additions

$$|\rho^{\alpha+1}\frac{|\cos\theta|^\alpha(\sin^2\theta)}{\sqrt{4\cos^2\theta+3\sin^2\theta}}| \leq \rho^{\alpha+1}\frac{|\cos\theta|^\alpha|\sin^2\theta|}{\sqrt{3}|\sin\theta|} = \frac{\rho^{\alpha+1}}{\sqrt{3}}|\cos\theta|^\alpha|\sin\theta| \leq \frac{\rho^{\alpha+1}}{\sqrt{3}}$$

uniformly in θ.

If $\alpha > 0$ then $L = 0$ uniformly in θ.

If $\alpha \leq -1$ then $L = \nexists$.

Remains the case $-1 < \alpha < 0$.

$$\lim_{\rho\to 0} \frac{\rho^{\alpha+1}|\cos\theta|^\alpha \sin^2\theta}{\sqrt{4\cos^2\theta+3\sin^2\theta}} = 0$$

this limit is equivalent to

$$\lim_{\rho\to 0} \rho^{\alpha+1}h(z) = 0 \qquad \forall \theta \neq \frac{\pi}{2} \quad \frac{3\pi}{2}$$

but the limit is not uniform since it is not possible to increase the function. Then for $-1 < \alpha < 0$ the limit goes to $+\infty$. The local boundedness theorem is violated.

Then f is continuous for $\alpha \geq 0$, f is continuous in $(0,b)$ for $\alpha \geq 0$.

Second: differentiability

We consider the points $(0,b)$ with $b \neq 0$ and with $\alpha > 0$.

$$\frac{\partial f}{\partial x}(0,b) = \lim_{t\to 0}\frac{f(t,b)+f(0,b)}{t} = \lim_{t\to 0}\frac{|t|^\alpha b^2}{\sqrt{4t^2+3b^2}}\frac{1}{t} = \begin{cases} \nexists & \text{if} \quad \alpha < 1 \\ \nexists & \text{if} \quad \alpha = 1 \\ 0 & \text{if} \quad \alpha > 1 \end{cases}$$

$$\frac{\partial f}{\partial y}(0,b) = \lim_{t\to 0}\frac{f(0,b+t)-f(0,b)}{t} = \lim_{t\to 0}\frac{0}{t} = 0.$$

Now we consider the case $\alpha > 1$ and show if f is differentiable. For $\alpha > 1$, $\nabla f(0,0) = [0,0]$.

$$\lim_{(h,k)\to(0,0)} \frac{f(h,b+k)-f(0,b)-\nabla f(0,b)\begin{bmatrix} h \\ k \end{bmatrix}}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{|h|^\alpha(b+k)^2}{\sqrt{4h^2+3(b+k)^2}\sqrt{h^2+k^2}}$$

$$= \lim_{\rho\to 0} \frac{\rho^{\alpha-1}|\cos\theta|^\alpha (b+\rho\sin\theta)^2}{\sqrt{4\rho^2\cos^2\theta + 3(b+\rho\sin\theta)^2}} = 0 \qquad \forall\theta.$$

Now we show if it is uniformly.

$$|\frac{\rho^{\alpha-1}|\cos\theta|^\alpha (b+\rho\sin\theta)^2}{\sqrt{4\rho^2\cos^2\theta + 3(b+\rho\sin\theta)^2}}| \le \rho^{\alpha-1}\frac{|b+\rho\sin\theta|}{\sqrt{3}} \le \rho^{\alpha-1}\frac{|b|+\rho}{\sqrt{3}}.$$

The right term doesn't depend on *theta* so the limit is uniform in θ. If $\alpha > 1$ f is differentiable in $(0,b)$ with $b \ne 0$. Now we consider the case $\alpha \ge 0$ in $(0,0)$.

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0}\frac{0}{t} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0}\frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0}\frac{0}{t} = 0$$

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - \nabla f(0,0)\begin{bmatrix} h \\ k \end{bmatrix}}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{f(h,k)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{|h|^\alpha k^2}{\sqrt{4h^2+3k^2}\sqrt{h^2+k^2}}$$

if $\alpha = 0$ the limit is

$$\lim_{(h,k)\to(0,0)} \frac{k^2}{\sqrt{4h^2+3k^2}\sqrt{h^2+k^2}} = \nexists.$$

If $\alpha > 0$ in polar coordinates:

$$\lim_{\rho\to 0} \frac{\rho^\alpha |\cos\theta|^\alpha \rho^2(\sin^2\theta)}{\rho\sqrt{4\cos^2\theta + 3\sin^2\theta}} = 0 \qquad \forall\theta.$$

Now we show if it is uniform in θ.

$$|\frac{\rho^\alpha|\cos\theta|^\alpha\sin^2\theta}{\sqrt{4\cos^2\theta+3\sin^2\theta}}| \le \rho^\alpha\frac{|\cos\theta|^\alpha\sin^2\theta}{\sqrt{3\sin^2\theta}} = \frac{\rho^\alpha|\cos\theta|^\alpha|\sin^2\theta|}{\sqrt{3}\sin\theta} = \rho^2\frac{|\cos\theta|^\alpha|\sin\theta|}{\sqrt{3}} \le \frac{\rho^2}{\sqrt{3}}.$$

It is uniform in θ. Then $L = 0$ uniformly in θ.

Then f is differentiable in the origin for $\alpha > 0$.

Exercise 20

Let $A \subseteq \mathbb{R}^n$ open and $f : A \to \mathbb{R}^m$ differentiable on A. Show that:

1. if A is convex and $\|Df(x)\| \leq L \qquad \forall x in A$ then f is Lip;
2. if f is Lip with constant L then $\|Df(x)\| \leq L \qquad \forall x \in A$.

Solution

Point 1
We apply the Finite Accretion Theorem. If A is convex, then

$$\forall x'', x' \in A \qquad \|f(x'') - f(x')\| \leq \sup_{\xi \in A} \|Df(\xi)\| \, \|x'' - x'\| \leq \star$$

since $\|Df(x)\| \leq L$ then also the sup $\leq L$, then

$$\star \leq L \, \|x'' - x'\|$$

then f is Lip.
Point 2
Let $x \in A$, $v \in \mathbb{R}^m$, $t \in \mathbb{R}$.

$$\|f(x + tv) - f(x)\| \leq L \, \|tv\| = L|t| \, \|v\|$$

$$\frac{\|f(x + tv) - f(x)\|}{t} \leq L \, \|v\| \, .$$

Since f is differentiable and derivable, the limit exists, so passing to the limit for $t \to 0$ we obtain:

$$\|Df(x) \cdot v\| \leq L \, \|v\| \, .$$

Now we take all the vectors for norm equal to 1:

$$w \in \mathbb{R}^n \qquad \|w\| = 1$$

then

$$\|Df(x) \cdot w\| \leq L,$$

if it is valid for all the previous vectors we obtain

$$\sup_{w \in \mathbb{R}^n, \quad \|w\|=1} \|Df(x)w\| \leq L$$

then

$$\|Df(x)\| \leq L.$$

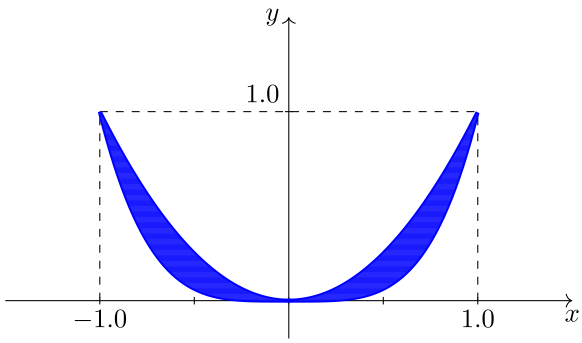
Exercise 21

Compute the value of the double integral

$$\int\int_T (x^3 e^{y^2} + x^2 y)dxdy$$

over the region T delimited by the curves $y = x^2$ and $y = x^4$.

Solution



The region T is sysmmetric with respect to the y-axis. Furthermore

$$\int\int_T x^3 e^{y^2} dxdy = 0$$

since $f(x, y) = x^3 y^2$ is an odd function with respect to the x-axis.

$$\int\int_T x^2 ydxdy$$

is even. Then

$$2\int_0^1 x^2 \int_{x^4}^{x^2} ydydx = 2\int_0^1 x^2 [\frac{y^2}{2}]_{x^4}^{x^2} dx = \int_0^1 x^2(x^4 - x^8)dx = [\frac{x^7}{7} - \frac{x^1 1}{11}]_0^2 = \frac{1}{7} - \frac{1}{11}.$$

Exercise 22

Let $f_n : \mathbb{R} \to \mathbb{R}$ given by:

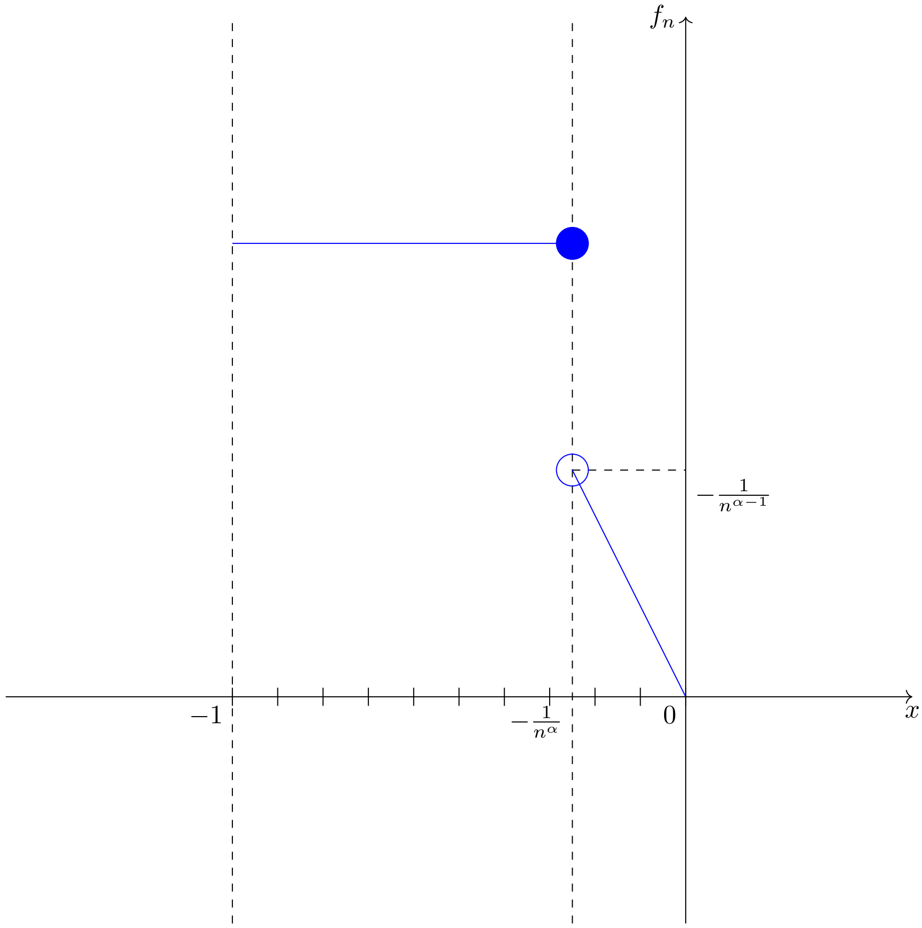
$$f_n(x) = \begin{cases} 1 & -1 \le x \le -\frac{1}{n^\alpha} \\ -nx & -\frac{1}{n^\alpha} < x \le 0 \end{cases}$$

with $\alpha > 0$. Study the pointwise and uniform convergences. Then show the result of the integrals

$$\int_{-1}^{0} f(x)dx \qquad \text{and} \qquad \int_{-1}^{0} f_n(x)dx.$$

Solution

First of all we can plot the graph of the function $f_n(x)$. We have that



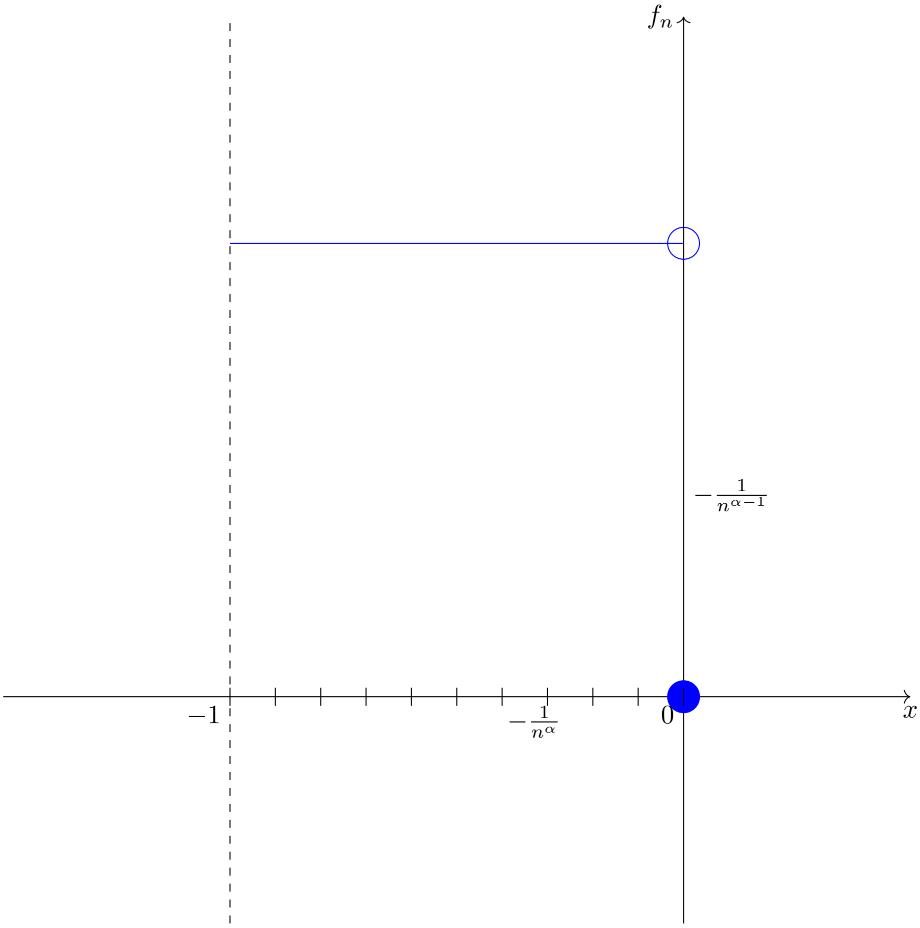
- if $\alpha - 1 > 0$ the functions grow up;
- if $\alpha - 1 >< 0$ the functions go down;
- if $\alpha - 1 = 0$ the functions are continuous;

Now we can consider the pointwise convergence.

If $x = 0$

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} f_n(0) = 0 \qquad \forall n \in \mathbb{N}$$

We now can make the following consideration: if $x < 0$ the interval $]-\frac{1}{n^\alpha};0[$ narrows for n that becomes greater and the variable x is always outside, so that the function x does one everywhere except in zero. The pointwise limit is: Now we consider the uniform convergence: what is the maximum distance between



the pointwise limit and the function?

$$\sup_{x\in[-1;0]} |f_n(x) - f(x)|$$

If we were to imagine putting the pointwise limit and the function in the same graph, we would see that the distance between these two functions would be equal to one even though it is never taken. Then

$$\lim_{n\to+\infty} \sup_{x\in[-1;0]} |f_n(x) - f(x)| = 1$$

and the convergence is not uniform, for no α.

Now we compute the integrals.

$$\int_{-1}^{0} f(x)dx = \int_{-1}^{0} 1dx = 1$$

and

$$\int_{-1}^{0} f_n(x)dx = 1 - \frac{1}{n^\alpha}\cdot 1 + \frac{1}{n^\alpha}\cdot\frac{1}{n^{\alpha-1}}\cdot\frac{1}{2} = 1 - \frac{1}{n^\alpha} + \frac{1}{n^{2\alpha-1}}\cdot\frac{1}{2}$$

$$\lim_{n\to+\infty} 1 - \frac{1}{n^\alpha} + \frac{1}{n^{2\alpha-1}}\cdot\frac{1}{2} =$$

- if $2\alpha - 1 > 0$, $l = 1$;
- if $2\alpha - 1 = 0$, $l = \frac{3}{2}$;
- if $2\alpha - 1 < 0$, $l = \nexists$.

Only in the first case does the passage of the limit under the sign of the integral apply, for

$$\alpha > \frac{1}{2}.$$

Exercise 23

How much is the distance between $f, g \in C^0(D;\mathbb{R})$, with D the closed circle of center $(-1,0)$ and radius 1 and $f(x,y) = 2x+2$, $g(x,y) = 2y$?

Solution

The norm of the space of continuous functions C^0 is:

$$d_\infty(f,g) = \sup_{x\in D} |g(x) - f(x)| = \max_{x\in D} |g(x) - f(x)|,$$

since f, g are continuous, then the function $|f-g|$ is continuous and applying the Weierstrass Theorem we have that $\sup = \max$.

Now we define

$$g(x) - f(x) = F(x)$$

as

$$F(x) = |2x - 2y + 2|$$

and we need to find the maximum of this function over a compact subset of $\mathbb{R}^2$. The problem is the modulus. We can divide the problem into two subproblems:

$$F(x)|_{D_1} = 2x - 2y + 2$$

$$F(x)|_{D_2} = 2y - 2x - 2.$$

On D_1 we have

$$F(x) = 2(x - y + 1)$$

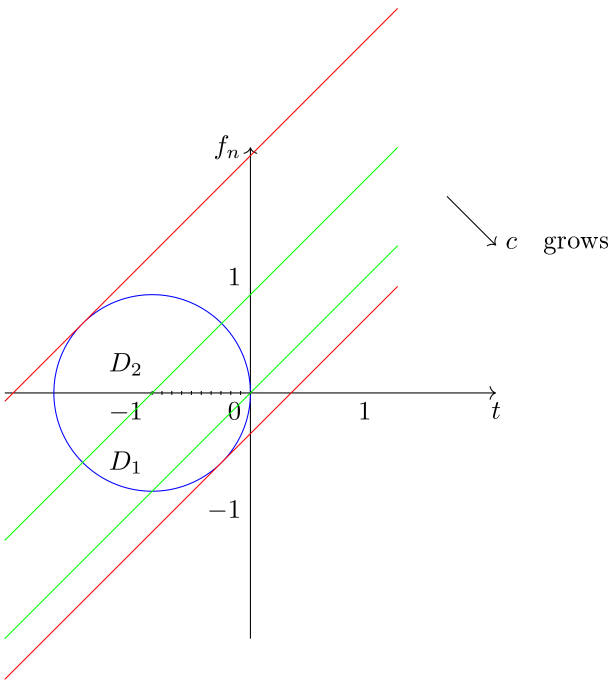
and on D_2 we have

$$F(x) = 2(y - x - 1)$$

so that $F(x)$ is constant along the curves

$$\gamma_c : x - y = c,$$

that are straight lines. For D_1 the maximum point will be the tangent point with the circonference and



for D_2 the minimum point will be the same. So to determine it we need to solve the following system.

$$\begin{cases} (x+1)^2 + y^2 = 1 \\ y = x - c \end{cases}$$

by substitution we have

$$(x+1)^2 + (x-c)^2 = 1$$

$$x^2 + 2x + 1 + x^2 - 2cx + c^2 = 1$$

we have a second order equation

$$2x^2 + 2x(1-c) + c^2 = 0$$

we can now apply the tangent condition $\Delta = 0$,

$$4(1-c)^2 - 8c^2 = 0$$

$$1 - 2c + c^2 - 2c^2 = 0$$

$$c^2 + 2c - 1 = 0$$

then

$$c_1 = -1 - \sqrt{2} \qquad c_2 = -1 + \sqrt{2}.$$

Then by substituting we have that the maximum is

$$\max_D f = 2\sqrt{2}$$

and the minimum is

$$\min_D f = -2\sqrt{2}.$$