

A collection of Functional Analysis exercises

June 15, 2024

Chapter 1

Functional Spaces

Exercise 1

Compute

$$\lim_{n \rightarrow +\infty} \int_1^{\infty} f_n(x) dx$$

where

$$f_n(x) = \frac{\sin(nx)}{x^3} e^{-n\sqrt{x}}$$

for all $x \geq 1$ and for all $n \in \mathbb{N}$.

Solution

This exercise is trivial using the Dominated Convergence Theorem.

First we calculate the **pointwise convergence**.

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{\sin(nx)}{x^3} e^{-n\sqrt{x}} = 0$$

for all $x \geq 1$ since $\lim_{n \rightarrow \infty} \frac{\sin(nx)}{x^3} = 0$ and $e^{-n\sqrt{x}}$ is bounded.

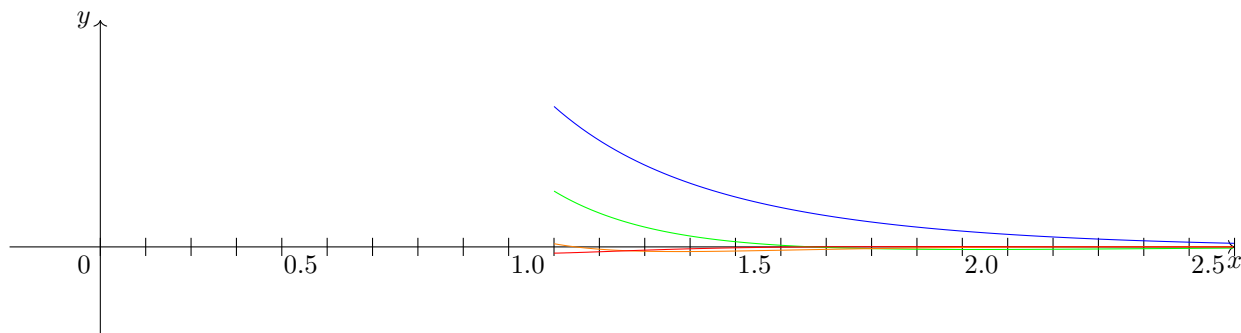


Figure 1.1: The sequence of functions $f_n(x)$.

$f(x) = 0 \quad \forall x \geq 1$ is the punctual limit. Now we search a dominant function.

$$|f_n(x)| = \left| \frac{\sin(nx)}{x^3} e^{-n\sqrt{x}} \right| \leq \star,$$

since the sine and $e^{-n\sqrt{x}}$ are bounded functions:

$$-1 \leq \sin(nx) \leq 1 \quad \forall n \in \mathbb{N} \quad \forall x \in \mathbb{R}$$

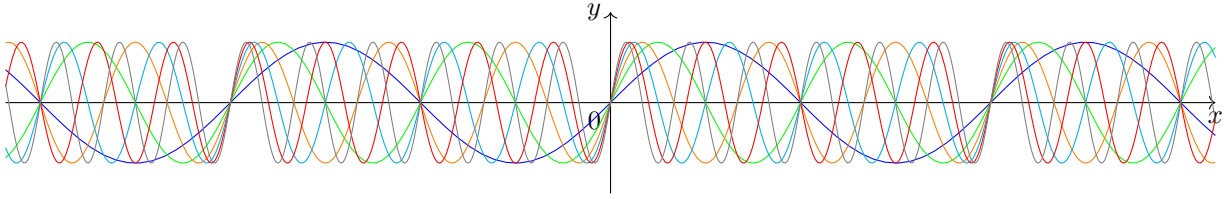
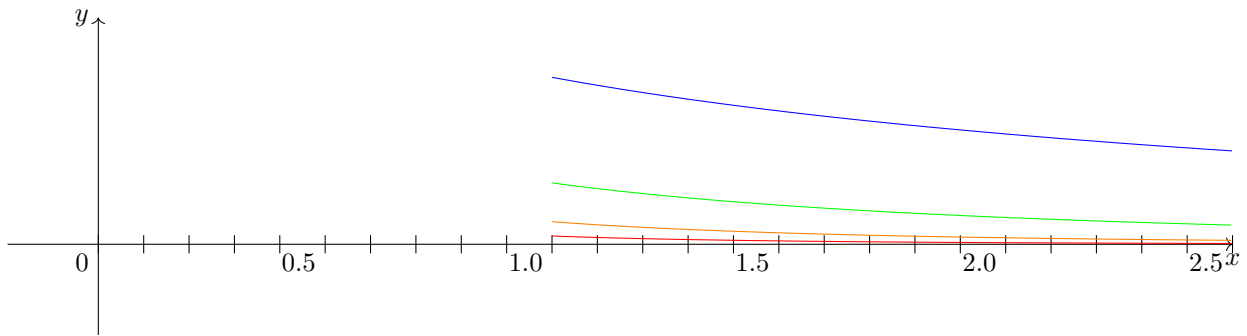


Figure 1.2: The sine function.

$$e^{-n\sqrt{x}} \leq 1 \quad \forall n \in \mathbb{N} \quad x \geq 1$$

Figure 1.3: The sequence of functions $e^{-n\sqrt{x}}$.

$$\star \leq \left| \frac{1}{x^3} \right| = \frac{1}{x^3} = g(x) \quad \forall n \in \mathbb{N}$$

since $x \in [0, +\infty)$. Now we need to verify if $g \in L^1([0, +\infty))$.

$$\int_1^{+\infty} |g(x)| dx = \int_1^{+\infty} \left| \frac{1}{x^3} \right| dx = \int_1^{+\infty} \frac{1}{x^3} dx < +\infty$$

since the summability criteria:

$$\int_1^{+\infty} \frac{1}{x^\alpha} dx = \begin{cases} \frac{1}{\alpha-1} & \text{if } \alpha > 1 \\ +\infty & \text{if } \alpha \leq 1 \end{cases}$$

Now we can apply the Dominated Convergence Theorem (or Lebesgue Theorem):

$$\lim_{n \rightarrow +\infty} \int_1^{+\infty} f_n(x) dx = \int_1^{+\infty} \lim_{n \rightarrow +\infty} f_n(x) dx = \int_1^{+\infty} \lim_{n \rightarrow +\infty} \frac{\sin(nx)}{x^3} e^{-n\sqrt{x}} dx = \int_1^{+\infty} 0 dx = 0.$$

Then the solution is

$$\lim_{n \rightarrow +\infty} \int_1^{+\infty} f_n(x) dx = \lim_{n \rightarrow +\infty} \int_1^{+\infty} \frac{\sin(nx)}{x^3} e^{-n\sqrt{x}} dx = 0 \quad \forall x \leq 1.$$

Exercise 2

Compute

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx$$

where

$$f_n(x) = \frac{x}{1+x^{2n}} \quad \text{with} \quad x \in (0, 1).$$

Solution

We need to apply the Dominated Converge Theorem.

First of all we analyze the **pointwise convergence**.

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{x}{1+x^{2n}},$$

if $x \in [0, 1)$ we have

$$\lim_{n \rightarrow \infty} \frac{x}{1+x^{2n}} = x$$

since $x^{2n} \rightarrow 0$ for $x \in [0, 1)$.

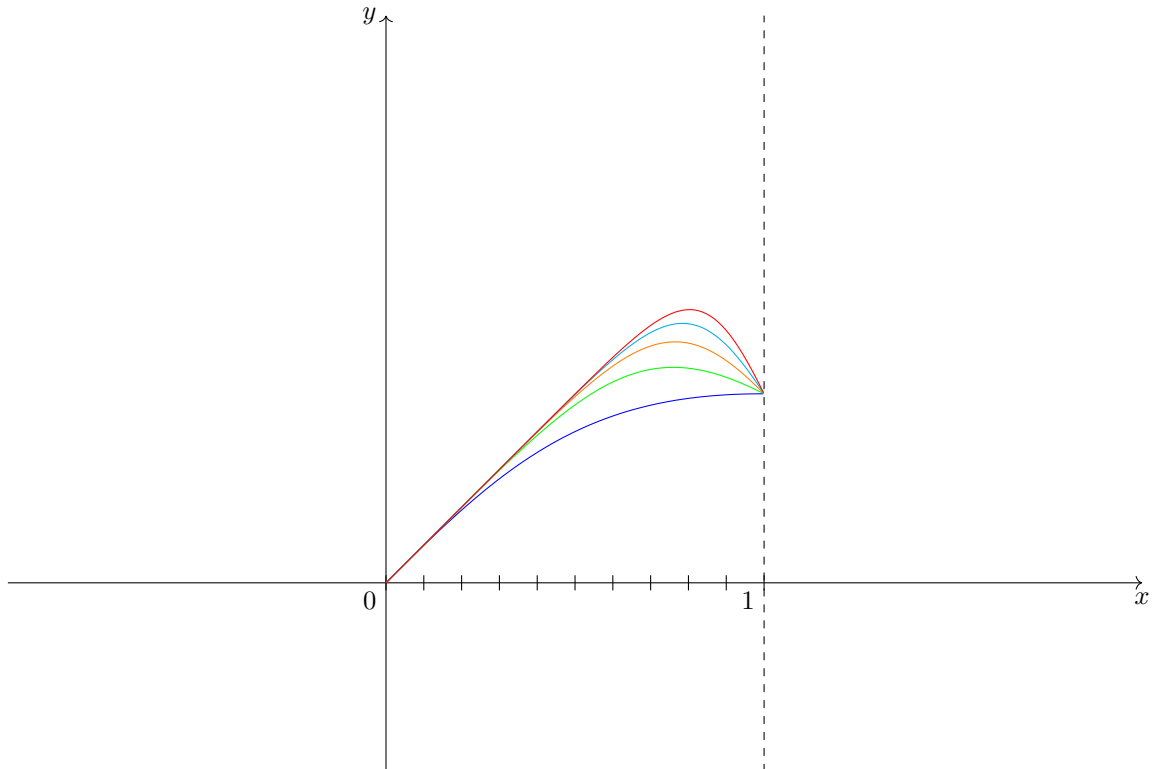
If $x = 1$,

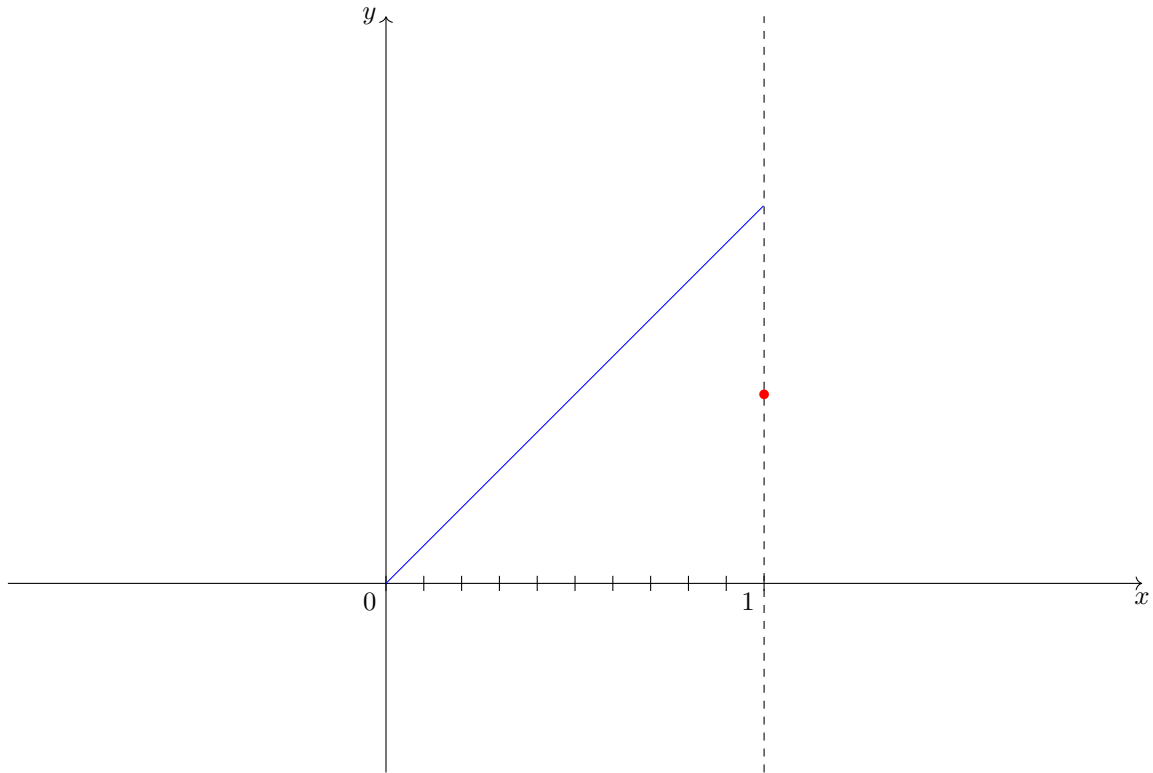
$$\lim_{n \rightarrow \infty} \frac{x}{1+x^{2n}} = \lim_{n \rightarrow \infty} \frac{1}{1+1^{2n}} = \frac{1}{2}.$$

The pointwise limit is

$$\lim_{n \rightarrow \infty} f_n(x) = \begin{cases} x & \text{if } x \in [0, 1) \\ \frac{x}{2} & \text{if } x = 1 \end{cases}$$

Now we find the dominant function.





$$\exists g \in L^1(0, 1) \quad \text{s. t.} \quad |f_n(x)| \leq g?$$

$$|f_n(x)| = \left| \frac{x}{1+x^{2n}} \right| \leq \frac{x}{1+x^{2n}} \leq x = g(x) \quad \forall n \in \mathbb{N}, \quad \forall x \in (0, 1).$$

$$\int_0^1 |g(x)| dx = \int_0^1 |x| dx = \int_0^1 x dx = \left[\frac{x^2}{2} \right]_0^1 = \frac{1}{2} < +\infty$$

so that

$$g \in L^1(0, 1).$$

We can now apply the Dominated Convergence Theorem.

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{x}{1+x^{2n}} dx = \int_0^1 \lim_{n \rightarrow \infty} \frac{x}{1+x^{2n}} dx = \int_0^1 x dx = \frac{1}{2}.$$

The solution is

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{x}{1+x^{2n}} dx = \frac{1}{2}.$$

Exercise 3

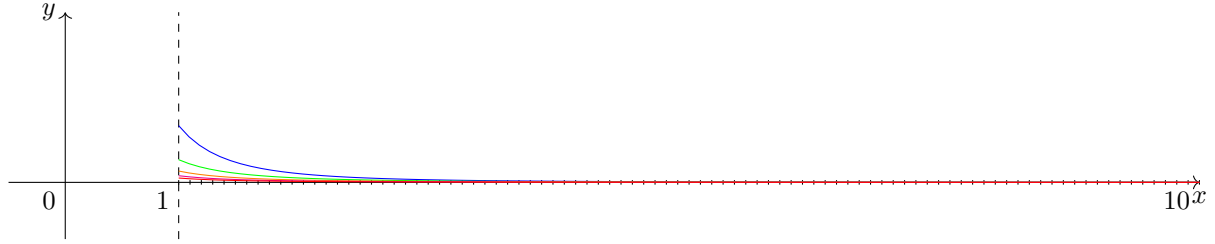
Studying convergence in $L^1([1, +\infty))$ of

$$f_n(x) = \frac{\cos(nx)}{n^2 + x} \frac{1}{x^2} \quad \forall x \geq 1 \quad \forall n \in \mathbb{N}.$$

Solution

Convergence in $L^1([0, +\infty))$:

$$\|f_n - f\|_{L^1([1, +\infty))} = \int_1^{+\infty} |f_n - f| dx \rightarrow 0$$



POINTWISE CONVERGENCE

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{\cos(nx)}{n^2 + x} \frac{1}{x^2} = \star$$

$$-1 \leq \cos(nx) \leq 1$$

$$\star = 0,$$

so the pointwise limit is

$$f(x) = 0 \quad \forall x \geq 1.$$

Now we need to find a dominant function:

$$\exists g \in L^1([1; +\infty)) \quad \text{s.t.} \quad |f_n(x)| \leq g \quad \forall x \geq 1 \quad \forall n \in \mathbb{N}$$

$$|f_n(x)| = \left| \frac{\cos(nx)}{n^2 + x} \frac{1}{x^2} \right| \leq \frac{1}{n^2 + x} \frac{1}{x^2} \leq \frac{1}{x} \frac{1}{x^2} = \frac{1}{x^3} = g(x) \quad \forall x \geq 1 \quad \forall n \in \mathbb{N}$$

$$\int_1^{+\infty} |g(x)| dx = \int_1^{+\infty} \left| \frac{1}{x^3} \right| dx = \int_1^{+\infty} \frac{1}{x^3} dx < +\infty$$

summability criteria:

$$\int_1^{+\infty} \frac{1}{x^\alpha} dx = \begin{cases} \frac{1}{\alpha-1} & \text{if } \alpha > 1 \\ +\infty & \text{if } \alpha \leq 1, \end{cases}$$

then

$$g \in L^1([1; +\infty)).$$

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{L^1([1; +\infty))} \rightarrow 0$$

$$\lim_{n \rightarrow \infty} \int |f_n - f| dx \rightarrow 0$$

iff

$$\lim_{n \rightarrow \infty} \int_1^{+\infty} |f_n| dx \rightarrow 0.$$

Now we can apply the Dominated Convergence Theorem:

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_1^{+\infty} |f_n| dx &= \int_1^{+\infty} \lim_{n \rightarrow +\infty} |f_n| dx = \int_1^{+\infty} \lim_{n \rightarrow +\infty} \left| \frac{\cos(nx)}{n^2 + x} \frac{1}{x^2} \right| dx \\ &= \int_1^{+\infty} \lim_{n \rightarrow +\infty} \left(\frac{\cos(nx)}{n^2 + x} \frac{1}{x^2} \right) dx = \int_1^{+\infty} 0 dx = 0, \end{aligned}$$

so that

$$f_n \rightarrow 0 \quad \text{in} \quad L^1([1; +\infty)).$$

Exercise 4

Let $f_n(x) = \sum_{n=1}^{\infty} \frac{|\sin(nx)|}{2^n}$ $x \in [0, \pi]$. Compute

$$\int_0^{\pi} f(x) dx.$$

Solution

Remember that a series is the limit of the partial sums. If the partial sums are formed by positive terms, then the series are monotone. We consider

$$f_h(x) = \sum_{n=1}^h \frac{|\sin(nx)|}{2^n}.$$

We have truncated the series up to the h term. If we consider the truncated series we have that $f_h(x)$ is monotone, in fact

$$0 \leq f_h \leq f_{h+1}.$$

Furthermore

$$f_h(x) = \sum_{n=1}^h \frac{|\sin(nx)|}{2^n} \rightarrow f(x) \quad \text{for} \quad h \rightarrow \infty.$$

Then we can apply the Beppo Levi's Theorem:

$$\int_0^{\pi} f(x) dx = \int_0^{\pi} \lim_{h \rightarrow \infty} f_h(x) dx = \lim_{h \rightarrow \infty} \int_0^{\pi} f_h(x) dx = \lim_{h \rightarrow \infty} \int_0^{\pi} \sum_{n=1}^h \frac{|\sin(nx)|}{2^n} dx = \lim_{h \rightarrow \infty} \sum_{n=1}^h \frac{1}{2^n} \int_0^{\pi} |\sin(nx)| dx.$$

Now we have to compute the integral. We know that

$$\int_0^{\infty} |\sin(y)| dy = n \int_0^{\pi} \sin(y) dy$$

so that

$$\int_0^{\pi} |\sin(nx)| dx = \int_0^{n\pi} |\sin y| \frac{dy}{n} = \int_0^{\pi} \sin y dy = [-\cos y]_0^{\pi} = 2.$$

Then

$$\int_0^{\pi} f(x) dx = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k 2 = \sum_{k=1}^{\infty} \frac{2}{2^k} = 2.$$

Then

$$\int_0^{\pi} f(x) dx = 2.$$

Exercise 5

Compute

$$\int_0^{+\infty} \frac{1}{\sqrt{x}} \sum_{k=0}^{\infty} e^{-(\alpha^k)\sqrt{x}} dx$$

as $\alpha \geq 0$ varies.

Solution

Consider

$$\sum_{k=0}^{\infty} e^{l(\alpha)^k \sqrt{x}},$$

it is a series with positive terms, then it converges or positively diverges, that is well defined. If we consider its truncated

$$\sum_{k=0}^n e^{-(\alpha^k)\sqrt{x}}$$

we can construct the functions

$$f_{\alpha,n}(x) = \sum_{k=0}^n e^{-(\alpha^k)\sqrt{x}}.$$

This is a sequence of functions $f_n \geq 0$ with

$$0 \leq f_n \leq f_{n+1} \quad \forall n \in \mathbb{N},$$

then it is monotone. We can use the Beppo Levi Theorem:

$$\int_0^{+\infty} \frac{1}{\sqrt{x}} \sum_{k=0}^{+\infty} e^{-(\alpha^k)\sqrt{x}} dx = \int_0^{+\infty} \frac{1}{\sqrt{x}} \lim_{n \rightarrow \infty} \sum_{k=0}^n e^{-(\alpha^k)\sqrt{x}} dx = \star$$

now we can apply Beppo Levi Theorem

$$\star = \lim_{n \rightarrow \infty} \sum_{k=0}^n \int_0^{+\infty} \frac{e^{-(\alpha^k)\sqrt{x}}}{\sqrt{x}} dx = \star$$

now we can make a change of variables,

$$y = \sqrt{x}$$

$$dy = \frac{1}{2} \frac{1}{y} dx$$

$$dx = 2y dy$$

$$\star = \lim_{n \rightarrow \infty} \sum_{k=0}^n \int_0^{\infty} \frac{e^{-(\alpha^k)y}}{y} 2y dy = \lim_{n \rightarrow \infty} \sum_{k=0}^n 2 \int_0^{\infty} e^{-(\alpha^k)y} dy$$

Now we have that

$$[e^{-(\alpha^k)y}]' = -(\alpha^k)e^{-(\alpha^k)y}$$

then

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=0}^n 2 \frac{1}{(-\alpha^k)} \int_0^{\infty} -(\alpha^k)e^{-(\alpha^k)y} dy &= \lim_{n \rightarrow \infty} \sum_{k=0}^n 2 \frac{1}{(-\alpha^k)} \int_0^{\infty} [e^{-(\alpha^k)y}]' dy \\ &= \sum_{k=0}^{+\infty} \frac{2}{-\alpha^k} (\lim_{c \rightarrow \infty} [e^{-(\alpha^k)y}]_0^c) = \sum_{k=0}^{\infty} \frac{2}{\alpha^k} = 2 \sum_{k=0}^{\infty} \left(\frac{1}{\alpha}\right)^k = \star \end{aligned}$$

this is a geometric series, then

$$\star = 2 \begin{cases} +\infty & \text{if } \frac{1}{\alpha} \geq 1 \\ \frac{1}{1-\frac{1}{\alpha}} & \text{if } \left|\frac{1}{\alpha}\right| < 1 \\ \text{indet.} & \text{if } \frac{1}{\alpha} \leq -1. \end{cases}$$

But we have that $\alpha \geq 0$, then

$$\int_0^\infty \frac{1}{\sqrt{x}} \sum_{k=0}^\infty e^{-(\alpha^k)\sqrt{x}} dx = \begin{cases} \frac{2\alpha}{\alpha-1} & \text{if } \frac{1}{\alpha} \in (0, 1) \\ \infty & \text{if } \frac{1}{\alpha} \geq 1 \end{cases} = \begin{cases} \frac{2\alpha}{\alpha-1} & \text{if } \alpha > 1 \\ \infty & \text{if } \alpha \in (0, 1]. \end{cases}$$

Chapter 2

L^p Spaces

Exercise 1

Analyze the convergence in $L^p([0, 1])$ with $1 \leq p < \infty$ of

$$f_n(x) = \frac{\cos(nx)e^{-nx}}{\sqrt[n]{x}} \quad \text{for } x \in [0, 1] \quad \forall n \in \mathbb{N}.$$

For which L^p the sequence converge to a certain function?

Solution

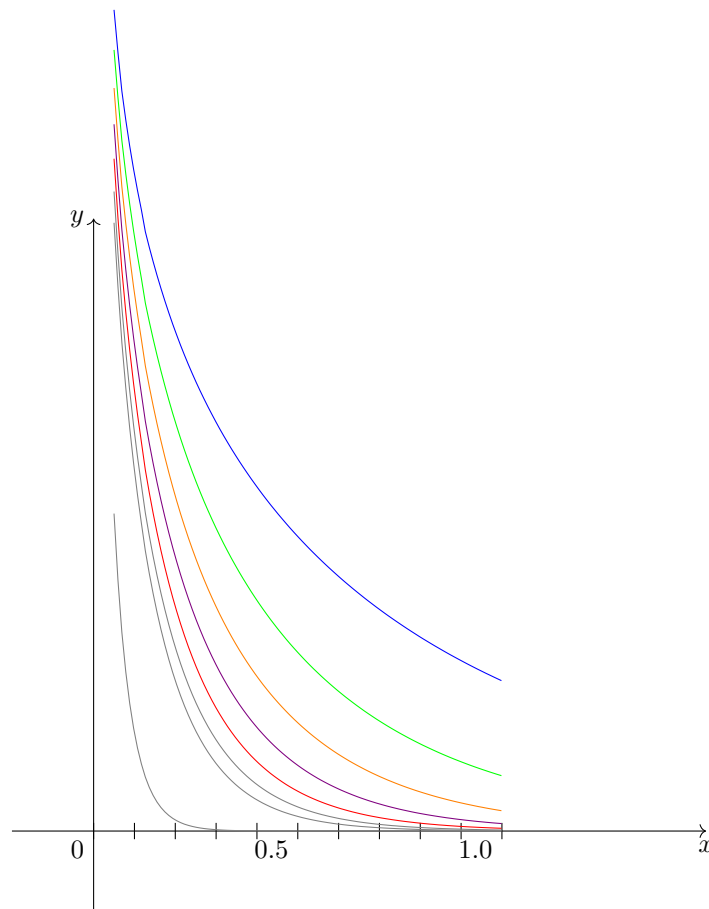


Figure 2.1: The sequence of functions $f_n(x)$.

First of all we search for which p this sequence belongs to some L^p , applying the Dominated Convergence Theorem.

$$|f_n(x)| = \left| \frac{\cos(nx)}{\sqrt[4]{x}} e^{-nx} \right| \leq \frac{1}{\sqrt[4]{x}} = g(x) \quad x \in [0, 1]$$

We know that $f_n(x)$ belongs to some L^p if and only if

$$\int_0^1 |f_n(x)|^p dx < +\infty.$$

The exponents p that satisfy this relations are the candidates.

$$\int_0^1 |f_n(x)|^p dx = \int_0^1 \left| \frac{\cos(nx)}{\sqrt[4]{x}} e^{-nx} \right|^p dx \leq \int_0^1 \left| \frac{1}{\sqrt[4]{x}} \right|^p dx = \int_0^1 \frac{1}{x^{\frac{p}{4}}} dx.$$

From the summability criteria:

$$\int_1^{+\infty} \frac{1}{x^\alpha} dx = \begin{cases} \frac{1}{\alpha-1} & \text{if } \alpha > 1 \\ +\infty & \text{if } \alpha \leq 1 \end{cases},$$

since

$$\left| \frac{\cos(nx)}{\sqrt[4]{x}} e^{-nx} \right|^p \leq \left(\frac{1}{\sqrt[4]{x}} \right)^p$$

we have that for $p \in [1, 4)$ $f_n(x) \in L^p([0, 1]) \quad \forall n \in \mathbb{N}$.

- $f_n \in L^1([0, 1])$;
- $f_n \in L^2([0, 1])$;
- $f_n \in L^3([0, 1])$.

Pointwise Convergence:

$$\lim_{n \rightarrow +\infty} f_n(x) = \lim_{n \rightarrow +\infty} \frac{\cos(nx)}{\sqrt[4]{x}} e^{-nx} = \lim_{n \rightarrow +\infty} \frac{\cos(nx)}{\sqrt[4]{x} e^{nx}} \rightarrow 0$$

so

$$f_n \rightarrow 0 \quad \text{pointwise} \quad \forall x \in [0, 1],$$

we can apply the comparison criterium.

$$\lim_{x \rightarrow 0^+} f_n(x) \sqrt[4]{x} = \lim_{x \rightarrow 0^+} \frac{\cos(nx)}{\sqrt[4]{x}} e^{-nx} \sqrt[4]{x} = 1$$

$$f_n(x) \sim \frac{1}{\sqrt[4]{x}} \quad \text{for } x \rightarrow 0^+.$$

Now we can analyze the convergence in $L^p([0, 1])$

$$\|f_n - f\|_{L^p}$$

$$\|f_n(x) - f(x)\|_{L^p([0,1])}^p = \|f_n(x)\|_{L^p([0,1])}^p = \left\| \frac{\cos(nx)}{\sqrt[4]{x} e^{nx}} \right\|_{L^p([0,1])}^p = \int_0^1 \left| \frac{\cos(nx)}{\sqrt[4]{x} e^{nx}} \right|_{L^p([0,1])}^p dx = \star$$

since

$$\left| \frac{\cos(nx)}{\sqrt[4]{x} e^{nx}} \right|_{L^p([0,1])}^p \leq g(x) = \frac{1}{x^{\frac{p}{4}}}$$

where

$$g \in L^p([0, 1]) \quad \text{for } 1 \leq p < 4,$$

we can apply the Dominated Convergence Theorem

$$\lim_{n \rightarrow +\infty} \|f_n(x) - f(x)\|_{L^p([0,1])}^p = \lim_{n \rightarrow +\infty} \int_0^1 \left| \frac{\cos(nx)}{\sqrt[4]{x}} e^{-nx} \right|^p dx = \int_0^1 \lim_{n \rightarrow +\infty} \left| \frac{\cos(nx)}{\sqrt[4]{x}} e^{-nx} \right|^p dx = 0.$$

So

$$\lim_{n \rightarrow +\infty} \|f_n(x) - f(x)\|_{L^p([0,1])} \rightarrow 0$$

$$f_n(x) \rightarrow 0 \quad \text{in} \quad L^p([0,1]) \quad \forall p \in [1; 4).$$

Since

$$\lim_{x \rightarrow 0^+} \frac{|f_n(x)|}{g(x)} = 1 \quad \forall n \in \mathbb{N}$$

we have

$$f_n \in L^p([0,1]) \leftrightarrow g \in L^p([0,1])$$

so that

$$f_n \notin L^p([0,1]) \quad \text{if} \quad p \geq 4.$$

The sequence $\{f_n(x)\}_{n \in \mathbb{N}}$ can't converge in $L^p([0,1])$ spaces if $p \geq 4$.

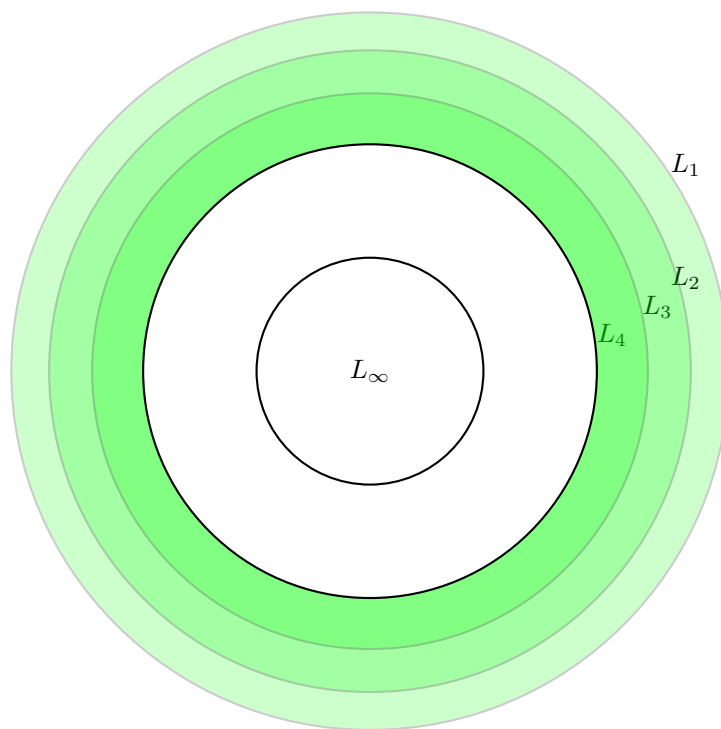
In the case $p = +\infty$, we have

$$\|f_n(x)\|_\infty = \operatorname{ess\,sup}_{x \in (0,1)} \left| \frac{\cos(nx)}{\sqrt[4]{x}} e^{-nx} \right| \leq \sup_{x \in (0,1)} \left| \frac{1}{\sqrt[4]{x}} \right| \rightarrow +\infty,$$

so

$$f_n \not\rightarrow 0 \quad \text{in} \quad L^\infty((0,1)).$$

Since $[0,1]$ is a bounded set we have the following embeddings: The sequence $f_n(x)$ lives in "green"



spaces.

Chapter 3

Hilbert Spaces

Exercise 1

Let $X = (C(0, 1); \|\cdot\|_\infty)$ and consider

$$K = \{f \in X : \int_0^{\frac{1}{2}} f(t)dt - \int_{\frac{1}{2}}^1 f(t)dt = 1\}.$$

Show that K is closed and not empty, and determine the projection of 0 over the set K .

Solution

K is not empty. To show that we can take:

$$f(t) = \frac{\pi}{2} \sin(2\pi t).$$

Now we can consider:

$$u(t) = \chi_{(0, \frac{1}{2})}(t) - \chi_{(\frac{1}{2}, 1)}(t)$$

and consider the following operator:

$$(Tf) = \int_0^1 f u dt$$

where

$$T : C(0, 1) \rightarrow \mathbb{R}$$

$$K = T^{-1}(\{1\})$$

since $\{1\}$ is a singleton then it is closed; the contrainage of a closed set must be closed, so K is closed.

$$|Tf| \leq C \|f\|_\infty$$

$$f \in C(0, 1) \quad \|f\|_\infty \leq 1$$

then

$$f \notin K,$$

that is the elements of K are of the form

$$\|\cdot\|_\infty > 1.$$

We have that

$$f(x) \leq \|f\|_\infty \leq 1 \quad \forall x \in (0, 1)$$

"by contradiction"

$$f \in K \quad \int_0^1 f u = 1$$

$$1 = \int_0^1 f u \leq \int_0^1 |f| |u| \leq \|f\|_\infty \int_0^1 dt = \|f\|_\infty \leq 1.$$

We have that

$$|fu| \leq 1 \quad \int_0^1 |fu| = 1$$

so that

$$\begin{aligned} |fu| &= 1 \quad a.e. \\ \int_0^1 (1 - |fu|) &= 0 \quad a.e. \end{aligned}$$

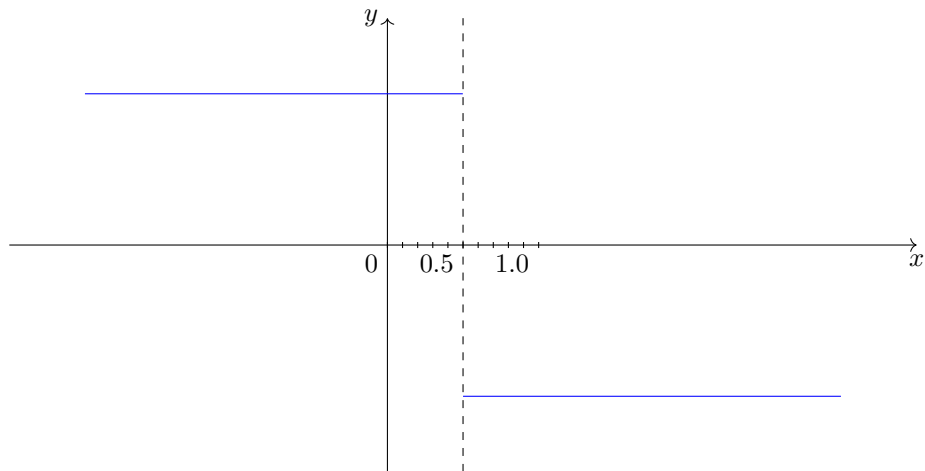
so that

$$fu = 1 \quad a.e..$$

So we obtain this contradiction:

$$\begin{cases} f = 1 & \text{if } x \in (0, \frac{1}{2}) \\ f = -1 & \text{if } x \in (\frac{1}{2}, 1) \end{cases}$$

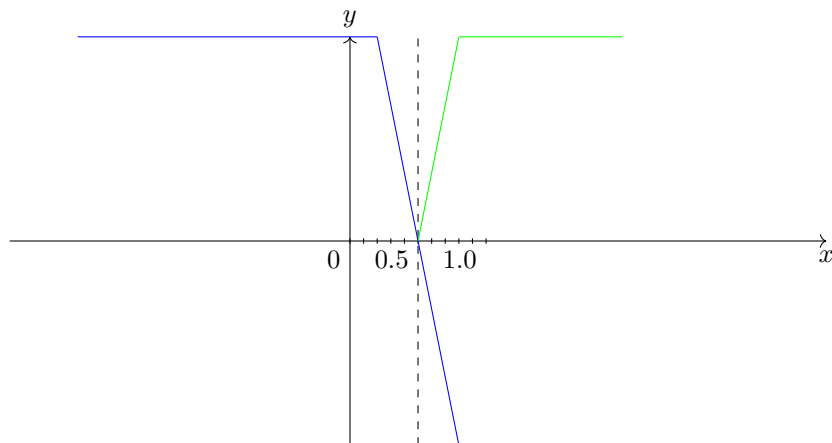
Now we consider:



$$\begin{aligned} d &= \inf\{\|f\|_\infty : f \in K\} \\ \|f\|_\infty &\leq 1 \quad \implies \quad f \notin K \\ d &= \inf\{\|f\|_\infty : f \in K\} \geq 1, \end{aligned}$$

we can take $1 < \alpha < 2$ and $\epsilon = \frac{\alpha-1}{\alpha}$.

$$f_\alpha = -\frac{\alpha}{\epsilon}\left(x - \frac{1}{2}\right)$$



Now we show that $f_\alpha = -\frac{\alpha}{\epsilon}(x - \frac{1}{2})$ belongs to K .

$$\int_0^{\frac{1}{2}} f_\alpha - \int_{\frac{1}{2}}^1 f_\alpha = 1 \quad \forall \alpha \in (1, 2)$$

$$\|f_\alpha\|_\infty = \alpha,$$

α is the supremum,

$$d = \inf\{\|f\|_\infty : f \in K\} \leq \|f_\alpha\|_\infty = \alpha$$

$$\alpha \in (1, 2)$$

$$\begin{cases} d \leq 2 \\ \forall \alpha \in (1, 2) \end{cases} \implies d \leq 1$$

so that

$$d = \inf\{\|f\|_\infty : f \in K\} = 1,$$

but this inf is not assumed, this is not a minimum, then

$$\nexists f \in K \quad \text{s.t.} \quad d = \|f\|_\infty = 1,$$

$$d = d(0, K)$$

$$0 \notin K.$$

Chapter 4

Operators

Exercise 1

Let

$$a(x) = \begin{cases} x & \text{if } x \in (0, \frac{1}{2}] \\ 0 & \text{if } x \in (\frac{1}{2}, 1] \end{cases}$$

and consider the operator

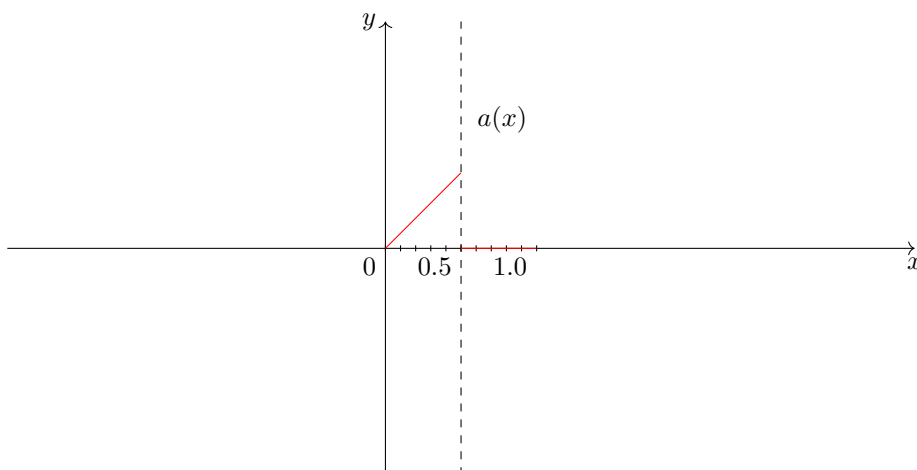
$$T : L^2(0, 1) \rightarrow L^2(0, 1)$$

given by

$$Tf(x) = a(x)f(x), \quad x \in (0, 1).$$

Show that $T \in \mathcal{L}(L^2[0, 1])$ and compute $\|T\|$.

Solution



$$\begin{aligned} f &\in L^2(0, 1) \\ \|Tf\|_{L^2(0,1)}^2 &= \int_0^1 a(x)^2 f(x)^2 dx \leq \frac{1}{4} \int_0^1 f(x)^2 dx = \frac{1}{4} \|f\|_{L^2(0,1)}^2 \end{aligned}$$

so that

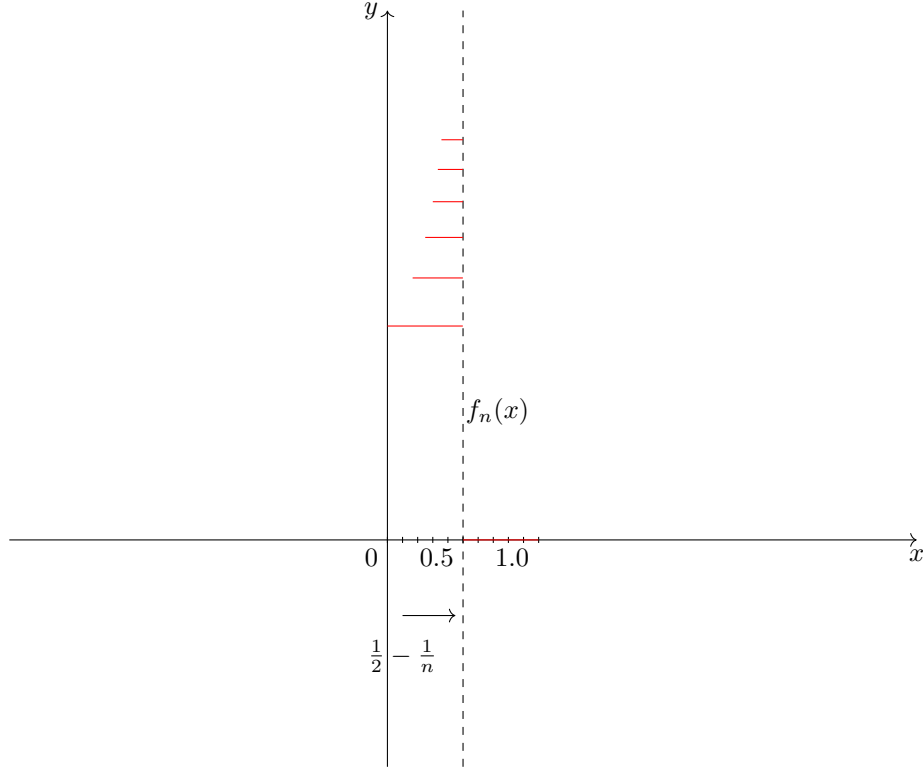
$$\|T\|_{\mathcal{L}(L^2(0,1))} \leq \frac{1}{2}.$$

Then we want to show that $\frac{1}{2}$ is the value of the norm, that is

$$\|T\| = \frac{1}{2}.$$

We define

$$f_n(x) = \begin{cases} \sqrt{n} & \text{if } x \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2}], \\ 0 & \text{otherwise} \end{cases}, \quad n \geq 2$$



$$\|f_n(x)\|_{L^2(0,1)}^2 = \int_0^1 f_n(x)^2 dx = 1$$

Now we compute the norm of the image:

$$\|Tf_n(x)\|_{L^2(0,1)}^2 = \int_0^1 a(x)^2 f_n(x)^2 dx \geq \star$$

we can minor the integral with

$$\star \geq \int_{\frac{1}{2} - \frac{1}{n}}^{\frac{1}{2}} n(\frac{1}{2} - \frac{1}{n})^2 dx = (\frac{1}{2} - \frac{1}{n})^2 \rightarrow \frac{1}{4} \quad \text{for } n \rightarrow +\infty$$

so that

$$\|T\|_{\mathcal{L}(L^2(0,1))} = \frac{1}{2}.$$

We can characterize the norm in various ways.

$$\|T\|_{\mathcal{L}(L^2(0,1))} = \sup_{f \in L^2(0,1)} \frac{\|Tf\|_{L^2(0,1)}}{\|f\|_{L^2(0,1)}} \leq \frac{1}{2}.$$

We have shown that

$$\|Tf_n(x)\|_{L^2(0,1)}^2 \geq \frac{1}{4},$$

that is

$$\|Tf_n(x)\|_{L^2(0,1)} \geq \frac{1}{2},$$

but at the same time we have

$$\|T\| \leq \frac{1}{2}$$

then

$$\|Tf_n(x)\| = \frac{1}{2}.$$

Exercise 2

Let $(f'_h)_{h \in \mathbb{N}} \in L^p$ for some p , with the following hypotheses:

- $(f'_h)_{h \in \mathbb{N}}$ is bounded in L^p for some p ;
- $f_h(0)$ is bounded.

Show that $(f_h)_{h \in \mathbb{N}}$ is compact (relatively) in $(C([0, 1]), \|\cdot\|_\infty)$.

Solution

From the hypotheses we can suppose that

$$f_h(x) = f_h(0) + \int_0^x f'_h(y) dy \quad x \in [0, 1].$$

Since we have to show the compactness in the set of continuous functions we need to utilize the Ascoli-Arzelà theorem.

Let $C > 0$ constant.

Equiboundedness

$$h \in \mathbb{N}, \quad x \in [0, 1]$$

$$|f_h(x)| \leq |f_h(0)| + \int_0^x |f'_h(y)| dy \leq \star$$

using the hypothesis 2 and the Hölder inequality

$$\star \leq C + \|f'_h\|_{L^p(0,1)} x^{\frac{1}{p'}} \leq M$$

so that

$$|f_h(x)| \leq M,$$

that is f_h is equibounded.

Equicontinuity

$$x, y \in [0, 1], \quad x < y$$

$$f_h(y) - f_h(x) \leq \int_x^y |f'_h(w)| dw \leq \star$$

using the Hölder inequality

$$\star \leq \|f'_h\|_{L^p(0,1)} |y - x|^{\frac{1}{p'}}.$$

This shows that the functions f_h are equi-hölder with exponent $\frac{1}{p'}$, in particular they are equicontinuous.

Then from the Ascoli-Arzelà theorem they are relatively compact.