# *A collection of Calculus of Variations exercises*

June 2, 2024

# Chapter 1

# Scalar Case

## Exercise 1

Let

$$I[u] = \int_a^b 2u'(x)^3 dx.$$

Find a certain function $u : [a, b] \to \infty$ that is a minimum of the given integral functional.

## Solution

$$F(x, u(x), u'(x)) = F(u'(x)) = 2u'(x)^3.$$

We can rewrite the integrand as

$$F(u'(x)) = F(p) = 2p^3$$

From the Euler-Lagrange equations:

$$\frac{d}{dx}\frac{\partial F}{\partial u'}(u'(x)) = \frac{\partial F}{\partial p}(p) = \frac{\partial F}{\partial u}(p)$$

we have

$$\frac{d}{dx}\frac{\partial F}{\partial p}(p) = 0.$$

Since

$$\frac{\partial F}{\partial p}(p) = \frac{\partial(2p^3)}{\partial p} = 6p^2 = 6u'(x)^2$$

then

$$\frac{d}{dx}\frac{\partial F}{\partial u'}(u'(x)) = 12u'(x)u''(x).$$

We obtain an ordinary differential equation of the second order

$$12u'(x)u''(x) = 0$$

$$u'(x) = C_1$$

$$u(x) = C_2 x.$$

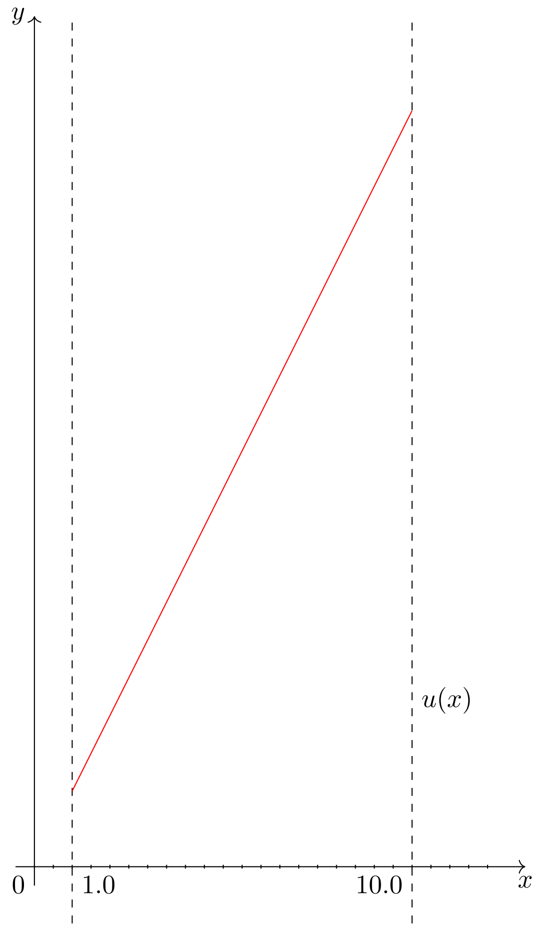
We now assume $a = 1$ and $b = 10$.

$$\begin{cases} u(x) = C_2 x \\ u'(x) = C_1 \\ u(1) = 2 \\ u'(10) = 4 \end{cases}$$

$$C_1 = 4$$

$$C_2 = 2$$

$$u(x) = 2x$$


y
u(x)
0
1.0
10.0
x