Quantile Regression with `quantreg`

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A bit of Theory

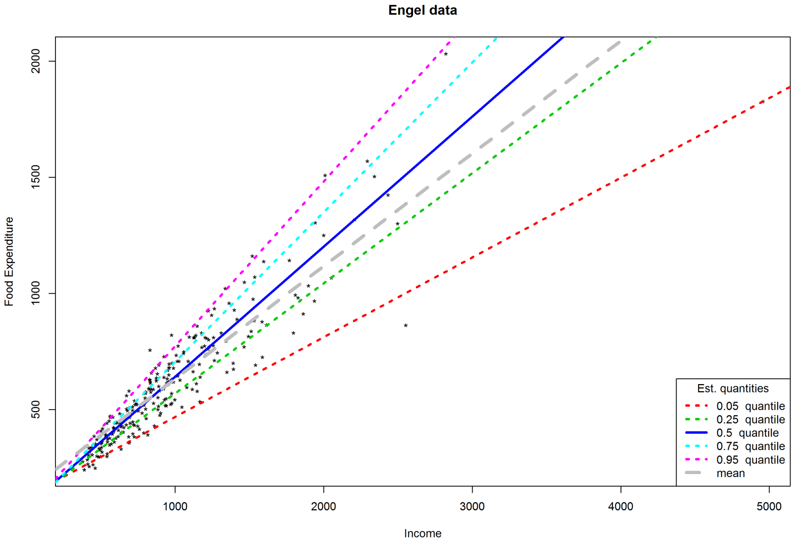
Expanding the usual approach

The objective of linear regression analysis is the estimation of the conditional *mean*

Quantile Regression aims at estimating specific conditional *quantiles* (e.g. median)

It was introduced by Roger Koenker and Gilbert Bassett: "Regression Quantiles", *Econometrica*, 46, 33-50 (1978)

Example from built-in data



Expanding the usual approach

- Linear regression is supported by a number of assumptions about the predictor variables, the response variables and their relationship (error independence, homoscedasticity,...)

The structure underlying this type problems allow them to be transformed into linear algebra equivalents

- On the other hand, Quantile regression makes no particular assumption on the distribution of the response nor about its variance

This leads instead to linear programming problems

LR-QR comparison

Table 1: A quick comparison between the two regressions

Linear Regression	Quantile Regression
Predicts the mean	Predicts conditional quantiles
Applies when n is small	Needs sufficient data
Based on many assumptions	Is distribution agnostic
Is sensitive to outliers	Is robust to response outliers
Is computationally inexpensive	Is computationally intensive

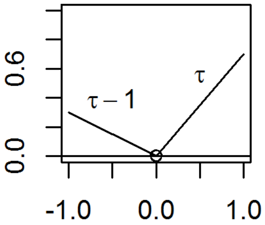
Under the hood

Called τ the objective quantile of a random variable Y, Quantile regression minimizes a sum that gives asymmetric penalties:

- $(\tau - 1)e_i$ for underestimates
- τe_i for overestimates

This is condensed in what's defined as the *loss function*

$$\rho_\tau(y) = (\tau - 1_{\{y<0\}})y$$



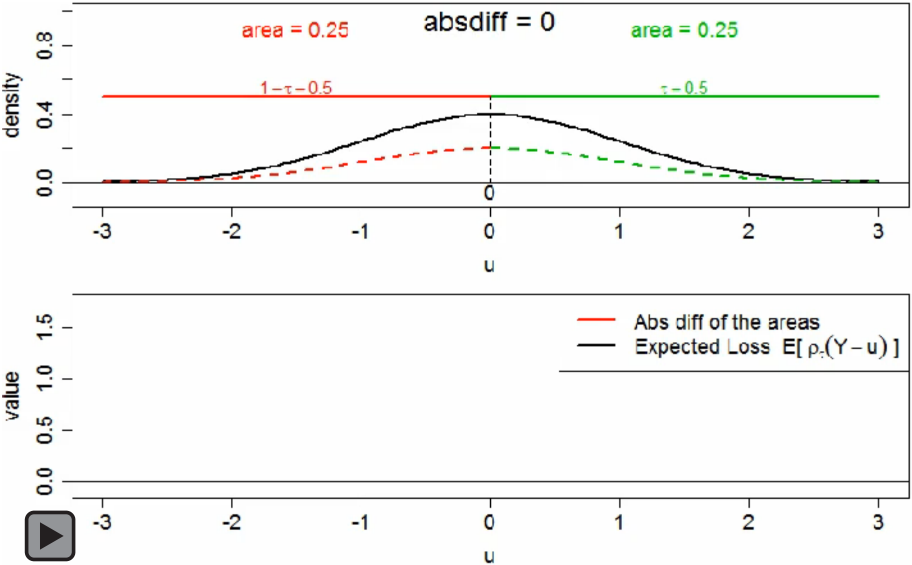
Under the hood

The wanted quantile for τ can be calculated by minimizing with respect to u the expected loss for the r.v. $Y - u$ that is

$$\min_u E[\rho_\tau(Y - u)] =$$
$$= \min_u \left[(\tau - 1) \int_{-\infty}^{u} (y - u) dF_Y(y) + \tau \int_{u}^{+\infty} (y - u) dF_Y(y) \right]$$

The concept is generalized to allow its use with distributions coming from observed samples

A visual hint



The dataset

California Housing Data

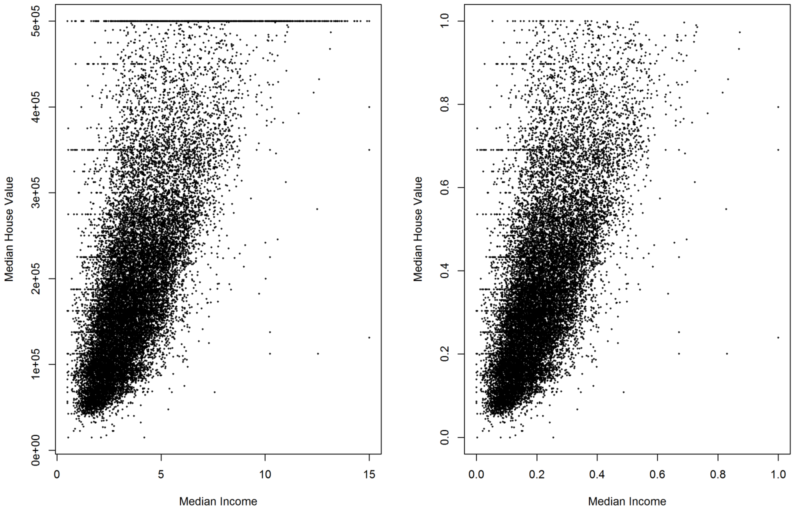
This dataset found on Kaggle gathers information from the census of the Californian districts.

It contains about 20600 observations of many different variables:

```r
house <- read.csv("house_from_gitsklearn.csv")
summary(house)
```

```
##    longitude         latitude     housing_median_age  total_rooms   
##  Min.   :-124.3   Min.   :32.54   Min.   : 1.00      Min.   :    2  
##  1st Qu.:-121.8   1st Qu.:33.93   1st Qu.:18.00      1st Qu.: 1448  
##  Median :-118.5   Median :34.26   Median :29.00      Median : 2127  
##  Mean   :-119.6   Mean   :35.63   Mean   :28.64      Mean   : 2636  
##  3rd Qu.:-118.0   3rd Qu.:37.71   3rd Qu.:37.00      3rd Qu.: 3148  
##  Max.   :-114.3   Max.   :41.95   Max.   :52.00      Max.   :39320  
##                                                                      
##  total_bedrooms     population      households     median_income    
##  Min.   :   1.0   Min.   :    3   Min.   :   1.0   Min.   : 0.4999  
##  1st Qu.: 296.0   1st Qu.:  787   1st Qu.: 280.0   1st Qu.: 2.5634  
##  Median : 435.0   Median : 1166   Median : 409.0   Median : 3.5348  
##  Mean   : 537.9   Mean   : 1425   Mean   : 499.5   Mean   : 3.8707  
##  3rd Qu.: 647.0   3rd Qu.: 1725   3rd Qu.: 605.0   3rd Qu.: 4.7432  
##  Max.   :6445.0   Max.   :35682   Max.   :6082.0   Max.   :15.0001  
##  NA's   :207                                                         
##  median_house_value   ocean_proximity
##  Min.   : 14999     <1H OCEAN :9136  
##  1st Qu.:119600     INLAND    :6551  
##  Median :179700     ISLAND    :   5  
##  Mean   :206856     NEAR BAY  :2290  
##  3rd Qu.:264725     NEAR OCEAN:2658  
##  Max.   :500001                      
## 
```

Cleaning the dataset



The package

quantreg

The R-package `quantreg` provides the estimation and inference methods for models of conditional quantiles

It was written by Roger Koenker

```
library(quantreg)
```

Some Examples

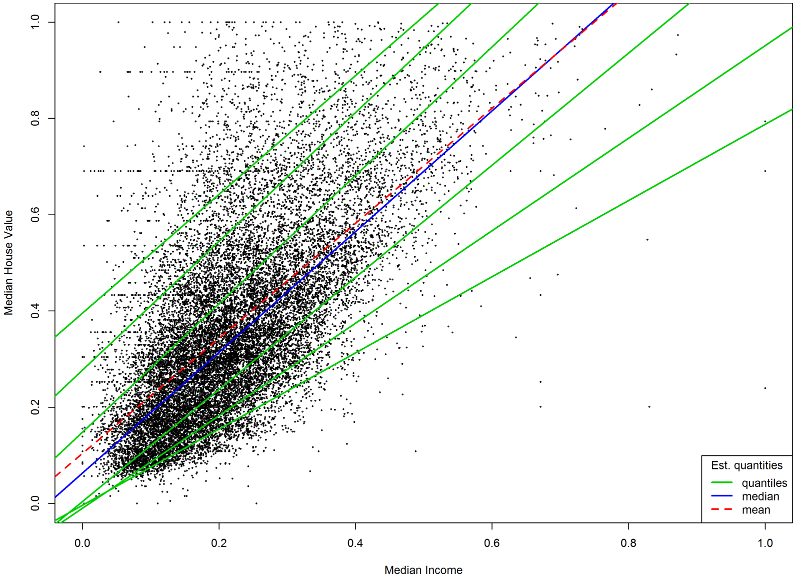
The function rq

to perform a Quantile regression one can call:

```
rq(formula, tau=.5, data, subset, weights, na.action,
   method="br", model = TRUE, contrasts, ...)
```

So for example:

```
taus <- c(.05,.1,.25,.50,.75,.90,.95)
for( i in 1:length(taus)){
  current.rq<- rq( median_house_value ~ median_income,
                   data=house,tau=taus[i])
  abline(current.rq)
}
```



Median House Value
Median Income
0.0
0.2
0.4
0.6
0.8
1.0
Est. quantities
quantiles
median
mean

The function summary.rqs

With the result given by `rq` one can print the summary with the function `summary.rqs`

```
summary.rqs(object, se = NULL, covariance=FALSE,
            hs = TRUE,  U = NULL, gamma = 0.7, ...)
```

the coefficients for each requested quantiles are shown

```
population.rq<- rq(population ~ households,
                   data=house, tau=taus)
population.summary<-summary.rqs(population.rq, ci='boot')
population.summary
```

```
##
## Call: rq(formula = population ~ households, tau = taus, data = house)
##
## tau: [1] 0.05
##
## Coefficients:
##             Value     Std. Error t value   Pr(>|t|)
## (Intercept)   0.00065   0.00016    4.18270   0.00003
## households    0.30683   0.00287  107.02170   0.00000
##
...
```

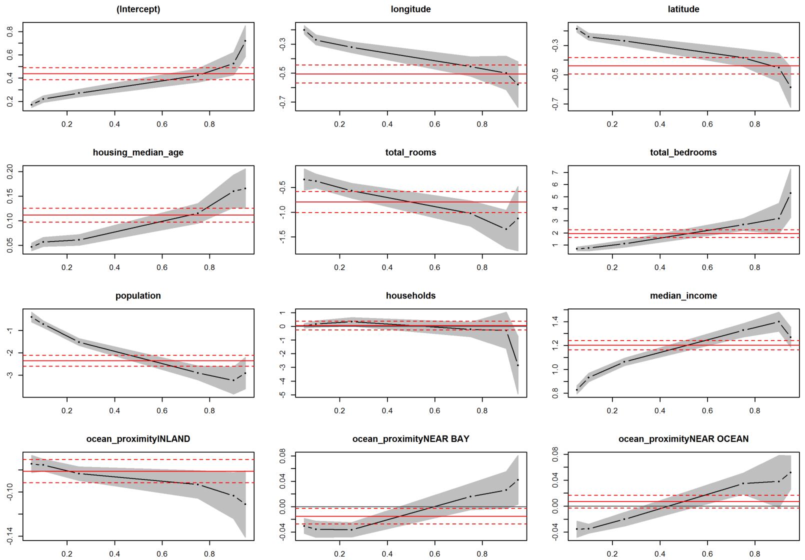
The function plot.summary.rqs

One can plot the summary with the following command:

```
plot.summary.rqs(x, parm = NULL, level = 0.9, ols = TRUE,
  mfrow = NULL, mar = NULL, ylim = NULL, main = NULL,
  col = gray(c(0, 0.75)), border = NULL, lcol = 2,
  lty = 1:2, cex = 0.5, pch = 20, type = "b", xlab = "",
  ylab = "", ...)
```

The coefficients for each requested quantiles are plotted together with their confidence intervals.

```r
set.seed(3)
res.house <- house[ sample(1:nrow(house), 5000), ]
fit <- rq(median_house_value ~ ., tau=taus, data=res.house)
plot.summary.rqs(summary.rqs(fit))
```



The function `anova.rq`

It is possible to perform the Anova of rq objects.

`anova.rq` comes in two forms:

1. when the fit is performed for a vector of quantiles the function perform a test on the hypothesis that all the coefficients are the same
2. compare nested models as in linear regression

anova.rqs to compare slopes

```r
anova.rqs(fit)
```

```
## Quantile Regression Analysis of Deviance Table
##
## Model: median_house_value ~ median_income
## Joint Test of Equality of Slopes: tau in {  0.05 0.1 0.25 0.5 0.75 0.9 0.95  }
##
##   Df Resid Df F value    Pr(>F)
## 1  6   136319  210.36 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

The results show indeed that it is unlikely that the lines have the same slope

anova.rq for nested models

```
fit.1 <- rq(median_house_value ~ median_income, tau=0.5, data=house)
fit.2 <- rq(median_house_value ~ median_income + total_rooms,
            tau=0.5, data=house)
fit.3 <- rq(median_house_value ~ median_income + total_bedrooms,
            tau=0.5, data=house)

anova.rq(fit.1, fit.2)
```

```
## Quantile Regression Analysis of Deviance Table
##
## Model 1: median_house_value ~ median_income + total_rooms
## Model 2: median_house_value ~ median_income
##   Df Resid Df F value Pr(>F)
## 1  1    19472  1.5544 0.2125
```

```
anova.rq(fit.1, fit.3)
```

```
## Quantile Regression Analysis of Deviance Table
##
## Model 1: median_house_value ~ median_income + total_bedrooms
## Model 2: median_house_value ~ median_income
##   Df Resid Df F value    Pr(>F)
## 1  1    19472  49.759 1.797e-12 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

The results shows that the additional covariate `total_rooms` is not significant while the `total_bedrooms` covariate does improve significantly the prediction

The function `predict.rq`

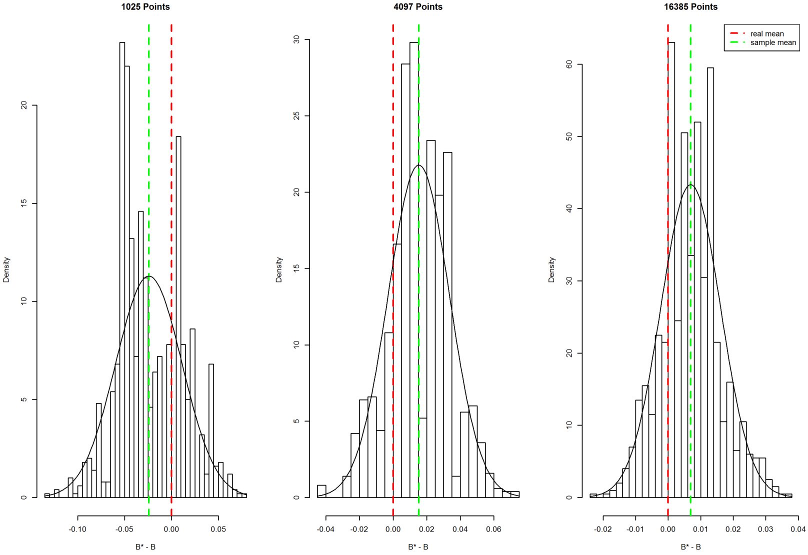
`quantreg` gives also the possibility to predict the quantiles given the fit

```
newdata <- house[3,]
yy<-predict.rq(object=fit.1,newdata=newdata,
               type="percentile", interval="confidence",
               se='boot')
yy
```

```
##         fit     lower    higher
## 3 0.6479275 0.6423845 0.6538685
```

The function boot.rq

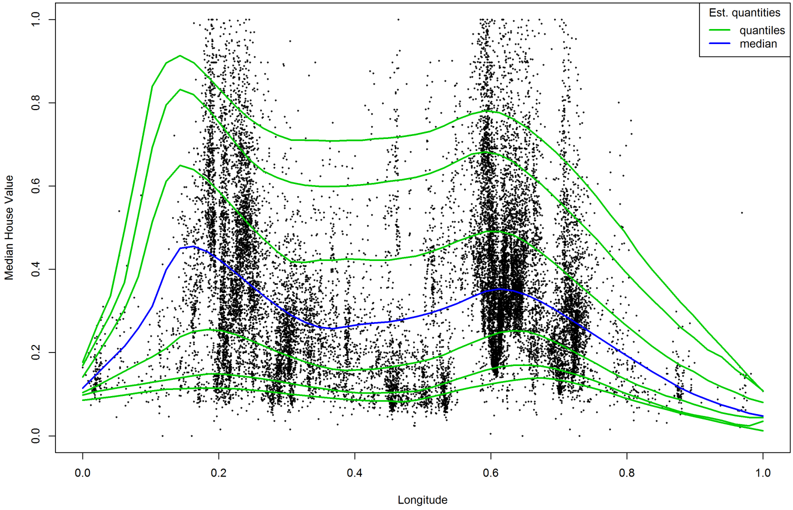
```
set.seed(4)
ns <- c(10, 12, 14)

par(mfrow = c(1, length(ns)))
for( i in ns ){
  n <- (2^i)+1
  x <- seq(0, 2, length=n);  y <- -x + rnorm(n)
  df <- data.frame(x, y)
  b <- boot.rq(x=x, y=y, tau=.5, R = 1000)
  main <- paste(n, "Points")
  m <- mean(b$B) + 1;  s <- sd(b$B)
  hist(b$B + 1, breaks = 30, prob=TRUE,
       xlab = "B* - B", main = main)
  abline(v=0, add=TRUE, col='red', lty=2, lwd=2)
  abline(v=m, add=TRUE, col='green', lty=2, lwd=2)
  curve(dnorm(x, m, s), add=TRUE)
}
```


1025 Points
4097 Points
16385 Points
real mean
sample mean
Density
B* - B

The function lprq

This function does *locally polynomial quantile regression* univariate smoothing

```
plot(house$longitude, house$median_house_value,
     xlab="Longitude", ylab="Median House Value",
     cex=.25, cex.lab=1,cex.axis=1, lheight=0.5)
for(tau in taus){
  fit <- lprq(house$longitude, house$median_house_value,
              tau=tau, h=0.1, m=50)
  if( tau == 0.5){
    lines(fit$xx, fit$fv, col="4", lwd=2)
  }
  else{
    lines(fit$xx, fit$fv, col="green3", lwd=2)
  }
}
```



Est. quantities
quantiles
median
Median House Value
0.0
0.2
0.4
0.6
0.8
1.0
Longitude
0.0
0.2
0.4
0.6
0.8
1.0

The function `rqss`

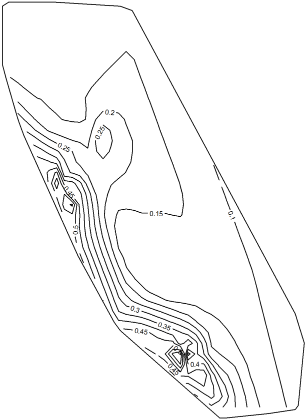
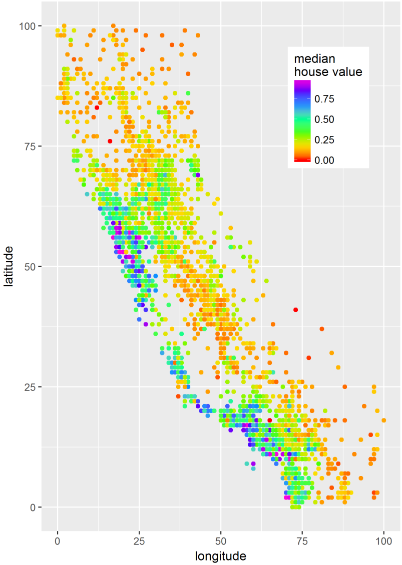
`rqss` fits *additive quantile regression* models with possible univariate and/or bivariate nonparametric terms.

```
rqss(formula, tau = 0.5, data = parent.frame(), weights,
     na.action, method = "sfn", lambda = NULL,
     contrasts = NULL, ztol = 1e-5, control, ...)
```

```
fhouse<-house
fhouse$longitude <- round(fhouse$longitude * 100)
fhouse$latitude <- round(fhouse$latitude * 100)
fit <- rqss(median_house_value ~ qss(cbind(longitude,latitude),
                                              lambda = 5),
            data = fhouse)
```

The resulting fit can be plotted with the function `plot.rqss`

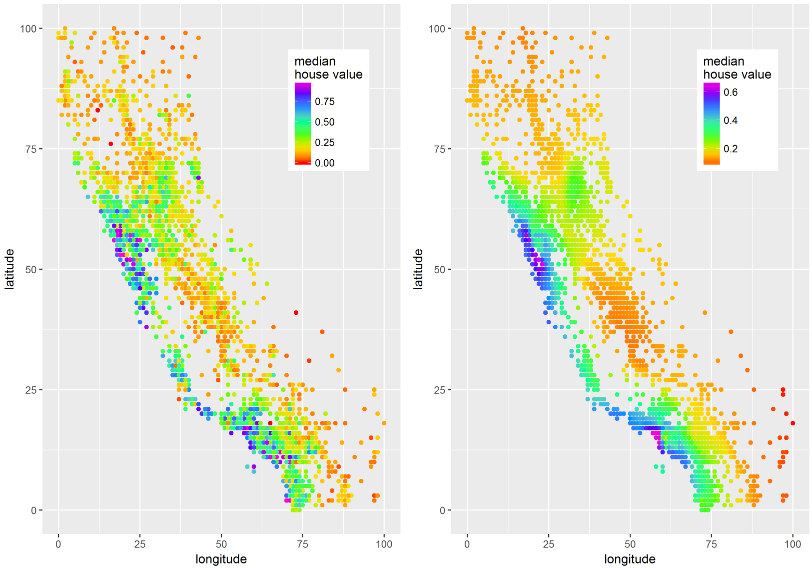
```r
plot.new()

gl <- grid.layout(nrow=1, ncol=2)
vp.1 <- viewport(layout.pos.col=1, layout.pos.row=1)
vp.2 <- viewport(layout.pos.col=2, layout.pos.row=1)

pushViewport(viewport(layout=gl))
pushViewport(vp.1)

par(new=TRUE, fig=gridFIG())
plot.rqss(fit, axes = FALSE, xlab = "", ylab = "",
          render="contour", bands="uniform")
popViewport()

pushViewport(vp.2)
ggplotted <- ggplot(fhouse) + geom_point(
  mapping=aes(x=longitude, y=latitude,
              col=median_house_value)) +
  scale_color_gradientn("median\nhouse value",
                        colours = rainbow(7)) +
  theme(legend.position = c(.8, .8))
print(ggplotted, newpage = FALSE)

popViewport(1)
```


0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5

median
house value
0.75
0.50
0.25
0.00
latitude
longitude
0
25
50
75
100

the command `predict.rqss` allows to predict the values

```
predict.rqss(object, newdata, interval = "none",
             level = 0.95, ...)
```

```
newdata1<-fhouse[c("longitude","latitude")]
newdata1$median_house_value<-
  predict.rqss(fit, newdata=newdata1, interval="none")
```



median house value
0.75
0.50
0.25
0.00
latitude
longitude
median house value
0.6
0.4
0.2
latitude
longitude

Thank you
for your attention!