

When is a function a fold, or an unfold?

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or, The Discipline of Thorsten

1. Fold

Recall `foldr` in Haskell:

```
foldr :: (a->b->b) -> b -> [a] -> b
foldr f e []          = e
foldr f e (x:xs)      = f x (foldr f e xs)
```

Abstractly:

$$\text{fold} : \text{Functor } F \Rightarrow (F A \rightarrow A) \rightarrow \mu F \rightarrow A$$

$$\text{fold}_F f (\text{in}_F x) = f (F (\text{fold}_F f) x)$$

For example, with $\mathbf{L}A = 1 + \text{Nat} \times A$ for lists of naturals,

$$\text{sum} : \mu \mathbf{L} \rightarrow \text{Nat}$$

$$\text{sum} = \text{fold}_{\mathbf{L}} \text{add} \quad \text{where} \quad \begin{aligned} \text{add}(\text{Inl}()) &= 0 \\ \text{add}(\text{Inr}(m, n)) &= m + n \end{aligned}$$

2. When is a function a fold?

Universal property:

$$h = \text{fold}_{\mathsf{F}} f \quad \Leftrightarrow \quad h \circ \text{in}_{\mathsf{F}} = f \circ \mathsf{F} h$$

for $h: \mu \mathsf{F} \rightarrow A$ and $f: \mathsf{F} A \rightarrow A$.

3. When is a function *not* a fold?

Consider $allEqual : \mu\mathbf{L} \rightarrow \mathbf{Bool}$:

$$allEqual[1] = \mathbf{True}$$

$$allEqual[2] = \mathbf{True}$$

$$allEqual[1, 1] = \mathbf{True}$$

$$allEqual[1, 2] = \mathbf{False}$$

3. When is a function *not* a fold?

Consider $allEqual : \mu\mathbb{L} \rightarrow Bool$:

$$allEqual[1] = True$$

$$allEqual[2] = True$$

$$allEqual[1, 1] = True$$

$$allEqual[1, 2] = False$$

There is no $f : \mathbb{L}Bool \rightarrow Bool$ such that $allEqual = fold_{\mathbb{L}} f$.

4. Non-answers

- the universal property says it all
but *intensionally*, not extensionally
- injections are folds
extensional, but only an *implication* (consider *sum*)

Neither is very helpful at showing that a function is *not* a fold.

5. Totally

Definition. The *kernel* $\ker p$ of total function $p: X \rightarrow Y$ is equivalence relation

$$\ker p = \{ (x, x') \mid x, x' \in X \wedge px = px' \}$$

Theorem (postfactors). Given $r: X \rightarrow Z$ and $p: X \rightarrow Y$ and $Y \rightarrow Z \neq \emptyset$,

$$(\exists q: Y \rightarrow Z \bullet r = q \circ p) \iff \ker r \supseteq \ker p$$

Proof. \Rightarrow : calculation. \Leftarrow : choose $qy = rx$ when $y = px$, else arbitrary.

Corollary. If initial algebra $(\mu F, \text{in}_F)$ exists, then for $h: \mu F \rightarrow A$,

$$(\exists f \bullet h = \text{fold}_F f) \iff \ker (h \circ \text{in}_F) \supseteq \ker (Fh)$$

In particular, $((1, [1]), (1, [2])) \in \ker (id \times \text{allEqual}) - \ker (\text{allEqual} \circ \text{cons})$

6. Dually

Definition. The *image* $\text{img } p$ of total function $p: X \rightarrow Y$ is

$$\text{img } p = \{y \in Y \mid \exists x \in X \bullet px = y\}$$

(Why dual? equivalence $\ker p = p^\circ \circ p$, coreflexive $\text{img } p = p \circ p^\circ$.)

Theorem (prefactors). Given $r: X \rightarrow Z$ and $q: Y \rightarrow Z$ and $X \rightarrow Y \neq \emptyset$,

$$(\exists p: X \rightarrow Y \bullet r = q \circ p) \iff \text{img } r \subseteq \text{img } q$$

Proof. \Rightarrow : calculation. \Leftarrow : case analysis on $X = \emptyset \vee Y \neq \emptyset$.

Corollary. If final coalgebra $(\nu F, \text{out}_F)$ exists, then for $h: A \rightarrow \nu F$,

$$(\exists g \bullet h = \text{unfold}_F g) \iff \text{img } (\text{out}_F \circ h) \subseteq \text{img } (F h)$$

7. Partially

Definition. The kernel $\ker p$ of partial function $p: X \rightarrow Y$ is equivalence

$$\begin{aligned} \ker p = & \{ (x, x') \mid x, x' \in \text{dom } p \wedge px = px' \} \\ & \cup \{ (x, x') \mid x, x' \in (X - \text{dom } p) \} \end{aligned}$$

Theorem. For partial functions $r: X \rightarrow Z$ and $p: X \rightarrow Y$,

$$(\exists q: Y \rightarrow Z \bullet r = q \circ p) \iff \ker r \supseteq \ker p \wedge \text{dom } r \subseteq \text{dom } p$$

Proof. More awkward, because involving a case analysis.

8. Allegorically

Definition. Relation R is *simple* if $R \circ R^\circ \subseteq id$.

Definition. *Left division* operator $T \subseteq S \backslash R \Leftrightarrow S \circ T \subseteq R$.

Definition. The kernel $\ker R$ of a relation $R: X \sim Y$ is equivalence

$$\ker R = (R \backslash R) \cap (R \backslash R)^\circ$$

Remark. Concretely, $\ker R = \{ (x, x') \mid \forall y \bullet (x, y) \in R \Leftrightarrow (x', y) \in R \}$.

Theorem. For simple $T: X \sim Z$ and $R: X \sim Y$,

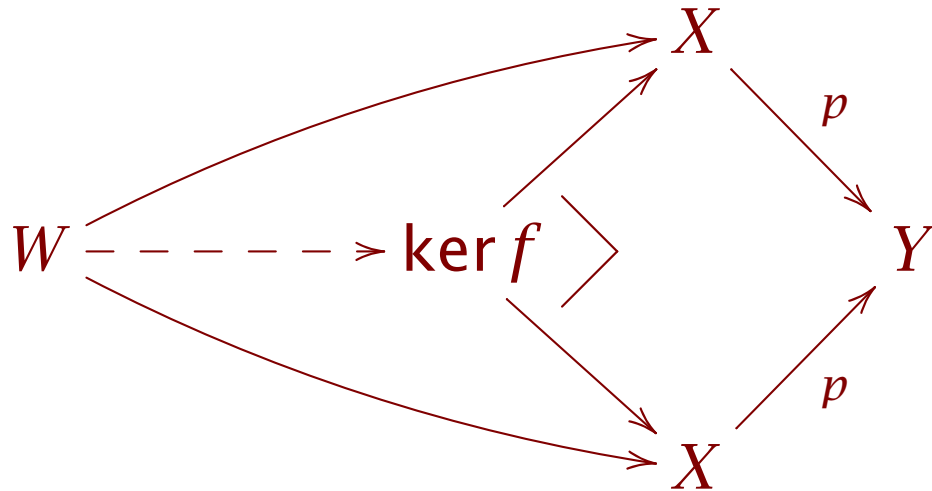
$$(\exists \text{ simple } S: Y \sim Z \bullet T = S \circ R) \Leftrightarrow \ker R \subseteq \ker T \wedge \text{dom } R \supseteq \text{dom } T$$

Proof. Nice and straightforward, by calculation.

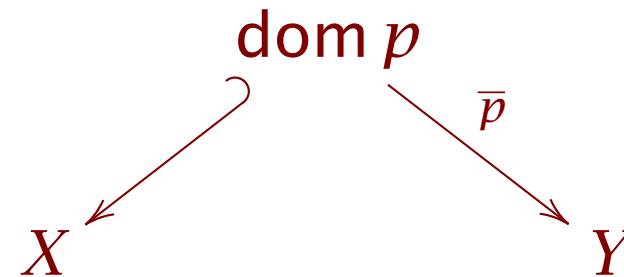
9. Categorically

Can that nice calculational proof be generalised?

Definition. The *kernel pair* of an arrow p is its pullback along itself:



Definition. A *partial map* $p: X \rightarrow Y$ is a span with a monomorphic left leg:

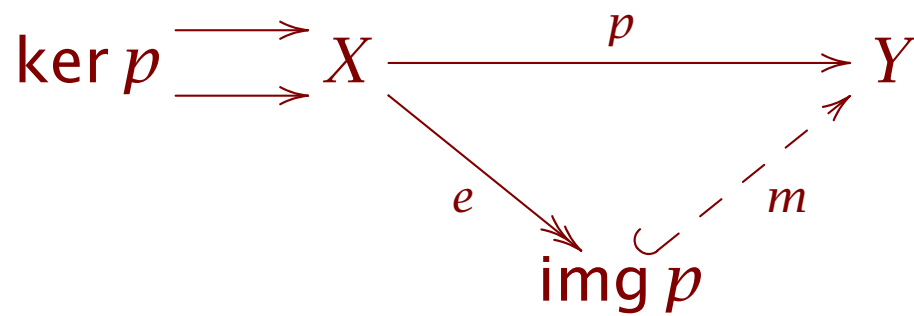


10. Regularly

Definition. A *regular category* is:

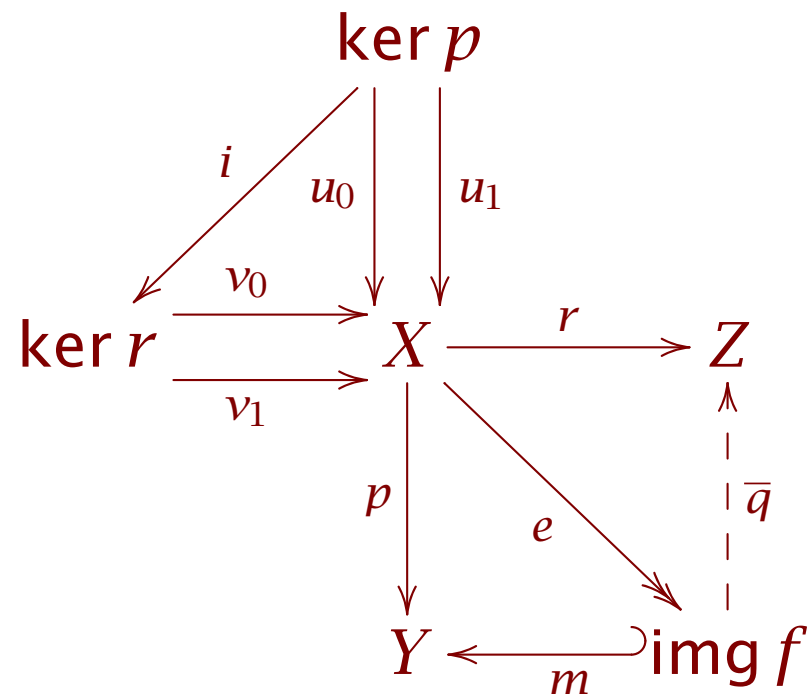
- finitely complete (so has all kernel pairs), such that
- every kernel pair has a coequaliser (a *regular epimorphism*), and
- regular epimorphisms are stable under pullbacks.

Lemma. In a regular category, every arrow $p: X \rightarrow Y$ *factorises* into a regular epimorphism followed by a monomorphism.



11. Postfactors

Theorem (Sam Staton). Given arrows $p: X \rightarrow Y$ and $r: X \rightarrow Z$ in a regular category, there exists a partial map $q: Y \rightarrow Z$ such that $r = q \circ p$ iff there exists an arrow $i: \ker p \rightarrow \ker r$ satisfying $u_0 = v_0 \circ i$ and $u_1 = v_1 \circ i$, where u_0, u_1 and v_0, v_1 are the kernel pairs of p and r respectively.



12. Discussion

A *relation* $R: X \sim Y$ is a jointly monic span $X \leftarrow R \rightarrow Y$.

Relational composition is defined in terms of *pullbacks of spans*.

Regular category is precisely the structure required for *associativity*.

Set is regular, so we get the story for total functions.

But any category of Eilenberg–Moore algebras over *Set* is also regular; so covers *partial functions* too ($\mathcal{P}fun \simeq \text{EM-algebras for } Maybe$).

What about other monads? Relations are Kleisli arrows for powerset.

Story here only for folds.

Unfolds not quite dual (probably don't want to dualise 'partial function').

13. Acknowledgements

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