CATEGORY THEORY
MIDLANDS GRADUATE SCHOOL 2023

EXERCISE 2 (3 APRIL)

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Reminder

Definition 1 (product diagram)**.** *For $A, B \in \mathcal{C}_0$, a* product diagram *is an object $P \in \mathcal{C}_0$ together with morphisms $\pi_1 \in \mathcal{C}(P, A)$ and $\pi_2 \in \mathcal{C}(P, B)$,*

$$A \xleftarrow{\pi_1} P \xrightarrow{\pi_2} B$$

with the following universal property*: Given any other diagram of this shape,*

$$A \xleftarrow{p_1} X \xrightarrow{p_2} B$$

there is a unique morphism $t \in \mathcal{C}(X, P)$ such that $p_1 = \pi_1 \circ t$ and $p_2 = \pi_2 \circ t$. One expresses these equations by saying that the following two triangles commute:

$$\begin{tikzcd} & X \arrow[dl, "p_1"'] \arrow[d, dashed, "t"] \arrow[dr, "p_2"] & \\ A & P \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

Exercise 3: Coproducts

Using the definition of a product diagram given in the reminder above, re-construct the definition of a coproduct diagram by "reversing all arrows". Check that, in the category SET, your definition describes the disjoint union of two sets. What objects are characterised by your definition in the other categories discussed in the lecture?

Note: A solution to the first part, i.e. the definition of a coproduct diagram, is on the next page!

Exercise 4: Isomorphisms

Show that, for a morphism $f \in \mathcal{C}(X, Y)$, there is at most one inverse. This means that, if we have two morphisms $g_1, g_2 \in C(Y, X)$ such that $g_1 \circ f = \mathsf{id}_X$ and $f \circ g_1 = \mathsf{id}_Y$ and $g_2 \circ f = \mathsf{id}_X$ and $f \circ g_2 = \mathsf{id}_Y$, then we can conclude that $g_1 = g_2$.

In other words, a morphism is an isomorphism in at most one way. (Note: Isomorphisms are defined on the previous exercise sheet.)

Exercise 5: Interaction of products, coproducts, initial and terminal objects

Assume that the category $\mathcal{C}$ has a terminal object, for which we write $\mathsf{1}$, and an initial object, for which we write $\mathsf{0}$. Moreover, assume that $\mathcal{C}$ has all products and coproducts. This means that, for every pair of objects $A, B \in \mathcal{C}_0$, we have a product diagram and a coproduct diagram. We write $A \times B$ for the "middle object" of the product diagram, and $A + B$ for the "middle object" of the coproduct diagram. (Note: the definition of a product diagram is above, the definition of a coproduct diagram is below, the definitions for initial and terminal are on the previous exercise sheet.)

Recall that we write $X \cong Y$ if there is an isomorphism in $\mathcal{C}(X, Y)$. For each of the following isomorphisms, either prove them in $\mathcal{C}$ or find a category $\mathcal{C}$ in which they don't hold:

a. $A \times \mathsf{1} \cong A$

b. $A \times \mathsf{0} \cong A$

c. $A \times \mathsf{0} \cong \mathsf{0}$

d. $A + \mathsf{0} \cong A$

e. $A \times B \cong B \times A$

f. $A \times (B + C) \cong (A \times B) + (A \times C)$

g. Can you write down an isomorphism that *never* holds, i.e., that doesn't hold in *any* category $\mathcal{C}$?

Partial solution to Exercise 3

The following is a copy of the definition of a product, with some arrows reversed.

Definition 2. *For $A, B \in \mathcal{C}_0$, a* coproduct diagram *is an object $S \in \mathcal{C}_0$ (S for "disjoint **s**um") together with morphisms $\mathsf{inj}_1 \in \mathcal{C}(S, A)$ and $\mathsf{inj}_2 \in \mathcal{C}(S, B)$,*

$$A \xrightarrow{\mathsf{inj}_1} S \xleftarrow{\mathsf{inj}_2} B$$

with the following universal property*: Given any other diagram of this shape,*

$$A \xrightarrow{i_1} X \xleftarrow{i_2} B$$

there is a unique morphism $t \in \mathcal{C}(S, X)$ such that $t \circ \mathsf{inj}_1 = i_1$ and $t \circ \mathsf{inj}_2 = i_2$. One expresses these equations by saying that the following two triangles commute:

$$\begin{tikzcd}
A \arrow[r, "\mathsf{inj}_1"] \arrow[dr, "i_1"'] & S \arrow[d, dashed, "t"] & B \arrow[l, "\mathsf{inj}_2"'] \arrow[dl, "i_2"] \\
& X &
\end{tikzcd}$$