

# CATEGORY THEORY

## MIDLANDS GRADUATE SCHOOL 2023

### EXERCISE 2 (3 APRIL)

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#### REMINDER

**Definition 1** (product diagram). For  $A, B \in \mathcal{C}_0$ , a product diagram is an object  $P \in \mathcal{C}_0$  together with morphisms  $\pi_1 \in \mathcal{C}(P, A)$  and  $\pi_2 \in \mathcal{C}(P, B)$ ,

$$A \xleftarrow{\pi_1} P \xrightarrow{\pi_2} B$$

with the following universal property: Given any other diagram of this shape,

$$A \xleftarrow{p_1} X \xrightarrow{p_2} B$$

there is a unique morphism  $t \in \mathcal{C}(X, P)$  such that  $p_1 = \pi_1 \circ t$  and  $p_2 = \pi_2 \circ t$ . One expresses these equations by saying that the following two triangles commute:

$$\begin{array}{ccccc} & & X & & \\ & p_1 \swarrow & \vdots t \searrow & p_2 \swarrow & \\ A & \xleftarrow{\pi_1} & P & \xrightarrow{\pi_2} & B \end{array}$$

#### EXERCISE 3: COPRODUCTS

Using the definition of a product diagram given in the reminder above, re-construct the definition of a coproduct diagram by “reversing all arrows”. Check that, in the category **SET**, your definition describes the disjoint union of two sets. What objects are characterised by your definition in the other categories discussed in the lecture?

#### EXERCISE 4: INTERACTION OF PRODUCTS, COPRODUCTS, INITIAL AND TERMINAL OBJECTS

Assume that the category  $\mathcal{C}$  has a terminal object, for which we write  $1$ , and an initial object, for which we write  $0$ . Moreover, assume that  $\mathcal{C}$  has all products and coproducts. This means that, for every pair of objects  $A, B \in \mathcal{C}_0$ , we have a product diagram and a coproduct diagram. We write  $A \times B$  for the “middle object” of the product diagram, and  $A + B$  for the “middle object” of the coproduct diagram.

Recall that we write  $X \cong Y$  if there is an isomorphism in  $\mathcal{C}(X, Y)$ . For each of the following isomorphisms, either prove them in  $\mathcal{C}$  or find a category  $\mathcal{C}$  in which they don’t hold:

- a.  $A \times 1 \cong A$
- b.  $A \times 0 \cong A$
- c.  $A \times 0 \cong 0$
- d.  $A + 0 \cong A$
- e.  $A \times B \cong B \times A$
- f.  $A \times (B + C) \cong (A \times B) + (A \times C)$

**g.** Can you write down an isomorphism that *never* holds, i.e., that doesn't hold in *any* category  $\mathcal{C}$ ?