CATEGORY THEORY
MIDLANDS GRADUATE SCHOOL 2023

EXERCISE 3&4 (4 AND 5 APRIL)

NICOLAI KRAUS

Reminder

Definition 1 (functor)**.** *Given categories $\mathcal{C}$ and $\mathcal{D}$, a* functor $F : \mathcal{C} \to \mathcal{D}$ *consists of:*

- *a function between the object parts, $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$; however, the index is often omitted and one only writes FX for $F_0(X)$.*
- *for any two objects $X, Y \in \mathcal{C}_0$, a function $F_{X,Y}$ from $\mathcal{C}(X,Y)$ to $\mathcal{D}(FX, FY)$; again, one usually just writes Fg instead of $F_{X,Y}(g)$ for $g \in \mathcal{C}(X,Y)$*

such that:

- *Identities are preserves: $F(\mathsf{id}_X) = \mathsf{id}_{FX}$*
- *Composition is preserved: $F(g \circ f) = Fg \circ Ff$.*

Exercise 6: Functors preserving structure

By definition, a functor between categories preserves identities and compositions. What else does it preserve?

a. Show that every functor preserves isomorphisms. This means that, if $F : \mathcal{C} \to \mathcal{D}$ is a functor and $k \in \mathcal{C}(X,Y)$ is an isomorphism, then $Fk \in \mathcal{D}(FX, FY)$ is an isomorphism.

b. Assume we have a functor $F : \mathcal{C} \to C$ and an object $A \in \mathcal{C}_0$ such that, for every $X \in \mathcal{C}_0$, the object FX is a product of A and X. We can write $(A \times \_)$ as a more suggestive name for this functor F. Show that this functor preserves coproducts.

c. Find an example of a functor that preserves neither products nor coproducts.

Exercise 7 (continues Exercise 1 from Sunday): functors out of the free category on a directed multigraph

For the definitions, please see yesterday's exercise sheet.

Let $G = (V, E)$ be a directed multigraph and $\mathcal{D}$ be a category. Show that the collection of functors $\mathcal{F}_G \to \mathcal{D}$ is in bijection with the collection of pairs (s, t), where $s : V \to \mathcal{D}_0$ is a function and t chooses, for each pair $a, b \in V$ and each edge $e \in E(a, b)$, a morphism in $\mathcal{D}(s(a), s(b))$.

Exercise 8: The category CAT

The goal of this exercise is to construct CAT, the "category of all categories". The objects of CAT are categories.[1] The morphisms between $\mathcal{C}$ and $\mathcal{D}$ are simply the functors from $\mathcal{C}$ to $\mathcal{D}$. Construct the remaining structure and prove the laws required to make CAT a category.

Bonus exercise: If you already know what a *natural transformation* and a *2-category* is, show that CAT is a 2-category.

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[1] Note that CAT cannot be an object of CAT, which would lead to Russel's paradox. A "smallness" condition is needed to avoid this. One usually requires that the objects of CAT are categories that have sets of objects, while the objects of CAT itself form a proper class. In type theory, this would correspond to saying that the objects of CAT live in the first universe, while CAT itself lives in the second.
For the purpose of this exercise, you can safely ignore this issue.