Counting on Thorsten: Representing Ordinals in Homotopy Type Theory

Fredrik Nordvall Forsberg

joint work with **Nicolai Kraus** and **Chuangjie Xu**

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Working with Thorsten

A Categorical Semantics for Inductive-Inductive Definitions

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Abstract. Induction-induction is a principle for defining data types in Martin-Löf Type Theory. An inductive-inductive definition consists of a set A, together with an A-indexed family $B : A \to \mathrm{Set}$, where both A and B are inductively defined in such a way that the constructors for A can refer to B and vice versa. In addition, the constructors for B can refer to the constructors for A. We extend the usual initial algebra semantics for ordinary inductive data types to the inductive-inductive setting by considering dialgebras instead of ordinary algebras. This gives a new and compact formalisation of inductive-inductive definitions, which we prove is equivalent to the usual formulation with elimination rules.

1 Introduction

Induction is an important principle of definition and reasoning, especially so in constructive mathematics and computer science, where the concept of inductively defined set and data type coincide. There are two well-established approaches to model the semantics of such data types: in Martin-Löf Type Theory [14], each set A comes equipped with an eliminator which at the same time represents reasoning by induction over A and the definition of recursive functions out of A. A more categorical approach [10] models data types as initial T-algebras for a suitable endofunctor T.

At first, it would seem that the eliminator approach is stronger, as it allows us to define dependent functions $(x : A) \to P(x)$, in contrast with the non-dependent arrows $A \to B$ given by the initiality of the algebra. However, Hermida and Jacobs [12] showed that an eliminator can be defined for every initial T-algebra, where T is a polynomial functor. Ghani et. al. [9] then extended this to arbitrary endofunctors. This covers many forms of induction and data type definitions such as indexed inductive definitions [5] and induction-recursion [7] (Dybjer and Setzer [8] also give a direct proof for induction-recursion).

There are, however, other meaningful forms of data types which are not covered by these results. One such example are inductive-inductive definitions [16], where a set A and a function $B : A \to \mathrm{Set}$ are simultaneously inductively defined (compare

* Supported by EPSRC grant EP/G033374/1.
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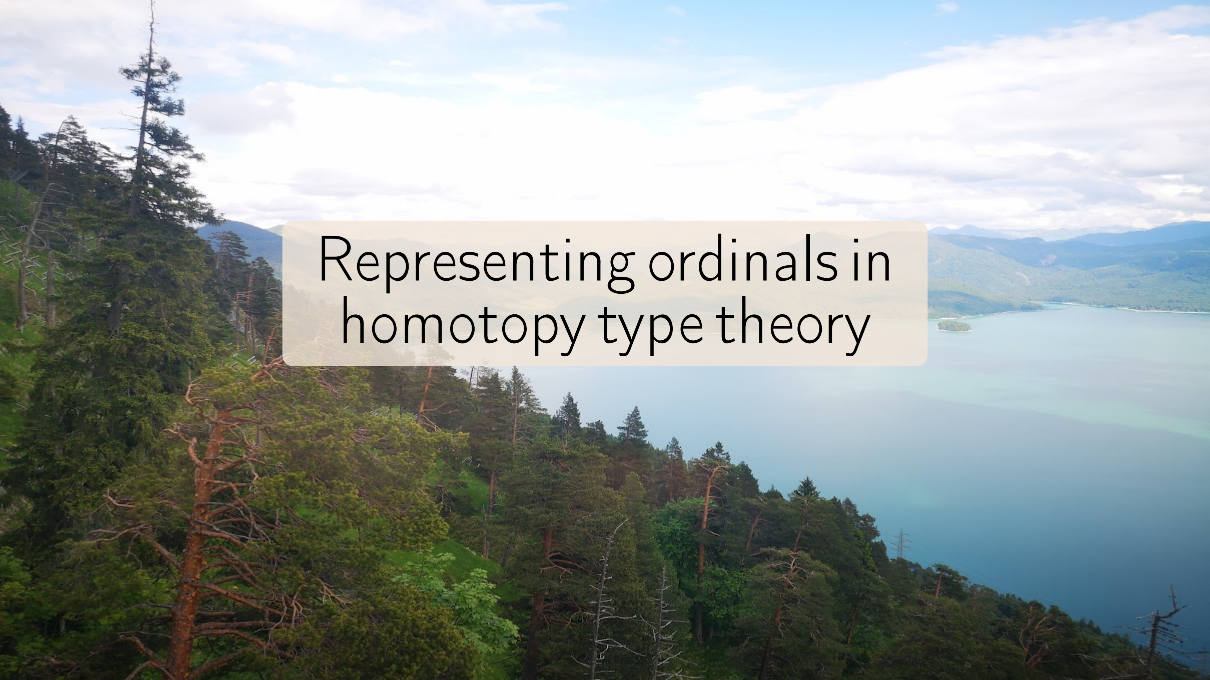
Representing ordinals in
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Ordinals in computer science

Bold claim: ordinals are one of the most important structures in logic and computer science.

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Unintuitive but amazing that classically all these uses can be reduced to the set-theoretic axioms for ordinals.

Set-theoretical ordinals

In classical set theory, an ordinal is a set with an order $\prec$ which is

- **transitive**: $(a \prec b) \to (b \prec c) \to (a \prec c)$
- **wellfounded**: every sequence $a_0 \succ a_1 \succ a_2 \succ a_3 \succ \ldots$ terminates
- **trichotomous**: $(a \prec b) \vee (a = b) \vee (b \prec a)$

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This definition can be adopted in type theory, but it is not the only candidate!

Different notions of ordinals in type theory

$$
\begin{tikzcd}
\text{decidable} & \text{Cantor normal forms} \arrow[d, hook] \\
\text{partially decidable} & \text{Brouwer trees} \arrow[d, hook] \\
\text{undecidable} & \text{Wellfounded, extensional, and transitive orders}
\end{tikzcd}
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N. Kraus, F. N-F., and C. Xu.
Connecting constructive notions of ordinals in homotopy type theory
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Type-Theoretic Approaches to Ordinals
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Cantor Normal Forms as an inductive-inductive definition

Motivation: $\alpha = \omega^{\beta_1} + \omega^{\beta_2} + \cdots + \omega^{\beta_n}$ with $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$

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Let Cnf be the type of *hereditary descending lists*:

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\omega^{-}+- &: (a : \mathsf{Cnf}) \to (b : \mathsf{Cnf}) \to \{p : a \geq \mathsf{fst}\, b\} \to \mathsf{Cnf}
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where $\mathsf{fst}\,(\omega^a + b) = a$.

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Equivalent implementations: (i) Sigma type of normal forms, and (ii) finite hereditary multisets (as a quotient inductive type).

Brouwer ordinal trees in constructive type theory

Inductive type $\mathcal{B}$ of Brouwer trees [Brouwer 1926; Martin-Löf 1970]:

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&\mathsf{data}\ \mathcal{B}\ \mathsf{where}\\
&\quad \mathsf{zero} : \mathcal{B}\\
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Our solution: induction-induction and path constructors, ensuring:

- Limits can only be taken of strictly increasing sequences;
- Bisimilar sequences have equal limits.

A refined type of Brouwer tree ordinals

```
data Brw where
  zero  : Brw
  succ  : Brw → Brw
  limit : (f : ℕ → Brw) → {f↑ : increasing f} → Brw
  bisim : ∀ f {f↑} g {g↑} →
          f ≈ g →
          limit f {f↑} ≡ limit g {g↑}
  trunc : isSet Brw

data _≤_ where
  ≤-zero      : ∀ {x} → zero ≤ x
  ≤-trans     : ∀ {x y z} → x ≤ y → y ≤ z → x ≤ z
  ≤-succ-mono : ∀ {x y} → x ≤ y → succ x ≤ succ y
  ≤-cocone    : ∀ {x} f {f↑ k} → (x ≤ f k) → (x ≤ limit f {f↑})
  ≤-limiting  : ∀ f {f↑ x} → ((k : ℕ) → f k ≤ x) → limit f {f↑} ≤ x
  ≤-trunc     : ∀ {x y} → isProp (x ≤ y)
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note: $x < y$
means succ $x \leq y$

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Implemented in Cubical Agda (along with everything else).

Transitive, wellfounded, and extensional orders as a large type

The type Ord consists of pairs $(X : \mathcal{U}, \prec : X \to X \to \mathsf{Prop})$ such that $\prec$ is transitive, extensional, wellfounded [HoTT book].

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Theorem (Escardo [2022])

The type Ord has a non-trivial decidable property if and only if weak excluded middle $\neg P \uplus \neg\neg P$ holds.

If it quacks like an ordinal

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Theorem

For each notion of ordinals $(A, <, \leq)$ considered:

- A *is a set and* $<$ *and* $\leq$ *are* Prop*-valued.*
- $<$ *is transitive, extensional and wellfounded.*
- $x < y \to x \leq y$, *and* $x < y \leq z \to x < z$ *(note:* $x \leq y < z \to x < z$ *equivalent to* LEM *for Ord).*
- *Can define* $+$, $\times$ *and prove they uniquely satisfy relational specification (for Cnf, Brw also exponentiation).*
- *Can compute limits of Brw and Ord, but not Cnf (implies* WLPO*).*
- *Cnf has decidable equality and* $<$*, whereas Brw and Ord do not (this is equivalent to* LPO *and* LEM *respectively).*
- *There are order-preserving embeddings Cnf* $\hookrightarrow$ *Brw* $\hookrightarrow$ *Ord.*
- ...

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Summary

- All notions support transfinite induction;
- All notions support ordinal arithmetic (in different ways!);
- Cnf: decidable $\leftrightarrow$ True; Brw: decidable $\leftrightarrow$ LPO; Ord: decidable $\leftrightarrow$ LEM.

- *Can compute limits of Brw and Ord, but not Cnf (implies* WLPO*).*
- *Cnf has decidable equality and $<$, whereas Brw and Ord do not (this is equivalent to* LPO *and* LEM *respectively).*
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One more notion of ordinal

Another notion of ordinal

Thorsten suggested the following variation coOrd:

Definition

- An ordinal is given by a colist of ordinals.

- $\alpha \leq \beta$ if $a_i < \beta$ for all $a_i \in \alpha$.
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Intuition: $[a_1, a_2, \ldots]$ is supremum of the successors of all a_i.

Similar to ideas by Coquand, Lombardi, and Neuwirth [2022].

Some simple ordinals

0 is represented by the empty colist $[]$.

The successor of α is represented by the colist $[\alpha]$.

The limit of $f : \mathbb{N} \to \mathsf{coOrd}$ is represented by the infinite colist $[f(0), f(1), \ldots]$.

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Proposition: There is an embedding $\alpha \mapsto \big(\Sigma\beta : \mathsf{coOrd}\big)\,(\beta < \alpha) : \mathsf{coOrd} \hookrightarrow \mathsf{Ord}$.

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$$\alpha + [] = \alpha$$
$$\alpha + (b_1 :: \beta) = (\alpha + b_1) :: (\alpha + \beta)$$

$<$ and $\leq$ should have expected properties (e.g. $<$ wellfounded, extensional, transitive, $\alpha < \beta \to \beta \leq \gamma \to \alpha < \gamma$, etc).

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However in terms of raw power: Classically each colist is either finite or infinite. Hence classically coOrd $\hookrightarrow$ Brw, i.e. coOrd $\cong$ Brw is consistent.

So our interest in coOrd should be constructive. It might be in good case study in combining coinduction and quotient inductive types?

Conclusions

There are many ways to represent ordinals in type theory; concepts from homotopy type theory help.

Meaningful distinctions deserve to be preserved: use the right representation for the job.

New representation of ordinals based on colists.

Happy birthday Thorsten!