CATEGORY THEORY
MIDLANDS GRADUATE SCHOOL 2023

EXERCISE 3&4 (4 AND 5 APRIL)

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Reminder

Definition 1 (functor)**.** *Given categories $\mathcal{C}$ and $\mathcal{D}$, a* functor $F : \mathcal{C} \to \mathcal{D}$ *consists of:*

- *a function between the object parts, $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$; however, the index is often omitted and one only writes FX for $F_0(X)$.*
- *for any two objects $X, Y \in \mathcal{C}_0$, a function $F_{X,Y}$ from $\mathcal{C}(X,Y)$ to $\mathcal{D}(FX, FY)$; again, one usually just writes Fg instead of $F_{X,Y}(g)$ for $g \in \mathcal{C}(X,Y)$.*

such that:

- *Identities are preserved: for all $X \in \mathcal{C}_0$, we have $F(\mathsf{id}_X) = \mathsf{id}_{FX}$*
- *Composition is preserved: for objects $X, Y, Z \in \mathcal{C}_0$ and morphisms $f \in \mathcal{C}(X,Y), g \in \mathcal{C}(Y,Z)$, we have: $F(g \circ f) = Fg \circ Ff$. Note that the first composition is composition in $\mathcal{C}$, while the second composition is composition in $\mathcal{D}$.*

Definition 2 (isomorphism)**.** *Given a category $\mathcal{C}$ and objects $X, Y \in \mathcal{C}_0$, a morphism $f \in \mathcal{C}(X,Y)$ is an isomorphism if there is a $g \in \mathcal{C}(Y,X)$ such that $g \circ f = \mathsf{id}_X$ and $f \circ g = \mathsf{id}_Y$.*

Exercise 6: Functors preserving structure

By definition, a functor between categories preserves identities and compositions. What else does it preserve?

a. Show that every functor preserves isomorphisms. This means that, if $F : \mathcal{C} \to \mathcal{D}$ is a functor and $k \in \mathcal{C}(X,Y)$ is an isomorphism, then $Fk \in \mathcal{D}(FX, FY)$ is an isomorphism.

b. Find an example of categories $\mathcal{C}$ and $\mathcal{D}$ an a functor $F : \mathcal{C} \to \mathcal{D}$ such that $\mathcal{C}$ has an initial object $\mathbf{0} \in \mathcal{C}_0$ and a terminal object $\mathbf{1} \in \mathcal{C}_0$, but such that $F\mathbf{0}$ is not initial in $\mathcal{D}$ and $F\mathbf{1}$ is not terminal in $\mathcal{D}$.

c. Construct the functor $\mathsf{List} : \mathsf{SET} \to \mathsf{SET}$. Check that the functor laws are satisfied.

d. Does the functor $\mathsf{List} : \mathsf{SET} \to \mathsf{SET}$ from the previous question preserve product diagrams? In more detail, the question is the following.

If

$$A \xleftarrow{\quad \pi_1 \quad} A \times B \xrightarrow{\quad \pi_2 \quad} B$$

is a product diagram in SET, it follows from the definition of a functor that we can define the objects and morphisms in

$$\mathsf{List}(A) \xleftarrow{\quad \mathsf{List}(\pi_1) \quad} \mathsf{List}(A \times B) \xrightarrow{\quad \mathsf{List}(\pi_2) \quad} \mathsf{List}(B)$$

How exactly do $\mathsf{List}(\pi_1)$ and $\mathsf{List}(\pi_2)$ work? Is the above a product diagram in SET?

e. Given a set S, define the product functor $(S \times \_) : \mathsf{SET} \to \mathsf{SET}$.

f. Does the product functor $(S \times \_)$ of part (e) above preserve coproduct diagrams?

Exercise 7: The category CAT

The goal of this exercise is to construct CAT, the "category of all categories". The objects of CAT are categories.[1] The morphisms between $\mathcal{C}$ and $\mathcal{D}$ are simply the functors from $\mathcal{C}$ to $\mathcal{D}$. Construct the remaining structure and prove the laws required to make CAT a category. (Is there an issue? We previously said that we don't talk about equality of objects. Do you need to do that here?)

Bonus exercise: If you already know what a *natural transformation* and a *bicategory* is, show that CAT is a bicategory.

Exercise 8 (continues Exercise 1 from Sunday): functors out of the free category on a directed multigraph

For the definitions, please see Sunday's exercise sheet.

a. Define a category of GRAPH directed multigraphs. Objects should be directed multigraphs. Given two such directed multigraphs (V_1, E_1) and (V_2, E_2), a morphisms between them should be a function $f : V_1 \to V_2$ together with, for every $a, b \in V_1$, a function $f_{a,b} : E_1(a, b) \to E_2(f\, a, f\, b)$. You thus need to define composition, identities, and check that the category laws hold.

b. Given a category, define a directed multigraph by forgetting some of the structure. Can you make a functor $\mathcal{U} : \mathsf{CAT} \to \mathsf{GRAPH}$ out of this? (Note: This is an example of a so-called *forgetful functor.*)

c. Let $G = (V, E)$ be a directed multigraph and $\mathcal{D}$ be a category. In Exercise 1 (Sunday), we have constructed a category $\mathcal{F}_G$. Show that the collection of functors $\mathcal{F}_G \to \mathcal{D}$ is in bijection with the collection of pairs (p, q), where $p : V \to \mathcal{D}_0$ is a function and q chooses, for each pair $a, b \in V$ and each edge $e \in E(a, b)$, a morphism in $\mathcal{D}(p(a), q(b))$.

Use this to construct a functor $\mathcal{F} : \mathsf{GRAPH} \to \mathsf{CAT}$.

(Note: $\mathcal{F}$ is a "free construction". Such constructions occur often in category theory. What you have proved above shows that $\mathcal{F}$ is a *left adjoint* to U, written $\mathcal{F} \dashv \mathcal{U}$, which implies that $\mathcal{U} \circ \mathcal{F}$ is a monad – a concept that you may be familiar with from functional programming.)

[1] Note that CAT cannot be an object of CAT, which would lead to Russel's paradox. A "smallness" condition is needed to avoid this. One usually requires that the objects of CAT are categories that have sets of objects, while the objects of CAT itself form a proper class. In type theory, this would correspond to saying that the objects of CAT live in the first universe, while CAT itself lives in the second.

For the purpose of this exercise, you can safely ignore this issue.