When is a function a fold, or an unfold?

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or, The Discipline of Thorsten

1. Fold

Recall `foldr` in Haskell:

```
foldr :: (a->b->b) -> b -> [a] -> b
foldr f e []        = e
foldr f e (x:xs)    = f x (foldr f e xs)
```

Abstractly:

$$\mathit{fold} : \mathit{Functor}\,\mathsf{F} \Rightarrow (\mathsf{F}\,A \to A) \to \mu\mathsf{F} \to A$$
$$\mathit{fold}_{\mathsf{F}}\,f\,(\mathrm{in}_{\mathsf{F}}\,x) = f\,(\mathsf{F}\,(\mathit{fold}_{\mathsf{F}}\,f)\,x)$$

For example, with $\mathsf{L}\,A = 1 + \mathit{Nat} \times A$ for lists of naturals,

$$\mathit{sum} : \mu\mathsf{L} \to \mathit{Nat}$$
$$\mathit{sum} = \mathit{fold}_{\mathsf{L}}\,\mathit{add} \quad \textbf{where} \quad \begin{aligned} \mathit{add}\,(\mathit{Inl}\,()) &= 0 \\ \mathit{add}\,(\mathit{Inr}\,(m,n)) &= m + n \end{aligned}$$

2. When is a function a fold?

Universal property:

$$h = \mathit{fold}_{\mathsf{F}}\, f \quad\Leftrightarrow\quad h \circ \mathrm{in}_{\mathsf{F}} = f \circ \mathsf{F}\, h$$

for $h : \mu\mathsf{F} \to A$ and $f : \mathsf{F}\,A \to A$.

3. When is a function *not* a fold?

Consider $allEqual : \mu\mathsf{L} \to Bool$:

$$
\begin{aligned}
allEqual\,[\,1\,] &= True \\
allEqual\,[\,2\,] &= True \\
allEqual\,[\,1,1\,] &= True \\
allEqual\,[\,1,2\,] &= False
\end{aligned}
$$

3. When is a function *not* a fold?

Consider $allEqual : \mu\mathsf{L} \to Bool$:

$$\begin{aligned} allEqual\,[\,1\,] &= True \\ allEqual\,[\,2\,] &= True \\ allEqual\,[\,1,1\,] &= True \\ allEqual\,[\,1,2\,] &= False \end{aligned}$$

There is no $f : \mathsf{L}\,Bool \to Bool$ such that $allEqual = fold_{\mathsf{L}}\,f$.

4. Non-answers

- the universal property says it all

  but *intensionally*, not extensionally

- injections are folds

  extensional, but only an *implication* (consider *sum*)

Neither is very helpful at showing that a function is *not* a fold.

5. Totally

Definition. The *kernel* $\mathsf{ker}\, p$ of total function $p : X \to Y$ is equivalence relation

$$\mathsf{ker}\, p = \{\, (x, x') \mid x, x' \in X \ \wedge \ p\, x = p\, x' \,\}$$

Theorem (postfactors). Given $r : X \to Z$ and $p : X \to Y$ and $Y \to Z \neq \varnothing$,

$$(\exists q : Y \to Z \ \bullet \ r = q \circ p) \quad \Leftrightarrow \quad \mathsf{ker}\, r \supseteq \mathsf{ker}\, p$$

Proof. $\Rightarrow$: calculation. $\Leftarrow$: choose $q\, y = r\, x$ when $y = p\, x$, else arbitrary.

Corollary. If initial algebra $(\mu\mathsf{F}, \mathit{in}_{\mathsf{F}})$ exists, then for $h : \mu\mathsf{F} \to A$,

$$(\exists f \ \bullet \ h = \mathit{fold}_{\mathsf{F}}\, f) \quad \Leftrightarrow \quad \mathsf{ker}\, (h \circ \mathit{in}_{\mathsf{F}}) \supseteq \mathsf{ker}\, (\mathsf{F}\, h)$$

In particular, $((1, [\,1\,]), (1, [\,2\,])) \in \mathsf{ker}\, (\mathit{id} \times \mathit{allEqual}) - \mathsf{ker}\, (\mathit{allEqual} \circ \mathit{cons})$

6. Dually

Definition. The *image* $\mathsf{img}\, p$ of total function $p : X \to Y$ is

$$\mathsf{img}\, p = \{ y \in Y \mid \exists x \in X \bullet p\, x = y \}$$

(Why dual? equivalence $\mathsf{ker}\, p = p^{\circ} \circ p$, coreflexive $\mathsf{img}\, p = p \circ p^{\circ}$.)

Theorem (prefactors). Given $r : X \to Z$ and $q : Y \to Z$ and $X \to Y \neq \emptyset$,

$$(\exists p : X \to Y \bullet r = q \circ p) \quad \Leftrightarrow \quad \mathsf{img}\, r \subseteq \mathsf{img}\, q$$

Proof. $\Rightarrow$: calculation. $\Leftarrow$: case analysis on $X = \emptyset \vee Y \neq \emptyset$.

Corollary. If final coalgebra $(\nu\mathsf{F}, \mathrm{out}_{\mathsf{F}})$ exists, then for $h : A \to \nu\mathsf{F}$,

$$(\exists g \bullet h = \mathit{unfold}_{\mathsf{F}}\, g) \quad \Leftrightarrow \quad \mathsf{img}\,(\mathrm{out}_{\mathsf{F}} \circ h) \subseteq \mathsf{img}\,(\mathsf{F}\, h)$$

7. Partially

Definition. The kernel $\mathsf{ker}\,p$ of partial function $p : X \twoheadrightarrow Y$ is equivalence

$$\begin{aligned}\mathsf{ker}\,p = {} & \{\,(x, x') \mid x, x' \in \mathsf{dom}\,p \;\wedge\; p\,x = p\,x'\,\} \\ & \cup \{\,(x, x') \mid x, x' \in (X - \mathsf{dom}\,p)\,\}\end{aligned}$$

Theorem. For partial functions $r : X \twoheadrightarrow Z$ and $p : X \twoheadrightarrow Y$,

$$(\exists q : Y \twoheadrightarrow Z \;\bullet\; r = q \circ p) \quad \Leftrightarrow \quad \mathsf{ker}\,r \supseteq \mathsf{ker}\,p \wedge \mathsf{dom}\,r \subseteq \mathsf{dom}\,p$$

Proof. More awkward, because involving a case analysis.

8. Allegorically

Definition. Relation R is *simple* if $R \circ R^{\circ} \subseteq id$.

Definition. *Left division* operator $T \subseteq S \backslash R \;\Leftrightarrow\; S \circ T \subseteq R$.

Definition. The kernel $\mathsf{ker}\,R$ of a relation $R : X \sim Y$ is equivalence

$$\mathsf{ker}\,R = (R \backslash R) \cap (R \backslash R)^{\circ}$$

Remark. Concretely, $\mathsf{ker}\,R = \{\,(x, x') \mid \forall y \bullet (x, y) \in R \Leftrightarrow (x', y) \in R\}$.

Theorem. For simple $T : X \sim Z$ and $R : X \sim Y$,

$$(\exists \text{ simple } S : Y \sim Z \bullet T = S \circ R) \quad \Leftrightarrow \quad \mathsf{ker}\,R \subseteq \mathsf{ker}\,T \wedge \mathsf{dom}\,R \supseteq \mathsf{dom}\,T$$

Proof. Nice and straightforward, by calculation.

9. Categorically

Can that nice calculational proof be generalised?

Definition. The *kernel pair* of an arrow p is its pullback along itself:

$$
\begin{tikzcd}
 & & X \arrow[dr, "p"] & \\
W \arrow[urr, bend left] \arrow[drr, bend right] \arrow[r, dashed] & \ker f \arrow[ur] \arrow[dr] \arrow[r, phantom, "\lrcorner"] & & Y \\
 & & X \arrow[ur, "p"'] &
\end{tikzcd}
$$

Definition. A *partial map* $p : X \nrightarrow Y$ is a span with a monomorphic left leg:

$$
\begin{tikzcd}
 & \operatorname{dom} p \arrow[dl, hook'] \arrow[dr, "\overline{p}"] & \\
X & & Y
\end{tikzcd}
$$

10. Regularly

Definition. A *regular category* is:

- finitely complete (so has all kernel pairs), such that
- every kernel pair has a coequaliser (a *regular epimorphism*), and
- regular epimorphisms are stable under pullbacks.

Lemma. In a regular category, every arrow $p : X \to Y$ *factorises* into a regular epimorphism followed by a monomorphism.

$$\begin{tikzcd}
\mathrm{ker}\, p \arrow[r, shift left] \arrow[r, shift right] & X \arrow[rr, "p"] \arrow[dr, "e"'] & & Y \\
& & \mathrm{img}\, p \arrow[ur, hook, dashed, "m"'] &
\end{tikzcd}$$

11. Postfactors

Theorem (Sam Staton). Given arrows $p : X \to Y$ and $r : X \to Z$ in a regular category, there exists a partial map $q : Y \nrightarrow Z$ such that $r = q \circ p$ iff there exists an arrow $i : \ker p \to \ker r$ satisfying $u_0 = v_0 \circ i$ and $u_1 = v_1 \circ i$, where u_0, u_1 and v_0, v_1 are the kernel pairs of p and r respectively.

$$
\begin{tikzcd}
 & \ker p \arrow[dl, "i"'] \arrow[d, shift right, "u_0"'] \arrow[d, shift left, "u_1"] & \\
\ker r \arrow[r, shift left, "v_0"] \arrow[r, shift right, "v_1"'] & X \arrow[r, "r"] \arrow[d, "p"'] \arrow[dr, two heads, "e"] & Z \\
 & Y & \operatorname{img} f \arrow[l, hook', "m"] \arrow[u, dashed, "\overline{q}"']
\end{tikzcd}
$$

12. Discussion

A *relation* $R : X \sim Y$ is a jointly monic span $X \leftarrow R \rightarrow Y$.

Relational composition is defined in terms of *pullbacks of spans*.

Regular category is precisely the structure required for *associativity*.

Set is regular, so we get the story for total functions.

But any category of Eilenberg–Moore algebras over *Set* is also regular;
so covers *partial functions* too ($\mathcal{P}\mathit{fun} \simeq$ EM-algebras for *Maybe*).

What about other monads? Relations are Kleisli arrows for powerset.

Story here only for folds.
Unfolds not quite dual (probably don't want to dualise 'partial function').

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