

# Totally separated types

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Types, Thorsten & Theories

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Question by Altenkirch, Auberté & Li (2012)

Let  $X$  be any definable type in MLTT,

let  $x_0, x_1 : X$  be two probably distinct elements.

Question Can we find  $P : X \rightarrow 2$  with  $P x_0 = 0$  &  $P x_1 = 1$  ?

Answer (M-HÉ 2012) There is a type  $X$  with elements  $x_0 \neq x_1$ ,  
such that for any  $P : X \rightarrow 2$ , if  $P x_0 = 0$  &  $P x_1 = 1$ ,  
then WLPO holds.

(WLPO = every infinite binary sequence is constantly zero or not.)

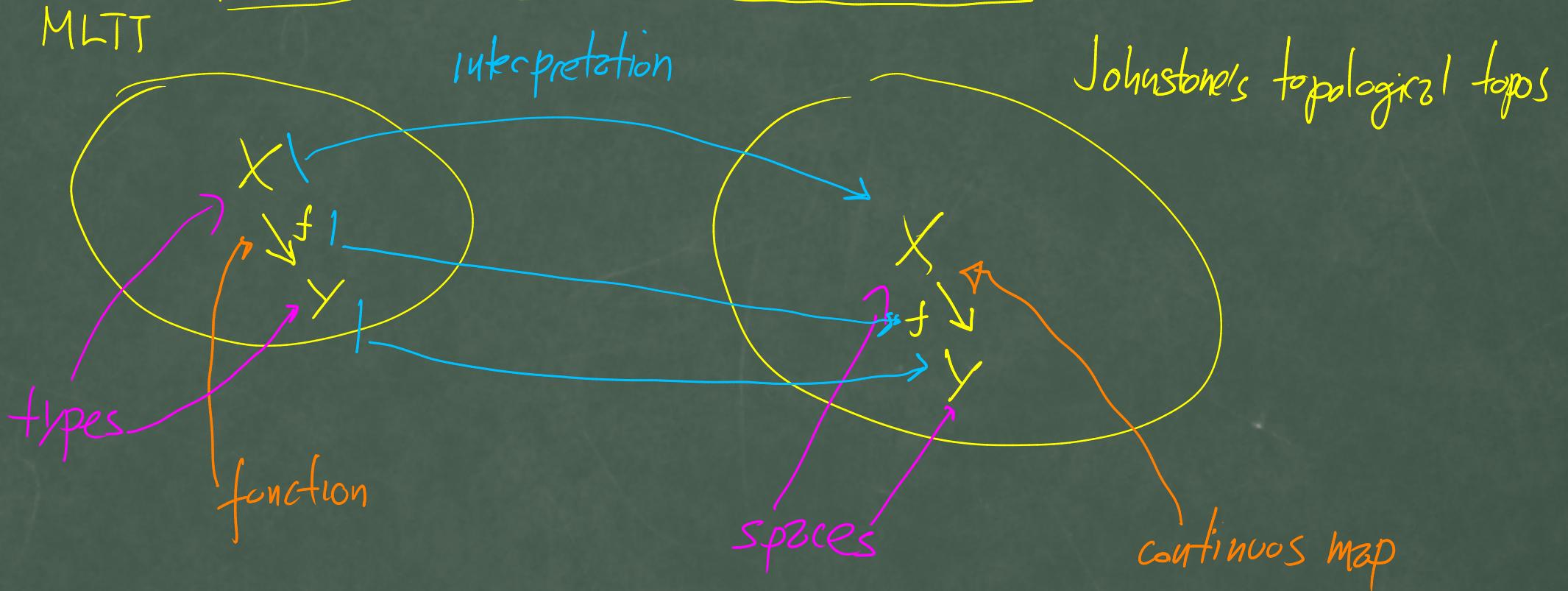
The inspiration for the answer comes from classical topology.

But our construction doesn't actually use topology.

1. clopen (closed and open) subsets of  $X$  correspond to continuous maps  $X \rightarrow 2$ .
2. Potentially, all maps are continuous in constructive mathematics.
3. WLPO is a discontinuity axiom (we'll come back to this).
4. A space is totally separated if the clopens separate the points.
5. There are plenty of spaces that fail to be totally separated.

→ Can we construct such spaces as types (with no extra structure)?

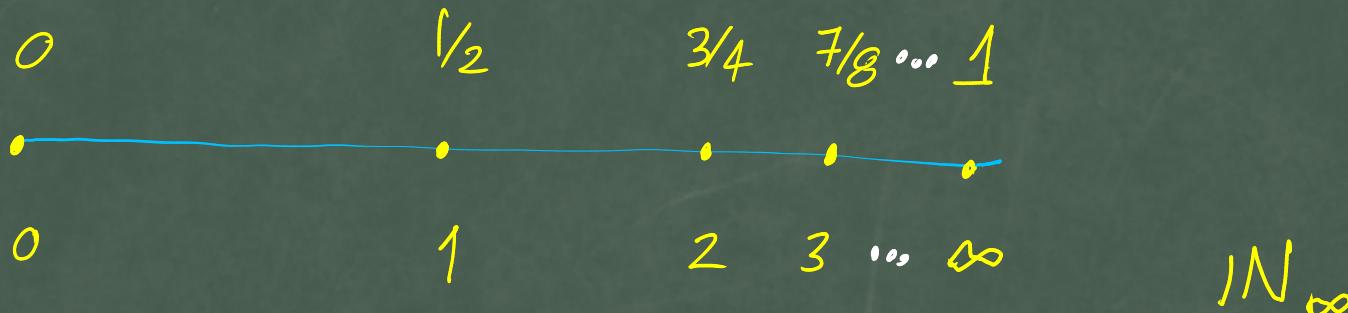
Have in mind something like this



But actually it is enough to just use the intuition of the above picture to construct and prove what we want.

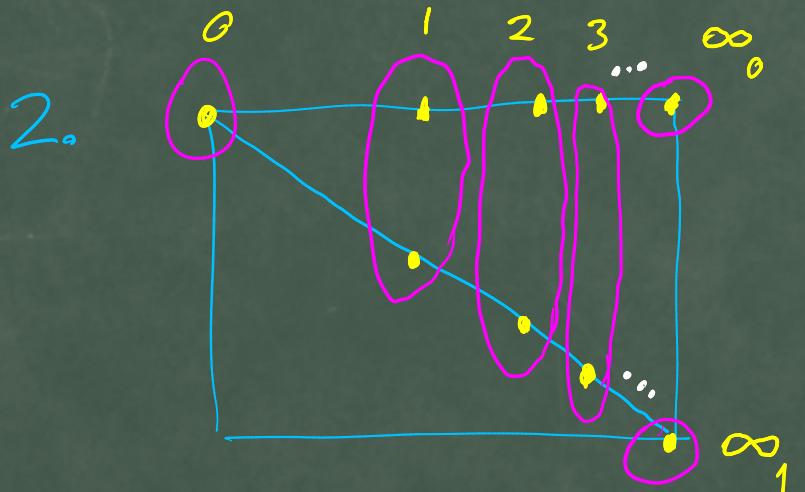
## | Two examples from topology |

1. One-point compactification of the discrete natural numbers.



This is compact  
& totally separated  
(Stone space)

A  $\mathbb{K}_2$  "generic convergent sequence".



Quotient by the equivalence relation.

$$\mathbb{N}_{\infty, \infty}$$

This is compact, but  
not Hausdorff & hence  
not totally separated.

# | Constructing the above two examples as types |

1.  $\mathbb{N}_\infty := \sum_{\alpha: \mathbb{N} \rightarrow 2} \prod_{i: \mathbb{N}} \alpha_i \geq \alpha_{i+1}$ . Decreasing binary sequences

$\sim$  constants

So  $1^n 0^\omega$  represents  $n: \mathbb{N}$ , (count how many 1's we have)  
 and  $1^\omega$  represents  $\infty$ .

2.  $\mathbb{N}_{\infty, \infty, 1} := \sum_{x: \mathbb{N}_\infty} x = \infty \rightarrow 2$

when  $x$  is  $\infty$ , this type has two elements-

identity type

we create two copies of  $\infty$ :  $\begin{cases} \lambda - .0 & \\ \lambda - .1 & \end{cases}$

when  $x$  is finite, this type has one element.

Theorem

Assuming function extensibility:

1. The elements  $\infty_0$  and  $\infty_1$  of  $\text{IN}_{\infty_0, \infty_1}$  are distinct.
2. For any  $p: \text{IN}_{\infty_0, \infty_1} \rightarrow 2$ , if  $p^{\infty_0} = 0$  and  $p^{\infty_1} = 1$  then WLPO holds.

What can we say if we don't assume funext? [Metz-theorem]

Answer  
to original  
question

1. The elements  $\infty_0$  and  $\infty_1$  are probably distinct.
2. There is no definable  $p: \text{IN}_{\infty_0, \infty_1} \rightarrow 2$  with  $p^{\infty_0} = 0$  &  $p^{\infty_1} = 1$ .

Because this cannot be defined with funext, and hence cannot be defined with fewer assumptions.

Totality separated types      T.F.A.E for any type  $X$ :

$$1. \quad \prod x, y : X, \underbrace{\left( \prod p : X \rightarrow 2, p_x = p_y \right)}_{x =_2 y} \rightarrow x = y$$

Boolean Leibniz principle

2. The apartness relation

$$\text{tight, i.e. } \neg(x \#_2 y) \rightarrow x = y.$$

3. The canonical map  $x \rightarrow ((X \rightarrow 2) \rightarrow 2)$  is an embedding.

$$x \mapsto \lambda p. p_x$$

4. The quasi component  $\sum y : X, x =_2 y$  of any  $x : X$  is a singleton.

## Properties and closure properties of T.S. types

1. T.S. types are sets in the sense of HoTT/UF.
2. T.S. types are  $\eta$ -separated:  $\eta(x=y) \rightarrow x=y$ .
3. T.S. types are closed under retracts.
4.  $\mathbb{1}, \mathbb{Z}, \text{Fin } n, \text{IN}$  are T.S.
5. If  $(X_i)_{i:I}$  is a family of T.S. types, then  $\prod_{i:I} X_i$  is T.S.  
(We don't require  $I$  to be T.S.)
6. So e.g.  $\text{IN} \rightarrow \mathbb{Z}, \text{IN} \rightarrow \text{IN}, (\text{IN} \rightarrow \text{IN}) \rightarrow \text{IN}$ , etc. are T.S.
7.  $\text{IN}_\infty$  is T.S. (theorem) but  $\text{IN}_{\infty, \infty}$  is not probably T.S. (meta-theorem).
8. So T.S. types are "not" closed under  $\Sigma$ .

cf d. Assuming `funext` & `proptrunc`:

Every type  $X$  has a T.S. reflection, given by the image  $\mathsf{TX}$  of the canonical map  $X \rightarrow ((X \rightarrow 2) \rightarrow 2)$ :

Any type  $X \xrightarrow{\gamma} \mathsf{TX}$  T.S. type

$f \downarrow \vdots \exists! \bar{f}$

$A$  T.S. type

I hope you enjoyed the answer to your question  
& the zoddends.

