# From Hedberg's Theorem to Truncation Elimination and Two-Level Type Theory

**Types and Topology**
**Martín Escardó's 60th birthday**

Nicolai Kraus

University of Nottingham

18 Dec 2025

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  Topic: “Higher Dimensional Type Theory”
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- Disclaimer: All/most research in this talk is over 10 years old.

# One of Martín's questions (from 2012)

Given $f : A \to B$ that is constant, i.e. $\mathsf{wconst}_f :\equiv \prod_{a_1, a_2 : A} f(a_1) = f(a_2)$, does it factor through $\|A\|$?

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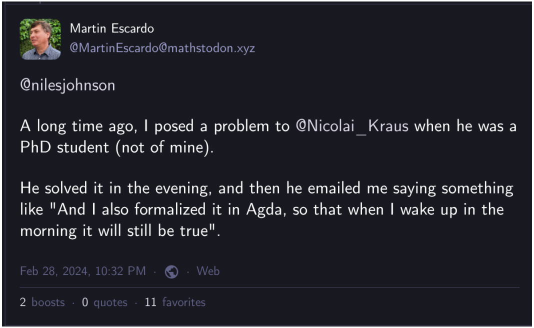


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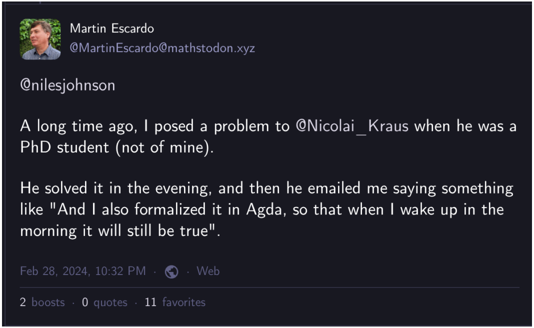
Martin Escardo
@MartinEscardo@mathstodon.xyz

@nilesjohnson

A long time ago, I posed a problem to @Nicolai_Kraus when he was a PhD student (not of mine).

He solved it in the evening, and then he emailed me saying something like "And I also formalized it in Agda, so that when I wake up in the morning it will still be true".

Feb 28, 2024, 10:32 PM · · Web

2 boosts · 0 quotes · 11 favorites

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Christian Sattler, some years later: Assume $(a_0, p_0) : \mathsf{fix}\, f$. Then $\mathsf{fix}\, f \simeq \Sigma(a : A). f\ a_0 = a$, which is contractible. $\square$

# Another case that works (Martín, 2012)

**Theorem.** If $B$ is a set, then the canonical map

$$(\|A\| \to B) \quad \longrightarrow \quad \Sigma\,(f : A \to B)\,.\,\mathsf{wconst}_f$$

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Unnecessarily complicated argument (not the original):

**Lemma 1.** Assume that $B$ is a set and $a : A$. Then,
$B \quad \longrightarrow \quad \Sigma\,(f : A \to B)\,.\,\mathsf{wconst}_f$
is an equivalence. $\square$

**Lemma 2.** If $X$ implies that $Y \xrightarrow{e} Z$ is an equivalence, then $(X \to Y) \xrightarrow{e\circ\_} (X \to Z)$ is an equivalence. $\square$

Proof of the theorem: In Lemma 1, weaken the assumption to $\|A\|$. Apply Lemma 2 with $X :\equiv \|A\|$ and observe that

$$(\|A\| \to \Sigma\,(f : A \to B)\,.\,\mathsf{wconst}_f)$$
$$\simeq (\Sigma\,(f : A \to B)\,.\,\mathsf{wconst}_f).$$ $\square$

# Generalizations (2014)

**Theorem.** If $B$ is a set, then the canonical map

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is an equivalence.

**Goal: Weaken the assumption on $B$.**

**Theorem.** If $B$ is a 1-type, then $\|A\| \to B$ is equivalent to the following data:

$$f : A \to B$$

$$c : \mathsf{wconst}_f :\equiv \Pi_{a_1, a_2 : A} \; f(a_1) = f(a_2)$$

$$d : \mathsf{coh}_{f,c} :\equiv \Pi_{a_1 a_2 a_3 : A} \; c(a_1, a_2) \cdot c(a_2, a_3) = c(a_1, a_3).$$

**Proof.** Show that $\mathfrak{a}_0 : A$ implies that

$$B \to \Sigma(f : A \to B) \,.\, \Sigma(c : \mathsf{wconst}_f) \,.\, \mathsf{coh}_{f,c}$$

is an equivalence, then same as above.

## Proof of $A \longrightarrow B \simeq \Sigma\,(f : A \to B)\,.\,\Sigma\,(c : \mathsf{wconst}_f)\,.\,\mathsf{coh}_{f,c}$

Assume $\mathfrak{a}_0 : A$. We transform $B$ into
$\Sigma\,(f : A \to B)\,.\,\Sigma\,(c : \mathsf{wconst}_f)\,.\,\mathsf{coh}_{f,c}$
by adding and removing contractible components:

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- Main result for HoTT: Fix $n \in \{0, 1, 2, \ldots\}$. If $B$ is an $n$-type, then $\|A\| \to B$ is equivalent to constant functions $A \to B$ with $n$ levels of coherence.

# Coherent constancy

A coherently constant function $A \to B$ is a map between the semisimplicial diagrams $\mathsf{cosk}_0(\mathsf{const}\, A)$ and $\mathsf{const}\, B$.

$$
\begin{tikzcd}
\cdots & & \cdots \\
A \times A \times A \arrow[rr, dashed, "\mathsf{coh}_{f,c}"] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & & \begin{array}{c} \Sigma\,(b_1, b_2, b_3 : B)\,. \\ \Sigma\,(p_{12} : b_1 = b_2)\,. \\ \Sigma\,(p_{23} : b_2 = b_3)\,. \\ \Sigma\,(p_{13} : b_1 = b_3)\,. \\ p_{12} \cdot p_{23} = p_{13} \end{array} \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\
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