CATEGORY THEORY
MIDLANDS GRADUATE SCHOOL 2023

EXERCISE 1 (2 APRIL)

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Reminder

Definition 1 (category)**.** *A* category $\mathcal{C}$ *consists of:*

- *a collection $\mathcal{C}_0$ of objects, written $X, Y, Z, \ldots$*
- *for any two objects X and Y, a collection $\mathcal{C}(X, Y)$ of morphisms, written $f, g, h, \ldots$*
- *a composition operation: for $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$, we have $g \circ f \in \mathcal{C}(X, Z)$*
- *for any object X, the identity morphism $\mathsf{id}_X \in \mathcal{C}(X, X)$*

such that:

- *Every identity morphism is left- and right-neutral. This means that, for $f \in \mathcal{C}(X, Y)$, we have $f \circ \mathsf{id}_X = f$ and $\mathsf{id}_Y \circ f = f$.*
- *Composition is associative, i.e. $(h \circ g) \circ f = h \circ (g \circ f)$.*

Definition 2 (initial and terminal object)**.** *An object X in a category $\mathcal{C}$ is* initial *if, for every object Y, there is exactly one morphism from X to Y. This means that $\mathcal{C}(X, Y)$ is the one-element set. An object Z is* terminal *if, for every object Y, there is exactly one morphism from Y to Z, i.e., $\mathcal{C}(Y, Z)$ is the one-element set.*

Definition 3 (isomorphism)**.** *A morphism $f \in \mathcal{C}(X, Y)$ is an* isomorphism *if there is a morphism $g \in \mathcal{C}(Y, X)$ such that $g \circ f = \mathsf{id}_X$ and $f \circ g = \mathsf{id}_Y$.*

Exercise 1: The free category on a directed multigraph

A *directed multigraph* (sometimes also called *quiver*) consists of:

- a set V of vertices
- for each pair (a, b) of vertices, a set $E(a, b)$ of edges from a to b.

Note that the edges are directed, i.e., $E(a, b)$ is not necessarily the same as $E(b, a)$, and many parallel edges are allowed.

Let $G = (V, E)$ be a directed multigraph. The *free category on* G, (here) written $\mathcal{F}_G$, has V as objects and a morphism $\mathcal{F}_G(X, Y)$ is a sequence of consecutive edges, starting in X and ending in Y.

a. Show that $\mathcal{F}_G$ is a category.

b. What are the isomorphisms in $\mathcal{F}_G$?

c. For which G does $\mathcal{F}_G$ have an initial object? And when does $\mathcal{F}_G$ have a terminal object?

Exercise 2: The category SPAN

The object of the category SPAN are sets. For sets X and Y, an element of $\mathsf{SPAN}(X, Y)$ consists of a set Z together with a function $f : Z \to X$ and a function $g : Z \to Y$.

a. Can you define a composition operation and prove that SPAN is a category?

b. What are the initial and terminal object of SPAN?

c. What are the isomorphisms?

d. How does SPAN compare to the category REL of sets and relations?