Indexed Categories with Families

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Types, Thorsten and Theories
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Indexed categories with families

Let $(\mathcal{C}, \mathcal{T})$ be a category with families (cwf). Then an *indexed cwf* is a functor

$$\mathrm{P} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cwf}$$

where $\mathbf{Cwf}$ is the category of cwfs with strict cwf-morphisms.

- Cf *indexed categories*

  $$\mathrm{P} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$$

  the basis for Lawvere's 1969 hyperdoctrines.
- Indexed cwfs are common in nature: they model predicate logic, system F, logic-enriched type theory, Makkai's 1995 FOLDS, type theory with universe-level judgments, cubical type theory, ...

Towards a uniform cwf-based approach to logic

- Cwfs as stepping stone between categorical logic and traditional approach to logic in terms of syntax and inference rules.
- Castellan, Clairambault, Dybjer 2021 (Lambek volume): *Categories with families: unityped, simply typed, dependently typed*. Uniform treatment of the categorical logic of untyped, simply typed, and dependently typed lambda calculi (type theory).
- Indexed cwfs (and simply typed cwfs) provide uniform cwf-based treatment of several other logical systems.
- *"Initiality conjecture"* approach to logical systems.
- Cwfs can be presented as models of a *generalized algebraic theory (gats)* in the sense of Cartmell 1978.
- Implementation in terms of Altenkirch and Kaposi's 2016 *quotient inductive-inductive types (qiits)*.

Ucwfs, scwfs, cwfs

All are based on a category of contexts $\mathcal{C}$ with a terminal object.

Ucwfs have only one type and one presheaf of terms

$$\mathrm{Tm} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$$

Scwfs have a set of types and one presheaf of terms

$$\mathrm{Tm}_A : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$$

for each type A.

Cwfs have a functor

$$\mathrm{T} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Fam}$$

or equivalently, two presheaves

$$\begin{array}{rcl}
\mathrm{Ty} & : & \mathcal{C}^{\mathrm{op}} \to \mathrm{Set} \\
\mathrm{Tm} & : & (\int^{\mathcal{C}} \mathrm{Ty})^{\mathrm{op}} \to \mathrm{Set}
\end{array}$$

Cf also Awodey's natural models.

Context comprehension

Ucwfs assign for each $n \in \mathcal{C}_0$ a representation $\mathrm{s}(n) \in \mathcal{C}_0$ of the presheaf

$$\mathcal{C}(-,n) \times \mathrm{Tm}(-) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$$

Scwfs assign for each $\Gamma \in \mathcal{C}_0$ and $A \in \mathrm{Ty}$ a representation $\Gamma.A \in \mathcal{C}_0$ of the presheaf

$$\mathcal{C}(-,\Gamma) \times \mathrm{Tm}_A(-) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$$

Cwfs assign for each $\Gamma \in \mathcal{C}_0$ and $A \in \mathrm{Ty}(\Gamma)$ a representation $\Gamma.A \in \mathcal{C}_0$ of the presheaf

$$\sum_{\gamma \in \mathcal{C}(-,\Gamma)} \mathrm{Tm}(-,A[\gamma]) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$$

Categories of cwfs

(Sub)categories of cwfs and strict cwf-morphisms

$$\mathbf{Ucwf} \subseteq \mathbf{Scwf} \subseteq \mathbf{Cwf}$$

Categories of cwfs with extra structure

Models of *Martin-Löf type theory* with $\Pi, \Sigma, +, 0, 1, \mathbb{N}, \mathrm{I}, \mathrm{U}$-types:

$$\mathbf{Cwf}^{\Pi,\Sigma,+,0,1,\mathbb{N},\mathrm{I},\mathrm{U}}$$

All structure is preserved strictly by cwf-morphisms. This yields a generalized algebraic presentation of Martin-Löf type theory. The formal system is an initial model.

Categories of cwfs with extra structure

- Models of generalized algebraic theories with presentation (sort symbols, operator symbols, and equations) in Σ:

  $$\mathbf{Cwf}_\Sigma$$

  The formal system generated by Σ is *defined abstractly* as an initial object $\mathcal{T}_\Sigma$.
- If Σ is the generalized algebraic theory of cwfs, then an object of $\mathbf{Cwf}_\Sigma$ is a cwf with an *internal cwf*.
- See BCDE (Bezem, Coquand, Dybjer, Escardó) 2021 (Hofmann volume), *On generalized algebraic theories and categories with families*.

Quotient inductive-inductive types

What is the relationship between qiits and gats? Are they the same?

Indexed categories with families occurring in nature

By varying the index cwf (ucwf, scwf, cwf, adding extra structure) $(\mathcal{C}, \mathcal{T})$ and the target cwfs (with extra structure) we capture several logical systems occurring in nature. For example,

- If $(\mathbf{L}, \mathrm{LTm})$ is a ucwf of universe levels (forming a semi-lattice with an inflationary endomorphism), then

  $$\mathcal{P} : \mathbf{L}^{\mathrm{op}} \to \mathbf{Cwf}^{\Pi,\Sigma,+,0,1,\mathbb{N},\mathrm{I},\mathrm{U}_l}$$

  models *type theory with universe levels judgments*, following BCDE 2022.
- If $(\mathcal{C}, \mathrm{Tm})$ is a ucwf of terms, then

  $$\mathcal{P} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Scwf}^{\to,\times,+,0,1}$$

  models *predicate logic*, provided $\forall$ and $\exists$ are also supported.

Happy birthday, Thorsten!

Predicate logic

This is modelled by **ucwf-indexed scwfs** $(\mathcal{C}, \mathrm{Tm}, \mathcal{P})$ with extra structure for the logical constants.

- $(\mathcal{C}, \mathrm{Tm})$ is a ucwf;
- $\mathcal{P} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Scwf}^{+}$ is a functor into the category of scwfs (with extra structure $^{+}$ for the logical connectives) and strict scwf-morphisms.
  - $\mathcal{P}(n)$ is the scwf of propositions and proofs in n term variables.
  - If $\gamma \in \mathcal{C}(n, m)$, then $\mathcal{P}(\gamma) : \mathcal{P}(m) \to \mathcal{P}(n)$ is the strict scwf-morphism which applies the substitution γ to the different components of the scwf $\mathcal{P}(m)$.
- There is extra structure for the quantifiers.

These are similar to Lawvere's **hyperdoctrines** (which are based on indexed categories), but closer to the usual formal systems of predicate logic. Note that they yield a generalized algebraic theory for predicate logic.

Type theory with universe level judgments

This is modelled by **ucwf-indexed scwfs** $(\mathcal{C}, \mathrm{Tm}, \mathcal{P})$ with extra structure for the typ formers of the type theory.

- $(\mathcal{C}, \mathrm{Tm})$ is a ucwf with extra structure for the universe level operations $(-)^{+}$ and $\vee$ making it a semi-lattice with an inflationary endomorphism ;
- $\mathcal{P} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cwf}^{+}$ is a functor into the category of cwfs (with extra structure $^{+}$ for the type formers) and strict cwf-morphisms.
  - $\mathcal{P}(n)$ is the cwf of types and terms in n universe level variables.
  - If $\gamma \in \mathcal{C}(n, m)$, then $\mathcal{P}(\gamma) : \mathcal{P}(m) \to \mathcal{P}(n)$ is the strict cwf-morphism which applies the level substitution γ to the different components of the cwf $\mathcal{P}(m)$.
- We may add extra structure for level-indexed products (analogous to the universal quantifier in predicate logic).

Universe level variables and judgments

We introduce new judgment forms

$$l \text{ level} \qquad l = l'$$

Contexts can contain level variables

$$\alpha \text{ level}$$

Some rules:

$$\frac{}{\Gamma \vdash \alpha \text{ level}}(\alpha\ in\ \Gamma) \qquad \frac{l \text{ level}}{l^+ \text{ level}} \qquad \frac{l \text{ level} \qquad l' \text{ level}}{l \vee l' \text{ level}}$$

No 0 level!

Level-indexed universes

Formation rules

$$\frac{l\ \text{level}}{\mathrm{U}_l\ \text{type}} \qquad \frac{A : \mathrm{U}_l}{\mathrm{T}_l(A)\ \text{type}}$$

Introduction rules

$$\frac{A : \mathrm{U}_l \qquad B : \mathrm{T}_l(A) \to \mathrm{U}_{l'}}{\Pi^{l,l'} A B : \mathrm{U}_{l \vee l'}} \qquad \cdots \qquad \mathrm{U}^l : \mathrm{U}_{l^+}$$

Conversion rules

$$\begin{array}{rcl} \mathrm{T}_{l \vee l'}\ (\Pi^{l,l'} A B) & = & \Pi(x : \mathrm{T}_l(A))\mathrm{T}_{l'}(B\ x) \\ & \vdots & \\ \mathrm{T}_{l^+}(\mathrm{U}^l) & = & \mathrm{U}_l \end{array}$$

The ucwf of universe levels: sort and operator symbols

Sort symbols:

$$
\begin{array}{rcl}
 & \vdash & \mathrm{lctx} \\
m,n : \mathrm{lctx} & \vdash & \mathrm{lhom}(m,n) \\
m : \mathrm{lctx} & \vdash & \mathrm{ltm}(m)
\end{array}
$$

Operator symbols:

$$
\begin{array}{rcl}
m : \mathrm{lctx} & \vdash & \mathrm{lid}_m : \mathrm{lhom}(m,m) \\
m,n,p : \mathrm{lctx}, \gamma : \mathrm{lhom}(n,p), \delta : \mathrm{lhom}(m,n) & \vdash & \gamma \circ \delta : \mathrm{lhom}(m,p) \\
\\
m,n : \mathrm{lctx}, \gamma : \mathrm{lhom}(n,m), l : \mathrm{ltm}(m,A) & \vdash & l[\gamma] : \mathrm{ltm}(n) \\
\\
 & \vdash & 0 : \mathrm{lctx} \\
m : \mathrm{lctx} & \vdash & \langle\rangle_m : \mathrm{lhom}(m,0) \\
\\
m : \mathrm{lctx} & \vdash & \mathrm{s}(m) : \mathrm{lctx} \\
m,n : \mathrm{ctx}, \gamma : \mathrm{lhom}(n,m), l : \mathrm{ltm}(n) & \vdash & \langle \gamma, l \rangle : \mathrm{lhom}(n, \mathrm{s}(m)) \\
m : \mathrm{ctx} & \vdash & \mathrm{p} : \mathrm{Hom}(\mathrm{s}(m), m) \\
m : \mathrm{ctx} & \vdash & \mathrm{q} : \mathrm{tm}(\mathrm{s}(m)) \\
\\
m : \mathrm{lctx}, l : \mathrm{ltm}(m) & \vdash & l^+ : \mathrm{ltm}(m) \\
m : \mathrm{lctx}, l, l' : \mathrm{ltm}(m) & \vdash & l \vee l' : \mathrm{ltm}(m)
\end{array}
$$

The ucwf of universe levels: equations

All ucwf-equations.

$$\vdots$$

Semi-lattice equations for $l \vee l'$

$$\begin{aligned} (l \vee l') \vee l'' &= l \vee (l' \vee l'') \\ l \vee l' &= l' \vee l \\ l \vee l &= l \end{aligned}$$

Equations for l^+.

$$\begin{aligned} l \vee l^+ &= l^+ \\ (l \vee l')^+ &= l^+ \vee l'^+ \end{aligned}$$

Commutativity of $\vee$ and $+$ with level substitution.

$$\vdots$$

Universe-level indexed cwfs

- Object part of the functor $\mathbf{L}^{\mathrm{op}} \to \mathbf{Cwf}^{\Pi,\mathrm{U}_l}$
  - Level-indexed contexts and context-morphisms
  - Level-indexed types and terms
- Arrow part of the functor $\mathbf{L}^{\mathrm{op}} \to \mathbf{Cwf}^{\Pi,\mathrm{U}_l}$
  - Level-substitution in contexts and context-morphisms
  - Level-substitution in types and terms

Level-indexed categories of contexts

There is a category above each level-context n:
Sort symbols:

$$\begin{array}{rcl} n : \mathrm{lctx} & \vdash & \mathrm{ctx}_n \\ n : \mathrm{lctx}, \Delta, \Gamma : \mathrm{ctx}_n & \vdash & \mathrm{Hom}_n(\Delta, \Gamma) \end{array}$$

Operator symbols:

$$\begin{array}{rcl} n : \mathrm{lctx}, \Gamma : \mathrm{ctx}_n & \vdash & \mathrm{id}_{n,\Gamma} : \mathrm{Hom}_n(\Gamma, \Gamma) \\ n : \mathrm{lctx}, \Xi, \Delta, \Gamma : \mathrm{ctx}_n, \gamma : \mathrm{Hom}_n(\Delta, \Gamma), \delta : \mathrm{Hom}_n(\Xi, \Delta) & \vdash & \gamma \circ \delta : \mathrm{Hom}_n(\Xi, \Gamma) \end{array}$$

Equations:

$$\begin{array}{rcl} \mathrm{id}_{n,\Gamma} \circ \gamma & = & \gamma \\ \gamma \circ \mathrm{id}_{n,\Delta} & = & \gamma \\ (\gamma \circ \delta) \circ \xi & = & \gamma \circ (\delta \circ \xi) \end{array}$$

Note: officially $\circ$ has six arguments rather than two.

Level substitution in contexts and context-morphisms

Operator symbols (overloaded notation):

$$
\begin{array}{rcl}
n, n' : \mathrm{lctx}, \sigma : \mathrm{lhom}(n, n'), \Gamma : \mathrm{ctx}_{n'} & \vdash & \Gamma[\sigma] : \mathrm{ctx}_n \\
n, n' : \mathrm{lctx}, \sigma : \mathrm{lhom}(n, n'), \Delta, \Gamma : \mathrm{ctx}_{n'}, \gamma : \mathrm{Hom}_{n'}(\Delta, \Gamma) & \vdash & \gamma[\sigma] : \mathrm{Hom}_n(\Delta[\sigma], \Gamma[\sigma])
\end{array}
$$

Equations:

$$
\begin{array}{rcl}
\Gamma[\mathrm{lid}_n] & = & \Gamma \\
\Gamma[\sigma \circ \sigma'] & = & \Gamma[\sigma][\sigma'] \\
\gamma[\mathrm{lid}_n] & = & \gamma \\
\gamma[\sigma \circ \sigma'] & = & \gamma[\sigma][\sigma']
\end{array}
$$

Etc, for level-indexing the other components of cwfs and of Π-types.

A level-indexed non-cumulative tower of universes

A finitary theory!

Operator symbols:

$$
\begin{array}{rcl}
n : \mathrm{lctx}, l : \mathrm{ltm}(n), \Gamma : \mathrm{lctx}_n & \vdash & (\mathrm{U}_l)_\Gamma : \mathrm{ty}_n(\Gamma) \\
n : \mathrm{lctx}, l : \mathrm{ltm}(n), \Gamma : \mathrm{ctx}_n, a : \mathrm{tm}_n(\Gamma, (\mathrm{U}_l)_\Gamma) & \vdash & \mathrm{T}_l(a) : \mathrm{ty}_n(\Gamma) \\
n : \mathrm{lctx}, l, l' : \mathrm{ltm}(n), \Gamma : \mathrm{ctx}_n, a : \mathrm{tm}_n(\Gamma, (\mathrm{U}_l)_\Gamma), b : \mathrm{tm}_n(\Gamma \cdot \mathrm{T}_l(a), (\mathrm{U}_{l'})_\Gamma)) & \vdash & \hat{\Pi}^{l,l'}(a,b) : \mathrm{tm}_n(\Gamma, (\mathrm{U}_{l \vee l'})_\Gamma) \\
n : \mathrm{lctx}, l : \mathrm{ltm}(n), \Gamma : \mathrm{ctx}_n & \vdash & \mathrm{U}^l_\Gamma : \mathrm{tm}_n(\Gamma, (\\
CwfUU_{l+})_\Gamma) & &
\end{array}
$$

Equations:

$$\vdots$$

Happy birthday, Thorsten!