Algorithms on Strings

Advanced Programming and Algorithmic Design

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String-Matching

Basic Definitions and Properties

An alphabet is a set of symbols e.g., $\{0, 1\}$ or $\{a, \ldots, z\}$

A string $S[1 \ldots |S|]$ is a finite sequence of symbols in an alphabet

Σ^* is the set of all strings built on Σ

ϵ is the empty string and belongs to Σ^*

E.g.,

"This is a string" or "This_is,a+string" or ϵ

Basic Definitions and Properties (Cont'd)

If $x \in \Sigma^*$ and $y \in \Sigma^*$, then $xy \in \Sigma^*$ is their concatenation

If $y = xw$:

- x is a prefix of y and we write $x \sqsubset y$
- w is a suffix of y and we write $x \sqsupset y$

If $x \in \Sigma^*$ and $q \in \mathbb{N}$, x_q will be the x's prefix of length q

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Lemma (Overlapping-suffix lemma)

Let x, y, and w s.t. $x \sqsupset w$ and $y \sqsupset w$.

- *if $|x| > |y|$, then $y \sqsupset x$*
- *if $|x| = |y|$, then $y = x$*

Basic Definitions and Properties (Cont'd)

Given :

- a finite alphabet Σ

c	a	b	a	b	b

Basic Definitions and Properties (Cont'd)

Given :

- a finite alphabet Σ
- a text $T[1 \ldots n]$

Text	c	a	b	a	b	b

Basic Definitions and Properties (Cont'd)

Given :

- a finite alphabet Σ
- a text $T[1 \ldots n]$
- a pattern $P[1 \ldots m]$ with $m \leq n$

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Pattern	b	a	b			

Basic Definitions and Properties (Cont'd)

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P occurs with shift s in T means $T[s+1 \ldots s+m] = P$

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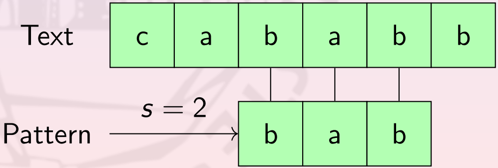

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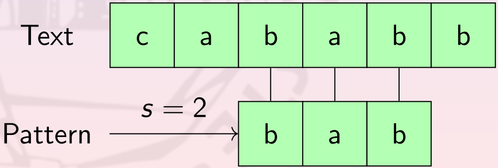

If P occurs with shift s in T, then s is a valid shift

The String-Matching Problem

Requires to find all the valid shifts for P in T

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E.g., For

Text	c	a	b	a	b	a	b	a	b	a	c
Pattern	b	a	b								

We get $\{$

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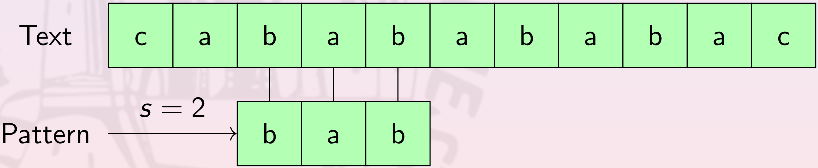

We get $\{2,$

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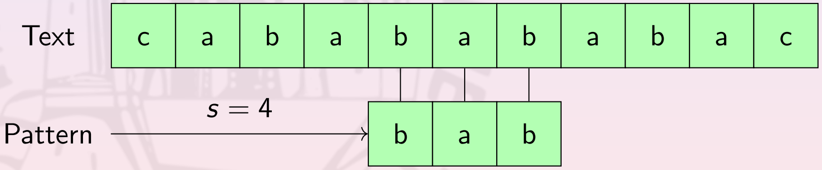

We get $\{2, 4,$

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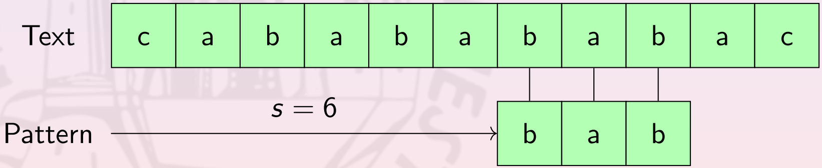
E.g., For



We get $\{2, 4, 6\}$

A Naïve Solution to String-Matching Problem

To solve the problem, we may try all the possible shifts for P in T

Text	c	a	b	a	b	a	b	a	c
Pattern	b	a	b						

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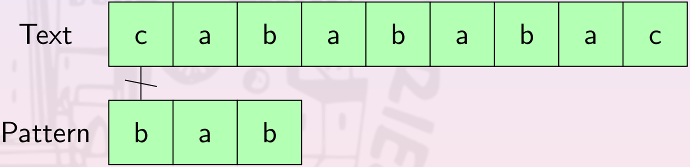

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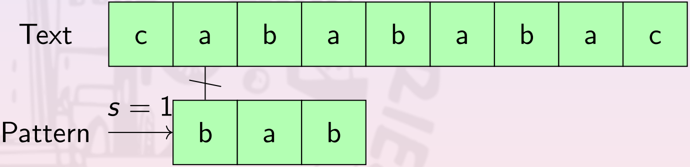

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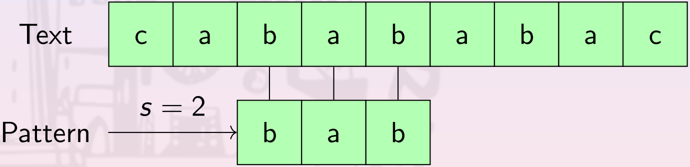

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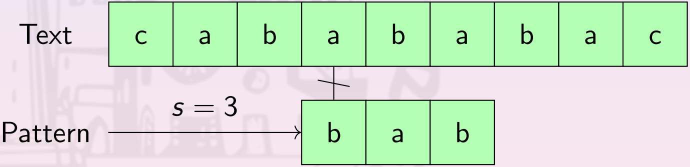

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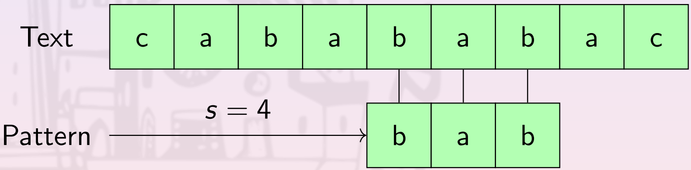

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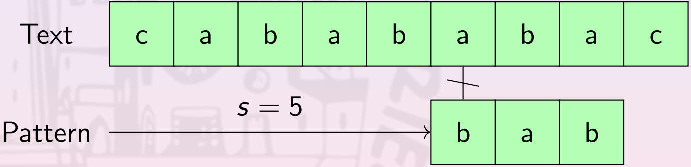

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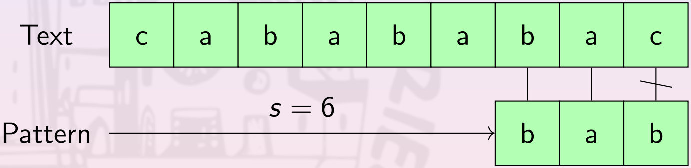


Naïve Solution: Pseudo-Code

```
def NAIVE_STRING_MATCHING(T, P):
  valid ← []
  for s ← 1 upto |T|−|P|+1:
    i ← 1
    while i ≤ |P| and T[i+s] = P[i]:
      i ← i+1
    endwhile

    if i > |P|:
      valid.append(s)
    endif
  endfor

  return valid
enddef
```

Naïve Solution: Complexity

A match is tested for all the possible $|T| - |P| + 1$ shifts

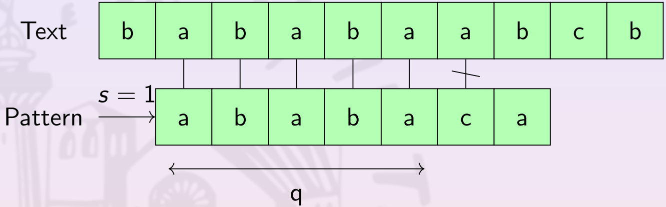
Each match test costs $O(|P|)$

Since $|P| \leq |T|$, the overall complexity is $O(|P| * |T|)$

E.g., to face a worst-case-scenario consider:

Text	a	a	a	a	a	a	a	a	a
Pattern	a	a	a	a	b				

A Better Idea



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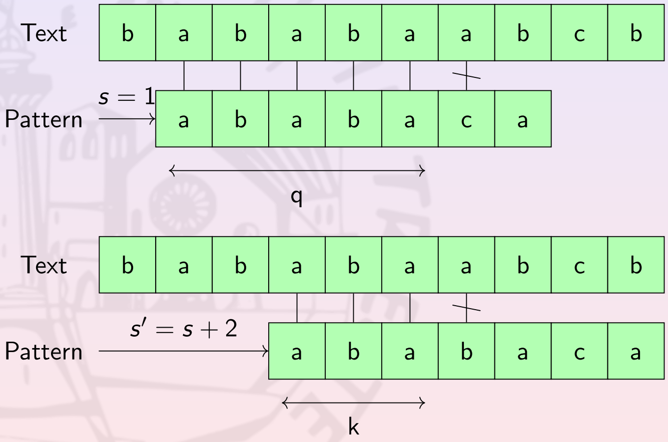

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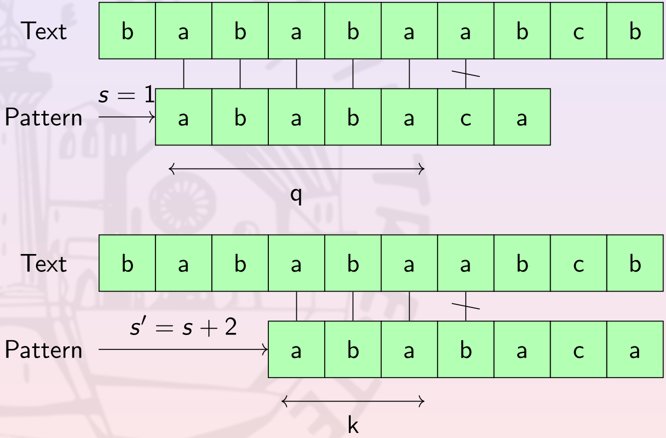

Thus, $P_k \sqsupset P_q$ beacuse $P_q \sqsupset T[2..q+1]$ and $P_k \sqsupset T[2..q+1]$

The Prefix Function

The prefix function for P is defined as

$$\pi[q] = \max\{k : k < q \text{ and } P_k \sqsupset P_q\}$$

P_5	a	b	a	b	a

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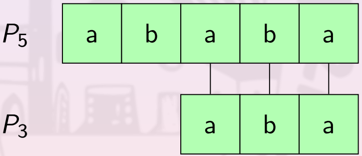

$\pi[5] = 3$

The Prefix Function

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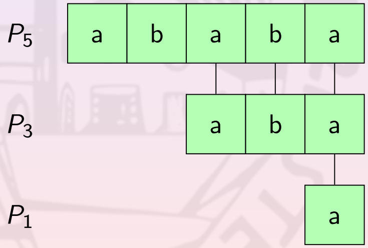

$\pi[5] = 3$

$\pi[3] = 1$

The Prefix Function

The prefix function for P is defined as

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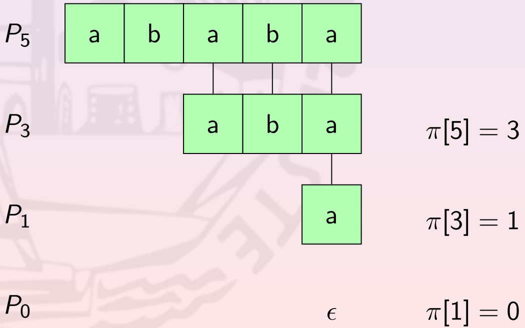


Computing the Prefix Function

Let $\pi^*[q]$ be $\{\pi[q], \pi^2[q], \ldots, \pi^{(t)}[q]\}$

Lemma (Prefix-function iteration lemma)

$\pi^*[q] = \{k : k < q \text{ and } P_k \sqsupset P_q\}$

Lemma

If $\pi[q] > 0$, *then* $\pi[q] - 1 \in \pi^*[q-1]$

Let E_q be $\{k \in \pi^*[q] : P[k+1] = P[q+1]\}$

Theorem

$$\pi[q] = \begin{cases} 0 & \text{if } E_{q-1} = \emptyset \\ 1 + \max\{k \in E_{q-1}\} & \text{otherwise} \end{cases}$$

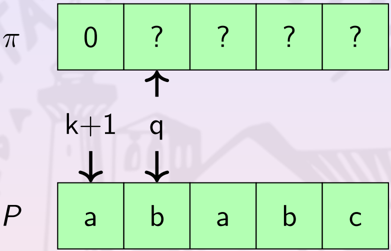
Computing the Prefix Function: Pseudo-Code

```
def COMPUTE_PREFIX_FUNCTION(P):
  π ← INIT_ARRAY(|P|)
  π[1] ← 0
  k ← 0
  for q ← 2 upto |P|:
    while k > 0 and P[k+1] ≠ P[q]:
      k = π[k]
    endwhile

    if P[k+1] = P[q]:
      k = k + 1
    π[q] ← k
  endfor

  return π
enddef
```

Computing the Prefix Function: an Example



At the begin of each **for**-loop iteration, $k = \pi[q-1]$

P_k is the largest proper suffix of P_{q-1} which is also a prefix for it i.e., $P_k \sqsupset P_{q-1}$ and $P_k \sqsubset P_{q-1}$

The initialization sets

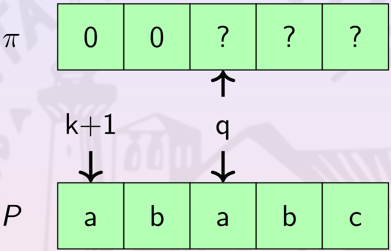
- $\pi[1] = 0$
- $q = 2$
- $k = 0$

Then no **while**-loop iterations

Since $P[k+1] \neq P[q]$, $P_{k+1} \not\sqsupset P_q$ and k is not updated

$\pi[q] \leftarrow 0$

Computing the Prefix Function: an Example



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When $q = 3$:

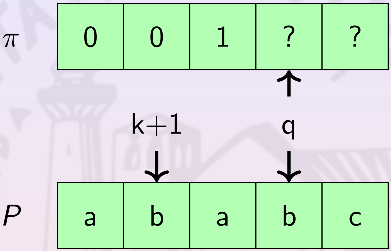
- $k = 0$
- $P[k+1] = P[q]$

Then no **while**-loop iterations

Since $P[k+1] = P[q]$, $P_{k+1} \sqsupset P_q$ and k is updated to 1

$\pi[q] \leftarrow 1$

Computing the Prefix Function: an Example



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P_k is the largest proper suffix of P_{q-1} which is also a prefix for it i.e., $P_k \sqsupset P_{q-1}$ and $P_k \sqsubset P_{q-1}$

When $q = 4$:

- $k = 1$
- $P[k+1] = P[q]$

Then no **while**-loop iterations

Since $P[k+1] = P[q]$, $P_{k+1} \sqsupset P_q$ and k is updated to 2

$\pi[q] \leftarrow 2$

Computing the Prefix Function: an Example

π	0	0	1	2	?
			k+1		q
P	a	b	a	b	c

At the begin of each **for**-loop iteration, $k = \pi[q-1]$

P_k is the largest proper suffix of P_{q-1} which is also a prefix for it i.e., $P_k \sqsupset P_{q-1}$ and $P_k \sqsubset P_{q-1}$

When $q = 5$:

- $k = 2$
- $P[k+1] \neq P[q]$

Since $P[k+1] \neq P[q]$, the 2nd largest prefix-suffix P_{q-1} is computed i.e., $\pi[k]$ and k is updated to 0

Since $P[k+1] \neq P[q]$, k is not updated

$\pi[q] \leftarrow 0$

Computing the Prefix Function: an Example

π	0	0	1	2	0

P	a	b	a	b	c

At the begin of each **for**-loop iteration, $k = \pi[q-1]$

P_k is the largest proper suffix of P_{q-1} which is also a prefix for it i.e., $P_k \sqsupset P_{q-1}$ and $P_k \sqsubset P_{q-1}$

The Prefix Function: Complexity

The **while**-loop condition holds only if $k > 0$

However, each iteration of the **while**-loop decreases k

k is initialized to 0 and is increased in the **for**-loop

So, the **while**-loop can be repeated $|P| - 1$ times at most

The overall asymptotic complexity is $\Theta(|P|)$

The Knuth-Morris-Pratt Algorithm

Once a mismatch has been identified after q matches

The algorithm uses the prefix function to avoid $\pi[q]$ useless character comparisons

The Knuth-Morris-Pratt Algorithm: Pseudo-Code

```
def KMP(T,P):
  valid = []
  π ← COMPUTE_PREFIX_FUNCTION(P)
  q ← 0
  for i ← 1 upto |T|:
    while q > 0 and P[q+1] ≠ T[i]:
      q = π[q]
    endwhile

    if P[q+1] = T[i]:
      q = q + 1
    if q = |P|:
      valid.append(i−q+1)
      q = π[q]
    endif
  endfor

  return valid
enddef
```

The Knuth-Morris-Pratt Algorithm: an Example

T	a	b	a	b	a	b	c	a
	↑ i							

P	a	b	a	b	c
	↑ q+1				

π	0	0	1	2	0

At the begin of each **for**-loop iteration, if $i > 1$, then $q = \pi[i-1]$

P_q is the largest proper suffix of P_{i-1} which is also a prefix for it i.e., $P_q \sqsupset P_{i-1}$ and $P_q \sqsubset P_{i-1}$

The initialization sets

- $i = 1$
- $q = 0$

Then no **while**-loop iterations

Since $P[q+1] = T[i]$, $P_{q+1} \sqsupset T_i$ and q is updated to 1

The Knuth-Morris-Pratt Algorithm: an Example

T								
	a	b	a	b	a	b	c	a
		$\uparrow$ i						

P					
	a	b	a	b	c
		$\uparrow$ q+1			

π					
	0	0	1	2	0

At the begin of each **for**-loop iteration, if $i > 1$, then $q = \pi[i-1]$

P_q is the largest proper suffix of P_{i-1} which is also a prefix for it i.e., $P_q \sqsupset P_{i-1}$ and $P_q \sqsubset P_{i-1}$

When $i = 2$:

- $q = 1$
- $P[q+1] = T[i]$

Then no **while**-loop iterations

Since $P[q+1] = T[i]$, $P_{q+1} \sqsupset T_i$ and q is updated to 2

The Knuth-Morris-Pratt Algorithm: an Example

T	a	b	a	b	a	b	c	a
			↑ i					
P	a	b	a	b	c			
			↑ q+1					
π	0	0	1	2	0			

At the begin of each **for**-loop iteration, if $i > 1$, then $q = \pi[i-1]$

P_q is the largest proper suffix of P_{i-1} which is also a prefix for it i.e., $P_q \sqsupset P_{i-1}$ and $P_q \sqsubset P_{i-1}$

When $i = 3$:

- $q = 2$
- $P[q+1] = T[i]$

Then no **while**-loop iterations

Since $P[q+1] = T[i]$, $P_{q+1} \sqsupset T_i$ and q is updated to 3

The Knuth-Morris-Pratt Algorithm: an Example

T	a	b	a	b	a	b	c	a
				↑ i				
P	a	b	a	b	c			
				↑ q+1				
π	0	0	1	2	0			

At the begin of each **for**-loop iteration, if $i > 1$, then $q = \pi[i-1]$

P_q is the largest proper suffix of P_{i-1} which is also a prefix for it i.e., $P_q \sqsupset P_{i-1}$ and $P_q \sqsubset P_{i-1}$

When $i = 4$:

- $q = 3$
- $P[q+1] = P[i]$

Then no **while**-loop iterations

Since $P[q+1] = T[i]$, $P_{q+1} \sqsupset T_i$ and q is updated to 4

The Knuth-Morris-Pratt Algorithm: an Example

T	a	b	a	b	a	b	c	a

i

P	a	b	a	b	c

q+1

π	0	0	1	2	0

At the begin of each **for**-loop iteration, if $i > 1$, then $q = \pi[i-1]$

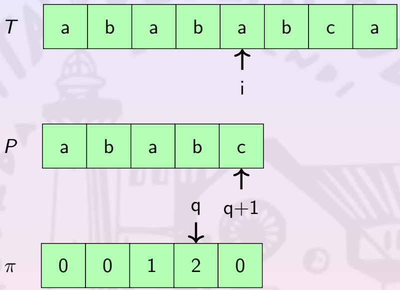
P_q is the largest proper suffix of P_{i-1} which is also a prefix for it i.e., $P_q \sqsupset P_{i-1}$ and $P_q \sqsubset P_{i-1}$

When $i = 5$:

- $q = 4$
- $P[q+1] \neq T[i]$

Since $P[q+1] \neq T[i]$,

The Knuth-Morris-Pratt Algorithm: an Example



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When $i = 5$:

- $q = 4$
- $P[q+1] \neq T[i]$

Since $P[q+1] \neq T[i]$, the 2nd largest prefix-suffix P_q is computed i.e., $\pi[q]$ and

The Knuth-Morris-Pratt Algorithm: an Example

T	a	b	a	b	a	b	c	a
					↑ i			

P	a	b	a	b	c
		q ↓	↑ q+1		
π	0	0	1	2	0

At the begin of each **for**-loop iteration, if $i > 1$, then $q = \pi[i-1]$

P_q is the largest proper suffix of P_{i-1} which is also a prefix for it i.e., $P_q \sqsupset P_{i-1}$ and $P_q \sqsubset P_{i-1}$

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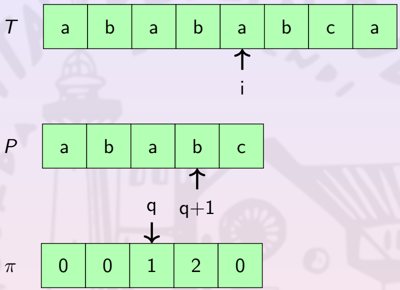
The Knuth-Morris-Pratt Algorithm: an Example



When $i = 5$:

- $q = 4$
- $P[q+1] \neq T[i]$

Since $P[q+1] \neq T[i]$, the 2nd largest prefix-suffix P_q is computed i.e., $\pi[q]$ and q is updated to 2

Since $P[q+1] = T[i]$, q is updated to 3

At the begin of each **for**-loop iteration, if $i > 1$, then $q = \pi[i-1]$

P_q is the largest proper suffix of P_{i-1} which is also a prefix for it i.e., $P_q \sqsupset P_{i-1}$ and $P_q \sqsubset P_{i-1}$

The Knuth-Morris-Pratt Algorithm: an Example

T	a	b	a	b	a	b	c	a
						↑ i		

P	a	b	a	b	c
				↑ q+1	

π	0	0	1	2	0

At the begin of each **for**-loop iteration, if $i > 1$, then $q = \pi[i-1]$

P_q is the largest proper suffix of P_{i-1} which is also a prefix for it i.e., $P_q \sqsupset P_{i-1}$ and $P_q \sqsubset P_{i-1}$

When $i = 6$:

- $q = 3$
- $P[q+1] = T[i]$

Then no **while**-loop iterations

Since $P[q+1] = T[i]$, $P_{q+1} \sqsupset T_i$ and q is updated to 4

The Knuth-Morris-Pratt Algorithm: an Example

T	a	b	a	b	a	b	c	a
							i	

P	a	b	a	b	c
					q+1

π	0	0	1	2	0

At the begin of each **for**-loop iteration, if $i > 1$, then $q = \pi[i-1]$

P_q is the largest proper suffix of P_{i-1} which is also a prefix for it i.e., $P_q \sqsupset P_{i-1}$ and $P_q \sqsubset P_{i-1}$

When $i = 7$:

- $q = 4$
- $P[q+1] = T[i]$

Then no **while**-loop iterations

Since $P[q+1] = T[i]$, $P_{q+1} \sqsupset T_i$ and

The Knuth-Morris-Pratt Algorithm: an Example

T	a	b	a	b	a	b	c	a
							↑ i	

P	a	b	a	b	c

					q ↓
π	0	0	1	2	0

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The Knuth-Morris-Pratt Algorithm: an Example

T	a	b	a	b	a	b	c	a
			s				i	

P	a	b	a	b	c

					q
π	0	0	1	2	0

At the begin of each **for**-loop iteration, if $i > 1$, then $q = \pi[i-1]$

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Since $q = |P|$, $s = i - q + 1$ is a valid shift

The Knuth-Morris-Pratt Algorithm: an Example

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							i	

					q
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							↑ i	

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	↑ q+1				

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Since $q = |P|$, $s = i - q + 1$ is a valid shift and q is updated to $\pi[q] = 0$

The Knuth-Morris-Pratt Algorithm: an Example

T	a	b	a	b	a	b	c	a
								↑ i

P	a	b	a	b	c
	↑ q+1				

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At the begin of each **for**-loop iteration, if $i > 1$, then $q = \pi[i-1]$

P_q is the largest proper suffix of P_{i-1} which is also a prefix for it i.e., $P_q \sqsupset P_{i-1}$ and $P_q \sqsubset P_{i-1}$

When $i = 8$:

- $q = 0$
- $P[q+1] = T[i]$

Then no **while**-loop iterations

Since $P[q+1] = T[i]$, $P_{q+1} \sqsupset T_i$ and q is updated to 1

The Knuth-Morris-Pratt Algorithm: Complexity

As for the prefix function computation, the **while**-loop condition holds only if $q > 0$

However, each iteration of the **while**-loop decreases q

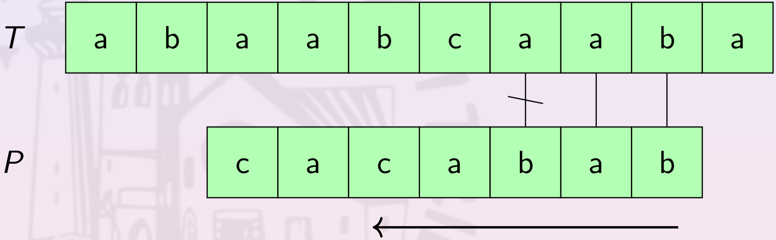
q is initialized to 0 and is increased in the **for**-loop

So, the **while**-loop can be repeated $|T|$ times at most

The overall asymptotic complexity is $\Theta(|P| + |T|)$

The Boyer-Moore-Galil Algorithm

Matches the pattern backward



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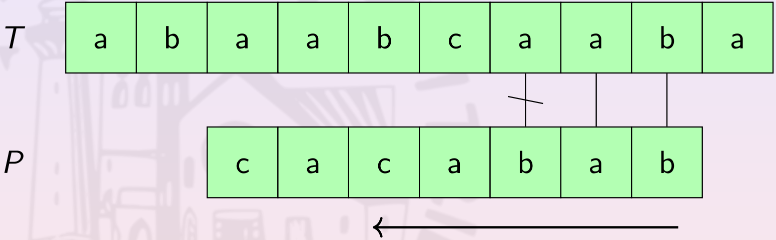
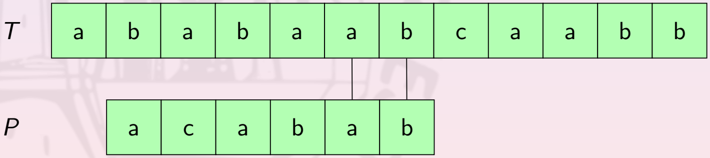


Uses 3 main ingredients:

- **good-suffix rule**
- **bad-character rule**
- **Galil's rule**

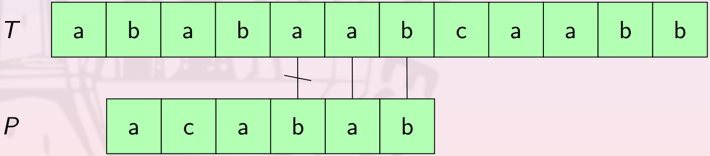
The Good-Suffix Rules

If $P[i \ldots |P|] = T[i + j \ldots |P| + j]$ and



The Good-Suffix Rules

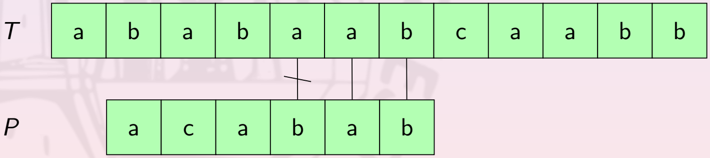
If $P[i \ldots |P|] = T[i+j \ldots |P|+j]$ and $P[i-1] \neq T[i+j-1]$



The Good-Suffix Rules

If $P[i \ldots |P|] = T[i+j \ldots |P|+j]$ and $P[i-1] \neq T[i+j-1]$

- align $T[i+j \ldots |P|+j]$ to its rightmost occurrence in P with a preceding character $\neq P[i-1]$



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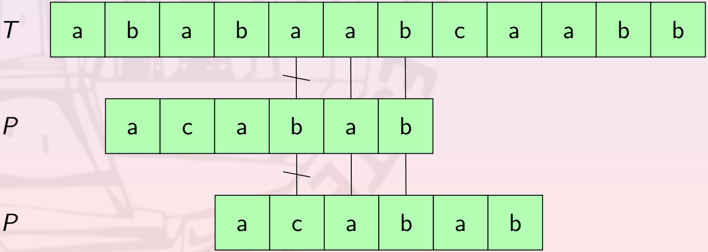

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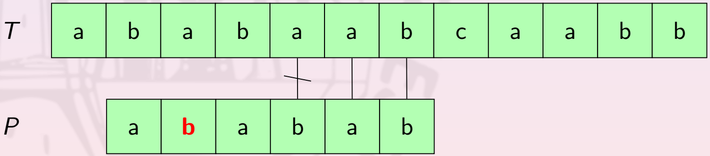

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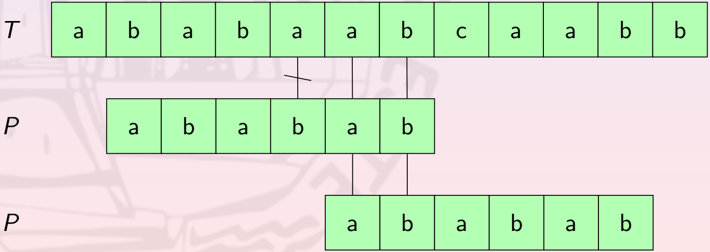
- align $T[i+j \ldots |P|+j]$ to its rightmost occurrence in P with a preceding character $\neq P[i-1]$
- if not exists, align the longest $P_q \sqsubset P$ to $T[|P|+j-q \ldots |P|+j]$



The Good-Suffix Rules: Computing it

It is almost like π^{-1} on the reversed pattern P^{-1}

P	a	c	a	b	a	b

The Good-Suffix Rules: Computing it

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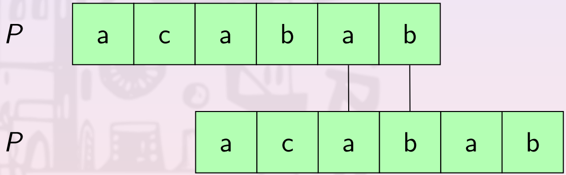

$P_2^{-1} \sqsupset P_4^{-1}$

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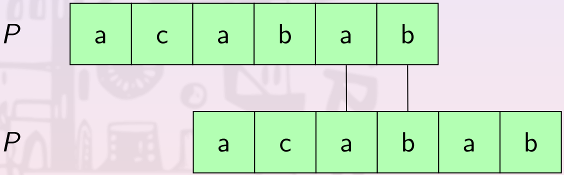

$P_2^{-1} \sqsupset P_4^{-1}$, $\pi[4] = 2$

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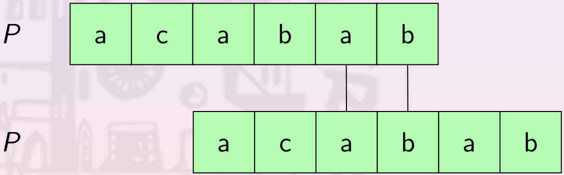

$P_2^{-1} \sqsupset P_4^{-1}$, $\pi[4] = 2$, and $\pi^{-1}[2] = 4$

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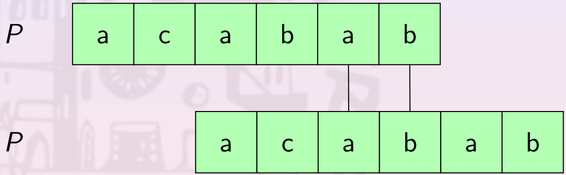

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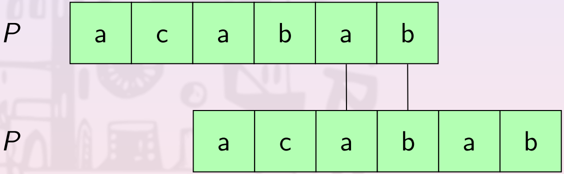
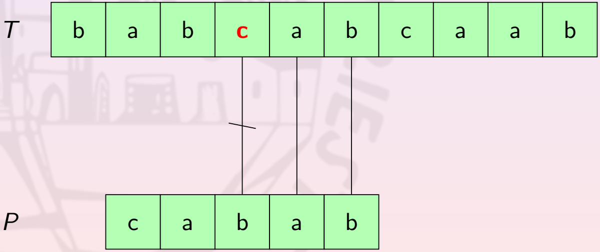


$P_2^{-1} \sqsupset P_4^{-1}$, $\pi[4] = 2$, and $\pi^{-1}[2] = 4$

You can guess a complexity $\Theta(|P|)$ to compute it

The Bad-Character Rules

If $P[i] \neq T[i+j]$



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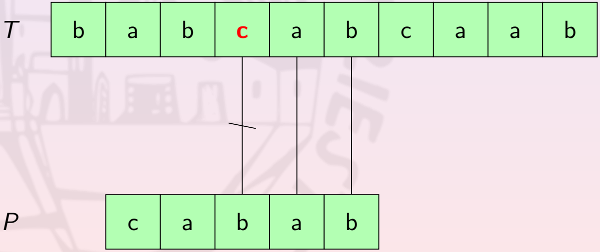
If $P[i] \neq T[i+j]$

- align $T[i+j]$ to its rightmost occurrence in P



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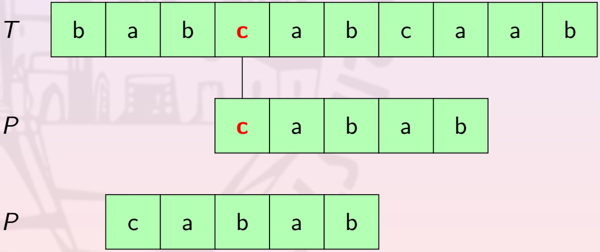

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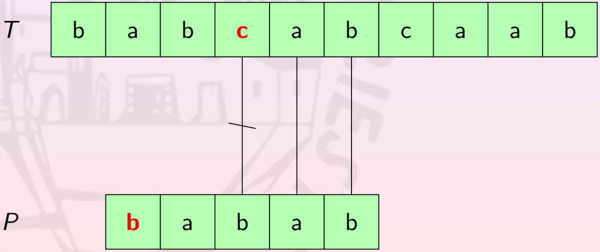

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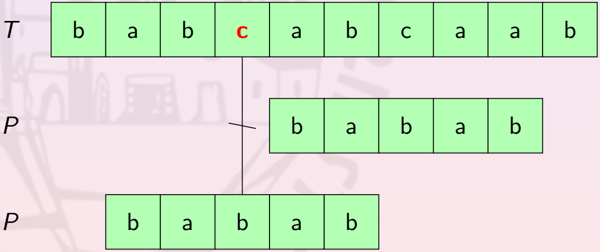

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- initialize an array C s.t. $|C| = |\Sigma|$

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The complexity is $\Theta(|P| + |\Sigma|)$

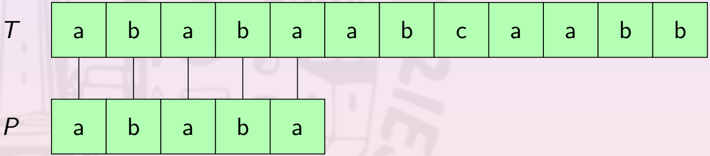
The Galil's Rules

If a valid match has been discovered

T	a	b	a	b	a	a	b	c	a	a	b	b
P	a	b	a	b	a							

The Galil's Rules

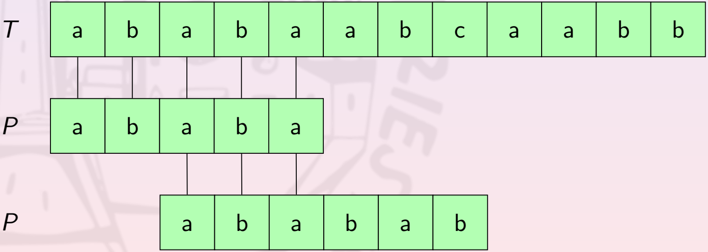
If a valid match has been discovered and P is k-periodic



The Galil's Rules

If a valid match has been discovered and P is k-periodic

P is shifted forward by k and $|P| - k$ comparisons avoided



The Boyer-Moore-Galil's Algorithm

- try to match P on T backward
- if a mismatch is found, then select the largest shift among those suggested by the good-suffix and the bad-character rules
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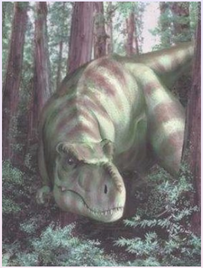
The overall asymptotic complexity is $O(|P| + |T|)$

In an average scenario is sub-linear w.r.t $|T|$.

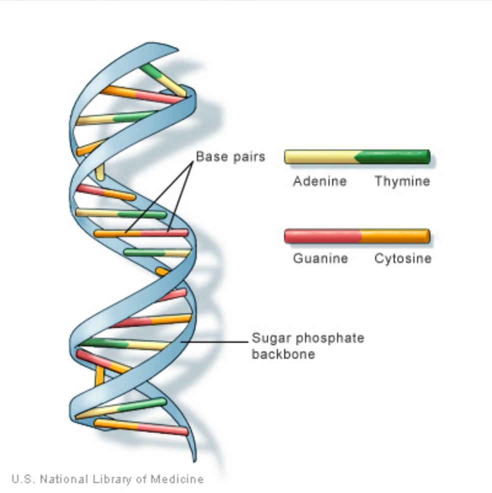
Multiple Patterns String Matching

Life's Code

Life's Code

All life forms share the same *code*: the DNA.



Why Studing DNA is Interesting?

- forecast/cure diseases
- threat genetic conditions

“Reading” DNA

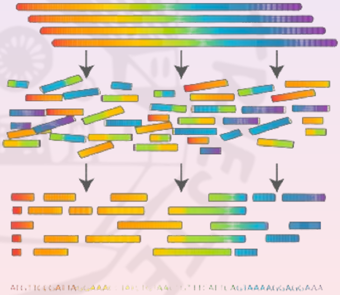
Sequencers are machines to read DNA molecules

But they cannot (yet) accurately read a full DNA molecule

The longer the reading, the higher the probability of errors

Sequencing and Assembling DNA

- DNA is fragmented in relative short pieces (about 800bps)
- the fragments are sequenced
- the sequencer reads are assembled like a 1-D puzzle



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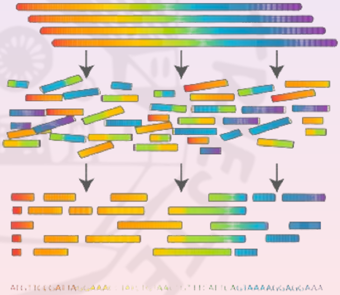

Still slow and expensive due to fragment lengths

Re-sequencing and Aligning

Should we repeat the process of each individual? No

- DNA is fragmented in smaller pieces (about 100bps)
- the fragments are sequenced
- the sequencer reads are aligned over the reference genome

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The Multiple Patterns Single Text Matching Problem

We have

- a text T
- a large set of patterns $\mathcal{P} = \{P_1, \ldots, P_l\}$

The Multiple Patterns Single Text Matching Problem

We have

- a text T
- a large set of patterns $\mathcal{P} = \{P_1, \ldots, P_l\}$

We want to find a valid shift for each P_i

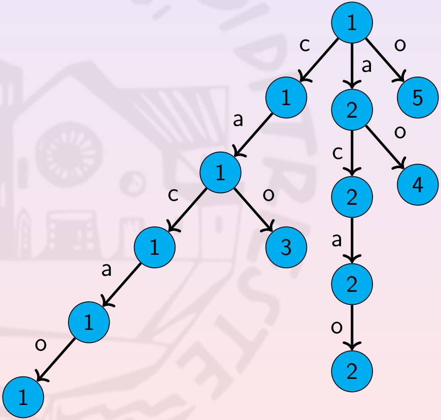
A Naïve Solution

For each P_i, compute BOYER_MOORE_GALIL(T, P_i)

Complexity: $O\left(|T| * \sum_{i=1}^{l} |P_i|\right)$

A Tree-Based Solution

E.g. $T = cacao$



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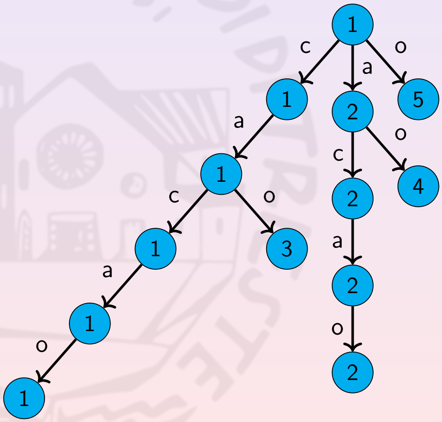
Searching for $P_1 = cao$



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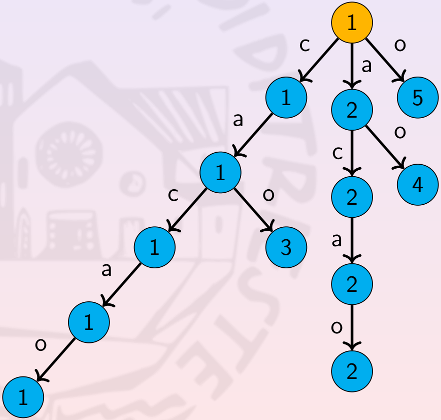

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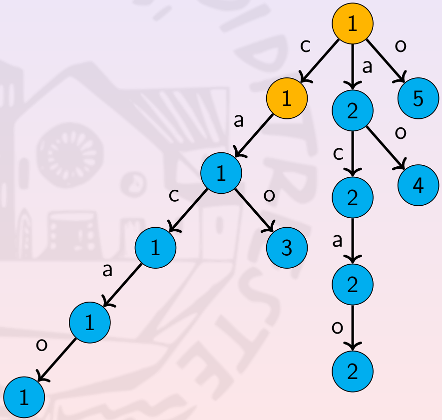

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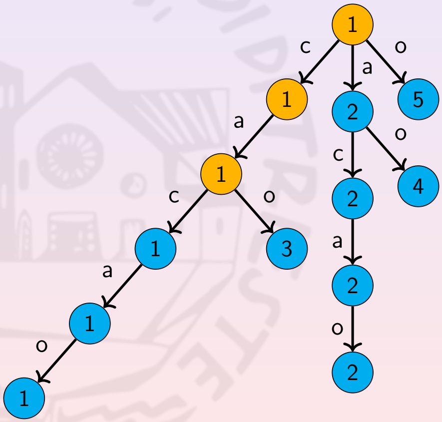

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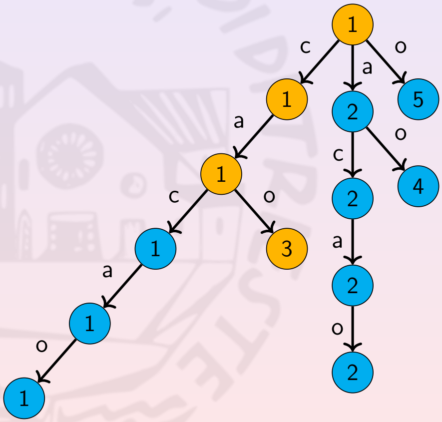

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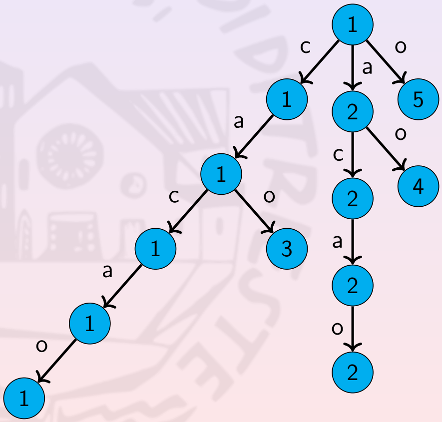
E.g. $T = cacao$

Searching for $P_1 = ac$



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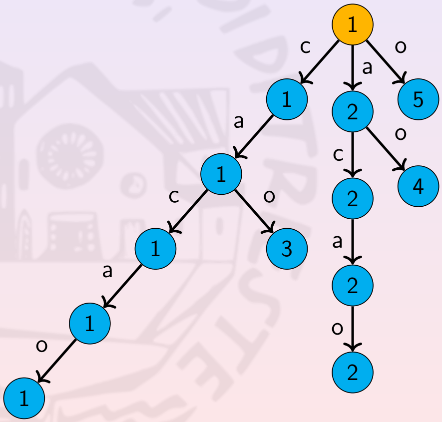

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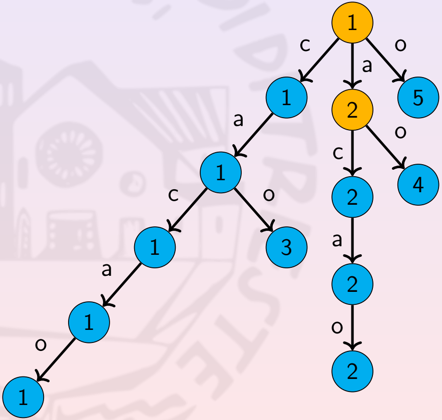

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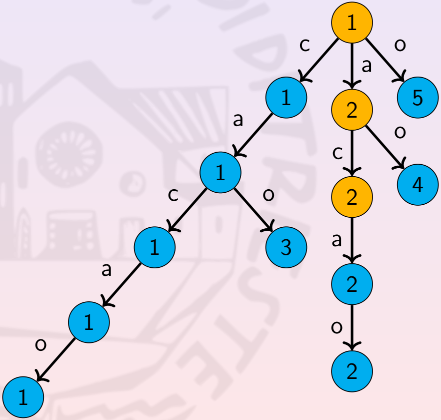

Searching Time

Searching for P_i in the of T's substrings costs $\Theta(|P_i|)$

Once it has been computed, solving our problem takes time

$$\Theta\left(\sum_{i=1}^{l} |P_i|\right)$$

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$$\Theta\left(\sum_{i=1}^{l} |P_i|\right)$$

How much does its computation cost?

Suffix Tries: A Formal Definition

Let $\sigma(T)$ be the set of all the substrings of T.

$STrie(T)$ of T is a tuple $(Q \cup \{\perp\}, \overline{\epsilon}, L, g, f)$ where:

- $Q = \{\overline{x} \mid x \in \sigma(T)\}$
- $\perp \notin Q$
- $L : Q \mapsto [1 \ldots |T|]$ is the shift label
- $g : (Q \cup \{\perp\}) \times \Sigma \mapsto Q$ is the transition function
  - $g(\overline{x}, a) = \overline{xa}$ for all $xa \in \sigma(T)$
  - $g(\perp, a) = \epsilon$ for all $a \in \Sigma$
- $f : Q \mapsto Q \cup \{\perp\}$ is the prefix function
  - $f(\overline{ax}) = \overline{x}$ for all $ax \in \sigma(T)$
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Let $\sigma(T)$ be the set of all the substrings of T.

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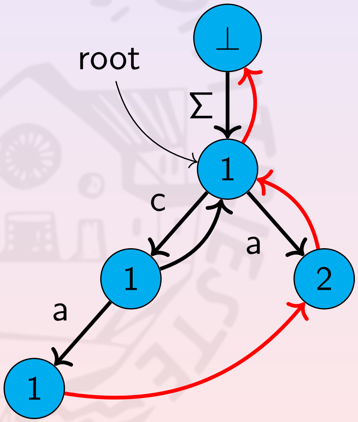
Suffix Tries: An Example

E.g. $T = ca$



Growing Tries by Appending Characters

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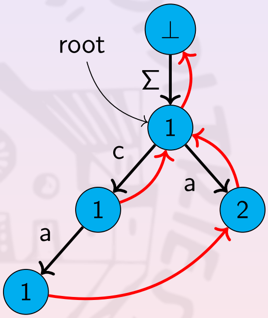

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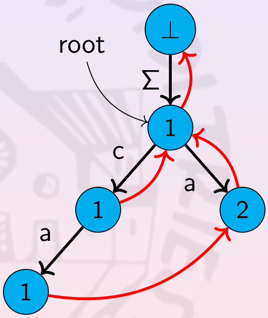

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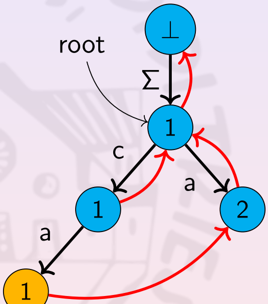

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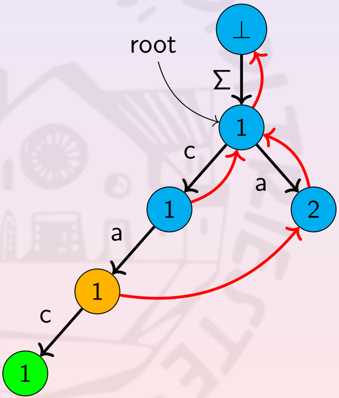

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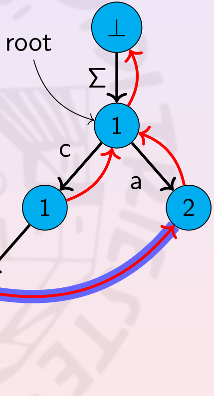

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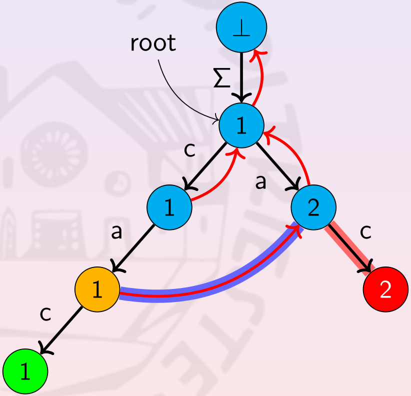

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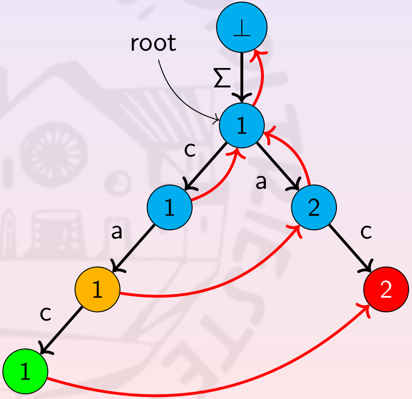

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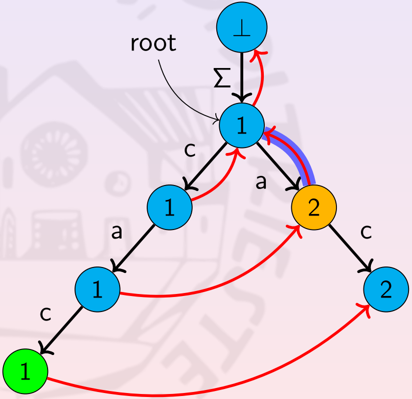

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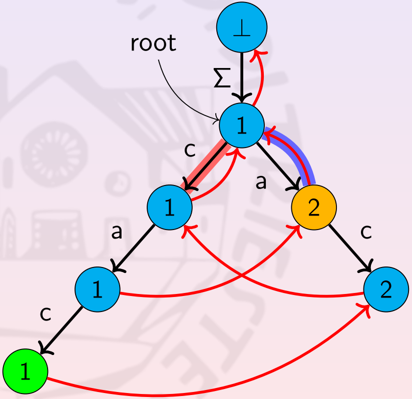

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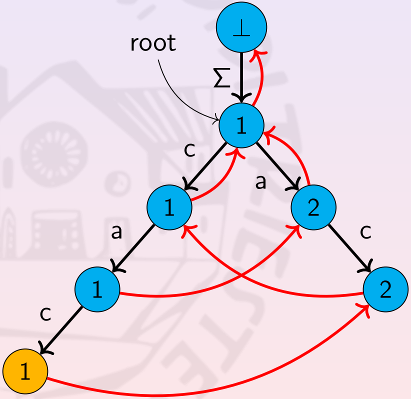

Boundary Path

Let T^i be $T[1 \ldots i]$

The boundary path of $STrie(T^i)$ is the sequence

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The active point is the first s_j that is a leaf

The end point is the first $s_{j'}$ having a $T[i+1]$-transition

How the Algorithm Works

It adds a $T[i+1]$-transition from s_h for all $h \in [1, j'-1]$

If $h \in [1, j-1]$, then it extends a branch

If $h \in [j, j'-1]$, then it creates a new branch

Building a Suffix Trie: Pseudo-Code

```
def UPDATE_SUFFIX_TRIE(S, T, i, top):
  r ← top
  old_s ← None
  while S.g(r, T[i]) = None:
    s ← CREATE_NEW_NODE()

    S.add_node(s)
    S.g(r, T[i]) ← s

    if old_s ≠ None:
      S.f(old_s) ← s
    endif
    old_s ← s

    r ← S.f(r)
  endwhile
  f(old_s) ← S.g(r, T[i])

  return S.g(top, T[i])
enddef
```

Building a Suffix Trie: Complexity

- each node is visited at most twice
- constant steps per node
- $|Q| = |\Sigma(T)|$

Building $STrie(T)$ costs $\Theta(\Sigma(T))$

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Lemma

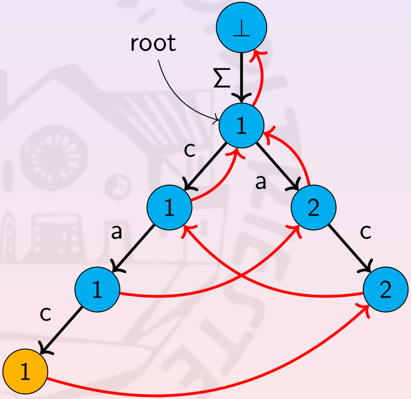
$|\Sigma(T)| \in O(|T|^2)$ *(e.g., $a^n b^n$)*

Theorem

Building a $STrie(T)$ costs $O(|T|^2)$

Reducing Complexity

Suffix tries are redundant



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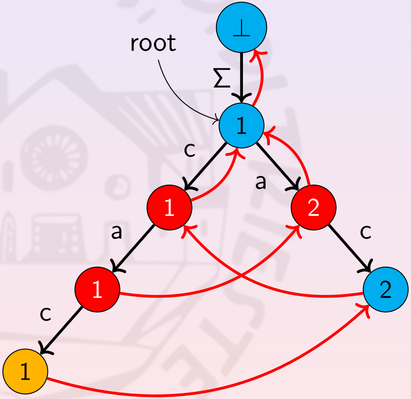

Reducing Suffix Trie Redundancy: Suffix Trees

- Q' containing branching nodes Q_b and leaves Q_l
- $g' : ((Q_b \cup \{\perp\}) \times \Sigma^*) \mapsto Q'$
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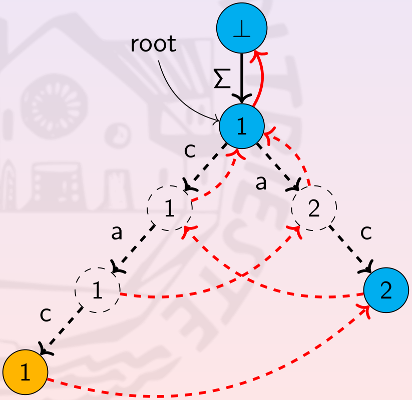

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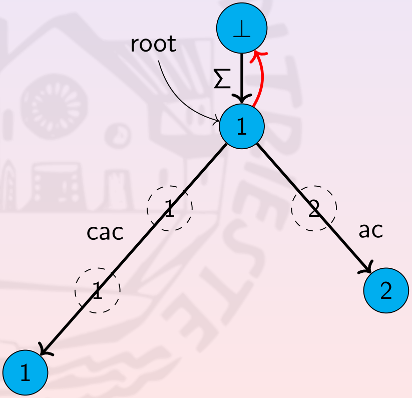

Counting Nodes

- the leaves represent some of the suffixes of T and they are at most $|T|$
- all the internal nodes are branching and they are at most $|T|$-1

These kind of trees has $\Theta(|T|)$ nodes

Substrings to Indexes Intervals

To save space g' labels are represented as T-index intervals

E.g., if $T = cacao$, then

- cao is represented by $[3, 5]$
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If $\Sigma = \{a_1, \ldots, a_k\}$, then

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We can also avoid L: look at the last matching label to infer shifts

A Suffix Tree Example

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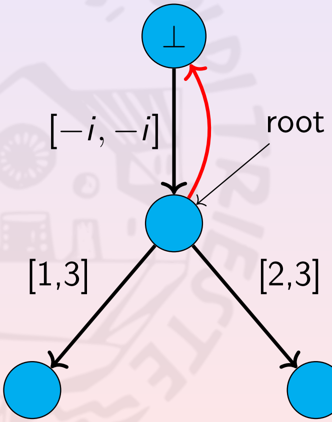

Implicit and Explicit Nodes

Not all the node of the suffix tries are explicitly represented

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If x is the closed ancestor of (x, w), then (x, w) is canonical

Branch Extensions in Suffix Trees

Branch extensions is avoided by labeling transition to leaf as $[h, \infty]$

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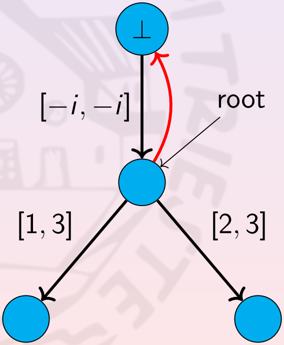

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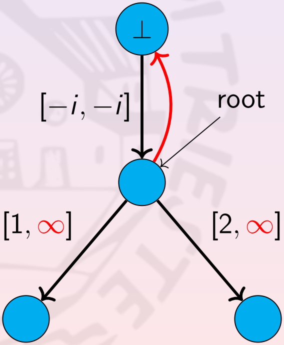

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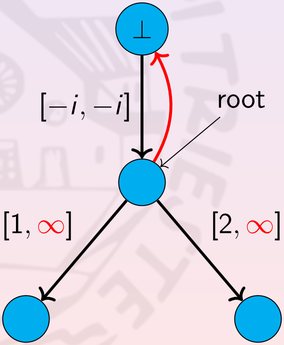

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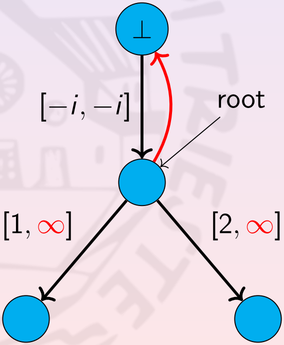

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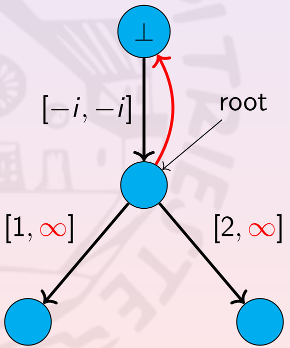

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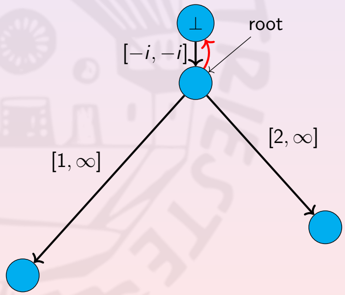

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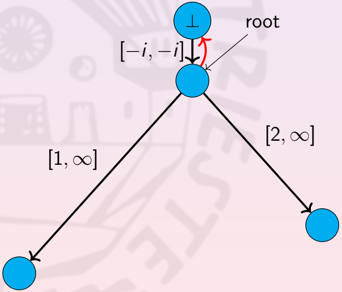

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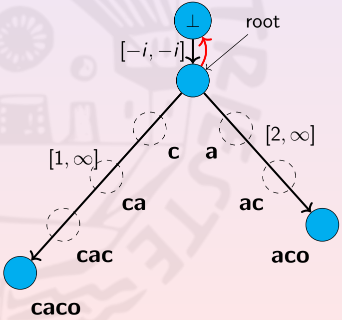

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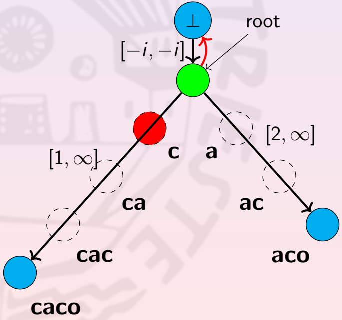

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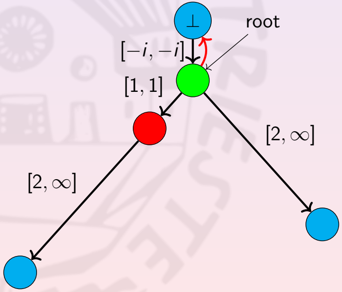

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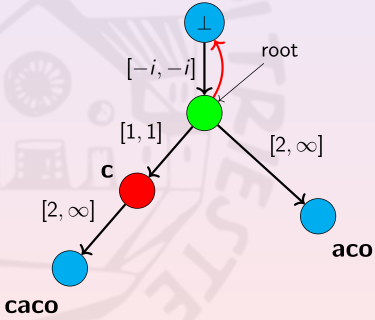

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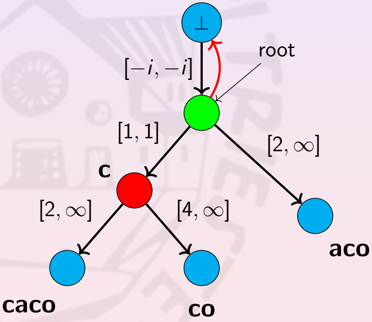

Following the Boundary Path

If $(s, [k, i])$ is on boundary path, the next node is $(f'(s), [k, i])$

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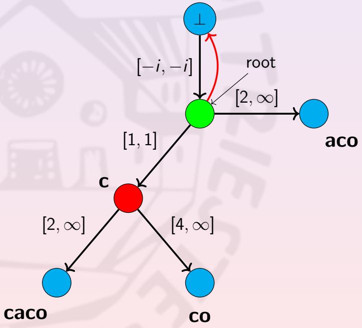

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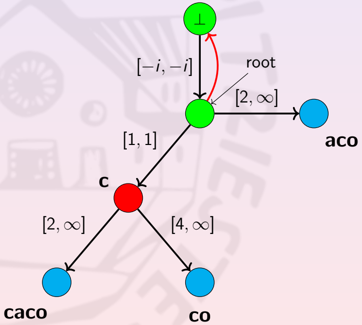

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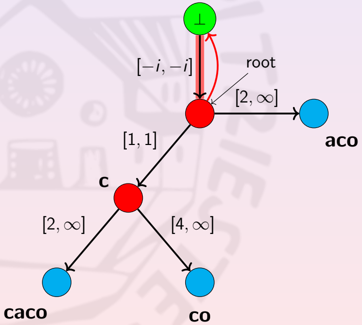

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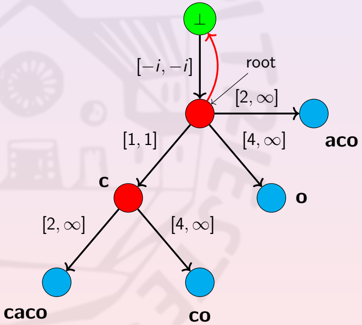

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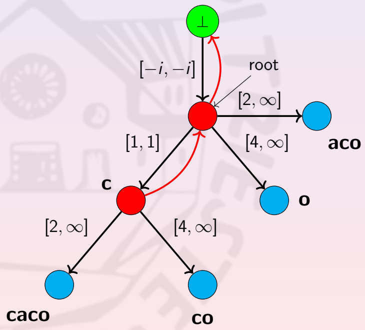

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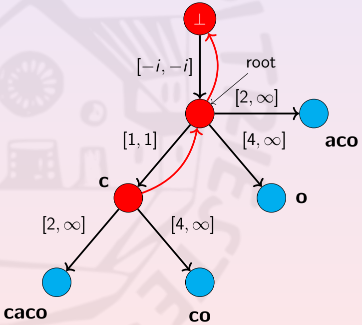

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Before any other procedure, we must canonize it.

Let $[k', i']$ the label of the $T[k]$-transition from s'

- if $[k, i]$ is shorter than $[k', i']$, it is canonical
- otherwise replace:
  - s' with $g'(s', [k', i'])$
  - $[k, i]$ with $[k + (i' - k') + 1, i]$

  and repeat

Finding Next Active Point

$\overline{T[j \ldots i]}$ is the active point of $STree(T^i)$ iff $T[j \ldots i]$ is the longest suffix of T^i that occurs twice

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Finding Next Active Point

$\overline{T[j \ldots i]}$ is the active point of $STree(T^i)$ iff $T[j \ldots i]$ is the longest suffix of T^i that occurs twice

$\overline{T[j' \ldots i]}$ is the end point of $STree(T^i)$ iff $T[j \ldots i]$ is the longest suffix of T^i s.t. $T[j \ldots i+1]$ is a substring of T^i

Theorem

If $(s, [k, i])$ is the end point of $STree(T^i)$, then $(s, [k, i+1])$ is the active point of $STree(T^{i+1})$.