

Graphs and Algorithms

Advanced Programming and Algorithmic Design

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a.a. 2018/2019

Is There a Unique Way to Represent. . .

- dynamic systems
- information flows
- infectious disease spread
- knowledge relations
- dependency relations
- computer network
- document network (e.g., WWW)
- money transfer tracking
- route systems

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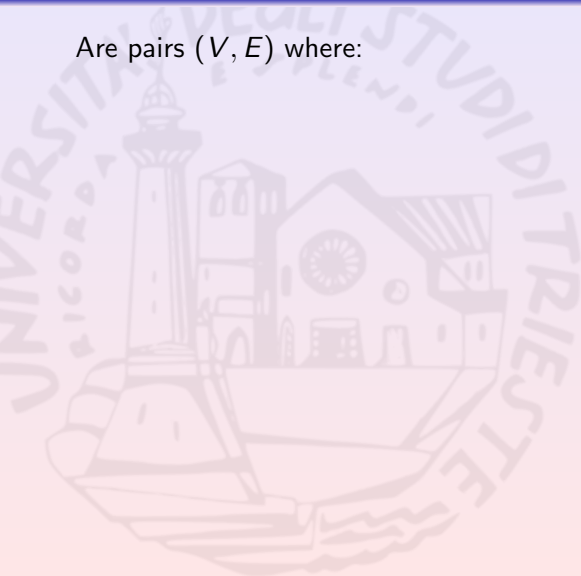
Yes, by using **graphs**

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Basics

Graphs (Graph Theory)

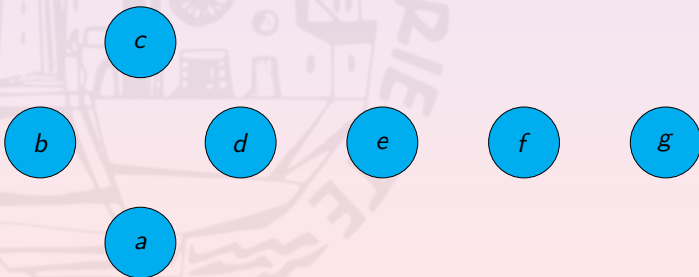
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V is a set of **nodes**

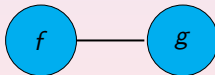
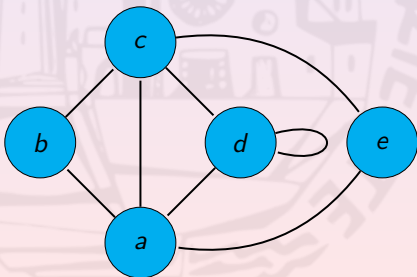


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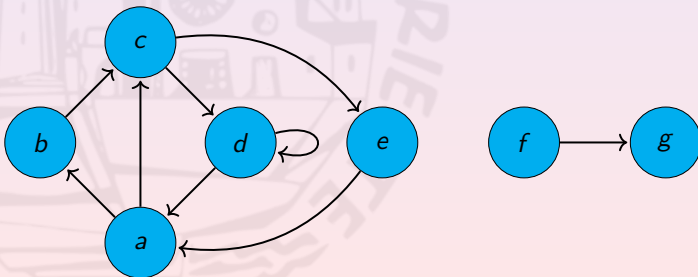
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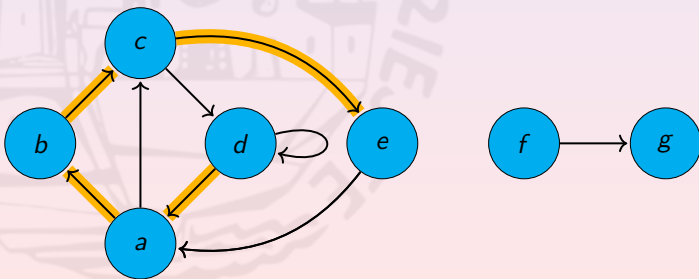
If the edges are **(un)directed**, the graph is (un)directed



Paths and Cycles

A **path** of length n between $a, b \in V$ is a sequence e_1, \dots, e_n s.t.

- e_1 involves a
- e_n involves b
- e_i and e_{i+1} involve a common node n_i

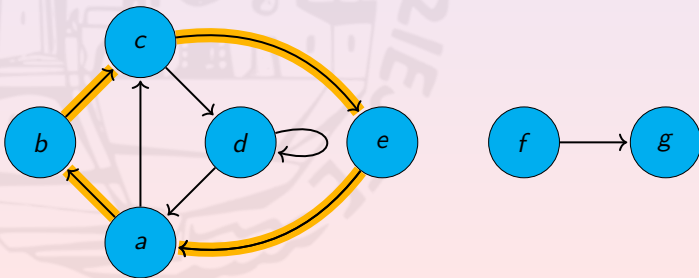


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A **cycle** is a path whose initial and final node coincide.



Connected and Acyclic Graphs (Graph Theory)

A graph is **connected** if there is a path between every pairs of nodes

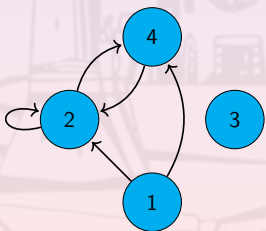
A **connected component** of a undirected graph G is a maximum connected sub-graph of G .

A graph is **acyclic** if it does not contains cycles

Directed Acyclic Graphs are also known as **DAGs**

Representing Graphs

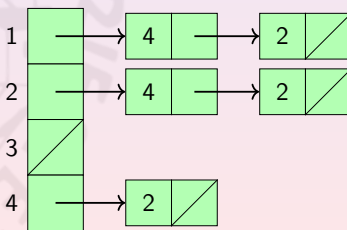
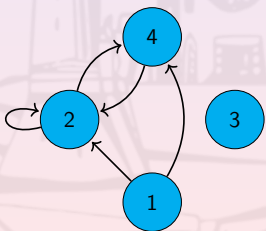
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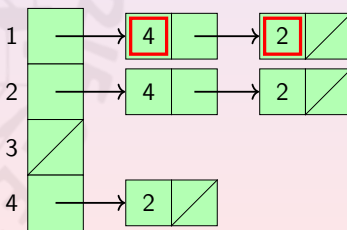
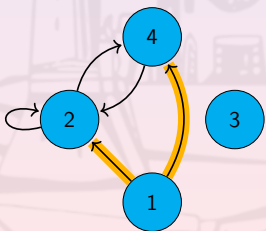
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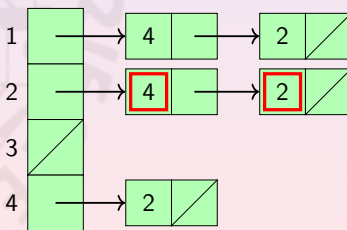
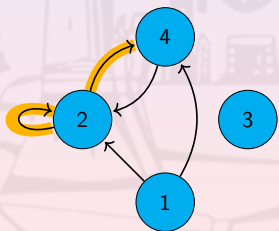
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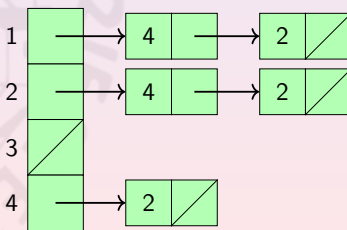
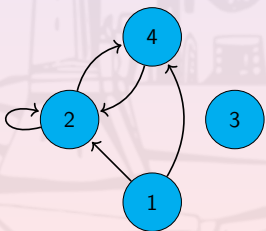
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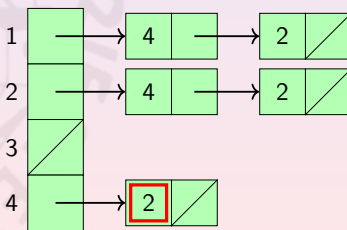
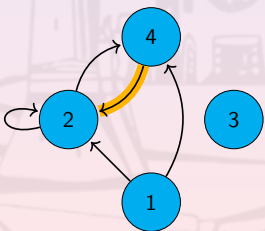
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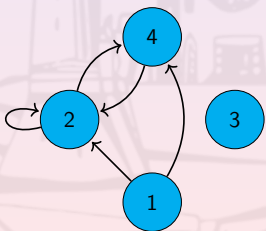
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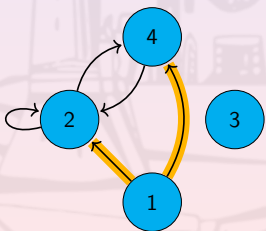


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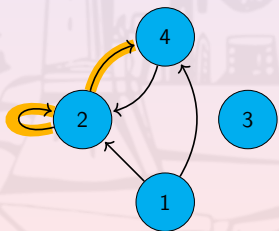


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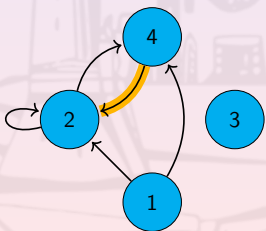


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Two main ways for both directed and undirected graphs:

Breadth-First-Search visits the graph as a “wave” from the source

Depth-First-Search search “deeper” in the graph whenever possible

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Breadth-First Search

Breadth-First-Search (BFS)

Visiting order is related to the distance from a **source** s : the lesser the distance of a node, the sooner it will be visited

Because of this, BFS is used to compute source-node distances

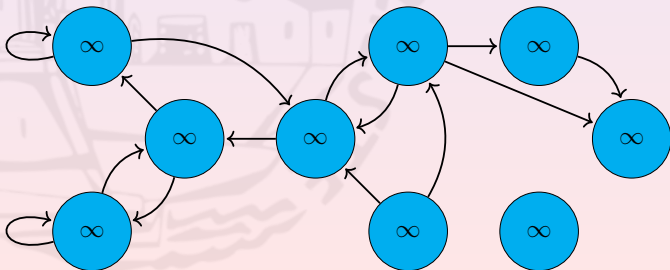
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It also produces the **breadth-first tree** i.e., the tree of shortest paths from s , and returns the shortest path from s to any reachable node

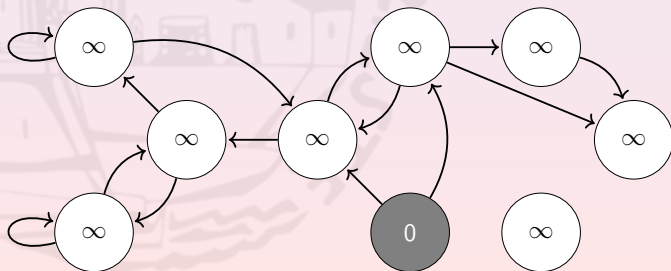
Breadth-First-Search (BFS): Coloring and Example



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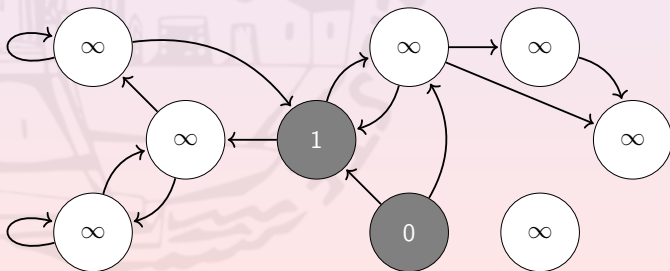
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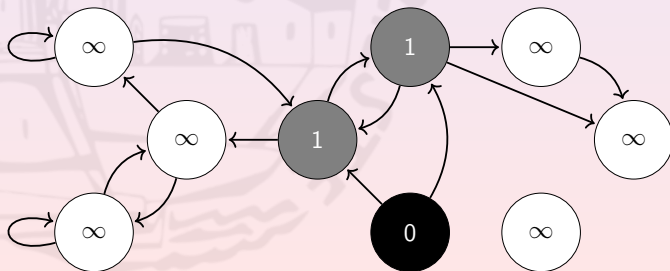
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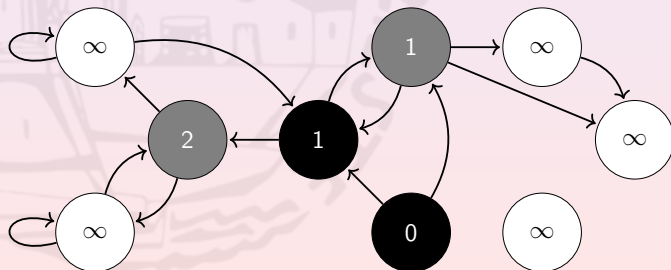
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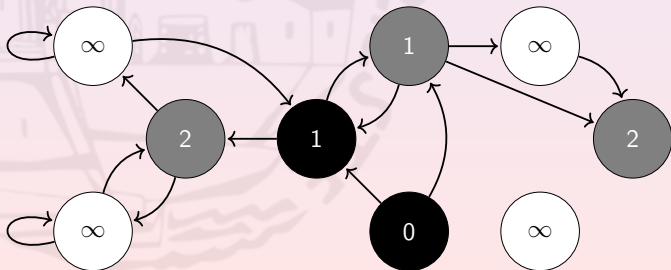
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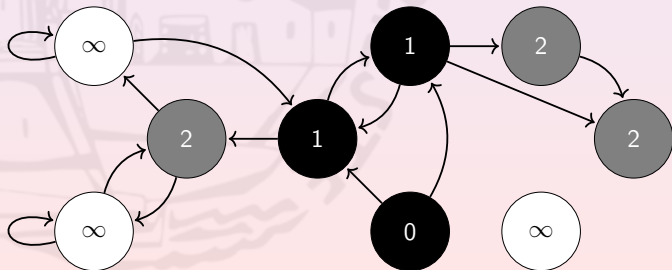


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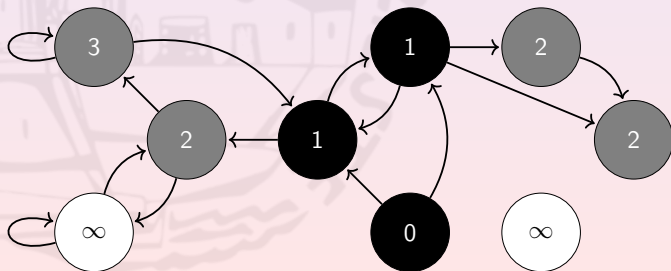




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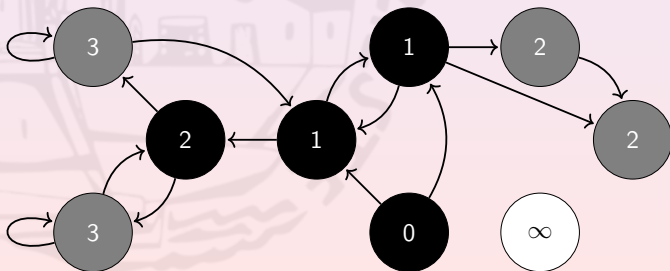
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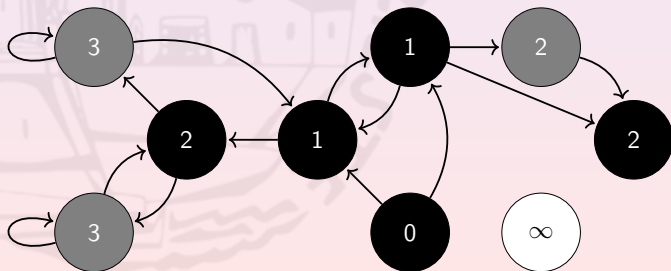


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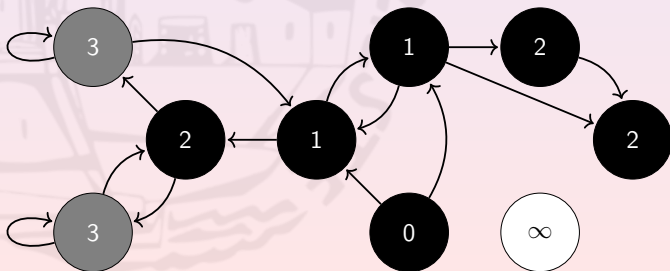




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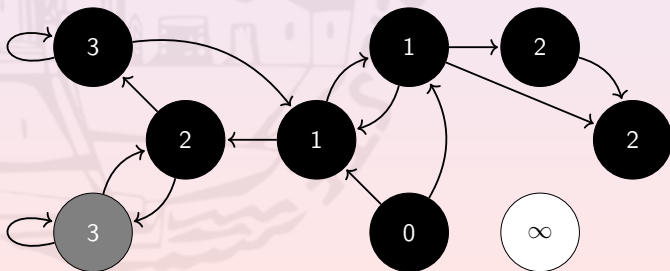
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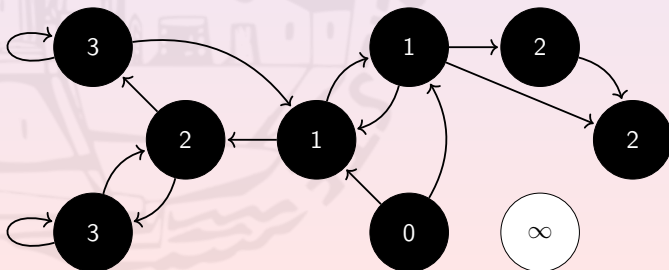
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Breadth-First-Search (BFS): Pseudo-Code

```
def BFS_SET(v, color, d, pred):  
    v.color ← color  
    v.d ← d  
    v.pred ← pred  
  
def BFS_INIT(G, s):  
    for v in G.V:  
        BFS_SET(v, WHITE, ∞, NIL)  
    endfor  
    BFS_SET(s, GRAY, 0, s)  
  
    return BUILD_QUEUE([s])
```

Breadth-First-Search (BFS): Pseudo-Code (Cont'd)

```
def BFS(G, s):  
    Q ← BFS_INIT(G, s)  
    while Q ≠ ∅:  
        u ← DEQUEUE(Q)  
  
        for v in G.Adj[u]:  
            if v.color = WHITE:  
                BFS_SET(v, GRAY, u.d+1, u)  
                ENQUEUE(Q, v)  
            endif  
        endfor  
        u.color ← BLACK  
    endwhile
```

Breadth-First-Search (BFS): Complexity

An iteration of the `while` extracts a u from Q and GREY colors it

The `for` loop costs $\Theta(|Adj[u]|)$ per `while` iteration

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BFG has asymptotic complexity $O(|V| + |E|)$

Breadth-First-Search (BFS): Code Properties

Lemma

Let $Q = [v_1, \dots, v_n]$ be the queue during BFS. Then $v_i.d \leq v_{i+1}.d$ for all $i \in [1, n-1]$ and $v_n.d \leq v_1.d + 1$.

Theorem

Let $\delta(s, v)$ be the distance from s to v . After BFS:

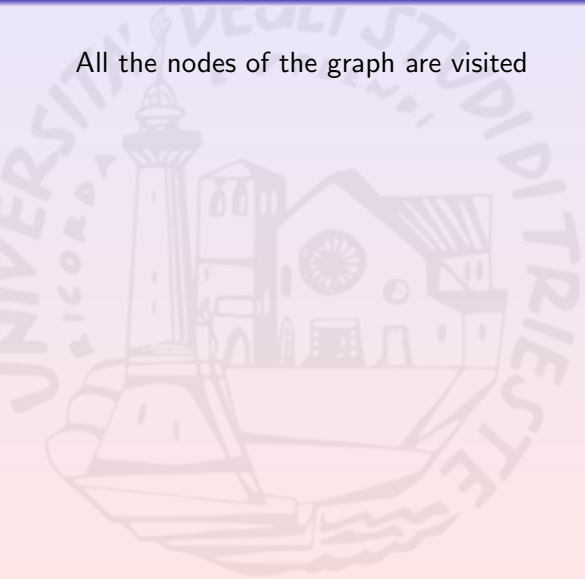
- $v.d \neq \infty$ iff v is reachable from s
- if $v.d \neq \infty$, then $v.d = \delta(s, v)$
- the shortest path from s to v ends with $(v.pred, v)$

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Depth-First Search

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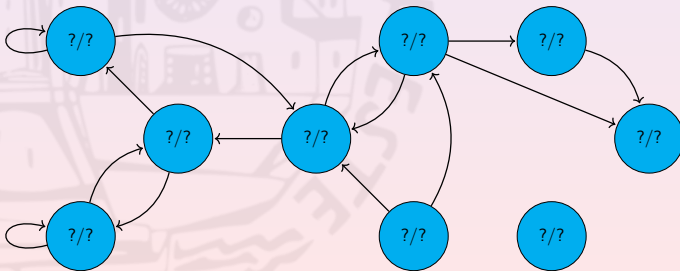
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DFS labels all the nodes with **discovery time** and **finishing time**

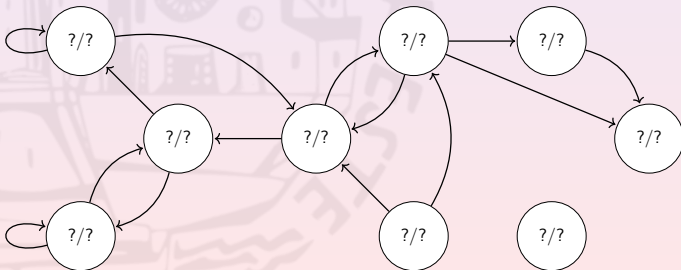
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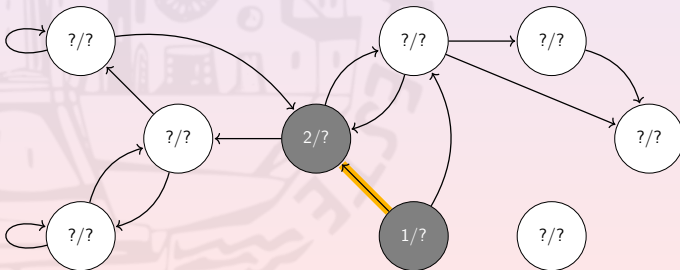




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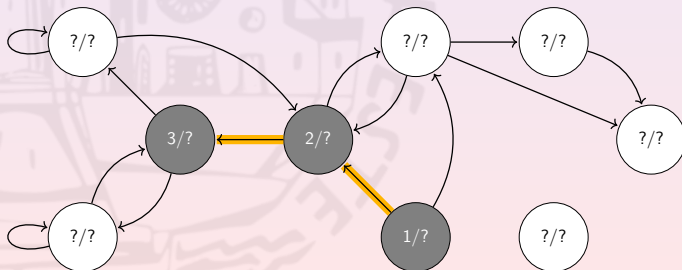
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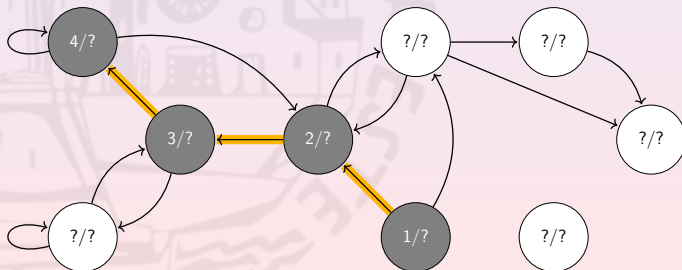
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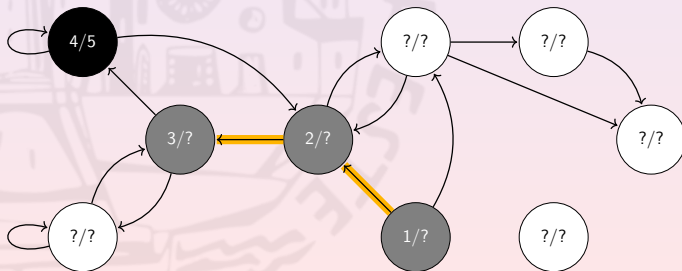
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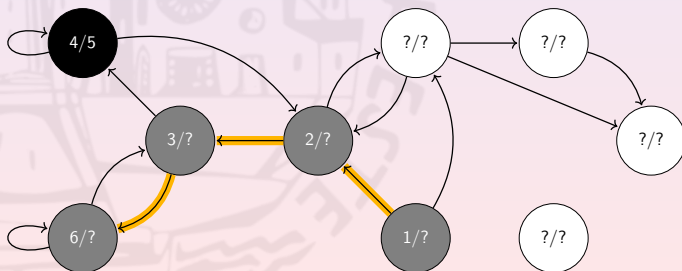
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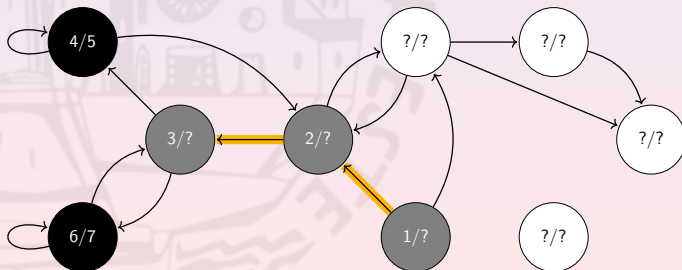
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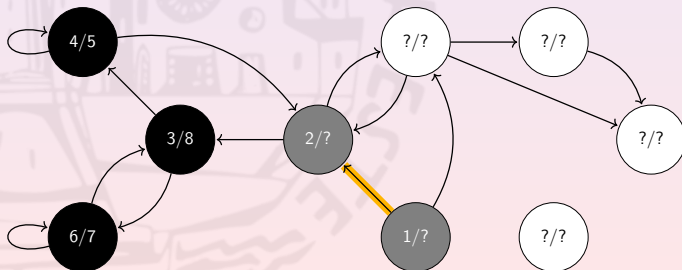
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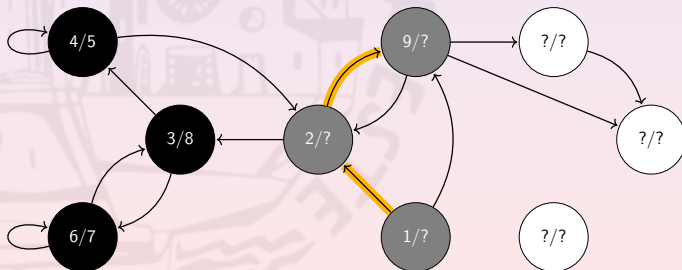
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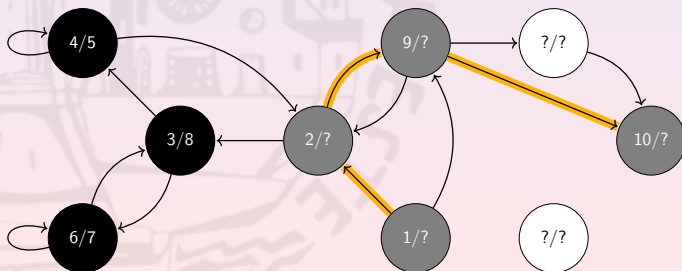
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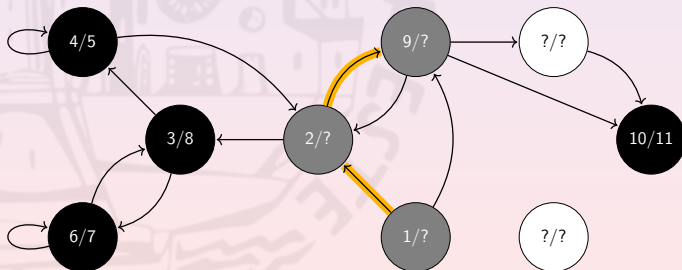
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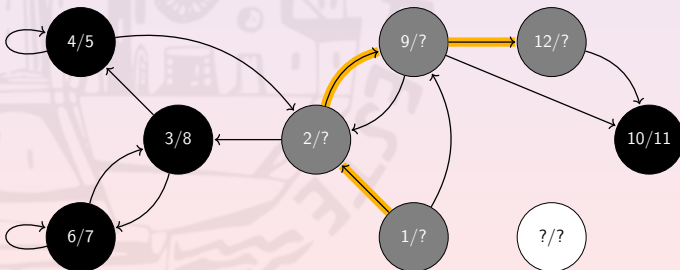
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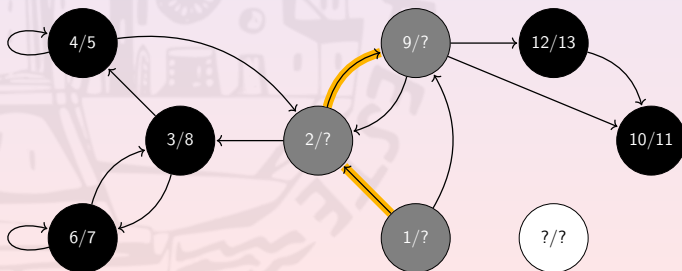
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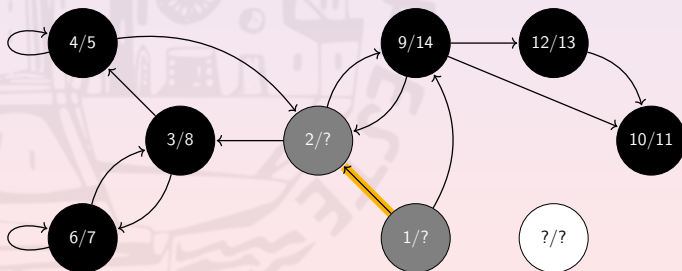
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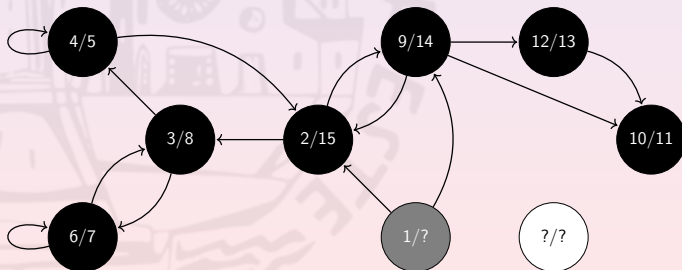
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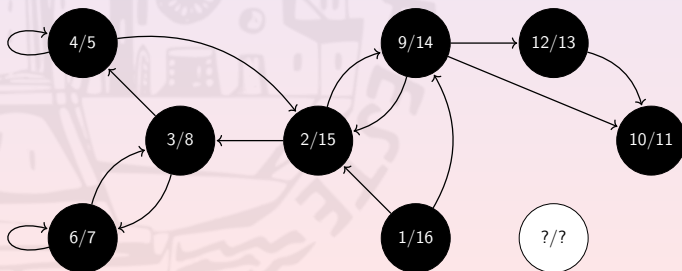
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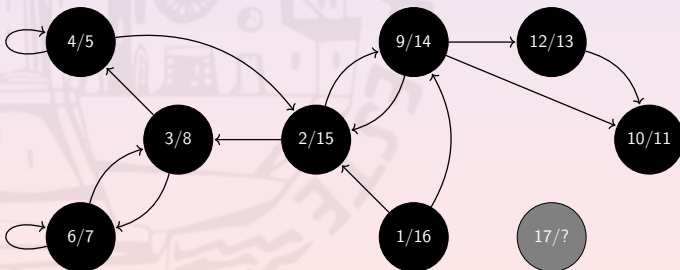
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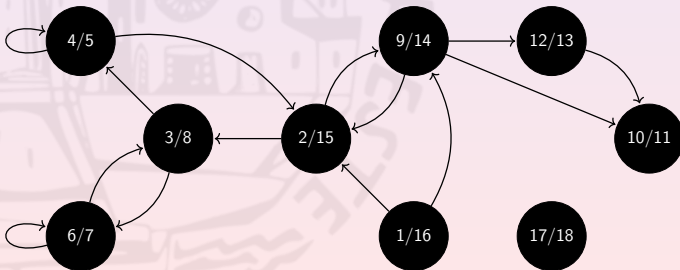
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Depth-First-Search (DFS): Pseudo-Code

```
def DFS(G):  
    for v in G.V:  
        v.color ← WHITE  
        v.pred ← NIL  
    endfor  
  
    time ← 0  
    for v in G.V:  
        if v.color = WHITE:  
            time ← DFS_VISIT(G,v,time)  
        endif  
    endfor  
enddef
```

Depth-First-Search (DFS): Pseudo-Code (Cont'd 2)

```
def DFS_VISIT(G, v, time):  
    # discovery  
    time  $\leftarrow$  time + 1  
    v.d  $\leftarrow$  time  
    v.color = GRAY  
  
    # search for WHITE neighbors  
    for u in G.Adj[v]:  
        if u.color = WHITE:  
            u.pred  $\leftarrow$  v  
            time  $\leftarrow$  DFS_VISIT(G, v, time)  
        endif  
    endfor
```

Depth-First-Search (DFS): Pseudo-Code (Cont'd 3)

finalization

$\text{time} \leftarrow \text{time} + 1$

$v.f \leftarrow \text{time}$

$v.\text{color} = \text{BLACK}$

return time

enddef

Depth-First-Search (DFS): Complexity

Each DFS_VISIT call GRAY color the node parameter

The for loop costs $\Theta(|Adj[u]|)$ per DFS_VISIT call

Each iteration of the for calls DFS_VISIT on $v \in Adj[u]$ if only if it is WHITE

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DFS has asymptotic complexity $\Theta(|V| + |E|)$

Depth-First-Search (DFS): Edges Classes

Tree Edges: edges of the depth first forest

Back Edges: connect a nodes to one of its ancestors

Forward Edges: non-tree edges connecting a node to one of its descendents

Cross Edges: the remaining edges

Depth-First-Search (DFS): Properties

Theorem (Parenthesis Theorem)

For any pair of nodes $v, u \in V$, either:

- $[u.d, u.f] \cap [v.d, v.f] = \emptyset$ and neither u is a descendant of v nor v of u
- $[u.d, u.f] \subsetneq [v.d, v.f]$ and u is a descendant of v
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Theorem (White-Path Theorem)

For any pair of nodes $v, u \in V$, u is a descendant of v iff at time $v.d - 1$ there exists a WHITE-only path from v to u .

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Topological Sort

How to Prepare... Tiramisù

It is quite simple

We have to:



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- beating the egg whites until still
- adding sugar
- soaking the Savoiardi (NOT Pavesini, please!) cookies in coffee
- preparing coffee
- incorporating mascarpone cheese
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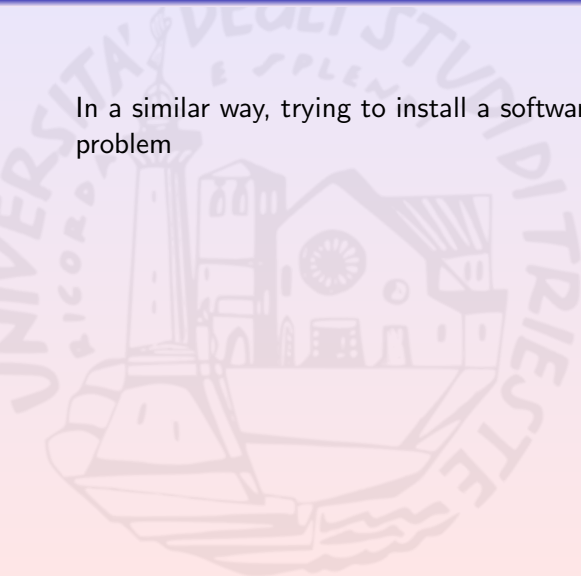
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Timing and sub-task ordering is fundamental

Installing Software

In a similar way, trying to install a software can be a “sorting” problem



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E.g., To install software A solve following dependencies

- Software A needs software B, C, D
- Software B depends on libraries E and F and on software G
- Software G depends on library E and on D
- ...

How to figure out what is needed and in which order?

Dependency Relations and Topological Sort

Dependency relations can be modeled by using directed graphs

Nodes represent sub-tasks, edges the needs, e.g., $(v, u) \in E$ iff v = “soaking cookies” needs u = “preparing coffee”

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On DAGs, there exists a **topological order**, \preceq_T , on nodes satisfying

If $(u, v) \in E$, then $u \preceq_T v$

Computing Topological Sort and Complexity

```
def TOPOLOGICAL_SORT(G)  
    call DFS(G)
```

```
enddef
```

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Asymptotic complexity $\Theta(|V| + |E|)$

Topological Sort: Correctness

Lemma

Let $[v_1, \dots, v_n]$ be the output of $\text{TOPOLOGICAL_SORT}(G)$. Then $v_i.f > v_{i+1}.f$ for all $i \in [1, n - 1]$

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Theorem

If G is a DAG, then $\text{TOPOLOGICAL_SORT}(G)$ produces G 's topological sort

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Strongly Connected Components

What Comes Before Egg or Hen?

When a graph is not a DAG, establishing a topological ordering is not possible

From the reachability point of view, all the nodes in a loop behave in the same way

If one of them is reachable from a node, so are the others

If one of them can reach a node, so do the others

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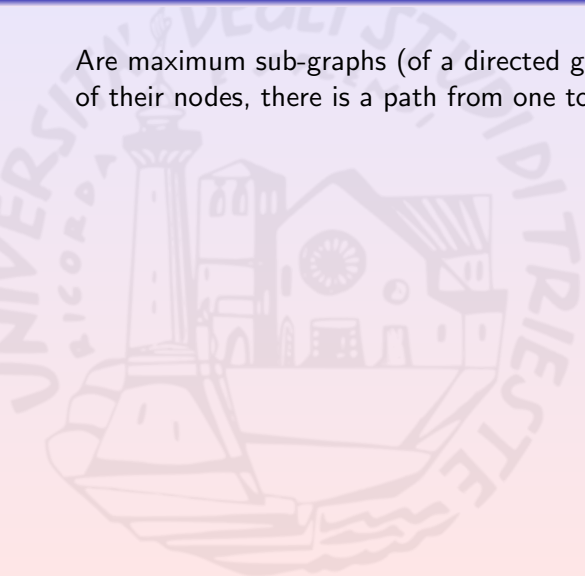
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Discovering equivalent nodes is a useful task

Strongly Connected Components (SCCs)

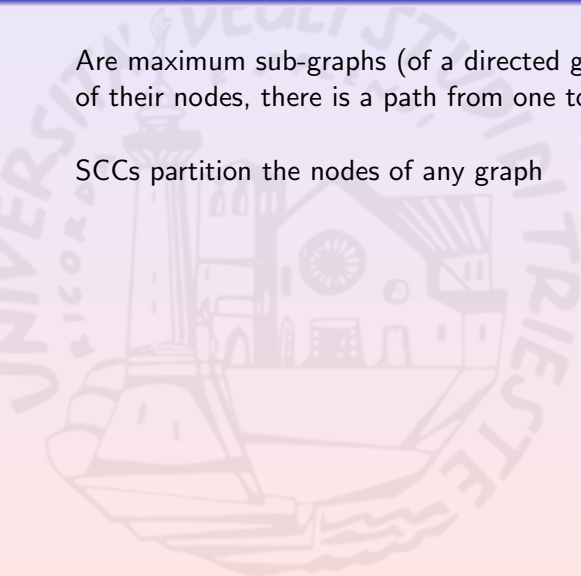
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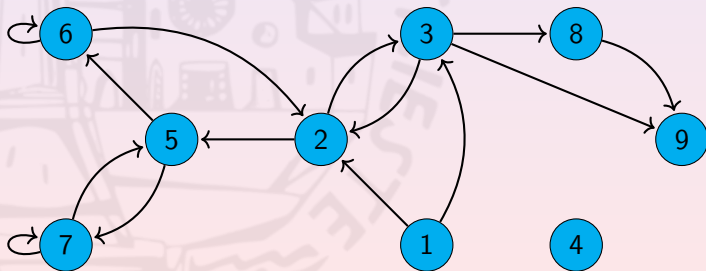
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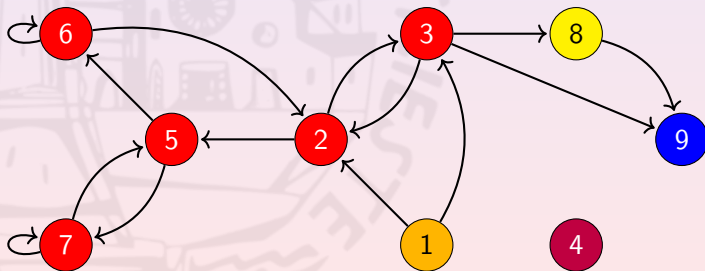
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Minimum Discovery Time from the Sub-Tree (lowlink): if it is smaller than the node discovery time, then either a forward or a back edge must be reachable from it

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Minimum Discovery Time from the Sub-Tree (lowlink): if it is smaller than the node discovery time, then either a forward or a back edge must be reachable from it

Is it relevant which of the two? No, we are interested in the SCC of the destination node

One DFS_VISIT Call to Rule Them All

One DFS_VISIT call discovers all the nodes of the SCCs it “touches”, i.e., no half visited SCC

DFS_VISITs do not interleave nodes of two distinct SCCs.

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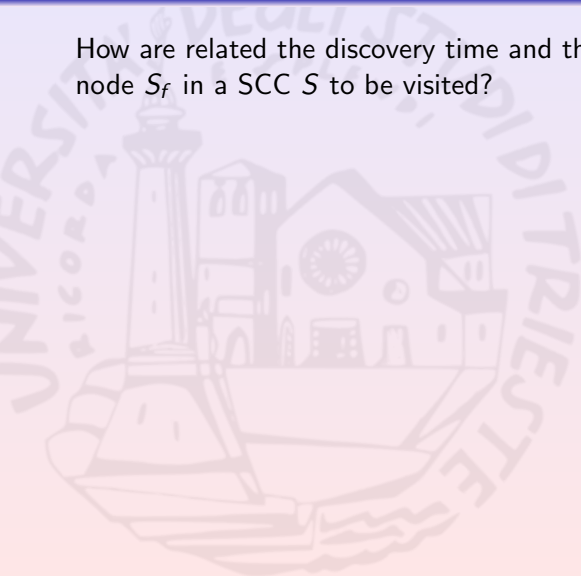
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then we have an algorithm to identify all the SCCs

Two Useful Intuitions

How are related the discovery time and the lowlink of the first node S_f in a SCC S to be visited?



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So, once S_f has been finished, all the nodes in S have too

Which nodes in the visit belong to the just identified SCC if we not consider the nodes of the already discovered SCCs?

The last ones!

We can use a stack to store finished nodes and use lowlink property to detect S_f

Tarjan's SCCs Algorithm: Pseudo-Code

```
def TARJAN_SCC(G):  
    for v in G.V:  
        v.color ← WHITE  
    endfor  
  
    time ← 0  
    S ← BUILD_STACK()  
    for v in G.V:  
        if v.color = WHITE:  
            time ← TARJAN_SCC_VISIT(G, v, S, time)  
        endif  
    endfor  
enddef
```

Tarjan's SCCs Algorithm: Pseudo-Code (Cont'd)

```
def TARJAN_SCC_VISIT(G, v, S, time):  
    time  $\leftarrow$  time + 1  
    v.d  $\leftarrow$  time  
    v.color = GRAY  
  
    v.lowlink  $\leftarrow$  time  
    S.push(v)  
    v.onStack  $\leftarrow$  True  
  
    for u in G.Adj[v]:  
        if u.color = WHITE:  
            time  $\leftarrow$  TARJAN_SCC_VISIT(G, v, S, time)
```

Tarjan's SCCs Algorithm: Pseudo-Code (Cont'd 2)

```
    v.lowlink  $\leftarrow$  min(v.lowlink , w.lowlink)
else if w.onStack:
    v.lowlink  $\leftarrow$  min(v.lowlink , w.d)
endif
endfor

if v.lowlink = v.d:
    yield EXTRACT_SCC_FROM_STACK(S,v)
endif

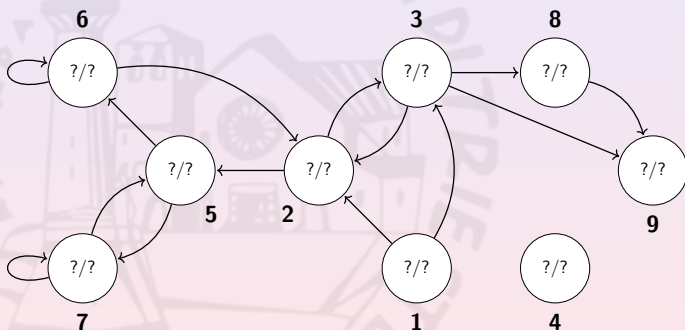
return time
enddef
```

Tarjan's SCCs Algorithm: Pseudo-Code (Cont'd 3)

```
def EXTRACT_SCC_FROM_STACK(S, v):  
    L  $\leftarrow$  EMPTY_LIST()  
  
    repeat :  
        w  $\leftarrow$  S.pop()  
        w.onStack  $\leftarrow$  False  
  
        L.append(w)  
    until v  $\neq$  S.top()  
  
    return L  
enddef
```

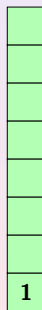
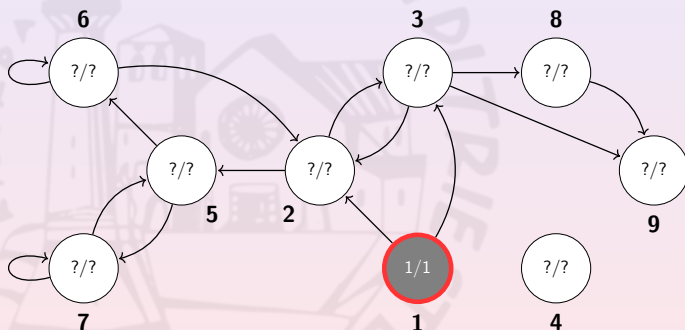
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Nodes are labeled by "Discovery Time" / "Lowlink"



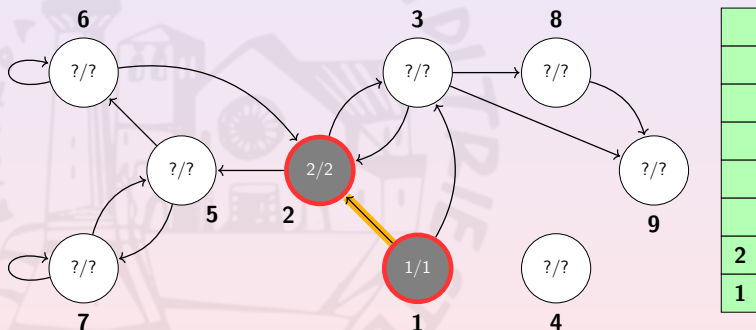
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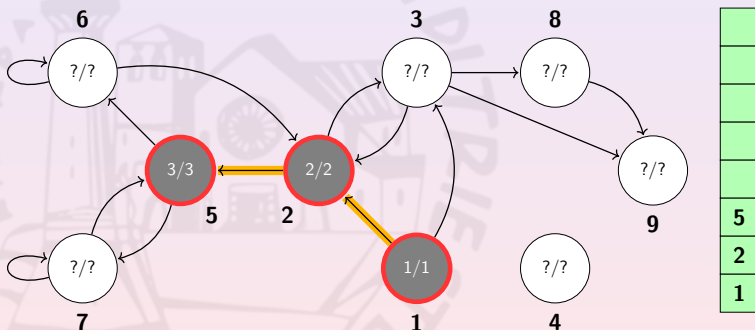
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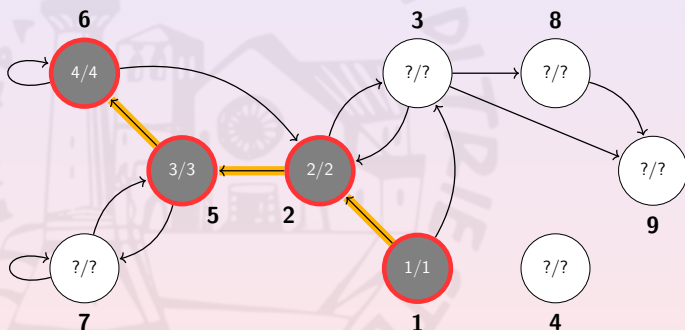
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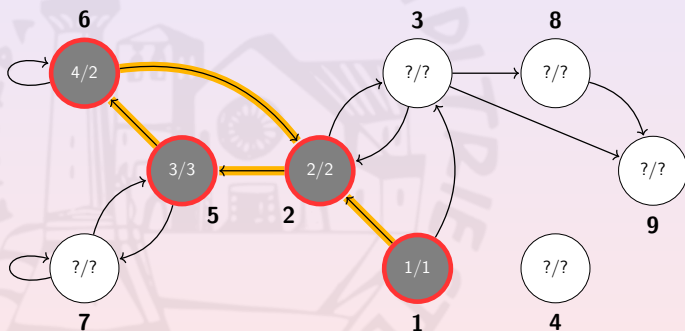
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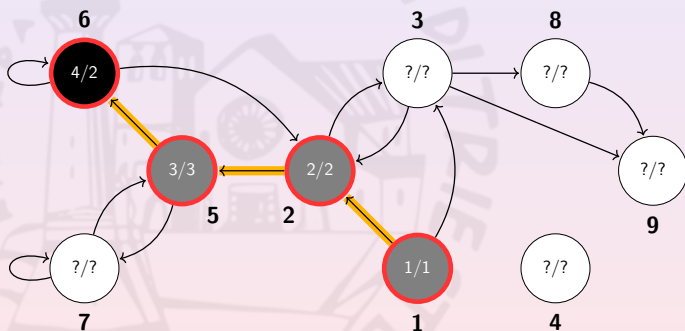
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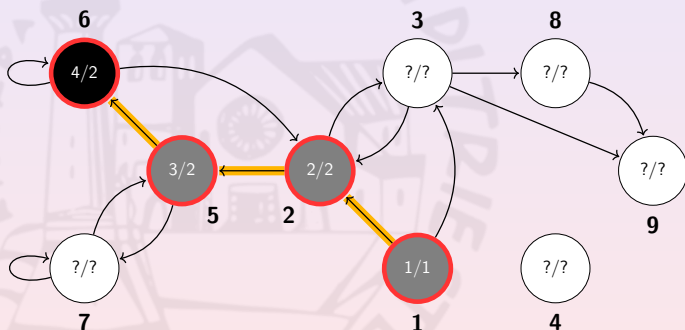
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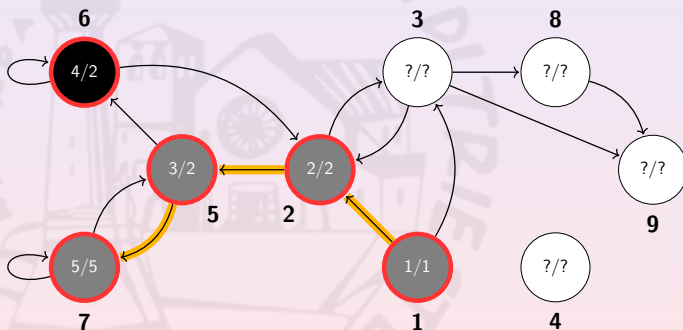
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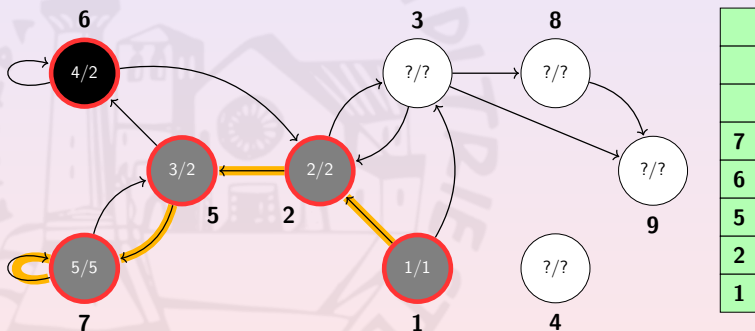
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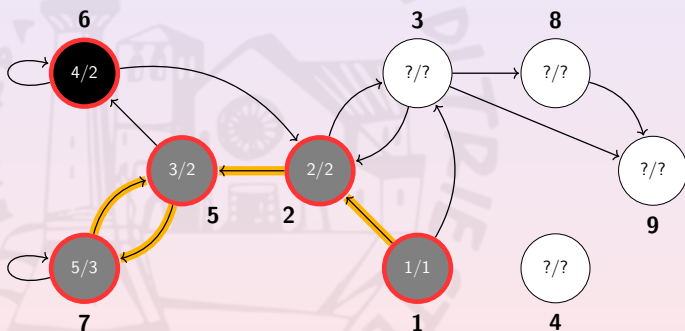
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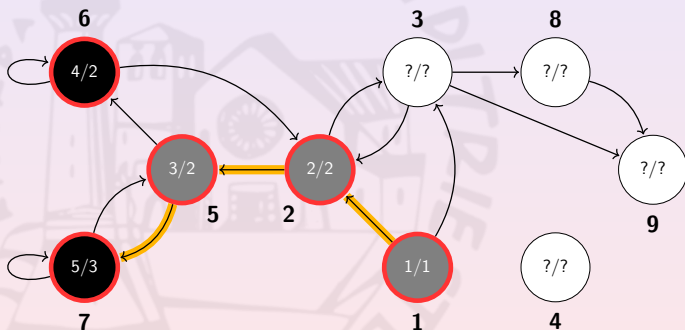
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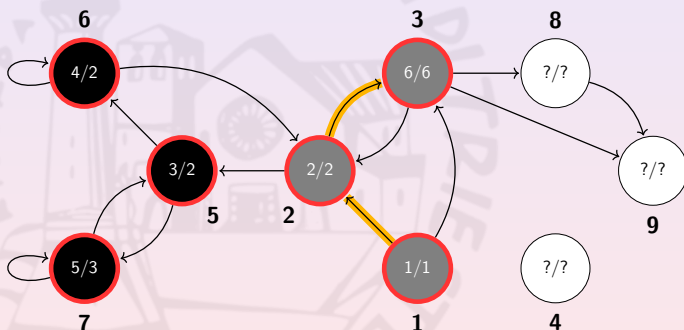
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Tarjan's SCCs Algorithm: Example

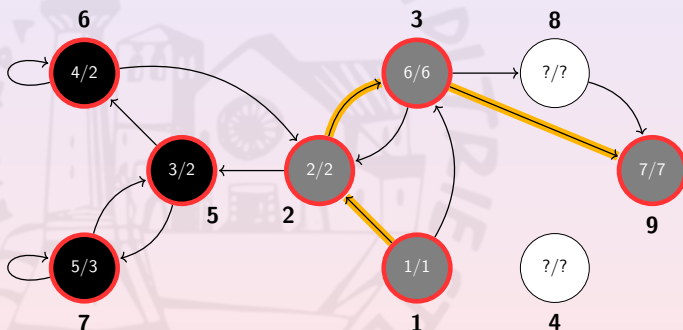
Nodes are labeled by "Discovery Time" / "Lowlink"



3
7
6
5
2
1

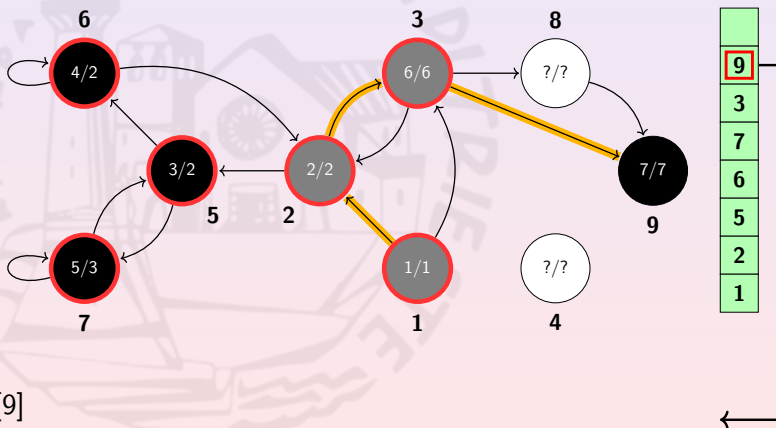
Tarjan's SCCs Algorithm: Example

Nodes are labeled by "Discovery Time" / "Lowlink"



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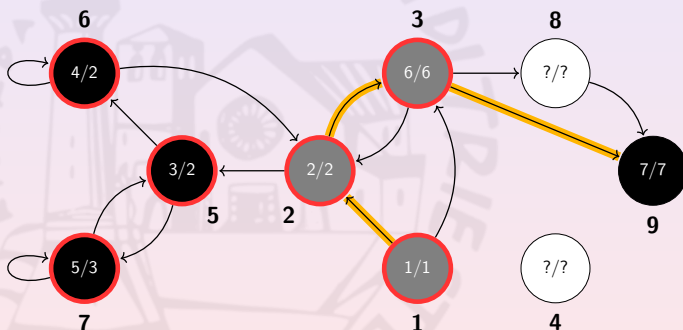
Nodes are labeled by "Discovery Time" / "Lowlink"



[9]

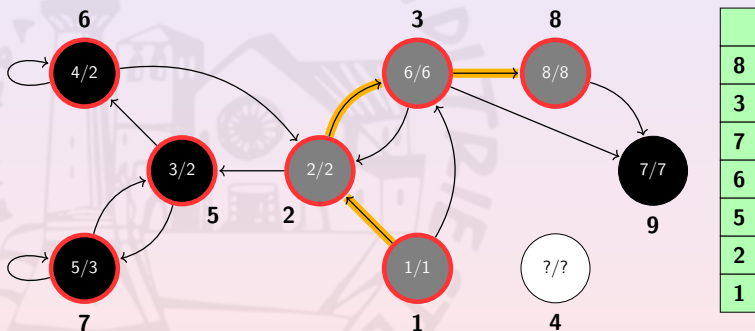
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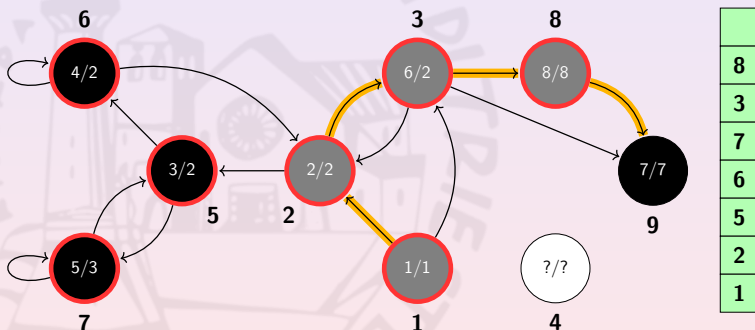
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Nodes are labeled by "Discovery Time" / "Lowlink"



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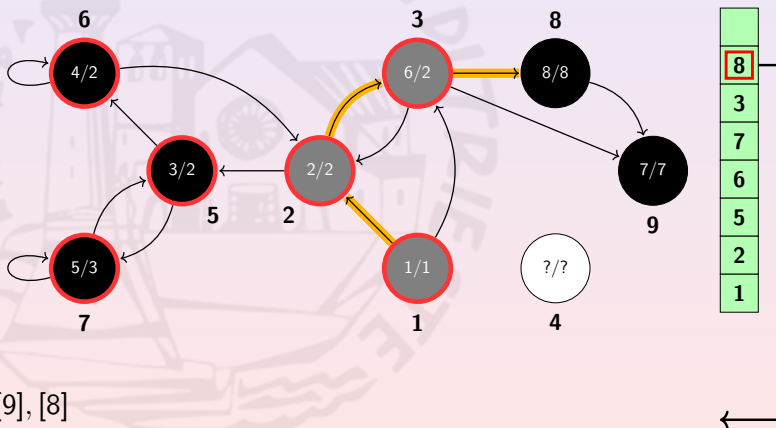
Nodes are labeled by "Discovery Time" / "Lowlink"



[9], [8]

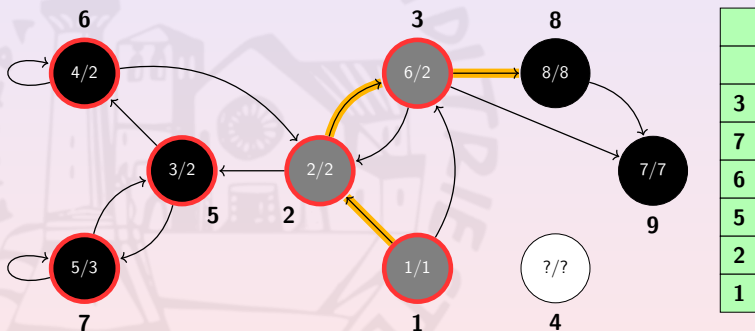
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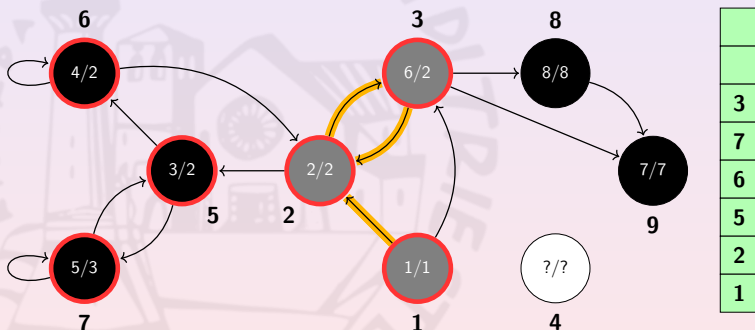
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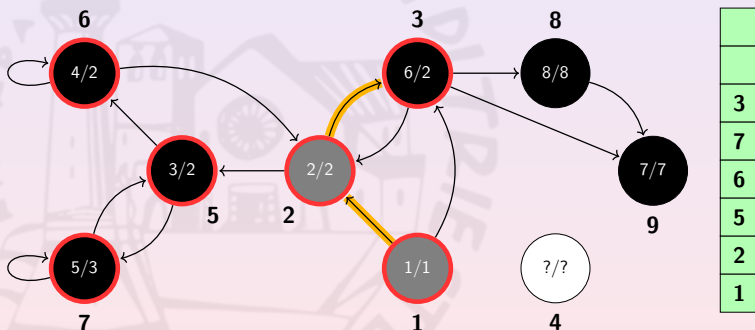
Nodes are labeled by "Discovery Time" / "Lowlink"



[9], [8]

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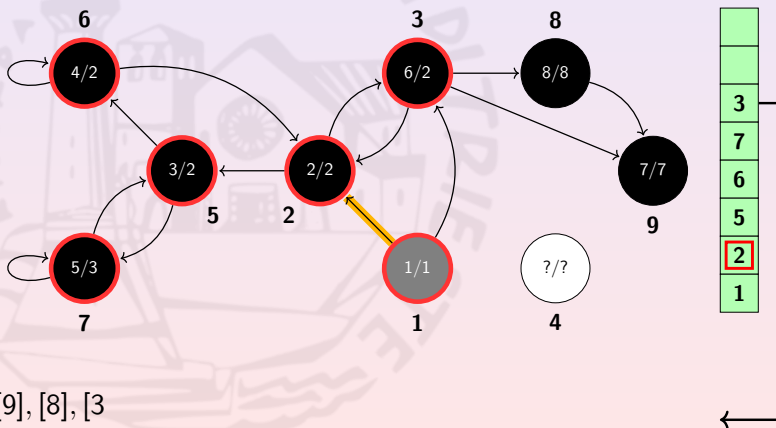
Nodes are labeled by "Discovery Time" / "Lowlink"



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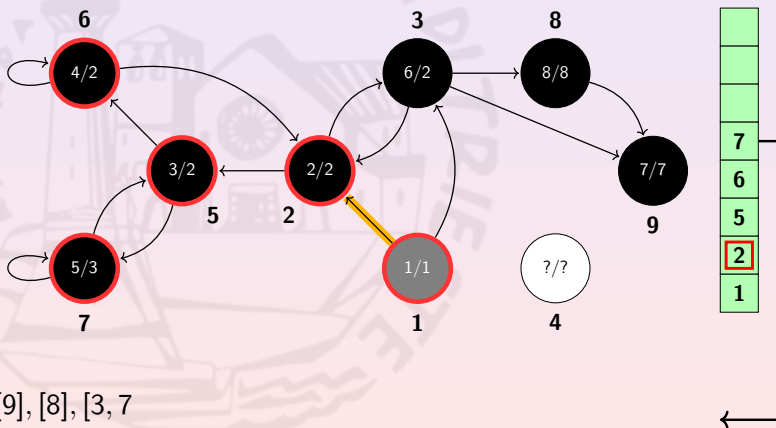
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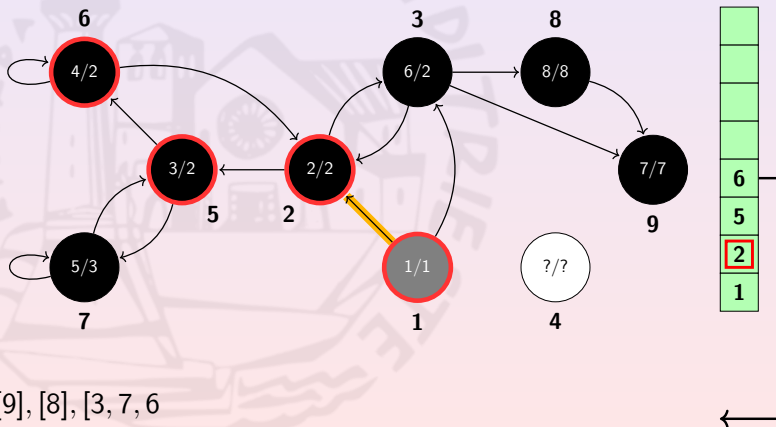
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[9], [8], [3], 7

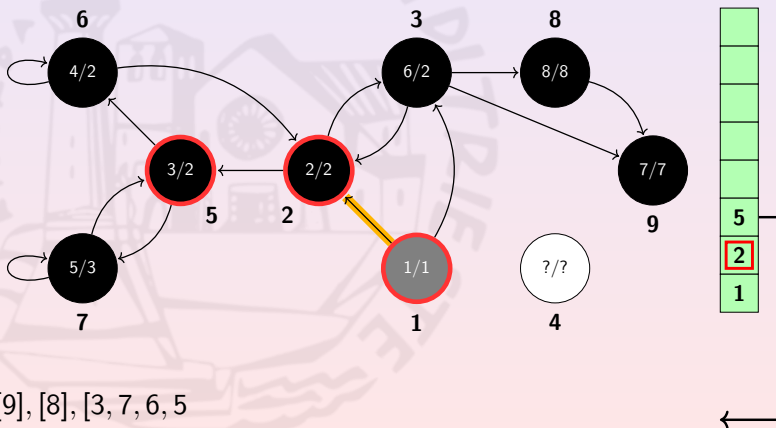
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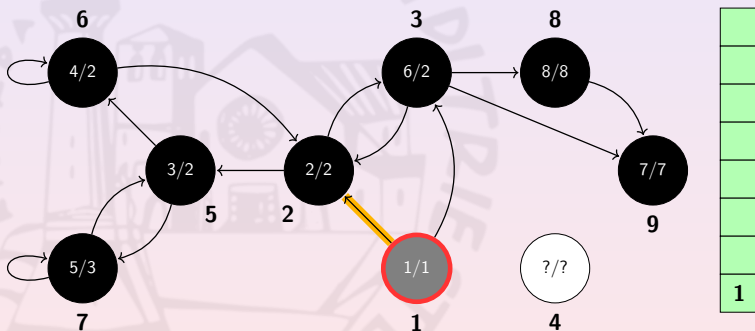
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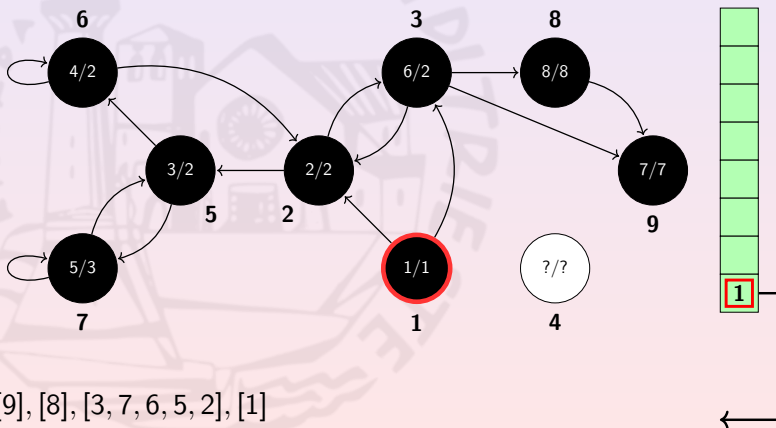
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[9], [8], [3, 7, 6, 5, 2]

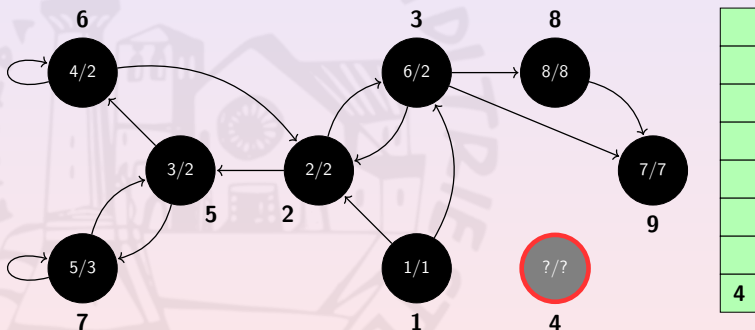
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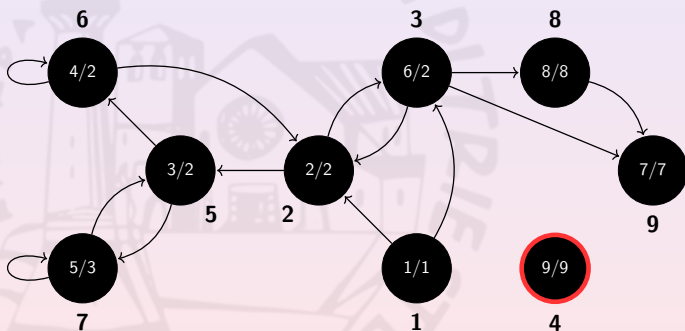
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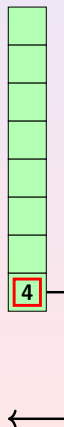
[9], [8], [3, 7, 6, 5, 2], [1]

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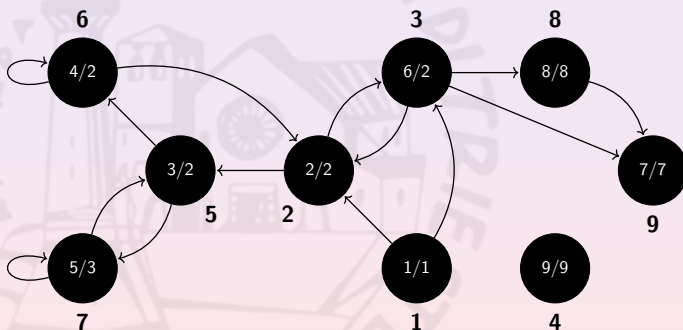


[9], [8], [3, 7, 6, 5, 2], [1], [4]



Tarjan's SCCs Algorithm: Example

Nodes are labeled by "Discovery Time" / "Lowlink"



[9], [8], [3, 7, 6, 5, 2], [1], [4]

Tarjan's SCCs Algorithm: Complexity

The algorithm performs a DFS-like visit + stack handling

One single node is pushed in S during each `TARJAN_SCC_VISIT` call

`TARJAN_SCC_VISIT` is called on WHITE nodes and sets them to GRAY

So, the # of node inserted into S at some point is $|V|$

All `EXTRACT_SCC_FROM_STACK` calls cumulatively cost $\Theta(|V|)$

Tarjan's algorithm costs $\Theta(|V| + |E|)$

The background of the slide features a large, faint watermark of the University of Trieste logo. The logo is circular and contains the text "UNIVERSITA' DEGLI STUDI DI TRIESTE" around the perimeter and "E SPLENDI" in the center. In the middle of the logo is a detailed illustration of a building, likely a university hall or library, with a dome and a tower.

Transitive Closure

Transitive Closure: Definition and Naïve Solution

For each pair of nodes v and w , we would like to know whether there is a path from v to w

How to solve this problem?

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Complexity? $|V| * O(|V| + |E|) = O(|V|^2 + |V| * |E|)$

Any other ideas?

Have a Look at the Matrix Formulation of the Problem

	1	2	3	4
1	1	0	0	0
2	1	1	0	0
3	0	0	1	0
4	0	1	0	1

- w has distance 1 from v iff $A[v][w] \neq 0$

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- w has distance n from v iff $A^n[v][w] > 0$

Transitive Closure By Matrix Multiplication

Every acyclic path has at most length $|V|$

We can solve the problem by using Strassen's algorithm:

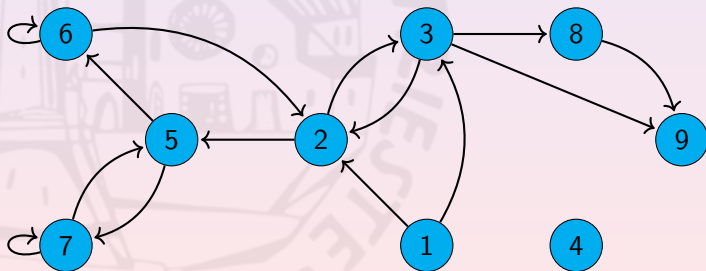
$$|V| * \Theta(|V|^{\log_2 7})$$

Which is worst than using BFS!!!

SCCs Are Unnecessarily Cumbersome

A.f.a. reachability concerns, all the nodes in a SCC are the same

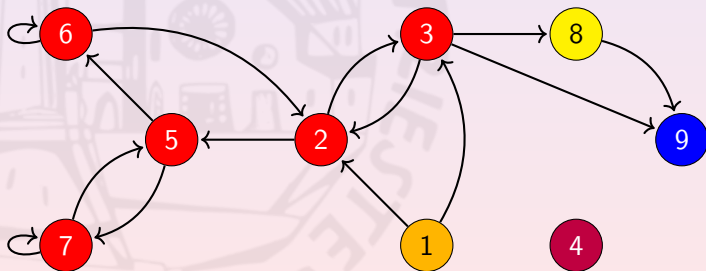
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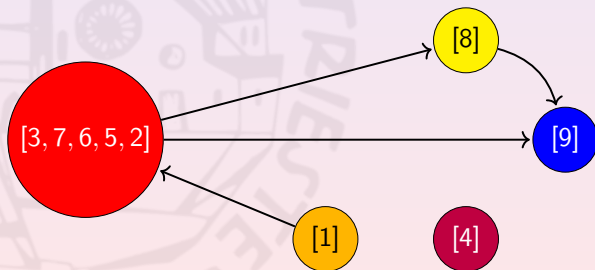
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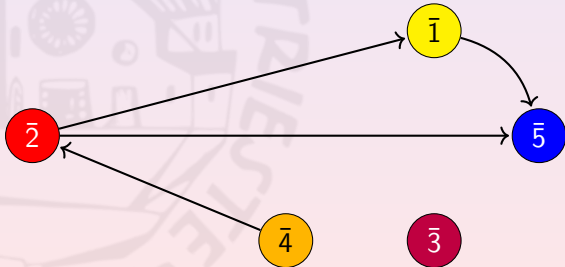
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And Then What?



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Topological Sort!!!



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Why? Have a look at the adjacency matrix of \bar{G}

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The new adjacency matrix is **upper-triangular**

And Then What?

Topological Sort!!! Tarjan's algorithm already did it!

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$$\bar{G}^* = \left(\begin{array}{c|c} \mathbf{A}^* & \mathbf{A}^* \times \mathbf{C} \times \mathbf{B}^* \\ \hline 0 & \mathbf{B}^* \end{array} \right)$$

The Complete Algorithm

- Tarjan's SCCs Algorithm on the input graph G : $\Theta(|V| + |E|)$
- Build the SCCs Graph \bar{G} of G : $\Theta(|E|)$
- Topological sort of \bar{G} : $\Theta(|V| + |E|)$
- Compute \bar{G}^* in time:

$$T(n) = 2 * T(n/2) + 2 * \Theta(n^{\log_2 7})$$

- Extend the transitive closure of \bar{G} to G : $O(|V|^2)$

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The overall asymptotic complexity is $\Theta(|E| + |V|^{\log_2 7})$