Diagonal Equivalence to Matrices with Prescribed Row and Column Sums

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and

$$\lim_{n\to\infty}\frac{f^{n+k+1}(x)-f^{n+k}(x)}{f^{n+1}(x)-f^n(x)}=\lim_{n\to\infty}\prod_{i=0}^{k-1}\frac{f^{n+i+2}(x)-f^{n+i+1}(x)}{f^{n+i+1}(x)-f^{n+i}(x)}=1. \tag{5}$$

Since $f'(x)$ is decreasing, it follows from (1) that $f'(x)\leqq 1$ in (a, b). Consequently the function $f(x)-x$ is decreasing in (a, b), i.e.

$$f(u)-u\leqq f(v)-v,\quad \text{whenever } v<u,\quad u, v\in(a, b). \tag{6}$$

Both the expressions in (6) are negative; hence

$$\frac{f(u)-u}{f(v)-v}\geqq 1,\quad \text{whenever } v<u,\quad u, v\in(a, b). \tag{7}$$

Setting $u=f^n(x)$ and $v=f^n(y)$ in (7) we get

$$\frac{f^{n+1}(x)-f^n(x)}{f^{n+1}(y)-f^n(y)}\geqq 1. \tag{8}$$

On the other hand, by (3), $f^{k+n}(x)<f^n(y)$ whence, by (6), $f^{n+1}(y)-f^n(y)\leqq f^{k+n+1}(x)-f^{k+n}(x)$ and

$$\frac{f^{n+1}(x)-f^n(x)}{f^{k+n+1}(x)-f^{k+n}(x)}\geqq\frac{f^{n+1}(x)-f^n(x)}{f^{n+1}(y)-f^n(x)}\,. \tag{9}$$

We now obtain (2) from (8), (9) and (5).

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DIAGONAL EQUIVALENCE TO MATRICES WITH PRESCRIBED ROW AND COLUMN SUMS

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The author shows in [1] that corresponding to each positive square matrix A there is a unique doubly stochastic matrix of the form D_1AD_2 where D_1 and D_2 are diagonal matrices with positive main diagonals. This doubly stochastic matrix can be obtained as a limit of the iterative process of alternately normalizing the rows and columns of A. It is the intent of this paper to generalize this result in the following way.

THEOREM. *Let $r_1, \cdots, r_m, c_1, \cdots, c_n$ be fixed positive numbers. Then, corresponding to each positive $m\times n$ matrix A, there is a unique matrix of the form D_1AD_2 with row sums $\mu r_1, \cdots, \mu r_m$ and column sums $c_1, \cdots, c_n$ where $\mu=\sum_j c_j/\sum_i r_i$. D_1 and D_2 are respectively $m\times m$ and $n\times n$ diagonal matrices* *with positive diagonals and are themselves unique up to a scalar multiple.*

The iterative process of alternately scaling the rows and columns of A to have row and column sums respectively r_i and c_j can be used to find D_1AD_2. The subsequence from the iteration in which column sums are scaled converges to D_1AD_2 while the subsequence in which the row sums are scaled converges to $(1/\mu)D_1AD_2$. In particular, if $\sum_i r_i = \sum_j c_j$ then the entire iteration converges to D_1AD_2.

The uniqueness part of the theorem is immediate. Suppose that D_1AD_2 and $D_1'AD_2'$ satisfy the requirements of the theorem where

$$D_1 = \operatorname{diag}(x_1, \cdots, x_m), \qquad D_1' = \operatorname{diag}(x_1', \cdots, x_m'),$$
$$D_2 = \operatorname{diag}(y_1, \cdots, y_n), \quad \text{and} \quad D_2' = \operatorname{diag}(y_1', \cdots, y_n').$$

Then, if $p_i = x_i'/x_i$ $(i=1, \cdots, m)$, and $q_j = y_j'/y_j$ $(j=1, \cdots, n)$,

$$\begin{aligned}\operatorname{Min} p_i = p_{i_1} &= \mu r_{i_1}\left(\textstyle\sum_j x_{i_1}a_{i_1j}q_jy_j\right)^{-1} \\ &\geqq \mu r_{i_1}(\operatorname{Max} q_j)^{-1}\left(\textstyle\sum_j x_{i_1}a_{i_1j}y_j\right)^{-1} = (\operatorname{Max} q_j)^{-1},\end{aligned}$$

and, similarly, $\operatorname{Max} q_j \leqq (\operatorname{Min} p_i)^{-1}$.

These inequalities are compatible only if each inequality is an equality. But equality in the former case forces $q_j = \operatorname{Max} q_j = q$, a positive constant for $j=1, \cdots, n$; in the latter case it forces $p_i = \operatorname{Min} p_i = p$, a positive constant for $i=1, \cdots, m$. Either equality then shows that $q = p^{-1}$. Thus $D_1' = pD_1$, $D_2' = p^{-1}D_2$ and $D_1AD_2 = D_1'AD_2'$.

The existence part of the theorem follows at once from the following lemma. The proof of the lemma justifies the statements regarding the iteration.

LEMMA. *Let $V \subseteq R^m \times R^n$ denote the set of pairs of vectors (x, y) with positive components such that $\sum_i x_ia_{ij}y_j = c_j$ $(j=1, \cdots, n)$, and $\|x\| = \operatorname{Max} |x_i| \leqq \delta^{1/2}a^{-1/2}$, $\|y\| = \operatorname{Max}|y_j| \leqq \delta^{1/2}a^{-1/2}$ where $\delta = \operatorname{Max}(r_i, c_j)$ and $a = \operatorname{Min} a_{ij}$, and let*

$$\phi(x, y) = \operatorname*{Max}_i r_i^{-1} \textstyle\sum_j x_ia_{ij}y_j - \operatorname*{Min}_i r_i^{-1} \textstyle\sum_j x_ia_{ij}y_j$$

have domain of definition V. Then $\phi(\hat{x}, \hat{y}) = 0$ for some $(\hat{x}, \hat{y}) \in V$.

Proof. First note that V is nonvoid since it contains (x^0, y^0) with components

$$x_{i0} = \delta^{1/2}a^{-1/2}, \quad y_{j0} = c_j\delta^{-1/2}a^{1/2}\left(\textstyle\sum_i a_{ij}\right)^{-1}.$$

Construct a sequence $(x^k, y^k) \in V$ as follows beginning with (x^0, y^0): Let $\alpha_{ik} = r_i^{-1}\sum_j x_{ik}a_{ij}y_{jk}$ and $\beta_{jk} = c_j^{-1}\sum_i \alpha_{ik}^{-1}x_{ik}a_{ij}y_{jk}$ and use

$$x_{i,k+1} = \delta^{1/2}a^{-1/2}M_k^{-1}\alpha_{ik}^{-1}x_{ik}$$

$$y_{j,k+1} = \delta^{-1/2}a^{1/2}M_k\beta_{jk}^{-1}y_{jk}$$

as the components of (x^{k+1}, y^{k+1}) where $M_k = \operatorname{Max}_i \alpha_{ik}^{-1}x_{ik}$.

Since for any j

$$\beta_{jk}^{-1} y_{jk} = \left(\sum_i \alpha_{ik}^{-1} x_{ik} a_{ij}\right)^{-1} c_j \leqq \left(\alpha_{ik}^{-1} x_{ik} a_{ij}\right)^{-1} c_j \leqq \left(\alpha_{ik}^{-1} x_{ik}\right)^{-1} a^{-1} \delta$$

for all i, it is true in particular that $\beta_{jk}^{-1} y_{jk} \leqq \mathrm{Min}_i (\alpha_{ik}^{-1} x_{ik})^{-1} a^{-1} \delta = M_k^{-1} a^{-1} \delta$ for all j. It readily follows that each $(x^k, y^k) \in V$.

Denoting Min α_i and Max α_i by α_{i_1} and α_{i_2} and Min β_j and Max β_j by β_{j_1} and β_{j_2} respectively, we see that

$$\begin{aligned}
\phi(x^{k+1}, y^{k+1}) &= \alpha_{i_2,k+1} - \alpha_{i_1,k+1} \\
&= r_{i_2}^{-1} \sum_j x_{i_2,k+1} a_{i_2 j} y_{j,k+1} - r_{i_1}^{-1} \sum_j x_{i_1,k+1} a_{i_1 j} y_{j,k+1} \\
&= r_{i_2}^{-1} \sum_j \alpha_{i_2 k}^{-1} x_{i_2 k} a_{i_2 j} \beta_{jk}^{-1} y_{jk} - r_{i_1}^{-1} \sum_j \alpha_{i_1 k}^{-1} x_{i_1,k} a_{i_1 j} \beta_{jk}^{-1} y_{jk} \\
&= r_{i_2}^{-1} \alpha_{i_2 k}^{-1} x_{i_2 k} a_{i_2 j_2} \beta_{j_2 k}^{-1} y_{j_2 k} + r_{i_2}^{-1} \sum_{j \neq j_2} \alpha_{i_2 k}^{-1} x_{i_2 k} a_{i_2 j} \beta_{jk}^{-1} y_{jk} \\
&\quad - r_{i_1}^{-1} \alpha_{i_1 k}^{-1} x_{i_1 k} a_{i_1 j_1} \beta_{j_1 k}^{-1} y_{j_1 k} - r_{i_1}^{-1} \sum_{j \neq j_1} \alpha_{i_1 k}^{-1} x_{i_1 k} a_{i_1 j} \beta_{jk}^{-1} y_{jk} \\
&\leqq r_{i_2}^{-1} \alpha_{i_2 k}^{-1} x_{i_2 k} a_{i_2 j_2} \beta_{j_2 k}^{-1} y_{j_2 k} + r_{i_2}^{-1} \beta_{j_1 k}^{-1} \sum_{j \neq j_2} \alpha_{i_2 k}^{-1} x_{i_2 k} a_{i_2 j} y_{jk} \\
&\quad - r_{i_1}^{-1} \alpha_{i_1 k}^{-1} x_{i_1 k} a_{i_1 j_1} \beta_{j_1 k}^{-1} y_{j_1 k} - r_{i_1}^{-1} \beta_{j_2 k}^{-1} \sum_{j \neq j_1} \alpha_{i_1 k}^{-1} x_{i_1 k} a_{i_1 j} y_{jk} \\
&= \beta_{j_2 k}^{-1} (\alpha_{i_2 k}^{-1} x_{i_2 k} a_{i_2 j_2} r_{i_2}^{-1} y_{j_2 k}) + \beta_{j_1 k}^{-1} (1 - \alpha_{i_2 k}^{-1} x_{i_2 k} a_{i_2 j_2} r_{i_2}^{-1} y_{j_2 k}) \\
&\quad - \beta_{j_1 k}^{-1} (\alpha_{i_1 k}^{-1} x_{i_1 k} a_{i_1 j_1} r_{i_1}^{-1} y_{j_1 k}) - \beta_{j_2 k}^{-1} (1 - \alpha_{i_1 k}^{-1} x_{i_1 k} a_{i_1 j_1} r_{i_1}^{-1} y_{j_1 k}) \\
&= (\beta_{j_1 k}^{-1} - \beta_{j_2 k}^{-1})(1 - \alpha_{i_2 k}^{-1} x_{i_2 k} a_{i_2 j_2} r_{i_2}^{-1} y_{j_2 k} - \alpha_{i_1 k}^{-1} x_{i_1 k} a_{i_1 j_1} r_{i_1}^{-1} y_{j_1 k}) \\
&\leqq (\beta_{j_1 k}^{-1} - \beta_{j_2 k}^{-1})(1 - 2d_k),
\end{aligned}$$

where $d_k = \mathrm{Min}_{i,j}\, \alpha_{ik}^{-1} x_{ik} a_{ij} r_i^{-1} y_{jk}$.

But

$$\alpha_{ik}^{-1} x_{ik} = \left(\sum_j a_{ij} y_{jk}\right)^{-1} r_i \geqq \left(\sum_j a_{ij}\right)^{-1} r_i a^{1/2} \delta^{-1/2} \geqq \rho R^{-1} a^{1/2} \delta^{-1/2}$$

and

$$y_{jk} = \left(\sum_i a_{ij} x_{ik}\right)^{-1} c_j \geqq \left(\sum_i a_{ij}\right)^{-1} c_j a^{1/2} \delta^{-1/2} \geqq \rho R^{-1} a^{1/2} \delta^{-1/2},$$

where $\rho = \mathrm{Min}(r_i, c_j)$ and $R = \sum_{i,j} a_{ij}$. Thus, for all k

$$d_k \geqq \rho R^{-1} a^{1/2} \delta^{-1/2} a \delta^{-1} \rho R^{-1} a^{1/2} \delta^{-1/2} = \rho^2 R^{-2} a^2 \delta^{-2}.$$

Furthermore, it is clear that $\alpha_{i_2 k}^{-1} \leqq \beta_{jk} \leqq \alpha_{i_1 k}^{-1}$ for all j and k. Hence

$$\begin{aligned}
\phi(x^{k+1}, y^{k+1}) &\leqq (\alpha_{i_2 k} - \alpha_{i_1 k})(1 - 2\rho^2 R^{-2} a^2 \delta^{-2}) \\
&= (1 - 2\rho^2 R^{-2} a^2 \delta^{-2}) \phi(x^k, y^k),
\end{aligned}$$

and it follows that $\phi(x^k, y^k) \to 0$.

Since $\|x^k\| \leqq \delta^{1/2}a^{-1/2}$ and $\|y^k\| \leqq \delta^{1/2}a^{-1/2}$ for all k, the sequence $\{(x^k, y^k)\}$ has a limit point $(\hat{x}, \hat{y}) \in R^m \times R^n$. The continuity of ϕ shows in particular that

$$\phi(\hat{x}, \hat{y}) = \lim_{k\to\infty} \phi(x^k, y^k) = 0.$$

It is seen that $\beta_{j_2k}^{-1} \leqq \alpha_{i,k+1} \leqq \beta_{j_1k}^{-1}$ for all i and k. Thus $\alpha_{i_1k} \leqq \alpha_{i_1,k+1} \leqq \alpha_{i_2,k+1} \leqq \alpha_{i_2k}$ and, since $\alpha_{ik}^{-1}x_{ik} \geqq \rho R^{-1}a^{1/2}\delta^{-1/2}$, $x_{ik} \geqq \alpha_{ik}\rho R^{-1}a^{1/2}\delta^{-1/2} \geqq \alpha_{i_10}\rho R^{-1}a^{1/2}\delta^{-1/2}$ $(i=1, \cdots, m; k=0, 1, \cdots)$. This along with $y_{jk} \geqq \rho R^{-1}a^{1/2}\delta^{-1/2}$ $(j=1, \cdots, n; k=0, 1, \cdots)$, shows that $\hat{x}$ and $\hat{y}$ have positive coordinates.

Then, since $\|x^k\| \leqq \delta^{1/2}a^{-1/2}$ and $\|y^k\| \leqq \delta^{1/2}a^{-1/2}$ for all k, and $\sum_i x_{ik}a_{ij}y_{jk} = c_j$ for all j and k, $\|\hat{x}\| \leqq \delta^{1/2}a^{-1/2}$, $\|\hat{y}\| \leqq \delta^{1/2}a^{-1/2}$, and $\sum_i \hat{x}_i a_{ij}\hat{y}_j = c_j$ for all j. Thus $(\hat{x}, \hat{y}) \in V$ and the lemma is proved.

The proof of the theorem may now be completed. Using the $(\hat{x}, \hat{y})$ of the lemma, set $D_1 = \operatorname{diag}(\hat{x}_1, \cdots, \hat{x}_m)$ and $D_2 = \operatorname{diag}(\hat{y}_1, \cdots, \hat{y}_n)$. Then $\sum_i \hat{x}_i a_{ij}\hat{y}_j = c_j$ $(j=1, \cdots, n)$, and, since $\phi(\hat{x}, \hat{y}) = 0$, $\sum_j \hat{x}_i a_{ij}\hat{y}_j = Kr_i$ $(i=1, \cdots, m)$, where K does not depend upon i. But

$$K \sum_i r_i = \sum_i \sum_j \hat{x}_i a_{ij}\hat{y}_j = \sum_j c_j = \mu \sum_i r_i;$$

hence $K = \mu$.

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A NOTE ON EXTREME POINTS

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Let $F: X \to Y$ be a mapping of a topological space X into another topological space Y. A point $x \in X$ will be said to be an extreme point of X with respect to the mapping $F: X \to Y$, if there exists an open neighborhood $N(x)$ of x such that $F(x) \in \text{Bdry } F(N(x))$, where Bdry $A = A - A^0$, A^0 standing for the interior of the set A. The set of all extreme points of X with respect to the mapping $F: X \to Y$ will be denoted by $E_F(X, Y)$ and the mapping F will be said to be extreme at every point $x \in E_F(X, Y)$ and the value of the mapping $F(x)$, $x \in E_F(X, Y)$ will be called an extreme value of the mapping $F: X \to Y$.

Lemma 1. *If x is an extreme point of X with respect to the mapping $F: X \to Y$, and $N(x)$ is an open neighborhood of x such that $F(x) \in$* Bdry *$F(N(x))$, then if $N'(x)$ is any other open subneighborhood of x such that $N'(x) \subset N(x)$, then $F(x)$ will also belong to* Bdry $F(N'(x))$.

Proof. We have by hypothesis $F(x) \in \text{Bdry } F(N(x))$. Let $N'(x)$ be an open subneighborhood of x such that $N'(x) \subset N(x)$. If possible, let $F(x) \in F(N'(x))^0$.