TWISTED MULTIPLICATIVITY OF THE T-TRANSFORM: TWO MORE PROOFS

KURUSCH EBRAHIMI-FARD, NICOLAS GILLIERS

Contents

1. Introduction **1**
1.1 Contribution 3
1.2 Outline 3
1.3 Basic notions and notations 3
2. Operads and brace algebras **4**
2.1 Algebraic planar operads 4
2.2 Operads of non-crossing partitions and non-crossing linked partitions 5
2.3 Operads with multiplication 6
3. Twisted multiplicativity of the T-transform **7**
3.1 Free cumulants of free products 8
3.2 Short proof of the twisted multiplicativity 10
3.3 Free product in an operad with multiplication 12
References **14**

1. Introduction

In [4], K. Dykema introduces and studies two central objects in free probability theory, i.e., the operator-valued R-transform, more precisely, the *unsymmetrised* R-transform, as well as the – interrelated – operator-valued *unsymmetrised* S- and T-transforms. Those transforms play a fundamental role in both scalar- as well as operator-valued free probability theory as they allow for the effective –algorithmic– calculation of the distribution of a sum respectively product of free random variables. In the scalar-valued case, they can be traced back to the seminal works by Voiculescu [12, 13]. Here, the R- and T-transforms with respect to a random variable a in a non-commutative probability space $(\mathcal{A}, \phi)$ are formal power series in one variable, $R_a(z), T_a(z) \in \mathbb{K}\langle\langle z\rangle\rangle$. They are characterised by linearity

$$R_{a+b}(z) = R_a(z) + R_b(z) \tag{1}$$

respectively multiplicativity

$$T_{ab}(z) = T_a(z) \cdot T_b(z), \tag{2}$$

where a and b are free random variables in $\mathcal{A}$ with $\phi(a) = \phi(b) = 1$ and the product on the right-hand side of (2) is the Cauchy product, defined for two series $f(z), g(z) \in \mathbb{K}\langle\langle z\rangle\rangle$ by:

$$(f \cdot g)_n = \sum_{\substack{k,q\geq 0,\\ k+q=n}} f_k g_q.$$

The S-transform is defined as the inverse (with respect to the Cauchy product) of the T-transform, $S_a(z) = T_a^{-1}(z)$. All three transforms are related through the *distribution series* $\Phi_a(z)$, $a \in \mathcal{A}$,

$$\Phi_a(z) = \sum_{n\geq 1} \phi(a^n) z^{n-1}, \tag{3}$$

Date: October 28, 2020

namely,

$$(z + z^2\Phi_a(z))^{\langle -1\rangle} = z(1 + zR_a(z))^{-1}$$
$$(z\Phi_a(z))^{\langle -1\rangle} = z(1 + z)^{-1}S_a(z). \tag{4}$$

The inversion on the left-hand sides of the equations above is with respect to composition $\circ$ of formal power series, defined by

$$(f \circ g)_n = \sum_{\substack{k,n_1,\dots,n_k\geq 1\\ n_1+\cdots+n_k=n}} f_k g_{n_1} \cdots g_{n_k}. \tag{5}$$

The so-called free additive and multiplicative convolution problems have been shown by Voiculescu to admit solutions by constructing canonical random variables. Dykema verified that in the operator-valued case, these results admit counterparts, where *multilinear function series* play the role of formal power series.

More specifically, let $(\mathcal{A}, \phi, B)$ be an operator-valued probability space [**?**]. A mutilinear function series $\alpha \in \mathrm{Mult}[[B]]$ is a sequence $(\alpha_n)_{n\geq 0}$ of multilinear maps on the algebra B, with $\alpha_n : B^{\otimes n} \to B$. Let $a \in \mathcal{A}$ be a random variable with $\phi(a) = 1$. The multlinear function series $\Phi_a \in \mathrm{Mult}[[B]]$ that replaces the distribution series in the scalar case (3) is defined through

$$\Phi_a(b_1, \dots, b_n) = \phi(ab_1ab_2 \cdots ab_na). \tag{6}$$

Given two multilinear function series $\alpha, \beta \in \mathrm{Mult}[[B]]$, we define their formal product by

$$(\alpha \centerdot \beta)_n(b_1, \dots, b_n) = \sum_{k=0}^{n} \alpha_k(b_1, \dots, b_k)\beta_{n-k}(b_{k+1}, \dots, b_n).$$

This product turns the space $\mathrm{Mult}[[B]]$ into an unital algebra, with unit $1 = (1_B, 0, \dots)$. We define a second product on $\mathrm{Mult}[[B]]$ given by *composition*, $\alpha \circ \beta$, if $\beta_0 = 0$, by $(\alpha \circ \beta)_0 = \alpha_0$, and

$$(\alpha\circ\beta)_n(b_1, \dots, b_n) = \sum_{k=1}^{n} \sum_{\substack{p_1,\dots,p_k\geq 1\\ p_1+\cdots+p_k=n}} \alpha_k\Big(\beta_{p_1}(b_1, \dots, b_{p_1}), \beta_{p_2}(b_{q_2+1}, \dots b_{q_2+p_2}), \dots, \beta_{p_k}(b_{q_k+1}, \dots, b_{q_k+p_k})\Big),$$

where $q_j := p_1 + \cdots + p_{j-1}$. Notice that composition is linear only in the left argument. The unit for this product is the multlinear function serie

$$I = (0, \mathrm{id}_B, 0, \dots).$$

Following an analogue approach as in (4) using Φ_a defined by (6), we obtain operator-valued counterparts of the S-, R- and T-transforms that are now multilinear function series. By constructing random variables with prescribed T-transform, Dykema showed in [4] that the multiplicativity (2) of the T-transform in scalar-valued case generalises in the operator-valued case to the following so-called twisted multiplicativity property:

$$T_{ab} = \big(T_a \circ (T_b \centerdot I \centerdot T_b^{-1})\big) \centerdot T_b \tag{7}$$

Non-crossing partitions underlie combinatorics of additive convolution. For multiplicative convolution, the same role is played by non-crossing linked partitions. In particular, Möbius inversion on a certain poset of *connected non-crossing linked partitions* $\mathrm{NCL}_D^{(1)}$ relates coefficient of the T transform to free cumulants,

$$\kappa_n(ab_1, \dots, b_na) = \sum_{\pi\in \mathrm{NCL}^1_{\mathrm{D}}(n)} t_a(\pi)(b_1, \dots, b_n), \tag{8}$$

see [4] for the definition of $t_a(\pi)$. For small values of n, one has

$$\kappa_2(ab_1, a) = t_1(a)(b_1),\ \ \kappa_3(ab_1, ab_2, a) = t_a(2)(b_1, b_2) + t_a(1)(b_1t_a(1)(b_2)),$$
$$\kappa_4(ab_1ab_2ab_3a) = t_a(3)(b_1, b_2, b_3) + t_a(2)(b_1t_a(1)(b_2), b_3) + t_a(2)(b_1, b_2t_a(1)(b_3)) + t_a(1)(b_1t_2(b_2, b_3))$$

The above relations (one for each integer $n \geq 1$) can be cast into a fixed point equation onto multilinear function series, with

$$K_a = 1 + \sum_{n\geq 1} k_a(n), \quad k_a(n)(b_1, \ldots, b_n) = \kappa_{n+1}(ab_1, ab_2, \ldots, b_n a),$$

Relations (8) is equivalent to $K_a = T_a \circ (I \cdot K_a)$.

1.1. **Contribution.** In this work we give a new (and short) operadically oriented proof of equation (7), without constructing canonical random variables. This yields to a conceptual understanding of equation (7) as resulting from a distributivity law of the product $\centerdot$ with respect to $\circ$ and constitutive of Gerstenhaber algebras. These algebras arises in particular if considering operads with a distinguished operator of arity 2 that is called *multiplication* (for reasons exposed below). The operad with multiplication at stake in our work is spanned by multilinear map on the algebra B and the multiplication coincides with the multiplication on B. It is worth noticing that these algebras admit an homotopical version, introduced in **??**. In this context, the multiplication m can be used to define a differential on the underlying collection of the operad. This construction yields for example the Hoschild cohomology of B, if one starts with the endomorphism operad of $\mathcal{B}$, that is with the collection of all multilinear map on B.

We introduce in the setting of operad with multiplication free product and define the T-transform. We give a third proof, more fundamental that the above mentioned one, of the twisted multiplicativity property for this abstract T-transform.

1.2. **Outline.** Excepted the introduction, the article is divided into two parts. In the first part (Sections 2 and 3) we provide the reader with the necessary background on operads and Brace algebras. We then define two operads we will be dealing with, the planar operad of non-crossing partitions (see Definition 4) and another one on the span of non-crossing linked partitions (see Definition 8). In Section **??**, we define operads with multiplication, the operad of non-crossing partitions is an example of.

In Section 3, we first address the problem of computing operator-valued free cumulants with products as entries and give a fixed point equation for computing the free cumulants of the product of two free random variables, see Section 2.3. From equations Fix. Pt. Eqn. I and Fix. Pt. Eqn. II, we give a short proof Theorem 15, pointing the relation 14 as the key property of the concactenation and composition products for the twisted multiplicativity property to hold. In Section 3.3, we define the free product and define the T-transform in this abstract setting.

1.3. **Basic notions and notations.** From operator-valued probability theory, we only recall the definition of an operator-valued non-commutative probability space [9].

Definition 1 (Operator-valued probability space)**.** An operator-valued probability space of a triple $(\mathcal{A}, \phi, B)$, such that

1. $\mathcal{A}$ is a von-Neumann algebra (or a $C^\star$ algebra, depending on the type of functional calculus we want to have access to),
2. B is a Banach algebra with an involution,
3. $\mathcal{A}$ is a B-B-bimodule, which means that B acts on the left and on the right of an element in $\mathcal{A}$ and
$$b_1 \cdot (a \cdot b_2) = (b_1 \cdot a) \cdot b_2, \quad (b_1 \cdot a)^\star = a^\star \cdot b^\star.$$
4. $\phi : \mathcal{A} \to B$, *the state*, is a B-B bimodule morphism, but not an algebra morphism, and is positive:
$$\phi(aa^\star) \in BB^\star, \ a \in \mathcal{A}.$$

For the present work, functional of random variables are restricted to polynomials one. As a consequence, we can drop all topological assumptions in the above definiton. In particular, $\mathcal{A}$ and B are assumed to be involutive algebras, no more.

2. Operads and brace algebras

2.1. **Algebraic planar operads.** The reader is directed to the monograph [8] for a detailed introduction to operads and planar operads. Operads are model for composing operators with multiple inputs and one input. They have been introduced by Boardman and Vogt in the 70's. A *collection* C is a sequence of vector spaces $(C(n))_{n\geq 1}$. A morphism between two collections is a sequence of linear morphisms $(\phi(n))_{n\geq 1}$ with $\phi_n : C(n) \to C(n)$, $n \geq 1$. The category of all collections is denoted Coll.

The tensor product, denoted $\bullet$, on the category Coll is the 2-functor from Coll $\times$ Coll to Coll defined by:

$$(C \bullet D)(n) = \bigoplus_{\substack{k\geq 1\\ n_1+\cdots+n_k=n}} C(k) \otimes D(n_1) \otimes \cdots \otimes D(n_k),$$
$$(f \bullet g)(n) = \bigoplus_{\substack{k\geq 1\\ n_1+\cdots+n_k=n}} f(k) \otimes g(n_1) \otimes \cdots \otimes g(n_k).$$

The unit element for the tensor product $\bullet$ is the collection denoted by $\mathbb{C}_\bullet$ such that $\mathbb{C}_\bullet(n) = \delta_{n=1}\mathbb{C}$. An operad $\mathcal{C}$ is a monoid in the monoidal category (Coll, $\bullet$, $\mathbb{C}$), i.e., a triple $(C, \gamma, \eta_{\mathcal{C}})$ with

$$C \in \text{Coll},\ \gamma : C \bullet C \to C,\ \eta_C : \mathbb{C} \to C,$$

satisfying $(\gamma \bullet \mathrm{id}_{\mathcal{C}}) \circ \gamma = (\mathrm{id}_{\mathcal{C}} \bullet \gamma) \circ \gamma$ and $(\eta_{\mathcal{C}} \bullet \mathrm{id}_{\mathcal{C}}) \circ \gamma = (\mathrm{id}_{\mathcal{C}} \bullet \eta_{\mathcal{C}}) \circ \gamma = \mathrm{id}_{\mathcal{C}}$. Note that we use the symbol $\bullet$ for the tensor product on collections to not confuse it with the composition of functions. It is common to use $\circ$ for an operadic composition:

$$\gamma(p \otimes (q_1 \otimes \cdots \otimes q_{|p|})) = p \circ (q_1 \otimes \cdots \otimes q_{|p|}) \tag{9}$$

Here $|p|$ denotes the degree of p. Accordingly, partial composition is denoted by $\circ_i$:

$$p \circ_i q = p \circ (1^{\otimes k-1} \otimes q \otimes 1^{\otimes |p|-k}),\ 1 \leq i \leq |p|. \tag{10}$$

We should use these notations when there is no risk of confusion.

Example. 1. The forgetful functor from the category of operads to the category of collections admits a left adjoint, i.e., the free functor $\mathcal{F}$. Given a collection C, the free operad $\mathcal{F}(C)$ on C, is spanned by planar rooted trees with internal nodes decorated by elements in the collection C. The degree of a decoration matches the number of inputs of an internal nodes. Composition is obtained by grafting the root of a tree to a leaf of another, and the identity is the tree with a single leaf.

2. Given a vector space V, we denote by End_V the operad whose underlying collection is the graded vector space of multilinear maps on V,

$$\mathrm{End}_V(n) = \mathrm{Hom}_{\mathrm{Vect}}(V^{\otimes n}, V),$$

and the composition is induced by composition of functions. If one replaces V by a set S, one can defines the category End_S of all set function with multiple arguments,

$$\mathrm{End}_S(n) = \mathrm{Hom}_{\mathrm{Set}}(S^{\times n}, S).$$

3. *Operad of words-insertions* $\mathcal{W}$. Set $T(\mathcal{A}) = \sum_{n\geq 1} \mathcal{A}^n$ the space of all non-commutative polynomials on elements in $\mathcal{A}$, and $\bar{T}(\mathcal{A}) = \mathbb{C} \cdot \emptyset \oplus T(\mathcal{A})$. Then $\bar{T}(\mathcal{A})$ is an unital algebra for the concatenation product $\cdot$ with unit $\emptyset$. The degree w of an element $w \in \mathcal{A}^{\otimes n}$ is $n+1$ and $|\emptyset| = 1$. Define on $\bar{T}(\mathcal{A})$ the operadic composition γ:

$$\gamma(w \otimes u_0 \otimes \cdots \otimes u_n) = u_0 \cdot w_1 \cdot u_1 \cdots u_{n-1} \cdot w_n \cdot u_n$$

The definition of a planar operad uses the monoidal structure of the category $\mathrm{Vect}_{\mathbb{C}}$ of all vector spaces. Replacy this monoidal category by another yields the notion of a planar operad in a monoidal category. For example one can take instead of $\mathrm{Vect}_{\mathbb{C}}$ the category Set of sets with bijections. It is a monoidal category for the cartesian product.

Definition 2 (Hadamard product)**.** Let $P = (\mathcal{P}, \gamma_P)$ and $Q = (\mathcal{Q}, \gamma_Q)$ two operads. The Hadamard product of P and Q is the operad $P \otimes_H Q = (\mathcal{P} \otimes_H \mathcal{Q}, \gamma_{P\otimes_H Q})$ defined by

$$\begin{aligned}&(\mathcal{P} \otimes_H \mathcal{Q})(n) = \mathcal{P}(n) \otimes \mathcal{Q}(n)\\ &\gamma_{P\otimes_H Q}(p \otimes q \otimes (p_1 \otimes q_1 \otimes \cdots \otimes p_n \otimes q_n)) = \gamma_P(p \otimes p_1 \otimes \cdots \otimes p_n) \otimes \gamma_Q(q \otimes q_1 \otimes \cdots \otimes q_n).\end{aligned}$$

Definition 3 (Free product)**.** Let $P = (\mathcal{P}, \gamma_P)$ and $Q = (\mathcal{Q}, \gamma_Q)$ two operads. The *free product* of P and Q is the operad $P \star Q$ obtained by quotienting the free operad on the collection $\mathcal{P} \oplus \mathcal{Q}$ by relations in the operad P, relations in the operad Q and no other.

It may happen that the operators we want to compose with each other have different inputs and output ranges. A model for composing such operators is called *coloured operad*. Given a set of coloured C, we consider coloured collections. In a coloured collection, the vector space C_n of operators with n inputs is split into a direct sum of spaces, $C^{c'}_{c_1,\ldots,c_n}$, $c_1, \ldots, c_n, c' \in C$ comprising all operators with sources spaces labeled by $c_1, \ldots, c_n$ and target labeled by c'. Formal composition of collection (the monoidal product $\bullet$) admits a coloured version, for which operators are formally composed provided that colourations of inputs and outputs match.

2.2. **Operads of non-crossing partitions and non-crossing linked partitions.**

2.2.1. *Non-crossing partitions.* We denote by $\mathrm{NC}(n)$ the set of all non-crossing (n-c) partitions of $[\![1, \ldots, n]\!]$. A $\pi \in \mathrm{NC}(n)$ is viewed as an operator with $n + 1$ inputs and the degree $|\pi| = n + 1$. These inputs are the gaps between the elements of the partitioned set, including the front gap before 1 and the back gap after n. We can insert $n + 1$ n-c partitions inside another n-c partition by stuffing them into the gaps of the latter. It is clear that the resulting partition is again non-crossing.

Definition 4 (Operad of non-crossing partitions)**.** We set $\mathcal{NC}(n) := \mathrm{NC}(n-1)$. In particular, we have $\mathcal{NC}(0) = \emptyset$ and $\mathcal{NC}(1) = \{\emptyset\}$. The empty partition is the operad unit. Let π be a n-c partition and $(\alpha_1, \ldots, \alpha_{|\pi|})$ a sequence of n-c partitions. The composition $\gamma_{\mathcal{NC}}(\pi \otimes \alpha_1 \otimes \cdots \otimes \alpha_{|\pi|})$ is obtained by inserting each partition α_i in between the two integers i and $i + 1$, $i \leq 1$. In symbols:

$$\gamma_{\mathcal{NC}}(\pi \otimes \alpha_1 \otimes \cdots \otimes \alpha_{|\pi|}) = \bigcup_{i=1}^{|\pi|} \{i - 1 + b,\ b \in \pi_i\} \cup \tilde{\pi}$$

where $\tilde{\pi}$ is the n-c partition of $\{|\pi_1|, |\pi_1| + |\pi_2|, \ldots, |\pi_1| + \cdots + |\pi_n|\}$ induced by π.

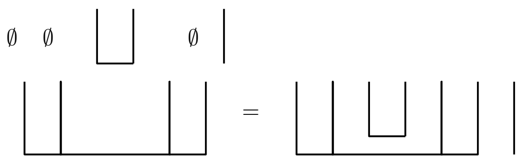

FIGURE 1. Example of a composition in the gap-insertion operad $\mathcal{NC}$.

The gap-insertion operad of non-crossing partitions admits the following presentation in terms of generators and relations.

Definition 5 (Coloured non-crossing partition)**.** A coloured non-crossing partition is a element $\pi \otimes w$, $|w| = |\pi|$ in the Hadamard tensor product $\mathcal{NC} \otimes_H \mathcal{W}$ (see Definition 3).

Lemma 6 (Proposition 3.1.4 in [5])**.** *For any $n \geq 1$, we put $1_{n+1} = \{[\![1, n]\!]\}$. Then the operad $(\mathcal{NC}, \rho_{\mathcal{NC}})$ is generated by the elements 1_n, $n \geq 1$ with the relation:*

$$\forall m, n \geq 1, \quad 1_m \circ_m 1_n = 1_n \circ_1 1_m.$$

The gap-insertion operad implements algebraically the *nesting* of blocks of a partition. The distribution of a random variable $a \in \mathcal{A}$ in a probability space $(\mathcal{A}, E, B)$ yields an operadic morphism E^a on the gap insertion operad with values in $\mathrm{Hom}(B)$ prescribed by

$$\mathsf{E}^a(1_n)(b_0, \ldots, b_n) = E(b_0 a b_1 a \cdots a b_n)$$

with $b_0, \dots, b_n \in B$. The morphism E^a is well defined. In fact, because E is B-B bimodule map, we get

$$\mathsf{E}^a(1_1)(b_0) = b_0$$
$$\mathsf{E}^a(1_n) \circ_n \mathsf{E}^a(1_m) = \mathsf{E}^a(1_m) \circ_1 \mathsf{E}^a(1_n).$$

To the free cumulants of a corresponds another operadic morphism $\mathsf{K}^a : \mathcal{NC} \to \mathrm{Hom}(B)$ such that

$$\mathsf{K}^a(1_n)(b_0, \dots, b_n) = \kappa_n(b_0 a b_1, \dots, a b_n).$$

We now give the definition of a non-crossing linked partition, which is more general than what is usually given in the literature.

2.2.2. *Non-crossing linked partitions.*

Definition 7 (Non-crossing linked partitions)**.** Let $n \geq 1$ be an integer. A *non-crossing linked partition* is a collection π of subsets (blocks) of $[\![1, n]\!]$ such that:

1. $\bigcup_{V \in \pi} V = [\![1, n]\!]$
2. for U and V two blocks of π, if $a < c < b < d$ with $a, b \in U$ and $c, d \in V$, then $V = U$,
3. if $U \neq V$ and $\sharp U, \sharp V > 1, \sharp U \cap V \geq 1$ then $U \cap V = \{x\}$ and $x = \min U$ or $x = \min V$.

For $n \geq 1$, we denote by $\mathrm{NCL}(n)$ the set of non-crossing linked (ncl) partitions of $[\![1, n]\!]$.

See Fig. 2 for examples of non-crossing linked partitions. Notice that we adopt a more general definition of a non-crossing linked partition, in comparison with the definition given in [4]. In particular two blocks are allowed to share their minimum.

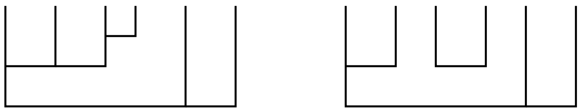

FIGURE 2. On the left, a connected non-crossing linked partition and on the right a non-crossing linked partition.

Definition 8 (Operad of non-crossing linked partitions)**.** Define the degree $|\pi|$ of a ncl partition by $|\pi| = n$ if $\pi \in \mathrm{NCL}(n)$. Given a ncl partition α in $\mathrm{NCL}(n)$ and $\beta_1, \dots, \beta_n$ a sequence of n ncl partitions, we define:

$$\gamma_{\mathrm{NCL}}(\alpha \otimes \beta_1 \otimes \cdots \otimes \beta_{|\alpha|}) = \bigcup_{i=1,\dots,n} \{|\beta_{i-1}| + V,\ V \in \beta_i\} \cup \tilde{\pi} \tag{11}$$

where $\tilde{\pi}$ is the non-crossing linked partition of the set $\{1, 1 + |\beta_1|, |\beta_1| + |\beta_2| + 1, \dots, |\beta_1| + |\beta_{n-1}| + 1\}$ induced by π. We denote by $|$ the unique element in $\mathrm{NCL}(1)$. The ncl partition $|$ is the unit for $\circ$.

Proposition 9 (Connected non-crossing linked partitions)**.** *Let $n \geq 1$ an integer and define the following subset of* $\mathrm{NCL}(n)$*:*

$$\mathrm{NCL}^{(1)}(n) = \{\pi \in \mathrm{NCL}(n) | \pi^{\vee} = 1_n\}$$

Then $\mathrm{NCL}^{(1)}$ *is an operad with composition the restriction of* γ_{NCL}*. In addition, it is a free operad on the collection of single block non-crossing linked partitions.*

2.3. **Operads with multiplication.**

Definition 10 (Brace algebras, see [1], [11], [2])**.** A brace algebra is a couple $(A, \{\ \})$, where A is a vector space and $\{-;-\}$ is a linear map from $A \otimes T(A)$ to A such that

$$\{\{x; y_1, \dots, y_n\}; z_1 \cdots z_p\} = \sum \{x; z_1 \cdots z_{i_1} \{y_1; z_{i_1+1} \cdots z_{j_1}\} z_{j_1+1} \cdots \{y_n; z_{i_n+1} \cdots z_{j_n}\} z_{j_n+1} \cdots z_p\}\} \tag{12}$$

with $x, y_1, \dots, y_n, z_1, \dots, z_p \in A$.

An operad $((\mathcal{P}(n))_{n\geq 1}, \gamma, \mathrm{id})$ yields a brace algebra on $\mathcal{P}$,

$$\{x; y_1 \cdots y_n\} = \sum \gamma(x, \mathrm{id}, \ldots, y_1, \mathrm{id}, \ldots, y_2, \mathrm{id}, \ldots, y_{n-1}, \mathrm{id}, \ldots, y_n, \mathrm{id}, \ldots). \tag{13}$$

Equation (12) of associativity for γ. Assume further existence of an operator $m \in \mathcal{P}(2)$, that we call *multiplication*, and satisfying

$$\gamma(m; \mathrm{id}, m) = \gamma(m; m, \mathrm{id}). \tag{14}$$

Example (Operad with a multiplication)**.** 1. The operad $\mathcal{NC}$ of non-crossing partitions is an operad with multiplication, with $m = |$. In fact, $\gamma(| \otimes \{\emptyset\} \otimes |)$ and $\gamma(| \otimes | \otimes \{\emptyset\})$ are both equal to the partition $||$.
2. The operad $\mathrm{Hom}(B)$ of multilinear maps on an algebra (B, μ_B) is an operad with multiplication with $m = \mu_B$.

In a graded context, that is when the general term of the summation (13) is affected with a sign, existence of a multiplication in an operad provides a very rich structure [6]. In particular, the multiplication m together with a certain *graded pre-lie product* yields a differential complex $(\mathcal{P}, d)$.

The ungraded counterpart of this graded pre-lie product, denoted $\circ$, is defined by:

$$x \circ y = x\{y\}, \; x, y \in \mathcal{P}.$$

In fact, $x \circ (y \circ z) - (x \circ y) \circ z$ is symmetric in y and z, being equal to the sum over all (operadic) compositions of the operator x with y and z. We denote by $[\, , \,]$ the Lie bracket induced by $\circ$:

$$[x, y] = x \circ y - y \circ x, \; x, y \in \mathcal{P}.$$

In [3], the author define a product $\times$ on a space of formal series $\hat{\mathsf{A}}_{\mathcal{P}}$,

$$\hat{\mathsf{A}}_{\mathcal{P}} = \prod_n \mathcal{P}(n), \quad a \times b = \sum_{n\geq 1} \{a_n; b_{m_1} \cdots b_{m_n}\}, \quad a, b \in \hat{\mathsf{A}}_{\mathcal{P}}.$$

Proposition 11 (see Proposition 4.1 in [3])**.** $(\hat{\mathsf{A}}_{\mathcal{P}}, \times, \mathrm{id})$ *is an associative monoid. Besides, $a \in \hat{\mathsf{A}}_{\mathcal{P}}$ is an invertible element if and only if $a_0 \neq 0$.*

The last proposition implies that the set $G \subset \hat{\mathsf{A}}_{\mathcal{P}}$,

$$G = \left\{ x \in \hat{\mathsf{A}}_{\mathcal{P}} : x_1 = \mathrm{id} \right\}$$

endowed with the composition $\times$ is a group. The multiplication m yields another group product that we define in section 3.3. But first, the multiplication m yields on the collection $\mathcal{P}$ a bilinear non-unital associative product $\cdot$ defined by

$$x \cdot y = \{m; xy\}, \; x, y \in \hat{\mathsf{A}}_{\mathcal{P}}. \tag{15}$$

Proposition 12. $(\mathcal{P}, \{\, ; \,\}, \cdot)$ *is a Gerstenhaber-Voronov algebra, which means that*

$$\{x \cdot y; z_1, \ldots, z_p\} = \sum_{k=0}^{p} \{x; z_1 \cdots z_k\} \cdot \{y; z_{k+1} \cdots z_p\}, \; x, y, z_1, \ldots, z_p \in \hat{\mathsf{A}}_{\mathcal{P}} \tag{16}$$

Equation (16) is key to the twisted multiplicativity property for the T-transform as explained in section 3.3.

3. Twisted multiplicativity of the T-transform

We give a short graphical proof of Theorem 7.18 in [4]. The starting point is a formula, written using the language of operads, for the operator-valued free cumulants with products as entries. We then show how non-crossing linked partitions are naturally brought up by degree reduction ie by filling inputs of a multilinear map with the unit of the algebra B.

3.1. **Free cumulants of free products.** In this section, we explain how to compute the multilinear function series corresponding to the free cumulants of the product of two free variables as the solution of a certain fixed point equation. The proof of the fixed point equation Fix. Pt. Eqn. I, new to the extend of our knowledge relies on a formula computing free cumulants of the product of two free random formula. This formula is well known in the scalar case but in the operad-valued case, the author has not been able to locate the latter in the literature. However, it is simple to *guess* what this formula should be.

Let us fix once and for all two free random variables a and b in the operator-valued probability space $(\mathcal{A}, B)$ with $\phi(a) = \phi(b) = 1$, and recall that we denote by K_a the multilinear function series

$$K_a = 1 + \sum_{n\geq 1} \kappa_{n+1}(ab_1a\cdots ab_na) \tag{17}$$

In the scalar case, that is when $B = \mathbb{C}$, one has [10]

$$\kappa_n(a) = \sum_{\pi\in \mathrm{NC}(n)} \kappa_\pi(a)\kappa_{Kr(\pi)}(b). \tag{18}$$

Let $\pi \in \mathrm{NC}(n)$. The non-crossing partition $Kr(\pi)$ is the Kreweras complement of π, first introduced in [7]. For two non-crossing partitions α and β in NC(n), one denotes by $\alpha\cup\beta$ the partition of $[\![1, 2n]\!]$ whose restriction to the odd integers, respectively to the even integers, coincides with α, respectively β.By definition, $Kr(\pi)$ is the maximal non-crossing partition (for the refinement order) such that $\pi\cup K(\pi)$ is a non-crossing partition of $\mathrm{NC}(2n)$.

In the operator-valued case, since the cumulants of a and b do not commute with each others (they are elements of the non-commutative algebra B), the right-hand side of equation (18) does not factorise over π and its Kreweras complement $Kr(\pi)$. We should maintain the linear order between random variables in the word $a\otimes b\otimes\cdots a\otimes b$.

Denote by $\tilde{\pi}$ the non-crossing partition $\pi\cup Kr(\pi)$. Each block of $\tilde{\pi}$ is coloured with 0 or 1, according to the parity of the elements in the block yielding an element of the free product $\mathcal{NC}\sqcup\mathcal{NC}$.

Recall that the free product $\mathsf{K}^a\sqcup\mathsf{K}^b$ is the unique operadic morphism on $\mathcal{NC}\sqcup\mathcal{NC}$ such that $\mathsf{K}^a\sqcup\mathsf{K}^b(\pi) = \mathsf{K}^a(\pi)$ if all blocks of π are coloured with 0 and $\mathsf{K}^a\sqcup\mathsf{K}^b(\pi) = \mathsf{K}^b(\pi)$ if all blocks of π are coloured with 1.

With these definitions, the operator-valued counterpart of the formula (18) reads

$$x_0\kappa_n(ay_1bx_1, ay_2bx_2, \ldots, ay_nb)x_n = \sum_{\pi\in\mathrm{NC}(n)} (\mathsf{K}^a\sqcup\mathsf{K}^b)(\tilde{\pi})(x_0, y_0, \ldots, y_n, x_n), \tag{19}$$

with $x_0, \ldots, x_n \in B$ and $y_1, \ldots, y_n \in B$.

More generally, the formula for free cumulants with products as entries for the scalar case extends to the operator-valued case as follows,

$$x_0\kappa_n(a_1^1y_1^1\cdots y_{p_1-1}^1a_{p_1}^1x_1, \ldots, x_{n-1}a_1^ny_1^n\cdots y_{p_n-1}^na_{p_n}^n)x_n = \sum_{\pi\in\mathrm{NC}(p_1+\cdots+p_n)\vee\sigma_{a^1,\ldots,a^n}} \mathsf{K}(\pi\otimes a_1^1\cdots a_{p_n}^n),$$

with $\mathsf{K} : \mathrm{NC}\otimes_{\mathrm{H}}\mathcal{W}\to\mathrm{Hom}(B)$ the unique operadic morphism with

$$\mathsf{K}(\pi\otimes a_1\cdots a_n)(x_0, \ldots, x_n) = x_0\kappa_n(a_1x_1, \ldots, a_n)x_n.$$

and $\sigma_{a^1,\ldots,a^o}$ is the interval partition of $NC(p_1+\cdots+p_n)$ with blocks $\{p_1+\cdots+p_{i-1}+1, \ldots, p_1+\cdots p_i\}$, $1\leq i\leq n$. To recover the free cumulants of ab, we set the $y's$ equal to $1\in B$ in the above formula (19). We explain how degree reduction leads to a sum over non-crossing *linked* partitions in place of the sum over non-crossing partitions in the right hand side of equation (19).

Pick a non-crossing partition $\pi\in\mathrm{NC}(n)$. In Fig. 3, we pictured the partition $\tilde{\pi}$ with the blocks coloured according to the parity of the elements in the latter. We symbolize evaluation to 1 of the $y's$ by crosses, since doing so reduces the number of inputs of the multilinear map associated to $\tilde{\pi}$. It is clear from the drawing in Fig. 3 that each block sees a cross either in its back or front gap.

A *black block* sees its *front gap* filled with a cross while for a *blue block* (except the outer blue block), it is the *back gap* which is filled with a cross.

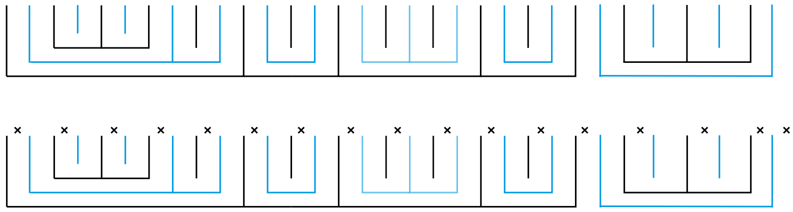

FIGURE 3. On the upper line, a partition $\tilde{\pi}$, the non-crossing partition π is drawn in black and its Kreweras complement is drawn in blue. On the second line, we symbolized with a cross evaluation to 1 of a variable in B (that fall within a gap)

The free cumulant associated to the partition $\tilde{\pi}$ after evaluation of the $y's$ to 1 can thus be obtained by composing in the operad Hom_B left B-linear extension of the free cumulants of a, $(k^a_L(n))_{n\geq 1}$, right B-linear extension of the free cumulants of a and b, $(k^a_R(n))_{n\geq 1}$ and $(k^b_R(n))_{n\geq 1}$, respectively,

$$k^a_L(n)(b_1,\dots,b_n) = \kappa_n(b_1a,\dots,b_na), \tag{20}$$

$$k^b_R(n)(b_1,\dots,b_n) = \kappa_n(bb_1,\dots,b)b_n,\ \ k^a_R(n)(b_1,\dots,b_n) = \kappa_n(ab_1,\dots,a)b_n \tag{21}$$

In Fig. 4, we associate a (coloured) non-crossing linked partition $\tilde{\pi}_\ell$ to the partition $\tilde{\pi}$ drawn in Fig. 3. A block b labeled R in $\tilde{\pi}_\ell$ corresponds to a block of the same size $\tilde{b}\in\tilde{\pi}$ with a cross in its front gap. Each leg, or input (in the operad of non-crossing linked partition), of b matches a gap of $\tilde{b}$, starting with the second. For a block $b\in\tilde{\pi}_\ell$ labeled L, a leg matches a gap of $\tilde{b}$, starting with the first one. Notice that there is no block in $\tilde{\pi}_\ell$ corresponding to a singleton in $\tilde{\pi}$. To the even outer block, the blue outer block in Fig. 3, and the only one that have crosses in its front and back gap corresponds a block in $\tilde{\pi}$ labeled "L". This is the only one with this property, to another even (blue) block in $\tilde{\pi}$ corresponds a block labeled "R" in $\tilde{\pi}_\ell$. All other blocks labelled L are coloured in black.

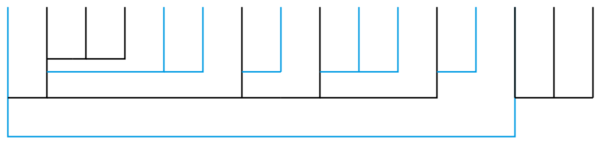

FIGURE 4

The outer block containing 1 of a connected non-crossing linked partition $\tilde{\pi}_\ell$ arising in this way is coloured in blue and marked "L". Blocks with a non-empty intersection with this block are all coloured in black and marked "L". A block meeting with one of the latter are coloured in blue and marked R, and so on. To a blue block tagged "R" corresponds a multilinear map in $(k^R_b(n))_{n\geq 1}$, to a blue block tagged "L" corresponds a multilinear map in $(k^L_b(n))_{n\geq 1}$. To a black block tagged "L" corresponds a multilinear map in $(k^L_a(n))_{n\geq 1}$. This correspondance yields a multilinear map $\mathsf{K}^{a,b}(\tilde{\pi}_\ell)$ in an obvious way and

$$x_0\kappa_n(abx_1,abx_2,\dots,ab)x_n = \sum_{\pi\in\mathrm{NC}(n)}\mathsf{K}^{a,b}(\tilde{\pi}_\ell). \tag{22}$$

Notice that the non-crossing linked partition $\tilde{\pi}$ does meet the requirement of Dykema to have no blocks sharing their minimum. This is the reason for we gave a more general diffusion of non-crossing linked partition.

Proposition 13. *The multilinear function series V whose homogeneous component of order n is the sum of the multilinear maps $\mathsf{K}^{a,b}(\pi)$ over all non-crossing linked partitions $\tilde{\pi}_\ell$ with block alternatively coloured with blue or black, starting with a black block, is the solution of the fixed point equation:*

$$V = K_a\times[(K_b\times[I\cdot V])\cdot I)] \tag{Fix. Pt. Eqn. I}$$

and K_{ab} is given by

$$K_{ab} = V\cdot(K_b\times[I\times V]). \tag{Fix. Pt. Eqn. II}$$

To recover the free cumulant serie K_{ab}, non-crossing linked partitions with an alternating colouration are composed with a single block partition coloured in blue and marked "L" resulting in the equation (Fix. Pt. Eqn. II) for K_{ab}.

3.2. **Short proof of the twisted multiplicativity.** In this section, we give a short graĥical proof for the twisted multiplicativity property of the T-transform. Following the work of K. Dykema, we define two subsets of multilinear function series,

$$\mathrm{Mult}[[B]]_0 = \{A \in \mathrm{Mult}[[B]] : A_0 = 0\}, \quad G = \mathrm{Mult}[[B]]_1 = \{A \in \mathrm{Mult}[[B]] : A_0 = 1\}.$$

We represent as rooted planar trees *concatenations* (the product $\centerdot$) and *compositions* (the product $\times$) of multilinear function series). The composition $A \times B$ of two series $A \in \mathrm{Mult}[[B]]$ and $B \in \mathrm{Mult}[[B]]_0$ by a two nodes graph with a single vertical edge (see Fig.). By doing so, we see associate to the multilinear function series A as an operator with a single input acting on $\mathrm{Mult}[[B]]_1$,

$$\mathrm{Mult}[[B]]_1 \ni B \mapsto B \times A. \tag{23}$$

The above operator is a set operator, in particular it is not linear. The outputed series belongs either to $\mathrm{Mult}[[B]]_0$ if $A \in \mathrm{Mult}[[B]]_0$ either to $\mathrm{Mult}[[B]]_1$ if $A \in \mathrm{Mult}[[B]]_1$.

More generally, if $W = E_1E_2\dots E_n$ is a *word* on multilinear function series, we associate to W set operators, with multiple inputs, acting on $\mathrm{Mult}[[B]]_1$. Each of these operators is drawn as a corolla decorated with W and with at most n leaves. Each leaf corresponds to composition of the inputed multilinear function serise with one of the letter E_i, followed by concatenation of the resulting multilinear function series. For example, in the case $W = E_1E_2$, we have drawn in Fig. 5 the associated operators.

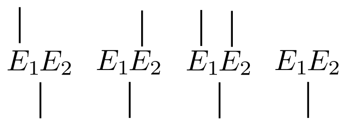

FIGURE 5. Operators associated with the word E_1E_2, from left to right, $A \mapsto (E_1 \circ 1)E_2$, $A \mapsto E_1(E_2 \circ A), (A, B) \mapsto (E_1 \circ A)(E_2 \circ B)$ and $E_1 \centerdot E_2$.

Notice that the edges of the corollas drawn in Fig. 5 should be coloured, with 1 for the inputs and with 0 for the output, respectively 1, if $E_1 \cdot \dots \cdot E_n{}^0 = 0$, respectively, $E_1 \cdot \dots \cdot E_n{}^1 = 1$. We omit these colourizations to lighten notations.

In Fig. 6, we represented graphically the defining relation of the T-transform and in Fig. 7 the two equations (Fix. Pt. Eqn. I) and (Fix. Pt. Eqn. II).

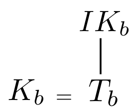

FIGURE 6. Equation satisfied by the T-transform and the free cumulants.

Proposition 14 (Proposition 2.3. in [4])**.** *Let A, B and C be three*

$$[A \cdot B] \times C = (A \times C) \cdot (B \times C),\ C_0 = 0.$$

Notice that since $1 = 1 \circ E$, one has $(A \circ E)^{-1} = A^{-1} \circ E$.

Theorem 15 (Theorem 7.18 in [4])**.** *Let a, b be two free random variables, then*

$$T_{ab} = (T_a \times [T_b \centerdot I \centerdot T_b^{-1}]) \centerdot T_b.$$

Proof. The proof of the statement is contained in Fig. 8 and Fig. 10.

We detail the computations of Fig 8. For the first equality, we use equation (Fix. Pt. Eqn. I) and for the second one equation (Fix. Pt. Eqn. II). The third one follows from inserting the defining equation for the T-transform of b (see Fig. 6). We then recognize the equation (Fix. Pt. Eqn. I) in the leftmost

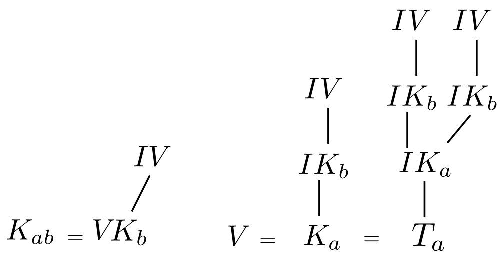

FIGURE 7. Graphical representation of equations (Fix. Pt. Eqn. I) and (Fix. Pt. Eqn. II).

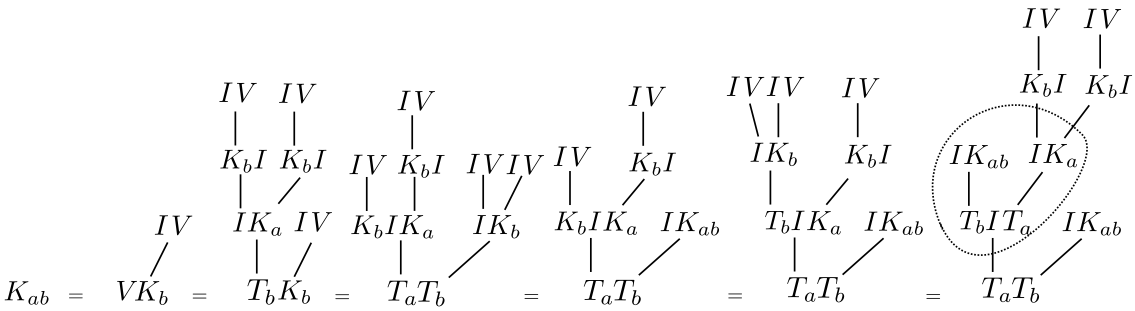

FIGURE 8. Proof of the multiplicativity property of the T-transform with respect to multiplication of free random variables

tree attached to the node T_aT_b. The fourth and fifth equality proceed from the same computations. To continue, we use the relation drawn in Fig. 9 for the expression circled with a dotted line.

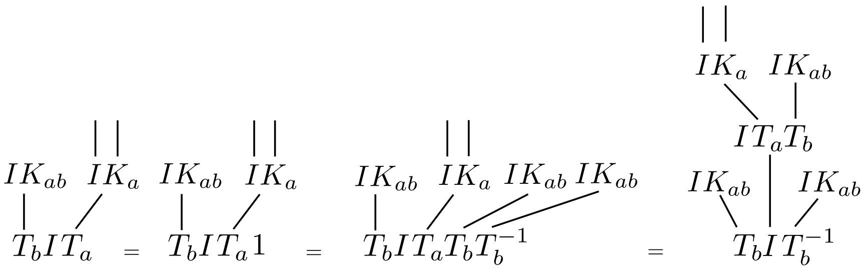

FIGURE 9. Direct corollary of the relation $1 = 1 \times E = (A \times E)(A^{-1} \times E)$.

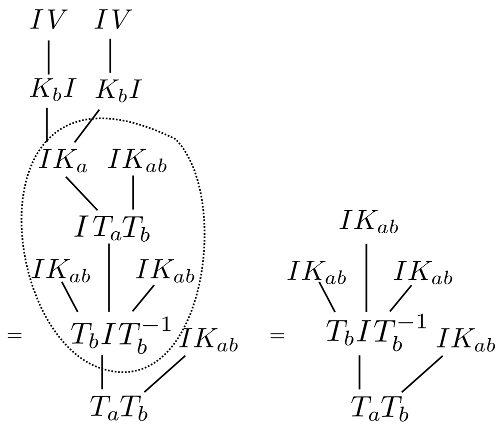

FIGURE 10. End of the proof of the multiplicativity property of the T-transform with respect to multiplication of free random variables

□

3.3. **Free product in an operad with multiplication.** In this section, we give a more conceptual of the twisted multiplicativity property. We consider a larger setting of an operad with multiplication $(\mathcal{P},\gamma,m)$.

Pick a and b two formal series in $\hat{\mathsf{A}}_{\mathcal{P}}$ (see 2.3) and define their concatenation product $a \centerdot b$ by

$$a \centerdot b = \sum_{n\geq 1} \sum_{(k,q):k+q=n} a_k \cdot b_q = \sum_{n\geq 1} \{m; a_k b_q\} \tag{24}$$

The following proposition is a direct consequence of associativity of m and the product γ.

Proposition 16. *$(\hat{\mathsf{A}}_{\mathcal{P}},\cdot)$ is an associative graded algebra.*

Set $\bar{\mathsf{A}}_{\mathcal{P}}$ equal to the unitization (for the product $\centerdot$) of $\hat{\mathsf{A}}_{\mathcal{P}}$; $\bar{\mathsf{A}}_{\mathcal{P}} = \mathbb{C}1 \oplus A$, deg $1 = 0$ and put $H = \{x \in \bar{\mathsf{A}}_{\mathcal{P}} : x_0 = 1\}$. Notice, that so far we have introduced two groups, $(G,\times)$ and $(H,\cdot)$.

Next, we define two actions, one of the group G on $\bar{\mathsf{A}}_{\mathcal{P}}$ and another one of the group H on the group G. The group G acts linearly on the right of an element in $\bar{\mathsf{A}}_{\mathcal{P}}$ by

$$a \curvearrowleft g = a \times g,\ 1 \curvearrowleft g = 1,\ a \in \hat{\mathsf{A}}_{\mathcal{P}},\ g \in G. \tag{25}$$

The group H acts on the left of an element of G by conjugation

$$h \curvearrowright g = h \centerdot g \centerdot h^{-1}, \quad h \in H,\ g \in G \tag{26}$$

Definition 17 (G-Algebra)**.** Let A be a complex unital algebra. We say that A is a G-algebra if G acts as a group of algebra automorphisms of A. Expressed otherwise, the action of G on A turns A into a right module over $\mathbb{C}[G]$.

Proposition 18. *The triple $(\bar{\mathsf{A}}_{\mathcal{P}},\curvearrowleft,G)$ is a G-algebra, in particular:*

$$(a \centerdot a') \curvearrowleft g = (a \curvearrowleft g) \centerdot (a' \curvearrowleft g), \quad 1 \curvearrowleft g = 1,\ (u \curvearrowleft g)^{-1} = u^{-1} \curvearrowleft g \tag{27}$$

This last proposition immediately implies that H is a stable set for the action of G. The two actions $\curvearrowleft$ and $\curvearrowright$ are compatible in the following sense.

Proposition 19. *Let $h \in H$, $g, g' \in G$, then*

$$(h \curvearrowright g') \times g = (h \curvearrowleft g) \curvearrowright (g' \times g). \tag{28}$$

The identity I of the operad $(\mathcal{P},\gamma)$ yields to two set-maps i^l and i^r, the left and right translations by I in $(\hat{\mathsf{A}}_{\mathcal{P}},\centerdot)$.

$$i^l : \begin{array}{ccc} \hat{\mathsf{A}}_{\mathcal{P}} & \to & \hat{\mathsf{A}}_{\mathcal{P}}, \\ h & \mapsto & I \centerdot h \end{array}, \quad i^r : \begin{array}{ccc} \hat{\mathsf{A}}_{\mathcal{P}} & \to & \hat{\mathsf{A}}_{\mathcal{P}} \\ h & \mapsto & h \centerdot I \end{array}$$

Restriction of i^l and i^h yields to H yields two set-maps from H to G. These maps are not group morphisms. Instead, they are injective and extends to co-algebras morphisms between the polynomial bialgebras of H and G with stable ranges for the product $\times$. In fact, the maps on the group algebra $\mathbb{C}[H]$ taking values in $\mathbb{C}[G]$ extending linearly the maps i^l and i^r satisfy

$$(i^l \otimes i^l) \circ \Delta^H = \Delta^G \circ i^l,\ (i^r \otimes i^r) \circ \Delta^H = \Delta^G \circ i^r$$

Thanks to Proposition 19 and the symmetry of the coproduct Δ^G, the set H can be endowed with two additional products, namely

$$h \star_l h' = h' \centerdot (h \curvearrowleft i^l(h')),\ h \star_r h' = (h \curvearrowleft i^r(h')) \centerdot h'$$

Proposition 20. *Let $h, h' \in H$ and $g \in G$,*

$$i^l(h) \times i^l(h') = i^l(h \star_l h'),\ i^r(h) \times i^r(h') = i^r(h \star_r h')$$

$$i^r(h) = h \curvearrowright i^l(h)$$

Proof. With $h, h' \in H$, one has

$$i^l(h) \times i^l(h') = (I \centerdot h) \times (I \centerdot h') = (I \centerdot h') \centerdot (h \times (I \centerdot h')) = I \centerdot (h' \centerdot (h \times (I \centerdot h')))).$$

The computations with i^r in place of i^l are similar. The second statement is obvious. $\square$

Proposition 21. *Pick h and h' in H, then*

$$i^l(h) \circ i^r(h') = h' \curvearrowright i^l(h \star_r h'), \quad i^r(h) \circ i^l(h') = (h')^{-1} \curvearrowright i^r(h \star_l h') \tag{29}$$

Proof. Let $h, h' \in H$, then

$$i^l(h) \times i^r(h') = h' \centerdot I \centerdot (h \times (h' \centerdot I)) = h' \centerdot (I \centerdot (h \times (h' \centerdot I)) \centerdot h') \centerdot (h')^{-1} \tag{30}$$

□

Next, we define the free product in this abstract setting.

Definition 22 (Free product)**.** Pick k_a and k_b in H and define the free product $k_a \boxtimes k_b \in H$ as

$$v = k_a \curvearrowleft i^r(k_b \curvearrowleft i^l(v))$$
$$k_a \boxtimes k_b = k_b \star_l v$$

Theorem 23. *Pick k_a and k_b in H and assume the following fixed point equations in H to hold*

$$k_a = u_a \curvearrowleft i^l(k_a), \quad k_b = u_b \curvearrowleft i^l(k_b), \tag{31}$$

with t_a and t_b in H, then one has

$$k_a \boxtimes k_b = \big([u_a \curvearrowleft (u_b \curvearrowright I)] \centerdot u_b\big) \curvearrowleft i^l(k_a \boxtimes k_b).$$

Proof. Set $k_{ab} = k_a \boxtimes k_b$. First, we use the fact that $\curvearrowleft$ is a right action of the group $(G, \times)$ to write

$$k_{ab} = v \centerdot (k_b \curvearrowleft i^l(v)) = v \centerdot (u_b \curvearrowleft (i^l(k_b) \times i^l(v))) = v \centerdot (u_b \curvearrowleft i^l(k_b \star_l v)) = v \centerdot (u_b \curvearrowleft i^l(k_{ab}))$$

We then use the fixed point equation satisfied by k_a and k_b,

$$\begin{aligned} v = u_a \curvearrowleft (i^l(k_a) \times i^r(k_b \curvearrowleft i^l(v))) &= u_a \curvearrowleft (i^l(k_a) \times i^r(u_b \curvearrowleft (i^l(k_b) \times i^l(v)))) \\ &= u_a \curvearrowleft (i^l(k_a) \times i^r(u_b \curvearrowleft i^l(k_b \star_l v))) \\ &= u_a \curvearrowleft (i^l(k_a) \times i^r(u_b \curvearrowleft i^l(k_{ab}))). \end{aligned}$$

By using equation (29) with $h' = u_b \curvearrowleft i^l(k_{ab})$ and $h = k_a$ we obtain

$$\begin{aligned} i^l(k_a) \times i^r(u_b \curvearrowleft i^l(k_{ab})) &= (k_b \curvearrowleft i^l(v)) \curvearrowright i^l(k_a \star_r (u_b \curvearrowleft i^l(k_{ab})) \\ &= (u_b \curvearrowleft i^l(k_{ab})) \curvearrowright i^l(k_a \star_r (u_b \curvearrowleft i^l(k_{ab})). \end{aligned}$$

Coming back to the fixed point equation satisfied by v and inserting the fixed point equation satisfied by k_b, we get also

$$v = k_a \curvearrowleft i^r(u_b \curvearrowleft i^l(k_b \star_l v)) = k_a \curvearrowleft i^r(u_b \curvearrowleft i^l(k_{ab})).$$

This last equation implies

$$\begin{aligned} i^l(k_a) \times i^r(u_b \curvearrowleft i^l(k_{ab})) &= (u_b \curvearrowleft i^l(k_{ab})) \curvearrowright (v \centerdot u_b \curvearrowleft i^l(k_{ab})) \\ &= (u_b \curvearrowleft i^l(k_{ab})) \curvearrowright i^l(k_{ab}) \end{aligned}$$

Thus we obtain, for v and k_{ab},

$$v = u_a \curvearrowleft \Big((u_b \curvearrowleft i^l(k_{ab})) \curvearrowright i^l(k_{ab})\Big),$$

$$k_{ab} = \Big[u_a \curvearrowleft \Big((u_b \curvearrowleft i^l(k_{ab})) \curvearrowright i^l(k_{ab})\Big)\Big] \centerdot \Big[u_b \curvearrowleft i^l(k_{ab})\Big]$$

It follows from equations (28) and (27) that

$$\begin{aligned} k_{ab} = \Big[u_a \curvearrowleft \Big((u_b \curvearrowright I) \centerdot i_l(k_{ab})\Big)\Big] \centerdot \Big[u_b \curvearrowleft i^l(k_{ab})\Big] &= \Big[\Big(u_a \curvearrowleft (u_b \curvearrowright I)\Big) \curvearrowleft i^l(k_{ab})\Big] \centerdot \Big[u_b \curvearrowleft i^l(k_{ab})\Big] \\ &= \Big((u_a \curvearrowleft (u_b \curvearrowright I)) \centerdot u_b\Big) \curvearrowleft i^l(k_{ab}) \end{aligned}$$

□

If one chooses for the the operad with multiplication the endomorphism operad of B, with m equal to the product in B, the above proof gives a third proof of the twisted multiplicativity for the T-transform in operator-valued free probability.

References

[1] Marcelo Aguiar. Infinitesimal bialgebras, pre-lie and dendriform algebras. *Lecture Notes in Pure and Applied Mathematics*, pages 1–34, 2004.

[2] Frédéric Chapoton. Un théorème de cartier–milnor–moore–quillen pour les bigèbres dendriformes et les algèbres braces. *Journal of Pure and Applied Algebra*, 168(1):1–18, 2002.

[3] Frédéric Chapoton and Muriel Livernet. Relating two hopf algebras built from an operad. *International Mathematics Research Notices*, 2007(9):rnm131–rnm131, 2007.

[4] Kenneth J Dykema. Multilinear function series and transforms in free probability theory. *Advances in Mathematics*, 208(1):351–407, 2007.

[5] Kurusch Ebrahimi-Fard, Loïc Foissy, Joachim Kock, and Frédéric Patras. Operads of (noncrossing) partitions, interacting bialgebras, and moment-cumulant relations. *arXiv preprint arXiv:1907.01190*, 2019.

[6] Murray Gerstenhaber and Alexander A Voronov. Homotopy g-algebras and moduli space operad. *arXiv preprint hep-th/9409063*, 1994.

[7] Germain Kreweras. Sur les partitions noncroisées d'un cycle. *Discrete Mathematics*, 1:333–350, 1972.

[8] Jean-Louis Loday and Bruno Vallette. *Algebraic operads*, volume 346. Springer Science & Business Media, 2012.

[9] James A Mingo and Roland Speicher. *Free probability and random matrices*, volume 35. Springer, 2017.

[10] Alexandru Nica and Roland Speicher. *Lectures on the combinatorics of free probability*, volume 13. Cambridge University Press, 2006.

[11] Maria Ronco. A milnor–moore theorem for dendriform hopf algebras. *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics*, 332(2):109–114, 2001.

[12] Dan Voiculescu. Addition of certain non-commuting random variables. *Journal of functional analysis*, 66(3):323–346, 1986.

[13] Dan Voiculescu. Multiplication of certain non-commuting random variables. *Journal of Operator Theory*, pages 223–235, 1987.