A SHUFFLE ALGEBRA POINT OF VIEW ON OPERATOR VALUED PROBABILITY THEORY

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ABSTRACT. We relate certain operadic morphisms of the words insertion operad and the gap insertion operad of non-crossing partitions through half-shuffle type structures. By introducing the notion of Hopf algebra in a duoidal category, we build a convolution algebra that contains, as subsets, both types of morphisms. We enrich this structure to an unshuffle Hopf algebra in a duoidal category of bi-graded collections and show that both types of operadic morphisms can be obtained as right and left half-shuffle exponentials. We discuss applications to operator valued probability theory.

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1. INTRODUCTION

In classical probability theory it is now well established that moment-cumulant relations are most accurately understood in the context of Möbius inversion on the lattice of set partitions and its associated incidence co-algebra. The combinatorial side of (scalar valued) Voiculescu's free probability theory finds its roots in the seminal work of Speicher [21] who pushed forward this idea, originally due to Rota, and showed that upon replacing set partitions by non-crossing set partitions, the same machinery can be used to define an equivalent notion of moment-cumulant relations in free probability.

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More precisely, in free probability moments and cumulants are seen as linear maps on the incidence coalgebra of the lattice of non-crossing partitions and the moment-cumulant relations are expressed in terms of the convolution product of the cumulant map with the zeta function. We refer the reader to [17, 18] for introductions to the theory of free probability.

When considering operator valued moments and cumulants, Speicher's results can be (partially) extended [22]. In fact, let $(\mathcal{A}, e, B)$ be an operator valued probability space. Recall that this means that B is an algebra acting from the right as well as from the left on the involutive algebra $\mathcal{A}$ and $e : \mathcal{A} \to B$ is B-B linear map [17]. As in the scalar-valued case ($B = \mathbb{C}$), the conditional expectation e can be extended to a multiplicative function $E = (e_\pi)_{\pi \in \mathrm{NC}}$ on the lattice of non-crossing partitions. In comparison, E does now depend not only on the sizes of the different blocks of π, but also on the *nesting* of the blocks. Still, the convolution of E with a *scalar valued* function makes sense and moment-cumulant relations are obtained. Operator valued cumulants depend as well on the nesting of the blocks of non-crossing partitions. Extracting algebraic structures encoding the nesting of blocks is then primordial to a better understanding of the properties of free cumulants. This may also add in particular to a concise description of relations with their boolean as well as monotone counterparts.

Recently, Ebrahimi-Fard and Patras proposed a rather different perspective on moment-cumulant relations in the scalar-valued case [7, 8, 9]. It does not involve Möbius inversion on lattices of set partitions. Instead, it is based on combinatorial Hopf algebraic techniques. More precisely, by describing a genuine shuffle algebra on words they encode (cumulants) moments as values taken by some Hopf algebra (infinitesimal) characters. This setting allows for unified picture of the three different types of cumulants; free, monotone and boolean as three faces of a single object, the unshuffle coproduct. Besides, this approach naturally gives rise to a (pre-)Lie theoretic description of the relations between the different cumulants in terms of shuffle adjoint transformations. It is critical to emphasise that the shuffle algebra setting does not involve at any point non-crossing partitions and that it has recently been successfully applied in the context of infinitesimal probability provided that field of complex number is replaced by a nilpotent algebra of strictly triangular matrices, see [3]. In the case of present interest, the target algebra of the morphisms we consider is non-commutative, so that the (pre-)Lie theoretic machinery developed in [9] fails to the extend to operator valued probability spaces.

Until recently, it was unclear how the two perspectives, i.e., Möbius inversion on the lattice of non-crossing set partitions on the one side and shuffle algebra on words on the other, can be related. In [6], the authors started to address this question. They showed that both the lattice and shuffle algebra approaches are governed each by their respective operad of non-crossing partitions and the associated incidence co-algebras. The shuffle algebra approach is associated with the so-called gap insertion operad of non-crossing partitions, which is going to be extensively used in this work, while the Möbius inversion formulation is encoded by the incidence coalgebra of a set partition-refinement operad. In particular, the incidence co-algebra of the gap insertion operad bears an unshuffle algebraic structure. The free moment and cumulant morphisms, M respectively k, are then seen to be related as solutions of the associated left half-shuffle fixed point equation:

$$M = \varepsilon + k \prec M.$$

The main objective of the present work is to extend the shuffle algebraic approach to free, boolean and monotone moment-cumulant relations to the setting of operator valued probability theory. Our approach heavily relies on the first part of reference [6]. We explain how considering moments and free cumulants of an operator valued probability space as multiplicative functions on the lattice of non-crossing partitions, naturally leads to an operadic perspective. Such a point of view encompasses as well the boolean cumulants seen as "almost" operadic morphisms on the word insertion operad. This result extends the picture developed in [6] half way to the operator valued case. In fact, the lacking part, i.e., the extension of the operadic point of view on Möbius inversion, can be more easily imported in our setting. This will be the subject of a forthcoming companion article.

As already mentioned, in this context both the nesting and the linear ordering of the blocks of a non-crossing partition are essential, since the moments associated with each block do not commute with each other, even if we consider moments of a single random variable. To accommodate this fact, elements of the incidence co-algebra should be considered as operators with multiple outputs and the co-algebraic structure should be replaced by a co-properadic structure. To be more precise, a word build from non-crossing set partitions (including the partition of the empty set) is an operator with as many inputs as gaps between the elements of the partitioned sets. A single output is associated with each partition in the word. The co-properadic structure (which is actually a simpler version of a plain non-symmetric co-properadic structure) is then dual to the gap insertion operad. A word on non-crossing set partitions should be seen as "a horizontal object" and applying the coproduct map on such a word results in two words that are vertically stacked.

We will show that this new insight finds a transparent description by means of a so-called duoidal structure on bi-graded collections (graded vector spaces with two gradings standing for the number of inputs and the number of outputs of an operator). A *duoidal category* is endowed with two tensor products (we use the symbols $\boxtimes$ and $\otimes$ throughout the article) satisfying a Lax property. The compatibility means essentially that composing horizontally and then vertically, or the other way around, four bi-graded collections, results in the same object drawn with the bi-graded collections at stake on the vertices. We shall use the terminology *vertically* and *horizontally* for the sub-categories of the categories of bigraded collections, or objects related to one or the other monoidal structure.

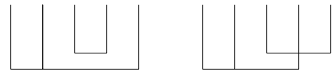
After having expounded the duoidal structure of the category of bi-graded collections, we proceed with the definition of the equivalent notion of a $\boxtimes\otimes$-Hopf algebra. The latter has both a vertical product and coproduct which are compatible through a horizontal algebraic structure. The associated convolution algebra of horizontal algebra morphisms valued in a properad of endomorphisms provides a unifying description of the gap insertion and word insertion operadic morphisms. This allows us, ultimately, to restore the Lie theoretic perspective on moments and cumulants in the operator-valued setting. Free and boolean cumulants are realised as horizontal algebra morphisms satisfying extra properties with respect to the properadic structure on words of set partitions. Besides, these two types of vertical morphisms are shown to be related by inversion in the convolution algebra.

We enrich the structure by introducing the notion of unshuffle Hopf algebra in a duoidal category. Once again, we show that the $\boxtimes\otimes$-Hopf algebra of words on non-crossing partitions can be endowed with such a structure. The free operator valued moments and cumulants, seen as operadic morphisms of the gap insertion operad, satisfy a left half-shuffle fixed point equation. The boolean cumulants, considered as operadic morphisms on the word insertion operad, are realised as a right half-shuffle exponential by seeing the word insertion operad as describing insertions of interval partitions into each other (in between the blocks).

As explained, the introduction of a second monoidal structure is supported by the fact the Lie theoretic perspective on moments and cumulants is restored by doing so. We emphasise that the picture is completely restored, as the notion of infinitesimal character makes sense.

In the context of non-commutative probability theory, various authors have used Hopf algebras and operads from many different perspectives. We mention the work of Friedrich–McKay [**?**], Hasebe–Lehner, [12], Mastnak–Nica [16] as well as the work of Gabriel [11]. In the latter, the author defines Hopf algebraic structures related to additive and multiplicative convolutions by using a geometric perpective on the space of (non-crossing) partitions. Operadic approaches to moment-cumulant relations have already been exploited by Joshuat-Vergès, Menous, Thibon and Novelli in [13] to obtain an operadic version of the shuffle point of view developed by Ebrahimi-Fard and Patras. Another perspective on moment-cumulant relations in an operadic framework was developed by Drummond-Cole in [4] and [5].

The paper is organised as follows. In section 2 we introduce non-crossing partitions. In section 3, we begin with a short reminder on operads and define the two operads that we consider in the present work the gap insertion operad and the word insertion operad. In 4, the category of bi-graded collections is endowed with a *duoidal* structure. In comparison with [6], these two monoidal structures we define in



section 4 and the Lax property they satisfy are the main ingredients that allows us to extend the results in [6]. In section 4 we define also the notion of unshuffle Hopf algebras in a duoidal settings and show that the space of words on non-crossing partitions can be endowed with such a structure. In section we proceed with the applications of the abstract settings with developed in the first sections to operator valued probability theory. Our main results are the following. In Lemma 39, we give an alternative view on the set of operator-valued moments associated with non-crossing partition. Proposition (25) is the central result of the present work, it states that any operadic morphism on the gap insertion operad is a solution of an half-shuffle fixed point equation. In Proposition 46 we prove a similar result for a class of morphisms we call boolean morphisms. In proposition 52, we compute explicitely the full shuffle exponential. Finally, in section 5.1.1 we relate our construction to the one expounded in [6] through thinning process.

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2. Set partitions

Let X be a finite, linearly ordered set. A *partition* of X into disjoint sets $\pi_i, 1 \leq i \leq k$, is denoted $\pi = \{\pi_1, \ldots, \pi_k\}$. An isomorphism between two set partitions is a monotone bijection of the underlying linearly ordered sets compatible with the block structure. Then, any partition is equivalent to a partition of the linearly ordered set $[\![1, n]\!]$ for some $n \in \mathbb{N}$ we call standard partition. It is convenient to work with the standard representative of each class.

For $k, n \in \mathbb{N}^\star$, we denote by $SP(k, n)$ the set of iso-classes of partitions of sets of n-elements into k blocks. The set $\mathrm{SP}(0, 0)$ contains only the empty partition. We put

$$\mathrm{SP} = \bigsqcup_{1\leq k\leq n} \mathrm{SP}(k, n),\ \mathrm{SP}(n) = \bigsqcup_{k\leq n} \mathrm{SP}(k, n),\ \mathrm{SP}_0 = \mathrm{SP}(0, 0) \sqcup \mathrm{SP}.$$

Given a monotone inclusion of linearly ordered sets $X \subset Y$ and given a partition π of Y, we write $\pi_{|X}$ for the trace of the partition of π on X.

Definition 1. Let X be a non-empty finite subset of $\mathbb{N}$ (or a linearly ordered set Y). The *convex hull* of X is by definition $\mathrm{Conv}(X)$=$[\![\min(X), \max(X)]\!]$. We shall say that X is convex if $\mathrm{Conv}(X) = X$. Any finite subset $X \subset \mathbb{N}$ decompose uniquely as

$$X = X_1 \sqcup \cdots \sqcup X_k$$

with each X_i convex and each $X_i \sqcup X_j$ *not* convex for $i \neq j$. The X_i are called the convex components.

Definition 2 (Non-crossing partitions)**.** A partition $\pi = \{\pi_1, \ldots, \pi_k\}$ is non-crossing if there are no $a, b \in \pi_i$ and no $c, d \in \pi_j$ with $i \neq j$ such that $a < c < b < d$.

For a detailed overview the algebraic structure of the set of non-crossing partitions, as well as an historical account, see [20]. The notion of non-crossing partitions has first been introduced by Kreweras in the seminal article [14]. For two integers $k, n \in \mathbb{N}^\star$, we denote by $\mathrm{NCP}(k, n)$ the set of iso-classes of non-crossing partitions of a set of n-elements into k blocks. The set $\mathrm{NCP}(0, 0)$ contains only the empty partition. We put

$$\mathrm{NCP} = \bigsqcup_{1\leq k\leq n} \mathrm{NCP}(k, n),\ \mathrm{NCP}(n) = \bigsqcup_{k\leq n} \mathrm{NCP}(k, n),\ \mathrm{NCP}_0 = \mathrm{NCP}(0, 0) \sqcup \mathrm{NCP}.$$

3. The gap insertion operad and the word insertion operad

In this section we settle all the basic algebraic structures that are used throughout this work. We start with a short reminder on collections and operads (both set and linear operads). Then, we formalize in this framework the idea of inserting a partition into the gaps of another partition. The reader is directed to [6] for a detailed exposition on operads, the gap-insertion operad and related structures and to the monograph [15] for an introduction to algebraic operad, both planar and symmetric and related concepts.

3.1. **Algebraic planar operads.** A collection P is a sequence of vector spaces $(P(n))_{n\geq 1}$. A morphism between two collections is a sequence of linear morphisms $(\phi(n))_{n\geq 1}$ with $\phi_n : P(n) \to P(n)$, $n \geq 0$. The category of all collections is denoted Coll. A tensor product on the category Coll is a 2-functor from Coll $\times$ Coll to Coll defined by:

$$(P \bullet Q)(n) = \bigoplus_{\substack{k\geq 1 \\ n_1+\cdots+n_k=n}} P(k) \otimes Q(n_1) \otimes \cdots \otimes Q(n_k),$$

$$(f \bullet g)(n) \quad \bigoplus_{\substack{k\geq 1 \\ n_1+\cdots+n_k=n}} f(k) \otimes g(n_1) \otimes \cdots \otimes g(n_k).$$

The unit element for the tensor product $\bullet$ is the collection denoted by $\mathbb{C}_\bullet$ such that $\mathbb{C}_\bullet(n) = \delta_{n=1}\mathbb{C}$. An operad $\mathcal{P}$ is a monoid in the monoidal category (Coll, $\bullet, \mathbb{C}$), i.e., a triple (P, ρ, η_P) with

$$P \in \text{Coll},\ \rho : P \bullet P \to P,\ \eta_P : \mathbb{C} \to \mathcal{P},$$

satisfying $(\rho \bullet \mathrm{id}_P) \circ \rho = (\mathrm{id}_P \bullet \rho) \circ \rho$ and $(\eta_P \bullet \mathrm{id}_P) \circ \rho = (\mathrm{id}_P \bullet \eta_P) \circ \rho = \mathrm{id}_P$. Note that we will use from time to time the notation $\circ$ for the operadic composition ρ to lighten notations. Accordingly, we use the same symbol $\circ$ with a subscript to denote partial compositions.

3.2. **Operad of partitions.** A partition $\pi \in \mathrm{SP}(n)$ is viewed as an operator with $n+1$ inputs. These inputs are the gaps between the elements of the partitioned set, including the front gap before 1 and the back gap after n. We can insert $n+1$ partitions inside these gaps. It is clear that if π is a non-crossing partition and we insert non-crossing partitions into the gaps of π then the resulting partition is again non-crossing.

Definition 3. We set $\mathcal{SP}(n) := \mathrm{SP}(n-1)$. In particular, we have $\mathcal{SP}(0) = \emptyset$ and $\mathcal{SP}(1) = \{\emptyset\}$. The empty partition is the operad unit. Let π be a partition and $(\alpha_1, \ldots, \alpha_{|\pi|})$ a sequence of set partitions. The composition $\rho_{\mathcal{SP}}(\pi \otimes \alpha_1 \otimes \cdots \otimes \alpha_{|\pi|})$ is obtained by inserting each partition α_i in between the two integers i and $i+1$, $i \leq 1$. In symbols:

$$\rho(\pi \otimes \alpha_1 \otimes \cdots \otimes \alpha_{|\pi|}) = \bigcup_{i=1}^{|\pi|} \{i-1+b,\ b \in \pi_i\} \cup \tilde{\pi}$$

where $\tilde{\pi}$ is the partition of $\{|\pi_1|, |\pi_1| + |\pi_2|, \ldots, |\pi_1| + \cdots + |\pi_n|\}$ induced by π.

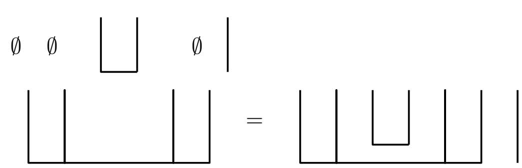

Figure 1. Example of a composition in the gap insertion operad $\mathcal{NC}$.

Lemma 4. *The sequence $\mathcal{NC} = (\mathcal{NC}(n))_{n\geq 0}$ with $\mathcal{NC}(n-1) = NC(n)$ defines a set operad called the non-crossing gap insertion operad when equipped with the composition law $\rho_{\mathcal{NC}} = \rho_{\mathcal{SP}|\mathcal{NC}\bullet\mathcal{NC}}$.*

The two set operadic structures $\rho_{\mathcal{SP}}$ and $\rho_{\mathcal{NC}}$ induce linear operadic structures on the free vector spaces spanned by SP, respectively NC. In the following, we shall not distinguish between them. The gap-insertion operad of non-crossing partitions admits the following presentation in terms of generators and relations.

Lemma 5 (Proposition 3.1.4 in [6])**.** *For any $n \geq 1$, we put $1_{n+1} = \{[\![1,n]\!]\}$. Then the operad $(\mathcal{NC}, \rho_{\mathcal{NC}})$ is generated by the elements 1_n, $n \geq 1$ with the relation:*

$$\forall m, n \geq 1, \quad 1_m \circ_m 1_n = 1_n \circ_1 1_m.$$

3.3. **Operad of word insertions.** We denote by $\langle \mathbb{N} \rangle$ the set of words on the alphabet $\{1, \ldots, n\}$. To avoid confusions, we use the symbol e_i for the letter $i \in \{1, \ldots, n\}$. Given a word $w = e_{w_1} \cdots e_{w_p} \in \langle \mathbb{N} \rangle$, we associate to it two numbers, its *insertion length* $\ell(w)$ and its *weight* $|w|$:

$$\ell(w) = p + 1, \; |w| = w_1 + \cdots + w_p + 1.$$

The word insertion operator formalizes the idea of inserting a word into another. Given two words $a = a_1 \cdots a_n$ and $b_1 \cdots b_m$, the word

$$a_1 \cdots a_{i-1} b_1 \cdots b_m a_i \cdots a_n$$

is obtained by *inserting the word b at the position $1 \leq i \leq \ell(a)$ in a.* We denote by $\langle \mathbb{N} \rangle^0$ the set $\langle \mathbb{N} \rangle \cup \{\emptyset\}$. The empty word $\{\emptyset\}$ will be the unit for the operation of insertion. By convention, the empty word has a length and a weight equal to 1.

Definition 6 (Word insertion operad)**.** For each integer $p \geq 1$, set $\mathcal{W}(p) = \{w \in \langle \mathbb{N} \rangle^0 : \ell(w) = p\}$. The empty word $\emptyset$ will be the unit for the word insertion operadic composition. Pick $a \in \mathcal{W}(p)$ and $b^1, \ldots, b^p$ a sequence of words of length p, we define

$$\rho_{\mathcal{W}}(a, b^1, \ldots, b^p) = b^1 a_1 b^2 a_2 \cdots a_{p-1} b^p.$$

The collection $(\mathbb{C}\mathcal{W}(p))_{p \geq 0}$ of vector spaces generated by the collection $\mathcal{W}$ is canonically endowed with a linear operadic structure. In the following we shall not distinguish between $\mathcal{W}$ and its linear counterpart. The word insertion operad is generated by letters, i.e., words with length equal to 2.

The word insertion operad $\mathcal{W}$ relates to the operad of non-crossing partition by mean of interval partitions.

Definition 7 (Interval partitions)**.** We say that a non-crossing partition $\pi \in \mathcal{NC}$ is an *interval partition* if and only if all the blocks of π are convex sets.

For each integer $n \geq 1$, let $I(n)$ the set of all interval partitions of $[\![1, n-1]\!]$. Set $I = \bigcup_{n \geq 0} I(n)$, then I is a sub-collection of $\mathcal{NC}$. However, the operadic structure on $\rho_{\mathcal{NC}}$ does not restrict to an operadic structure on I since, for example, we do not get an interval partition if we insert the interval $\mathbf{1}_2$ in the second slot of the interval $\mathbf{1}_3$. Instead, we push forward the operadic structure of $\mathcal{W}$ to define an operad on I.

We write an interval partition $\pi \in I(n)$ as $\pi = I_1 \cdots I_p$, with I_i the interval of $[\![1,n]\!]$ of length equal to the cardinal of the *ith* block of π for the lexicographic order. Notice that given a sequence of intervals $I_i = [\![1, n_i]\!]$ for some integer $n_i \geq 1$, there exists an unique interval partition π in $I(n_1 + \cdots + n_p + 1)$ such that $\pi = I_1 \cdots I_p$. The define the liner map $\Phi : \mathcal{W} \to I$ as

$$\begin{array}{cccc} \Phi : & \mathcal{W} & \longrightarrow & I \\ & e_{w_1} \ldots e_{w_p} & \mapsto & I_{w_1} \cdots I_{w_p}. \\ & \emptyset & \mapsto & \{\emptyset\}. \end{array} \tag{1}$$

Notice that Φ is neither an operadic morphism nor a collection morphism. In the following proposition, for an integer $m \geq 0$, we denote by $\mathcal{W}_m$ the set of words of weight m:

$$\mathcal{W}_m = \{w \in \langle \mathbb{N} \rangle : |w| = m\}, \tag{2}$$

and denote by $\circ_i$ the partial compositions in the gap insertion operad. The expression

$$\pi \circ_{i_1,\ldots,i_p} (\pi_1, \ldots, \pi_p),\ 1 \leq i_1 < \cdots < i_p \leq |\pi|$$

means that each of the partition π_j is inserted at position i_j in π and we insert the unit element $\{\emptyset\}$ in every other positions.

Proposition 8. *Let $m \geq 1$ be an integer, letters $e_{w_1}, \ldots, e_{w_{p-1}} \in \langle N \rangle$ and $b_1, \ldots, b_p$ words in $\mathcal{W}$. Then, the map Φ is an injective map such that $\Phi(\mathcal{W}_m) = I(m)$ and*

$$\Phi(\rho(e_{w_1} \cdots e_{w_{p-1}}, (b_1, \ldots, b_p))) = \Phi(e_{w_1} \cdots e_{w_p}) \circ_{1,w_1,w_1+w_2,\ldots,w_1+\cdots+w_{p-1}} (\Phi(b_1), \ldots, \Phi(b_p)).$$

Proof. The first two statements of the proposition follow easily from the definition of the map Φ. For the last one, we simply notice that

$$\Phi(\rho(e_{w_1} \cdots e_{w_{p-1}}, (b_1, \ldots, b_p))) = \Phi(b_1) I_{w_1} \cdots I_{w_{p-1}} \Phi(b_p).$$

□

4. The duoidal category of bi-graded collections

Elements of a collection are seen as operations with many inputs and a single output. The operadic structure models compositions between all these operations. In many branches of mathematics, ranging from probability theory, both classical and non-commutative, to gauge theory and quantum groups, algebraic structures with products and co-products that stand for merging respectively cutting processes have become popular. The framework of operads is however too narrow to treat such structures completely. Indeed, it turns to be quite fundamental to be able to handle operations with multiple in- and out-puts. After the work of JP. Serre [19] and B. Vallette [23], the right algebraic framework appears to be the one of properads (props). The construction we expose in the section is reminiscent of the props setting, but is in fact much simpler.

We introduce now the pro-eminent algebraic structure relevant to the present work, i.e., the category of bigraded collections endowed with two balanced monoidal structures. In the literature, such a category is called a *duoidal*[1] category. This section focuses on the so-called *laxity property* stated in (4). It is beyond the scope for the present work to provide the reader with a detailed account on the notion of duoidal category. Nevertheless, for the sake of completeness, we will give the definition of such a category, without fully commenting on it.

Definition 9 (Duoidal category)**.** *A duoidal category*, or 2-monoidal category, is a category endowed with a monoidal structure $(\otimes, E_\otimes)$, together with an additional monoidal structure $(\boxtimes, E_\boxtimes)$ such that $\boxtimes : C \times C \to C$ and $E_\boxtimes : 1 \to C$ are lax monoidal functors with respect to $(\otimes, E_\otimes)$ and the coherence axioms of $(\boxtimes, E_\boxtimes)$ are monoidal natural transformation with respect to $(\otimes, E_\otimes)$. The laxity of $(\boxtimes, E_\boxtimes)$ consists of natural transformations

$$(C_1 \boxtimes C_2) \otimes (C_3 \boxtimes C_4) \overset{R_{C_1,C_2,C_3,C_4}}{\longrightarrow} (C_1 \otimes C_3) \boxtimes (C_2 \otimes C_4)$$

together with morphisms $E_\otimes \to E_\otimes \boxtimes E_\otimes$, $\quad E_\boxtimes \otimes E_\boxtimes \to E_\boxtimes$, $\quad E_\otimes \to E_\boxtimes$.

The commutative diagram in Fig. 6 pictures $\boxtimes$-monoidal property of the associator $\alpha^\otimes$ of the monoidal structure $\otimes$ while in Fig. 5 we have drawn the commutative diagram that expresses naturality of R. We extensively use the terminology *horizontal* to refer to the monoidal category $(C, \otimes, E_\otimes)$ and *vertical* to refer to the monoidal category $(C, \boxtimes, E_\boxtimes)$. A duoidal category for which the natural transformation R is an isomorphism is a called *normal*. This property does not hold for the full category of bi-graded collections, but instead for a smaller category of objects we call infinitely divisible bi-graded collections. To get acquainted with duoidal categories, we draw examples. Denote by $\mathcal{F}$ be the vector space generated by all planar rootes forests ie words on planar rooted trees. The are two ways to

[1]https://ncatlab.org/nlab/show/duoidal+category

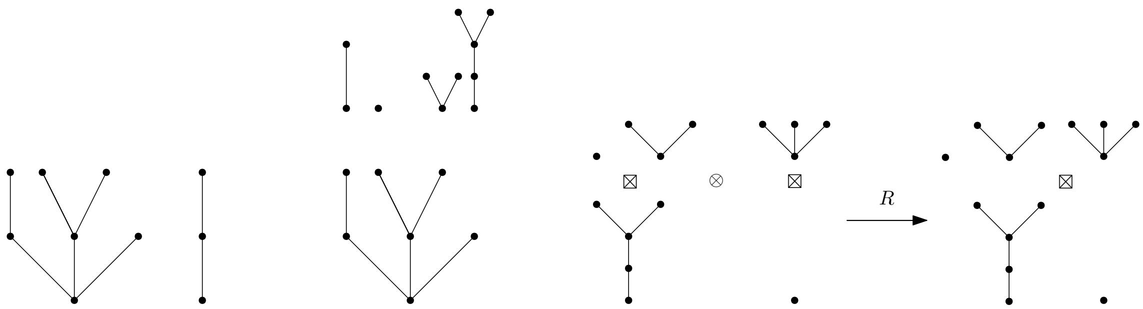

FIGURE 2. On the left, we stack horizontally two trees in the middle, with stack vertically two forests. On the right we draw an example of the action of the natural transformation R on forests.

compose forests, either we concatenate them, either we stack them vertically. For this last operation to be well defined, the number of leaves of the bottom forest should match the number of trees of the top forest, see Figure 2

We proceed in the last part of this section with the definition of ($\boxtimes$-co)/($\otimes$-al)gebras as either a coalgebra in the vertical monoidal category of horizontal algebras or as an algebra in the horizontal category of vertical coalgebras. The main result of this section is the following one.

Proposition. *The category of bigraded collection is duoidal.*

4.1. **The category of bi-graded collections and $\boxtimes\otimes$-Hopf algebras.** We formalize the idea of considering operators with multiple in- and outputs in the following definition.

Definition 10 (Bi-graded collection)**.** A bi-graded collection is a two parameter family of vector spaces $\boldsymbol{P} = (\boldsymbol{P}(n,m))_{n,m\geq 0}$. A morphism between two bi-graded collections $\boldsymbol{P}$ and $\boldsymbol{Q}$ is a sequence of linear morphisms $\phi(n,m) : \boldsymbol{P}(n,m) \to \boldsymbol{Q}(n,m)$. The category of all bi-graded collections is denoted Coll_2.

Example. *Let $Pl(n,m)$ be the linear span of planar forests (made of planar trees) with n trees and m leaves. Then $Pl = (Pl_{n,m})_{n,m\geq 1}$ is a bi-graded collection.*

Definition 11 (Horizontal tensor products)**.** The horizontal tensor product $\otimes$ is the functor $\otimes_H$: $\mathrm{Coll}_2 \times \mathrm{Coll}_2 \to \mathrm{Coll}_2$ defined by:

$$(\boldsymbol{P} \otimes_H \boldsymbol{Q})(n,m) = \bigoplus_{\substack{n_1+n_2=n \\ m_1+m_2=m}} \boldsymbol{P}(n_1,m_1) \otimes \boldsymbol{Q}(n_2,m_2),$$

$$(f \otimes_H g)(n,m) = \bigoplus_{\substack{n_1+n_2=n \\ m_1+m_2=m}} f(n_1,m_1) \otimes g(n_2,m_2).$$

The identity element for the horizontal tensor product is the bi-graded collection $\mathbb{C}_{\otimes}(n,m) = \delta_{n,m=0}\mathbb{C}$.

Definition 12 (Tensor product of bi-collections)**.** The tensor product $\boxtimes$ on the category Coll_2 is defined by:

$$\boldsymbol{P} \boxtimes \boldsymbol{Q}(n,m) = \bigoplus_{k\geq 0} \boldsymbol{P}(n,k) \otimes \boldsymbol{Q}(k,m),$$

$$(f \boxtimes g)(n,m) = \bigoplus_{k\geq 0} f(n,k) \otimes g(k,m).$$

The identity element for the tensor product $\boxtimes$ is the bigraded collection $\mathbb{C}_{\boxtimes}(n,m) = \delta_{n=m}\mathbb{C}$.

Fundamental examples of bi-graded collections are given by sets of words on a graded collection. If $P = (P_n)_{n\geq 0}$ is a graded collection, then:

$$\boldsymbol{P}(m,n) = \bigoplus_{\substack{n\geq 1 \\ k_1+\cdots+k_n=m}} P_{k_1}\otimes\cdots\otimes P_{k_n} \tag{3}$$

is the homogeneous component of degree m, n of a bigraded collection $\boldsymbol{P}$. With B an unital complex algebra, the case $P = (\mathrm{Hom}(B^{\otimes n}, B))_{n\geq 0}$ plays a proeminent role in the sequel.

We have already encountered another example of bi-graded collection whose homogeneous components are generated by forests with a certain number of forests and leaves.

In Fig. 3 the reader will find a pictorial description of elements in the horizontal and vertical tensor products. In the vertical tensor product, the number if inputs of the operator on the lower level matches the number of ouputs of the operator on the upper level. In comparison with the vertical tensor product introduced in [23], the tensor product $\boldsymbol{P}\boxtimes\boldsymbol{Q}$ we introduce is a sum over planar 2-levels diagrams with only one vertex on each levels, over *bundles*. It is easy to design a generalization of the vertical tensor product: we sum over planar diagrams that connect vertices placed on integer points of the lines $\mathbb{R}\times\{0\}$ to vertices placed on the line $\mathbb{R}\times\{1\}$.

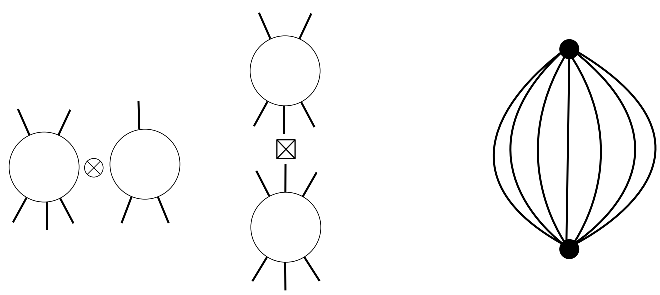

FIGURE 3. On the left, we have elements in the horizontal tensor product $\otimes_H$ and in the vertical one $\boxtimes$. On the right, we have a bundle.

Proposition 13. *The categories* $(Coll_2, \otimes_H, \mathbb{C}_H)$ *and* $(Coll_2, \boxtimes_H, \mathbb{C}_H)$ *are strict monoidal categories.*

Proof. Let $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R}$ be three bi-graded collections, because the categorie $(Vect_{\mathbb{C}}, \otimes)$ is strict,

$$\begin{aligned}((\boldsymbol{P}\boxtimes\boldsymbol{Q})\boxtimes\boldsymbol{R})(n,m) &= \bigoplus_{k,l}((\boldsymbol{P}(n,k)\otimes\boldsymbol{Q}(k,l))\otimes\boldsymbol{R}(l,m)) \\ &= \bigoplus_{k,l}(\boldsymbol{P}(n,k)\otimes(\boldsymbol{Q}(k,l)\otimes\boldsymbol{R}(l,m)) \\ &= (\boldsymbol{P}\boxtimes(\boldsymbol{Q}\boxtimes\boldsymbol{R}))(n,m)\end{aligned}$$

The same argument applies for $\otimes_H$. □

Remark 1.
- The tensor product $\otimes$ is a symmetric tensor product, whereas $\boxtimes$ is not. Neither the horizontal nor the vertical tensor product come with injections and the units for these two tensor products are not initial objects.
- $T_{\otimes}(\mathbb{C}) = \mathbb{C}$.

The following lemma is crucial and is the cornerstone for defintion of the notion Hopf-like structure in the context of duoidal category.

Lemma 14. *Let* $C_i,\ 1\leq i\leq 4$ *be four bigraded collections, then*

$$(C_1\boxtimes C_2)\otimes_H(C_3\boxtimes C_4)\hookrightarrow(C_1\otimes_H C_3)\boxtimes(C_2\otimes_H C_4)\,. \tag{4}$$

And if C is a collection,

$$(C_1 \boxtimes T_\otimes(C)) \otimes_H (C_3 \boxtimes T_\otimes(C)) \simeq (C_1 \otimes_H C_2) \boxtimes T_\otimes(C). \tag{5}$$

We denote by R_{C_1,C_2,C_3,C_4} this morphism.

Proof. Let C_1, C_2, C_3 and C_4 be four bi-graded collections. Let p_1, p^2, p^3, p^4 be elements of respectively, C_1, C_2, C_3 and C_4. Define R_{C_1,C_2,C_3,C_4} by the equation

$$\begin{aligned}R_{C_1,C_2,C_3,C_4}(p^1 \boxtimes (p_1^2 \otimes \cdots \otimes p^2_{|p^1|}) \otimes_H (p^3 \boxtimes (p_1^4 \otimes \cdots \otimes p^4_{|p^3|})))\\ = (p_1 \otimes_H p_3) \boxtimes (p_1^2 \otimes \cdots \otimes p^2_{|p^1|} \otimes_H p_1^4 \otimes \cdots \otimes p^4_{|p^3|})\end{aligned}$$

It is easy to see that R_{C_1,C_2,C_3,C_4}, if extended linearly, is a morphism of bigraded collections. Moreover it is injective, but is not surjective. In particular, the image of R is span by the elements $(p^1 \otimes p^3) \boxtimes (p^2 \boxtimes p^4)$ with a perfect match between the inputs of p^3 and the outputs of p^4 on one side and the inputs of p^1 and the outputs of p^2 on the other side. Hence, if $C_2 = C_4 = T_\otimes(C)$ for a certain collection C, we can define

$$R^{-1}_{C_1,C_2,C_3,C_4}(p^1 p^3 \boxtimes (p_1 \cdots p_{|p^1|+|p^3|})) = p^1 \boxtimes p_1 \cdots p_{|p^1|} \otimes_H p^3 \boxtimes p_{|p^1|+1} \cdots p_{|p^1|+|p^3|}.$$

□

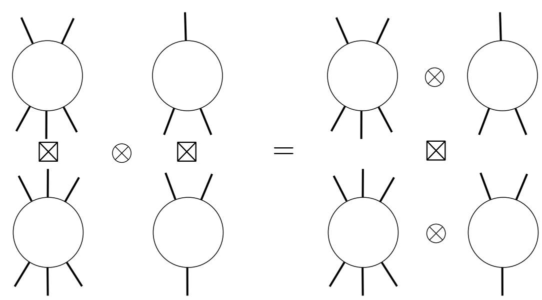

Figure 4. Drawing of the action of the natural functor R.

The family of morphisms $\{R_{C_1,C_2,C_3,C_4}, C_i \in \mathrm{Coll}_2\}$ define a natural transformation between the two functors $\otimes \circ \boxtimes \times \boxtimes$ and $\boxtimes \circ \otimes_H \times \otimes_H$. In fact, pick four morphisms $f_i : C_i \to D_i$, $1 \leq i \leq 4$, the diagram in Fig. 5 is a commutative diagram.

$$\begin{tikzcd}
(C_1 \boxtimes C_2) \otimes_H (C_3 \boxtimes C_4) \arrow[r, "(f_1 \boxtimes f_2) \otimes (f_3 \boxtimes f_4)"] \arrow[d, "R_{C_1,C_2,C_3,C_4}"] & (D_1 \boxtimes D_2) \otimes_H (D_3 \boxtimes D_4) \arrow[d, "R_{D_1,D_2,D_3,D_4}"] \\
(C_1 \otimes_H C_3) \boxtimes (C_2 \otimes C_4) \arrow[r, "(f_1 \otimes f_3) \boxtimes (f_2 \otimes f_4)"'] & (D_1 \otimes_H D_3) \boxtimes (D_2 \otimes_H D_4)
\end{tikzcd}$$

Figure 5. Naturality for R.

We denote by $\mathrm{Alg}_\boxtimes$ the category of complex associative algebras in the monoidal category $(\mathrm{Coll}_2, \boxtimes)$ and $\mathrm{Alg}_\otimes$ the category of complex associative algebras in the monoidal category $(\mathrm{Coll}_2, \otimes$. We do not require the existence of a unit. We write an an horizontal algebra as a pair $(A, m^A_\otimes)$ and a vertical algebra as a pair $(A, m^A_\boxtimes)$ with:

$$m^A_\otimes : A \otimes A \to A, \quad m^A_\boxtimes : A \boxtimes A \to A.$$

We consider the sub-category $\mathrm{Alg}^\iota_\otimes \subset \mathrm{Alg}_\otimes$ whose objects are the algebras $T_\otimes(C)$ with C a collection. The unit $\mathbb{C}_\boxtimes$ of the vertical tensor product $\boxtimes$ is an algebra in the monoidal category $(\mathrm{Coll}_2, \otimes, \mathbb{C}_\otimes)$, since $T_\otimes(\mathbb{C}_\otimes) = \mathbb{C}_\boxtimes$.

Corollary 1. *The category* $(Alg_{\otimes}, \boxtimes, \mathbb{C}_{\boxtimes})$ *is a monoidal category. If* $(A, m^A_{\otimes})$ *and* $(B, m^B_{\otimes})$ *are horizontal algebras, then the horizontal product* $m^{A\boxtimes B}_{\otimes} : (A \boxtimes B) \otimes (A \boxtimes B)$ *on* $A \boxtimes B$ *is defined by :*

$$m^{A\boxtimes B}_{\otimes} = (m^A_{\otimes} \boxtimes m^B_{\otimes}) \circ R_{A,B,A,B}. \tag{6}$$

Proof. We drop the subscript H from the symbol denoting the horizontal tensor product. In the following we denote by $\alpha^{\otimes}$ the associativity constraints for the tensor product $\otimes$. For each triple of bi-graded collections (C_1, C_2, C_3) we have an isomorphism $\alpha_{C_1,C_2,C_3} : (C_1 \otimes C_2) \otimes C_3 \to C_1 \otimes (C_2 \otimes C_3)$ of bi-graded collectionq. Pick (A, ρ_A) and (B, ρ_B) two algebras in $\mathrm{Alg}_{\otimes}$. We need first to define an associative product law $\rho_{A\otimes B}$ on $A \boxtimes B$. Set

$$\rho_{A\boxtimes B} = (\rho_A \boxtimes \rho_B) \circ R_{A,B,A,B} : (A \boxtimes B) \otimes (A \boxtimes B) \to A \otimes B$$

First, we show that $\rho_{A\boxtimes B}$ is associative, namely:

$$\rho_{A\boxtimes B} \circ (\rho_{A\boxtimes B} \boxtimes \mathrm{id}_{A\boxtimes B}) = \rho_{A\boxtimes B} \circ (\mathrm{id}_{A\boxtimes B} \boxtimes \rho_{A\boxtimes B}).$$

Its is a simple consequence of the commutative diagrams in Fig. 6 and the naturality of R expressed in Fig 5. We have to show that center face of the diagram in Fig. 7 is commutative. We recognize the diagram in Fig. 5 with $C_1 = C_3 = A$ and $C_2 = C_4 = B$ in the rightmost and leftmost faces of the diagram in Fig. 7. Besides, the outer face of the same diagram is the commutative diagram in Fig. 6. From associativity property of the two products ρ_A and ρ_B we deduce that the lower face (with the dotted arrow) is commutative. Hence, the central face must be commutative. We proved that given two horizontal algebras their vertical tensor is naturally endowed with a horizontal algebra structure.

$$\begin{tikzcd}
(C_1 \boxtimes C_2) \otimes ((C_3 \boxtimes C_4) \otimes (C_5 \boxtimes C_6)) \arrow[d, "\mathrm{id}_{1\otimes 2} \otimes R_{3,4,5,6}"'] & ((C_1 \boxtimes C_2) \otimes (C_3 \boxtimes C_4)) \otimes (C_5 \boxtimes C_6) \arrow[l, "\alpha^{\otimes}_{C_1\boxtimes C_2, C_3\boxtimes C_4, C_5\boxtimes C_6}"'] \arrow[d, "R_{1,2,3,4} \otimes \mathrm{id}_{5\otimes 6}"] \\
(C_1 \boxtimes C_2) \otimes ((C_3 \otimes C_5) \boxtimes (C_4 \otimes C_6)) \arrow[d, "R_{1,2,3\otimes 5,4\otimes 6}"'] & (C_1 \otimes C_3) \boxtimes (C_2 \otimes C_4)) \otimes (C_5 \boxtimes C_6) \arrow[d, "R_{1\otimes 3,5,2\otimes 4,6}"] \\
(C_1 \otimes (C_3 \otimes C_5))) \boxtimes ((C_2 \otimes (C_4 \otimes C_6)) & ((C_1 \otimes C_3) \otimes C_5) \boxtimes ((C_2 \otimes C_4) \otimes C_6) \arrow[l, "\alpha^{\otimes}_{1,3,5} \boxtimes \alpha^{\otimes}_{2,4,6}"']
\end{tikzcd}$$

FIGURE 6. The associator $\alpha^{\otimes}$ is a $\boxtimes$ monoidal natural transformation (between the functors $\boxtimes \circ (\boxtimes \times \mathrm{id})$ and $\boxtimes \circ (\mathrm{id} \times \boxtimes)$).

Next, we have to show that given two algebra morphisms $\alpha : A \to A'$ and $\beta : B \to B'$, $A, A', B, B' \in \mathrm{Coll}_2$ the bi-graded collection morphism $\alpha \boxtimes \beta$ is an algebra morphism from $(A \boxtimes B, \rho_{A\otimes B})$ to $(A' \boxtimes B', \rho_{A'\boxtimes B'})$.

$$\begin{aligned}
(\alpha \boxtimes \beta) \circ \rho_{A\otimes B} &= (\alpha \boxtimes \beta) \circ (\rho_A \boxtimes \rho_B) \circ R_{A,B,A,B} \\
&= (\alpha \circ \rho_A) \boxtimes (\beta \circ \rho_B) \circ R_{A,B,A,B} && \text{(Functoriality of } \boxtimes) \\
&= (\rho_{A'} \circ \alpha \otimes \alpha) \boxtimes (\rho_{B'} \circ \beta \otimes \beta) \circ R_{A,B,A,B} && (\alpha \text{ and } \beta \text{ are morphisms}) \\
&= (\rho_{A'} \boxtimes \rho_{B'}) \circ ((\alpha \otimes \alpha) \boxtimes (\beta \otimes \beta)) \circ R_{A,B,A,B} \\
&= (\rho_{A'} \boxtimes \rho_{B'}) \circ R_{A',B',A',B'} \circ (\alpha \boxtimes \beta) \otimes (\alpha \boxtimes \beta) && \text{(Naturality of } R) \\
&= \rho_{A'\boxtimes B'} \circ (\alpha \boxtimes \beta) \otimes (\alpha \boxtimes \beta)
\end{aligned}$$

□

Remark 2. The monoidal structure on $Alg_{\otimes}$ restricts to a monoidal structure on the sub-category of unital algebras. However, the resulting tensor product does not come with injections.

$$\begin{tikzcd}
(A\boxtimes B)\otimes((A\boxtimes B)\otimes(A\boxtimes B)) \arrow[d, "\mathrm{id}_{A\otimes B}\otimes R_{A,B,A,B}"] & & ((A\boxtimes B)\otimes(A\boxtimes B))\otimes(A\boxtimes B) \arrow[ll, "\alpha^{\otimes}_{A\boxtimes B,A\boxtimes B,A\boxtimes B}"'] \arrow[d, "R_{A,B,A,B}\otimes\mathrm{id}_{A\otimes B}"'] \\
(A\boxtimes B)\otimes((A\otimes A)\boxtimes(B\otimes B)) \arrow[d, "\mathrm{id}_{A\otimes B}\otimes(\rho_A\boxtimes\rho_B)"] \arrow[dd, bend right, "R_{A,B,A\otimes A,B\otimes B}"'] & & ((A\otimes A)\boxtimes(B\otimes B))\otimes(A\boxtimes B) \arrow[d, "\rho_A\boxtimes\rho_B\otimes\mathrm{id}_{A\otimes B}"'] \arrow[dd, bend left, "R_{A\otimes A,B\otimes B,A,B}"] \\
(A\boxtimes B)\otimes(A\boxtimes B) \arrow[dr, "R_{A,B,A,B}"] & & (A\boxtimes B)\otimes(A\boxtimes B) \arrow[dl, "R_{A,B,A,B}"'] \\
(A\otimes(A\otimes A))\boxtimes(B\otimes(B\otimes B)) \arrow[r, bend right, "(\mathrm{id}_A\otimes\rho_A)\boxtimes(\mathrm{id}_B\otimes\rho_B)"'] & (A\otimes A)\boxtimes(B\otimes B) \arrow[d, "\rho_A\boxtimes\rho_B"] & ((A\otimes A)\otimes A)\boxtimes((B\otimes B)\otimes B) \arrow[l, bend left, "(\rho_A\otimes\mathrm{id}_A)\boxtimes(\rho_B\otimes\mathrm{id}_B)"] \arrow[ll, dotted, bend left=60, "\alpha_{A,A,A}\boxtimes\alpha_{B,B,B}"] \\
& A\boxtimes B &
\end{tikzcd}$$

FIGURE 7. Associativity of the product $\rho_{A\boxtimes B}$.

By dualizing the last proposition, we get an equivalent statement for coalgebras. In fact, let $\mathrm{coAlg}_{\boxtimes}$ be the category of coalgebras in the monoidal category $(\mathrm{Coll}_2, \boxtimes, \mathbb{C}_{\boxtimes})$. Note that the identity object $\mathbb{C}_{\otimes}$ for $\otimes$ is a $\boxtimes$-coalgebra with co-product and co-unit:

$$\Delta : \mathbb{C}_{\otimes} \to \mathbb{C}_{\otimes} \boxtimes \mathbb{C}_{\otimes},\ 1_0 \to 1_0 \boxtimes 1_0,\ \epsilon(1_0 \boxtimes 1_0) = 1_0 \in \mathbb{C}. \tag{7}$$

Corollary 2. *The category $(coAlg_{\boxtimes}, \otimes, \mathbb{C}_{\otimes})$ is a monoidal category. If $(A, \Delta^{\boxtimes}_A)$ and $(B, \Delta^{\boxtimes}_B)$ are two vertical co-algebra then*

$$\Delta^{\boxtimes}_{A\otimes B} = R \circ_{A\otimes B} \circ (\Delta^{\boxtimes}_A \otimes \Delta^{\boxtimes}_B)$$

defines a coproduct on $A \otimes B$.

Example. *We process with a fundamental example of a bi-graded collection endowed with an horizontal product and a vertical product. Let P be a (single) graded collection. We set $T_{\otimes}(P) = \bigoplus_{n\geq 1} P^{\otimes_H n}$. There exists a canonical isomorphism of bi-graded collection $\phi : T_{\otimes}(P \bullet P) \longrightarrow T_{\otimes}(P) \boxtimes T_{\otimes}(P)$, defined by*

$$\begin{aligned}\phi\Big(\left[p^1 \bullet (q_1^1 \otimes \cdots \otimes q^1_{|p^1|})\right] \otimes_H \cdots \otimes_H \left[p^n \bullet (q_1^n \otimes \cdots \otimes q^n_{|p^n|})\right] \Big) \\ = p^1 \otimes_H \cdots \otimes_H p^n \boxtimes (q_1^1 \otimes_H \cdots \otimes_H q^n_{p^n}).\end{aligned}$$

As a consequence, the tensor product $P \boxtimes P$ is endowed with an algebra product obtained by pushing forward by mean of ϕ the concatenation product on $T_{\otimes}(P)$. Assume next $(\mathcal{P})$ is endowed with an operadic composition ρ $T_{\otimes}(P) \boxtimes T_{\otimes}(P) \to T_{\otimes}(P)$. Then ρ induces an horizontal morphism *(for the concatenation product $\otimes$) denoted $T_{\otimes}(\rho) : T_{\otimes}(P \bullet P) \to T_{\otimes}(P)$.*

We can consider objects in the category of bigraded collections that are co-algebras (respectively algebras) in $(\mathrm{Alg}_{\otimes}, \boxtimes)$, we call such objects ($\boxtimes$-co|$\otimes$-al)gebras (respectively ($\boxtimes$-al|$\otimes$-al)gebras).

We can also look at objects that are algebras (respectively coalgebras) in the category $(\mathrm{coAlg}_{\boxtimes}, \otimes)$, we call them ($\otimes$-al|$\boxtimes$-co)gebra (respectively ($\otimes$-co|$\boxtimes$-co)gebra).

Proposition 15. *A bi-graded collection C is a ($\boxtimes$-co/$\otimes$-al)gebra if and only if ($\otimes$-al/$\boxtimes$-co)gebra.*

Proof. The two diagrams expressing compatibility between the multiplication map $m^{\otimes}$ and the coproduct $\Delta^{\boxtimes}$ (saying either that $m^{\otimes}$ is $\Delta^{\boxtimes}$ coproduct morphism either that $\Delta^{\boxtimes}$ is a $m^{\otimes}$ algebra map) are both equal to the diagram in Fig. 8. $\square$

$$\begin{tikzcd} C\otimes C \arrow[r, "\Delta\otimes\Delta"] \arrow[dd, "m^{\otimes}"] & (A\boxtimes A)\otimes(A\boxtimes A) \arrow[dr, "R_{A,A,A,A}"] & \\ & & (A\otimes A)\boxtimes(A\otimes A) \arrow[dl, "m^{\otimes}\boxtimes m^{\otimes}"] \\ A \arrow[r, "\Delta"] & A\boxtimes A & \end{tikzcd}$$

FIGURE 8. Compatibility between the multiplication and comultiplication for $\boxtimes\otimes$-bialgebras.

Definition 16. An augmented object in $Alg_{\otimes}$ is the data of bi-graded collection C in $\mathrm{Alg}_{\otimes}$ with $C(n,m)=0$ if $n\neq m$ and $C(n,n)=\mathbb{C}$. We sometimes use the terminology vertical coaugmented horizontal algebras.

Given a co-augmented horizontal algebra we call $\eta:\mathbb{C}_{\boxtimes}\to C$ the co-augmentation map. We denote by $(\boxtimes\text{-Aug})\mathrm{Alg}_{\otimes}$ the sub-category of $\mathrm{Alg}_{\otimes}$ of co-augmented horizontal algebras. The vertical tensor product of two vertical augmented horizontal aglebras is again a vertically augmented horizontal algebras. The restriction of $\boxtimes$ to $(\boxtimes\text{-Aug})\mathrm{Alg}_{\otimes}$ comes with a richer stucture. Let p_1 and p_2 be the two projections $(\boxtimes\text{-Aug})\mathrm{Alg}_{\otimes}\times(\boxtimes\text{-Aug})\mathrm{Alg}_{\otimes}\to(\boxtimes\text{-Aug})\mathrm{Alg}_{\otimes}$. There exists two natural transformations $\iota_1:p_1\to\boxtimes$ and $\iota_2:p_2\to\boxtimes$ such the diagrams in Fig 9 are commutative diagrams.

$$\begin{tikzcd} C_1 \arrow[r, "\iota_1"] \arrow[d, "\alpha"'] & C_1\boxtimes C_2 \arrow[d, "\alpha\boxtimes\beta"] & C_2 \arrow[l, "\iota_2"'] \arrow[d, "\beta"] \\ D_1 \arrow[r, "\iota_1"'] & D_1\boxtimes D_2 & D_2 \arrow[l, "\iota_2"] \end{tikzcd}$$

FIGURE 9. The tensor product $\boxtimes$ on $(\boxtimes\text{-Aug})\mathrm{Alg}_{\otimes}$ is a tensor product with injections.

Definition 17 (Co-nilpotent $\boxtimes\otimes$-bialgebras)**.** A bi-graded $\boxtimes\otimes$-bialgebras (C,Δ,ε) is said *co-nilpotent* if

1. $C(n,n)=\mathbf{C}(n,n)=\mathbb{C}1_n,\ n\geq 0,$
2. $\bar{\Delta}(c)=\Delta(c)+1_m\boxtimes c+c\boxtimes 1_n,\ c\in C(n,m),\ n\neq m$, with
$$\bar{\Delta}:C\to C\boxtimes C,\ \Delta(C(n,m))\subset\bigoplus_{\substack{k\geq 0\\ k\neq n,m}}C(n,k)\otimes C(k,m),\ \Delta(C(n,n))=0,$$
3. Δ is point-wise nilpotent.

Remark 3. In the settings of the previous definition, $\varepsilon(c)=0$ if $c\in C(m,n),\ m\neq n$ and $\varepsilon(1_n)=1$.

Definition 18 ($\boxtimes\otimes$ Hopf algebras)**.** A bi-graded $\boxtimes\otimes$-Hopf algebra is a tuple $(C,\Delta,\varepsilon,S,\eta)$ of objects and morphisms in the category $\mathrm{Alg}_{\otimes}$ such that (C,Δ,ε) is a ($\boxtimes$-co)($\otimes$-al)gebra and (C,∇,η) an unital algebra in $(Alg_{\otimes},\boxtimes,\mathbb{C}_{\boxtimes})$ with a morphism $S:C\to C$ of horizontal algebras such that
$$(S\boxtimes\mathrm{id}_C)\circ\Delta=(\mathrm{id}_C\boxtimes S)\circ\Delta=\eta\circ\varepsilon.$$

The $\boxtimes\otimes$ Hopf algebra is said *connected* if $\bigoplus_{n\geq 0}C(n,n)\overset{\eta}{\simeq}\mathbf{C}$.

Remark 4. The map S is called an *antipode*. In the definition of a $\boxtimes\otimes$ Hopf algebra we do not assume any compatibility conditions between Δ and ∇. These two morphisms are algebra morphisms with respect to the horizontal algebraic product structure we have on the underlying bi-graded collection, but nothing more. In particular, we can not require for Δ to be ∇ morphisms, this stems from the fact that $C\boxtimes C$ is *not* an $\boxtimes$-algebra, even if C is.

The map S do not enjoy the same properties as the antipodal map of a plain usual commutative Hopf algebra. In particular, it is not a morphism with respect to the product ∇, nor an anti-comorphism with respect to Δ nor an unipotent morphism ($S^2 = \mathrm{id}$). We should see later that in the case of the $\boxtimes\otimes$ Hopf algebra canonically associated with the gap-insertion operad, the square of the antipode is a projector.

Proposition 19. *A co-nilpotent bi-graded $\boxtimes\otimes$ bi-algebra, $(C, \bar{\Delta}, \varepsilon)$ endowed with a product ∇ such that (C, ∇, η) is an unital algebra is a $\boxtimes\otimes$ Hopf algebra.*

Proof. We set $DC = \bigoplus_{\substack{n,m\geq 0\\ n\neq m}} C(n,m)$ the diagonal part of the bigraded collection C. Let $n \geq 1$ an integer. We denote by $N(n)$ the vector space generated by all elements in C with a co-nilpotency order less that n. By definition of a co-nilpotent bi-algebra, $\bigcup_{n\geq 1} N(n) = C$. For each $m \geq 0$, set $S(\eta(1_m)) = 1_m$. If $c \in C$, we denote by $o(c)$ the number of outputs of c and $i(c)$ the number of inputs of c.

Pick an element c in $N(1)$ with nilpotency order equal to one then $\bar{\Delta}(c) = \Delta(c) + 1_{o(c)} \boxtimes c + c \boxtimes 1_{i(c)} = 1_{o(c)} \boxtimes c + c \boxtimes 1_{i(c)}$. As a consequence, $S(c) = -c$. We compute recursively S on the vector space $N(n)$. To that end, we will show that

$$\Delta(N(n)) \subset N(n-1) \boxtimes N(n-1),\ n \geq 1. \tag{8}$$

First, assume that this property holds. Then, if c is an element in $N(n)$ the formula

$$\nabla \circ (S \boxtimes \mathrm{id}) \circ \Delta(c) = S(c) + c + \nabla \circ (S \boxtimes \mathrm{id}) \circ \Delta = 0 \tag{9}$$

gives a way to compute compute S on c. To show (8), we simply write:

$$\Delta^n(p) = ((\Delta^{n-1} \boxtimes \mathrm{id}) \circ \Delta)(p) = \sum_k \Delta^{n-1}(c^k_{(1)}) \boxtimes c^k_{(2)} = 0 \tag{10}$$

For each k, $c^k_{(2)} \in C(k, i(c))$ and $\Delta^{n-1}(c^k_{(1)}) \in C^{n-1}(o(c), k) = \Delta^{n-1}(C(o(c), k))$. Besides, the spaces $C^{(n-1)}(o(c), k) \boxtimes C(k, i(c)),\ k \geq 1$ are in direct sum in $C^{\boxtimes n}$. In particular, (10) implies $\bar{\Delta}^{n-1}(c^k_{(1)}) \boxtimes c^k_{(2)} = 0$ for each $k \geq 0$, which in turn implies $\bar{\Delta}^{n-1}(c^k_{(1)}) = 0$ for each $k \geq 0$. The same argument works for the second component also. □

For $C \in (\mathrm{Coll}_2, \nabla)$ with $\nabla : C \boxtimes C \to C$ and in integer $m \geq 2$, we denote by $\nabla^{m-1} : C^{\boxtimes m} \to C$ the map defined recursively by:

$$\nabla^m = \nabla^{m-1} \circ (\nabla \boxtimes \mathrm{id}^{\boxtimes (m-2)})$$

Proposition 20. *Let $(C, \bar{\Delta}, \epsilon, \nabla, \eta, S)$ be a co-nilpotent Hopf algebra, with the convention $\nabla^0 \circ \Delta^0 = id$, we have:*

$$S = \sum_{n\geq 0} (-1)^{n+1} \nabla^n \circ \Delta^n + \eta \circ \epsilon. \tag{11}$$

Proof. Pick an off-diagonal element $x \in C$ (x is not in DC), using the notation introduced in the last proof), since $\epsilon(y) = 0$ for every off-diagonal elements, one has

$$\begin{aligned}
\nabla \circ (S \boxtimes \mathrm{id}) \circ \bar{\Delta}(x) &= \nabla \circ (S \boxtimes \mathrm{id}) \circ \Delta(x) + x + S(x) \\
&= \sum_{n\geq 0} (-1)^{n+1} \nabla^{n+1} \circ \Delta^{n+1}(x) + S(x) + \nabla \circ \eta \circ \epsilon \boxtimes \mathrm{id} \circ \Delta(x) + x \\
&= -\nabla^0 \circ \Delta^0(x) + x + \nabla \circ \eta \circ \epsilon \boxtimes \mathrm{id} \circ \Delta(x) = 0.
\end{aligned}$$

□

4.2. **Monoïd of Horizontal morphisms.** In this section we define a monoidal structure on the class of bi-graded collection morphism from a $\boxtimes\otimes$ bi-algebras to a ($\boxtimes$-al)($\otimes$-al)gebra. Then we prove that this monoidal structure restricts to a group structure on the class of horizontal algebra morphisms provided that the source bi-algebra is an Hopf algebra. So first, pick a $\boxtimes\otimes$-bi algebra $(B, \Delta, \epsilon, m_{\otimes}^{B})$ and an unital ($\boxtimes$-al)($\otimes$-al)gebra $(A, \nabla, m_{\otimes}^{A}, \eta)$. Recall that it means that $\nabla : B \boxtimes B, \to B$ is an horizontal morphism and $\eta : \mathbb{C}_{\boxtimes} \to A$.

Definition 21 (Convolution product)**.** Let α, β be two morphisms of bi-graded collection from B to the bi-collection A. We define the *convolution product* $\alpha \star \beta : B \to A$ as the bi-collection map:

$$\alpha \star \beta = \nabla \circ (\alpha \boxtimes \beta) \circ \Delta.$$

Proposition 22. *With the notation introduced so far, the product $\star$ defines a monoidal structure on $Hom_{Coll_2}(B, A)$. The unit element for the convolution product $\star$ is the morphism $\eta\circ\varepsilon$. $(Hom_{Alg_{\otimes}}(B, A), \star)$ is also a monoid. Besides, if B is an $\boxtimes\otimes$ Hopf algebra, then a ($\boxtimes$-al)($\otimes$-al)gebra morphism α is invertible in $(Hom_{Alg_{\otimes}}(B, A), \star)$, $\alpha^{-1} = \alpha \circ S$.*

Proof. The fact that $\mathrm{Hom}_{\mathrm{Coll}_2}(B, A)$ is a monoid follows from coassociativity and associativity property of δ and ∇. Let α and β be two morphisms. Since $(\mathrm{Alg}_{\otimes}, \boxtimes, \mathbb{C}_{\boxtimes})$ is a monoid implies that $\alpha \boxtimes \beta$ is a $\otimes$- algebra morphism. Then, $\nabla \circ (\alpha \boxtimes \beta) \circ \Delta$ is a $\otimes$-algebra morphism, as being a composition of such morphisms.

$$\nabla \circ ((\alpha \circ S) \boxtimes \alpha) \circ \Delta = \nabla \circ (\alpha \boxtimes \alpha) \circ (S \boxtimes \mathrm{id}) \circ \Delta = \alpha \circ \nabla \circ (S \boxtimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon.$$

□

In the sequel, we take for B a $\boxtimes \otimes$ Hopf algebra of words on non-crossing partition and A will the vertical algebra in $Alg_{\otimes}$ associated with the endomorphism operad of an unital algebra B.

If H is a $\boxtimes\otimes$ Hopf algebra, we denote by $\mathrm{Hom}_{Alg_{\otimes}}(H, \cdot)$ the set of convolution-invertible elements. The next equations sums up the class of homomorphisms at stakes:

$$\mathrm{Hom}_{\text{Bi-graded}}(H, \cdot) \supset \mathrm{Hom}_{Alg_{\otimes}}(H, \cdot) \supset \mathrm{Hom}^{\times}_{Alg_{\otimes}}(H, \cdot) \begin{array}{l} \supset \mathrm{Hom}_{op}(H, \cdot) \\ \supset \mathrm{Hom}_{op}(H, \cdot) \circ S \end{array} \tag{12}$$

4.3. **Shuffles and Unshuffles $\boxtimes\otimes$-bialgebras.** We recall the basic definition of a shuffle algebra and an unshuffle algebra. Afterwards, we will see how to adapt the definition of an un-shuffle co-algbras to our setting that involves bi-graded collections and a duoidal structure.

4.3.1. *Background on shuffles and unshuffles.* We recall various classical results and definitions related to shuffle algebras. The terminology shuffle refers actually to different kind of objects. In the literature, the first meaning to shuffle arises from products of iterated integrals. As such it designates a commutative binary product. The second meaning refers to topological shuffles, the latter being non-commutative. These notions can be traced back at least to the 1950's, when these two notions were axiomatized in the work of Eilerberg–Maclane and Schützenberger. In this section, shuffle will always refer to the non-commutative case.

A *shuffle* algebra or *dendrimorphic* algebra is a $\mathbb{K}$ vector space D together with two bilinear compositions $\prec$ and $\succ$ subject to the following three axioms

$$\begin{aligned}(a \prec b) \prec c &= a \prec (b \prec c + b \succ c), \\ (a \succ b) \prec c &= a \succ (b \prec c), \\ a \succ (b \succ c) &= (a \succ b) \succ c.\end{aligned}$$

These three relations yield the following associative shuffle algebra product $a \sqcup b = a \prec b + a \succ b$ on D. The products $\prec$ and $\succ$ are called, respectively, *left shuffle* and *right shuffle*. The standard example of a commutative shuffle algebra (meaning that $a \sqcup\!\sqcup b = b \sqcup\!\sqcup a$) is provided by the tensor algebra $\bar{T}(V)$ over a K vector space V endowed with a left half-shuffle recursively defined by

$$(x_1 \otimes \cdots x_n) \prec (y_1 \otimes \cdots y_n) = x_1 \otimes (x_2 \otimes \cdots \otimes x_n \sqcup\!\sqcup y_1 \otimes \cdots \otimes y_m).$$

Shuffle algebras are not naturally unital. This is because it is impossible to split the unit equation $1 \sqcup\!\sqcup a = a \sqcup\!\sqcup 1 = a$, into two equations involving the half-shuffles products $\succ$ and $\prec$. This issue is circumvented by using the "Schützenberger" trick, that is, for D a shuffle algebra, $\bar{D} = D \oplus \mathbb{K}1$. denotes the shuffle algebra augemented by a unit $\mathbf{1}$ such that

$$a \prec \mathbf{1} = a = \mathbf{1} \succ a,\ \mathbf{1} \prec a = 0 = a \succ \mathbf{1}.$$

implying $1 \sqcup\!\sqcup a = a \sqcup\!\sqcup 1 = a$. By convention, $\mathbf{1} \sqcup\!\sqcup \mathbf{1} = \mathbf{1}$, but $\mathbf{1} \prec \mathbf{1}$ and $\mathbf{1} \succ \mathbf{1} = 0$ cannot be defined consistently. The following set of left and right half-shuffle words in $\bar{D}$ are defined recursively for fixed elements $(x_1, \ldots, x_n) \in D$, $n \in \mathbb{N}$

$$\begin{aligned}
w_{\prec}^{(0)}(x_1, \ldots, x_n) &= 1 = w^{(0)}(x_1, \ldots, x_n)\\
w_{\prec}^{(n)}(x_1, \ldots, x_n) &= x_1 \prec w^{(n-1)}(x_2, \ldots, x_n)\\
w_{\succ}^{(n)}(x_1, \ldots, x_n) &= w^{(n-1)}(x_1, \ldots, x_{n-1}) \succ x_n.
\end{aligned}$$

In the case $x_1 = \ldots = x_n = x$, we simply write $x^{\prec n} = w_{\prec}^{(n)}(x, \ldots, x)$ and $x^{\succ n} = w_{\succ}^{(n)}(x, \ldots, x)$. In the unital algebra $\bar{D}$, both the exponential and logarithm maps are defined in terms of the associative product $\sqcup\!\sqcup$:

$$\exp_{\sqcup\!\sqcup}(x) = 1 + \sum_{n \geq 1} \frac{x^{\sqcup\!\sqcup n}}{n!},\ \log(1+x) = -\sum_{n \geq 1} (-1)^n \frac{x^{\sqcup\!\sqcup n}}{n!}.$$

In general, the two sums that appears in the last equation are infinite. However, in many cases of interest, we are able to identify a subset of elements of D for which these two sums are finite sums. The half-shuffle exponentials also called "time-ordered" exponentials are defined by mean of the two shuffles $\prec$ and $\succ$:

$$\exp_{\prec}(x) = \mathbf{1} + \sum_{n \geq 1} x^{\prec n},\ \exp_{\succ}(x) = \mathbf{1} + \sum_{n \geq 1} x^{n \succ}.$$

Notice that the two *half-shuffle* exponentials are solutions of the following fixed point equations:

$$X = \mathbf{1} + x \prec X,\ X = \mathbf{1} + X \succ x.$$

These two time ordered exponentials and the shuffle exponential map are the key ingredients to the Hopf-algebraic approach of moment cumulants relation in non-commutative probability theory.

Lemma 23. *Let A be a shuffle algebra, and $\bar{A}$ its augmentation by a unit 1. For $x \in A$, we have*

$$\exp_{\succ}(-x) \sqcup\!\sqcup \exp_{\prec}(x) = \mathbf{1}.$$

The notion dual to the shuffle coproduct is contained in the following definition. The axiomatic definition of the unshuffle coproduct appears well after the notion of shuffle algebra in the litterature. It has first been considered by L. Foissy, in its seminal work on the Duchamp-Hivert-Thibon "free Lie algebra" conjecture.

Definition 24. A co-unital unshuffle co-algebra is a coaugmented coassociative coalgebra $\bar{C} = C \oplus \mathbb{K}$ with coproduct

$$\bar{\Delta}(c) = \Delta(c) + c \otimes 1 + 1 \otimes c,\ c \in C,$$

such that, on C, the reduced coproduct $\Delta = \Delta_{\prec} + \Delta_{\succ}$ with

$$\begin{aligned}
&(\Delta_{\prec} \otimes I) \circ \Delta_{\prec} = (I \otimes \Delta) \circ \Delta_{\succ}, \quad (\Delta \otimes I) \circ \Delta_{\succ} = (I \otimes \Delta_{\succ}) \circ \Delta_{\succ},\\
&(\Delta_{\succ} \otimes I) \circ \Delta_{\prec} = (I \otimes \Delta_{\prec}) \circ \Delta_{\succ}.
\end{aligned}$$

The maps $\Delta_{\succ}$ and $\Delta_{\prec}$ are called respectively left and right half-unshuffles.

Definition 25. An unshuffle (or codendrimorphic) bialgebra is a unital and counital bialgebra $\bar{B} = B \oplus \mathbb{K}1$ with product $\cdot$ and coproduct Δ. At the same time $\bar{B}$ is a counital coalgebra with $\Delta = \Delta_{\prec} + \Delta_{\succ}$. The following compatibility relations holds:

$$\Delta^{+}_{\prec}(a \cdot b) = \Delta^{+}_{\prec}(a) \cdot \Delta(b),\ \Delta^{+}_{\succ}(a \cdot b) = \Delta^{+}_{\succ}(a) \cdot \Delta(b), a, b \in A.$$

where $\Delta^{+}_{\prec}(a) = \Delta^{+}_{\prec}(a) + a \otimes 1, \Delta^{+}_{\succ}(a) = \Delta^{+}_{\succ}(a) + 1 \otimes a$.

4.3.2. *Shuffling and unshuffling in* $(Coll_2, \otimes, \boxtimes)$. In the following definition, we denote by 1_m the unit element of $\mathbb{C}$ of degree (m, m) in the bi-graded collection $\mathbb{C}$, $m \geq 1$.

Definition 26. An unshuffle co-algebra in Coll_2 is a coaugmented coassociative coalgebra $(\bar{\boldsymbol{C}} = \boldsymbol{C} \oplus \mathbb{C}, \Delta)$ in the monoidal category $(\mathrm{Coll}_2, \boxtimes)$ with coproduct

$$\Delta : \bar{\boldsymbol{C}} \to \bar{\boldsymbol{C}} \boxtimes \bar{\boldsymbol{C}},\ \Delta \in \mathrm{Hom}_{\mathrm{Coll}_2}(\bar{\boldsymbol{C}}, \bar{\boldsymbol{C}} \boxtimes \bar{\boldsymbol{C}})$$

such that for any $c \in \boldsymbol{C}$ $\Delta(c) = \bar{\Delta}(c) + c \boxtimes 1_m + 1_n \boxtimes c$ and the reduced coproduct $\Delta = \Delta_{\prec} + \Delta_{\succ}$ satisfies the three following equation

$$(\Delta_{\prec} \boxtimes I) \circ \Delta_{\prec} = (I \boxtimes \bar{\Delta}) \circ \Delta_{\succ},\ (\bar{\Delta} \boxtimes I) \circ \Delta_{\succ} = (I \boxtimes \Delta_{\succ}) \circ \Delta_{\succ} \tag{13}$$

$$(\Delta_{\succ} \boxtimes I) \circ \Delta_{\prec} = (I \boxtimes \Delta_{\prec}) \circ \Delta_{\succ}. \tag{14}$$

Remark 5. In the last definition, because Δ is a morphism of bigraded collection, $\Delta : \bar{\boldsymbol{C}}(m, n) \to \bigoplus_{k \geq 1} \bar{\boldsymbol{C}}(m, k) \otimes \bar{\boldsymbol{C}}(k, n)$. By the very definition of Δ, the diagonal part of $\bar{C}$ (the sum $\bigoplus_{n \geq 1} \bar{\boldsymbol{C}}(n, n)$) is a vector space of group like elements for $\bar{\Delta}$.

In the following definition, given a bigraded collection map $\rho : \bar{\boldsymbol{C}} \otimes_H \bar{\boldsymbol{C}} \to \bar{\boldsymbol{C}}$, we denote by $\rho^{\boxtimes}$ the map $(\rho \boxtimes \rho) \circ R$.

Definition 27. *An unshuffle (⊠-co)(⊗-al)gebra* is a coaugmented (⊠-co)(⊗-al)gebra $(\bar{\boldsymbol{C}} = \boldsymbol{C} \oplus \mathbb{C}_{\boxtimes}, \Delta, \rho)$ with

$$\Delta(c) = \bar{\Delta}(c) + c \boxtimes 1_m + 1_n \boxtimes c, c \in C,$$

and $\bar{\Delta} = \Delta_{\prec} + \Delta_{\succ}$ is an unshuffle coproduct (see Definition 26), satisfying such that the following compatibility conditions hold:

$$\Delta \circ \rho = \rho^{\boxtimes} \circ (\Delta \otimes \Delta) \tag{15}$$

$$(\Delta^{+}_{\prec} \circ \rho)(p \otimes q) = \rho^{\boxtimes} \circ (\Delta^{+}_{\prec} \otimes \Delta)(p \otimes q),\ (\Delta^{+}_{\succ} \circ \rho)(p \otimes q) = \rho^{\boxtimes} \circ (\Delta^{+}_{\succ} \boxtimes \Delta)(p \otimes q) \tag{16}$$

$$p \notin \mathbb{C}_{\boxtimes},\ q \in C,$$

$$\Delta^{+}_{\prec}(\rho(1_m \otimes q)) = \rho^{\boxtimes}((1_m \boxtimes 1_m) \otimes \Delta^{+}_{\prec}(q)), \Delta^{+}_{\succ}(\rho(1_m \otimes q)) = \rho^{\boxtimes}((1_m \boxtimes 1_m) \otimes \Delta^{+}_{\prec}(q)) \tag{17}$$

with $\Delta^{+}_{\prec}(c) = \Delta_{\prec}(c) + c \boxtimes 1_n$, $\Delta^{+}_{\prec}(c) = \Delta_{\prec}(c) + 1_m \boxtimes c$, $c \in C(m, n)$.

Remark 6. Equation (17) have no counterpart in comparison with the standard notion of unshuffle bialgebra (see Definition 25). It essentially tells that the half-unshuffle co-products are bimodule maps for the canonical bi-module structure on the unit $\mathbb{C}_{\boxtimes}$ induced by the horizontal algebraic structure. In the settings of Definition 25, two units are equal, the one associated with the monoidal structure used to define the algebraic structure and the one used for co-augmentation. This is not the case here, $\mathbb{C}_{\boxtimes} \neq \mathbb{C}_{\otimes}$.

Definition 28. *An unshuffle (⊠-co)(⊗-al)gebra Hopf algebra* is an unshuffle (⊠-co)(⊗-al)gebra and a (⊠-co)(⊗-al)gebra Hopf algebra.

Remark 7. In particular, a unshuffle bi-algebra is a co-algebra in the category $(\mathrm{Alg}_{\otimes}, \boxtimes, \mathbb{C})$.

4.4. **The $\boxtimes\otimes$ unshuffle bialgebra of the gap-insertion operad of non-crossing partitions.**

Definition 29 (Horizontal co-algebra structure on $\mathcal{NC}$)**.** We define two horizontal morphisms $\varepsilon : T_{\otimes}(\mathcal{NC}) \to \mathbb{C}$, $\Delta : T_{\otimes}(\mathcal{NC}) \to T_{\otimes}(\mathcal{NC}) \boxtimes T_{\otimes}(\mathcal{NC})$ by

$$\Delta(\pi) = \sum_{\substack{p,q_1,\dots,q_{|p|}\in NC \\ p\circ(q_1\otimes\cdots\otimes q_{|p|})=\pi}} p \boxtimes (q_1 \otimes \cdots \otimes q_{|p|}),\ \pi \in NC,$$
$$\varepsilon(\emptyset^{\otimes p}) = 1 \in \mathbb{C}(p,p).$$

Let π be a partition of a linearly ordered set X. The set of blocks of π carries a preorder defined by declaring $V_1 \to_{\pi} V_2$ to mean that $\mathrm{Conv}(\pi) \cap \rho \neq \emptyset$. In plain words, $V_1 \to_{\pi} V_2$ means that either V_2 is nested in V_1 or that the two blocks cross. In the latter case we also $V_1 \to_{\pi} V_2$, which shows that in general the relation $\to_{\pi}$ is not antisymmetric.

Definition 30 (Upperset and lowerset)**.** An *upperset* of a partition π is a subset U of the set of blocks of π such that if $\pi \in U$ and $V \to_{\pi} W$ in π then also $W \in U$. In plain words, if a block is in π then all nested blocks and all blocks crossing it are in ρ. Similary, a *lowerset* of π is a subset of the set of blocks such that if $\pi \in U$ and $V \to_{\pi} W$ in π then also $W \in U$. In plain words, if a block is in L then all englobing blocks and all blocks crossing it are also in L.

We summarize in the next proposition all results about upperset and lowerset that can otherwise be found with their proofs in [6].

Proposition 31. *Let π be a partition.*

- *If π is non crossing, then its lowersets and uppersets are noncrossing,*
- *If U is an upperset of π then the complement set of blocks U^c is a lowerset. If L is a lowerset of π then the complement set of blocks L^c is an upperset.*

Definition 32 (Cut)**.** A cut of a partition π is a splitting of the set of blocks into a lowerset L and an upperset U. We write $(L,U) \in cut(\pi)$.

Proposition 33. *Let π be a partition.*

- *If V and W are two crossing blocks of π and $(L,U) \in cut(\pi)$ then either $V,W \in U$ pr $V,W \in L$,*
- *If π' have the same non-crossing closure as π then $cut(\pi) \simeq cut(\pi')$,*
- *A lowerset L of π defines cannonically a list of uppersets whose union is L^c.*

Definition 34 (Gap monomial)**.** For U an upperset, we define the *gap monomial* $U^{\otimes}$ to be the monomial in $T_{\otimes}(\mathcal{NC})$ given by

$$U^{\otimes} = U_0 \otimes \cdots \otimes U_k,$$

with $U_0,\dots,U_k$ is the list of upperset defined by the lowerset U^c as in proposition 33.

Proposition 35. *Let π be a non-empty partition, then*

$$\Delta(\pi) = \sum_{(L,U)\in cut(\pi)} L \boxtimes U^{\otimes}, \tag{18}$$

where in the right hand side of the $\boxtimes$ factor we have the gap monomial, see definition 34.

Define $D \subset T_{\otimes}(\mathrm{SP})$ as being the vector space spanned by words on the partition of the emptyset, $D = \mathbb{C}\langle\{\emptyset\}^p,\ p \geq 1\rangle$. By definition we have $\mathbb{C}_{\boxtimes} \simeq D$. Then $T_{\otimes}(\mathrm{SP})$ splits as the sum $T_{\otimes}(\mathrm{SP}) = \bar{T}_{\otimes}(\mathrm{SP}) \oplus D$. A similar splitting holds for Δ:

$$\Delta(w) = \bar{\Delta}(w) + \{\emptyset\}^m \boxtimes w + w \boxtimes \{\emptyset\}^n,\ w \in T_{\otimes}(\mathrm{SP})(m,n).$$

In the following definition, we write $1 \in L$ if the block of π that contain 1 is in the lowerset L.

Definition 36 (Half unshuffles)**.** We define two bi-collection maps $\Delta^+_{\prec} : \bar{T}_{\otimes}(\mathrm{SP}) \to T_{\otimes}(\mathrm{SP})^{\boxtimes 2}$, $\Delta^+_{\prec} : \bar{T}_{\otimes}(\mathrm{SP}) \to T_{\otimes}(\mathrm{SP})^{\boxtimes 2}$. Let $\pi \in SP$ be a non-empty partition and set

$$\Delta^+_{\prec}(\pi) = \sum_{\substack{(L,U)\in cut(\pi)\\ 1\in L}} L \boxtimes U, \quad \Delta^+_{\succ}(\pi) = \sum_{\substack{(L,U)\in cut(\pi)\\ 1\notin L}} L \boxtimes U \tag{19}$$

We extend $\Delta^+_{\prec}$ and $\Delta^+_{\succ}$ by setting for a word $w \in \bar{T}_{\otimes}(SP)$ and a partition $p \in \mathrm{SP}$ and integer $q \geq 0$:

$$\Delta_{\prec}(\{\emptyset\}^q p_1 w) = (\{\emptyset\}^q \boxtimes \{\emptyset\}^q)\Delta_{\prec}(p)\Delta(w), \;\; \Delta_{\succ}(\{\emptyset\}^q p w) = (\{\emptyset\}^q \boxtimes \{\emptyset\}^q)\Delta_{\succ}(p)\Delta(w). \tag{20}$$

From the very definition of the two left/right half shuffles $\Delta^+_{\prec}$ and $\Delta^+_{\succ}$, its holds that $\Delta = \Delta^+_{\prec} + \Delta^+_{\succ}$.

Proposition 37. *$(T_{\otimes}(SP), \Delta, \Delta_{\prec}, \Delta_{\succ})$ is an unshuffle bi-algebra in $Coll_2$.*

Proof. We expose briefly, and for the sake of completeness, the arguments of exposed in [6], proposition 3.4.3. Let π be a partition. It is sufficient to check the relations (13) for a single partition, because of the equations (20). Even so, we need to define what mean by lowersets, uppersets and cuts for words on partitions. Let $w = p_1 \cdots p_s$ is a word on partitions with $p_i \in \mathcal{SP}(k_i)$ with $k_i \geq 0$. A lowerset of w is a disjoint union of lowersets of the partitions p_i. The notion of an upperset for w is defined similarly. The notion of cut for partitions is then downwardly transferred to words on partitions.

We say that (L, M, U) is a compatible pair of cuts of w if L is a lowerset of w, U is an upperset of w with complements $L^c = U \sqcup M$ and $U^c = L \sqcup M$ with the condition that $\tilde{L}, \tilde{M}, \tilde{U} \neq \{\emptyset\}$.

To the word w, we associate a partition $\tilde{w} \in \mathrm{SP}(k_1 + \cdots + k_s)$ defined by translating the non-empty blocks of the partition p_i:

$$\tilde{w} = \bigcup_{\substack{1\leq i\leq s\\ \pi_i \neq \{\emptyset\}}} \{k_i + b, b \in \pi_i\}$$

From the definition, we have a bijection between the set of lower/upper sets of π and the set of lower/upper set of L. With obvious notation, we write $1 \in M$ if M is an upper/lowerset of w if the blocks that contains 1 is in the lower/upper set $\tilde{M}$ of $\tilde{w}$. Then, it holds that

$$(\Delta_{\prec} \boxtimes \mathrm{id}) \circ \Delta_{\prec}(\pi) = \sum_{\substack{(U,M,L)\in cut_2(\pi)\\ 1\in L}} L \boxtimes M^{\otimes} \boxtimes U^{\otimes} = \mathrm{id} \boxtimes \Delta \circ \Delta_{\prec}(\pi),$$
$$(\Delta_{\succ} \boxtimes \mathrm{id}) \circ \Delta_{\prec}(\pi) = \sum_{\substack{(U,M,L)\in cut_2(\pi)\\ 1\in M}} L \boxtimes M^{\otimes} \boxtimes U^{\otimes} = \mathrm{id} \boxtimes \Delta_{\prec} \circ \Delta_{\succ}(\pi)$$
$$(\Delta \boxtimes 1) \circ \Delta_{\succ}(\pi) = \sum_{\substack{(U,M,L)\in cut_2(\pi)\\ 1\in U}} L \boxtimes M^{\otimes} \boxtimes U^{\otimes} = \mathrm{id} \boxtimes \Delta_{\succ} \circ \Delta_{\prec}(\pi)$$

□

This splitting of Δ induces two bi-linear (non associative) compositions maps on the vector space of bi-collection maps from $T_{\otimes}(\mathrm{SP})$ to $T_{\otimes}(\mathrm{SP})$:

$$f \prec g = (f \boxtimes g) \circ \Delta_{\prec}, \;\; f \succ g = (f \boxtimes g) \circ \Delta_{\succ}.$$

By definition, we have $f \prec T_{\otimes}(\eta_{\mathrm{Hom}(B)}) \circ \varepsilon = T_{\otimes}(\eta_{\mathrm{Hom}(B)}) \circ \varepsilon \succ f = g$ and $\eta_{\mathrm{Hom}(B)} \circ \varepsilon \prec f = f \succ T_{\otimes}(\eta_{\mathrm{Hom}(B)}) \circ \varepsilon = 0$.

Proposition 38. *The $\boxtimes\otimes$ bialgebra $(T_{\otimes}(SP), \Delta, \Delta_{\prec}, \Delta_{\succ})$ endowed with the vertical product $T_{\otimes}(\rho_{SP})$ is a Hopf unshuffle algebra.*

5. Operator valued non-commutative probability theory

We start a with a small (historical) account on free probability theory and its operator valued version. Free probability theory was created in 1985 by Dan Voiculescu to understand free factors of vonNeumann algebras. Originally developed in the vicinity of operator algebras theory, it appears soon after that *freeness* is the right algebraic framework to compute the asymptotic distribution of large random matrices. Creating a common notion encompassing (finite dimensional) distribution of random matrices and their asymptotic requires a step of abstraction on what is understood as a *probability space*. In a nutshell, such a space allows for computation of *means* and has a notion of *positivity*, so important in classical probability. The very first example of a probability space is the algebra of essentially bounded random variables endowed with the usual expectation. One can push further the process of abstracting classical concepts of probability theory, in particular independence of random variables. Boolean and monotone independence finds counterparts in the theory of non-commutative probability theory while the notion of classical independence covers strong assumptions on (non-commutative) random variables that are otherwise trivial in classical probability. The conditional expectation is a map acting on a space of random variables measurable with respect to a σ-field $\mathcal{F}_1$ valued in a smaller algebra of random variables measurable with respect to a sub σ-field $\mathcal{F}_2 \subset \mathcal{F}_1$. Conditional mean of a random variable with respect to the sigma field $\mathcal{F}_3$ is not scalar valued but algebra valued. Still, it enjoys the same positivity property as the mean does. Besides, it is linear with respect to left and right multiplication by random variable measureables with respect to the small sigma field. These properties in the settings of non-commutative probability translates as follows.

An operator valued probability space $\mathcal{A}$ is a bi-module involutive complex unital algebra $\mathcal{A}$ over an unital involutive algebra B, $\phi : \mathcal{A} \to B$ is a B-bimodule positive unital morphism. In symbols, with $a \in \mathcal{A},\ b_1, b_2 \in B$

1. $(b_1 \cdot a) \cdot b_2 = b_1 \cdot (a \cdot b_2)$, $(ba)^\star = a^\star b^\star$,
2. $\phi(b_1 a b_2) = b_1 \phi(a) b_2$, $\phi(aa^\star) \in BB^\star$

We define boolean, free and monotone conditional cumulants using Möbius inversion. As for the scalar case, conditional free, boolean and monotone independence is characterized by the nullity of mixed cumulants. The reader is directed to the monograph [22] for a detailed introduction on the combinatorial perspective on operator valued probability theory.

The objectives of this section are three folds, first we will see the extension of the mean map $E \to B$ to a map on (coloured) non-crossing partitions as defined by Speicher is most accurately defined as a morphisms of the gap insertion of non-crossing partition. Secondly, we will see that this morphism is the solution of the left half-shuffle fixed point equation. The same result holds for the extension of E on the set of interval partitions. At the end of the section, we compute the full shuffle exponential.

For simplicity, we pick a single random variable $a \in \mathcal{A}$. The B-valued distribution of the random variable a is the collection of elements in B:

$$E(b_0 a b_1 a \cdots a b_n),\ b_0, \ldots, b_n,\ n \geq 1. \tag{21}$$

Recall that $\mathbf{1}_n$ denotes the partition with only one block in $NC(n)$. Speicher's original recursive definition of $e_\pi : B\,[a]^{\otimes_B |\pi|} \to B,\ \pi \in NC$ is as follows:

$$\begin{aligned}
&e_{\mathbf{1}_n}(a_1 \otimes \cdots \otimes a_n) = E(a_1 \cdots a_{n+1}),\\
&e_\pi(a_1 \otimes \cdots \otimes a_n) = e_{\pi_1}(a_1 \otimes \cdots \otimes a_{k-1} \otimes E(a_k \otimes \cdots \otimes a_l) a_{l+1} \otimes \cdots \otimes a_n).
\end{aligned}$$

with $[\![k, l]\!]$ is an interval in π and π_1 is the restriction of π to $[\![1, n]\!] \backslash [\![k, l]\!]$. In this perspective, e_π is a map that takes as inputs random variables, which can pictured as sitting on the legs of the partition π. In our perspective, these random variables are considered as fixed parameters or as defining a colourization of the partition. We explain in the forthcoming section how to construct for each partition π a map E_π from $B^{\otimes |\pi|}$ to B that if considered alltogether comprise the distribution of a.

5.1. **Gap insertion morphisms and left half-shuffle.** In the settings exposed in the introduction we define a graded sequence $(E^a_n)_{n\geq 1} \in (\mathrm{Hom}(B))_{n\geq 1}$ as follows:

$$E_n(b_0 \otimes \cdots \otimes b_n) = E(b_0 a b_1 a \cdots a b_n), b_0, \ldots, b_n \in B. \tag{22}$$

The sequence $(E_n)_{n\geq 1}$ satisfies the following equation:

$$E_1(b_0) = b_0 \tag{23}$$

$$E_n \circ_{n+1} E_m = E_m \circ_1 E_n. \tag{24}$$

In fact, we have, for $b_1, \ldots, b_{n+m} \in B$

$$\begin{aligned}
(E_n \circ_{n+1} E_m)(b_0, \ldots, b_{n+m}) &= E(b_0 a b_1 \cdots a E_m(b_n, \ldots, b_{n+m})) \\
&= E(b_0 a b_1 \cdots a) E(b_n a \cdots b_{n+m}) \\
&= E(E(b_0 a b_1 \cdots a) b_n a \cdots b_{n+m}) \\
(E_m \circ_1 E_n)(b_0, \ldots, b_{n+m}) &= E(E(b_0 a b_1 \cdots a b_n) a \cdots b_{n+m})
\end{aligned}$$

Lemma 39. *Given a sequence $(\phi_n)_{n\geq 1}$ satisfying equation (23), there exists an unique operadic morphism $\Phi : \mathcal{NC} \to Hom(B)$ such that $\Phi(1_n) = \phi_n$.*

Proof. A presentation of the operad $\mathcal{NC}$ is given by the set of elements $(1_n)_{n\geq 1}$ together with the relation $1_n \circ_{n+1} 1_m = 1_m \circ_1 1_n$. □

We denote Φ the operadic morphism extending the values (22). This map is, essentially, the same as the one introduced in Defintion 2.1.1 in [22]. By using the morphism Φ and Möbius inversion, we can define the free cumulative functionals $(k_n)_{n\geq 1} \in \mathrm{Hom}_{+1}(B)$ as the unique graded sequence whose associated morphism K satisfies:

$$\Phi(\pi) = \sum_{\alpha \leq \pi} K(\alpha),\ K(\pi) = \sum_{\alpha \leq \pi} \mu(\alpha, \pi) \Phi(\alpha).$$

The goal of this section is to show that any operadic morphism on $\mathcal{NC}$ is the solution of a half-shuffle fixed point equation.

Definition 40 (An infinitesimal character)**.** An infinitesimal character $\underline{k} : T_\otimes(\mathcal{NC}) \to T_\otimes(\mathcal{NC})$ is a bi-collection morphism equal to zero everywhere except on elements of the form $\emptyset^q\, \pi\, \emptyset^p$, $p, q \geq 0$, with $\pi \in NC \backslash \{\emptyset\}$ on which it is equal to $\mathrm{id}_B^p \underline{k}(\pi) \mathrm{id}_B^q$

Given an infinitesimal character $\underline{k}$, we define the half exponential $\exp_\prec(\underline{k})$ by setting:

$$\exp_\prec(\underline{k})(\pi) = T_\otimes(\eta_{\mathrm{Hom}(B)}) \circ \varepsilon + \sum_{p\geq 1} \underline{k}^{\prec p},$$

with $k^{\prec p} = k \prec k^{\prec (p-1)}$ and $k^{\prec 1} = k$. It is easily seen that $\exp_\prec(\underline{k})$ is the unique solution of the following fixed point equation:

$$K = T_\otimes(\eta_{\mathrm{Hom}(B)}) \circ \varepsilon + \underline{k} \prec K. \tag{25}$$

We denote by $\phi_{|1} : \mathcal{NC} \to \mathrm{Hom}(B)$ the restriction of a bi-collection map $\pi : T_\otimes(\mathcal{NC}) \to T_\otimes(\mathrm{Hom}(B))$. Note that if ϕ is an horizontal morphism then $\phi = T_\otimes(\phi_{|1})$.

Proposition 41. *With the notation introduced so far, the collection map K with K the solution of the fixed point equation (25) is an horizontal morphisms. Besides the map $K_{|1}$ is an operadic morphism if and only if $k(\pi) = 0$ if $\sharp\pi > 1$ and $k(1_n) \circ_{n+1} k(1_m) = k(1_m) \circ_1 k(1_n)$.*

Proof. We show that the solution K of (25) is an horizontal morphism. We do it recursively. Let $\tilde{K}$ be the horizontal morphism extending the values of $K_{|1}$. The two maps K and $\tilde{K}$ agree on words on partitions with 0 non empty blocks, since in that case $K(\{\emptyset\}^q) = \tilde{K}(\{\emptyset\}^q) = T_\otimes(\otimes_{\mathrm{Hom}(b)}) \circ \varepsilon(\{\emptyset^q\})$. Assume next that K and $\tilde{K}$ agree on words of partitions with a total number of non-empty blocks at most equal to $N \geq 1$. Pick a word on partitions with $N+1$ blocks and write $w = \emptyset^p \pi \tilde{w}$, with $\pi \neq \{\emptyset\}$. Let V be the block of the partition associated with π that contains 1. Then by definition of an infinitesimal cumulant, we get

$$K(w) = (\underline{k} \prec K)(w) = \sum_{\substack{(L,U)\in Cut(\pi)\\ 1\in L}} T_{\otimes}(\rho_{\mathrm{Hom}(B)})\Big(\{\emptyset\}^p \underline{k}(L)\{\emptyset\}^{|\tilde{w}|} \boxtimes K(\{\emptyset\}^p U\tilde{w})\Big)$$

Since the number of non empty blocks of U and $U\tilde{w}$ is less than the number of empty blocks of w, we get

$$K(w) = \sum_{\substack{(L,U)\in Cut(\pi)\\ 1\in L}} \mathrm{id}_B\, T_{\otimes}(\rho_{\mathrm{Hom}(B)})\Big(\underline{k}(L) \boxtimes \tilde{K}(U)\Big)\, \tilde{K}(\tilde{w}) = \mathrm{id}_B(k \prec K)(\pi)\tilde{K}(\tilde{w}) = \tilde{K}(w).$$

Next, Assume $k(\pi) = 0$ if $\sharp\pi > 1$ and $k(1_n) \circ_{n+1} k(1_m) = k(1_m) \circ_1 k(1_n)$. Let $\phi : \mathcal{NC} \to \mathrm{Hom}(B)$ be the operadic morphism extending the values $k(1_n),\ n \geq 1$. If π is a partition with only one block, then

$$K(\pi) = (T_{\otimes}(\eta_{\mathrm{Hom}(B)}) \circ \varepsilon)(\pi) + (\underline{k} \prec K)(\pi) = 0 + \underline{k}(\pi) \circ (K(\emptyset^{|\pi|})) = \underline{k}(\pi) = \phi(\pi). \tag{26}$$

Assume that the result holds for words on partitions with at most N blocks, $K(\pi_1 \cdots \pi_p) = \phi(\pi_1 \cdots \pi_p)$ for every element $\pi_1 \cdots \pi_p \in T_{\otimes}(\mathcal{NC})$ with $\sharp\pi_1 + \cdots + \sharp\pi_p \leq N$. Let π be a partition with $N+1$ blocks. We denote by V the block of π that contains 1. With this notation, we have

$$\begin{aligned} K(\pi) &= (T_{\otimes}(\eta_{\mathrm{Hom}(B)}) \circ \varepsilon)(\pi) + \underline{k} \prec K(\pi) \\ &= \sum_{L\in \mathsf{LowerCut}(\pi)} T_{\otimes}(\rho_{\mathrm{Hom}(B)})(\underline{k}(L) \boxtimes K(U)) \\ &= T_{\otimes}(\rho_{\mathrm{Hom}(B)})(\underline{k}(1_{\sharp V}) \boxtimes \phi(U)) = \phi(\pi). \end{aligned}$$

The last equality follows by application of the recurive hypothesis since $\sharp U \leq N$. Now assume that the solution $K_{|1}$ is an operadic morphism. Let $\pi \neq \emptyset$ be a non-crossing partition.

$$\begin{aligned} K(\pi) &= K(1_{\sharp V}) \circ (K(\pi_0), \ldots, K(\pi_{|V|})) = (\underline{k} \prec K)(\pi) \\ &= \sum_{L\in \mathsf{LowerCut}(\pi)} T_{\otimes}(\rho_{\mathrm{Hom}(B)})(\underline{k}(L) \boxtimes K(U))(\pi) \\ &= \rho_{\mathrm{Hom}(B)}(k(1_{\sharp V}) \boxtimes (K(\pi_1) \otimes \cdots \otimes K(\pi_{|V|}))) + \sum_{L\neq\{V\}} T_{\otimes}(\rho_{\mathrm{Hom}(B)})(\underline{k}(L) \boxtimes K(U))(\pi) \end{aligned}$$

This last equality implies $\sum_{L\neq\{V\}} T_{\otimes}(\rho_{\mathrm{Hom}(B)})(\underline{k}(L) \boxtimes K(U))(\pi) = 0$. A simple recursive argument on the number of blocks ends the proof. $\square$

5.1.1. *Thinning and co-incidence Hopf algebra of the operad of non-crossing partition.* We are now even more careful on the monoidal structures at stakes. Hence, we use the heavier notation $\otimes_H$ for the horizontal tensor product in the category of bi-graded collections, and $\otimes$ for the tensor product of vector spaces. We denote by $\tau^{A,B}$ the isomorphism $\tau^{A,B} : A \otimes B \to B \otimes A,\ A, B \in \mathrm{Vect}_{\mathbb{C}}$. First, there exists a canonical natural transformation U between the two functors $\otimes_H : \mathrm{Coll}_2 \times \mathrm{Coll}_2 \to \mathrm{Vect}_{\mathbb{C}}$ and $\otimes : \mathrm{Coll}_2 \times \mathrm{Coll}_2 \to \mathrm{Vect}_{\mathbb{C}}$. The next proposition shows that such a natural transformation exists with $\boxtimes$ in place of $\otimes_H$.

Proposition 42. *Let A and B be two bi-graded collections. Define the thinning map $F^{A,B}$ by*

$$F^{A,B} : \begin{array}{ccc} A \boxtimes B & \to & A \otimes B \\ a \boxtimes b & \mapsto & a \otimes b \end{array}. \tag{27}$$

The map $F^{A,B}$ is a linear map (not a bi-graded map). Besides, if (A, m_A) and (B, m_B) are two horizontal algebras then $F^{A,B}$ is an algebra morphism from $(A, m_{A\otimes B})$ to $(A \otimes B, m_A \otimes m_B \circ (id_A \otimes \tau^{B,A} \otimes id_B)$.

Proof. Let A, B be two collections. The map $F^{A,B}$ is well defined as a linear map. Assume that A and B are horizontal algebras and let $a \boxtimes b, a_1 \boxtimes b_1$ then $F^{A,B}((a \boxtimes b) \cdot (a_1 \boxtimes b_1)) = F^{A,B}((a \cdot a_1) \boxtimes (b \cdot b_1)) = (a \cdot a_1) \boxtimes (b \cdot b_1) = (a \otimes b) \cdot (a_1 \cdot b_1)$. □

The maps $F^{A,B}$ defines two natural transformations. The first one is between the two functors $\boxtimes : \mathrm{Coll}_2 \times \mathrm{Coll}_2 \to \mathrm{Vect}_{\mathbb{C}}$ and $\otimes : \mathrm{Coll}_2 \times \mathrm{Coll}_2 \to \mathrm{Vect}_{\mathbb{C}}$. The second natural tranformation is between the two functors $\boxtimes : \mathrm{Alg}_{\otimes} \times \mathrm{Alg}_{\otimes} \to \mathrm{Alg}$ and $\otimes : \mathrm{Alg}_{\otimes} \times \mathrm{Alg}_{\otimes} \to \mathrm{Alg}$, where Alg denotes the category of algebras over $\mathbb{C}$. In the proof of proposition 42, we implicitely used commutativity of the following diagram in Fig. 10. The same diagram leads to a more abstract interpretation of the two natural transformations F and U. Consider the category whose objects are functors between the two categories Coll_2 and $\mathrm{Vect}_{\mathbb{C}}$. A morphism between two functors is a natural transformation. We this such category $\mathrm{Coll}_2 \to \mathrm{Vect}_{\mathbb{C}}$. A general construction allows for considering objects and morphisms of a category as objects of another categorie. Applying this construction to $\mathrm{Coll}_2 \to \mathrm{Vect}_{\mathbb{C}}$, the diagram in Fig. 10 is the condition for the pair (F, U) to define a morphism of this category between the two objects

$$(\boxtimes \circ \otimes_H \times \otimes_H, R) \text{ and } (\otimes \circ \otimes \times \otimes, S)$$

with S the natural transform that swap the second and third factors in $A \otimes B \otimes C \otimes D$.

$$\begin{tikzcd}
(A \boxtimes B) \otimes_H (A_1 \boxtimes B_1) \arrow[rr, "F^{A,B} \otimes F^{A_1,B_1}"] \arrow[d, "R^{A,B,A_1,B_1}"] & & A \otimes B \otimes A_1 \otimes B_1 \arrow[d, "\mathrm{id}_A \otimes \tau^{B,A_1} \otimes \mathrm{id}_B"] \\
(A \otimes_H A_1) \boxtimes (B \otimes_H B_1) \arrow[dr, "F^{A_1 \otimes_H B_1, B \otimes_H B_1}"] & & A \otimes A_1 \otimes B \otimes B_1 \\
& (A \otimes_H A_1) \otimes (B \otimes_H B_1) \arrow[ur, "U^{A,A_1} \otimes U^{B,B_1}"] &
\end{tikzcd}$$

FIGURE 10

What we call *thinning*, boldly speaking, is the data of the two natural transformations:

$$\otimes_H \overset{U}{\Longrightarrow} \otimes, \ \boxtimes \overset{F}{\Longrightarrow} \otimes \tag{Thinning}$$

Using *thinning*, we can relate the Hopf $\boxtimes\otimes$ unshuffle algebra we defined with the coincidence algebra associated with the operad of gap insertion on non-crossing partition introduced in [6]. Let $\left(H, \Delta, \tilde{\Delta}_{\prec}, \tilde{\Delta}_{\succ}\right)$ be the unshuffle bi-algebra defined in [6], Definition 3.4.1 and Proposition 3.4.3. Let $p : T_{\otimes}(\mathrm{NC}) \to H$ be the surjective horizontal algebra map defined by:

$$p : \emptyset \mapsto \mathbf{1}, \ \pi \mapsto \pi, \ \pi \in \mathrm{NC}$$

The following proposition is a downward consequence of the Proposition 35 and Definition 36.

Proposition 43. *The following commutation relations holds*

$$\begin{tikzcd}
T_{\otimes}(NC) \boxtimes T_{\otimes}(NC) \arrow[r, "F"] & T_{\otimes}(NC) \otimes T_{\otimes}(NC) \arrow[r, "p \otimes p"] & H \otimes H \\
T_{\otimes}(NC) \arrow[u, "{\Delta, \Delta_{\prec}, \Delta_{\succ}}"] \arrow[rr, "p"'] & & H \arrow[u, "{\tilde{\Delta}, \tilde{\Delta}_{\prec}, \tilde{\Delta}_{\succ}}"']
\end{tikzcd}$$

5.2. **Word insertions morphisms and right half-shuffle.**

5.2.1. *Right half-shuffle exponential.*

Definition 44 (Boolean morphism)**.** A boolean morphism $B : T_{\otimes}(\mathcal{NC}) \to T_{\otimes}(\mathrm{Hom}(B))$ is a bi-graded collection map such that

- $B(\pi) = 0,\ \ \pi \in \mathrm{NC}\backslash I$,
- $B \circ \Phi$ satisfies, for a word $a \in \mathcal{W}$ of length $p-1$ and words $b_1, \ldots, b_p \in \mathcal{W}$:

$$(B \circ \Phi)(\rho(a, (b_1, \ldots, b_p))) = (B \circ \Phi)(a) \circ_{1, w_1, w_1 + w_2, \ldots, w_1 + \cdots + w_{p-1}} ((B \circ \Phi)(b_1), \ldots, (B \circ \Phi)(b_p)).$$

Remark 8. To put it in words, a boolean morphism is a B morphism is equal to zero on any non-crossing partition whose blocks are nested. Otherwise stated, the adjacency forest of the blocks of π has at least one tree not equal to a root. The second condition in equation (44) simply states that if inserting a interval partition into a second one, *in between the blocks* of this partition, then we B on the *resulting interval partition* by mean of the operadic composition on $\mathrm{Hom}(B)$. We can not simply say that $B \circ \Phi$ is an operadic morphism,since a letter $e_n \in \mathcal{W}$ of arity 2 is send to $B(e_n)$ of arity n.

Proposition 45. *Let $(b_n)_{n \geq 1}$ be a graded sequence in $Hom_{+}(b)$, $b_n \in Hom(B^{\otimes(n+1)}, B),\ n \geq 1$ satisying the balance property (23). There exists an unique Boolean morphism B such that $(B \circ \Phi)(e_n) = b_n,\ n \in \mathbb{N}$.*

Proof. Recall that the word insertion operad $\mathcal{W}$ admits the following presentation

$$e_n,\ e_n \circ_1 e_m = e_m \circ_2 e_1. \tag{28}$$

Let $\hat{b}_n$ be the element b_n, but considered as an element with arity 2. Let $\mathcal{F}(\hat{b})$ be the free operad on the collection $\hat{b} = (\hat{b}_n)_{n \geq 1}$ and $\mathcal{F}(b)$ be the free operad on $b = (b_n)_{n \geq 1}$. The correspondence $\hat{b} \to b$ induces a linear injection $k : \mathcal{F}(\hat{b}) \to \mathcal{F}(b)$ defined recursively by

$$k(\hat{b}_n) = b_n,\ \ k([\tau_1, \tau_2]_{\hat{b}_n}) = b_n \circ_1 k(\tau_1) \circ_{n+1} k(\tau_2)$$

We denote by p the canonical operadic projection from $\mathcal{F}(b)$ to $End(B)$.

Let $\pi : \mathcal{F}(\hat{b}) \to \mathrm{End}(B)$ be the linear map defined by $\pi(\tau) = (p \circ k)(\tau))$. The collection $\pi(\mathcal{F}(\hat{b}))$ is endowed with a natural operadic structure, coming from the canonical one on $Hom(B)$, that turns π into an operadic map. Since b satisfies the balanced relation (23), p factorizes over the ideal I generated by the relation $\tau \circ_1 \alpha = \alpha \circ_{\alpha} \tau,\ \alpha, \tau \in b \subset \mathcal{F}(b)$. Then k factorizes over the ideal $k^{-1}(I)$, which is the ideal generated by $\tau \circ_1 \alpha = \alpha \circ_2 \tau,\ \alpha, \tau \in \hat{b} \subset \mathcal{F}(\hat{b})$. We get a linear map $\hat{k}$ and an operadic map $\hat{p}$. We obtain $B \circ \phi$ as the operadic morphism $\hat{\pi} : \mathcal{W} \simeq \mathcal{F}\backslash k^{-1}(I) \to \mathrm{Hom}(B)$. □

The goal of this section is to show that any Boolean morphism on $\mathcal{NC}$ is the solution of a (right) half-shuffle fixed point equation. Given an infinitesimal character $\underline{b} : T_{\otimes}(\mathrm{NC}) \to T_{\otimes}(End(B))$, we define the right half exponential $\exp_{\succ}(\underline{b})$ by:

$$\exp_{\succ}(\underline{b})(\pi) = T_{\otimes}(\eta_{\mathrm{Hom}(B)}) \circ \varepsilon + \sum_{p \geq 1} \underline{b}^{\succ p},$$

with $\underline{b}^{\succ p} = \underline{b} \succ \underline{b}^{\succ (p-1)}$ and $\underline{b}^{\succ 1} = b$. It is easily seen that $\exp_{\succ}(\underline{b})$ is the unique solution of the following fixed point equation:

$$B = T_{\otimes}(\eta_{\mathrm{Hom}(B)}) \circ \varepsilon + B \succ \underline{b}. \tag{29}$$

Proposition 46. *With the notation introduced so far, the bi-graded collections morphisms B solution of the fixed point equation (29) is a horizontal morphism. Besides, B is boolean if and only if $\underline{b}(1_n) \circ_{n+1} \underline{b}(1_m) = \underline{b}(1_m) \circ_1 \underline{b}(1_n)$ and $\underline{b}(\pi) = 0$ if $\sharp\pi > 1$.*

Proof. The proof is very similar to the free case. Let $\tilde{B}$ be the horizontal morphism extending the values of $\bar{b}$. The two maps B and $\tilde{B}$ agree on words on partitions with at most 1 non-empty blocks. Assume that B and $\tilde{B}$ agree on words on partitions with at most N non-empty blocks.

Pick a word on partitions with $N+1$ blocks and write $w = \emptyset^p \pi \tilde{w}$, with $\pi \neq \{\emptyset\}$. Let V be the block of the partition associated with π that contains 1. Then by definition of an infinitesimal cumulant, we get

$$B(w) = (\underline{k} \prec K)(w) = \sum_{\substack{(L,U)\in Cut(\pi)\\ 1\notin L}} T_{\otimes}(\rho_{\mathrm{Hom}(B)})\Big(B(\{\emptyset\}^p L\tilde{w}) \boxtimes \underline{b}(\{\emptyset\}^p U \{\emptyset\}^{|\tilde{w}|})\Big)$$

Since the number of non-empty blocks of $L\tilde{w}$ and L is less than the number of non-empty blocks in w, we get:

$$B(w) = \sum_{\substack{(L,U)\in Cut(\pi)\\ 1\notin L}} \mathrm{id}_B^p T_{\otimes}(\rho_{\mathrm{Hom}(B)})\Big(\tilde{B}(L) \boxtimes \underline{b}(U)\Big)\tilde{B}(\tilde{w}) = \mathrm{id}_B^p (B \succ b)(\pi)\tilde{B}(\tilde{w}) = \tilde{B}(w)$$

We assume that $\underline{b}(\pi) = 0$ if $\sharp\pi > 1$. Let ϕ be the boolean morphism that extends the values $\underline{b}(1_n)$, $n \geq 1$. We show recursively on the number of blocks of a multi-partition in $T_{\otimes}(\mathcal{NC})$ that $B = T_{\otimes}(\phi)$. First, the two maps coincide on multiparitions with at most one blocks. Let $N \geq 1$ and assume that $T_{\otimes}(\phi)$ and B are equal on multi-partition with at most N blocks. Pick π a partition with $N+1$ blocks. Assume first that the *adjacency forest* of π contains at least one tree not equal to the root.

$$B(\pi) = (B \succ \underline{b})(\pi) = \sum_{1\notin L \in \mathsf{LoCut}(\pi)} T_{\otimes}(\rho_{\mathrm{Hom}(B)})(B(L) \boxtimes \underline{b}(U)) \tag{30}$$

A cut of the partition π corresponds to an admissible cut of its adjacency tree. Since $\overline{b}(U) = 0$ if U is a word on partitions either containing at least two non-empty partition either equal to some $\emptyset^p, p \geq 1$, the cuts that contribute to the sum in the right hand side of (30) extract one and only one leaf of the adjacency forest. Hence, if the block V of π that contains 1 contains at least another block in its convex hull, $B(\pi) = 0$. Assume the opposite. It implies that the partition $\pi\backslash V$ is not an interval partition (and is not empty). Besides,

$$B(\pi) = T_{\otimes}(\rho_{\mathrm{Hom}(B)})(\underline{b}(V) \boxtimes B(\emptyset \otimes_H \pi\backslash V))$$

The induction hypothesis implies $B(\emptyset \otimes_H \pi\backslash V)) = 0$. Now suppose that $\pi = I_1 \cdots I_p$ is an interval partition.

$$B(\pi) = T_{\otimes}(\rho_{\mathrm{Hom}(B)})(\underline{b}(I_1) \boxtimes B(I_2 \cdots I_p)).$$

We apply the recursive hypothesis on $B(I_2 \cdots I_p)$ to end the proof. □

Proposition 47. *Let $\underline{k}$ be an infinitesimal morphism such that relation (23) holds. Then,*

$$\exp_{\prec}(\underline{k})^{-1} = \exp_{\succ}(-\underline{k}).$$

We should empasize that proposition 47 does not state a result about the existence of an inverse for $\exp_{\prec}(\underline{k})$, but gives a way on how to actually compute such an inverse. In particular, the proof of proposition 47 relies on the axiomatic definition of $\boxtimes\otimes$ unshuffle bialgebra, which can not be used to prove invertibility of the time ordered exponential.

Proof. We prove the result stated in Lemma 23 in our settings.

$$\begin{aligned}\exp_{\succ}(-\underline{k}) ⧢ \exp_{\prec}(\underline{k}) - T_{\otimes}(\eta_{\mathrm{Hom}B}) \circ \epsilon &= \sum_{n+m\geq 1} \underline{k}^{n\succ} \succ \underline{k}^{\prec m} + \sum_{n+m\geq 1} \underline{k}^{n\succ} \prec \underline{k}^{\prec m} \\ &= \sum_{n\geq 0, m\geq 1} (-1)^n \underline{k}^{n\succ} \succ \underline{k}^{\prec m} + \sum_{n\geq 1, m\geq 0} (-1)^n \underline{k}^{n\succ} \prec \underline{k}^{\prec m}\end{aligned}$$

Now since for $n \geq 1$, $\underline{k}^{\succ n} \prec \underline{k}^{\prec m} = (\underline{k}^{(n-1)\succ} \succ \underline{k}) \prec \underline{k}^{\prec m} = \underline{k}^{(n-1)\succ} \succ (\underline{k} \prec \underline{k}^{\prec m}) = \underline{k}^{(n-1)\succ} \succ (\underline{k}^{\prec m+1})$ we get the result. □

We would like to sum up what we have proved so far and give an overlook of the overall picture. We focus on two kinds of horizontal morphisms from $T_{\otimes}(\mathrm{NC})$ to $T_{\otimes}(\mathrm{End}B)$: operadic and boolean morphisms. We prove that operadic morphisms are invertible in the convolution algebra of horizontal morphisms, the inverse of such a morphism α is $\alpha \circ S$. In addition we proved that an operadic morphism can be written as a left half-shuffle exponential. The proposition 47 shows that the inverse can *also* be computed as right half-shuffle exponential, which is a boolean morphism. As a consequence of the same proposition, a boolean morphism is invertible with inverse an operadic morphismn. We insist on the fact that the inverse of a boolean morphism *can not* be computed by right composition with the antipode S.

These two last propositions admit the following corollary. Define the $\otimes$-algebra morphism $S : T_{\otimes}(\mathrm{NC}) \to T_{\otimes}(\mathrm{NC})$ by $S(\pi) = 0$ if π is not an interval partition and $S(\pi) = (-1)^{\mathsf{nc}(\pi)}\pi$ if π is an interval partition.

Corollary 3. $(T_{\otimes}(\mathcal{NC}), \Delta, T_{\otimes}(\rho_{\mathcal{NC}}), S)$ *is an unshuffle ($\boxtimes$-co)($\otimes$-al)gebra Hopf algebra.*

Let $T_{\otimes}(I)$ be the bi-graded collection of words on interval partitions. It easily seen that the coproduct Δ restricts to a vertical coproduct $\Delta : T_{\otimes}(I) \to T_{\otimes}(I) \boxtimes T_{\otimes}(I)$.

A the beginning of the section, we defined a map Φ that associate to a word an interval partition. To compare our construction with the double construction, we want to transfer the $\boxtimes\otimes$ Hopf algebra structure we have on $T_{\otimes}(I)$ to $T_{\otimes}(\mathcal{W})$.

To do so, we introduce an horizontal algebra morphism $\Phi^{\boxtimes} : T_{\otimes}(\mathcal{W}) \boxtimes T_{\otimes}(\mathcal{W}) \to T_{\otimes}(I) \boxtimes T_{\otimes}(I)$, whose action is described in Fig. 12

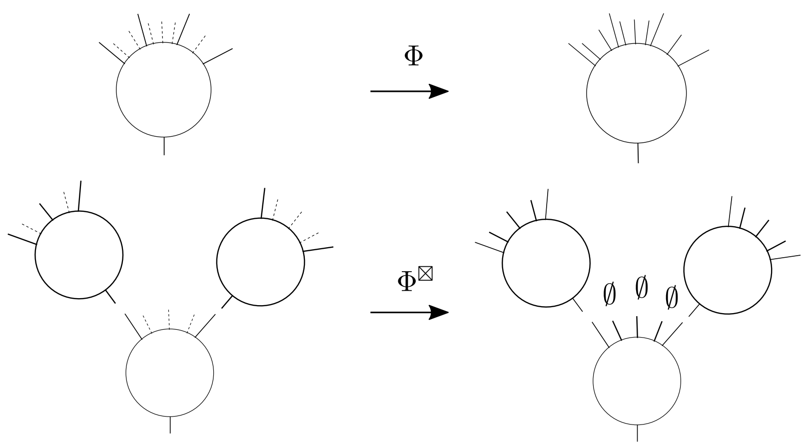

FIGURE 11. The maps ϕ and $\phi^{\boxtimes}$. On the first line, on the left, the word $e_3e_4e_2$ is drawn as an operator with 4 inputs. The dashed inputs stands for the extra inputs the intervals $I_3 = \Phi(e_3), \Phi(e_4) = I_4, \Phi(e_2) = I_2$ have in NC. On the bottom line, the action of $\Phi^{\boxtimes}$, on the left hand side, the element $e_4 \boxtimes (e_2e_2 \otimes e_4)$ is send to $I_4 \boxtimes (I_2I_2 \otimes \emptyset \otimes \emptyset \otimes \emptyset \otimes I_4)$.

Notice that both ϕ and $\Phi^{\boxtimes}$ are in fact injective horizontal algebras morphisms. Since $S(I) = I$, by pulling back all the structural morphisms of $T_{\otimes}(I)$ with the help of Φ and $\Phi^{\boxtimes}$, we get the following result.

Corollary 4. *There exists an unique unshuffle $\boxtimes\otimes$ Hopf algebraic structure $(T_{\otimes}(\mathcal{W}), \hat{\Delta}, \hat{\Delta}_{\prec}, \hat{\Delta}_{\succ})$ such that the following diagrams commute.*

Proof. The image of horizontal morphism $\Phi^{\boxtimes}$ contains the image of the co-product Δ. Since the map $\Phi^{\boxtimes}$ is injective there exists an unique horizontal morphism $\hat{\Delta} : T_{\otimes}(\mathcal{W}) \to T_{\otimes}(\mathcal{W}) \boxtimes T_{\otimes}(\mathcal{W})$ such that $\Phi^{\boxtimes} \circ \hat{\Delta} = \Delta \circ \Phi$. To show coassociativity for $\hat{\Delta}$, we introduce the map $\Phi^{2\otimes}$ which acts on the three fold vertical tensor product of $T_{\otimes}(\mathcal{W})$ and valued in the the three tensor product of $T_{\otimes}(I)$. The injective

$$\begin{tikzcd}
T_\otimes(\mathcal{W}) \arrow[r, "\Phi"] \arrow[d, "\hat{\Delta},\hat{\Delta}_{\prec},\hat{\Delta}_{\succ}"] & T_\otimes(I) \arrow[d, "\Delta,\Delta_{\prec},\Delta_{\succ}"] \\
T_\otimes(\mathcal{W}) \boxtimes T_\otimes(\mathcal{W}) \arrow[r, "\Phi^{\boxtimes}"] & T_\otimes(I) \boxtimes T_\otimes(I)
\end{tikzcd}
\qquad
\begin{tikzcd}
T_\otimes(\mathcal{W}) \boxtimes T_\otimes(\mathcal{W}) \arrow[r, "\Phi^{\boxtimes}"] \arrow[d, "T_\otimes(\rho)"'] & T_\otimes(I) \boxtimes T_\otimes(I) \arrow[d, "T_\otimes(\rho_{NC})"] \\
T_\otimes(\mathcal{W}) \arrow[r, "\Phi"] & T_\otimes(I)
\end{tikzcd}$$

$$\begin{tikzcd}
T_\otimes(\mathcal{W}) \arrow[r, "\Phi"] \arrow[d, "\hat{S}"] & T_\otimes(I) \arrow[d, "S"] \\
T_\otimes(\mathcal{W}) \arrow[r, "\Phi"] & T_\otimes(I)
\end{tikzcd}$$

map $\Phi^{2\boxtimes}$ has a similar action than the map $\Phi^{\boxtimes}$ (see fig. 12), it fills the extra inputs showing up if considering a letter e_n as the interval I_n with the empty partition. Since $\Delta(\emptyset) = \emptyset \boxtimes \emptyset$ we get

$$\begin{aligned}
\Phi^{2\boxtimes} \circ ((\hat{\Delta} \boxtimes \mathrm{id}) \circ \Delta) &= (\Delta \boxtimes \mathrm{id}) \circ \Delta \circ \Phi \\
&= (\mathrm{id} \boxtimes \Delta) \circ \Delta \circ \Phi = \Phi^{2\boxtimes} \circ (\mathrm{id} \boxtimes \hat{\Delta}) \circ \hat{\Delta}
\end{aligned}$$

□

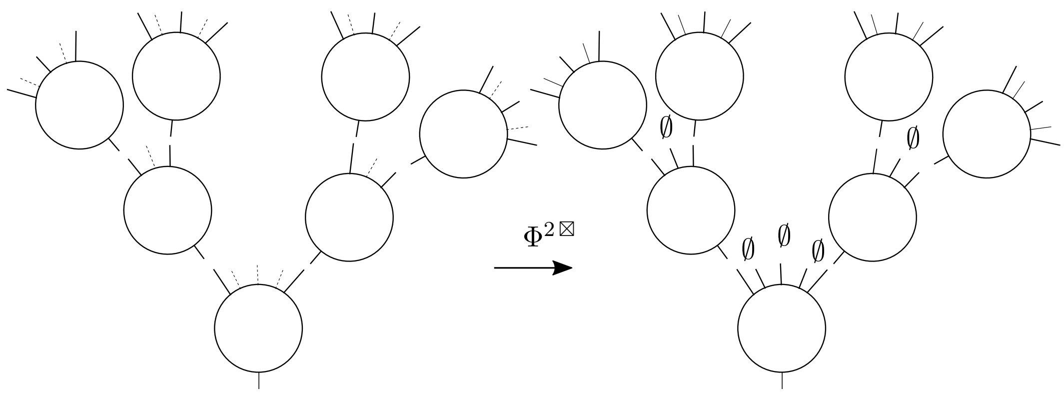

FIGURE 12. Action of the map $\Phi^{2\boxtimes}$ on the element $e_4 \boxtimes (e_2 \otimes_H e_2) \boxtimes (e_2e_2 \otimes e_2e_2 \otimes e_2e_2 \otimes e_2e_2)$ send to $I_4 \boxtimes (I_2 \otimes_H \emptyset \otimes_H \emptyset \otimes_H \emptyset \otimes_H I_2) \boxtimes (I_2I_2 \otimes_H \emptyset \otimes_H I_2I_2 \otimes_H \emptyset \otimes_H \emptyset \otimes_H \emptyset \otimes_H I_2I_2 \otimes_H \emptyset I_2I_2)$

5.2.2. *Thinning and double bar construction.* In this section, we relate the $\boxtimes\otimes$-unshuffle Hopf algebra of corollary 4 to a (standard) unital unshuffle Hopf algebra obtained by the double bar construction. To do so, we introduce a *thining map* F. First, set

$$\mathbb{K}^\infty = \bigoplus_{n\geq 1} \mathbb{K}.$$

We denote by $\left(\bar{T}T(\mathbb{R}^\infty), \Delta^\flat, \Delta^\flat_\prec, \Delta^\flat_\succ, S^\flat\right)$ the double bar construction on $\mathbb{R}^\infty$. Recall the formulas for the unshuffle coproducts $\Delta^\flat, \Delta^\flat_\prec, \Delta^\flat_\succ$ and the antipode $S^\flat$.

$$\Delta(a_1 \cdots a_n) = \sum_{S\subset[n]} a_S \otimes a_{J_1}|\cdots|a_{J_k},$$

$$\Delta^+_\prec(a_1 \cdots a_n) = \sum_{1\in S\subset[n]} a_S \otimes a_{J_1}|\cdots|a_{J_k}, \qquad \Delta^+_\succ(a_1 \cdots a_n) = \sum_{1\notin S\subset[n]} a_S \otimes a_{J_1}|\cdots|a_{J_k},$$

$$S(w) = (-1)^{\ell(w)-1} w.$$

It is convenient to introduce a non-unital version of the double construction. We denote by $T(T(\mathbb{R}^\infty))$ the algebra of polynomials on $T(\mathbb{R}^\infty)$. We obtain $\bar{T}(T(\mathbb{R}^\infty))$ as a quotient of $T(T(\mathbb{R}^\infty))$ by the relations

$\emptyset w = w\emptyset = w,\ w \in T(\mathbb{R}^\infty)$. Notice that, as horizontal algebras $T_\otimes(\mathcal{W}) \simeq T(T(\mathbb{R}^\infty))$. Hence, we define the algebra morphism $p : T_\otimes(\mathcal{W}) \to \bar{T}(T(\mathbb{R}^\infty))$ by

$$\begin{array}{rccc} p: & T_\otimes(\mathcal{W}) & \to & \bar{T}(T(\mathbb{R}^\infty)) \\ & e_{w_1}\cdots e_{w_p} & \mapsto & e_{w_1}\cdots e_{w_p} \\ & \emptyset & \mapsto & \mathbf{1} \end{array}.$$

Proposition 48. *The diagram in Fig. 13 is a commutative diagrams in the category of algebras and the diagram in Fig. 14 is a commutative diagram in the category of vector spaces.*

$$\begin{tikzcd} \bar{T}(T(\mathbb{R}^\infty)) \otimes \bar{T}(T(\mathbb{R}^\infty)) & T(T(\mathbb{R}^\infty)) \otimes T(T(\mathbb{R}^\infty)) \arrow[l, "p\otimes p\circ F^{T_\otimes(\mathcal{W}),T_\otimes(\mathcal{W})}"'] \\ \bar{T}(T(\mathbb{R}^\infty)) \arrow[u, "\Delta^\flat"] & T_\otimes(\mathcal{W}) \arrow[l, "p"] \arrow[u, "\Delta"'] \end{tikzcd}$$

FIGURE 13. Thinning of the coproduct Δ and the antipode.

$$\begin{tikzcd} \bar{T}(T(\mathbb{R}^\infty)) \otimes \bar{T}(T(\mathbb{R}^\infty)) & T(T(\mathbb{R}^\infty)) \otimes T(T(\mathbb{R}^\infty)) \arrow[l, "p\otimes p\circ F^{T_\otimes(\mathcal{W}),T_\otimes(\mathcal{W})}"'] \\ \bar{T}(T(\mathbb{R}^\infty)) \arrow[u, "\Delta_\prec^{+,\flat}{,}\Delta_\succ^{+,\flat}"] & T_\otimes(\mathcal{W}) \arrow[l, "p"] \arrow[u, "\Delta_\prec^{+}{,}\Delta_\succ^{+}"] \end{tikzcd}$$

FIGURE 14. Thinning of the half-shuffle coproducts $\Delta_\prec^+, \Delta_\succ^+$.

Proof. Pick $e_{w_1}\cdots e_{w_n} \in \mathcal{W}$. For $S \subset [\![n]\!]$, we set $e_S = e_{w_{s_1}}\cdots e_{w_{s_p}}$ and $e_\emptyset = \{\emptyset\}^n$. We denote by $\hat{J}_1^S, \ldots, \hat{J}_q^S$ the subsets of $[\![1,n]\!]$ defined as follows. Denote by $J_1^S, \ldots, J_k^S$ the (non-empty) connected components of $[\![1,n]\!]\backslash S$. There exists an unique elements $(e_{\hat{J}_1^S} \otimes_H \cdots \otimes_H e_{\hat{J}_{p+k}^S}) \in \mathcal{W}^{p+1}$ such that $\rho(e_S \otimes (e_{\hat{J}_1^S} \otimes_H \cdots \otimes_H e_{\hat{J}_{p+k}^S})) = e_{w_1}\cdots e_{w_n}$. A word $e_{\hat{J}_k^S}$ is either equal to a word $e_{J_l^S}$ either equal to a word on the empty letter $\emptyset$.

$$\Delta(e_{w_1}\cdots e_{w_n}) = \sum_{S\subset[n]} e_S \boxtimes (e_{\hat{J}_1^S} \otimes_H \cdots \otimes_H e_{\hat{J}_{p+k}^S}).$$

Notice that by the very definition of p, it holds that $p(e_{\hat{J}_1^S} \otimes_H \cdots \otimes_H e_{\hat{J}_k^S}) = e_{J_1}|\cdots|e_{J_k}$. The result follows. The statement abouts the antipodes S is straightforward to prove. To prove the assertion about the half-shuffles we first notice, owing to the relation (17), taht

$$\begin{aligned}(p\otimes p)\circ F^{T_\otimes(\mathcal{W}),T_\otimes(\mathcal{W})} \circ \Delta_\prec^+(\emptyset w w') &= (p\otimes p)\circ F^{T_\otimes(\mathcal{W}),T_\otimes(\mathcal{W})} \circ \Delta_\prec^+(w w') \\ &= (p\otimes p)\circ F^{T_\otimes(\mathcal{W}),T_\otimes(\mathcal{W})}(\Delta_\prec^+(w)\Delta(w'))\end{aligned}$$

Since both $p\otimes p$ and $F^{T_\otimes(\mathcal{W}),T_\otimes(\mathcal{W})}$ are algebra morphisms we get:

$$\begin{aligned}&(p\otimes p)\circ F^{T_\otimes(\mathcal{W}),T_\otimes(\mathcal{W})} \circ \Delta_\prec^+(\emptyset w w') = \\ &\qquad (p\otimes p)\circ F^{T_\otimes(\mathcal{W}),T_\otimes(\mathcal{W})} \circ \Delta_\prec^+(w) | (p\otimes p)\circ F^{T_\otimes(\mathcal{W}),T_\otimes(\mathcal{W})} \circ \Delta(w')\end{aligned}$$

Hence to show that $\Delta_\prec^{\flat,+}\circ p$ and $(p\otimes p)\circ F^{T_\otimes(\mathcal{W}),T_\otimes(\mathcal{W})}\circ \Delta_\prec^+$, we only need to show that the two maps are equal on $\mathcal{W}$. The computation are verbatim the ones we did for the maps $\Delta^\flat$ and Δ. $\square$

5.3. **Shuffle exponential.** In this section we compute the shuffle exponential (31). The bi-graded map on the operad of non-crossing partitions which corresponding to the restriction of this horizontal morphism to the space of operators with one output does not enjoy, to the extend of our knowledge, any compatibility properties with respect to the operadic structure. This boils down to the fact the tree factorial defines hereafter is not multiplicative with respect to an operation of grafting of trees.

Definition 49 (Monotone partition)**.** Let π a partition with k blocs. An admissible labeling of the blocs by integers in $[\![1,k]\!]$ is an injective labeling which is increasing with respect to the nesting preoder on the blocs: If a bloc $b \in \pi$ is contained in the convex hull of a block c in π then the label of b is less than the label of c. A partition with an admissible labeling of its blocks is called a monotone partition. The set of all monotone partitions is denoted $\mathcal{NC}_m$.

Definition 50 (Tree factorial, [1], Definition 3.2)**.** The tree factorial $t!$ of a rooted tree t is recursively defined as follows. Let t be a rooted tree with $n > 0$ vertices. It t consists of a single vertex, set $t! = 1$. Otherwise t can be decomposed into its root vertex and branches $t_1, \ldots, t_r$ and we defined recursively the number

$$t! = n \cdot t_1! \cdots t_k!$$

The tree factorial of a forest is the product of the factorials of the constituting trees.

Proposition 51 ([1], Proposition 3.3)**.** *The number $m(\pi)$ of monotone labelings of a non-crossing partition depends only its adjacency forest $\tau(\pi)!$ and is given by $m(\pi) = \frac{|\pi|!}{\tau(\pi)!}$*

Let $\overline{m} : T_\otimes(\mathrm{NC}) \to T_\otimes(\mathrm{Hom}(B))$ be an infinitesimal cumulants and define the shuffle exponential by

$$\exp_\star(\underline{m}) = T_\otimes(\eta_{\mathrm{Hom}(B)}) \circ \varepsilon + \sum_{p\geq 1} \underline{m}^{\star p}. \tag{31}$$

Proposition 52. *Let $(m_n)_{n\geq 1}$ be a graded sequence in $Hom_{+1}(B)$ satisfying condition (23). Let $\underline{m} : T_\otimes(\mathcal{NC}) \to T_\otimes(\mathcal{NC})$ be the infinitesimal morphism associated with $(m_n)_{n\geq 1}$. Then,* $\exp_\star$ *is an horizontal morphism and*

$$\exp_\star(\underline{m})(\pi) = \frac{1}{\tau(\pi)!}\exp_\prec(\underline{m})(\pi),\ \pi \in \mathcal{NC}.$$

Proof. Let π be a non-crossing partition with k blocks. The number of admissible labeling of the partition π is equal to $\frac{k!}{\tau(\pi)!}$. Hence, to prove the statement, it is sufficient to show that

$$\exp_\star(\underline{m})(\pi) = \frac{1}{k!}\sum_{\pi\in\mathcal{NC}_m} \exp_\prec(\underline{m})(\pi).$$

To that end, we show first that there exists a natural embedding of the set of admissible labelings of a partition into the set of multiple admissible cuts of a partition. A multiple cuts of a partition π is a sequence $(L_1, \ldots, L_s)$ of (possibly empty) subsets of blocks of π such that L_i is a lower cut of L_{i-1} with the convention $L_0 = \pi$. For such a multiple cut of π, we denote by $L_i \backslash L_{i-1}$ the words on partition in $T_\otimes(\mathcal{NC})$ such that

$$\rho_{NC}(L_i \boxtimes (L_{i-1}\backslash L_i)) = L_{i-1}.$$

Let (π, ℓ) a monotone partition. We associate to the labelling ℓ of the block a multiple cut $\mathsf{L}(\pi, \ell)$ of π as follows. For each integer $i \in [\![1,k]\!]$, we denote by V_i the block of π labeled with the integer i. We define recursively $\mathsf{L}(\pi, \ell)$ by the following rule:

$$\mathsf{L}(\pi,\ell)_0 = \pi,\ \mathsf{L}(\pi,\ell)_i = \mathsf{L}(\pi,\ell)_{i-1}\backslash V_i.$$

Because the labelling ℓ is monotone, we obtain indeed a multiple cut of π. Next, from the definition of the coproduct Δ, we see that:

$$\exp_\star(\underline{m})(\pi) = \sum_{s\geq 1}\sum_{(L_1,\ldots,L_s)} \frac{1}{s!} T_\otimes(\rho_{\mathrm{Hom}(B)}^{\boxtimes s})(\underline{m}(L_{s-1}\backslash L_s) \boxtimes \cdots \boxtimes \underline{m}(L_0\backslash L_1)),$$

with $\rho^{\boxtimes s}$ defined recursively by $\rho^{\boxtimes 1}_{\mathrm{Hom(B)}} = \rho_{\mathrm{Hom(B)}}$ and $\rho^{\boxtimes(s+1)}_{\mathrm{Hom}(B)} = \rho^{\boxtimes s}_{\mathrm{Hom(B)}} \boxtimes \mathrm{id} \circ \rho_{\mathrm{Hom}(B)}$. From the definition of an infinitesimal character, the sum on the right hand side of the last equation reduces to

$$\exp_\star(\underline{m})(\pi) = \sum_{(\pi,\ell)\in\mathcal{NC}_m} \frac{1}{k!} T_\otimes(\rho^{\boxtimes s}_{\mathrm{Hom}(B)})(\underline{m}(\mathsf{L}(\pi,\ell)_{s-1}\backslash\mathsf{L}(\pi,\ell)_s) \boxtimes \cdots \boxtimes \underline{m}(\mathsf{L}(\pi,\ell)_0\backslash\mathsf{L}(\pi,\ell)_1)).$$

The result follows from the last equation. □

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