THE NON-COMMUTATIVE SIGNATURE OF A PATH

CARLO BELLINGERI * AND NICOLAS GILLIERS †

Abstract. We introduce a class of operators associated to the signature of a smooth path X with values in a $C^{\star}$ algebra $\mathcal{A}$. These operators serve as basis of Taylor expansions of solutions to controlled differential equations of interest in non-commutative probability. They are defined by fully contracting iterated integrals of X, seen as tensors, with the product of $\mathcal{A}$. Were it considered that partial contractions should be included, we explain how these operators yield a trajectory on a group of representations of a combinatorial Hopf monoid. To clarify the role of partial contractions, we build an alternative group-valued trajectory whose increments embody full-contractions operators alone. We obtain therefore a notion of signature, which seems more appropriate in non-commutative probability.

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1. Introduction

This work intends to explore a direction suggested in [6] and aiming rough paths principles to study the following class of differential equations

$$\mathrm{d}Y_t = a(Y_t)\cdot \mathrm{d}X_t \cdot b(Y_t)\,, \quad Y_0 \in \mathcal{A}\,. \tag{1.1}$$

In the above equation, the driving path $X\colon [0,1] \to \mathcal{A}$ takes values in a $C^{\star}$-algebra $(\mathcal{A}, \cdot, \star, \|\cdot\|)$ and $a, b : \mathcal{A} \to \mathcal{A}$ are two polynomial functions or Fourier transforms of regular measures with exponential moments, see [2, 6].

This paper is the first of two whose objectives are to introduce a new notion of geometric rough paths, tailored to the class of equations (1.1). In this work, we focus on the algebra

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underlying Taylor expansions of solutions to equations (1.1), discarding other crucial aspects (such as measurability).

1.1. **The rough paths approach.** In the nineties [12], T.J. Lyons proposed the appropriate mathematical framework to study differential equations

$$\mathrm{d}Y_t = \sigma(Y_t)\,[\mathrm{d}X_t]\,, \quad Y_0 = y_0 \in \mathbb{R}^d. \tag{1.2}$$

In (1.2), the solution Y is a continuous path in $\mathbb{R}^d$, $\sigma\colon \mathbb{R}^d \to \mathsf{End}(\mathbb{R}^n, \mathbb{R}^d)$ is a smooth vector field and the driving path X is Hölder continuous. If X is smooth, standard differential calculus provides a rigourous interpretation to (1.2). For paths with lower regularity, Young's theory of integration [16] gives sense to equation (1.2) driven by a Hölder regular path X with exponent greater than $\frac{1}{2}$. Interesting stochastic driving paths are too irregular for Young integration. For instance, Brownian trajectories are only $\frac{1}{2} - \varepsilon$, $\varepsilon > 0$ Hölder continuous. Classical Itô integration supplements limitations of Young's theory and defines integrals driven by continuous semi-martingales as limits **in probability** of Riemann sums.

Rough path theory extends the standard rules of differential and integral calculus to Hölder paths X and provides a **pathwise** interpretation to (1.2). Let us add more details. Given a smooth field σ and $Y_0 \in \mathbb{R}^n$ the solution map $\Phi : X \mapsto Y$ to equation (1.2) is continuous with respect to the Lipschitz norm on the space of smooth driving paths X. A fundamental observation is the following one: by applying Picard's iterations to (1.2), one quickly reckons that the solution map Φ is a linear functional of the entire **signature** of X, that is the infinite collection of tensors.

$$\mathbb{X}_{st} = \left(1, X_t - X_s, \int_{\Delta^2_{st}} \mathrm{d}X_{t_1} \otimes \mathrm{d}X_{t_2}\,, \cdots, \int_{\Delta^n_{st}} \mathrm{d}X_{t_1} \otimes \cdots \otimes \mathrm{d}X_{t_n}\,, \cdots\right), \tag{1.3}$$

where $\Delta^m_{st} := \{t > t_1 > \cdots > t_m > s\}$ is the m-dimensional simplex. Now, signatures of smooth paths support a one parameter family of topologies with respect to Φ is continuous. Complete spaces for these topologies contain Hölder paths together with the additional data of an **abstract signature**. These abstract signatures are called **rough paths** and can alternatively be characterized by a set of algebraic and analytical properties. Indeed, a rough path is in particular the data (along with Hölder estimates) for each pair of times s, t of an element $\mathbb{X}_{st}$ of a **group** $(G, \star)$ included in the completed tensor space of $\mathbb{R}^n$ with the property that for each triple of times $s, u, t \in [0, 1]^3$

$$\mathbb{X}_{st} = \mathbb{X}_{su} \star \mathbb{X}_{ut}\,. \tag{1.4}$$

The relations (1.4) are usually called **Chen's relation** after Kuo-Tsai Chen [5] and its secular work on homology of loop spaces. We refer the reader to the monograph [7] for a detailed exposition on rough path theory.

1.2. **Motivation and previous works.** We choose to have a intrinsic –coordinate-free– approach to (1.1) and to work consistently with the specific class of fields we consider, that is with the algebra product. Rough paths theory on infinite-dimensional spaces is more intricate because of several notions of tensor products between two Banach algebras, see [8]. Considering the class of equations (1.1) the **projective tensor product** is the only reasonable one, since the algebra product is always continuous with respect to this topology. This is not true for the spatial (or injective) topology. This limitation strikes with the results obtained in [15, 4]. In these works, the author defines a rough path (in fact, a Lévy area) over the free Brownian motion in the **spacial tensor product** by using free Itô calculus. Whereas it is possible [11] to show existence of a free Lévy area (up to a infinitesimal loss in regularity) in the projective tensor product, an explicit procedure is missing.

To circumvent this issue, A. Deya and R. Schott introduced in [6] a weaker notion of Lévy area tailored to the class of equations (1.1) when the Hölder scale lies in $(\frac{1}{3}, \frac{1}{2}]$: the **product Lévy area**. This object embodies the data on the small scale behaviour of the driving path X only in the directions required to give sense to (1.1). The starting point to define it is a fine analysis of (1.2) with X smooth and the expansion of the solution Y obtained by applying Picard iterations. Pick $A, B \in \mathcal{A}$ and consider the following example

$$\mathrm{d}Y_t = (A \cdot Y_t) \cdot \mathrm{d}X_t \cdot (Y_t \cdot B), \quad Y_0 = 1_{\mathcal{A}}\,. \tag{1.5}$$

Writing the first two steps of the Picard Iteration, we obtain

$$Y_t = 1_{\mathcal{A}} + \int_{\Delta^1_{st}} A \cdot \mathrm{d}X_{t_1} \cdot B + \int_{\Delta^2_{st}} A^2 \cdot \mathrm{d}X_{t_1} \cdot B \cdot \mathrm{d}X_{t_2} \cdot B + \int_{\Delta^2_{st}} A \cdot \mathrm{d}X_{t_2} \cdot A \cdot \mathrm{d}X_{t_1} \cdot B^2 + R_{st}\,,$$

where R_{st} is a remainder term satisfying $|R_{st}| \lesssim |t-s|^3$. The above equation hints at a controln, at any order, of the small variations of Y by the following expressions

$$\int_{\Delta^n(s,t)} A_0 \cdot \mathrm{d}X_{t_{\sigma^{-1}(1)}} \cdot \dots \cdot \mathrm{d}X_{t_{\sigma^{-1}(n)}} \cdot A_n\,, \quad A_0, \dots, A_n \in \mathcal{A}. \tag{1.6}$$

where σ is a permutation of $[\![1, n]\!]$. The expressions in (1.6) are values of a mulitlinear operator $\mathbb{X}^{\sigma}_{st}$, that we call full contraction operator, depending on a choice of a permutation σ. The solution of the equation (1.1) expands over the contracted iterated integrals (1.6) in the way alluded to above under the constraints that the Fourier transforms of a and b are bounded measures on the real line. A product Lévy area is an abstraction of the order two full contraction operators, the ones indexed by permutations of $\{1, 2\}$.

We elaborate on the observation of A. Deya and R. Schott and extract important algebraic and analytical properties of the multilinear operators (1.6) with the objective of developing a rough theory for the class of equations (1.2) with driving noise X of arbitrary low Hölder regularity. To put it shortly, the main outcome of this work is "yes, it is possible" and we explain why by associating to the operators (1.6) a smooth trajectory over a group of triangular morphisms on a algebra of operators we introduce.

The main difficulties lie in writing a Chen relation for the operators (1.6) understood as a certain "algebraic rule" for computing (1.6) over an interval knowing the values of (1.6) over a subdivision of this interval. Consider for instance the full contraction operator

$$\mathbb{X}^3_{st}(A_0, A_1, A_2, A_3) := \int_{\Delta^3_{st}} A_0 \cdot \mathrm{d}X_{t_1} \cdot A_1 \cdot \mathrm{d}X_{t_3} \cdot A_2 \cdot \mathrm{d}X_{t_2} \cdot A_3 \tag{1.7}$$

Then the Chasles identity implies the following deconcatenation formula:

$$\begin{aligned}\mathbb{X}^3_{st}(A_0, A_1, A_2, A_3) &= \mathbb{X}^3_{st}(A_0, A_1, A_2, A_3) + \mathbb{X}^3_{ut}(A_0, A_1, A_2, A_3) \\ &\quad + \int_{t_3 \in \Delta^1_{ut}} \int_{(t_1,t_2) \in \Delta^2_{su}} A_0 \cdot \mathrm{d}X_{t_1} \cdot A_1 \cdot \mathrm{d}X_{t_3} \cdot A_2 \cdot \mathrm{d}X_{t_2} \cdot A_3 \\ &\quad + \int_{(t_2,t_3) \in \Delta^2_{ut}} \int_{t_1 \in \Delta^1_{su}} A_0 \cdot \mathrm{d}X_{t_1} \cdot A_1 \cdot \mathrm{d}X_{t_3} \cdot A_2 \cdot \mathrm{d}X_{t_2} \cdot A_3\,.\end{aligned}$$

The term on the second line above can not be expressed by composing order two full contraction operators. Instead, we can obtaine it by **composing** the operator,

$$(A_0, \dots, A_4) \mapsto \int_{\Delta^2_{su}} A_0 \cdot \mathrm{d}X_{t_1} \cdot A_1 \otimes A_2 \cdot \mathrm{d}X_{t_2} \cdot A_3 \in \mathcal{A} \otimes \mathcal{A} \tag{1.8}$$

with the following full contraction one

$$(A_0, A_1) \mapsto \int_{\Delta^1_{ut}} A_0 \cdot \mathrm{d}X_{t_1} \cdot A_1.$$

Thus a naive approach leads in fact to relations involving not only full contractions but also partial contractions.

The main result of the paper is loosely stated in the following Theorem.

Theorem 1.1. *Let $X\colon [0,1] \to \mathcal{A}$ be a smooth path. Then there exists a group $(\mathcal{G}, \circ)$, a bi-graded set $\mathcal{F}$ and a function $\mathbb{X}\colon [0,1]^2 \to \mathcal{G}$ with the following properties:*

- *For any $s,u,t\in[0,1]^3$ one has*

$$\mathbb{X}_{st}=\mathbb{X}_{ut}\circ\mathbb{X}_{su}\,. \tag{1.9}$$

- *For any $s,t\in[0,1]^2$, $\mathbb{X}_{st}$ has components a family of multilinear operators, $\{X^f_{st}\}_{f\in\mathcal{F}}$,*
- *The set $\mathcal{F}$ contains all permutations σ and $\mathbb{X}^f_{st}$ coincides with (1.6) when $f=\sigma$.*
- *Given two elements $\mathbb{X},\mathbb{Y}\in\mathcal{G}$,*

$$\mathbb{X}=\mathbb{Y}\Leftrightarrow\mathbb{X}^\sigma=\mathbb{Y}^\sigma,\text{for all permutations }\sigma.$$

We call the element $\mathbb{X}_{st}$ the ***non-commutative signature*** *of the path X and the relations (1.9)* ***non-commutative Chen's relations****.*

The group $\mathcal{G}$ is identified in Proposition 3.26 and the function $\mathbb{X}$ is introduced in Definiton 3.23. The countable set $\mathcal{F}$ is the set of planar leveled forets we introduce in Section 2.1.

1.3. **Outline.** Besides the introduction, this article is divided in two additional sections. In Section 2, we introduce a Hopf monoid of levelled forests, reminiscent of the Malvenuto Reuteneauer Poirier Hopf algebra.

- In Section 3.1, we define the partial and full contractions operators we alluded to, see Definitions 3.1 and 3.3.
- In Section 3.2 we prove a Chen relation for these operators, see Proposition 3.4. Next, we explain how this yields a path on a group of triangular algebra morphisms on an algebra spanned by couples of a tree and a word.
- In Section 3.4 we associate to the full and partial contractions operators a path of representations on the Hopf monoid of levelled forests we introduced in Section 2, see Theorem 3.14.
- In Section 3.5, we adopt a slightly different point of view and let the iterated integrals of a path acting on a set of operators we call **faces contractions**, see Definition 3.15. This yields a certain triangular algebra morphism, see Definition 3.23 that we relate to the one introduced in Section 3.2. In Proposition 3.25, we relate partial to full contractions operators.

In a forthcoming article, we continue to develop the theory. In particular, we introduce geometric non-commutative rough paths, geometric non-commutative controlled rough path, the operations of integration and composition and prove existence of solutions to (1.2).

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Preliminaries.

- In the following we denote by $\mathcal{A}$ a generic complex $C^\star$ algebra with product μ, unity $\mathbf{1}$, norm $\|\cdot\|$ and involution $\star$. In order to deal with a topology on the algebraic tensor product $\otimes$ which behaves correctly with μ, we will use the projective tensor product (see e.g. [13]). Given two Banach spaces $(E,\|\cdot\|_E)$ and $(F,\|\cdot\|_F)$, the projective norm of an element $x\in E\otimes F$ is defined by

$$\|x\|_\vee=\inf\left\{\sum_i\|a_i\|_E\|b_i\|_F\colon x=\sum_i a_i\otimes b_i\right\}.$$

We denote by $E\check{\otimes}F$ the completion of $E\otimes F$ for the projective norm. One can check the following properties

$$\|a\otimes b\|_\vee=\|a\|_E\|b\|_F,\ \ \|a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(n)}\|_\vee=\|a_1\otimes\cdots\otimes a_n\|_\vee\,, \tag{1.10}$$

for any permutation σ of the interval $[\![1,n]\!]$ and $a_1,\cdots,a_n\in E$. The definition of projective norm yields immediately that the multiplication μ extends to a continuous map $\mathcal{A}\check{\otimes}\mathcal{A}\to\mathcal{A}$ and, more generally, for any given pair of $C^\star$ algebras $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}\check{\otimes}\mathcal{B}$ is again a $C^\star$ algebra. Looking this operation from a very general point of view, the projective tensor product makes the category of complex $C^\star$ algebras a monoidal category (see the Appendix). In order to lighten the notation, we will adopt the symbol $\otimes$ to denote both the the projective tensor product between $C^\star$ algebras and the algebraic tensor product for pure tensors. Similarly, we will replace the product μ with a dot $\cdot$.

• For $n \geq 1$ an integer, we denote by $\mathcal{S}_n$ the set of permutations of $[n] = \{1, \cdots, n\}$, which is denoted by $(\sigma(1) \cdots \sigma(n))$. Given two integers a, b, we denote by $\mathsf{Sh}(a, b)$ the set of all shuffles of the two intervals $[\![1, a]\!]$ and $[\![a + 1, a + b]\!]$, that is

$\sigma \in \mathsf{Sh}(a, b)$ if and only if σ is non-decreasing on $[\![1, a]\!]$ and on $[\![a + 1, a + b]\!]$.

2. Hopf monoid of leveled binary forests

2.1. **Leveled trees and forests.** The objective of the present section is to introduced the combinatorial tool that will be used through this work; the algebra of leveled trees, isomorphic to the Malvenuto-Reutenauer-Poirier algebra. In the literature, one broadly finds two equivalent representations of a permutation, either as a word on integers or as a bijection from a certain interval of integers. We use a third – tree like – graphical representation of a permutation introduced in [9] by Ronco and Loday.

First, recall that a **planar rooted tree** is a planar graph with no cycles and one distinguished vertex we call the root. A tree is oriented from top to bottom : the target of edge is the vertex closest to the root (for the graph distance). In this orientation, each vertex of a tree as at most one outgoing edge (the root is the only vertex with no outgoing edge) and several inputs. A **leaf** of a tree is a vertex with no incoming edges. An **internal vertex** of a tree is a vertex that is not a leaf. A **planar binary tree** is a tree for which every internal node has two inputs.

Pick t a planar rooted tree. The set of internal vertices of τ is equipped with a partial order $\prec_\tau$. Pick v, w two vertices of t, we write $v \prec_t w$ if there is an oriented path of edges of t from w to v.

The set of internal vertices of a tree t is denote $\mathbb{V}(t)$ and we set $|\mathbb{V}(t)| = \|t\|$. Notice that for binary trees we have $\|t\| = |t| - 1$.

Definition 2.1 (**leveled planar binary tree LT**)**.** A **leveled planar binary tree** (or simply a leveled tree) is a pair (t, g) with t a planar binary tree and g an increasing function

$$g : (\mathbb{V}(\tau), \prec_\tau) \to [\|\tau\|]$$

We denote the set of planar binary tree by LT.

Notice that the root tree has no internal vertices and correspond to leveled tree $(\bullet, \emptyset)$ where here $\emptyset$ denotes the unique function from the empty set to the empty set. The **degree** of a leveled planar tree $\tau \in \mathsf{LT}$ is the number of its leaves. The complex span of LT is a graded vector space, its homogeneous component of degree $n \geq 1$ is the linear span of trees with n leaves. By definition, a leveled tree with degree one is the root tree (see Fig 1). We denote by LT_n the set of leveled trees with n generations and $\mathsf{LT}(n)$ the set of leveled trees with n leaves.

Proposition 2.2 ([9])**.** *Let n be an integer greater than one. The set of leveled planary rooted binary trees LT_{n+1} is in bijection with the set of permutations $\mathcal{S}_n$.*

We use throughout this work a convenient graphical representation of a leveled binary tree $\tau = (t, f)$. It consists in associating to τ a planar tree τ' that is obtained by vertically ordering the internal vertices of t, according to t by adding straight edges. A generation of such a tree is the set of all internal vertices at the same distance from the root. Each generation of τ' has exactly one vertex with two inputs, see Fig. 1. This graphical presentation turns effective to describe certain operations that we introduced in the next sections on leveled planar forests.

Leveled trees are not sufficient for our purposes, we need leveled planar **forests** that we introduce now.

A **planar forest** is a word (a non-commutative monomial) on planar trees. In the following, we denote by $\mathsf{nt}(f)$ the number of trees in the forest f, $|f|$ the total number of leaves in the

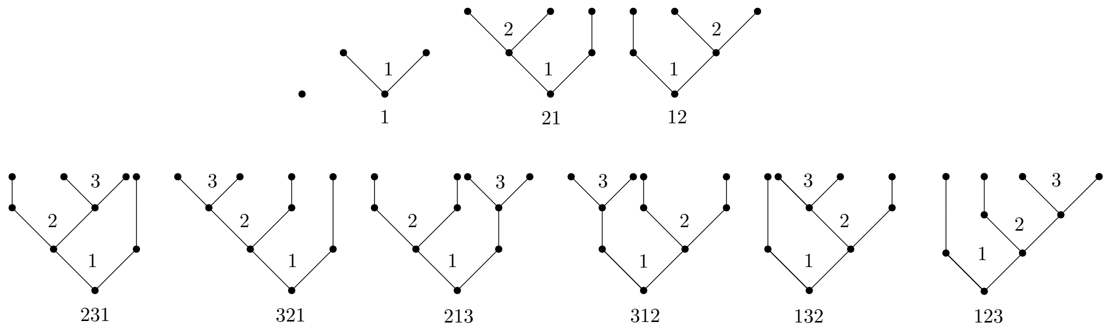

FIGURE 1. Examples of leveled trees $\tau = (t, f)$ (we have drawn τ', see above) in LT and their associated word. We obtain the corresponding binary trees t by contracting all the straight edges.

forest and we set $\|f\|$ equal to the number of internal vertices of the forests. If all trees of f are binary trees, then $\|f\| = |f| - \mathsf{nt}(f)$. The poset $(\mathbb{V}(f), \prec_f)$ of ordered internal vertices of f is the union of the posets of internal vertices of the trees in f.

Definition 2.3 (**leveled planar binary forests** LF)**.** A leveled planar binary forest f (or simply a leveled forest) is a pair (t_f, ℓ_f) of a planar forest t_f and a increasing bijection

$$g : \mathbb{V}(f) \to [||f||],\ v \prec_f w \Leftrightarrow g(v) < g(w).$$

We denote the set of planar binary tree by LF.

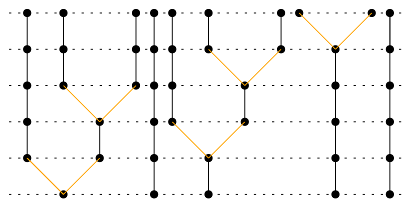

FIGURE 2. A leveled forest with five trees.

The **degree** of a leveled planar forest $f \in \mathsf{LT}$ is the number of leaves of t_f. The set LF is bi-graded, the homogeneous set of degrees $n \geq 1$ and $m \geq 1$ is $\mathsf{LT}(n, m)$ of leveled forests with n leaves and m trees. Notice that the leveled forests with the same numbers of leaves and trees are the leveled forests f with t_f a forest of root trees. Finally, we denote by LF_n the set of leveled trees with n generations.

A leveled forest $f = (t_f, t_g)$ can be pictured as planar forest in the same way as a explained before for leveled trees, the internal vertices are ordered vertically by adding straight edges according to g, see Fig. 2. We denote this forest by f'.

Above, we settled a bijection between leveled trees in LT and permutations. We can thus transfer the actions of a permutation σ in $\mathcal{S}_n$ by right and left multplication on S_n to left and right actions on leveled trees with n generations. We now explain how this action extends to leveled planar forests.

First, we define bijection between leveled forests in LF and pairs of a permutation and an interval partition (that may contain empty blocks).

Pick a leveled forest $f = (t_f, \ell_f)$. To each planar decorated forest with n internal vertices $\tau = (t, \ell_{|\mathbb{V}(\tau))}$ with t a tree in the planar forest f, we associate a (possibly empty) word $[f]_\tau$ with entries $[n]$; it is obtained by reading from left to right the labels of the tree.

If τ is the root tree then we associate to this tree the empty word $\emptyset$. In that way we build a word on words $[f] = [f]_{\tau_1} \cdots [f]_{\tau_{\mathsf{nt}(f)}}$. For example, the word on words associated with the leveled planar forest in Fig. 2 is

$$13 \,|\, \emptyset \,|\, 24 \,|\, 5 \,|\, \emptyset.$$

It is easily seen that this correspondance between leveled forests with n internal vertices a words on words, each with entries in $[n]$ such that each integer appears only once is bijective.

A permutation $\sigma \in \mathcal{S}_{\|f\|}$ left acts on the word $[f]$ by evaluating this permutation on each letter of each word in $[f]$. We denote by $\sigma \cdot f$ the forest associated with the word $\sigma \cdot [f]$. For example, the permutation $(1,2)(3,4)$ acts on the word on words representing the leveled forest in Fig 2 by

$$(1,2)(3,4) \cdot (13 \,|\, \emptyset \,|\, 24 \,|\, 5 \,|\, \emptyset) = 23 \,|\, \emptyset \,|\, 13 \,|\, 5 \,|\, \emptyset.$$

We define a right action of σ on f commuting with the left one in the next section.

We thus have three presentation of a leveled forest f, either a couple (t_f, ℓ_f), either as a binary planar forest with straight edges, f', either as a word $[f]$. With $t_f = t_1 \cdots t_{\mathsf{nt}(f)}$, we write $f = f_1 \cdots f_p$ with $f_i = (t_i, \ell_{f|\mathbb{V}(t_i)})$.

2.2. **Operations on leveled forests.** If $\alpha = (t_\alpha, f_\alpha)$ and $\beta = (t_\beta, f_\beta)$ are two leveled trees, we write $\beta \subset \alpha$ and say that β is a **leveled subtree** of α if

1. t_β contains the root and is a subtree of t_α, in particular $\mathbb{V}(t_\beta) \subset \mathbb{V}(t_\alpha)$,
2. $t_\alpha(v) = t_\beta(v), v \in \mathbb{V}(t_\beta)$.

Similarly, if β and α are two leveled forests, we write $\beta \subset \alpha$ if the two forests have the sum number of trees, so $\alpha = \alpha_1 \cdots \alpha_p$, $\beta = \beta_1 \cdots \beta_p$ for a certain integer $p \geq 1$ and $\beta_i \subset \alpha_i$ for $1 \leq i \leq p$.

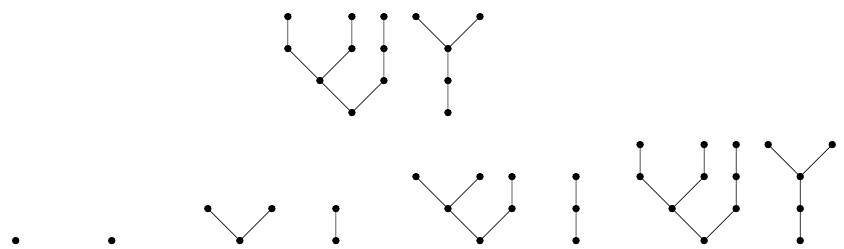

FIGURE 3. On the first line a leveled forest and on the second line the leveled subforests included.

Definition 2.4 (**Vertical cutting and gluing of forests**). • For α and β two leveled forests as above, with $\beta \subset \alpha$, we define **the cut of** α **by** β as the leveled forest $\alpha \setminus \beta = (t_{\alpha\setminus\beta}, \ell_{\alpha\setminus\beta})$ with

1. $t_{\alpha\setminus\beta}$ is the planar forest obtained by erasing all edges and internal vertices that belongs to t_β from t_α,
2. noticing that $\mathbb{V}(t_{\alpha\setminus\beta}) = \mathbb{V}(t_\alpha)\setminus\mathbb{V}(t_\beta)$, one defines $\ell_{\alpha\setminus\beta}(v) = \ell_{t_\alpha}(v) - \|\beta\|$ for $v \in \mathbb{V}(t_{\alpha\setminus\beta})$.

• The operation opposite to cutting is gluing and is denoted $\#$. Pictorially, we **glue a leveled forest** α **up to** β, by stacking the forest α up to the forest β if the number of trees in t_α matches the numbers of leaves of t_β. More formally, we define $\beta\#\alpha = (t_{\beta\#\alpha}, \ell_{\beta\#\alpha}$ if $\mathsf{nt}(\alpha) = |\beta|$ by

1. $t_{\beta\#\alpha}$ is the grafting of the forest t_α up to the forest t_β,
2. noticing that $\mathbb{V}(t_{\beta\#\alpha}) = \mathbb{V}(t_\alpha) \sqcup \mathbb{V}(t_\beta)$, $\ell_{\beta\#\alpha}(v) = \ell_\alpha(v)$ for $v \in \mathbb{V}(\alpha)$, $\ell_{\beta\#\alpha}(v) = \ell_\beta(v)$ for $v \in \mathbb{V}(t_\beta)$

Above, we introduced binary operations on leveled forests, that are vertical cutting ang gluing. We introduce their horizontal counterpart, that are unary operations.

Definition 2.5 (**Horizontal gluing and cutting of forests**). • Let $f \in \mathsf{LF}$ a leveled forest, we denote by $f^\flat \in \mathsf{LT}$ the leveled tree corresponding to the planar forest (with straight edges) obtained by concatenating all together the trees in the forest f' along their external edges. Otherwise stated, the word representation $[f^\flat]$ of $f^\flat$ is obtained by concatenating all words in $[f]$.

• Let τ be a leveled tree and $(n_1,\ldots,n_k)$ a composition of n, $n_i \geq 0$ and $\sum_{i=1}^{k} = n$. We define $\tau_{n_1,\ldots,n_k}$ the leveled forest which is represented by the word

$$[\tau]_1 \cdots [\tau]_{n_1} \mid \cdots \mid [\tau]_{n_1+n_{k-1}+1} \cdots [\tau]_{n_1+\cdots+n_k}.$$

with the convention that the chunk $[\tau]_{n_1+\cdots+n_{i-1}} \cdots [\tau_{n_1+\cdots+n_i}] = \emptyset$ if $n_i = 0$.

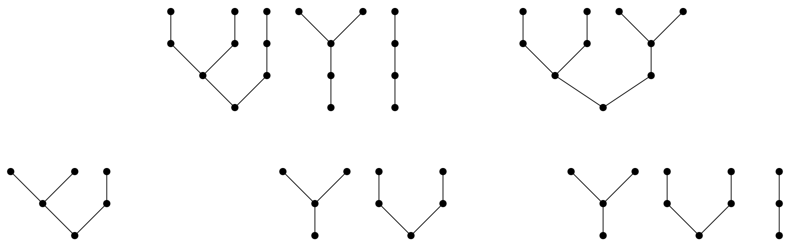

FIGURE 4. On the first line, the right tree is obtained by gluing all trees in the forest f along their external edges. On the second line, the two rightmost forest are horizontal cuts of the leftmost forests with parameters $(1,1)$ and $(1,1,0)$, respectively.

Let $f = f_1 \cdots f_p$ a leveled forest. Then a simple drawing show that for all subforest $f' \subset f$, one has

$$f \backslash f' = \left[f^{\flat} \backslash (f'^{\flat}) \right]_{\|f_1\|,\ldots,\|f_p\|}, \; f' \subset f.$$

We can design more general cutting and gluing operations, such as for example gluing of a subset of trees of a leveled forests. These operations will be needed only in the last section, where they are defined.

2.3. **Shuffle product on leveled forests.** In this section, we denote by $\langle\langle n \rangle\rangle$ the set of words on words with entries in $[n]$, for example $1|1, 12|1, 12|\emptyset \in \langle\langle n \rangle\rangle$. A permutation $\sigma \in \mathcal{S}_n$ acts on the left of a word w with entries in $[n]$: one simply applies σ on each letter in w. The action of σ extends as a morphism of $\langle\langle n \rangle\rangle$ for the concatenation product $\cdot$.

Owing to the correspondance between leveled forests and words on words, the above defined action of the permutation σ induces an action on the set LF_n of leveled forests with n generations. For example, the permutation $(1,2)(3,4)$ acts on the word on words representing the leveled forest in Fig 2 by

$$(1,2)(3,4) \cdot (13 \mid \emptyset \mid 24 \mid 5 \mid \emptyset) = 23 \mid \emptyset \mid 13 \mid 5 \mid \emptyset.$$

With $f = f_1 \cdots f_p$ and $g = g_1 \cdots g_q$ two leveled forests, we denote by $f \times g$ the leveled forest defined by

1. $t_{f\times g}$ is the planar forest that start (when read from the left) with the trees $t_{f_1},\ldots,t_{f_{p-1}}$ followed by the tree obtained by grafting t_{g_1} to the rightmost leaf of f_p and end with the trees $g_2 \cdots g_p$,
2. $\ell_{f\times g}(v) = f(v)$,$v \in \mathbb{V}(f)$, $\ell_{f\times g}(v) = g(v) + \|\alpha\|$, $v \in \mathbb{V}(g)$.

The operations $\times$ is better understood with the help of the word representation of a leveled forest, $[f \times g]$ is the word $[f]_1 \cdots |([f]_{\mathsf{nt}(f)} \cdot [g]_1)| \cdots [g]_q$. The binary operations $\times$ extends to as an **unital bilinear associative product** on $\mathbb{C}[\mathsf{LT}]$, its unit is the empty word $\emptyset$.

Definition 2.6 (**Shuffle product of leveled planar forests).** Let f and g be two leveled forests, define the shuffle product of f and g by

$$f \sqcup\!\sqcup g = \sum_{s \in \mathsf{Sh}(\|f\|,\|f'\|)} s \cdot (f \times g).$$

and extends it bilinearly to $\mathbb{C}[\mathsf{LF}]$.

Remark 2.7. The product $\sqcup$ restricted to LT is the shuffle product of Malvenuto-Reutenauer. The product $\sqcup$ crosses degrees : the number of leaves of $f \sqcup g$ is $|f|+|g|-1$ and the number of trees is $\mathsf{nt}(f)+\mathsf{nt}(g)$. From associativity of $\times$ and well known facts about shuffle, $\sqcup$ is associative and unital; its unit is the root tree.

In addition to the left action of a permutation $\sigma \in \mathcal{S}_n$ on a leveled forest with n generations, one can also define a right action of σ that consists in permuting horizontally the generations, instead of vertically. In terms of the word on word representing a forest f, this means

$$[f]\cdot\sigma = [[f^\flat]\cdot\sigma]_{\|f_1\|,\dots,\|f_{\mathsf{nt}(f)}\|} := [f^\flat_{\sigma(1)}\cdots f^\flat_{\sigma(\|f\|)}]_{\|f_1\|,\dots,\|f_{\mathsf{nt}(f)}\|} \tag{2.1}$$

We denote by c_n the permutation $(n,1)(n-1,2)\cdots(n-\lfloor\frac{n}{2}\rfloor,\lfloor\frac{n}{2}\rfloor)$. We use the right action of c_n to define a involution on LF_n, which is the **horizontal mirror symmetric of a forest**

$$\begin{array}{cccc} \bullet: & \mathbb{C}[\mathsf{LF}] & \to & \mathbb{C}[\mathsf{LF}] \\ & f & \mapsto & (f^\flat\cdot c_{\|f\|})_{\|f_{(\mathsf{nt}(f)}\|,\dots,\|f_1\|} \end{array} \tag{2.2}$$

Proposition 2.8. $(\mathbb{C}[\mathsf{LF}],\sqcup,\bullet)$ *is an involutive algebra.*

Proof. It is a direct a consequence of the following two facts : the left and right actions of $\mathcal{S}_n$ on LF_n commute and $c_{n+m}=\tau_{n,m}\circ c_n\otimes c_m$ where $\tau_{n,m}$ is the shuffle in $\mathsf{Sh}(n,m)$ determines by $\tau_{n,m}(1)=m+1, \tau_{n,m}(n)=m+n$. □

We will sometimes refer to $\bullet$ as the **horizontal involution**, for obvious reasons, to distinguish it from a second involution permuting vertically the generations of a forest that we define below.

2.4. **Hopf monoid of leveled forests.** In this section we introduce a Hopf algebraic structure on the bicollection of spanned by leveled forests and denoted $\mathcal{LF}$,

$$\mathcal{LF}(n,m)=\mathbb{C}[\mathcal{LF}(n,m)], n,m\geq 1.$$

In addition, we set $\mathcal{LF}(0,0)=\mathbb{C}$, $\mathcal{LF}(0,n)=\mathcal{LF}(m,0)=0$, $n,m\geq 1$ and $\mathcal{LT}$ for the collection spanned by leveled binary trees. This Hopf algebra is an object in the category of bicollections endowed with the vertical tensor product $\varobar$. In general, as it is briefly explained in the Appendix, the two-folded vertical tensor product $A\varobar A$ of a monoid A in the monoidal category $(\mathrm{Coll}_2,\ \varobar)$ is not a monoid in the same category. Owing to the fact that the monoid generated by $\mathcal{LF}$ in $(\mathrm{Coll}_2,\ \varobar)$ is **symmetric**, in particular $\mathcal{LF}\varobar\mathcal{LF}$ is a monoid in a natural way, it makes sense to require compatibility between a product and a co-product on $\mathcal{LF}$. We write the unit $\mathbb{C}_{\varobar}$ for the vertical tensor product as

$$\mathbb{C}_{\varobar}=\bigoplus_{n\geq 0}\mathbb{C}1_n.$$

Recall that we denote by $|f|$ the number of leaves of a leveled forest f and $\mathsf{nt}(f)$ the number of trees in f.

We begin with the definition of the coproduct acting on the bicollection $\mathcal{LF}$ of leveled forests. Let f be a leveled forest. Let f' be a leveled subforest of f (recall that f' contains the roots of all trees in f). By definition of the forest $f\backslash f'$, the number of outputs of the forest $f\backslash f'$ is equal to the number of inputs of the forest f' (the number of trees of $f\backslash f'$ matches the number of leaves of f'), the following makes senses

$$\Delta(f)=\sum_{f'\subset f} f'\varobar f\backslash f',\ f\in\mathcal{LF} \tag{2.3}$$

Proposition 2.9 (Coproduct)**.** *The bicollections morphism $\Delta:\mathcal{LF}\to\mathcal{LF}\varobar\mathcal{LF}$ is coassociative:*

$$(\Delta\varobar\mathrm{id}_{\mathcal{LF}})\circ\Delta=(\mathrm{id}_{\mathcal{LF}}\varobar\Delta)\circ\Delta \tag{co-ass.}$$

$$\varepsilon:\mathcal{LF}\to\mathbb{C}_{\varobar},\varepsilon(\bullet^n)=1_n,\ \varepsilon(f)=0,\ f\neq\bullet^n, \tag{counit}$$

$$(\varepsilon \varobar \mathrm{id}_{\mathcal{LF}}) \circ \Delta = (\mathrm{id}_{\mathcal{LF}} \varobar \varepsilon) \circ \Delta = \mathrm{id}.$$

Proof. Let f be a leveled forest, to show coassociativity we notice that:

$$((\Delta \varobar \mathrm{id}_{\mathcal{LF}}) \circ \Delta)(g) = \sum_{\substack{f'', f', f \\ f'' \# f' \# f = g}} f'' \varobar f' \varobar f = ((\mathrm{id}_{\mathcal{LF}} \varobar \Delta) \circ \Delta)(g)\,. \tag{2.4}$$

Hence the above properties follow. □

We proceed now with the definition of a vertical product $\nabla : \mathcal{LF} \varobar \mathcal{LF} \to \mathcal{LF}$. Given two forests f and f' with $\mathsf{nt}(f') = |f|$, we define $\nabla(f \varobar f')$ as the sum of forests obtained by first stacking f' up to f and then shuffling the generations of f' with the generations of f (see Section 2.1 for the definition of the action of a permutation on the generations of a forest),

$$\nabla(f \varobar f') = \sum_{s \in \mathsf{Sh}(\|f\|, \|f'\|)} s \cdot (f \# f'). \tag{2.5}$$

Associativity of the product ∇ is easily checked. The unit $\eta : \mathbb{C}_{\varobar} \to \mathcal{LF}$ is defined by $\eta(1_m) = \bullet^m$. Let $n \geq 1$, recall that we denote by c_n the maximal element for the Bruhat order in $\mathcal{S}_n$:

$$c_n = \prod_{p \in [\![1, \lfloor \frac{n}{2} \rfloor]\!]} (p, n-p).$$

For example, $c_1 = (1)$, $c_2 = (12)$, $c_3 = (1,3)$, $c_4 = (14)(23)$. Given these notion, we state the main theorem of the section

Theorem 2.10. $(\mathcal{LF}, \nabla, \eta, \Delta, \varepsilon)$ *is a conilpotent Hopf algebra in the category* $(\mathrm{Coll}_2, \varobar, \mathbb{C}_{\varobar})$.

To achieve this result we introduce an explicit antipode map.

Definition 2.11. Pick $n, m \geq 1$ two integers. Let $f \in \mathcal{LF}(n,m)$ be a leveled forest and define its **vertical mirror symmetric** $f^\star \in \mathcal{LF}(n,m)$ by

$$f^\star = c_{\|f\|} \cdot f.$$

We extend $\star$ as a conjugate-linear morphism on the bicollection $\mathcal{LF}$.

Proposition 2.12. *Let f be a leveled forest. The map $S : \mathcal{LF} \to \mathcal{LF}$ defined by*

$$S(f) = (-1)^{\|f\|} f^\star \tag{2.6}$$

is an antipode: $\nabla \circ (S \varobar \mathrm{id}_{\mathcal{LF}}) \circ \Delta = \nabla \circ (\mathrm{id}_{\mathcal{LF}} \varobar S) \circ \Delta = \varepsilon \circ \eta.$

Proof. Let a, b be two integers greater than one. Set $n = a + b$. The set of shuffles $\mathsf{Sh}(a,b)$ is divided into two mutually disjoint subsets, the set of shuffles sending a (the subset $\mathsf{Sh}(a,b)_+$) to n and the set of shuffles that do not (resp. $\mathsf{Sh}(a,b)_-$).

Recall that if f is a forest then f_-^k denotes the forest obtained by extracting the k first lowest generations of f and f_+^k denotes the forest obtained by extracting the k highest generations of f. By definition, one has:

$$\nabla(f' \varobar f \backslash f') = \sum_{s \in \mathsf{Sh}(\|f'\|, \|f \backslash f'\|)} s \cdot (f' \# (f \backslash f')^\star), \; f^\star = c_{\|f\|} \cdot f.$$

The following relation is easily checked and turn to be the cornerstone of the proof:

$$\tilde{s} \circ (c_n \otimes \mathrm{id}_m) = s \circ (c_{n+1} \otimes \mathrm{id}_{m-1}), \; s \in \mathsf{Sh}(m-1, n+1)_-, \tag{2.7}$$

with $\tilde{s}$ the unique shuffle in $\mathsf{Sh}(m,n)_+$ such that $\tilde{s}(m) = n + m$, $\tilde{s}(i) = s(i)$. Set $\bar{S}(f) = (-1)^{\|f\|} f^\star$. We prove by induction that $S = \bar{S}$. Assume that $S(f) = \bar{S}(f)$ for any forest f with at most $N \geq 1$ generations and pick a forest f with $N+1$ generations. Then, from the induction hypothesis we get:

$$S(f) + f + (\mathrm{id} \varobar \bar{S}) \circ \bar{\Delta}(f) = 0.$$

$$\begin{aligned}\nabla\circ(\mathrm{id}\,\varobar\,\bar{S})\circ\bar{\Delta}(f) &= \sum_{f'\subset f}(-1)^{\|f\backslash f'\|}\sum_{s\in\mathsf{Sh}(\|f'\|,\|f\backslash f'\|)} s\cdot\left[f'\,\#\,(f\backslash f')^{\star}\right]\\ &= \sum_{k=1}^{\|f\|-1}(-1)^k\sum_{s\in\mathsf{Sh}(\|f\|-k,k)} s\cdot\left[(f_-^{\|f\|-k}\,\#\,(f_+^k)^{\star}\right].\end{aligned}$$

We divide the sum over the set $\mathsf{Sh}(\|f\|-k,k)$ into two sums. The first sums ranges over the subset $\mathsf{Sh}(\|f\|-k,k)_+$ and the second one ranges overs $\mathsf{Sh}(\|f\|-k,k)_-$. Then, we gather the sums over $\mathsf{Sh}(\|f\|-k,k)_+$ and $\mathsf{Sh}(\|f\|-k+1,k-1)_-$:

$$\begin{aligned}\nabla\circ(\mathrm{id}\,\varobar\,\bar{S})\circ\bar{\Delta} = \sum_{k=2}^{\|f\|-2}(-1)^k &\sum_{s\in\mathsf{Sh}(\|f\|-k,k)_+} s\cdot\left[f_-^{\|f\|-k}\,\#\,(f_+^k)^{\star}\right]\\ -(-1)^k &\sum_{s\in\mathsf{Sh}(\|f\|-k+1,k-1)_-} s\cdot\left[f_-^{\|f\|-k+1}\,\#\,(f_+^{k-1})^{\star})\right]\\ +(-1)\sum_{s\in\mathsf{Sh}(1,\|f\|-1)_-} s\cdot\left[f_-^{\|f\|-1}\,\#\,(f_+^1)^{\star}\right] &+(-1)^{\|f\|-1}\sum_{s\in\mathsf{Sh}(\|f\|-1,1)_+} s\cdot\left[f_-^1\#(f_+^{\|f\|-1})^{\star}\right]\end{aligned}$$

Using equation (2.7), the right hand side of the last equation is equal to:

$$\begin{aligned}\nabla\circ(\bar{S}\,\varobar\,\mathrm{id})\circ\bar{\Delta} &= 0-\sum_{s\in\mathsf{Sh}(1,\|f\|-1)_-} s\cdot\left[f_-^1\,\#\,(f_+^{\|f\|-1})^{\star}\right]\\ &\qquad+(-1)^{\|f\|-1}\sum_{s\in\mathsf{Sh}(1,\|f\|-1)_+} s\cdot\left[f_-^{\|f\|-1}\,\#\,(f_+^1)^{\star}\right]\\ &= -f+(-1)^{\|f\|-1}f^{\star}\end{aligned}$$

This ends the proof. □

We defined the three structural morphisms ∇,Δ,S. To turn LF into a Hopf monoid, we have to check compatibility between the coproduct Δ and the product ∇; the coproduct Δ should be an morphism of the monoid $(\mathcal{LF},\nabla)$. This only makes sense provided that we can define a product on the tensor product $\mathcal{LF}\varobar\mathcal{LF}$.

Recall that if f is a leveled forest and $0\leq k\leq\|f\|$, one denotes by f_-^k the leveled subforest of f corresponding to the k generations at the bottom of f' : $t_{f_-^k}$ is the planar subforest of t_f with set of internal vertices the set of internal vertices of f labeled by an integer less than k and for leaves the vertices (including the leaves) of t_f connected to one of the latter internal vertices. The leveled forest f_+^k is obtained similary by extracting the k top generations of f'.

With $p,q\geq 1$ two integers, we denote by $\tau_{p,q}$ the shuffle in $\mathsf{Sh}(p,q)$ satisfying $\tau_{p,q}(1)=q+1$ and $\tau_{p,q}(p)=p+q$.

Definition 2.13. Define the braiding map

$$\mathsf{K}:\mathcal{LF}\varobar\mathcal{LF}\to\mathcal{LF}\varobar\mathcal{LF}$$

by, for g and f leveled forests such that $f\varobar g\in\mathcal{LF}\varobar\mathcal{LF}$,

$$\mathsf{K}(f\varobar g)=\left(\tau_{\|f\|,\|g\|}\cdot(f\#g)\right)_-^{\|g\|}\varobar\left(\tau_{\|f\|,\|g\|}\cdot(f\#g)\right)_+^{\|f\|}.$$

We pictured in Fig. 5 examples of the action of the braiding map on pairs of leveled forests.

We defined the braiding map K as acting on $\mathcal{LF}\varobar\mathcal{LF}$. We extend K as a 2-functor on the product of the monoid generated by $\mathcal{LF}$ in $(\mathrm{Coll}_2,\varobar)$. This means in particular that for integers $p,q\geq 1$, we define a bicollection morphism

$$\mathsf{K}_{p,q}:\mathcal{LF}^{\varobar p}\varobar\mathcal{LF}^{\varobar q}\to\mathcal{LF}^{\varobar q}\varobar\mathcal{LF}^{\varobar p}.$$

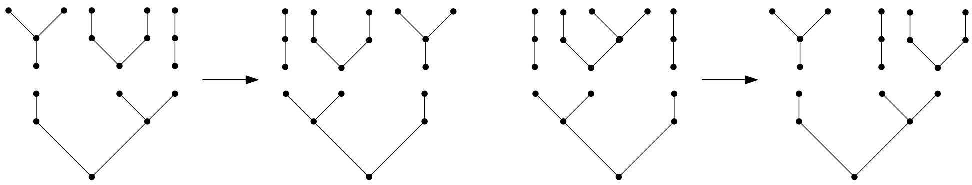

FIGURE 5. Actions of the braiding map

Pick $f_1 \obar \cdots \obar f_p \in \mathcal{LF}^{\obar p}$ and $g_1 \obar \cdots \obar g_q \in \mathcal{LF}^{\obar q}$. We first **stack vertically** the forests $f_1, \dots, g_q$ yielding a leveled forest $f_1 \# \cdots \# g_q$. The permutation $\tau_{\|f_1\|_1+\cdots+\|f_p\|,\|g_1\|+\cdots+\|g_q\|}$ then act on the generation of $f_1 \# \cdots \# g_q$. Finally, we define $h^{n_1,\dots,n_p}$ with $n_1 + \cdots + n_p = \|h\|$ to be the element in $\mathcal{LF}^{\obar p}$ obtained by recursively extracting generations of f, starting with the first n_1 generations at the bottom, continuing with the next n_2 generations and so on, we set

$$\mathsf{K}_{p,q}(f_1 \obar \cdots \obar f_p \obar g_1 \obar \cdots \obar g_q) = (\tau_{\|f_1\|+\cdots+\|f_p\|,\|g_1\|+\cdots+\|g_q\|} \cdot f_1 \# \cdots \# g_q)^{\|g_1\|,\dots,\|g_p\|,\|f_1\|,\dots,\|f_q\|} \tag{2.8}$$

Proposition 2.14. *The monoid generated by the bicollection $\mathcal{LF}$ in $(\mathrm{Coll}_2, \obar)$ is a symmetric monoidal category with symmetry constraints $(\mathsf{K}_{p,q})_{p,q \geq 0}$,*

$$\mathsf{K}_{p,q} \circ \mathsf{K}_{q,p} = \mathrm{id} \text{ and } (\mathrm{id}_{\mathcal{LF}^{\obar q}} \obar \mathsf{K}_{p,r}) \circ (\mathsf{K}_{p,q} \obar \mathrm{id}_{\mathcal{LF}^{\obar r}}) = \mathsf{K}_{p,q+r}$$

Proof. Both assertions are trivial and rely on the following relations between the permutations $\tau_{p,q}$, $p, q \geq 0$:

$$\tau_{p,q} \circ \tau_{q,p} = \mathrm{id}, (\mathrm{id}_q \otimes \tau_{p,r}) \circ (\tau_{p,q} \otimes \mathrm{id}_r) = \tau_{p,q+r}, \; p, q, r \geq 0.$$

□

Using the above defined symmetry constraint K, we can endow the two-folds tensor product $\mathcal{LF} \obar \mathsf{LT}$ with an algebra product:

$$(\nabla \obar \nabla) \circ (\mathrm{id} \obar \mathsf{K} \obar \mathrm{id}) : \mathcal{LF}^{\obar 4} \to \mathcal{LF}^{\obar 2}$$

Proposition 2.15. *The two bicolletion morphisms $\Delta : \mathcal{LF} \to \mathcal{LF} \obar \mathcal{LF}$ and $\nabla : \mathcal{LF} \obar \mathcal{LF} \to \mathcal{LF}$ are vertical algebra morphisms. With $\nabla^{(2)} = \nabla \circ (\nabla \obar \mathrm{id}) = \nabla \circ (\mathrm{id} \obar \nabla)$, this means that*

$$\nabla^{(2)} = \nabla^{(2)} \circ (\mathrm{id} \obar \mathsf{K} \obar \mathrm{id}), \quad (\nabla \obar \nabla) \circ (\mathrm{id} \obar \mathsf{K} \obar \mathrm{id}) \circ (\Delta \obar \Delta) = \Delta \circ \nabla.$$

Remark 2.16. We can rephrase the fact that ∇ is an algebra morphism by saying that $(\mathcal{LF}, \nabla)$ is, in fact, a commutative algebra.

Proof. We begin with the first assertion. Pick f_1, f_2, f_3, f_4 compatible leveled forests (the number of inputs of f_i matches the number of outputs of f_{i+1}, $1 \leq i \leq 3$),

$$\begin{aligned}
&(\nabla^{(2)} \circ \mathsf{K})(f_1 \obar f_2 \obar f_3 \obar f_4) \\
&\quad = \sum_{s \in \mathsf{Sh}(\|f_1\|,\|f_3\|,\|f_2\|,\|f_4\|)} s \cdot \Big[f_1 \# \big(\tau_{\|f_2\|,\|f_3\|} \cdot (f_2 \# f_3)\big)_-^{\|f_3\|} \# \big(\tau_{\|f_2\|,\|f_3\|} \cdot (f_2 \# f_3)\big)_+^{\|f_2\|} \# f_4 \Big] \\
&\quad = \sum_{s \in \mathsf{Sh}(\|f_1\|,\|f_3\|,\|f_2\|,\|f_4\|)} \big(s(\mathrm{id} \otimes \tau_{\|f_2\|,\|f_3\|})\big) \cdot \Big(f_1 \# f_2 \# f_3 \# f_4\Big) \\
&\quad = \sum_{s \in \mathsf{Sh}(\|f_1\|,\|f_2\|,\|f_3\|,\|f_4\|)} s \cdot \Big(f_1 \# f_2 \# f_3 \# f_4\Big) = \nabla^{(2)}(f_1 \obar f_2 \obar f_3 \obar f_4).
\end{aligned}$$

For the second assertion, we write first:

$$(\Delta\circ\nabla)(f \varobar g)=\sum_{1\leq k\leq \|f\|+\|g\|}\ \sum_{s\in \mathsf{Sh}(k,\|f\|+\|g\|-k)}(s\cdot (f\#g))^{k}_{-}\varobar (s\cdot (f\#g))^{\|f\|+\|g\|-k}_{+}$$

For each integer $1\leq k\leq \|f\|$, we split the set of shuffles $\mathsf{Sh}(\|f\|,\|g\|)$ according to the cardinal q of the set $s^{-1}(⟦1,k⟧)\cap ⟦\|f\|+1,\|f\|+\|g\|⟧$. Then a shuffle $s\in \mathsf{Sh}(\|f\|,\|g\|)$ $s=(s_1\otimes s_2)\circ\tilde{\tau}_{k,q}$ with $\tilde{\tau}_{k,q}$ the unique shuffle that sends the interval $⟦\|f\|+1,\|f\|+q⟧$ to the interval $⟦k-q+1,k⟧$ and fixes the interval $⟦\|f\|+q+1,\|f\|+\|g\|⟧$.

$$\sum_{\substack{1\leq k\leq \|f\|, 1\leq q\leq \|g\|,\\ 1\leq q\leq k}}\ \sum_{\substack{s_1\in\mathsf{Sh}(k-q,q),\\ s_2\in \mathsf{Sh}(\|f\|-(k-q)\|g\|-q)}}((s_1\otimes s_2)\circ\tilde{\tau}_{k,q})\cdot(f\#g))^{k}_{-}\varobar((s_1\otimes s_2)\circ\tilde{\tau}_{k,q}\cdot(f\#g))^{\|f\|+\|g\|-k}_{+}$$

Notice that $\tilde{\tau}_{k,q}=\tau_{k-q,q}$ and

$$\tilde{\tau}_{k,q}\cdot(f\,\#\,g)=f^{k-q}_{-}\#(\tau_{\|f\|-(k-q),q}\cdot(f_{+}^{\|f\|-(k-q)}\,\#\,g_{-}^{q}))\#g_{+}^{\|g\|-q}.$$

It follows that

$$\begin{aligned}(s_1\otimes s_2)\circ\tilde{\tau}_{k,q}\cdot(f\#g))^{k}_{-}&=\Big((s_1\otimes \mathrm{id})\cdot f^{k-q}_{-}\#(\tau_{\|f\|-(k-q),q}\cdot(f_{+}^{\|f\|-(k-q)}\,\#\,g_{-}^{q}))\#g_{+}^{\|g\|-q}\Big)^{k}_{-}\\ &=s_1\cdot f^{k-q}_{-}\#(\tau_{\|f\|-(k-q),q}\cdot f_{+}^{\|f\|-(k-q)}\#g_{-}^{q})^{q}_{-}.\end{aligned}$$

Similar computations show that

$$((s_1\otimes s_2)\circ\tilde{\tau}_{k,q}\cdot(f\#g))^{\|f\|+\|g\|-k}_{+}=s_2\cdot(\tau_{\|f\|-(k-q),q}\cdot(f_{+}^{\|f\|-(k-q)}\,\#\,g_{-}^{q}))_{+}^{\|f\|-(k-q)}\#g_{+}^{\|g\|-q}).$$

The case $\|f\|+1\leq k\leq \|f\|+\|g\|$ is similar, we split the set of shuffles $\mathsf{Sh}(\|f\|,\|g\|)$ according to the cardinal of the set $s^{-1}(⟦k+1,\|f\|+\|g\|⟧)\cap⟦1,\|f\|⟧)$ and omitted for brevity. Finally, we obtain for $\Delta\circ\nabla(f\varobar g)$ the expression:

$$\sum_{\substack{1\leq k\leq \|f\|,\\ 1\leq q\leq \|g\|}}\ \sum_{\substack{s_1\in\mathsf{Sh}(k,q)\\ s_2\in\mathsf{Sh}(\|f\|-k,\|g\|-q)}}s_1\cdot(f^{k}_{-}\#(\tau_{\|f\|-k,q}\cdot(f_{+}^{\|f\|-k}\,\#\,g_{-}^{q}))^{q}_{-})\varobar s_2\cdot(\tau_{k,q}\cdot(f_{+}^{\|f\|-k}\,\#\,g_{-}^{q}))_{+}^{\|f\|-k}\#g_{+}^{\|g\|-q})$$

which is easily seen to be equal to $(\nabla\varobar\nabla)\circ(\mathrm{id}\varobar\mathsf{K}\varobar\mathrm{id})\circ(\Delta\varobar\Delta)(f\varobar g)$. ☐

3. Iterated integrals of a path as operators

For the entire section, $X:[0,1]\to\mathcal{A}$ denotes a smooth path. We introduce a set of multilinear functions on $\mathcal{A}$ called **partial-** and **full-contraction** operators, indexed by leveled forests and pairs of times $s<t$. These operators are obtained by contracting a tuple of elements of the algebra using the multiplication with an iterated integral of the path X, see Definition 3.1.

3.1. Full and partial contractions operators.

Definition 3.1. Let $n\geq 1$ an integer and σ a permutation in $\mathcal{S}_n$. We introduce the map $\mathbb{X}^{\sigma}\colon[0,1]^2\to Hom(\mathcal{A}^{\otimes(n+1)},\mathcal{A})$

$$\mathbb{X}^{\sigma}_{st}(A_0,\ldots,A_n)=\int_{\Delta^{n}_{st}}A_0\cdot \mathrm{d}X_{t_{\sigma(1)}}\cdots \mathrm{d}X_{t_{\sigma(n)}}\cdot A_n\,. \tag{3.1}$$

If linearly extended to the vector space spanned by all leveled trees (or equally permutations), $\sigma\mapsto\mathbb{X}^{\sigma}_{st}$ is a collection morphism, see Appendix 4. The above definition may be misleading, recall that $\otimes$ denotes the projective tensor product, not the algebraic one. It contains the algebraic tensor product as a dense subspace and $\mathbb{X}^{\sigma}_{st}$ is the unique continuous operator extending the values prescribed by the above definition. The partial contraction operators correspond to leveled forests. We denote by $\mathrm{End}^{2}_{\mathcal{A}}$ the non-commutative polynomials on multilinear maps on $\mathcal{A}$ with values in $\mathcal{A}$. More precisely, with $\mathrm{End}_{\mathcal{A}}$ the collection

$$\mathrm{End}_{\mathcal{A}}(1)=\mathbb{C}\cdot\mathrm{id}_{\mathcal{A}},\ \mathrm{End}_{\mathcal{A}}(n)=\mathrm{Hom}(\mathcal{A}^{\otimes n},\mathcal{A}),n\geq 2$$

one has with the notation in use in the Appendix 4,

$$\mathrm{End}^2_{\mathcal{A}} = \bar{T}(\mathrm{End}_{\mathcal{A}}).$$

As such End is endowed with a product ∇_{End^2} this is induced by the functional composition and extends the canonical operadic structure on $\mathrm{End}_{\mathcal{A}}$ as horizontal monoid morphism (see Appendix 4). We have for two words $u_1 \cdots u_n$ and $v_1 \cdots v_p$, where $v_i \in \mathrm{End}(\mathcal{A}^{\otimes n_i}, \mathcal{A})$, $n = n_1 \cdots n_p$,

$$\nabla_{\mathrm{End}^{(2)}_{\mathcal{A}}}(v \varobar u) = (v_1 \circ (u_1 \otimes \cdots \otimes u_{n_1})) \cdots (v_p \circ (u_{n_1+\cdots+n_{i-1}} \otimes \cdots \otimes u_{n_1+\cdots+n_i}))$$

were $\circ$ stand for the functional composition.

Recall that $\bar{T}(\mathcal{A}) = \mathbb{C}\emptyset \oplus \bigoplus_{n\geq 1} \mathcal{A}^{\otimes n}$. We use the notation $|$ to denote the concatenation product on

$$\bar{T}((\mathcal{A})) := \bar{T}\bar{T}(\mathcal{A}) = \mathbb{C} \cdot 1 \oplus \bigoplus_{n\geq 1} \bar{T}(\mathcal{A})^{\otimes n}.$$

Definition 3.2 (**Representation of the algebra $\bar{T}((\mathcal{A}))$**)**.** We define a representation

$$\mathrm{Op} : \bar{T}((\mathcal{A})) \to \mathrm{End}^2_{\mathcal{A}}$$

as extending the following values, for $A_1 \otimes \ldots \otimes A_n \in T(\mathcal{A}),\ X_i \in \mathcal{A}$,

$$\mathrm{Op}(A_1 \cdots A_n)(X_0 \otimes \cdots \otimes X_n) = X_0 \cdot A_1 \cdot \cdots \cdot A_n \cdot X_n.$$

Let w be a word in $T(\mathcal{A})$ of length n. Let $(n_1, \ldots, n_k)$ be a composition of n; $n_i \geq 0$ and $n = \sum_i n_i$. The composition $(n_1, \ldots, n_k)$ yields a splitting of w: we define the element $[w]_{(n_1,\ldots,n_k)} \in T((\mathcal{A}))$ by

$$[w]_{(n_1,\ldots,n_k)} = w_1 \cdots w_{n_1} | w_{n_1+1} \cdots w_{n_1+n_2} | \cdots | w_{n_1+\cdots+n_{k-1}+1} \cdots w_{n_1+\cdots+n_k},$$

with the convention $w_{n_1+\cdots+n_{i-1}+1} \cdots w_{n_1+\cdots+n_{i-1}+n_i} = \emptyset$ if $n_i = 0$.

Definition 3.3. Let f be a leveled planar forest and $0 < s, t < T$ two times. Define $X^f_{s,t} \in \bar{T}((\mathcal{A}))$ and the **partial contractions operators** $\mathbb{X}^f_{st}$ by

$$X^f_{st} = \left[X^{f^\flat}_{st} \right]_{(\|f_1\|,\ldots,\|f_{\mathsf{nt}(f)}\|)}, \ \mathbb{X}^f_{st} = \mathrm{Op}(X^f_{st}).$$

Let us finish with a remark on the representation Op. By definition, Op is compatible with the concatenation product on $\bar{T}((\mathcal{A}))$. As explained, $\mathrm{End}^2_{\mathcal{A}}$ is endowed with a $\varobar$ monoidal structure $\nabla_{\mathrm{End}^{(2)}_{\mathcal{A}}}$. The same kind of structure exists on $\bar{T}((\mathcal{A}))$. In fact, $\bar{T}(\mathcal{A})$ can be endowed with an operadic structure $\circ$, that we call words-insertions. Given a word $a_1 \cdots a_n \in \bar{T}(\mathcal{A})$ and $w_0, \ldots, w_n \in \bar{T}(\mathcal{A})$,

$$a_1 \cdots a_n \circ (w_0 \otimes \cdots \otimes w_n) = w_0 a_1 w_1 \cdots w_{n-1} a_n w_n \tag{3.2}$$

One can check that $\circ$ satisfy asssociatiy and unitaly constraints of an operadic composition. We then extend this operadic composition as an $\ominus$-monoidal morphism and define in this way an associative product on $\nabla_{\bar{T}((\mathcal{A}))} : \bar{T}((\mathcal{A})) \varobar \bar{T}((\mathcal{A})) \to \bar{T}((\mathcal{A}))$. Then Op is compatible with respect to the products $\nabla_{\mathrm{End}^2_{\mathcal{A}}}$ and $\nabla_{\bar{T}((\mathcal{A}))}$:

$$(\nabla_{\mathrm{End}^2_{\mathcal{A}}} \varobar \nabla_{\mathrm{End}^2_{\mathcal{A}}}) \circ (\mathrm{Op} \varobar \mathrm{Op}) = Op \circ \nabla_{\bar{T}((\mathcal{A}))}. \tag{3.3}$$

3.2. **Chen relation.** In this section, we study first how concatenation of paths lift to the full and partial contractions operators, that is we write a Chen identity for the latters. In addition, we introduce a two parameters family of endomorphisms, constitutive of a model in the meaning of Hairer's theory of regularity structure, acting on a direct sum over leveled trees (or equivalently permutations). In this section, the symbol $\circ$ denotes alternatively the $\varobar$-composition $\nabla_{\bar{T}((\mathcal{A}))}$ or $\nabla_{\mathrm{End}^2_{\mathcal{A}}}$.

Proposition 3.4 (Chen relation). *Let $X : [0,1] \to \mathcal{A}$ be a smooth n path. Let $0 < s < u < t < T$ three times. Let f be a forest in* LF*. Then,*

$$\mathbb{X}^f_{st} = \sum_{f' \subset f} \mathbb{X}^{f'}_{ut} \circ \left[\mathbb{X}^{f \backslash f'}_{su}\right].$$

Proof. The statement of the proposition is implied by the same statement but for the iterated integrals X^f_{st}, $f \in \mathsf{LF}$ since ρ is a representation of the word-insertions operad. The initialization is done for forests with 0 generations. Assume that the results as been proved for forests having at most N generations and let f be a forest with $N+1$ generations.

$$X^f_{st} = \int_s^u dX_{t_1} \circ X^{f\backslash f_1}_{st_1} + \int_u^t dX_{t_1} \circ X^{f\backslash f_1}_{st_1} = X^f_{su} + \int_u^t dX_{t_1} \circ X^{f\backslash f_1}_{st_1}. \tag{3.4}$$

In the above formula, we use operadic composition in the coloured operad associated with the word insertion operad. We use the short notations:

$$\mathrm{d}X_{t_1} = \emptyset^{\otimes i-1} \otimes \mathrm{d}X_{t_1} \otimes \emptyset^{|f|-i} \in \mathsf{W}_1 \otimes \cdots \otimes \mathsf{W}_2 \otimes \cdots \otimes \mathsf{W}_1 = \hat{\mathsf{W}}_{1,\ldots 2,\ldots 1} \tag{3.5}$$

with i the index of the tree in the forest f which has two nodes at its first generation. Also, $X^{f\backslash f_1}_{st_1}$ is seen as an element of $\hat{\mathsf{W}}(n_1) \otimes \cdots \otimes \hat{\mathsf{W}}_{n_i^1, n_i^2} \otimes \cdots \otimes \hat{\mathsf{W}}_{n_k}$, where n_i^1 and n_i^2 are the two trees left out by cutting out the root of *ith* tree in the forest f. For any subforest f' of $f \backslash f_1$, we use $f'_{n_i^1, n_i^2}$ to denote the subforest of f obtained by adding a root connecting together the trees at position n_1 and n_1^2. We apply the recursive hypothesis to the forest $f \backslash f_1$ to get:

$$X^{f\backslash f_1}_{st_1} = \sum_{f' \subset f\backslash f_1} X^{f'}_{ut_1} \circ \left[X^{(f\backslash f_1)\backslash f'}_{su}\right] = \sum_{f' \subset f\backslash f^1} X^{f'}_{ut_1} \circ X^{f\backslash (f')_{n_i^1, n_i^2}}_{su}.$$

We insert this last relation into equation (3.4) to get the result since, with (3.5),

$$\int_u^t \mathrm{d}X_{t_1} \circ X^{f'}_{u,t_1} = X^{(f')_{n_i^1, n_i^2}}_{u,t}.$$

Thereby obtaining the thesis. □

If we choose for the leveled tree f a comb tree, that is a tree obtained by grafting corollas with two leaves with each others, always on the rightmost node, we find back the classical Chen identity. In fact, by cutting such a tree we obtain a smaller comb tree and a leveled forest with only straight trees, except for the last one which is a comb tree. To the family of operators $\{\mathbb{X}^f_{st}, s < t, f \in \mathsf{LT}\}$, we now associate a triangular endomorphism on

$$\mathsf{LT}(\mathcal{A}) = \bigoplus_{\tau \in \mathsf{LT}} \mathcal{A}^{\otimes |\tau|} \otimes \mathbb{C}[\tau].$$

For the remaining part of the article, we use the lighter notation

$$a \cdot \tau = a \otimes \tau \in \mathcal{A}^{\otimes |\tau|} \otimes \mathbb{C}[\tau]$$

In classical rough path theory, the signature of path x yields a path $\mathbb{X} : [0,1] \to G$ on the group of characters G on the shuffle Hopf algebra $(H, \Delta, \sqcup\!\sqcup, S)$ as explained in the introduction. To such a path, we associate a path of invertible triangular endomorphisms of H,

$$\bar{\mathbb{X}}_{\cdot} = \mathrm{id} \otimes \mathbb{X}_{\cdot} \circ \Delta.$$

Whereas it is not clear yet if it is possible to associate to the full and partial contractions operators a path on a certain convolution group of representations, our statement of the Chen relation makes clear that any prospective deconcatenation product Δ should act on a tree by cutting it in all possible ways, generations after generations. In section 2.4 we prove this cutting operation yields a comonoid in $(\mathcal{LF}, \oplus)$. For the time being, we simply write down the following formula for the model,

$$\begin{array}{rccc} \bar{\mathbb{X}}_{st}: & \bigoplus_{\tau \in \mathsf{LT}} \mathcal{A}^{\otimes|\tau|} & \longrightarrow & \bigoplus_{\tau \in \mathsf{LT}} \mathcal{A}^{\otimes|\tau|}, \\ & a \cdot \tau & \longmapsto & \sum_{\tau' \subset \tau} \mathbb{X}_{st}^{\tau \backslash \tau'}(a) \cdot \tau' \end{array} \tag{3.6}$$

Of course, the map $\bar{\mathbb{X}}_{st}$ crosses degrees, and we write $\bar{\mathbb{X}}_{st} = \mathrm{id} + \sum_{k=1}^{\infty} \bar{\mathbb{X}}_{st}^{(k)}$, with

$$\begin{array}{rccc} \bar{\mathbb{X}}_{st}^{(k)}: & \mathsf{LT}(\mathcal{A}) & \to & \mathsf{LT}(\mathcal{A}) \\ & a \cdot \tau & \mapsto & \sum_{\substack{\tau' \subset \tau \\ \|\tau \backslash \tau'\| = k}} \mathbb{X}_{st}^{\tau \backslash \tau'}(a) \cdot \tau' \end{array} \tag{3.7}$$

Proposition 3.4 immediately implies the following one.

Proposition 3.5. *Let $s < u < t < T$ be three times, then*

1. *for every leveled tree $\tau \in \mathsf{LT}$*
$$(\bar{\mathbb{X}}_{st}(\mathcal{A}_\tau) - \mathrm{id}) \subsetneq \bigoplus_{\tau' \subset \tau} \mathcal{A}_{\tau'}\,;$$
2. *for any s, u, t the so-called* ***non-commutative Chen's relations*** *holds*
$$\bar{\mathbb{X}}_{st} = \bar{\mathbb{X}}_{ut} \circ \bar{\mathbb{X}}_{su} \tag{3.8}$$
3. *for any $k \geq 1$ there exist a constant $C > 0$ such that*
$$\|\bar{\mathbb{X}}_{st}^{k}\| \leq \frac{\|X\|_{Lip}^{k}}{k!} |t-s|^{k}$$

Proof. We prove point 2, the Chen relation, the others are trivial. Let $s < u < t$ be three times and $A_f \cdot f \in \mathsf{LF}(\mathcal{A})$. Pursuant to the Chen's relation (Proposition 3.4),

$$\begin{aligned} \bar{\mathbb{X}}_{st}(A_f \cdot f) = \sum_{f' \subset f} \mathbb{X}_{st}^{f \backslash f'}(A_f) \cdot f' &= \sum_{f' \subset f} \sum_{f'' \subset f \backslash f'} \mathbb{X}_{ut}^{f''}(\mathbb{X}_{su}^{(f \backslash f') \backslash f''}(A_1 \otimes \cdots \otimes A_{|f|})) \cdot f' \\ &= \sum_{f' \subset f} \sum_{f'' \subset f \backslash f'} \mathbb{X}_{ut}^{f''}(\mathbb{X}_{su}^{(f \backslash (f'' \sharp f')}(A_1 \otimes \cdots \otimes A_{|f|})) \cdot f' \end{aligned}$$

We perform the change of variable $g = f'' \sharp f'$, $g' = f'$ and we obtain

$$\overline{\mathbb{X}}_{st}(A_1 \otimes \cdots \otimes A_{|f|} \cdot f) = \sum_{g \subset f} \sum_{g' \subset g} \mathbb{X}_{ut}^{g \backslash g'}(\mathbb{X}_{su}^{f \backslash g}(A_1 \otimes \cdots \otimes A_{|f|})) \cdot g' = (\overline{\mathbb{X}}_{ut} \circ \overline{\mathbb{X}}_{st})(A_1 \otimes \cdots \otimes A_{|f|} \cdot f)\,.$$

This concludes the proof. □

3.3. **Geometric properties.** In this section, we investigate consequences of the integration by part formula, in terms of relation between full- and partial-contraction operators associated with the smooth path X over the same time interval and on the endomorphisms (a model) $\bar{\mathbb{X}}_{st}, s < t$.

To set the ground for the second part of our work in which we define composition of n-c. controlled rough path with smooth functions on $\mathcal{A}$, we introduce a new operadic composition L on a collection of words with entries in $\mathcal{A}$ that is different from the composition of the words-insertions operad. This operad encodes operations brought up by the Chain' rule for a certain class of functions (the field a and b are part of). We should elaborate on this in a forthcoming article.

Definition 3.6. We define the collection of vector spaces $\mathcal{FS} = (\mathcal{FS}(0), \mathcal{FS}(1), \mathcal{FS}(2), \ldots)$ by

$$\mathcal{FS}(n) = \mathcal{A}^{\otimes n+1},\ n \geq 0.$$

Next, define $L : \mathcal{FS} \circ \mathcal{FS} \to \mathcal{FS}$ as follows. Pick a word $U \in \mathcal{A}^{\otimes n}$ and words $A^i \in \mathcal{A}^{\otimes m_i}$, $1 \leq i \leq p$ and set

$$L(U \otimes A^1 \otimes \cdots \otimes A^p) =$$
$$\Big(U_{(1)} \cdot A^1_{(1)}\Big) \otimes A^1_{(2)} \otimes \cdots \otimes \Big(A^1_{(m_1)} \cdot U_{(2)} \cdot A^2_{(1)}\Big) \otimes \cdots \otimes \Big(A^p_{(m_p)} \cdot U_n\Big).$$

The word $1 \otimes 1$ acts as the unit for L.

We denote by FS the graded vector space equal to the direct sum of all vector spaces in the collection $\mathcal{FS}$. Notice that elements of $\mathcal{A}$ are 0-ary operators in the collection $\mathcal{FS}$ and for example, the above formula gives $L(U_1 \otimes U_{(2)} \circ A) = U_1 \cdot A \cdot U_2 \in \mathcal{A}$. The following proposition holds and relies on associativity of the product on $\mathcal{A}$.

Proposition 3.7. $\mathrm{FS} = (\mathcal{FS}, L, 1 \otimes 1)$ *is an operad.*

In the collection $\mathcal{FS}$, a word with length n is an operator with $n-1$ entries, the inner gaps between the letters. So far, a leveled tree was considered as an operator with as much inputs as it has leaves. However, there is an alternative way to see such a tree as an operator : by considering the **faces** of the tree as inputs. A face is a region enclosed between two consecutive leaves and delimited by the two paths of edges meeting at the least common ancestor, see Fig. 6. We denote by $\mathsf{LT}^{\#}$ the set of leveled trees graded by the numbers of faces, $\mathsf{LT}^{\#}(n)$ the set of leveled trees with n faces, and $\mathsf{LT}^{\sharp}(\mathcal{A})$ the space $\mathsf{LT}(\mathcal{A})$ seen as a graded vector space with $\mathsf{LT}^{\#}(\mathcal{A})(n) = \mathbb{C}\left[\mathsf{LT}^{\#}(n)\right] \otimes \mathcal{FS}(n)$. Notice that the endomorphism $\bar{\mathbb{X}}_{st}$ we defined in

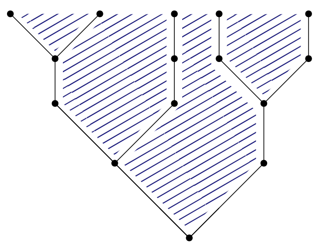

FIGURE 6. Faces of a leveled tree.

the previous section satisfies:

$$(\bar{\mathbb{X}}_{st} - \mathrm{id})(\mathsf{LT}^{\#}(\mathcal{A})(n)) \subset \bigoplus_{k<n} \mathsf{LT}^{\#}(\mathcal{A})(k).$$

Notation. The graded vector space $\mathsf{LT}^{\#}(\mathcal{A})$ yields a collection $\mathcal{LT}^{\#}(\mathcal{A})$, simply by setting the space n operators of $\mathcal{LT}^{\#}(\mathcal{A})$ equal to $\mathsf{LT}^{\#}(\mathcal{A})(n)$. We set, abusively,

$$\mathsf{LT}^{\#}(\mathcal{A}) \circ \mathsf{LT}^{\#}(\mathcal{A}) = \bigoplus_{n \geq 0} \big(\mathcal{LT}^{\sharp}(\mathcal{A}) \circ \mathcal{LT}^{\sharp}(\mathcal{A})\big)(n) \tag{3.9}$$

For any $U, A \in \mathsf{LT}^{\#}(\mathcal{A})$, we set

$$U \circ A = \sum_{\tau, \tau_1, \ldots, \tau_{\|\tau\|}} U^{\tau} \cdot \tau \otimes A^{\tau_1} \cdot \tau_1 \otimes \cdots \otimes A^{\tau_{\|f\|}} \cdot \tau_{\|\tau\|} \in \mathsf{LT}^{\#}(\mathcal{A}) \circ \mathsf{LT}^{\#}(\mathcal{A}).$$

Definition 3.8. Define the morphism of collections $\mathsf{L} : \mathsf{LT}^{\#}(\mathcal{A}) \circ \mathsf{LT}^{\#}(\mathcal{A}) \to \mathsf{LT}^{\#}(\mathcal{A})$

$$\mathsf{L}(U \otimes (A_1 \otimes \cdots \otimes A_{\|\alpha\|})) = \sum_{\substack{\alpha \\ \tau_1, \ldots, \tau_{\|\alpha\|}}} L(U^{\alpha} \otimes A_1^{\tau_1} \otimes \cdots \otimes A_{\|\alpha\|}^{\tau_{\|\alpha\|}}) \cdot \tau_1 \sqcup \cdots \sqcup \tau_{\|\alpha\|},$$

where we have the definition

$$U = \sum_{\alpha \in \mathsf{LT}^{\#}} U^{\alpha} \cdot \alpha, \;\; A_i = \sum_{\tau_i \in \mathsf{LT}^{\#}} A_i^{\tau_i} \cdot \tau_i\,.$$

Lemma 3.9. *Let α and β be two leveled trees in LT, and pick $A \in \mathcal{A}^{\otimes |\alpha|+|\beta|-1}$.*

$$\bar{\mathbb{X}}_{st}(A \cdot \alpha \sqcup\!\sqcup \beta) = \sum_{\tau_\alpha \subset \alpha, \tau_\beta \subset \beta} \mathbb{X}_{st}^{(\alpha \backslash \tau_\alpha) \sqcup\!\sqcup (\beta \backslash \tau_\beta)}(A) \cdot \tau_\alpha \sqcup\!\sqcup \tau_\beta, \tag{3.10}$$

Proof. The proof consists essentially in a re-summation. It stems from the definition of the map $\bar{\mathbb{X}}_{st}$ that:

$$\bar{\mathbb{X}}_{st}(A \cdot \alpha \sqcup\!\sqcup \beta) = \sum_{\substack{\tau \in \alpha \sqcup\!\sqcup \beta \\ \tau' \subset \tau}} \mathbb{X}_{st}^{\tau \backslash \tau'}(A) \;\cdot\; \tau'. \tag{3.11}$$

Let $\tau \in \alpha \sqcup\!\sqcup \beta$ a tree obtained by shuffling vertically the generations of α and β and pick $\tau' \subset \tau$ a subtree. Let s be the shuffle in $\mathsf{Sh}(\sharp\alpha, \sharp\beta)$ such that $\tau^{-1} = (\alpha \otimes \beta) \circ s^{-1}$. We associate to the pair (τ, τ') a triple which consists in the tree τ, and two others trees $\tau'_\alpha \subset \alpha$ and $\tau'_\beta \subset \beta$ satisfying

$$\tau' = (\tau'_\alpha \otimes \tau'_\beta) \circ \tilde{s}^{-1},$$

where $\tilde{s}$ is a shuffle in $\mathsf{Sh}\left(\|\tau'_\alpha\|, \|\tau'_\beta\|\right)$. Such a permutation $\tilde{s}$ is unique, in fact it is obtained from s^1 by extracting the first $\|\tau'\|$ letters of the word representing s^{-1}, followed by standardization and finally inversion. Recall that **standardization** means that we translate the first $\|\tau'\|$ letters representing s^{-1}, while maintaining their relative order to obtain a word on integers in the interval $[\![1, \sharp\tau']\!]$.

It is clear that the map $\phi : (\tau, \tau') \mapsto (\tau, \tau'_\alpha, \tau'_\beta)$ is injective. Now, given $\tau_\alpha \subset \alpha, \tau_\beta \subset \beta$, and two shuffles $s_- \in \mathsf{Sh}(\|\tau'_\alpha\|, \|\tau'_\beta\|)$, $s_+ \in \mathsf{Sh}(\|\alpha \backslash \tau'_\alpha\|, \|\beta \backslash \tau'_\beta\|)$, we define a third shuffle s_{-+} in $\mathsf{Sh}(\|\alpha\|, \|\beta\|)$ by requiring

$$s_{-+}(i) = s_-(i),\; 1 \leq i \leq \|\tau'_\alpha\|, \;\; s_{-+}(\|\tau'_\alpha\| + i) = s_+(i) + s_-(\|\tau'_\alpha\|),\; 1 \leq i \leq \|\tau_\alpha \backslash \tau'_\alpha\|$$

The map $\delta : (\tau'_\alpha, \tau'_\beta, s_+, s_-) \mapsto (\tau, \tau'_\alpha, \tau'_\beta)$ with $\tau^{-1} = \alpha \otimes \beta \circ s_{-+}^{-1}$ is a bijection between the image of ϕ and

$$\mathcal{S} = \{(\tau_\alpha, \tau_\beta, s_+, s_-),\; \tau_\alpha \subset \alpha, \tau_\beta \subset \beta, s_- \in \mathsf{Sh}(\|\tau_\alpha\|, \|\tau_\beta\|), s_+ \in \mathsf{Sh}(\|\alpha \backslash \tau_\alpha\|, \|\beta \backslash \tau_\beta\|)\}.$$

We can thus rewrite the sum in the right hand side of (3.11) as follows:

$$\sum_{\substack{\tau \in \alpha \sqcup\!\sqcup \beta \\ \tau' \subset \tau}} \mathbb{X}_{st}^{\tau \backslash \tau'}(A)\tau' = \sum_{\tau_\alpha, \tau_\beta, s_+, s_- \in \mathcal{S}} \mathbb{X}_{s,t}^{(\alpha \otimes \beta) \circ s_{-+} \backslash (\tau'_\alpha \otimes \tau'_\beta) \circ s_-^{-1}} (\tau'_\alpha \otimes \tau'_\beta) \circ s_-^{-1}$$

Now, we observe that the forest $(\alpha \otimes \beta) \circ s_{-+} \backslash (\tau'_\alpha \otimes \tau'_\beta) \circ s_-^{-1}$ does only depend on the trees τ_α, τ_β and the shuffle s_+. Summing over all shuffles s_+, we get $\alpha \backslash \tau_\alpha \sqcup\!\sqcup \beta \backslash \tau_\beta$. The statement of the Lemma follows by computing the sum over s_-. □

Definition 3.10. Next, with $A \in \mathsf{FS}(n)$ and $B \in \mathsf{FS}(m)$, we define their product $A \cdot B$

$$A \cdot B = A_{(1)} \otimes \cdots \otimes (A_{(n+1)} \cdot B_{(1)}) \otimes \cdots \otimes B_{(m+1)}.$$

The product $\cdot$ is a graded product on the collection FS with unit $1 \in \mathsf{FS}(0)$,

$$A \cdot B \in \mathsf{FS}(n+m),\;\; A \in \mathsf{FS}(n), B \in \mathsf{FS}(m)$$

Remark 3.11. The product $\cdot$ has a very special form, namely:

$$A \cdot B = (1 \otimes 1 \otimes 1) \circ (A \otimes B) = L((1 \otimes 1 \otimes 1) \otimes (A \otimes B)).$$

and the relation $L(m \otimes (id_{\mathrm{FS}} \otimes m)) = L(m \otimes (m \otimes id_{\mathrm{FS}})$ with $m = 1 \otimes 1 \otimes 1$ entails associativity of the product $\cdot$. We say that $m \in \mathcal{FS}(2)$ is a **multiplication** in the operad $(\mathcal{FS}, L)$. In addition, associativity of the operadic composition L results in the following distributivity law

$$(A \cdot B) \circ C = (A \circ B) \cdot (B \circ C),\ \ A, B, C \in \mathsf{FS}.$$

Conjointly with the shuffle product on leveled trees, the product $\cdot$ brings in a graded algebra product $\sqcup\!\sqcup : \mathsf{LT}^{\#}(\mathcal{A}) \otimes \mathsf{LT}^{\#}(\mathcal{A}) \to \mathsf{LT}^{\#}(\mathcal{A})$, namely

$$(A \cdot \alpha) \sqcup\!\sqcup (B \cdot \beta) = (A \cdot B) \cdot \alpha \sqcup\!\sqcup \beta, \tag{sh}$$

with unit $1 \cdot \bullet$. Let f, g two leveled forests and $A \in \mathcal{A}^{\otimes |f|}, B \in \mathcal{A}^{\otimes |g|}$, the integration by part formula

$$\int_{\Delta^2_{st}} \mathrm{d}X_{t_1} \otimes \mathrm{d}X_{t_2} + \int_{\Delta^2_{st}} \mathrm{d}X_{t_2} \otimes \mathrm{d}X_{t_1} = (X_t - X_s) \otimes (X_t - X_s)$$

implies for the iterated integrals of X:

$$\int_{\Delta^n_{st}} \mathrm{d}X_{\sigma_1 \cdot t} \otimes \int_{\Delta^m_{st}} \mathrm{d}X_{\sigma_2 \cdot t} = \int_{Delta^{n+m}_{st}} \mathrm{d}X_{\sigma_1 \sqcup\!\sqcup \sigma_2 \cdot t}$$

which implies for the contractions operators the following relation

$$\mathbb{X}^f_{st}(A_1 \otimes \ldots \otimes A_{|f|}) \cdot \mathbb{X}^g_{st}(B_1 \otimes \ldots \otimes B_{|g|}) = \mathbb{X}^{f \sqcup\!\sqcup g}_{st}((A_1 \otimes \ldots \otimes A_{|f|}) \cdot (B_1 \otimes \ldots \otimes B_{|g|})) \tag{3.12}$$

Proposition 3.12. *Let α and β be two leveled forests and pick $A \in \mathcal{A}^{|\alpha|}, B \in \mathcal{A}^{\otimes |\beta|}$,*

$$\bar{\mathbb{X}}_{st}((A \cdot \alpha) \sqcup\!\sqcup (B \cdot \beta)) = \bar{\mathbb{X}}_{st}(A \cdot \alpha) \sqcup\!\sqcup \bar{\mathbb{X}}_{st}(B \cdot \beta)$$

Proof. It is a simple consequence of the previous Proposition 3.9 and the shuffle relation for the partial contraction operators (3.12) In fact, one has the trivial identities

$$\begin{aligned}\bar{\mathbb{X}}_{st}((A \cdot B) \cdot \alpha \sqcup\!\sqcup \beta) &= \sum_{\tau_\alpha \subset \alpha, \tau_\beta \subset \beta} \mathbb{X}^{\alpha \setminus \tau_\alpha \sqcup\!\sqcup \beta \setminus \tau_\beta}_{st}(A \cdot B) \cdot \tau_\alpha \sqcup\!\sqcup \tau_\beta \\ &= \sum_{\tau_\alpha \subset \alpha, \tau_\beta \subset \beta} \mathbb{X}^{\alpha \setminus \tau_\alpha}(A) \cdot \mathbb{X}^{\beta \setminus \tau_\beta}_{st}(B) \cdot \tau_\alpha \sqcup\!\sqcup \tau_\beta = \bar{\mathbb{X}}_{st}(A) \cdot \bar{\mathbb{X}}_{st}(B)\end{aligned}$$

Thereby obtaining the desired identity. □

Corollary 3.13 (Geometricity)**.** *For all times $0 < s < t < T$, it holds that:*

$$\mathsf{L} \circ (\mathrm{id} \circ \bar{\mathbb{X}}_{st}) = \bar{\mathbb{X}}_{st} \circ \mathsf{L}. \tag{3.13}$$

Proof. For the proof, we rely solely on Proposition 3.12.

$$\begin{aligned}&\mathsf{L}(U^\alpha \otimes \bar{\mathbb{X}}_{st}(A^{\beta_1} \cdot \beta_1) \otimes \cdots \otimes \bar{\mathbb{X}}_{st}(A^{\beta_{\sharp\alpha}} \cdot \beta_{\sharp\alpha})) \\ &= \bar{\mathbb{X}}_{st}(U^\alpha_{(1)} \bullet) \sqcup\!\sqcup \bar{\mathbb{X}}_{st}(A^{\beta_1} \cdot \beta_1) \sqcup\!\sqcup \bar{\mathbb{X}}_{st}(U^\alpha_{(2)} \bullet) \cdots \bar{\mathbb{X}}_{st}(A^{\beta_{\sharp\alpha}} \cdot \beta_{\sharp\alpha}) \sqcup\!\sqcup \bar{\mathbb{X}}_{st}(U^\alpha_{(|\alpha|)} \bullet) \\ &= \bar{\mathbb{X}}_{st}\big((U^\alpha_{(1)} \bullet) \sqcup\!\sqcup (A^{\beta_1} \cdot \beta_1) \sqcup\!\sqcup (U^\alpha_{(2)} \bullet) \cdots (A^{\beta_{\sharp\alpha}} \cdot \beta_{\sharp\alpha}) \sqcup\!\sqcup (U^\alpha_{(|\alpha|)} \bullet)\big) \\ &= \bar{\mathbb{X}}_{st}(\mathsf{L}(U^\alpha \cdot \alpha \otimes A^{\beta_1} \cdot \beta_1 \otimes \cdots \otimes A^{\beta_{\sharp\alpha}} \cdot \beta_{\sharp\alpha}))\end{aligned}$$

□

We denote by $G(\mathcal{A})$ the group of triangular invertible algebra morphisms on $\mathsf{LT}^{\#}(\mathcal{A})$,

$$G(\mathcal{A}) = \{\mathbb{X} \in \mathrm{Hom}_{\mathrm{Alg}}(\mathsf{LT}(\mathcal{A}), \mathsf{LT}(\mathcal{A})) : (\mathbb{X} - \mathrm{id})(\mathsf{LT}(\mathcal{A})(\tau)) \subset \bigoplus_{\tau' \subset \tau} \mathsf{LT}(\mathcal{A})(\tau')\} \tag{3.14}$$

and denote $\bar{\mathbf{X}}_{st} := \mathbb{X}^{|\bullet}_{st}$. Then, for all pairs of times $s < t$, $\mathbb{X}_{st} \in G(\mathcal{A})$, and $\bar{\mathbf{X}}_{st} = \bar{\mathbf{X}}_{ut} \circ \mathbb{X}_{su}$.

3.4. **Representations of the monoid of leveled forests.**

Theorem 3.14. *Let $X : [0,1] \to \mathcal{A}$ be a smooth path. With the notation introduced so far, define $\mathbb{X}_{st} : \mathsf{LF} \to \mathrm{End}^2_{\mathcal{A}}$ by $\mathbb{X}_{st}(f) = \mathbb{X}^f_{st}$, $f \in \mathsf{LF}$, then*

$$\mathbb{X}_{st} = \nabla_{\mathrm{End}^2(\mathcal{A})} \circ (\mathbb{X}_{ut} \varobar \mathbb{X}_{su}) \circ \Delta,\ \nabla_{\mathrm{End}^2(\mathcal{A})} \circ (\mathbb{X}_{st} \varobar \mathbb{X}_{st}) = \mathbb{X}_{st} \circ \nabla$$

Proof. The first assertion follows directly from Proposition 3.4 and the definition of the co-product Δ. The second one follows from the shuffle identity for iterated integrals of X (seen as tensors) that we recall here, with $\sigma = \sigma_1 \otimes \sigma_2$, $\sigma_1 \in \mathcal{S}_k, \sigma_2 \in \mathcal{S}_l$,

$$\begin{aligned}\int_{\Delta^k_{st}} \int_{\Delta^l_{st}} \mathrm{d}X_{u_{s^{-1}(\sigma(1))}} \otimes \cdots \otimes \mathrm{d}X_{u_{s^{-1}(\sigma(k+l))}} \\ = \sum_{v \in \mathsf{Sh}(k,l)} \int_{\Delta^{k+l}_{st}} \mathrm{d}X_{u_{(v \circ s^{-1} \circ \sigma)(1)}} \otimes \cdots \otimes \mathrm{d}X_{u_{(v \circ s^{-1} \circ \sigma)(k+l))}}\end{aligned}$$

□

3.5. **Operators of faces contractions.** The Taylor expansion of a solution Y of an equation in the class we consider only involves the full contraction operators, the operators built on iterated integrals associated to trees. However, we explain in the previous section that in order to write the Chen relation for these operators, we have to consider partial contraction operators indexed by forests. These operators appear as coefficients of an endomorphism $\bar{\mathbb{X}}$ acting on $\mathsf{LT}(\mathcal{A})$. These "coefficients" associated with forests can not be related to the "coefficients" associated with trees if $\mathcal{A}$ is truly infinite dimensional. More formally,

$$\mathbb{X}_{st} \mapsto \mathbb{X}_{st}|^{\bullet} \tag{3.15}$$

is not injective. In this section, we explain how to remedy to this problem, and make operators associated to forests "technical proxys" for the Chen relation, that can be constructed from the operators associated to leveled trees.

To achieve this, we define a new collections $\mathcal{FC}$ of operators, that we call **faces contractions** and remember that for such a collection we denote by FC the graded vector space

$$\mathsf{FC} = \sum_{n \geq 0} \mathcal{FC}(n).$$

Finally, to the iterated integrals of X we associate a trajectory on a subgroup $F(\mathsf{FC})$ of invertible triangular **algebra morphisms** (with respect to a product we define) on $\mathsf{LT}^{\#}(\mathsf{FC})$.

Recall that to keep notations contained, we identify leveled trees in LT and permutations.

Definition 3.15. Let $\tau \in \mathsf{LT}$ be a leveled tree with at least one generation and pick $A_1 \otimes \cdots \otimes A_{(|\tau|)} \in \mathcal{A}^{|\tau|}$ and define the continuous linear map

$$\sharp(A_1 \otimes \cdots \otimes A_{|\tau|} \cdot \tau) : \mathcal{A}^{\otimes |\tau|} \to \mathcal{A}$$

by, for $X_1, \ldots, X_{\|\tau\|} \in \mathcal{A}$,

$$\sharp((A_1 \otimes \cdots \otimes A_{|\tau|}) \cdot \tau)(X_1, \ldots, X_{\|\tau\|}) = A_1 \cdot X_{\tau(1)} \cdots \cdot X_{\tau(\|\tau\|)} \cdot A_{|\tau|}.$$

In addition, we set $\mathsf{FC}(\bullet) = \mathcal{A}$. We denote by $\mathsf{FC}(\tau)$ the closure, with respect to the operator norm, in the Banach space of all multilinear maps on $\mathcal{A}$ of the space of all **τ-faces contractions**, specifically

$$\mathsf{FC}(\tau) = \mathrm{Cl}\left(\left\{\sharp((A_1 \otimes \cdots \otimes A_{|\tau|}) \cdot \tau),\ A_1 \otimes \cdots \otimes A_{|\tau|} \in \mathcal{A}^{|\tau|}\right\}\right).$$

and set

$$\mathsf{LT}^{\#}(\mathsf{FC}) = \bigoplus_{\tau \in \mathsf{LT}} \mathsf{FC}(\tau).$$

Remark 3.16. Notice that the operator $\sharp(A_1 \otimes \cdots \otimes A_{|\tau|})$ has $\|\tau\| = |\tau| - 1$ inputs. It can be pictorially represented by drawing the leveled tree τ and placing the $A^{'s}$ up to the leaves of τ and the $X's$ in the faces of τ; X_i is located on the i^{th} generation of τ. Whereas in the previous section, arguments of the multilinear operators we considered where located on the leaves, in this section they are located on the faces of a tree.

Let $n \geq 1$ an integer and pick σ a permutation in $\mathcal{S}_n$. With τ_σ the leveled tree associated to σ^{-1}, $\sigma \cdot \tau_\sigma$ is a comb tree associated to the identity permutation. The permutation σ acts on faces contractions operators by sending $\sharp(A_1 \otimes \cdots \otimes A_n \cdot \tau)$ in $\mathsf{FC}(\tau)$ to $\sharp(A_1 \otimes \cdots \otimes A_n \cdot \sigma \cdot \tau)$ in $\mathsf{FC}(\sigma \cdot \tau)$ with $\|\tau\| = n$. Thus

$$\sigma : \bigoplus_{\substack{\tau \in \mathsf{LT} \\ \|\tau\| = n}} \mathsf{FC}(\tau) \to \bigoplus_{\substack{\tau \in \mathsf{LT} \\ \|\tau\| = n}} \mathsf{FC}(\sigma \cdot \tau)$$

is a continuous invertible operator. Hence, the linear map

$$\begin{array}{rccc} \phi: & \mathsf{LT}^{\#}(\mathsf{FC}) & \to & \bigoplus_{\tau \in \mathsf{LT}^{\#}(\mathsf{FC})} \mathsf{FC}(\mathrm{id}_{\|\tau\|}) \otimes \tau \\ & \sum_{\tau \in \mathsf{LT}} m_\tau & \mapsto & \sum_{\tau \in \mathsf{LT}} \tau^{-1}(m_\tau) \otimes \tau \end{array}$$

is a continuous ismorphism too. Hence, if we define the collection $\mathcal{FC}$ by setting for $n \geq 0$ an integer

$$\mathcal{FC}(n) = \mathsf{FC}(\mathrm{id}_n),$$

we sees that $\mathsf{LT}^{\#}(\mathsf{FC})$ is in fact the graded vector space associated to the Hadamard product of $\mathcal{LT}^{\#}$ with $\mathcal{FC}$. In the following, we denote by $\sharp$ the endormorphsim of graded vector spaces

$$\begin{array}{rccc} \sharp: & \mathsf{LT}^{\#}(\mathsf{FC}) & \to & \mathsf{LT}^{\#}(\mathcal{A}) \\ & A_1 \otimes \cdots \otimes A_{(|\tau|)} & \mapsto & \sharp(A_1 \otimes \cdots \otimes A_{(|\tau|)}) \end{array} \tag{3.16}$$

Notation. Pick $\mathbb{X}$ an endomorphism of $\mathsf{LT}^{\#}(\mathsf{FC})$ and define the **components of** $\mathbb{X}$,

$$\mathbb{X}(\tau', \tau) : \mathsf{FC}(\|\tau\|) \to \mathsf{FC}(\|\tau'\|),\ \tau' \subset \tau$$

by requiring that

$$\mathbb{X}(m^\tau) = \sum_{\tau', \tau} \phi\Big(\mathbb{X}_{st}(\tau', \tau)\Big(\phi^{-1}(m^\tau)\Big) \otimes \tau'\Big),\ m^\tau \in \mathsf{FC}(\tau). \tag{3.17}$$

It will be convenient in the following to discuss either on the components of X, either on the restrictions - corestrictions of X

$$\mathbb{X}_{|\alpha}^{|\beta} : \mathsf{FC}(\alpha) \to \mathsf{FC}(\beta),\ \alpha, \beta \in \mathsf{LT}$$

We continue by defining a product on $\mathsf{LT}^{\#}(\mathsf{FC})$.

Proposition 3.17. *Let $\alpha, \beta \in \mathsf{LT}$ two leveled trees and $A \in \mathcal{A}^{\otimes|\alpha|}$, $B \in \mathcal{A}^{\otimes|\beta|}$, then for any tuple $X_1, \ldots, X_{\|\alpha\|+\|\beta\|}$ one has*

$$\begin{aligned} &\sharp((A \cdot \alpha) \sqcup (B \cdot \beta))(X_1, \ldots, X_{||\alpha||+||\beta||}) \\ &= \sum_{s \in \mathsf{Sh}(\|\alpha\|, \|\beta\|)} \sharp(A \cdot \alpha)(X_{s(1)}, \ldots, X_{s(\|\alpha\|)}) \cdot \sharp(B \cdot \beta)(X_{s(\|\alpha\|+1)}, \ldots, X_{s(\|\alpha\|+\|\beta\|)}) \end{aligned}$$

Besides, the following estimates holds

$$\|\sharp(A \cdot \alpha \sqcup B \cdot \beta)\| \leq \frac{(\sharp\alpha + \sharp\beta)!}{\sharp\alpha! \sharp\beta!} \|\sharp A \cdot \alpha\| \|\sharp A \cdot \beta\|.$$

Thanks to Proposition 3.17, there exists a product $\sqcup$ on $\mathsf{LT}^{\#}(\mathsf{FC})$ for which $\sharp$ is an algebra morphism.

Definition 3.18 (Shuffle product on faces contractions operators). Pick $m_\alpha \in \mathsf{FC}(\alpha)$ and $m_\beta \in \mathsf{FC}(\beta)$ two faces contractions operators and define

$$m^\alpha \sqcup m^\beta := \sum_{s\in \mathsf{Sh}(\|\alpha\|,\|\beta\|)} m^\alpha(X_{s(1)},\dots,X_{s(\|\alpha\|)})\cdot m^\beta(X_{s(\|\alpha\|+1)},\dots,X_{s(\|\alpha\|+\|\beta\|)})$$

we call the operation **shuffle product on faces contractions**

The involution $\star$ on $\mathcal{A}$ induces an involution on $\mathsf{LT}^{\#}(\mathsf{FC})$ turning $\sharp$ into a morphism of involutive algebras. Pick α a leveled tree. Recall that $\bullet(\alpha)$ denotes the tree obtained by horizontal mirror symmetry of α. Define for any contraction operator $m^\alpha \in \mathsf{FC}(\alpha)$ the faces contractions operator $\star(m^\alpha)$ in $\mathsf{FC}(\bullet(\alpha))$ by

$$\begin{aligned}\star(m^\alpha)(X_1\otimes\ldots\otimes X_{\|\alpha\|}) &= \star_{\mathcal{A}}(m^\alpha(\star_{\mathcal{A}^{\otimes\|\alpha\|}}(X_1\otimes\cdots\otimes X_{\|\alpha\|})))\\ &= \star_{\mathcal{A}}(m^\alpha(\star_{\mathcal{A}^{\otimes\|\alpha\|}}(X^\star_{\|\alpha\|}\otimes\cdots\otimes X^\star_1))),\ X_1\otimes\cdots\otimes X_{\|\alpha\|}\in\mathcal{A}^{\otimes\|\alpha\|}\end{aligned}$$

Proposition 3.19. *The quadruple* $(\mathsf{LT}^{\#}(\mathsf{FC}),\sqcup,\star,\|\ \|)$ *is a* $C^\star$*- algebra.*

Proof. We only check compatibility between the involution $\star$ and the product $\sqcup$. Let α,β two leveled trees and pick $m^\alpha\in\mathsf{FC}(\alpha)$ and $m^\beta\in\mathsf{FC}(\beta)$. Then

$$\begin{aligned}&\star(m^\alpha\sqcup m^\beta)(X_1\otimes\cdots\otimes X_{\|\alpha\|+\|\beta\|})\\ &\quad=\sum_{s\in\mathsf{Sh}(\|\alpha\|,\|\beta\|)} m^\beta(X^\star_{s(\|\alpha\|+\|\beta\|)},\dots,X^\star_{s(\|\alpha\|+1)})^\star\cdot m^\alpha(X^\star_{s(\|\alpha\|)},\dots,X^\star_1)^\star\\ &\quad=\sum_{s\in\mathsf{Sh}(\|\alpha\|,\|\beta\|)}\star(m^\beta)(X_{s(\|\alpha\|+1)},\dots,X^\star_{s(\|\alpha\|+\|\beta\|)})\cdot\star(m^\alpha)(X_{s(1)},\dots,X^\star_{\|\alpha\|})\\ &\quad=\star(m^\beta)\sqcup\star(m^\alpha)\,.\end{aligned}$$

Therefore we prove the result. ∎

Remark 3.20. Definition 3.8 introduces an operadic composition on the collection $\mathcal{FS}$. We define an operadic composition, that we denote by the symbol $\tilde{L}$, on the collection $\mathcal{FC}$ of faces contractions operator induced by the canonical operadic structure on $\mathrm{End}_{\mathcal{A}}$,

$$\tilde{L}(V\circ(W_1\otimes\cdots\otimes W_p)) = V\circ(W_1\otimes\cdots\otimes W_p), \tag{3.18}$$

where $V\in\mathsf{FC}(p)$, $W_i\in\mathsf{FC}(n_i)$ $1\le i\le p$ and the symbol $\circ$ in the right hand side of the above equation stands for the composition in $\mathrm{End}_{\mathcal{A}}$. We set FC$=(\mathcal{FC},\tilde{L},\mathrm{id}_{\mathcal{A}})$. Notice that with this definition, the $\sharp$ operator is a morphism between the operads FS and FC, namely, for $A_1,\dots,A_{p+1}\in\mathcal{A}$ and $W_1,\dots,W_p\in\mathcal{FS}$

$$\tilde{L}(\sharp(A_1\otimes\cdots\otimes A_{p+1})\circ\sharp W_1\otimes\cdots\otimes\sharp W_p) = \sharp L(A_1\otimes\cdots\otimes A_p\circ(W_1\otimes\cdots\otimes W_p)) \tag{3.19}$$

We use the same formula (3.8) to define the endomorphism $\tilde{\mathsf{L}}:\mathsf{LT}^{\#}(\mathsf{FC})\to\mathsf{LT}(\mathsf{FC})$.

Denote by $T(\mathsf{FC})$ **the group of invertible triangular algebra morphisms** on $(\mathsf{LT}^{\#}(\mathsf{FC}),\sqcup)$.

$$T(\mathsf{FC}) = \{\alpha\in\mathrm{End}_{\mathrm{Alg}}(\mathsf{LPBT}(\mathsf{FC})) : (\alpha-\mathrm{id})(\mathsf{FC}(\tau))\subset\bigoplus_{\tau'\subsetneq\tau}\mathsf{FC}(\tau')\},$$

Let $k\ge1$ an integer and pick k elements of $\mathcal{A}$, $A_1,\dots,A_k\in\mathcal{A}$. Define the following operators acting on $\mathsf{LT}^{\#}(\mathsf{FC})$:

$$L_{A_1,\dots,A_k}:\mathsf{LT}^{\#}(\mathsf{FC})\to\mathsf{LT}^{\#}(\mathsf{FC}) \tag{3.20}$$

by, for m^τ a faces-contraction operator in $\mathsf{FC}(\tau)$, $\tau\in\mathsf{LT}$ and $\tau'\in\mathsf{LT}$,

$$\begin{aligned}&L_{A_1,\dots,A_k}|^{\tau'}_{\tau}(m)(X_1,\dots,X_{\|\tau'\|}) = m(X_1,\dots,X_{\|\tau'\|},A_1,\dots,A_k) &&\text{if } \|\tau\|=\|\tau'\|+k,\\ &L_{A_1,\dots,A_k}|^{\tau'}_{\tau}(m)(X_1,\dots,X_{\|\tau'\|}) = 0 &&\text{otherwise}\end{aligned}\quad.$$

Notice that the norm of such an operator satisfies $\|L_{A_1,\ldots,A_k}\| \leq \|A_1 \otimes \cdots \otimes A_k\|$. Hence,

$$L^k : \mathcal{A}^{\otimes k} \ni (A_1 \otimes \cdots \otimes A_k) \mapsto L_{A_1,\ldots,A_k}$$

is well defined and continuous. We call $\mathcal{P}$ the closure for the operator norm of the direct sum of the ranges of the operators L^k :

$$\mathcal{P} = C\ell(\bigoplus_{k\geq 0} \mathrm{Im}(L^k)) \tag{3.21}$$

The space $\mathcal{P}$ is a Banach algebra, since $L_{A_1,\ldots,A_k} \circ L_{B_1,\cdots,B_q} = L_{A_1,\ldots,B_q}$. In addition, from the very definition of $L_{A_1,\ldots,A_k}$, $L_{A_1,\ldots,A_q}(\tau',\tau)$ depends only on the forest $\tau\backslash\tau'$.

Lemma 3.21. *Pick $\tilde{\mathbb{X}}$ and $\tilde{\mathbb{Y}}$ two operators in $\mathcal{P}$. Then, by setting for all pairs of leveled forest $f, f' \in \mathsf{LF}$ such that $f \oplus f' \in \mathsf{LF} \oplus \mathsf{LF}$ is well defined,*

$$((\tilde{\mathbb{X}} \otimes \tilde{\mathbb{Y}}), f \oplus f') = \tilde{\mathbb{X}}(f) \circ \tilde{\mathbb{X}}(f').$$

one has

$$((\tilde{\mathbb{X}} \otimes \tilde{\mathbb{Y}}), f \oplus f') = ((\tilde{\mathbb{Y}} \otimes \tilde{\mathbb{X}}), \mathsf{K}(f \oplus f')), \quad f \oplus f' \in \mathsf{LF} \oplus \mathsf{LF}\,. \tag{3.22}$$

Proof. Let $A_1, \ldots, A_{|f'|} \in \mathcal{A}$ and call σ (resp. σ') the permutation associated with $f_\flat$ (resp. $f'_\flat$). We use the notation cb_n for the right-comb tree associated with the identity permutation id_n. Next, define s the permutation in $\mathcal{S}_{\|f\|+\|f'\|}$ by

- $s_{f \oplus f'}(k) = i$, if the k^{th} face of $cb_{\mathsf{nt}(f)})\#f\#f'$ (reading the faces from left to right) is the i^{th} face of f,
- $s_{f \oplus f'}(k) = \|f\| + i$ if the k^{th} face of $f\#f'$ is the i^{th} face of f',
- $s_{f \oplus f'}(k) = \|f\| + \|f'\| + i$ if the k^{th} face is the i^{th} face of $cb_{\mathsf{nt}(f)}$.

With $\mathsf{K}(f \oplus f') = f'_{(1)} \oplus f_{(1)}$, with $\|f'_1\| = \|f'\}$ and $\|f\| = \|f_{(1)}\|$ notice that

$$f'^{\ \flat}_{(1)} = f'^{\flat},\ f_{(1)\flat} = f_\flat,\ s_{\mathsf{K}(f \oplus f')} = s_{f \oplus f'}.$$

Notice that $((\tilde{\mathbb{X}} \otimes \tilde{\mathbb{Y}}), f \oplus f')$ is non-zero only of pair of forests $f \boxtimes f'$ with $\|f\| = p$ and $\|f'\| = q$. Pick two such forests f, f'. Pick $U_1, \ldots, U_{\mathsf{nt}(f)-1} \in \mathcal{A}$. Pick $\mathbb{X} = L_{X_1,\ldots,X_p}$ and $\mathbb{Y} = L_{Y_1,\ldots,Y_q}$ two operators in $\mathcal{P}$ and put $Z = (U_1, \ldots, U_{\mathsf{nt}(f)}, X_1, \ldots, X_p, Y_1, \ldots, Y_q)$. One has

$$\begin{aligned}
&((\tilde{\mathbb{X}} \otimes \tilde{\mathbb{Y}}), f \oplus f')(\sharp(A_1 \otimes \cdots \otimes A_{|f'|}))(U_1, \ldots, U_{\mathsf{nt}(f)-1}) \\
&\quad = A_1 \cdot Z_{s^{-1}_{f \oplus f'}(1)} \otimes \cdots \otimes Z_{s^{-1}_{f \oplus f'}(\|f\|+\|f'\|)} \cdot A_{|f'|} \\
&\quad = ((\tilde{\mathbb{Y}}_{st} \otimes \tilde{\mathbb{X}}_{st}), \mathsf{K}(f \oplus f'))(\sharp(A_1 \otimes \cdots \otimes A_{|f'|}))(U_1, \ldots, U_{\mathsf{nt}(f)-1})
\end{aligned}$$

□

In the following, we use the notation

$$T_{\mathcal{P}}(\mathsf{FC}) = T(\mathsf{FC}) \cap \mathcal{P}. \tag{3.23}$$

for the set of triangular algebra morphisms of $(\mathsf{LT}^{\#}(\mathsf{FC}), ⊔⊔)$ in the Banach algebra $\mathcal{P}$.

Proposition 3.22. *Endow $T_{\mathcal{P}}(\mathsf{FC})$ with the involution:*

$$\star\,(E) = \star \circ E \circ \star \tag{3.24}$$

Then, first, $\star : T_{\mathcal{P}}(\mathsf{FC}) \to T_{\mathcal{P}}(\mathsf{FC})$ is well defined and $\star$ is an algebra morphism.

Proof. Direct consequence of the definition of the poduct ⊔⊔ . □

Definition 3.23. Pick $X : [0,1] \to \mathcal{A}$ a smooth path. Let $0 < s < t < 1$ be two times and define a triangular endomorphism

$$\tilde{\mathbb{X}}_{st} : \mathsf{LT}^{\#}(\mathsf{FC}) \to \mathsf{LT}^{\#}(\mathsf{FC})$$

in $\mathcal{P}$ determined by, for $m^\alpha \in \mathsf{FC}(\alpha)$ and $\alpha \in \mathsf{LT}$,

$$\tilde{\mathbb{X}}_{st}(m^\alpha) = \sum_{\beta \subset \alpha} \tilde{\mathbb{X}}_{st}|^{\beta}_{\alpha}(m^\alpha),\ \tilde{\mathbb{X}}_{st}|^{\beta}_{\alpha} : \mathsf{FC}(\alpha) \to \mathsf{FC}(\beta),$$

$$\tilde{\mathbb{X}}_{st}|^{\beta}_{\alpha}(m^\alpha)(X_1,\ldots,X_{\|\beta\|}) = \int_{s<t_1<\cdots t_{\|\alpha\|-\|\beta\|}<t} m^\alpha(X_1,\ldots,X_{\|\beta\|},\mathrm{d}X_{t_1},\ldots,\mathrm{d}X_{t_{\|\alpha\|-\|\beta\|}}) \tag{3.25}$$

with $X_1,\ldots,X_{\|\tau\|} \in \mathcal{A}$. See Fig. 7 for a picture representing the action of $\tilde{\mathbb{X}}_{st}$.

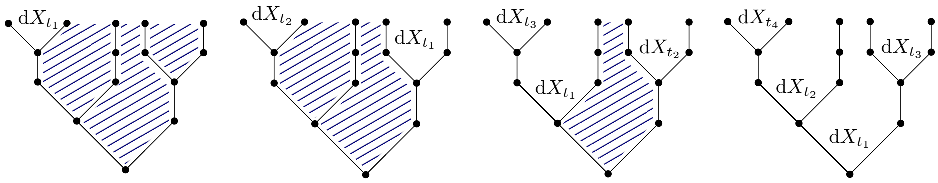

FIGURE 7. Faces contractions of a leveled forest

It is immediate to infer from equation (3.25) that the components of $\tilde{\mathbb{X}}_{st}(\alpha,\beta)$ and $\tilde{\mathbb{X}}_{st}(\beta',\alpha')$ are equal provided that $\beta\backslash\alpha = \beta'\backslash\alpha'$ and we use the notation $\tilde{\mathbb{X}}_{st}(\alpha\backslash\beta)$ for the common value. To be complete, if f is a leveled forest, then

$$\tilde{\mathbb{X}}_{st}(f) : \mathsf{FC}(\|f\| + \mathsf{nt}(f) - 1) \to \mathsf{FC}(\mathsf{nt}(f) - 1) \tag{3.26}$$

Set for any time $0 < t < 1$,

$$\tilde{\mathbf{X}}_{st} = \sum_{\tau \in \mathsf{LT}} \tilde{\mathbb{X}}_{st}|^{\bullet}_{\tau}.$$

Theorem 3.24. *Let X be a smooth path. Then $\tilde{\mathbb{X}}_{st}$ is an algebra morphism for any pair of times $0 < s < t < 1$ and*

$$\tilde{\mathbb{X}}_{st} \in T_{\mathcal{P}}(\mathsf{FC}),\ \mathbb{X}_{st} = \tilde{\mathbb{X}}_{ut} \circ \tilde{\mathbb{X}}_{su}\,. \tag{3.27}$$

Besides, $\sharp \circ \bar{\mathbb{X}}_{st} = \tilde{\mathbb{X}}_{st} \circ \sharp,\ s,t \in [0,1]^2$ and if X is a trajectory of self-adjoint operators then $\tilde{\mathbb{X}}_{st}$ is a morphism of $C^\star$-algebras.

Proof. The two first assertion are direct consequence of the equivariance property $\sharp \circ \bar{\mathbb{X}}_{st} = \tilde{\mathbb{X}}_{st} \circ \sharp,\ s,t \in [0,1]$ which is downward to check. $\square$

Notice that Corollary 3.24 implies, together with 3.12 that

$$\tilde{\mathbb{X}}_{st} \circ \tilde{\mathsf{L}} = \tilde{\mathsf{L}} \circ \mathrm{id} \circ \tilde{\mathbb{X}}_{st},\ s,t \in [0,1]. \tag{3.28}$$

Also, Lemma 3.21 implies that for two smooth paths $X, Y : [0,1] \to \mathcal{A}$

$$((\tilde{\mathbb{X}}_{st} \otimes \tilde{\mathbb{Y}}_{st}), f \oplus f') = ((\tilde{\mathbb{Y}}_{st} \otimes \tilde{\mathbb{X}}_{st}), \mathsf{K}(f \oplus f')),\quad f \oplus f' \in \mathsf{LF} \oplus \mathsf{LF}, \tag{3.29}$$

Let f be a leveled forest with n generations and $\mathsf{nt}(f)$ forests. Denote by n_i the number of generations of the i^{th} tree in f and set $\ell^f_j = \sum_{i=1}^{j} n_i + j$ for $1 \leq j \leq \mathsf{nt}(f) - 1$. In the following proposition, we denote by $\circ$ the operadic composition $\tilde{L}$. Recall that elements of $\mathcal{A}$ are considered as faces contractions operators with 0 inputs. Pick m is a faces contraction operator of arity p, $m_1,\ldots,m_q$ faces contraction operators and a sequece of integers $1 \leq i_1 < \ldots < i_q \leq p$, we denote by

$$m \circ_{i_1,\ldots,i_q} m_1 \otimes \cdots \otimes m_q \tag{3.30}$$

the operator obtained by connecting m_j at the i_j^{th} input of m. Recall that we denote by $f^\flat$ the forest obtained by gluing together along their external edges the trees of f. We define a partial gluing operations which consists in gluing a subset of trees of f together, and index by interval of integers in $[\![1, \mathsf{nt}(f) - 1]\!]$. We denote by f^I, $I \subset [\![1, \mathsf{nt}(f) - 1]\!]$ the forest obtained by gluing the trees of f with index contained in I along their external edges.

Proposition 3.25. *Let $m \in \mathsf{FC}(n)$ be a faces contraction operator, f a forest satisfying $n = \|f\| + \mathsf{nt}(f) - 1$, $I = \{i_1 < \cdots < i_p\} \subset [\![1, \mathsf{nt}(f) - 1]\!]$ an interval of integers and $A_1, \ldots, A_{\mathsf{nt}(f)-1}$ elements of $\mathcal{A}$. Then for any pair of times $0 < s < t < 1$,*

$$\tilde{\mathbb{X}}_{st}(f)(m) \circ_{i_1,\ldots,i_p} (A_1, \ldots, A_{i_p}) = \tilde{\mathbb{X}}_{st}(f^I)(m \circ_{\ell_{i_1},\ldots,\ell_{i_p}} A_I) \tag{3.31}$$

In particular,

$$\tilde{\mathbb{X}}_{st}(f)(m)(A_1, \ldots, A_{\mathsf{nt}(f)-1}) = \tilde{\mathbb{X}}_{st}(f^\flat)(m \circ_{\ell_1,\ldots,\ell_{\mathsf{nt}(f)-1}} A_1 \otimes \cdots \otimes A_{\mathsf{nt}(f)-1}) \tag{3.32}$$

The above proposition implies that the endomorphism $\tilde{\mathbb{X}}_{st}$ is characterized by its values on the leveled trees. Partial contractions are thus technical proxys required to write the Chen relation for the operator $\tilde{\mathbb{X}}_{st}$ but bear no additional data on the small scale behaviour of X. This is compliant with the simple observation that expansion of a solution of an equation in the class (1.2) does only involve full contractions.

Besides the fact that for all pair of times $s < t$, $\tilde{\mathbb{X}}_{st}$ is a triangular algebra morphisms, we observed to other properties : the first one, equation (3.22), stipulates exchange relations between operators $\tilde{\mathbb{X}}$. The second one is equation (3.31). Whereas it is immediate to define an abstract set of operators (without referring to the path X) satisfying (3.31), it is more difficult when it comes to (3.22).

Denote by $G(\mathsf{FC})$ the set of all triangular operators $\tilde{\mathbb{X}} : \mathsf{LT}^{\#}(\mathsf{FC}) \to \mathsf{LT}^{\#}(\mathsf{FC})$ in $T_{\mathcal{P}}(\mathsf{FC})$ satisfying equation (3.31) and by $G_\star(\mathsf{FC})$ the subgroup of self-adjoint operators in $G(\mathsf{FC})$.

Proposition 3.26. *The sets $G(\mathsf{FC})$ and $G_\star(\mathsf{FC})$ are sub-groups of the group of triangular invertible endomorphisms of $\mathsf{LT}^{\#}(\mathsf{FC})$.*

Proof. Pick two endomorphisms $\tilde{\mathbb{X}}$ and $\tilde{\mathbb{Y}}$ in $G(\mathsf{FC})$. Pick f a leveled forest and $I = \{i_1 < \cdots i_p\} \subset [\![1, \mathsf{nt}(f) - 1]\!]$. We show that $\tilde{\mathbb{X}} \circ \tilde{\mathbb{Y}} \in G(\mathsf{FC})$. One has

$$\begin{aligned}
((\tilde{\mathbb{X}} \circ \tilde{\mathbb{Y}})(f))(m) \circ_{i_1,\ldots,i_p} (A_I) &= \sum_{f' \subset f} \tilde{\mathbb{X}}(f')\Big(\tilde{\mathbb{Y}}(f \setminus f')(m)\Big) \circ_{i_1,\ldots,i_p} (A_I) \\
&= \sum_{f' \subset f} \tilde{\mathbb{X}}(f'^I)\Big(\tilde{\mathbb{Y}}(f \setminus f')(m) \circ_{\ell^{f'}_{i_1},\cdots,\ell^{f'}_{i_p}} A_I\Big) \\
&= \sum_{f' \subset f} \tilde{\mathbb{X}}(f'^I)(\tilde{\mathbb{Y}}(((f \setminus f')^{\ell^{f'}_{i_1},\cdots,\ell^{f'}_{i_p}}))\Big(m \circ_{\ell^f_{\ell^{f'}_{i_1}},\ldots,\ell^f_{\ell^{f'}_{i_p}}} (A_I)\Big)
\end{aligned}$$

Owing to associativity of $\circ$, we have $\ell^f_{\ell^{f'}_{i_1}}, \ldots, \ell^f_{\ell^{f'}_{i_p}} = \ell_{i_1}^{f' \bigoplus f}, \ldots, \ell_{i_p}^{f' \bigoplus f}$. The statement follows by noticing that

$$\{\Big(f'^I, (f \setminus f')^{\ell^{f'}_{i_1},\ldots,\ell^{f'}_{i_p}}\Big),\ f' \subset f\} = \{(f', f \setminus f'),\ f' \subset f^I\}.$$

$\square$

4. Appendix

We recall some definitions from the theory of operads and more generally, we underline here the categorical notions we use in this work. The reader will find below, among other things, definitons of collections, operads, bi-collections and PROSs. All of concepts are standard in the algebraic literature, see e.g. the monographies [10, 1], but not very known between non-algebraists. Hence the need of this small appendix. For further details we refer to [14, 3].

At the base of these definitions above lies the concept of **monoidal category**. In loose words, it is a category $\mathbf{C} = (\mathrm{Ob}(\mathbf{C}), \mathrm{Mor}(\mathbf{C})))$ equipped with an operation $\bullet$ and a unity element $I \in \mathrm{Ob}(\mathbf{C})$. The operation $\bullet$ associates to any couple of objects $A, B \in \mathrm{Ob}(\mathbf{C})$ a object $A \bullet B \in \mathrm{Ob}(\mathbf{C})$ and to any couple of morphisms $f\colon A \to A'$, $g\colon B \to B'$ a morphism $f \bullet g\colon A \bullet B \to A' \bullet B'$ in a functorial way. In order that $(\mathbf{C}, \bullet, I)$ is a monoidal category, the operation $\bullet$ must satisfy two main properties, which emulate the tensor product operation on finite dimensional vector spaces:

1. (Associativity constraints) for any triple of objects $A, B, C \in \mathrm{Ob}(\mathbf{C})$ one has that the object $(A \bullet B) \bullet C$ is isomorphic to $A \bullet (B \bullet C)$ in **in a functorial way**, that is there exists a natural isomorphism between the two functors $\bullet \circ (id \times \bullet)$ and $\bullet \circ (\bullet \times id)$;
2. (Unitaly constraints) for any object $A \in \mathrm{Ob}(\mathbf{C})$ the objects $A \bullet I$ and $I \bullet A$ are (naturaly) isomorphic to A.

The prototypical example is the category of finite dimensional vector spaces with monoidal product given by the tensor product of vector spaces. Another example is the category Set, the category of all sets with functions between sets as morphisms, with monoidal product given by the cartesian product of sets.[1] Of interest in the present work is the 2-monoidal category of collections and bicollections that we now define.

A monoid in a monoidal category is a categorical abstraction of a binary product on a set.

Definition 4.1 (Monoid)**.** A **monoid** in a monoidal category $(\mathcal{C}, \otimes, I)$ is a triple (C, ρ, η) with $C \in \mathbf{Ob}(\mathcal{C})$, $\rho : C \otimes C \to C$, $\eta : I \to C$ meeting the constraints

1. $\rho \circ (\rho \otimes \mathrm{id}) = \rho \circ \mathrm{id} \otimes \rho$,
2. $\rho \circ (\eta \otimes \mathrm{id}) = \mathrm{id}$

Definition 4.2 (Comonoid)**.** A **comonoid** in a monoidal category $(\mathcal{C}, \otimes, I)$ is a triple (C, Δ, ε) with $C \in \mathbf{Obj}(\mathcal{C})$, $\Delta : C \to C \otimes C$, $\varepsilon : C \to I$ meeting the constraints:

1. $\Delta \otimes \mathrm{id} \circ \Delta = \mathrm{id} \otimes \Delta \circ \Delta$,
2. $\varepsilon \otimes \mathrm{id} \circ \Delta = \mathrm{id} \otimes \varepsilon \circ \Delta$

Definition 4.3. We call a (reduced) **collection** P a sequence of complex vector spaces[2] $\{P(n)\}_{n\geq 1}$. A morphism between two collections P, Q is a sequence of linear maps $\{\phi(n)\}_{n\geq 1}$ with $\phi(n) : P(n) \to Q(n)$,$n \geq 1$. For any couple of morphisms between collections we define the composition of morphisms by composing each component. We denote the category of collections by Coll.

The category Coll has a natural monoidal structure $\odot$ over it: for any couple of collections P and Q and morphisms f, g we define

$$(P \odot Q)(n) := \bigoplus_{\substack{k\geq 1 \\ n_1+\cdots+n_k=n}} P(k) \otimes Q(n_1) \otimes \cdots \otimes Q(n_k)\,,$$

$$(f \odot g)(n) := \bigoplus_{\substack{k\geq 1 \\ n_1+\cdots+n_k=n}} f(k) \otimes g(n_1) \otimes \cdots \otimes g(n_k)\,.$$

Denoting by $\mathbb{C}_\odot$ the collection

$$\mathbb{C}_\odot = \begin{cases} \mathbb{C} & \text{if } n = 1 \\ 0 & \text{otherwise,} \end{cases}$$

it is straightforward to check that the triple $(\mathrm{Coll}, \odot, \mathbb{C}_\odot)$ is a monoidal category. If the vectors spaces of the collections P and Q above are Banach algebras, then we might use in place of the algebraic tensor product $\otimes$ the projective one $\hat{\otimes}$.

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[1] This monoidal category is particular in the sense that the monoidal product coincides with the cateogrial product. Such categories are called cartesian monoidal.

[2] The original definition involves vector spaces over a generic field but we consider only complex vector spaces, in accordance with the structures presented so far.

An operad is a monoid in the monoidal category $(\mathrm{Coll}, \odot, \mathbb{C}_\odot)$:

Definition 4.4. A non-symmetric **operad** (or simply an operad) is a monoid in the monoidal category $(\mathrm{Coll}, \odot, \mathbb{C}_\odot)$, i.e. a triple (P, ρ, η_P) of the following objects

$$P \in \mathrm{Ob}(\mathrm{Coll}), \quad \rho\colon P \odot P \to P, \quad \eta_P\colon \mathbb{C}_\odot \to P,$$

satisfying the properties $(\rho \odot \mathrm{id}_P) \circ \rho = (\mathrm{id}_P \odot \rho) \circ \rho$ and $(\eta_P \odot \mathrm{id}_P) \circ \rho = (\mathrm{id}_P \odot \eta_P) \circ \rho = \mathrm{id}_P$.

We keep the notation $\odot$ for the monoidal operation. It is common in the literature to denote the morphism ρ by $\circ$, i.e. for every $k \geq 1$, $p \in P(k)$ and $q_i \in Q(n_i)$ for $i = 1, \cdots, k$

$$p \circ (q_1 \otimes \cdots \otimes q_n) := \rho(n_1 + \cdots + n_k)(p \otimes q_1 \cdots \otimes q_k).$$

Moreover, for any $1 \leq i \leq k$ and $q_i \in Q(n_i)$ we use also the notation $\circ_i$ to denote partial composition

$$p \circ_i q := p \circ (\eta_P(1)(1)^{\otimes i-1} \otimes q \otimes \eta_P(1)(1)^{\otimes k-i}),$$

where $\eta_P(1)\colon \mathbb{C} \to P(1)$. Since the maps $\rho(n)_{n\geq 0}$ carry multiple inputs and give back one output, it is common in the literature to call them many-to-one operators.

In fact, it is possible to generalise the notion of an operad to model composition between many-to-many operators, that is operators with multiple in- and outputs. This leads us to define the category of bicollections.

Definition 4.5. We call a **bicollection** a two parameters family of complex vector spaces

$$P = \{P(n, m)\}_{n,m\geq 0}.$$

A morphism between two bicollections P, Q is a sequence of linear maps $\{\phi(n, m)\}_{n,m\geq 0}$ with $\phi(n, m) : P(n, m) \to Q(n, m)$. For any couple of morphisms between bicollections we define the composition of morphisms by composing each component. We denote the category of bicollections by Coll_2.

The category of bicollections is endowed with two compatible monoidal structures.

Definition 4.6. For any couple of bicollections P and Q and morphisms f, g we define the **horizontal tensor product** $\ominus$ as follows

$$\begin{aligned} (P \ominus Q)(n, m) &:= \bigoplus_{\substack{n_1+n_2=n \\ m_1+m_2=m}} P(n_1, m_1) \otimes Q(n_2, m_2), \\ (f \ominus g)(n, m) &:= \bigoplus_{\substack{n_1+n_2=n \\ m_1+m_2=m}} f(n_1, m_1) \otimes g(n_2, m_2). \end{aligned} \tag{4.1}$$

together with the horizontal unity

$$\mathbf{C}_\ominus = \mathbf{C}_\ominus(m, n) = \begin{cases} \mathbb{C} & \text{if } n = m = 0 \\ 0 & \text{otherwise.} \end{cases}$$

We define also the **vertical tensor product** $\varobar$

$$\begin{aligned} (P \varobar Q)(n, m) &:= \bigoplus_{k=0}^{+\infty} P(n, k) \otimes Q(k, m), \\ (f \varobar g)(n, m) &:= \bigoplus_{k=0}^{+\infty} f(n, k) \otimes g(k, m). \end{aligned} \tag{4.2}$$

together with the vertical unity

$$\mathbf{C}_\varobar = \mathbf{C}_\varobar(m, n) = \begin{cases} \mathbb{C} & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases}$$

We refer to the triple $(\mathsf{Coll}_2, \ominus, \mathbb{C}_\ominus)$ and $(\mathsf{Coll}_2, \varobar, \mathbb{C}_{\varobar})$ respectively as the category of **horizontal bicollections** and the **vertical bicollections**.

Lemma 4.7. $(\mathsf{Coll}_2, \ominus, \mathbb{C}_\ominus)$ *and* $(\mathsf{Coll}_2, \varobar, \mathbb{C}_{\varobar})$ *are monoidal categories.*

Proof. This is simple computations, based on the fact that $(\mathrm{Vect}_\mathbb{C}, \otimes, \mathbb{C})$ is monoidal. □

Remark 4.8. We point at some core differences and similarities between the tensor product of vector spaces, and the two tensor products we defined on bicollections. If V and W are two vector spaces, there exists an isomorphism of vector spaces $S_{V,W} : V \otimes W \to W \otimes V$. The set $\{S^{\otimes}_{V,W}, V, W \in \mathrm{Vect}_\mathbb{C}\}$ defines a natural transformation, called a symmetry contraint. The vertical tensor product $\varobar$ does not have such symmetry constraints, though we constructed such one but for the monoid generated by the bicollection $\mathcal{LF}$. The horizontal tensor product is symmetric, if V and W are bicollections,

$$S_{V,W} : V \ominus W \to W \ominus V,\ S^{\ominus}(V_n \otimes W_m) = S^{\otimes}(V_n \otimes W_m)$$

Call a category $\mathcal{C}$ a **closed** if for all objects $A, B \in \mathcal{C}$ the set of morphisms

$$\mathrm{Hom}_\mathcal{C}(A, B)$$

is an object of $\mathcal{C}$. The **internal hom** functor denoted $[A, -] : \mathcal{C} \to \mathcal{C}$ is defined by

$$[A, B] = \mathrm{Hom}_\mathcal{C}(A, B),\ [A, f](g) = f \circ g,\ f : B \to C, g : A \to B.$$

A category $\mathcal{C}$ is a **closed monoidal category** if it is closed, monoidal and if the following compatibility holds: for all objects $A, B, C \in \mathcal{C}$

$$\mathrm{Hom}_\mathcal{C}(A, \mathrm{hom}_\mathcal{C}(B, C)) \cong \mathrm{Hom}_\mathcal{C}(A \otimes B, C),$$

with the isomorphism being natural in all three arguments. The category of finite dimensional vector spaces with the usual tensor product is closed monoidal, owing to the fact that the set of linear maps between vector spaces is again a vector space and then using usual identification of bilinear maps with linear maps on the tensor product. Now, neither $(\mathsf{Coll}_2, \ominus, \mathbb{C}_\ominus)$ nor $(\mathsf{Coll}_2, \varobar, \mathbb{C}_{\varobar})$ are closed monoidal. Indeed, they are not even closed, since there is no canonical bigrading on the set of morphisms.

There exists a functor from the category of collections to the category of bicollections, that is the free horizontal monoid functor $T : \mathsf{Coll} \to \mathsf{Coll}_2$, adjoint to the forgetful functor associating to a a monoid (P, γ, η) for the horizontal tensor product $\ominus$ the collection $(P(1, n))_{n\geq 1}$.

Definition 4.9. Let $P = (P_n)_{n\geq 1}$ be a collection, we define the **word bicollection** $T(P)$ by

$$T(P)(m, n) = \bigoplus_{k_1+\cdots+k_n=m} P_{k_1} \otimes \cdots \otimes P_{k_n}\,, \tag{4.3}$$

when $n \geq 1$ and $m \geq 1$. Moreover we set $T(P)(0, 0) = \mathbb{C}$ and $T(P)(m, 0) = T(P)(0, n) = 0$.

Proposition 4.10. *Let C_i, $1 \leq i \leq 4$ be four bicollections, then there exists an explicit morphism*

$$R_{C_1,C_2,C_3,C_4} \colon \big(C_1 \varobar C_2\big) \ominus \big(C_3 \varobar C_4\big) \to (C_1 \ominus C_3) \varobar (C_2 \ominus C_4)\ .$$

We call R_{C_1,C_2,C_3,C_4} the ***exchange law****. Besides, if the bicollections C_2 and C_3 are equal and in the image of the free functor T, one has*

$$\big(C_1 \varobar \mathrm{W}(C)\big) \ominus \big(C_2 \varobar \mathrm{W}(C)\big) \simeq (C_1 \ominus C_2) \varobar \mathrm{W}(C). \tag{4.4}$$

The family of morphisms $\{R_{C_1,C_2,C_3,C_4}, C_i \in \mathrm{Coll}_2\}$ defines a **natural transformation** (which is, in general, not an isomorphism) between the functors $\ominus \circ \varobar \times \varobar$ and $\varobar \circ \ominus \times \ominus$. In particular, for any quadruplet of morphisms $f_i : C_i \to D_i$, $1 \leq i \leq 4$, one has the following commutative diagram We denote by $\mathrm{Alg}_\ominus$ (resp. $\mathrm{CoAlg}_\ominus$) the category of all monoids (resp. comonoids) in $(\mathrm{Coll}_2, \ominus, \mathbb{C}_\ominus)$, $\mathrm{Alg}_{\varobar}$ (resp. $\mathrm{CoAlg}_{\varobar}$ the category of monoids (resp. comonoids) in $(\mathrm{Coll}_2, \ominus, \mathbb{C}_{\varobar})$.

$$\begin{tikzcd}
(C_1 \obar C_2) \ominus (C_3 \obar C_4) \arrow[r, "(f_1 \obar f_2)\ominus(f_3 \obar f_4)"] \arrow[d, "R_{C_1,C_2,C_3,C_4}"] & (D_1 \obar D_2) \ominus (D_3 \obar D_4) \arrow[d, "R_{D_1,D_2,D_3,D_4}"] \\
(C_1 \ominus C_3) \obar (C_2 \ominus C_4) \arrow[r, "(f_1 \ominus f_3)\obar(f_2 \ominus f_4)"'] & (D_1 \ominus D_3) \obar (D_2 \ominus D_4)
\end{tikzcd}$$

FIGURE 8. R is a natural transformation

Proposition 4.11. [1, Prop. 6.3.5]

- *The category* $(\mathrm{Alg}_{\ominus}, \obar, \mathbb{C}_{\obar})$ *is a monoidal category. Indeed for any couple of horizontal algebra* $(A, m^A_{\ominus}, \eta_A)$ *and* $(B, m^B_{\ominus}, \eta_B)$*, the product* $m^{A \obar B} : A \obar B \to A \obar B$ *is defined*

$$m^{A \obar B}_{\ominus} := (m^A_{\ominus} \obar m^B_{\ominus}) \circ R_{A,B,A,B}\,, \quad \eta_{A \obar B} = \eta_A \obar \eta_B\,. \tag{4.5}$$

Moreover the bicollection $\mathbb{C}_{\obar}$ *is a an horizontal monoid*

$$m^{\obar}_{\ominus} : \mathbb{C}_{\obar} \ominus \mathbb{C}_{\obar} \to \mathbb{C}_{\obar}\,, \quad \eta^{\obar}_{\ominus} : \mathbb{C}_{\ominus} \to \mathbb{C}_{\obar}\,, \tag{4.6}$$

which are respectively a horizontal algebra and a horizontal unity.

- *The category* $(\mathrm{CoAlg}_{\obar}, \ominus, \mathbb{C}_{\ominus})$ *is a monoidal category. Indeed for any couple of vertical comonoid* $(A, m^A_{\obar}, \eta_A)$ *and* $(B, m^B_{\obar}, \eta_B)$*, the product* $\Delta^{A \ominus B} : A \to A \ominus B$ *is defined*

$$\Delta^{A \obar B}_{\ominus} := R_{A,B,A,B} \circ \Delta^A_{\obar} \ominus \Delta^B_{\obar}\,, \quad \eta_{A \obar B} = \eta_A \obar \eta_B\,. \tag{4.7}$$

Moreover the bicollection $\mathbb{C}_{\obar}$ *is a an horizontal monoid*

$$m^{\obar}_{\ominus} : \mathbb{C}_{\obar} \ominus \mathbb{C}_{\obar} \to \mathbb{C}_{\obar}\,, \quad \eta^{\obar}_{\ominus} : \mathbb{C}_{\ominus} \to \mathbb{C}_{\obar}\,, \tag{4.8}$$

which are respectively a horizontal algebra and a horizontal unity.

Definition 4.12. We call PROS a monoid in the monoidal category $(\mathrm{Alg}_{\ominus}, \obar, \mathbb{C}_{\obar})$. That is an horizontal monoid $(C, m^C_{\ominus}, \eta^C_{\ominus})$, endowed with a couple of bicollections morphisms

$$m^C_{\obar} : C \obar C \to C\,, \quad \eta^C_{\obar} : \mathbb{C}_{\obar} \to C\,.$$

defining a vertical monoidal structure on C. In addition, These morphisms are horizontal morphisms.

We recall that the same structure takes also the name of double monoid in the literature, see e.g. [1].

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