ROUGH INTEGRATION FOR BANACH ALGEBRAS VALUED PATHS

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ABSTRACT. We foster the work initiated in 2013 on integration against trajectories with low Hölder regularity valued in an infinite dimensional Banach algebra by A. Deya and R. Schott. In particular, we introduce a new notion of non-commutative rough path above an Hölder path, weaker than the usual notion from Lyon's rough path theory, with the objective of constructing a solution theory for a certain class of differential equations arising in non-commutative probability. We show that, in contrary to the finite dimensional setting, there is no underlying Hopf algebra which convolution group accomodates truly irregular n-c rough paths, instead, we use the language of Hairer's theory of regularity structures.

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1. INTRODUCTION

1.1. **Statement of the problem.** This work intends to broaden a path, opened by Deya and Schott towards utilization of rough path principles for studying solutions of a class of controlled differential equations driven by paths valued in infinite dimensional von Neumann algebras.

In the middly nineties, T.J. Lyons introduced [9] the appropriate mathematical framework to express and study coupling between evolving systems or, more formally, to study controlled differential equations

$$dY_t = \sigma(Y_t)\,[dX_t]\,,\ Y_0 = y_0 \in \mathbb{R}^n. \tag{1}$$

In equation (1), Y is a continuous paths in a finite dimensional state space $\mathbb{R}^d$, σ (the field) is a function from $\mathbb{R}^d$ to the space $\mathsf{End}(\mathbb{R}^n, \mathbb{R}^d)$ of endomorphisms between the state space of the *driving noise* X and the state space of the solution Y of (1). For a driving path X with bounded variations, the equation (7) has a rigourous interpretation: the symbol dX stands for the (signed) measure whose repartition function is X itself. Notice that up to time-change reparametrization such a path X is a Lipschitz function. In that case equation (1) can be cast into its *integral form*,

$$Y_t = Y_s + \int_s^t \sigma(Y_u)\mathrm{d}X_u \tag{2}$$

If X has Hölder regularity greater than $\frac{1}{2}$, the equation above is meaningful with the integral in the right hand side interpreted in the Young' sense , thereby providing an interpretation to (1) for driving noises with such regularity.

Equations (1) with irregular noises are common in probability theory where X is a stochastic process (a brownian motion for examle). These stochastic driving noises are too irregular for Young integration; brownian paths are $\frac{1}{2} - \varepsilon$ Hölder continuous. Classical itò integration overcomes these limitations, providing a rigourous interpretation to (1) if X is a continuous semi-martingale. This construction is probabilistic in nature: approximations to the integral in the right hand side of 2 converge in probability.

If we ventur to squeeze in a sentence rough path theory, we would say that it extends rules of ordinary differential and integral calculus to provide a meaningful interpretation to (7) for irregular Hölder paths X throught identification of the relevant complete metric spaces, containing the smooth driving noises and solutions as strict subspaces. Then, equation (1) yields continuous map between spaces of smooth datas that are extended to the larger englobing spaces.

The solution Y does not depend continuously on the driving noise X: the latter may converge uniformly to 0 without driving with him the solution. However, continuity is restored provided that we take into account all the iterated integrals of X. Lyons introduced in his early work the key concept of rough path theory by pushing further the notion of *signature of a path* introduced by Chen. The signature $S(X)$ summarizes local properties of a path X with bounded 1-variation by means of iterated integrals of the path [9, 10]

$$S(X)(s,t) = 1 + \int_s^t \mathrm{d}X_u + \int_{s<u_1<u_2<t} \mathrm{d}X_{u_1} \otimes \mathrm{d}X_{u_2} + \sum_{k=3}^{+\infty} \int_{s<u_1<\cdots<u_k<t} \mathrm{d}X_{u_1} \otimes \cdots \otimes \mathrm{d}X_{u_k}. \tag{3}$$

The signature is a two parameter family of elements in the completed tensor algebra over the ambient space of X or, equivalently, of the algebraic dual of the tensor algebra. Two algebraic relations are satisfied by the iterated integrals of a path. The generalization to the signature of the Chasles identity is named Chen identity [10] and reads

$$S(X)(s,t) = S(X)(s,u) \otimes S(X)(u,t). \tag{4}$$

Being an element of the completed tensor product, the signature can be seen as a linear form $\mathbb{X}_{st}$ on $T(V)$, namely

$$T(V) \ni v_1 \otimes \cdots \otimes v_n \mapsto \int_{s<u_1<\cdots<u_n<t} \langle v_1, \mathrm{d}X_{u_1}\rangle \cdots \langle v_n, \mathrm{d}X_{u_n}\rangle$$

The tensor algebra $T(V)$ has a rich algebraic structure: it is a conilpotent cofree coaugemented Hopf algebra. This means that $T(V)$ can be endowed with two structural maps that are a coproduct $\Delta : T(V) \to T(V) \otimes T(V)$ and a product $\sqcup\!\sqcup : T(V) \otimes T(V) \to T(V)$. The coprodcut map yields a group structure on the set of characters of $(T(V), \sqcup\!\sqcup)$ that we denotes $\star$. Using integration by part formula, one can prove that $\mathbb{X}_{st}$ is a character on the shuffle algebra [10]. The Chen's identity can then be rewritten

$$\mathbb{X}_{st} = \mathbb{X}_{su} \star \mathbb{X}_{ut}$$

A (geometric) rough path is an abstract notion of signature for an irregular path. It can be concisely defined as follows : it is a trajectory over the group of character of $(T(V), \sqcup\!\sqcup)$ satisfying (crucial) analytical assumptions.

With time, people started bouncing against boundaries of the theory. Applying principles of rough path theory when the state space V is infinite dimensional is challenging. In particular, it is not always possible to give a rigorous interpretatin to (7) for infinite dimensional state spaces, whereas owing to Lyons-Victoir's extension Theorem, it is always possible to do so in the finite dimensional setting. There are many non-equivalent tensor products of infinite dimensional Banach spaces. The choice of a topology is constrained by continuity of the field at hand and whether it is possible to lift an irregular to a rough path in the completed tensor algebra for the chosen tensor product.

In the area of expertise of the authors, example of infinite dimensional driving noise are found amongst non-commutative stochastic processes.

In the following paragraph we motivate the introduction of a class of controlled differential equations we intend to study. Next, we sort out the central observation of [5] that pushes us to introduce a refined notion of rough paths tailored to the class of equations we study.

Consider a standard hermitian browian motion $H_N = (H_N(t))_{0\leq t\leq 1}$ on $\mathcal{M}_N(\mathbb{C})$: it is a matrix filled with independent complex brownian motions (up to symmetries) with a variance scaling as the inverse of the dimension N. The non-commutative distribution of H_N encompasses all the moments

$$\frac{1}{N}\mathbb{E}\left[\mathsf{Tr}(H_N^k(t_1)H_N^k(t_2)\cdots H_N^k(t_p))\right],\ 0\leq t_1,\ldots,t_p\leq 1 \tag{5}$$

General results proved by Voiculescu et al. in [14, 15] entails convergence of the non-commutative distribution as the dimension N tends to infinity. In addition, the limit is the non-commutative distribution of the celebrated *free Brownian motion* that we denote by w: for each time $t\geq 0$, w_t is in particular a self-adjoint element of a von Neumann algebra.

In [2, 7] the authors state convergence in non-commutative distribution of the unitary brownian motion U_N (and the orthogonal, compact symplectic Unitary Brownian motion), a matrix valued stochastic process solution of the following Stratonovitch differential equation:

$$\mathrm{d}U_N(t) = U_N(t)\circ \mathrm{d}\mathcal{H}_N(t).$$

Using free stochastic calculus [3], the limiting distribution of the unitary brownian motion can be apprehended as the non-commutative distribution of the so-called *free unitary Brownian motion* $u = (u_t)_{t\geq 0}$,

$$\mathrm{d}u_t = u_t\cdot \mathrm{d}w_t - \frac{1}{2}\mathrm{d}t,\ u_0 = 1,\ t\geq 0. \tag{6}$$

The above discussion reveals the following class of differential equations

$$\mathrm{d}Y_t = \sum_{\substack{i=1\ldots n\\ j=1\ldots p}} a^i_j(Y_t)\cdot \mathrm{d}X^j_t\cdot b^i_j(Y_t),\ t\geq 0, \tag{7}$$

where $a^i_j, b^i_j : \mathcal{A}\to\mathcal{A}$ polynomials (or more generally Fourier transform a regular measure supported on the real line). The processes X^j are Hölder paths of self-adjoint operators and $\cdot$ is the product in a von-Neumann algebra $\mathcal{A}$ accomodating the paths X^j.

The very first investigations on equations (7) with X a free brownian motion can be traced back to the work of Biane and Speicher [3]. Later, Donati-Martin extended this approach to define stochastic integrals of a large class of processes against q-deformed brownian motions. In 2001, M. Capitain and C. Donati-Martin took the first leap to apply rough path principles to equations (7) by introducting a (spatial) Lévy area above the free Brownian motion yielding existence of solutions in the one sided case (if all the b^i_j are constant equal to 1).

In the two papers [5, 6] Deya and Schott hammered the general case ($b\neq 1$) for X a q-deformed Brownian motion. Central in their work is the notion of product Lévy area.

Let $(\mathcal{A},\eta,\|\cdot\|)$ be a Banach algebra. In contrary to the finite dimensional case, the algebraic tensor product $\mathcal{A}\otimes\mathcal{A}$ can be endowed with several non-Lipschitz equivalent norms. We will be interested in two normed complete topologies on $\mathcal{A}\otimes\mathcal{A}$: the *spatial* topology and the *projective* topology (see []). The projective topology is the finer topology among complete metric topologies on $\mathcal{A}\otimes\mathcal{A}$ which associated norm factorizes for pure tensor, whereas the spatial topology is the coarser one.

The multiplication map μ is continuous (and Fréchet differentiable) for the projective topology $\mathcal{A}\hat{\otimes}\mathcal{A}$ of $\mathcal{A}\otimes\mathcal{A}$. For $a:\mathcal{A}\mapsto\mathcal{A}$ and $b:\mathcal{A}\mapsto\mathcal{A}$ two smooth function, define the smooth function

$$\begin{array}{rccc} \sigma_{a,b}: & \mathcal{A} & \to & \mathsf{End}(\mathcal{A},\mathcal{A}) \\ & y & \mapsto & x\mapsto a(y)xb(y). \end{array}$$

Proposition 1 (Proposition 3.6, [5])**.** *Let $\mathcal{A}$ be a von Neumann algebra accomodating a free Brownian motion. The multiplication map $\mu:\mathcal{A}\otimes\mathcal{A}\mapsto A$ is not continuous with respect to the spatial norm.*

The above proposition rules out the possibility to adress existence of solutions to (7) with X a free brownian motion and field $\sigma_{a,b}$ using the spatial Lévy area. The adequate topology to work with would be the projective one. However, Victoir proved in [13], that no Lévy area above the free brownian motion exists in the projective tensor product. This is the initial motivation to devise an approach to a strong solution theory for based on the roots of rough paths theory, initiated in [5].

As explained, it is not always possible to construct a rough path in the projective tensor product but one can wonder if a *weaker* object embodying the datas needed to give a meaning to (7) exists. The starting is then a fine analysis of (7) if X is a smooth path and in particular of the expansion of the solution obtained by applying Picard iterations. Let us do it on a simple example,

$$\mathrm{d}Y_t = A\cdot Y_t\cdot \mathrm{d}X_t + \mathrm{d}X_t\cdot Y_t\cdot B \tag{8}$$

The first two steps of the Picard Iteration provides:

$$\begin{aligned} Y_t = Y_s &+ (X_t-X_s)A + B(X_t-X_s) \\ &+\int_{s<t_1<t_2<t} B\cdot\mathrm{d}X_{t_1}\cdot B\cdot\mathrm{d}X_{t_2} + \int_{s<t_1<t_2<t}\mathrm{d}X_{t_2}\cdot A\cdot\mathrm{d}X_{t_1}\cdot A \\ &+\int_{s<t_1<t_2<t}\mathrm{d}X_{t_2}\cdot B\cdot\mathrm{d}X_{t_1}\cdot A + \int_{s<t_1<t_2<t}\mathrm{d}X_{t_1}\cdot A\cdot\mathrm{d}X_{t_2}\cdot B \\ &+ R_{st} \end{aligned} \tag{9}$$

with R_{st} a remainder term. The above equation hints at a control of the small variations of Y involvin the iterated integrals

$$\int_{s<t_1<\cdots<t_n<t} A_0\cdot\mathrm{d}X_{t_{\sigma^{-1}(1)}}\cdot\cdots\cdot\mathrm{d}X_{t_{\sigma^{-1}(n)}}\cdot A_n,\ A_0,\ldots,A_n\in\mathcal{A}, \tag{10}$$

where σ is a permutation of $[n]$. In this articlar, we elaborate on this observation and extract the important algebraic and analytical properties of the operators (10) with the objective to define an abstract notion of non-commutative rough path embodying these properties.

1.2. **Contributions.** In this article, we put aside probabilistic considerations and restrict ourselves to the equations in the class (7) driven by one noise X. This means that instead of considering a von-Neumann algebra endowed with a filtration for the state space of the driving noises X, we work with a Banach algebra $(\mathcal{A},\|\cdot\|)$.

First we study closely the algebraical and analytical properties of the operators (10). We show that to the operators (10) corresponds a path on a convolution group of (controlled) representations of a certain Hopf algebra of leveled forests in the category of bicollections. In comparison with rough path theory, it is then tempting to define n-c rough paths as Hölder trajectories on the same convolution group. We explain why this can not be and relevancy of regularity structures.

Secondly, we define a regularity structure with index set a partially order set of permutations, that is the image of $\mathcal{A}$ by the Schur functor associated with a certain collection of permutations. We explain how a smooth trajectory on the convolution group mentioned above yields a smooth model for this regularity structure. We define next a second regularity structure with a product, with same index set but with smaller model space and prove the two reg. stct. relate by mean of a surjective morphism that we call faces contractions. *We define a non-commutative rough path as a model for this reg. stct.*

Thirdly, we define controlled n-c rough paths and construct the *rough integral*.

1.3. **Outline.** In Section 3, we pick a path X of bounded variation valued in a Banach algebra $\mathcal{A}$. We define leveled forests, the indexing sets for *partial* contraction operators. The full partial contraction operators are indexed by leveled trees and we give a bijection between the set of such trees and permutations. This material is contained in Sections 2.1, 2.2 and 3.1.

In Section 3.2, we interpret the Chen relation for the iterated integrals of X as elements of the complete tensor space over $\mathcal{A}$ but for the contractions operators. In the same section, we introduce also a reg. stct. and a model on it associated with the contraction operators.

In Section 3.3, we endow the model space with a product encoding integration by parts for the contractions operators.

In Section 3.4 we give a Hopf-algebraic interpretation of the model we defined in Section 3.2, by taking advantage of the fact that the monoid generated by the bicollection spanned by planar leveled forest a symmetric. In Section 4 we define non-commutative rough paths, n-c controlled rough path and the rough integral. The adequate reg. stct. and the faces contractions morphism between the latter and the reg. stct. defined in Section 3.2 can be found in Section 4.1.

2. Forests and contractions operators

2.1. Basic definitions.

2.2. **Leveled trees and forests.** The objective of the present section is to introduced the combinatorial tool that will be used through this work; the algebra of leveled trees, isomorphic to the Malvenuto-Reutenauer-Poirier algebra. In the literature, one broadly finds two equivalent representations of a permutation, either as a word or as a bijection from a certain interval of integers. We use a third graphical representation of a permutation introduced in [8] by Ronco and Loday. In particular it allows a tractable formulation of the Chen's relation for the contraction operators we define below.

First, recall that a rooted tree is a graph with no cycles and one distinguished vertex we call the root. Each vertex of a tree as at most one output and several inputs. A leaf of a tree is a vertex with no inputs. A planar rooted tree is a rooted tree together with a homotopy class of embeddings of the tree in the plane. It can equivalently be given as a rooted tree with a labeling of the vertices of the tree by finite sequences of integers as follows. The root of the tree is labeled with 0, and if a vertex v is labeled with a sequence s then its inputs are labeled with sequences (v, i) with i an integer ranges in the interval $[\![1, s]\!]$.

A *path* p in a tree is a sequence of vertices $(v_1, \ldots, v_n)$ such that $v_i \neq v_{i+1}$ and v_{i+1} is a neighbour of v_i, we call the integer n the *length* of p. The distance between two vertices in a tree is the minimal length of a path connecting those two vertices in the tree.

Definition 2.1 (Leveled Planar Binary Tree (LPBT))**.** An *planar rooted almost binary tree* is a planar tree for which each node has *at most* two inputs. We use the symbol LPBT for the set of all planar rooted almost binary trees, that are leveled in the following sense. A planar tree is in LPBT if all the leaves are at the same distance d from the root and there are exactly $i+1$ nodes at a distance $i \leq d$ from the root. Equivalenty, in the set of all nodes at a distance i from the root, only one has two inputs, the other nodes have only one input.

The degree of a tree $\tau \in \mathsf{LPBT}$ is the number of its leaves, which we denote by $|\tau|$. The complex span of LPBT is a graded vector space, its homogeneous component of degree $n \geq 1$ is the linear span of trees with n leaves. By definition, a leveled tree with only one leaf is the root (see Fig 1).

A generation of a tree is a set of nodes at the same distance from the root. We number a generations with the distance of the nodes to the root.

Notation. The number of generations of a tree is denoted $\|\tau\|$. From the leveling property of τ, we deduce that $\|\tau\| = |\tau| - 1$. For any integer $n \geq 1$, we denote by $\mathsf{LPBT}(n)$ the subset of LPBT consisting of trees with n leaves and LPBT_n the subset of trees with n generations.

Proposition 2.2. *Let n be an integer greater than one. The set of planary rooted binary trees LPBT_{n+1} is in bijection with the set of permutations $\mathcal{S}_n$.*

Proof. We pick a permutation $\sigma \in \mathcal{S}_n$ (seen as a word) and construct inductively its associated leveled tree as follows. Assume that the tree up to the l^{th} generation has been constructed. To build the next generation, we seek for the position of the letter $l+1$ in the gaps between letters of the word obtained from σ after having erased letters greater than l. This indicates to which leaf we add two inputs to build the tree up to the l^{th} generation (to the other leaves we add a single input).

Reciprocally, let τ be a leveled tree. We construct the permutation σ associated with τ by labelling, in each generation, the only node with two inputs by the number of its generation (see Fig 1). We then read the word representing the permutation σ by concatenating the label from left to right. □

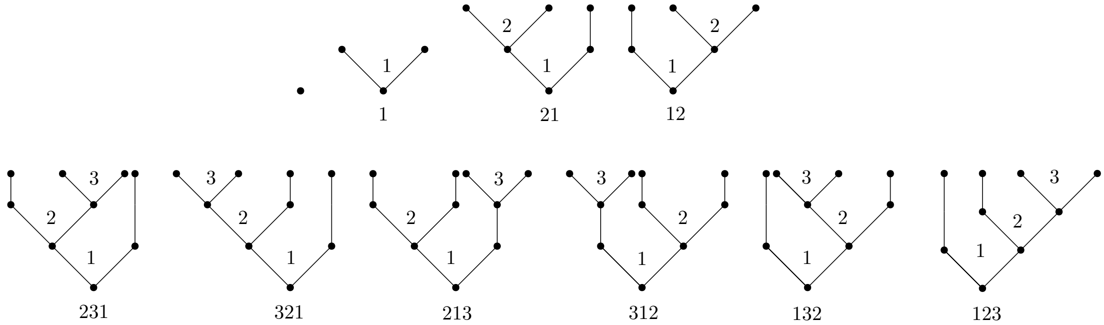

FIGURE 1. Examples of trees in LPBT a their associated word

To a leveled one associated in the next section an operator that acts on a word with entries in $\mathcal{A}$ and outputs an element in $\mathcal{A}$. A node with two inputs on a generation corresponds to multiplication of the arguments flowing along the incoming edges with a third element in $\mathcal{A}$ labeling the node. Leveled trees are not sufficient for our purposes, we need leveled planar *forests* that will encode partial contractions operators.

A *planar forest* is a word (non-commutative monomials) on planar trees. A *generation of a planar forest* is the set of all vertices of all the trees in the forest at the same distance from a root.

Notation. In the following, we denote by $\mathsf{nt}(f)$ the number of trees in the forest f, $|f|$ the total number of leaves in the forest and $\|f\| = |f| - \mathsf{nt}(f)$ the number of generations.

Definition 2.3 (Leveled Planar Binary Forests (LPBF))**.** A leveled planar binary forest is a planar forest f such that the generation at distance $0 \leq i \leq |f| - \mathsf{nt}(f)$ of the forest has exactly $\mathsf{nt}(f) + i$ elements. We denote by LPBF the set of all planar binary leveled forets. See Fig. 2 for an example a leveled planar forest.

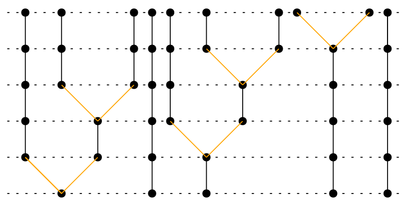

FIGURE 2. A leveled forest with five trees. For each generation, the node with two inputs is coloured in orange

Above, we settled a correspondance between leveled trees in LPBT and permutations. A similar statement holds for leveled forests in LPBF provided that the permutations are replace by partitioned permutations, that is are certain words on words with integers as entries.

Pick a forest $f \in \mathsf{LPBF}$. Each generation of the forest has only vertex with two inputs. We label this vertex according to its distance to the root plus one. These vertices are horizontally ordered according to the relative positions of the regions above these vertices.

To each tree τ in the forest f we associate a word $[f]_\tau$ obtained by reading from left to right the labels of the tree. If a tree is a straight tree (a vertical line, see the last tree in Fig. 2), then we associate to this tree the empty word $\emptyset$. In that way we build a word on words $[f] = [f]_{\tau_1} \cdots [f]_{\tau_{\mathsf{nt}(f)}}$.

For example, the word on words associated with the leveled planar forest in Fig. 2 is

$$13 \,|\, \emptyset \,|\, 24 \,|\, 5 \,|\, \emptyset.$$

A permutation $\sigma \in \mathcal{S}_{||f||}$ left acts on the word $[f]$ by evaluating this permutation on each letter of each word in $[f]$. We denote by f^σ the forest associated with the word $\sigma \cdot [f]$. For example, the permutation $(1,2)(3,4)$ acts on the word on words representing the leveled forest in Fig 2 by

$$(1,2)(3,4) \cdot (13 \,|\, \emptyset \,|\, 24 \,|\, 5 \,|\, \emptyset) = 23 \,|\, \emptyset \,|\, 13 \,|\, 5 \,|\, \emptyset.$$

2.3. **Operations on leveled forests.** If α and β are two leveled trees, we write $\beta \subset \alpha$ if β contains the root and is a leveled subtree of α. Similarly, if β are two leveled forets, we write $\beta \subset \alpha$ if β contains all the roots of α and is a leveled sub-forest of α.

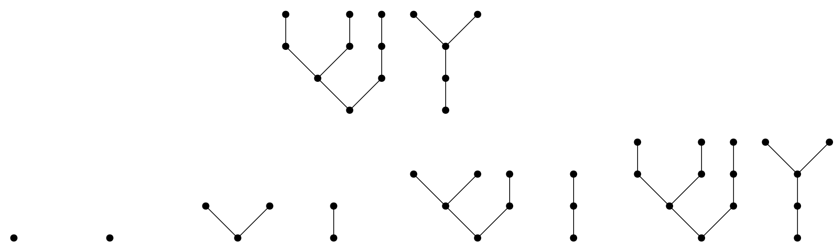

FIGURE 3. On the fires line a leveled forest. On the second line the leveled forests included in that forest

Definition 2.4 (Vertical cutting and gluing of forests)**.** With α and β as above, the cut of α by β is a leveled forest $\alpha \backslash \beta$ obtained by erasing all edges of β (and the isolated nodes we incidentally created).

The inverse operation of cutting, denoted $\sharp$, is gluing : it stacks a forest up on a tree if the number of trees in the forest matches the numbers of leaves of the tree.

Definition 2.5 (Horizontal cutting and gluing of forests)**.** With $f \in \mathsf{LPBF}$ a leveled forest, we denote by $f^\flat \in \mathsf{LPBT}$ the leveled tree obtained by gluing alltogether the trees in the forest along their external edges. Let τ be a leveled tree and $(n_1, \ldots, n_k)$ a composition of n, we define $[\tau]_{n_1,\ldots,n_k}$ the leveled forest associated with the word

$$[\tau]_1 \cdots [\tau]_{n_1} \mid \cdots \mid [\tau]_{n_1+n_{k-1}+1} \cdots [\tau]_{n_1+\cdots+n_k}.$$

with the convention that the chunk $[\tau]_{n_1+\cdots+n_{i-1}} \cdots [\tau_{n_1+\cdots+n_i}] = \emptyset$ if $n_i = 0$.

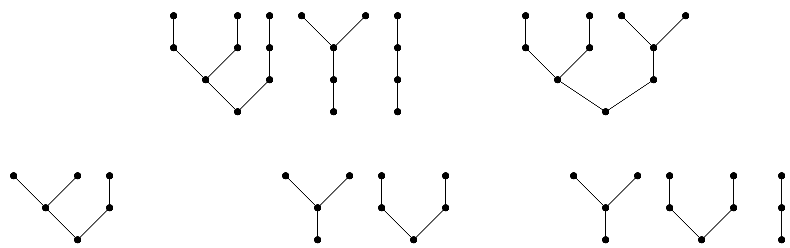

FIGURE 4. On the first line, the right tree is obtained by gluing all trees in the forest f along their external edges. On the second line, the two rightmost forest are horizontal cuts of the leftmost forest with parameters $(1,1)$ and $(1,1,0)$, respectively.

Let $f = \tau_1 \cdots \tau_p$ be a leveled forest. It holds that

$$f \backslash f' = \left[f^\flat \backslash (f'^\flat) \right]_{||f_1||,\ldots,||f_p||}, \; f' \subset f.$$

2.4. **Shuffle product on leveled forests.** In this section, we denote by $\langle\langle n\rangle\rangle$ the set of words on words with entries in $[n]$. A permutation $\sigma \in \mathcal{S}_n$ acts on the left of a word w with entries in $[n]$: one simply applies σ on each letter in w. The action of σ extends as a morphism of $\langle\langle n\rangle\rangle$ for the concatenation product.

Owing to the correspondance between leveled forests and words on words, the above defined action of the permutation σ induces an action on the set LPBF_n of leveled forests with n generations. For example, the permutation $(1,2)(3,4)$ acts on the word on words representing the leveled forest in Fig 2 by

$$(1,2)(3,4)\cdot(13\,|\,\emptyset\,|\,24\,|\,5\,|\,\emptyset) = 23\,|\,\emptyset\,|\,13\,|\,5\,|\,\emptyset.$$

With f and f' two forests, we denote by $f \times f'$ the leveled forest obtained by:

- pushing up all the generations of f' by adding $\|f\|$ generations at the bottom (all the added nodes have only one in- and out-put) at the bottom of f',
- adding $\|f'\|$ generations at the top of f (again, the added node have only one in- and out-put),
- gluing the two forests resulting from these operations along their external edgs : the right-most edges (resp. the left-most) for the forest f and left-most for f'.

Definition 2.6 (Shuffle product of leveled planar forests)**.** Let f and f' be two leveled forests, define the shuffle product of f and f' by

$$f \sqcup f' = \sum_{s\in\mathsf{Sh}(\|f\|,\|f'\|)} s\cdot(f\otimes f').$$

Remark 2.7. The product $\sqcup$ extends the shuffle product on permutations [11] defined by Malvenuto, Reutenauer and studied by Poirier to leveled forests.

3. Iterated integrals of non-commutative paths

For the entire section, $X : [0,T] \to \mathcal{A}$ denotes a path with bounded variations. We introduce partial- and full-contraction operators as a set of multilinear maps build upon the iterated integrals of X. These operators generate representations of a pros which underlying bicollection is spanned by leveled forests. Besides, we show that these operators can be organized as a path on a convolution group of representations of this pros (which is, in fact, a Hopf algebras in the category of bicollections endowed with the vertical monoidal structure).

We begin with a (very) brief reminder on the projective topology of the tensor product $E\otimes F$ of two Banach spaces $(E,\|\cdot\|_E)$ and $(F,\|\cdot\|_F)$. The projective norm $\|x\|$ of an element $x\in E\otimes F$ is defined by

$$\|x\| = \inf_{x=\sum_i a_i\otimes b_i}\sum_i \|a_i\|_E\|b_i\|_F.$$

One can check the following properties for the projective norm:

$$\|x\otimes y\|_{E\hat{\otimes}F} = \|x\|_E\|y\|_F,\ \|x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(n)}\|_{E^{\hat{\otimes}n}} = \|x_1\otimes\cdots\otimes x_n\|_{E^{\hat{\otimes}n}}. \tag{11}$$

In the following, we denote by $E\hat{\otimes}F$ the completion of $E\otimes F$ for the projective norm. Because the associativity constraints of the category $\mathrm{Vect}_{\mathbb{C}}$ of vector spaces are continuous with respect to the projective norm and owing to the equalities in (11), the category Banach of all Banach spaces is a symmetric monoidal category for the projective tensor product $\hat{\otimes}$.

Recall that $\mu : \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ is the multiplication map on the Banach algebra $\mathcal{A}$. It is easily checked that if $x,y\in\mathcal{A}\otimes\mathcal{A}$ and $A\otimes A$ is endowed with its natural algebraic structure, then:

$$\|x\cdot_{\mathcal{A}\otimes\mathcal{A}} y\| \leq \|x\|\|y\|,\ \|\mu(x)\|\leq\|x\|.$$

As a consequence, the multiplication map μ extends to $\mathcal{A}\hat{\otimes}\mathcal{A}$ of $\mathcal{A}\otimes\mathcal{A}$.

Let $\mathcal{B}$ be another Banach algebra. Because $\|a\otimes b\| = \|a\|\|b\|, a\in\mathcal{A}, b\in\mathcal{B}$, the two injections $\iota_1 : \mathcal{A}\to\mathcal{A}\otimes\mathcal{B}$ and $\iota_2 : \mathcal{B}\to\mathcal{A}\otimes\mathcal{B}$ defined by;

$$\iota_1 : a\mapsto a\otimes 1_{\mathcal{B}},\ \iota_2 : b\mapsto 1_{\mathcal{A}}\otimes b,\ b\in\mathcal{B}.$$

are continuous for the projective norm. In other words, the category BanachAlg of all Banach algebras is a monoidal category for the projective tensor product $\hat{\otimes}$. Recall that we denote by T : Banach $\rightarrow$ Banach the free functor, $T(E) = \bigoplus_{n\geq 1} E^{\otimes n}$ and by $\bar{T}$ the augmentation, $\bar{T}(E) = \mathbb{C}\cdot\emptyset \oplus \bigoplus_{n\geq 1} E^{\otimes n}$. In the remaining of the article, we use the symbol $\otimes$ in place of $\hat{\otimes}$.

Definition 3.1 (Pros of words on words with entries in E)**.** Let E be a Banach space. We define an operadic composition on $\bar{T}(E)$ as follows. The space of operators with arity $n+1$ is the n fold tensor product $E^{\otimes n}$ ($\emptyset$ as arity one) and

$$\begin{array}{rlcl} \gamma: & T(E)\circ T(E) & \rightarrow & T(E) \\ & x_1\cdots x_n \circ (y_0\otimes\cdots\otimes y_n) & \mapsto & y_0x_1y_1\cdots y_{n-1}x_ny_n \end{array}$$

We denote by WI the pros $\mathbb{C}1\oplus T(\bar{T}(E))$.

Recall that if E is a Banach space, its endomorphism (non-symmetric) operad is the operad with underlying collection the multilinear maps on E with values in E, and the operadic composition is induced by composition of functions.

Definition 3.2 (Pros of endomorphisms End_E)**.** Let $(E, \|\cdot\|)$ be a Banach space. We denote by End_E the pros of all non-commutative polynomials with entries operators in the endomorphism operad of E.

3.1. **Full and partial contractions operators.**

Definition 3.3 (Full contractions operators)**.** Let $n\geq 1$ an integer, $0<s<t<T$ two times and σ a permutation in $\mathcal{S}_n$. Define

$$X^{\sigma}_{s,t} = \int_s^t\int_s^{t_1}\cdots\int_s^{t_{n-1}} \mathrm{d}X_{(\sigma\cdot\mathbf{t})_1}\otimes\cdots\otimes \mathrm{d}X_{(\sigma\cdot\mathbf{t})_n},\quad \sigma\in\mathcal{S}_n,\quad 0\leq s<t\leq T. \tag{12}$$

To each iterated integral $\mathbb{X}^{\sigma}_{st}$, $\sigma\in\mathcal{S}_n$ we associate an operator of arity $n+1$ in $\mathrm{Hom}(\mathcal{A}^{\otimes(n+1)},\mathcal{A})$:

$$\mathbb{X}^{\sigma}_{st}(A_0,\dots,A_n) = \int_s^t\int_s^{t_1}\cdots\int_s^{t_{n-1}} A_0\cdot\mathrm{d}X_{(\sigma\cdot\mathbf{t})_1}\cdots\mathrm{d}X_{(\sigma\cdot\mathbf{t})_n}\cdot A_n,\quad A_0,\dots,A_n\in\mathcal{A}. \tag{13}$$

The multiplication of the algebra $\mathcal{A}$ yields a representation of the word-insertions pros, as stated in the next definition.

Definition 3.4 (Representation of the pros WI)**.** We define a representation $\mathrm{Op}:\mathsf{WI}\rightarrow\mathrm{End}_{\mathcal{A}}$ of the words insertions pros, extending the following values on words in $T(\mathcal{A})$,

$$\mathrm{Op}(A_1\cdots A_n)(X_0\otimes\cdots\otimes X_n) = X_0\cdot A_1\cdot\cdots\cdot A_n\cdot X_n,\quad A_1\otimes\ldots\otimes A_n\in T(\mathcal{A}),\ X_i\in\mathcal{A}.$$

Let w be a word in $T(\mathcal{A})$ of length n. Let $(n_1,\dots,n_k)$ be a composition of n; $n_i\geq 0$ and $n=\sum_i n_i$. The composition $(n_1,\dots,n_k)$ yields a splitting of w: we define the element $[w]_{(n_1,\dots,n_k)}\in T((\mathcal{A}))$ by

$$[w]_{(n_1,\dots,n_k)} = w_1\cdots w_{n_1}|w_{n_1+1}\cdots w_{n_1+n_2}|\cdots|w_{n_1+\cdots+n_{k-1}+1}\cdots w_{n_1+\cdots+n_k},$$

with the convention $w_{n_1+\cdots+n_{i-1}+1}\cdots w_{n_1+\cdots+n_{i-1}+n_i}=\emptyset$ if $n_i=0$.

Definition 3.5 (Partial contractions operators)**.** Let f be a leveled planar forest and $0<s,t<T$ two times. Define $X^f_{s,t}\in T(T(\mathcal{A}))$ and the *partial contractions operators* $\mathbb{X}^f_{st}$ by

$$X^f_{st} = \left[X^{f_\flat}_{st}\right]_{(\|f_1\|,\dots,\|f_{\mathtt{nt}(f)}\|)},\ \mathbb{X}^f_{st} = \mathrm{Op}(X^f_{st}).$$

3.2. **Chen relation.** In this section, we study first how concatenation of paths lift to the full and partial contractions operators, that is we write a Chen identity for the latters. In addition, we introduce a two parameters family of endomorphisms, constitutive of a model in the meaning of Hairer's theory of regularity structure, acting on a direct sum over leveled trees (or equivalently permutations).

In this section, the symbol $\circ$ denotes alternatively the vertical composition on the PROS of multilinear maps on $\mathcal{A}$, $\mathrm{End}_{\mathcal{A}}$ or the composition on PROS of words with entries in $\mathcal{A}$, $\mathcal{W}(\mathcal{A})$.

Proposition 3.6 (Chen identity)**.** *Let $X : [0,T] \to \mathcal{A}$ be a bounded variation path. Let $0 < s < u < t < T$ three times. Let f be a forest in* LPBF*. Then,*

$$\mathbb{X}^{f}_{st} = \sum_{f' \subset f} \mathbb{X}^{f'}_{ut} \circ \left[\mathbb{X}^{f \backslash f'}_{su}\right].$$

Proof. The statement of the proposition is implied by the same statement but for the iterated integrals X^{f}_{st}, $f \in \mathsf{LPBF}$ since ρ is a representation of the word-insertions operad. The initialization is done for forests with 0 generations. Assume that the results as been proved for forests having at most N generations and let f be a forest with $N+1$ generations.

$$X^{f}_{st} = \int_s^u dX_{t_1} \circ X^{f\backslash f_1}_{st_1} + \int_u^t dX_{t_1} \circ X^{f\backslash f_1}_{st_1} = X^{f}_{su} + \int_u^t dX_{t_1} \circ X^{f\backslash f_1}_{st_1}. \tag{14}$$

In the above formula, we use operadic composition in the coloured operad associated with the word insertion operad. We use the short notations:

$$dX_{t_1} = \emptyset^{\otimes i-1} \otimes dX_{t_1} \otimes \emptyset^{|f|-i} \in \mathsf{W}_1 \otimes \cdots \otimes \mathsf{W}_2 \otimes \cdots \otimes \mathsf{W}_1 = \hat{\mathsf{W}}_{1,\ldots 2,\ldots 1} \tag{15}$$

with i the index of the tree in the forest f which has two nodes at its first generation. Also, $X^{f\backslash f_1}_{st_1}$ is seen as an element of $\hat{\mathsf{W}}(n_1) \otimes \cdots \otimes \hat{\mathsf{W}}_{n_i^1,n_i^2} \otimes \cdots \otimes \hat{\mathsf{W}}_{n_k}$, where n_i^1 and n_i^2 are the two trees left out by cutting out the root of ith tree in the forest f. For any subforest f' of $f\backslash f_1$, we use $f'_{n_i^1,n_i^2}$ to denote the subforest of f obtained by adding a root connecting together the trees at position n_1 and n_1^2. We apply the recursive hypothesis to the forest $f\backslash f_1$ to get:

$$X^{f\backslash f_1}_{st_1} = \sum_{f' \subset f\backslash f_1} X^{f'}_{ut_1} \circ \left[X^{(f\backslash f_1)\backslash f'}_{su}\right] = \sum_{f' \subset f \backslash f^1} X^{f'}_{ut_1} \circ X^{f\backslash (f')_{n_i^1,n_i^2}}_{su}.$$

We insert this last relation into equation (14) to get the result since, with (15),

$$\int_u^t dX_{t_1} \circ X^{f'}_{u,t_1} = X^{(f')_{n_i^1,n_i^2}}_{u,t}.$$

For the map $\tilde{X}_{s,t}$, the Chen's relation reads:

$$\tilde{X}^{f}_{s,t} = \sum_{\tau \subset f} \tilde{X}^{\tau}_{su} \circ \tilde{X}^{f\backslash \tau}_{ut}.$$

□

If we choose for the leveled tree f a combe tree, that is a tree obtained by grafting corollas with two leaves with each others, always on the rightmost node, we find back the classical Chen identity. In fact, by cutting such a tree we obtain a smaller comb tree and a leveled forest with only straight trees, except for the last one which is a comb tree.

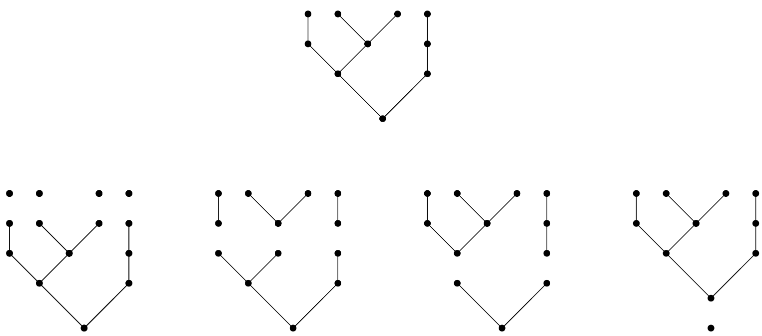

FIGURE 5. All admissible cuts of a leveled tree.

We introduce next the regularity structure and a model for this structure that will be key to develop a solution theory for the class of controlled differential equations we are interested in. As a matter of fact, any algebra-valued path with bounded variation yields a model on this regularity structure, but the latter is still two large to introduce n-c rough paths as Hölder trajectories on the group of invertible triangular endomorphisms of this reg. stct.

The model space is obtained by a fairly general algebraic construction from the theory of operads: it is the image by a certain functor, the Schur functor associated to the collection with basis leveled trees,

$$\mathsf{LPBT}(\mathcal{A}) = \bigoplus_{\tau \in \mathsf{LPBT}} \mathcal{A}^{\otimes|\tau|} \otimes \mathbb{C}\,[\tau]\,.$$

For the remaining part of the paper, we use the lighter notation $a \cdot \tau$ for $a \otimes \tau$ in $\mathcal{A}^{\otimes|\tau|} \otimes \mathbb{C}\,[\tau]$.

In classical rough path theory, the signature of path x yields a path $\mathbb{X} : [0,T] \to G(H)$ on the group of characters $G(H)$ on the shuffle Hopf algebra $(H, \Delta, \sqcup\!\sqcup, S)$ as explained in the introduction. To such a path, we associate a path of invertible triangular endomorphisms of H,

$$\bar{\mathbb{X}}_{.} = \mathrm{id} \otimes \mathbb{X}_{.} \circ \Delta,$$

then $(H, \bar{\mathbb{X}}, \bar{\mathbb{X}}_{st} = \bar{\mathbb{X}}_t \circ \bar{\mathbb{X}}_s^{-1})$ is a model. Whereas it is not clear yet if it is possible to associate to the full and partial contractions operators a path on a certain convolution group of representations, our statement of the Chen relation makes clear that any prospective deconcatenation product Δ should act on a tree by cutting it in all possible ways, generations after generations. In section 3.4 we prove this operation provides a comonoid in a certain category. For the time being, we simply write down the followng formula for the model,

$$\begin{array}{rcccc} \bar{\mathbb{X}}_{st} : & \bigoplus\limits_{\tau \in \mathsf{LPBT}} \mathcal{A}^{\otimes|\tau|} & \longrightarrow & \bigoplus\limits_{\tau \in \mathsf{LPBT}} \mathcal{A}^{\otimes|\tau|}, \\ & a \cdot \tau & \longmapsto & \sum\limits_{\tau' \subset \tau} \mathbb{X}_{st}^{\tau\backslash\tau'}(a) \cdot \tau' \end{array} \tag{16}$$

Of course, the map $\bar{\mathbb{X}}_{st}$ crosses degrees, and we write $\bar{\mathbb{X}}_{st} = \mathrm{id} + \sum_{k=1}^{\infty} \bar{\mathbb{X}}_{st}^{(k)}$, with

$$\begin{array}{rcccc} \bar{\mathbb{X}}_{st}^{(k)} : & \mathsf{LPBT}(\mathcal{A}) & \to & \mathsf{LPBT}(\mathcal{A}) \\ & a \cdot \tau & \mapsto & \sum\limits_{\substack{\tau' \subset \tau \\ \|\tau\backslash\tau'\| = k+1}} \mathbb{X}_{st}^{\tau\backslash\tau'}(a) \cdot \tau' \end{array} \tag{17}$$

Proposition 3.6 immediately implies the following one.

Proposition 3.7. *Let $s < u < t < T$ be three times, then*

1. $(\bar{\mathbb{X}}_{st}(\mathcal{A}_\tau) - \mathrm{id}) \subsetneq \bigoplus_{\tau' \subset \tau} \mathcal{A}_{\tau'}$ *for every leveled tree* $\tau \in \mathsf{LPBT}$
2. *(Chen relations)* $\bar{\mathbb{X}}_{st} = \bar{\mathbb{X}}_{ut} \circ \bar{\mathbb{X}}_{su}$
3. $\|\bar{\mathbb{X}}_{st}^{k}\| \prec |t-s|^k$

Proof. We prove point 2, the Chen relation, the others are trivial. Let $s < u < t$ be three times and $A_f \cdot f \in \mathsf{LPBF}(\mathcal{A})$. Pursuant to the Chen's relation (Proposition 3.6),

$$\begin{aligned} \overline{\mathbb{X}}_{st}(A_f \cdot f) = \sum_{f' \subset f} \mathbb{X}_{st}^{f\backslash f'}(A_f) \cdot f' &= \sum_{f' \subset f} \sum_{f'' \subset f\backslash f'} \mathbb{X}_{ut}^{f''}(\mathbb{X}_{su}^{(f\backslash f')\backslash f''}(A_1 \otimes \cdots \otimes A_{|f|})) \cdot f' \\ &= \sum_{f' \subset f} \sum_{f'' \subset f\backslash f'} \mathbb{X}_{ut}^{f''}(\mathbb{X}_{su}^{(f\backslash (f'' \sharp f')}(A_1 \otimes \cdots \otimes A_{|f|})) \cdot f' \end{aligned}$$

We perform the change of variable $g = f'' \sharp f'$, $g' = f'$ and obtain:

$$\overline{\mathbb{X}}_{st}(A_1 \otimes \cdots \otimes A_{|f|} \cdot f) = \sum_{g \subset f} \sum_{g' \subset g} \mathbb{X}_{ut}^{g\backslash g'}(\mathbb{X}_{su}^{f\backslash g}(A_1 \otimes \cdots \otimes A_{|f|})) \cdot g' = (\overline{\mathbb{X}}_{ut} \circ \overline{\mathbb{X}}_{st})(A_1 \otimes \cdots \otimes A_{|f|} \cdot f)$$

$\square$

3.3. **Geometric properties.** In this section, we investigate consequences of the integration by part formula, stated as shuffle relatons for the iterated integrals of X, on full- and partial-contraction operators and on the model $\bar{\mathbb{X}}$. To set the ground for the second part of our work in which we define composition of n-c. controlled rough path with smooth functions, we introduce an operadic composition L on a collection of words with entries in $\mathcal{A}$ that is different from the composition on the words insertions operad. This operad encodes operations on derivatives of certain functions (the field a and b are part of) that occur in the chain's rule. We should elaborate on this in Section **??**

Definition 3.8. We define the collection of vector spaces $\mathcal{W} = (\mathcal{W}(0), \mathcal{W}(1), \mathcal{W}(2), \ldots)$ by

$$\mathcal{W}(n) = \mathcal{A}^{\otimes n+1},\ n \geq 0.$$

Next, define $L : \mathcal{W} \circ \mathcal{W} \to \mathcal{W}$ as follows. Pick a word $U \in \mathcal{A}^{\otimes n}$ and words $A^i \in \mathcal{A}^{\otimes m_i}$, $1 \leq i \leq p$ and set

$$L(U \otimes A^1 \otimes \cdots \otimes A^p) = \Big(U_{(1)} \cdot A^1_{(1)}\Big) \otimes A^1_{(2)} \otimes \cdots \otimes \Big(A^1_{(m_1)} \cdot U_{(2)} \cdot A^2_{(1)}\Big) \otimes \cdots \otimes \Big(A^p_{(m_p)} \cdot U_n\Big).$$

Notice that elements of $\mathcal{A}$ are 0-ary operators in the collection $\mathcal{W}$ and for example, the above formula gives $L(U_1 \otimes U_{(2)} \circ A) = U_1 A U_2 \in \mathcal{A}$. The following proposition holds thanks to associativity of the product on $\mathcal{A}$.

Proposition 3.9. $\mathsf{W} = (\mathcal{W}, L, 1 \otimes 1)$ *is an operad.*

In the collection $\mathcal{W}$, a word with length n is an operator with $n-1$ entries, the inner gaps between the letters. So far, a leveled was considered as an operator with as much inputs as leaves. However, there is an alternative way to see such a tree as an operator : by considering the *faces* of the tree as inputs. A face is a region enclosed between two consecutive leaves and delimited by the two paths of edges meeting at the least common ancestor, see Fig. 6. We denote by $\mathsf{LPBT}_\sharp$ the collection spanned by leveled trees with grading the numbers of faces.

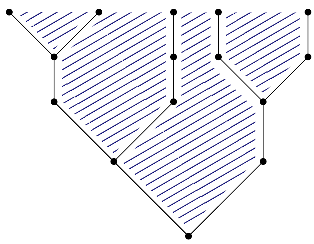

FIGURE 6. Faces of a leveled tree

With $\otimes_H$ the Hadamard tensor product on collections,

$$\mathsf{LPBT}(\mathcal{A}) = \bigoplus_{n \geq 0} (\mathsf{LPBT}_\sharp \otimes_H \mathcal{W})(n)$$

and we denote with the same symbol the collection $\mathsf{LPBT}(\mathcal{A}) = \mathsf{LPBT}_\sharp \otimes_H \mathcal{W}$. Notice that the endomorphism $\bar{\mathbb{X}}_{st}$ we defined in the previous section satisfies:

$$(\bar{\mathbb{X}}_{st} - \mathrm{id})(\mathsf{LPBT}(\mathcal{A})(n)) \subset \bigoplus_{k<n} \mathsf{LPBT}(\mathcal{A})(k).$$

Notation. For any $U, A \in \mathsf{LPBT}(\mathcal{A})$, we set

$$U \circ A = \sum_{\tau, \tau_1, \ldots, \tau_{\|\tau\|}} U^\tau \cdot \tau \otimes A^{\tau_1} \cdot \tau_1 \otimes \cdots \otimes A^{\tau_{\|f\|}} \cdot \tau_{\|\tau\|} \in \mathsf{LPBT}(\mathcal{A}) \circ \mathsf{LPBT}(\mathcal{A}).$$

Definition 3.10. Define the morphism of collections $\mathsf{L} : \mathsf{LPBT}(\mathcal{A}) \circ \mathsf{LPBT}(\mathcal{A}) \to \mathsf{LPBT}(\mathcal{A})$

$$\mathsf{L}(U \otimes (A_1 \otimes \cdots \otimes A_{\|\alpha\|})) = \sum_{\substack{\alpha \\ \tau_1, \ldots, \tau_{\|\alpha\|}}} L(U^\alpha \otimes A_1^{\tau_1} \otimes \cdots \otimes A_{\|\alpha\|}^{\tau_{\|\alpha\|}}) \cdot \tau_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup \tau_{\|\alpha\|},$$

with $U = U^\alpha \cdot \alpha \in \mathsf{LPBT}(\mathcal{A})$, $A_i = \sum_{\tau_i \in \mathsf{LPBT}(\mathcal{A})} A_i^{\tau_i}$.

Lemma 3.11. *Let α and β be two leveled trees in* LPBT, *and pick* $A \in \mathcal{A}^{\otimes|\alpha|+|\beta|-1}$.

$$\bar{\mathbb{X}}_{st}(A \cdot \alpha \sqcup\!\sqcup \beta) = \sum_{\tau_\alpha \subset \alpha, \tau_\beta \subset \beta} \mathbb{X}_{st}^{(\alpha\backslash\tau_\alpha)\sqcup\!\sqcup(\beta\backslash\tau_\beta)}(A) \cdot \tau_\alpha \sqcup\!\sqcup \tau_\beta, \tag{18}$$

Proof. The proof consists essentially in a re-summation. It stems from the definition of the map $\bar{\mathbb{X}}_{st}$ that:

$$\bar{\mathbb{X}}_{st}(A \cdot \alpha \sqcup\!\sqcup \beta) = \sum_{\substack{\tau \in \alpha \sqcup\!\sqcup \beta \\ \tau' \subset \tau}} \mathbb{X}_{st}^{\tau\backslash\tau'}(A) \cdot \tau'. \tag{19}$$

Let $\tau \in \alpha \sqcup\!\sqcup \beta$ a tree obtained by shuffling vertically the generations of α and β and pick $\tau' \subset \tau$ a subtree. Let s be the shuffle in $\mathsf{Sh}(\sharp\alpha, \sharp\beta)$ such that $\tau^{-1} = (\alpha \otimes \beta) \circ s^{-1}$. We associate to the pair (τ, τ') a triple which consists in the tree τ, and two others trees $\tau'_\alpha \subset \alpha$ and $\tau'_\beta \subset \beta$ satisfying

$$\tau' = (\tau'_\alpha \otimes \tau'_\beta) \circ \tilde{s}^{-1},$$

where $\tilde{s}$ is a shuffle in $\mathsf{Sh}\left(\|\tau'_\alpha\|, \|\tau'_\beta\|\right)$. Such a permutation $\tilde{s}$ is unique, in fact it is obtained from s^1 by extracting the first $\|\tau'\|$ letters of the word representing s^{-1}, followed by standardization and finally inversion. Recall that *standardization* means that we translate the first $\|\tau'\|$ letters representing s^{-1}, while maintaining their relative order to obtain a word on integers in the interval $[\![1, \sharp\tau']\!]$.

It is clear that the map $\phi : (\tau, \tau') \mapsto (\tau, \tau'_\alpha, \tau'_\beta)$ is injective. Now, given $\tau_\alpha \subset \alpha, \tau_\beta \subset \beta$, and two shuffles $s_- \in \mathsf{Sh}(\|\tau'_\alpha\|, \|\tau'_\beta\|)$, $s_+ \in \mathsf{Sh}(\|\alpha\backslash\tau'_\alpha\|, \|\beta\backslash\tau'_\beta\|)$, we define a third shuffle s_{-+} in $\mathsf{Sh}(\|\alpha\|, \|\beta\|)$ by requiring

$$s_{-+}(i) = s_-(i),\ 1 \leq i \leq \|\tau'_\alpha\|,\quad s_{-+}(\|\tau'_\alpha\| + i) = s_+(i) + s_-(\|\tau'_\alpha\|),\ 1 \leq i \leq \|\tau_\alpha\backslash\tau'_\alpha\|$$

The map $\delta : (\tau'_\alpha, \tau'_\beta, s_+, s_-) \mapsto (\tau, \tau'_\alpha, \tau'_\beta)$ with $\tau^{-1} = \alpha \otimes \beta \circ s_{-+}^{-1}$ is a bijection between the image of ϕ and

$$\mathcal{S} = \{(\tau_\alpha, \tau_\beta, s_+, s_-),\ \tau_\alpha \subset \alpha, \tau_\beta \subset \beta, s_- \in \mathsf{Sh}(\|\tau_\alpha\|, \|\tau_\beta\|), s_+ \in \mathsf{Sh}(\|\alpha\backslash\tau_\alpha\|, \|\beta\backslash\tau_\beta\|)\}.$$

We can thus rewrite the sum in the right hand side of (19) as follows:

$$\sum_{\substack{\tau \in \alpha \sqcup\!\sqcup \beta \\ \tau' \subset \tau}} \mathbb{X}_{st}^{\tau\backslash\tau'}(A)\tau' = \sum_{\tau_\alpha, \tau_\beta, s_+, s_- \in \mathcal{S}} \mathbb{X}_{s,t}^{(\alpha\otimes\beta)\circ s_{-+}\backslash\,(\tau'_\alpha\otimes\tau'_\beta)\circ s_-^{-1}} (\tau'_\alpha \otimes \tau'_\beta) \circ s_-^{-1}$$

Now, we observe that the forest $(\alpha \otimes \beta) \circ s_{-+} \backslash\, (\tau'_\alpha \otimes \tau'_\beta) \circ s_-^{-1}$ does only depend on the trees τ_α, τ_β and the shuffle s_+. Summing over all shuffles s_+, we get $\alpha\backslash\tau_\alpha \sqcup\!\sqcup \beta\backslash\tau_\beta$. The statement of the Lemma follows by computing the sum over s_-. □

Next, with $A \in \mathcal{A}^{\otimes n}$ and $B \in \mathcal{A}^{\otimes m}$, we define their product $A \cdot B$

$$A \cdot B = A_{(1)} \otimes \cdots \otimes (A_{(n)} \cdot B_{(1)}) \otimes \cdots \otimes B_{(m)}.$$

The product $\cdot$ is a graded product on the collection $\mathcal{W}$ with unit $1 \in \mathcal{W}(0)$,

$$A \cdot B \in \mathcal{W}(n+m),\ A \in \mathcal{W}(n), B \in \mathcal{W}(m)$$

Remark 3.12. The product $\cdot$ has a very special form, namely:

$$A \cdot B = (1 \otimes 1 \otimes 1) \circ (A \otimes B) = L((1 \otimes 1 \otimes 1) \otimes (A \otimes B)).$$

and the relation $m \circ (1 \otimes m) = m \circ (m \otimes 1)$ with $m = 1 \otimes 1 \otimes 1$ entails associativity of $\cdot$. We say that $m \in \mathcal{W}(2)$ is a *multiplication* in the operad $(\mathcal{W}, L)$. In addition, associativity of the operadic composition L results in the following distributivity law

$$(A \cdot B) \circ C = (A \circ B) \cdot (B \circ C),\ A, B, C \in \mathcal{W}.$$

Conjointly with the shuffle product on leveled trees, the product $\cdot$ brings in a graded algebra product $\sqcup\!\sqcup : \mathsf{LPBT}\mathcal{A} \otimes \mathsf{LPBT}(\mathcal{A}) \to \mathsf{LPBT}(\mathcal{A})$, namely

$$(A \cdot \alpha) \sqcup\!\sqcup (B \cdot \beta) = (A \cdot B) \cdot \alpha \sqcup\!\sqcup \beta, \tag{sh}$$

with unit $1 \cdot \bullet$. Let f, g two leveled forests and $A \in \mathcal{A}^{\otimes |f|}, B \in \mathcal{A}^{\otimes |g|}$, the integration by part formula

$$\int_{s<t_1<t_2<t} \mathrm{d}X_{t_1} \otimes \mathrm{d}X_{t_2} + \int_{s<t_1<t_2<t} \mathrm{d}X_{t_2} \otimes \mathrm{d}X_{t_1} = (X_t - X_s) \otimes (X_t - X_s)$$

implies for the iterated integrals of X:

$$\int_{s<t_1<\cdots<t_n<t} \mathrm{d}X_{\sigma_1 \cdot t} \otimes \int_{s<t_1<\cdots t_m<t} \mathrm{d}X_{\sigma_2 \cdot t} = \int_{s<t_1 \cdots t_{m+n}<t} \mathrm{d}X_{\sigma_1 \sqcup\!\sqcup \sigma_2 \cdot t}$$

which implies for the contractions operators the following relation

$$\mathbb{X}^{f}_{st}(A_1 \otimes \ldots \otimes A_{|f|}) \cdot \mathbb{X}^{g}_{st}(B_1 \otimes \ldots \otimes B_{|g|}) = \mathbb{X}^{f \sqcup\!\sqcup g}_{st}((A_1 \otimes \ldots \otimes A_{|f|}) \cdot (B_1 \otimes \ldots \otimes B_{|g|})) \tag{20}$$

Proposition 3.13. *Let α and β be two leveled forests and pick $A \in \mathcal{A}^{|\alpha|}, B \in \mathcal{A}^{\otimes |\beta|}$,*

$$\bar{\mathbb{X}}_{st}((A \cdot \alpha) \sqcup\!\sqcup (B \cdot \beta)) = \bar{\mathbb{X}}_{st}(A \cdot \alpha) \sqcup\!\sqcup \bar{\mathbb{X}}_{st}(B \cdot \beta)$$

Proof. It is a simple consequence of the previous Proposition 3.11 and the shuffle relation for the partial contraction operators (20) In fact, one has

$$\begin{aligned}
\bar{\mathbb{X}}_{st}((A \cdot B) \cdot \alpha \sqcup\!\sqcup \beta) &= \sum_{\tau_\alpha \subset \alpha, \tau_\beta \subset \beta} \mathbb{X}^{\alpha \backslash \tau_\alpha \sqcup\!\sqcup \beta \backslash \tau_\beta}_{st} (A \cdot B) \cdot \tau_\alpha \sqcup\!\sqcup \tau_\beta \\
&= \sum_{\tau_\alpha \subset \alpha, \tau_\beta \subset \beta} \mathbb{X}^{\alpha \backslash \tau_\alpha}(A) \cdot \mathbb{X}^{\beta \backslash \tau_\beta}_{st}(B) \cdot \tau_\alpha \sqcup\!\sqcup \tau_\beta = \bar{\mathbb{X}}_{st}(A) \cdot \bar{\mathbb{X}}_{st}(B)
\end{aligned}$$

□

Corollary 3.14 (Geometricity)**.** *For all times $0 < s < t < T$, it holds that:*

$$\mathsf{L} \circ (\mathrm{id} \circ \bar{\mathbb{X}}_{st}) = \bar{\mathbb{X}}_{st} \circ \mathsf{L}. \tag{21}$$

Proof. For the proof, we rely solely on Proposition 3.13.

$$\begin{aligned}
&\mathsf{L}(U^\alpha \otimes \bar{\mathbb{X}}_{st}(A^{\beta_1} \cdot \beta_1) \otimes \cdots \otimes \bar{\mathbb{X}}_{st}(A^{\beta_{\sharp\alpha}} \cdot \beta_{\sharp\alpha})) \\
&\quad = \bar{\mathbb{X}}_{st}(U^\alpha_{(1)} \bullet) \sqcup\!\sqcup \bar{\mathbb{X}}_{st}(A^{\beta_1} \cdot \beta_1) \sqcup\!\sqcup \bar{\mathbb{X}}_{st}(U^\alpha_{(2)} \bullet) \cdots \bar{\mathbb{X}}_{st}(A^{\beta_{\sharp\alpha}} \cdot \beta_{\sharp\alpha}) \sqcup\!\sqcup \bar{\mathbb{X}}_{st}(U^\alpha_{(|\alpha|)} \bullet) \\
&\quad = \bar{\mathbb{X}}_{st}((U^\alpha_{(1)} \bullet) \sqcup\!\sqcup (A^{\beta_1} \cdot \beta_1) \sqcup\!\sqcup (U^\alpha_{(2)} \bullet) \cdots (A^{\beta_{\sharp\alpha}} \cdot \beta_{\sharp\alpha}) \sqcup\!\sqcup (U^\alpha_{(|\alpha|)} \bullet)) \\
&\quad = \bar{\mathbb{X}}_{st}(\mathsf{L}(U^\alpha \cdot \alpha \otimes A^{\beta_1} \cdot \beta_1 \otimes \cdots \otimes A^{\beta_{\sharp\alpha}} \cdot \beta_{\sharp\alpha}))
\end{aligned}$$

□

We collect in the next proposition the salient results of the last two sections. First, we set of leveled trees LPBT is a poset for the following order

$$\tau \prec \tau' \Leftrightarrow \tau \subset \tau', \tau, \tau' \in \mathsf{LPBT}.$$

Secondly, we denote by $G(\mathcal{A})$ the group of triangular invertible algebra morphisms on $\mathsf{LPBT}(\mathcal{A})$,

$$G(\mathcal{A}) = \{\mathbb{X} \in \mathrm{Hom}_{\mathrm{Alg}}(\mathsf{LPBT}(\mathcal{A}), \mathsf{LPBT}(\mathcal{A})) : (\mathbb{X} - \mathrm{id})(\mathsf{LPBT}(\mathcal{A})(\tau)) \subset \bigoplus_{\tau' \subset \tau} \mathsf{LPBT}(\mathcal{A})(\tau')\} \tag{22}$$

Proposition 3.15 (Regularity structure S_1)**.** *The triple $(\mathsf{LPBT}, \mathsf{LPBT}(\mathcal{A}), G(\mathcal{A}))$ is a regularity structure and $(\bar{\mathbb{X}}_{st}, \bar{\mathbb{X}}_{0t})$ is a model for this regularity structure.*

3.4. **A convolution group of representations.** In this section we define the Hopf algebra H the full and partial contraction operators associated with bounded variations paths yield representations of. This Hopf algebra is bigraded; the underlying vector space is a bi-collection. To define H, we use the vertical tensor product $\boxtimes$. Thanks to the fact that the monoid generated by H in $(\mathrm{Coll}_2, \boxtimes)$ is *symmetric*, which is not true for the category $(\mathrm{Coll}_2, \boxtimes)$ itself, it makes sense to require compatibility between the product and co-product on H, in particular $H \boxtimes H$ is an algebra.

Recall that we denote by $|f|$ the number of leaves of a leveled forest f and $\mathsf{nt}(f)$ the number of trees in f. We use the same symbol LPBF for the bicollection spanned by leveled forests and refer to the annexe for the definition of a bicollection. We start by defining the coproduct acting on the bi-graded collection LPBF of leveled forests. Let f be a leveled forest. Let f' be a leveled subforest of f (recall that f' contains the roots of all trees in f). By definition of the forest $f\backslash f'$, the number of outputs of the forest $f\backslash f'$ is equal to the number of inputs of the forest f', we can thus define $\Delta(f) \in \mathsf{LPBF}\boxtimes\mathsf{LPBF}$ by

$$\Delta(f) = \sum_{f' \subset f} f' \boxtimes f\backslash f',\ f \in \mathsf{LPBF} \tag{23}$$

Proposition 3.16 (Coproduct)**.** *The bi-graded map $\Delta : \mathsf{LPBF} \to \mathsf{LPBF} \boxtimes \mathsf{LPBF}$ is coassociative:*

$$(\Delta \boxtimes \mathrm{id}_{\mathsf{LPBF}}) \circ \Delta = (\mathrm{id}_{\mathsf{LPBF}} \boxtimes \Delta) \circ \Delta \tag{co-ass.}$$

$$\varepsilon : \mathsf{LPBF} \to \mathbb{C}_{\boxtimes}, \varepsilon(\bullet^n) = 1_n,\ \varepsilon(f) = 0,\ f \neq \bullet^n, \tag{counit}$$

$$(\varepsilon \boxtimes \mathrm{id}_{\mathsf{LPBF}}) \circ \Delta = (\mathrm{id}_{\mathsf{LPBF}} \boxtimes \varepsilon) \circ \Delta = \mathrm{id}.$$

Proof. Let f be a leveled forest, to show coassociativity we notice that:

$$((\Delta \boxtimes \mathrm{id}_{\mathsf{LPBF}}) \circ \Delta)(g) = \sum_{\substack{f'', f', f \\ f'' \# f' \# f = g}} f'' \boxtimes f' \boxtimes f = ((\mathrm{id}_{\mathsf{LPBF}} \boxtimes \Delta) \circ \Delta)(g) \tag{24}$$

□

We proceed now with the definition of a vertical product $\nabla : \mathsf{LPBF} \boxtimes \mathsf{LPBF} \to \mathsf{LPBF}$. Given two forests f and f' with $\mathsf{nt}(f') = |f|$, we define $\nabla(f \boxtimes f')$ as the sum of forests obtained by first stacking f' up to f and then shuffling the generations of f' with the generations of f (see Section 2.2 for the definition of the action of a permutation on the generations of a forest)

$$\nabla(f \boxtimes f') = \sum_{s \in \mathsf{Sh}(\|f\|, \|f'\|)} s \cdot (f \# f'). \tag{25}$$

Associativity of the product ∇ is easily checked. Let $n \geq 1$, we denote by c_n the maximal element for the Bruhat order in $\mathcal{S}_n$:

$$c_n = \prod_{p \in [\![1, \lfloor \frac{n}{2} \rfloor]\!]} (p, n-p)$$

Let f be a leveled forest and define $f^\star = c_{\|f\|} \cdot f$.

Proposition 3.17. *Let f be a leveled planar forest, then the map $S : \mathsf{LPBF} \to \mathsf{LPBF}$ defined by*

$$S(f) = (-1)^{\|f\|} f^\star \tag{26}$$

is an antipode: $\nabla \circ (S \boxtimes \mathrm{id}_{\mathsf{LPBF}}) \circ \Delta = \nabla \circ (\mathrm{id}_{\mathsf{LPBF}} \boxtimes S) \circ \Delta = \varepsilon \circ \eta$ with $\eta : \mathbb{C}_{\boxtimes} \to \mathsf{LPBF}$ defined by $\eta(1_m) = \bullet^m$.

Proof. Let a, b be two integers greater than one. Set $n = a + b$. The set of shuffles $\mathsf{Sh}(a, b)$ is divided into two mutually disjoint subsets, the set of shuffles sending a (the subset $\mathsf{Sh}(a, b)_+$) to n and the set of shuffles that do not (resp. $\mathsf{Sh}(a, b)_-$).

Recall that if f is a forest then f_{-}^{k} denotes the forest obtained by extracting the k first lowest generations of f and f_{+}^{k} denotes the forest obtained by extracting the k highest generations of f. By definition, one has:

$$\nabla(f' \boxtimes f \backslash f') = \sum_{s \in \mathsf{Sh}(\|f'\|, \|f \backslash f'\|)} s \cdot (f' \# (f \backslash f')^{\star}), \ f^{\star} = c_{\|f\|} \cdot f.$$

The following relation is easily checked and turn to be the cornerstone of the proof:

$$\tilde{s} \circ (c_n \otimes \mathrm{id}_m) = s \circ (c_{n+1} \otimes \mathrm{id}_{m-1}), \ s \in \mathsf{Sh}(m-1, n+1)_{-}, \tag{27}$$

with $\tilde{s}$ the unique shuffle in $\mathsf{Sh}(m,n)_{+}$ such that $\tilde{s}(m) = n+m, \ \tilde{s}(i) = s(i)$. Set $\bar{S}(f) = (-1)^{\|f\|} f^{\star}$. We prove by induction that $S = \bar{S}$. Assume that $S(f) = \bar{S}(f)$ for any forest f with at most $N \geq 1$ generations and pick a forest f with $N+1$ generations. Then, from the induction hypothesis we get:

$$S(f) + f + (\mathrm{id} \boxtimes \bar{S}) \circ \bar{\Delta}(f) = 0.$$

$$\begin{aligned}
\nabla \circ (\mathrm{id} \boxtimes \bar{S}) \circ \bar{\Delta}(f) &= \sum_{f' \subset f} (-1)^{\|f \backslash f'\|} \sum_{s \in \mathsf{Sh}(\|f'\|, \|f \backslash f'\|)} s \cdot [f' \# (f \backslash f')^{\star}] \\
&= \sum_{k=1}^{\|f\|-1} (-1)^k \sum_{s \in \mathsf{Sh}(\|f\|-k, k)} s \cdot \left[(f_{-}^{\|f\|-k} \# (f_{+}^{k})^{\star}\right].
\end{aligned}$$

We divide the sum over the set $\mathsf{Sh}(\|f\|-k, k)$ into two sums. The first sums ranges over the subset $\mathsf{Sh}(\|f\|-k, k)_{+}$ and the second one ranges overs $\mathsf{Sh}(\|f\|-k, k)_{-}$. Then, we gather the sums over $\mathsf{Sh}(\|f\|-k, k)_{+}$ and $\mathsf{Sh}(\|f\|-k+1, k-1)_{-}$:

$$\begin{aligned}
&\nabla \circ (\mathrm{id} \boxtimes \bar{S}) \circ \bar{\Delta} = \\
&\quad \sum_{k=2}^{\|f\|-2} (-1)^k \sum_{s \in \mathsf{Sh}(\|f\|-k, k)_{+}} s \cdot \left[f_{-}^{\|f\|-k} \# (f_{+}^{k})^{\star}\right] - (-1)^k \sum_{s \in \mathsf{Sh}(\|f\|-k+1, k-1)_{-}} s \cdot \left[f_{-}^{\|f\|-k+1} \# (f_{+}^{k-1})^{\star})\right] \\
&\qquad + (-1) \sum_{s \in \mathsf{Sh}(1, \|f\|-1)_{-}} s \cdot \left[f_{-}^{\|f\|-1} \# (f_{+}^{1})^{\star}\right] + (-1)^{\|f\|-1} \sum_{s \in \mathsf{Sh}(\|f\|-1, 1)_{+}} s \cdot \left[f_{-}^{1} \# (f_{+}^{\|f\|-1})^{\star}\right]
\end{aligned}$$

Using equation (27), the right hand side of the last equation is equal to:

$$\begin{aligned}
\nabla \circ (\bar{S} \boxtimes \mathrm{id}) \circ \bar{\Delta} &= 0 - \sum_{s \in \mathsf{Sh}(1, \|f\|-1)_{-}} s \cdot \left[f_{-}^{1} \# (f_{+}^{\|f\|-1})^{\star}\right] + (-1)^{\|f\|-1} \sum_{s \in \mathsf{Sh}(1, \|f\|-1)_{+}} s \cdot \left[f_{-}^{\|f\|-1} \# \, (f_{+}^{1})^{\star}\right] \\
&= -f + (-1)^{\|f\|-1} f^{\star}
\end{aligned}$$

This ends the proof. □

We defined the three strucutral morphisms ∇, Δ, S. To turn LPBF into a Hopf algebra, we need to compatibility between the coproduct Δ and the product ∇; the coproduct Δ should be an algebra morphisms. This only makes sense provided that we can define a product on $\mathsf{LPBF} \boxtimes \mathsf{LPBF}$. This turns out to be the case, thanks to the fact that the monoid generated by LPBF is symmetric as we will prove in the next proposition.

Recall that if f is a forest, one denotes by f_{-}^{k} the subforest of f corresponding to the k generations at the bottom of f and by f_{+}^{k} the forest corresponding to the k top generations of f.

In addition, if p, q are two integers greater than one, we denote by $\tau_{p,q}$ the shuffle in $\mathsf{Sh}(p,q)$ such that $\tau_{p,q}(1) = q+1$ and $\tau_{p,q}(p) = p+q$.

Definition 3.18 (Braiding map on $\mathsf{LPBF} \boxtimes \mathsf{LPBF}$)**.** Define the braiding map

$$\mathsf{K} : \mathsf{LPBF} \boxtimes \mathsf{LPBF} \to \mathsf{LPBF} \boxtimes \mathsf{LPBF}$$

by, for g and f leveled forests such that $f \boxtimes g \in \mathsf{LPBF} \boxtimes \mathsf{LPBF}$,

$$\mathsf{K}(f \boxtimes g) = \left(\tau_{\|f\|, \|g\|} \cdot (f \# g)\right)_{-}^{\|g\|} \boxtimes \left(\tau_{\|f\|, \|g\|} \cdot (f \# g)\right)_{+}^{\|f\|}.$$

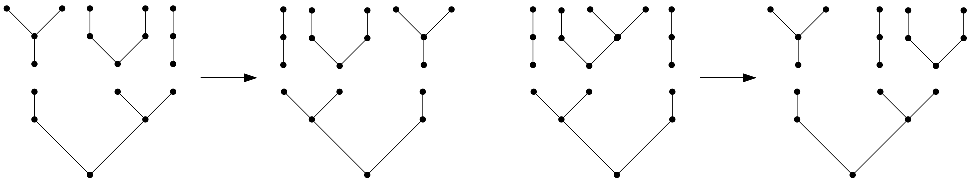

FIGURE 7. Actions of the braiding map

We pictured in Fig. 7 examples of the action of the braiding map on pairs of leveled forests.

We defined the Braiding map K as acting on $\mathsf{LPBF} \boxtimes \mathsf{LPBF}$. We can also exchange two elements $f_1 \boxtimes \cdots \boxtimes f_p$ and $g_1 \boxtimes \cdots \boxtimes g_q$ with the f's and g's forests in LPBF as follows. We first glue all together the forests to obtain another forest $f_1 \# \cdots \# g_q$. Then, we let the permutation $\tau_{\|f_1\|_1+\cdots+\|f_p\|,\|g_1\|+\cdots+\|g_q\|}$ to act on $f_1 \# \cdots \# g_q$. Finally, if we define $h^{n_1,\ldots,n_p}$ with $n_1 + \cdots + n_p = \|h\|$ to be the element in $\mathsf{LPBF}^{\boxtimes p}$ obtained by recursively extracting generations of f, starting with the first n_1 generations at the bottom, continuing with the next n_2 generations and so on, we set

$$\mathsf{K}_{p,q}(f_1 \boxtimes \cdots \boxtimes f_p \boxtimes g_1 \boxtimes \cdots \boxtimes g_q) = (\tau_{\|f_1\|+\cdots+\|f_p\|,\|g_1\|+\cdots+\|g_q\|} \cdot f_1 \# \cdots \# g_q)^{\|g_1\|,\ldots,\|g_p\|,\|f_1\|,\ldots,\|f_q\|}$$

Proposition 3.19. *The monoid generated by the bicollection* LPBF *in* $(\mathrm{Coll}_2, \boxtimes)$ *is a symmetric monoidal category with symmetry constraints* $(\mathsf{K}_{p,q})_{p,q\geq 0}$. *In particular,*

$$\mathsf{K}_{p,q} \circ \mathsf{K}_{q,p} = \mathrm{id} \text{ and } (\mathrm{id}_{\mathsf{LPBF}^{\boxtimes q}} \boxtimes \mathsf{K}_{p,r}) \circ (\mathsf{K}_{p,q} \boxtimes \mathrm{id}_{\mathsf{LPBF}^{\boxtimes r}}) = \mathsf{K}_{p,q+r}$$

Proof. Both assertions are trivial and rely on the following facts for the permutation $\tau_{p,q}$,$p, q \geq 0$,

$$\tau_{p,q} \circ \tau_{q,p} = \mathrm{id}, (\mathrm{id}_q \otimes \tau_{p,r}) \circ (\tau_{p,q} \otimes \mathrm{id}_r) = \tau_{p,q+r},\ \ p, q, r \geq 0.$$

□

Proposition 3.20. *The two maps* Δ *and* ∇ *are vertical algebra morphisms for the vertical product on* $\mathsf{LPBF} \boxtimes \mathsf{LPBF}$*:*

$$(\nabla \boxtimes \nabla) \circ (\mathrm{id} \boxtimes \mathsf{K} \boxtimes \mathrm{id}) : \mathsf{LPBF}^{\boxtimes 4} \to \mathsf{LPBF}^{\boxtimes 2}$$

Formally, with $\nabla^{(2)} = \nabla \circ (\nabla \boxtimes \mathrm{id}) = \nabla \circ (\mathrm{id} \boxtimes \nabla)$

$$\nabla^{(2)} = \nabla^{(2)} \circ (\mathrm{id} \boxtimes \mathsf{K} \boxtimes \mathrm{id}), \quad (\nabla \boxtimes \nabla) \circ (\mathrm{id} \boxtimes \mathsf{K} \boxtimes \mathrm{id}) \circ (\Delta \boxtimes \Delta) = \Delta \circ \nabla,$$

Remark 3.21. We can rephrase the fact that ∇ is a morphism by saying that (LPBF, ∇) is a commutative algebra.

Proof. We begin with the first assertion. Pick f_1, f_2, f_3, f_4 compatible leveled forests (the number of inputs of f_i matches the number of outputs of f_{i+1}, $1 \leq i \leq 3$),

$$\begin{aligned}
&(\nabla^{(2)} \circ \mathsf{K})(f_1 \boxtimes f_2 \boxtimes f_3 \boxtimes f_4) \\
&\quad= \sum_{s \in \mathsf{Sh}(\|f_1\|,\|f_3\|,\|f_2\|,\|f_4\|)} s \cdot \left[f_1 \# \left(\tau_{\|f_2\|,\|f_3\|} \cdot (f_2 \# f_3)\right)_-^{\|f_3\|} \# \left(\tau_{\|f_2\|,\|f_3\|} \cdot (f_2 \# f_3)\right)_+^{\|f_2\|} \# f_4 \right] \\
&\quad= \sum_{s \in \mathsf{Sh}(\|f_1\|,\|f_3\|,\|f_2\|,\|f_4\|)} \left(s (\mathrm{id} \otimes \tau_{\|f_2\|,\|f_3\|})\right) \cdot \left(f_1 \# f_2 \# f_3 \# f_4\right) \\
&\quad= \sum_{s \in \mathsf{Sh}(\|f_1\|,\|f_2\|,\|f_3\|,\|f_4\|)} s \cdot \left(f_1 \# f_2 \# f_3 \# f_4\right) = \nabla^{(2)}(f_1 \boxtimes f_2 \boxtimes f_3 \boxtimes f_4).
\end{aligned}$$

For the second assertion, we write first:

$$(\Delta \circ \nabla)(f \boxtimes g) = \sum_{1 \leq k \leq \|f\|+\|g\|} \sum_{s \in \mathsf{Sh}(k,\|f\|+\|g\|-k)} (s \cdot (f \# g))_-^k \boxtimes (s \cdot (f \# g))_+^{\|f\|+\|g\|-k}$$

For each integer $1 \leq k \leq \|f\|$, we split the set of shuffles $\mathsf{Sh}(\|f\|, \|g\|)$ according to the cardinal q of the set $s^{-1}([\![1, k]\!]) \cap [\![\|f\| + 1, \|f\| + \|g\|]\!]$. Then a shuffle $s \in \mathsf{Sh}(\|f\|, \|g\|)$ $s = (s_1 \otimes s_2) \circ \tilde{\tau}_{k,q}$ with $\tilde{\tau}_{k,q}$

the unique shuffle that sends the interval $[\![\|f\|+1, \|f\|+q]\!]$ to the interval $[\![k-q+1,k]\!]$ and fixes the interval $[\![\|f\|+q+1, \|f\|+\|g\|]\!]$.

$$\sum_{\substack{1\leq k\leq \|f\|, 1\leq q\leq \|g\|,\\ 1\leq q\leq k}} \sum_{\substack{s_1\in \mathsf{Sh}(k-q,q),\\ s_2\in \mathsf{Sh}(\|f\|-(k-q)\|g\|-q)}} ((s_1\otimes s_2)\circ \tilde{\tau}_{k,q})\cdot (f\# g))^k_- \boxtimes ((s_1\otimes s_2)\circ \tilde{\tau}_{k,q}\cdot (f\# g))^{\|f\|+\|g\|-k}_+$$

Notice that $\tilde{\tau}_{k,q} = \tau_{k-q,q}$ and $\tilde{\tau}_{k,q}\cdot (f\,\#\,g) = f_-^{k-q}\#(\tau_{\|f\|-(k-q),q}\cdot (f_+^{\|f\|-(k-q)}\,\#\,g_-^q))\# g_+^{\|g\|-q}$. It follows that $(s_1\otimes s_2)\circ\tilde{\tau}_{k,q}\cdot (f\# g))^k_- = \Big((s_1\otimes \mathrm{id})\cdot f_-^{k-q}\#(\tau_{\|f\|-(k-q),q}\cdot (f_+^{\|f\|-(k-q)}\,\#\,g_-^q))\# g_+^{\|g\|-q}\Big)^k_- = s_1\cdot f_-^{k-q}\#(\tau_{\|f\|-(k-q),q}\cdot f_+^{\|f\|-(k-q)}\# g_-^q)^q_-$. Similar computations show that

$$((s_1\otimes s_2)\circ\tilde{\tau}_{k,q}\cdot (f\# g))^{\|f\|+\|g\|-k}_+ = s_2\cdot (\tau_{\|f\|-(k-q),q}\cdot (f_+^{\|f\|-(k-q)}\,\#\,g_-^q))_+^{\|f\|-(k-q)}\# g_+^{\|g\|-q}).$$

The case $\|f\|+1\leq k\leq \|f\|+\|g\|$ is similar, we split the set of shuffles $\mathsf{Sh}(\|f\|,\|g\|)$ according to the cardinal of the set $s^{-1}([\![k+1,\|f\|+\|g\|]\!])\cap [\![1,\|f\|]\!])$ and omitted for brevity. Finally, we obtain for $\Delta\circ\nabla(f\boxtimes g)$ the expression:

$$\sum_{\substack{1\leq k\leq \|f\|,\\ 1\leq q\leq \|g\|}} \sum_{\substack{s_1\in \mathsf{Sh}(k,q)\\ s_2\in \mathsf{Sh}(\|f\|-k,\|g\|-q)}} s_1\cdot (f_-^k\#(\tau_{\|f\|-k,q}\cdot (f_+^{\|f\|-k}\,\#\,g_-^q))^q_- \boxtimes s_2\cdot (\tau_{k,q}\cdot (f_+^{\|f\|-k}\,\#\,g_-^q))_+^{\|f\|-k}\# g_+^{\|g\|-q})$$

which is easily seen to be equal to $(\nabla\boxtimes\nabla)\circ(\mathrm{id}\boxtimes \mathsf{K}\boxtimes\mathrm{id})\circ(\Delta\boxtimes\Delta)(f\boxtimes g)$. □

We have proved the following theorem.

Theorem 3.22. $(\mathsf{LPBF},\nabla,\Delta,S)$ *is a Hopf algebra in the category* $(\mathrm{Coll}_2,\boxtimes,\mathbb{C}_\boxtimes)$.

Theorem 3.23. *Let* $X:[0,1]\to\mathcal{A}$ *be a bounded variation path. With the notation introduced so far, define* $\mathbb{X}_{st}:\mathsf{LPBF}\to\mathrm{End}_{\mathcal{A}}$ *by* $\mathbb{X}_{st}(f)=\mathbb{X}^f_{st}$, $f\in\mathsf{LPBF}$, *then*

$$\mathbb{X}_{st} = \nabla_{\mathrm{End}(\mathcal{A})}\circ \mathbb{X}_{ut}\boxtimes\mathbb{X}_{su}\circ\Delta,\ \ \nabla_{\mathrm{End}(\mathcal{A})}\circ\mathbb{X}_{st}\boxtimes\mathbb{X}_{st} = \mathbb{X}_{st}\circ\nabla$$

Proof. The first assertion follows directly from Proposition 3.6 and the definition of the coproduct Δ. The second one follows from the shuffle identity for iterated integrals of X (seen as tensors) that we recall here, with $\sigma=\sigma_1\otimes\sigma_2$, $\sigma_1\in\mathcal{S}_k,\sigma_2\in\mathcal{S}_l$,

$$\begin{aligned}\int_{s<u_1<\cdots<u_k<t}\int_{s<u_{k+1}<\cdots<u_{k+l}<t} \mathrm{d}X_{u_{s^{-1}(\sigma(1))}}\otimes\cdots\otimes \mathrm{d}X_{u_{s^{-1}(\sigma(k+l))}}\\ = \sum_{v\in\mathsf{Sh}(k,l)}\int_{s<u_1<\cdots<u_{k+l}} \mathrm{d}X_{u_{(v\circ s^{-1}\circ\sigma)(1)}}\otimes\cdots\otimes\mathrm{d}X_{u_{(v\circ s^{-1}\circ\sigma)(k+l))}}\end{aligned}$$

□

We draw the connection between models on regularity structures and paths on the convolution group of representations of the algebra (LPBF,∇). The coproduct Δ and the counit ε induces the structure of a comonad on the Schur functor $\mathsf{LPBF}\boxtimes$ (see Section 5).

This implies that for any collection V, $\mathsf{LPBF}\boxtimes V$ is a LPBF cofree conilpotent coalgebra; there exists a coassociative map Δ^V

$$\Delta^V:\mathsf{LPBF}\boxtimes V\to\mathsf{LPBF}\boxtimes\mathsf{LPBF}\boxtimes V, \tag{28}$$

explicitely $\Delta^V=\Delta\boxtimes\mathrm{id}$. For each pair of times $s<t$, the representation $\mathbb{X}_{st}$ of LPBF induces a morphism of collections (that we denote the same way)

$$\mathbb{X}_{st}:(\mathsf{LPBF}(\mathcal{A})\simeq)\mathsf{LPBF}\boxtimes T(\mathcal{A})\to T(\mathcal{A})$$
$$\mathbb{X}_{st}(f\boxtimes A_{(1)}\otimes\cdots\otimes A_{(|f|)}) = \mathbb{X}_{st}(f)(A_{(1)}\otimes\cdots\otimes A_{(|f|)}),$$

with $f\boxtimes A_{(1)}\otimes\cdots\otimes A_{(f)}$, $A_{(i)}$, $1\leq i\leq |f|$. Thanks to the fact that $\mathsf{LPBF}\boxtimes T(\mathcal{A})$ is a conilpotent cofree coalgebra, $\mathbb{X}_{st}$ coextends to a morphism $\bar{\bar{\mathbb{X}}}_{st}:\mathsf{LPBF}\boxtimes T(\mathcal{A})\to\mathsf{LPBF}\boxtimes T(\mathcal{A})$ of LPBF-coalgebras, namely

$$\bar{\bar{\mathbb{X}}}_{st} = \mathrm{id}\boxtimes\mathbb{X}_{st}\circ\Delta^{T(\mathcal{A})}.$$

Now owing to Theorem 3.23, $\bar{\bar{\mathbb{X}}}_{st} = \bar{\bar{\mathbb{X}}}_{ut} \circ \bar{\bar{\mathbb{X}}}_{su}$. Furthermore, with the product $\sqcup\!\sqcup$ on $\mathsf{LPBF} \boxtimes T(\mathcal{A})$ extending the product denoted with the same symbol we defined in equation (sh) and the corresponding endomorphism $\mathsf{L} : \mathsf{LPBF} \boxtimes T(\mathcal{A}) \to \mathsf{LPBF} \boxtimes T(\mathcal{A})$ introduced in Definition 3.10, Propositions 3.13 and Corollaire 3.14 hold with $\bar{\bar{\mathbb{X}}}$ in place of $\bar{\mathbb{X}}$. With the partial order $\prec$ defined on leveled forests by $f \prec f' \Leftrightarrow f \subset f'$ and the group $\tilde{G}(\mathcal{A})$ of triangular invertible algebra morphism of $\mathsf{LPBF}(\mathcal{A})$, the triple $(\mathsf{LPBF}, \mathsf{LPBF}(\mathcal{A}), \tilde{G}(\mathcal{A}))$ is a regularity structure that we call S_2. Besides, $(\bar{\bar{\mathbb{X}}}_{st}, \bar{\mathbb{X}}_{0t}$ is a model on S_2). Notice that for a tree $\tau \in \mathsf{LPBT}$,

$$\bar{\mathbb{X}}_{st}(\tau) = \mathrm{id} \boxtimes \mathbb{X}_{st} \circ \Delta^{T(\mathcal{A})}(\tau), \;\; s,t < T.$$

4. Non commutative rough paths

In the previous section we associated two objects to iterated integrals of a path X with bounded variations. The first one is a path $\mathbb{X}.$ of representations of a Hopf algebra on the vector space of all leveled forests LPBF and the second one is a model $\bar{\mathbb{X}}_{\cdot,\cdot}$ on a regularity structure. Besides, we proved that the two are, in essence, equivalent.

It is then tempting to give an abstract definition of a non-commutative rough path above an algebra-valued path as a Hölder path (with respect to a topology we have not yet defined) in the convolution group of representations of the Hopf algebra LPBF (with values in the PROS endomorphism of $\mathcal{A}$).

Comparing to classical rough path theory, the n-c rough path should be entirely determined by restrictions to a finite number of homogeneous components of LPBF, depending on the Hölder regularity α of X. These homogeneous components span the vector space with basis trees with at most $\lfloor \frac{1}{\alpha} \rfloor$ generations and the forests contained in these trees (obtained by cutting out contiguous generations anywhere in the trees). Conversely, a path of restricted representations – a truncated n-c. rough path– should yield a path of representation of LPBF using classical Young integration.

There is a major drawback for adopting such definition of a (truncated) n-c rough path. In fact, with these definitions we can not expect a truncated rough path to extend to a genuine n-c rough path.

If X is a path with Hölder regularity greater than $\frac{1}{3}$, the authors in [5] observed that for solving equations in the class we are interested in, one only needs contractions of the Lévy area with the product in the algebra. Deya and Schott proved existence of this product Lévy area, as they call it, above the free Brownian motion w, whereas Victoir proved that no Lévy area in the projective tensor product exists above the free Brownian motion. This product Lévy area defines a truncated n-c rough path $\mathbb{W}^{(2)}$ as defined above. This truncated rough path associates multilinear maps to the three forests on the left in Fig. 8.

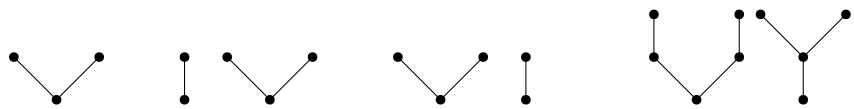

Figure 8

Next, assume this truncated rough path to extend to a genuine rough path $\mathbb{W}$. In particular $\mathbb{W}(\tau)$ exists, with τ the forest f in the right of Fig. 8. But then, $\mathbb{W}(\tau)(1,1,1,1)$ is a Lévy area in the projective tensor product above w! This indicates that, working with infinite dimensional algebras $\mathcal{A}$, the Hopf-algebraic perspective on rough paths dramatically fails. We should look for non-commutative rough paths in a space strictly comprising operadic representations of LPBF. Notice, in addition, that leveled forest, partial contractions, never appear in the expansion of a solution of an equation in the class we are interested in, only in the writing of the Chen relation.

To build the larger group that will contain our non-commutative rough path, we build another regularity structure with same index set, the planar leveled forests but with a smaller model space $\mathsf{LPBT}(\mathsf{FC})$ and a surjective map $\sharp : \mathsf{LPBT}(\mathcal{A}) \to \mathsf{LPBT}(\mathsf{FC})$. We next show that the model space $\mathsf{LPBT}(\mathsf{FC})$ is an algebra turning $\sharp$ into a an algebra morphism. Finally, to the iterated integrals of X we associate a model on $(\mathsf{LPBT}, \mathsf{LPBT}(\mathsf{FC}), G(\mathsf{FC}))$, where $G(\mathsf{FC})$ is a group of invertible triangular algebra morphisms on $\mathsf{LPBT}(\mathsf{FC})$,

$$G(\mathsf{FC}) = \{\alpha \in \mathrm{End}_{\mathrm{Alg}}(\mathsf{LPBT}(\mathsf{FC})) : (\alpha - \mathrm{id})(\mathsf{FC}(\tau)) \subset \bigoplus_{\tau' \subsetneq \tau} \mathsf{FC}(\tau')\}, \tag{29}$$

4.1. **Definition of a n-c. rough path.** We start with the definition of the model space $\mathsf{LPBT}(\mathsf{FC})$. To that end we introduce operators that we call faces contractions. These operators are multilinear maps on $\mathcal{A}$ with values in $\mathcal{A}$ (elements of the pros of endomorphisms of $\mathcal{A}$). Such an operator depends on a choice of a word and a permutation. Recall that to keep notations contained, we identify leveled trees in LPBT and permutations.

Definition 4.1 (Faces contractions)**.** Let $\tau \in \mathsf{LPBT}$ be a leveled tree and pick $A_1 \otimes \cdots \otimes A_{(|\tau|)} \in \mathcal{A}^{|\tau|}$ and define

$$\sharp(A_1 \otimes \cdots \otimes A_{|\tau|} \cdot \tau) : \mathcal{A}^{\otimes |\tau|} \to \mathcal{A}$$

by

$$\sharp((A_1 \otimes \cdots \otimes A_{|\tau|}) \cdot \tau)(X_1, \ldots, X_{\|\tau\|}) = A_1 \cdot X_{\tau(1)} \cdots X_{\tau(\|\tau\|)} \cdot A_{|\tau|},\ X_1, \ldots, X_{\|\tau\|} \in \mathcal{A}.$$

We designate by $\mathsf{FC}(\tau)$ the closure, with respect to the operator norm, of the space of all *τ-faces contractions*, specifically

$$\mathsf{FC}(\tau) = \mathrm{Cl}(\{\sharp((A_1 \otimes \cdots \otimes A_{|\tau|}) \cdot \tau),\ A_1 \otimes \cdots \otimes A_{|\tau|} \in \mathcal{A}^{|\tau|}\}) \subset \mathrm{End}_{\mathcal{A}}$$

and set

$$\mathsf{LPBT}(\mathsf{FC}) = \bigoplus_{\tau \in \mathsf{LPBT}} \mathsf{FC}(\tau).$$

Proposition 4.2. *Let $\alpha, \beta \in \mathsf{LPBT}$ two leveled trees and $A \in \mathcal{A}^{\otimes|\alpha|}$, $B \in \mathcal{A}^{\otimes|\beta|}$, then*

$$\begin{aligned}&\sharp((A \cdot \alpha) \sqcup\!\sqcup (B \cdot \beta))(X_1, \ldots, X_{||\alpha||+||\beta||}) \\ &\qquad = \sum_{s \in \mathsf{Sh}(\|\alpha\|,\, \|\beta\|)} \sharp(A \cdot \alpha)(X_{s(1)}, \ldots, X_{s(\|\alpha\|)}) \cdot \sharp(B \cdot \beta)(X_{s(\|\alpha\|+1)}, \ldots, X_{s(\|\alpha\|+\|\beta\|)})\end{aligned}$$

Besides,

$$\sharp(A \cdot \alpha \sqcup\!\sqcup B \cdot \beta) \leq \frac{(\sharp\alpha + \sharp\beta)!}{\sharp\alpha!\sharp\beta!} \|\sharp A \cdot \alpha\|\|\sharp A \cdot \beta\|.$$

Thanks to proposition 4.2, there exists a product $\sqcup\!\sqcup$ on $\mathsf{LPBT}(\mathsf{FC})$ for which $\sharp$ is an algebra morphism. Explicitely, if $m_\alpha \in \mathsf{FC}(\alpha)$ and $m_\beta \in \mathsf{FC}(\beta)$,

$$m^\alpha \sqcup\!\sqcup m^\beta = \sum_{s \in \mathsf{Sh}(\|\alpha\|,\|\beta\|)} m^\alpha(X_{s(1)}, \ldots, X_{s(\|\alpha\|)}) \cdot m^\beta(X_{s(\|\alpha\|+1)}, \ldots, X_{s(\|\alpha\|+\|\beta\|)})$$

We introduce now a model on the regularity structure $(\mathsf{LPBT}, \mathsf{LPBT}(\mathsf{FC}), G(\mathsf{FC}))$.

Definition 4.3. Let $s < t$ be two times and define the endomorphism $\tilde{\mathbb{X}}_{st} : \mathsf{LPBT}(\mathsf{FC}) \to \mathsf{LPBT}(\mathsf{FC})$ in $G(\mathsf{FC})$ determined by, $m^\alpha \in \mathsf{FC}(\alpha),\ \alpha \in \mathsf{LPBT}$,

$$\tilde{\mathbb{X}}_{st}(m^\alpha) = \sum_{\tau \subset \alpha} \tilde{\mathbb{X}}_{st}|^\tau_\alpha(m^\alpha),\ \tilde{\mathbb{X}}_{st}|^\tau_\alpha : \mathsf{FC}(\alpha) \to \mathsf{FC}(\tau),$$

$$\tilde{\mathbb{X}}_{st}|^\tau_\alpha(m^\alpha)(X_1, \ldots, X_{\|\tau\|}) = \int_{s<t_1<\cdots t_{\|\alpha\|-\|\tau\|}<t} m^\alpha(X_1, \ldots, X_{\|\tau\|}, \mathrm{d}X_{t_1}, \ldots, \mathrm{d}X_{t_{\|\alpha\|-\|\tau\|}}) \tag{30}$$

with $X_1, \ldots, X_{\|\tau\|} \in \mathcal{A}$. See Fig. 9 for a picture representing the action of $\tilde{\mathbb{X}}_{st}$.

The following proposition is an easy consequence of equation (30) and Proposition 3.7.

Proposition 4.4. *$(\tilde{\mathbb{X}}_{st}, \tilde{\mathbb{X}}_{0t})$ is a model for the regularity structure $(\mathsf{LPBT}, \mathsf{LPBT}(\mathsf{FC}), G(\mathsf{FC}))$ and*

$$\sharp \circ \bar{\mathbb{X}}_{st} = \tilde{\mathbb{X}}_{st} \circ \sharp,\ s < t.$$

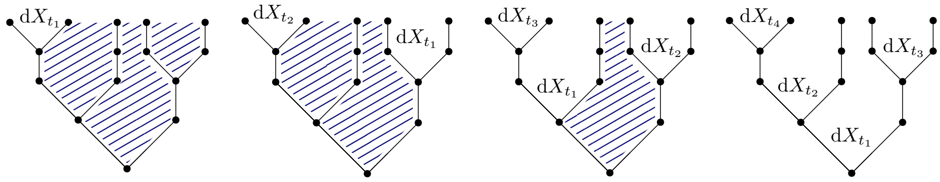

FIGURE 9. faces contractions of leveled forest

We are now in position to give an abstract definition of a non-commutative rough path. First, if $\mathbb{X}$ is a triangular endomorphism of $\mathsf{LPBT}(\mathsf{FC})$, that is if $\mathbb{X} \in G(\mathsf{FC})$, we define the k^{th} homogeneous component of $\mathbb{X}$, $\mathbb{X}^{(k)} : \mathsf{LPBT}(\mathsf{FC}) \to \mathsf{LPBT}(\mathsf{FC})$ by, with $m^\tau \in \mathsf{FC}(\tau)$,

$$\mathbb{X}^{(k)}(m^\tau) = \mathbb{X}_{st}|_{\tau}^{\tau'}(m^\tau), \tau' \subset \tau,\ \sharp\tau' = \sharp\tau - k,$$

Definition 4.5 (Non-commutative geometric rough path)**.** Let $0 < \alpha < 1$ a real number. We call *α-non-commutative geometric rough path* the data of two parameters family $\{\mathbb{X}_{st}, s < t\}$ of invertible continuous operators of $\mathsf{LPBT}(\mathsf{FC})$ subject to, for any pair of times $s < t$

1. (Triangularity) $\mathbb{X}_{st} \in G(\mathsf{FC})$,
2. (Chen relation) For any triple of times $s < u < t$, $\mathbb{X}_{st} = \mathbb{X}_{ut} \circ \mathbb{X}_{su}$,
3. (Hölder estimate) $\|\mathbb{X}^{(k)}_{st}\| \prec |t-s|^{k\alpha}$
4. (Geometricity) $\mathbb{X}_{st}(m^\alpha \sqcup m^\beta) = \mathbb{X}_{st}(A\cdot\alpha) \sqcup \mathbb{X}_{st}(B\cdot\beta)$.

If $\mathbb{X}$ is a non-commutative rough path, owing to the Chen's relation, $\mathbb{X}_{st} = \mathbb{X}_{0t} \circ \mathbb{X}_{0s}^{-1}$. Hence, a n-c. rough path is the set of increments of a trajectory on the group $G(\mathsf{FC})$ satisfying equation (4.1). We give next the definition of a truncated n-c rough path. To that end, we define a truncation of the group $G(\mathsf{FC})$ for each integer N. First, set

$$\mathsf{LPBT}(\mathsf{FC})_N = \bigoplus_{\substack{\tau \in \mathsf{LPBT} \\ \|\tau\| \le N}} \mathsf{FC}(\tau).$$

and a subgroup $G(\mathsf{FC})_N$ of the group of invertible triangular endomorphisms of $\mathsf{LPBT}(\mathsf{FC})_N$ as

$$G(\mathsf{FC})_N = \{\mathbb{X} \in \mathrm{End}_{\mathrm{Vect}_{\mathbb{C}}}(\mathsf{LPBT}(\mathsf{FC})_N) : (\mathbb{X} - \mathrm{id})(\mathsf{FC}(\tau)) \subset \bigoplus_{\tau' \subsetneq \tau} \mathsf{FC}(\tau'),$$
$$\alpha(m^\alpha \sqcup m^\beta) = \alpha(m^\alpha) \sqcup \alpha(m^\beta),\ \text{if } \|\alpha\| + \|\beta\| \le N\}$$

We define next a metric $d_N : G(\mathsf{FC})_N \times G(\mathsf{FC})_N \to \mathbb{R}^+$ on the group $G(\mathsf{FC})_N$ as follows,

$$d_N(\mathbb{X}, \mathbb{Y}) = \max_{k=1\ldots N}(k!\|(\mathbb{Y}^{-1}\circ\mathbb{X})^{(k)}\|)^{\frac{1}{k}} + \max_{k=1\ldots N}(k!\|(\mathbb{X}^{-1}\circ\mathbb{Y})^{(k)}\|)^{\frac{1}{k}} \tag{31}$$

Proposition 4.6. *The distance d_N is a left-invariant distance on the group $G(\mathsf{FC})_N$ and $(G(\mathsf{FC})_N, d_N)$ is a complete metric space.*

Proof. We show first the triangular inequality for d_N. For $\mathbb{X} \in G(\mathsf{FC})$, put $|\mathbb{X}| = \max_{k=1\ldots N}(k!\|\mathbb{X}^{(k)}\|)^{\frac{1}{k}}$. Let $\mathbb{X}, \mathbb{Y}, \mathbb{Z} \in G(\mathsf{FC})$, then

$$\begin{aligned} d_N(\mathbb{X}, \mathbb{Z}) &= \max_{k=1\ldots N}(k!\|(\mathbb{Z}^{-1}\circ\mathbb{X})^{(k)}\|)^{\frac{1}{k}} + \max_{k=1\ldots N}(k!\|(\mathbb{X}^{-1}\circ\mathbb{Z})^{(k)}\|)^{\frac{1}{k}} \\ &= \max_{k=1\ldots N}(k!\|(\mathbb{Z}^{-1}\circ\mathbb{Y}\circ\mathbb{Y}^{-1}\circ\mathbb{X})^{(k)}\|)^{\frac{1}{k}} + \max_{k=1\ldots N}(k!\|(\mathbb{X}^{-1}\circ\mathbb{Y}\circ\mathbb{Y}^{-1}\circ\mathbb{Z})^{(k)}\|)^{\frac{1}{k}} \end{aligned}$$

but then

$$\begin{aligned}\|(\mathbb{Z}^{-1}\circ\mathbb{Y}\circ\mathbb{Y}^{-1}\circ\mathbb{X})^{(k)}\| &\leq \sum_{q=0}^{k}\|(\mathbb{Z}^{-1}\circ\mathbb{Y})^{(q)}\|\|(\mathbb{Y}^{-1}\circ\mathbb{X})^{(k-q)}\|\\ &\leq k!\sum_{q=0}^{k}\binom{k}{q}|\mathbb{Z}^{-1}\circ\mathbb{Y}|^{q}|\mathbb{Y}^{-1}\circ\mathbb{X}|^{k-q}\\ &\leq k!(|\mathbb{Z}^{-1}\circ\mathbb{Y}|+|\mathbb{Y}^{-1}\circ\mathbb{X}|)^{k}\end{aligned}$$

□

Definition 4.7 (Truncated non-commutative geometric rough path)**.** A truncated geometric rough path is an Hölder path on the group $G(\mathsf{FC})_{\mathsf{N}}$.

Let $n\geq 1$ an operator and pick σ a permutation in $\mathcal{S}_n$. With τ_σ is the leveled tree associated to σ^{-1}, $\sigma\cdot\tau_\sigma$ is a comb tree associated with the identity permutation. The permutation σ acts on faces contractions operators by sending $\sharp(A_1\otimes\cdots\otimes A_n\cdot\tau)$ in $\mathsf{FC}(\tau)$ to $\sharp(A_1\otimes\cdots\otimes A_n\cdot\sigma\cdot\tau)$ in $\mathsf{FC}(\sigma\cdot\tau)$ with $\|\tau\|=n$ and thus

$$\sigma:\bigoplus_{\substack{\tau\in\mathsf{LPBT}\\ \|\tau\|=n}}\mathsf{FC}(\tau)\to\bigoplus_{\substack{\tau\in\mathsf{LPBT}\\ \|\tau\|=n}}\mathsf{FC}(\sigma\cdot\tau)$$

is a continuous invertible operator. Hence, the linear map

$$\begin{aligned}\phi:\ \mathsf{LPBT}(\mathsf{FC}) &\to \bigoplus_{\tau\in\mathsf{LPBT}(\mathsf{FC})}\mathsf{FC}(\mathrm{id}_{\|\tau\|})\otimes\tau\\ \sum_{\tau\in\mathsf{LPBT}}m_\tau &\mapsto \sum_{\tau\in\mathsf{LPBT}}\tau^{-1}(m_\tau)\otimes\tau\end{aligned}$$

is a continuous ismorphism too. In the following, we set $\mathsf{FC}(n)=\mathsf{FC}(\mathrm{id}_n)$. The definition we introduced for a non-commutative rough path is fairly general but nevertheless encompasses the properties of the model $\mathbb{X}_{st}$ (we used the heavier notation $\bar{\mathbb{X}}_{st}$ above for a model) associated with BV paths required for construction of the rough integral and for classical rule of differential calculus to hold (see below).

We define an operadic composition, that we also call L (see 3.10), on faces contractions. Operators with arity n are the faces contractions in $\mathsf{FC}(n)$ and we set

$$L(\sharp(A_1\otimes\cdots\otimes A_p)\circ(\sharp W_1\otimes\cdots\otimes\sharp W_p)=\sharp(A_1\otimes\cdots\otimes A_p)\circ(\sharp W_1\otimes\cdots\otimes\sharp W_p), \tag{32}$$

where $A_1\otimes\cdots\otimes A_p\in\mathcal{A}$, $W_i\in\mathsf{FC}(n_i)$ and the symbol $\circ$ in the right hand side of the above equation stands for the composition in the pros $\mathrm{End}_{\mathcal{A}}$ of endomorphisms of $\mathcal{A}$. We use the same formula (3.10) to define the endomorphism $\mathsf{L}:\mathsf{LPBT}(\mathsf{FC})\to\mathsf{LPBT}(\mathsf{FC})$ for the model space $\mathsf{LPBT}(\mathsf{FC})$.

With these definitions, a n-c geometric rough path $\mathbb{X}$ satisfies

$$\mathbb{X}_{st}\circ\mathsf{L}=\mathsf{L}\circ\mathrm{id}\circ\mathbb{X}_{st},\ s<t. \tag{33}$$

In classical rough paths theory, we distinguish weak geometric rough paths to strong ones which sit in the adherence (for the relevant topology, see) of rough paths above bounded-variations paths. However, weakly geometric rough paths, with Hölder regularity α can still be approximated by rough paths above bounded variations paths, true iterated integrals to put it otherwise, but with respect to norms that are weaker than the norm on α-rough path. The notion of geometric n-c. rough paths we gave is really weak; it comprises trajectories of operators on the model space that are certainly not limits of models associated with bounded variation paths, for any topology that implies point-wise convergence at least. In fact for a BV path the associated model is not only triangular but have also constant coefficients on the diagonals. More precisely, if one defines the linear maps

$$\tilde{\mathbb{X}}_{st}(\tau',\tau):\mathsf{FC}(\|\tau\|)\to\mathsf{FC}(\|\tau'\|),\tau'\subset\tau$$

requiring that

$$\tilde{\mathbb{X}}_{st}(m^\tau)=\sum_{\tau'\subset\tau}\phi\Big(\mathbb{X}_{st}(\tau',\tau)\Big(\phi^{-1}(m^\tau)\Big)\otimes\tau'\Big),\ m^\tau\in\mathsf{FC}(\tau), \tag{34}$$

then $\tilde{\mathbb{X}}_{st}(\tau',\tau) = \tilde{\mathbb{X}}_{st}(\tau_1',\tau_1)$ if $\tau\backslash\tau' = \tau_1\backslash\tau_1'$. We call the maps $\mathbb{X}_{\cdot,\cdot}(\tau\backslash\tau') := \tilde{\mathbb{X}}_{st}(\tau',\tau)$, $\tau'\subset\tau$ the components (or coefficients) of $\mathbb{X}_{\cdot,\cdot}$.

Other relations hold among components $(\tilde{\mathbb{X}}_{st}(f))_{f\in\mathsf{LPBF}}$, $(\tilde{\mathbb{Y}}_{st}(f))_{f\in\mathsf{LPBF}}$ of two models associated with BV paths, namely

$$((\tilde{\mathbb{X}}_{st}\otimes\tilde{\mathbb{Y}}_{st}), f\boxtimes f') = ((\tilde{\mathbb{Y}}_{st}\otimes\tilde{\mathbb{X}}_{st}), \mathsf{K}(f\boxtimes f')),\quad f\boxtimes f'\in\mathsf{LPBF}\boxtimes\mathsf{LPBF}, \tag{35}$$

with $((\tilde{\mathbb{X}}_{st}\otimes\tilde{\mathbb{Y}}_{st}), f\boxtimes f') = \tilde{\mathbb{X}}_{st}(f)\circ\tilde{\mathbb{X}}_{st}(f')$ and f,f' two compatible forests. Let $A_1,\ldots,A_{|f'|}\in\mathcal{A}$ and call σ (resp. σ') the permutation associated with $f_\flat$ (resp. $f'_\flat$). We use the notation cb_n for the comb tree associated with the identity permutation id_n. Next, define s the permutation in $\mathcal{S}_{\|f\|+\|f'\|}$ by

- $s_{f\boxtimes f'}(k) = i$, if the k^{th} face of $cb_{\mathsf{nt}(f)})\#f\#f'$ (reading the faces from left to right) is the i^{th} face of f,
- $s_{f\boxtimes f'}(k) = \|f\|+i$ if the k^{th} face of $f\sharp f'$ is the i^{th} face of f',
- $s_{f\boxtimes f'}(k) = \|f\|+\|f'\|+i$ if the k^{th} face is the i^{th} face of $cb_{\mathsf{nt}(f)}$.

With $\mathsf{K}(f\otimes f') = f'_{(1)}\boxtimes f_{(1)}$, notice that

$$f'_{(1)\flat} = f'_\flat,\ f_{(1)\flat} = f_\flat,\ s_{\mathsf{K}(f\boxtimes f')} = s_{f\boxtimes f'}$$

Pick $U_1,\ldots,U_{\mathsf{nt}(f)-1}\in\mathcal{A}$. Putting $dZ_{t_1}\otimes\cdots\otimes\mathrm{d}Z_{t_{\|f\|+\|f'\|}} = \mathrm{d}X_{t_{\sigma(1)}}\otimes\cdots\otimes\mathrm{d}X_{t_{\sigma(\|f\|)}}\otimes\mathrm{d}Y_{t_{\sigma'(1)}}\otimes\cdots\otimes\mathrm{d}Y_{t_{\sigma'(\|f'\|)}}\otimes U_1\otimes\cdots\otimes U_{\mathsf{nt}(f)-1}$, one has

$$\begin{aligned}
&((\tilde{\mathbb{X}}_{st}\otimes\tilde{\mathbb{Y}}_{st}), f\boxtimes f')(\sharp(A_1\otimes\cdots\otimes A_{|f'|}))(U_1,\ldots,U_{\mathsf{nt}(f)-1})\\
&\qquad= \int_{s<t_1<\cdots<t_{\|f'\|}<t}\int_{s<u_1<\cdots<u_{\|f\|}<t} A_1\cdot\mathrm{d}Z_{t_{s^{-1}_{f\boxtimes f'}(1)}}\otimes\cdots\otimes\mathrm{d}Z_{t_{s^{-1}_{f\boxtimes f'}}(\|f\|+\|f'\|)}\cdot A_{|f'|}\\
&\qquad= ((\tilde{\mathbb{Y}}_{st}\otimes\tilde{\mathbb{X}}_{st}), \mathsf{K}(f\boxtimes f'))(\sharp(A_1\otimes\cdots\otimes A_{|f'|}))(U_1,\ldots,U_{\mathsf{nt}(f)-1})
\end{aligned}$$

Whereas it is immediate to impose the first relation to weak geometric n-c rough paths with arbitrary Hölder regularity, it is not the case for the second one. These relations hold because the model associated with a BV path sits in the adherence of a space of partial-faces-contractions operators that we now define. Pick $A_1,\ldots,A_k\in\mathcal{A}$ and define

$$L_{A_1,\ldots,A_k}:\mathsf{LPBT}(\mathsf{FC})\to\mathsf{LPBT}(\mathsf{FC}) \tag{36}$$

by $L_{A_1,\ldots,A_k}|_\tau^{\tau'}(m)(X_1,\ldots,X_{\|\tau'\|}) = m(X_1,\ldots,X_{\|\tau'\|},A_1,\ldots,A_k)$ if $\|\tau\|-k>0$ and $L_{A_1,\ldots,A_k}|_\tau^{\tau'}(m)=0$ otherwise. Notice that the norm of such an operator satisfies:

$$\|L_{A_1,\ldots,A_k}\|\leq\|A_1\otimes\cdots\otimes A_k\|.$$

Hence, $L^k:\mathcal{A}^{\otimes k}\ni(A_1\otimes\cdots\otimes A_k)\mapsto L_{A_1,\ldots,A_k}$ is well defined and continuous. We call $\mathcal{P}$ the closure for the operator norm of sum of the ranges of the operators L^k :

$$\mathcal{P} = C\ell(\bigoplus_{k\geq0}\mathrm{Im}(L^k)) \tag{37}$$

The space $\mathcal{P}$ is a Banach algebra, since $L_{A_1,\ldots,A_k}\circ L_{B_1,\cdots,B_q} = L_{A_1,\ldots,B_q}$. In addition, from the very definition of $L_{A_1,\ldots,A_k}$, $L_{A_1,\ldots,A_q}(\tau',\tau)$ depends only on the forest $\tau\backslash\tau'$ and

$$(L_{A_1,\ldots,A_k}\otimes L_{B_1,\ldots,B_q}, f\boxtimes f') = (L_{B_1,\ldots,B_q}\otimes L_{A_1,\ldots,A_k}, \mathsf{K}(f\boxtimes f')) \tag{38}$$

Thus the relations 34 and 35 hold for operators in $\mathcal{P}$.

Definition 4.8 (Strong geometric rough path)**.** We call a strong geometric rough path a weak geometric rough path $\mathbb{X} = (\mathbb{X}_{st})_{s<t}$ such that for all $s<t$, $\mathbb{X}_{st}\in\mathcal{P}$.

If $\mathbb{X}$ is a triangular operator in $\mathcal{P}$ (with identity operators on the diagonal, $\mathbb{X}(\tau,\tau) = \mathrm{id}_{\|\tau\|}$, then its inverse is also in $\mathcal{P}$ for the simple reason that

$$\mathbb{X}^{-1} = \sum_{n=0}^{N}(-1)^n(\mathbb{X}-\mathrm{id})^n$$

for some integer $N>0$. Incidentally, the intersection $G(\mathsf{FC})^{\mathrm{str.}}=G(\mathsf{FC})\cap\mathcal{P}$ is a complete group (for the distance induced by the operator norm) and a strong geometric rough path can equivalently be given as the increments of a trajectory over the group $G(\mathsf{FC})_{\mathrm{str.}}$ satisfying the required Hölder estimates. We set $G(\mathsf{FC})^{\mathrm{str.}}_N=G(\mathsf{FC})_N\cap\mathcal{P}$.

Definition 4.9 (Truncated strong geometric rough path)**.** Let $0<\alpha<1$ a real number and set $N=\lfloor\frac{1}{N}\rfloor$. A truncated strong geometric rough path $\mathcal{P}$ of order N as a Hölder path on $G(\mathsf{FC})^{\mathrm{str.}}_N$.

4.2. **Controlled rough paths.** Let $0<\alpha<1$ be a real number and set $N=\lfloor\frac{1}{\alpha}\rfloor$. Pick $\mathbb{X}$ a n-c rough truncated rough path of order N with time horizon $T>0$ (see Definition 4.7) and set $\mathbb{X}_{st}=\mathbb{X}_{0t}\circ\mathbb{X}_{0s}^{-1}$. In this section, we introduce the space of integrands for our n-c. rough integral, the space of n-c rough paths controlled by $\mathbb{X}$. Such a controlled path Z takes values in the subspace of $\mathsf{LPBT}(\mathcal{A})$ spanned by words together with trees with at least one generations. We require a certain Taylor-like series expansion for Z over the components of the rough path $\mathbb{X}$. We define finally the rough integral, that is an endomorphism of the space of controlled paths.

We denote by LPBT_N^2 the subset of trees of LPBT with at most N generations and at least one generation. Recall that the set of trees with at most N generations is denoted LPBT_N.

Such a tree is entirely determined by the forest above his first generation. Recall the definition 4.1 of the map $\sharp:\mathsf{LPBT}(\mathcal{A})\to\mathsf{LPBT}(\mathsf{FC})$. The component of an element $A\in\mathsf{LPBT}(\mathcal{A})$ on the basis element τ is denoted $A^{\tau}\in\mathcal{A}$, $A=\sum_{\tau}A_{\tau}\cdot\tau$.

Definition 4.10 (Controlled n-c. rough path)**.** In the setting introduce above,

1. A $\mathbb{X}$-*1-controlled* n-c rough path Z is a α-Hölder path
$$Z:[0,T]\to\bigoplus_{\tau\in\mathsf{LPBT}_{N-1}}\mathcal{A}^{\otimes|\tau|}\cdot\tau$$
such that for any pair or time $s<t$, one has $\sharp Z_t^{\tau}=\mathbb{X}_{st}(\sharp Z_s)^{\tau}+R^{\tau}_{s,t},\ \tau\in\mathsf{LPBT}_{N-1}$, where $R^{f}_{s,t}$ is an element of $\mathcal{C}_2^{(N-\|\tau\|)\alpha}$. The space of all $\mathbb{X}$-controlled is denoted $\mathcal{C}_1(\mathbb{X})$. We endow this space with the following norm:
$$\|Z\|_{\mathcal{C}_1(\mathbb{X})}=\|Z_0\|+\sum_{\tau\in\mathsf{LPBT}_{N-1}}\|R^{\tau}_{\cdot,\cdot}\|_{\mathcal{C}_2^{(N-\|\tau\|)\alpha}}$$
2. A $\mathbb{X}$-*2-controlled* n-c rough path Z is a α-Hölder path
$$Z:[0,T]\to\bigoplus_{f\in\mathsf{LPBT}^2_N}\mathcal{A}^{\otimes|f|}\cdot f$$
such that for any pair or time $s<t$, one has $\sharp Z_t^{f}=\mathbb{X}_{st}(\sharp Z_s)^{f}+R^{f}_{s,t},\ f\in\mathsf{LPBT}^2_N$, where $R^{f}_{s,t}$ is an element of $\mathcal{C}_2^{(N+1-\|f\|)\alpha}$. The space of all $\mathbb{X}$-controlled is denoted $\mathcal{C}_2(\mathbb{X})$. We endow this space with the following norm:
$$\|Z\|_{\mathcal{C}_2(\mathbb{X})}=\|Z_0\|+\sum_{f\in\mathsf{LPBT}^2_N}\|R^{f}_{\cdot,\cdot}\|_{\mathcal{C}_2^{(N+1-\|f\|)\alpha}}$$

4.3. **Examples of controlled rough paths.** In this section, we restrict our the setting to the one of $C^{\star}$ algebras, in particular $\mathcal{A}$ is endowed with an involution $\star$ compatible in a certain sense with the norm. In addition, we suppose that $\mathbb{X}$ is an α- Hölder n-c. geometric rough path ($0<\alpha<1$) above a path X of *self-adjoint* elements of $\mathcal{A}$. We show that certain transformations of the path X of the form $a(X_.)$ yield controlled rough paths.

Let $N\geq1$ an integer. We denote by $\mathbb{F}_N$ the vector space of complex valued functions on the real line whose Fourier transform is a measure with finite moments up to order N:
$$f\in\mathbb{F}_N\Leftrightarrow f(x)=\int_{\mathbb{R}}e^{ix\xi}\mu_f(\mathrm{d}\xi),\ \ \text{with }\mu_f(|x|^l)<+\infty,\ l\leq k.$$
A function f in $\mathbb{F}_N$ is in particular N times continuously differentiable. We denote by $\mathcal{P}$ the space of polynomial functions on $\mathbb{R}$. Notice that $p(A)$ for any $A\in\mathcal{A}$, $p\in\mathcal{P}$ is well defined and $f(A)$

for any $F \in \mathbb{F}_0$ and any self-adjoint element $A \in \mathcal{A}$ is well defined too. The norm of $f \in \mathbb{F}_N$ is $\|f\| = \sum_{i=0}^{k} \mu_f(|x|^l)$. In the sequel, we denote by $\mathcal{A}_\star$ the vector space of self-adjoint elements of $\mathcal{A}$. We study the differentiability properties of $f \in \mathbb{F}_N$ as a function from $\mathcal{A}_\star \to \mathcal{A}_\star$.

Definition 4.11. Let $N \geq 1$ an integer and f be a function in $\mathbb{F}_N$. Define $\partial_X f \in \mathsf{LPBT}(\mathcal{A})$ for any $X \in \mathcal{A}_\star$ by

$$\partial^\tau_X f = \int_{\mathbb{R}} \mu_f(\mathrm{d}\xi)(i\xi)^{\|\tau\|} \int_{(\alpha_0,\ldots,\alpha_{\|\tau\|})\in[0,1]^{\|\tau\|}} e^{\mathrm{i}\alpha_0\xi X} \otimes \cdots \otimes e^{\mathrm{i}\alpha_{\|\tau\|}\xi X} \otimes e^{\mathrm{i}(1-\sum_{i=0}^{\|\tau\|}\alpha_i)\xi X} \mathrm{d}\alpha_0 \cdots \mathrm{d}\alpha_{\|\tau\|}$$

Proposition 4.12. *Pick $n \leq N$ integers and let f be a function in $\mathbb{F}_N$, X a self-adjoint element, then*

$$\mathrm{d}^n_X f(Y_1,\ldots,Y_n) = \sum_{\tau\in\mathsf{LPBT}_n} \sharp(\partial^\tau_X f)(Y_1,\ldots,Y_n),\ Y_1,\ldots,Y_n \in \mathcal{A}_\star.$$

Besides, with $X, Y \in \mathcal{A}_\star$ are self-adjoint elements, we have for all tree τ with at most n leaves:

$$\|\partial^\tau_X f - \partial^\tau_Y f\| \leq \|f\|_n \|X - Y\|_{\mathcal{A}}.$$

Proof. The proof is a simple induction on the order n using the two following equations

$$d^n_X f(Y_1,\ldots,Y_n) = d_X(d^{n-1}_{\cdot} f(Y_1,\ldots,Y_{n-1})(Y_n)) = \sum_{\tau\in\mathsf{LPBT}_{n+1}} d_X(\partial^\tau_{\cdot} f \,\sharp\, (Y_1,\ldots,Y_{n-1}))(Y_n).$$

$$e^{x+h} - e^x = \int_0^1 e^{\alpha(x+h)} h e^{(1-\alpha)x} \mathrm{d}\alpha.$$

□

The following proposition is a particular case of a more general result that we state in the forthcoming paper.

Proposition 4.13. *Let $\mathbb{X}$ be a truncated rough path with Hölder regularity α of order $N = \lfloor \frac{1}{\alpha} \rfloor$ above a trajectory X. Pick a function $a \in \mathbb{F}_{N+1}$, then $t \mapsto \partial_{X_t} f$ is a $\mathbb{X}$-controlled rough path.*

Proof. Recall the definition of the map L introduced in Section 3.3. Then it holds that

$$\partial_{x_t} a = \mathsf{L}(U^a_{x(t)} \circ (1 \otimes 1 \cdot \curlyvee)), \tag{39}$$

with $U^a = \sum_{\tau\in\mathsf{LPBT}} \dfrac{1}{\|\tau\|!} \partial^\tau_{x(t)} a$. Then, Corollary 3.14 implies that

$$\mathbb{X}_{st}\Big(\sharp\partial_{x_s} a\Big) = \mathbb{X}_{st}\Big(\mathsf{L}(\sharp U^a_{x(s)} \circ \sharp(1 \otimes 1 \cdot \curlyvee))\Big) = \mathsf{L}\Big(\sharp U^a_{x(s)} \circ \mathbb{X}_{st}\sharp(1 \otimes 1 \cdot \curlyvee)\Big)$$

Since $\mathbb{X}_{st}\sharp(1 \otimes 1 \cdot \curlyvee) = x(t) - x(s) + 1 \otimes 1 \cdot \curlyvee$ and ${U^a_{x(t)}}^\tau = {U^a_{x(s)}}^\tau + R^\tau_{st}$, with $R^\tau_{st} \in \mathcal{C}_2^{\alpha(N-\|\tau\|)}$, we obtain

$$\mathbb{X}_{st}(\sharp\partial_{x(s)} a) = \mathsf{L}(\sharp U^a_{x(t)} \circ (x(t) - x(s) + 1 \otimes 1 \cdot \curlyvee)) + \mathsf{L}(R_{st} \circ (x(t) - x(s) + 1 \otimes 1 \cdot \curlyvee))$$

Call E_{st} the second term in the right hand side of the above equation. Pick a leveled tree τ. The component E^τ_{st} is a sum over leveled trees τ' with $\|\tau'\| \geq \|\tau\|$ of terms obtained by composing $R^{\tau'}_{st}$ (in the operad FC) with a polynomial with $\|\tau'\| - \|\tau\|$ entries equal to $(x(t) - x(s))$ and the other entries equal to $1 \otimes 1 \cdot \curlyvee$. It clearly results in a term in $\mathcal{C}_2^{\alpha(N-\|\tau\|)}$. A similar reasoning leads to the same bounds for the coefficiens of the first term in the right hand side of the above equation. □

4.4. **Rough Integral.** Pick a $\mathbb{X}$-controlled process Z and define the germ $A = (A_{s,t})_{s<t\in[0,T]} \in \mathcal{A}^{[0,1]\times[0,1]}$ by

$$A^Z_{st} = \mathbb{X}_{st}(\sharp Z_s)^\bullet = \sum_{\tau\in\mathsf{LPBT}_N} \mathbb{X}^{\|\tau\|}_{st}(\sharp Z^\tau_{st} \cdot \tau)^\bullet. \tag{40}$$

Proposition 4.14. *We have $\delta.A^Z \in C_3^{(N+1)\gamma}$.*

Proof. let $s < u < t$ three times, then

$$
\begin{aligned}
\delta_{sut}A &= \mathbb{X}_{st}(\sharp Z_{st})^{\bullet} - \mathbb{X}_{su}(\sharp Z_{su})^{\bullet} - \mathbb{X}_{ut}(\sharp Z_{ut})^{\bullet} \\
&= (\mathbb{X}_{ut} \circ \mathbb{X}_{su})(\sharp Z_s)^{\bullet} - \mathbb{X}_{su}(\sharp Z_s)^{\bullet} - \mathbb{X}_{ut}(\sharp Z_u)^{\bullet} \\
&= \sum_{\tau \in \mathsf{LPBT}_N^2} \sum_{k \geq 1} (\mathbb{X}_{ut}^{(k)} \circ \mathbb{X}_{su}^{(\|\tau\|-k)})(\sharp Z_s^{\tau})^{\bullet} - \sum_{\tau \in \mathsf{LPBT}_N^2} \mathbb{X}_{ut}^{(\|\tau\|)}(\sharp Z_u^{\tau})^{\bullet} \\
&= \sum_{\tau \in \mathsf{LPBT}_N^2} \sum_{k \geq 1} \mathbb{X}_{ut}^{(k)} \Big(\sum_{\alpha \in \mathsf{LPBT}_{N+1}} \sum_{\alpha \subset \tau} \mathbb{X}_{su}^{(\|\tau\|-\|\alpha\|)}(\sharp Z_s^{\tau}) \Big)^{\bullet} - \sum_{\tau \in \mathsf{LPBT}_N^2} \mathbb{X}_{ut}^{(\|\tau\|)}(\sharp Z_u^{\tau})^{\bullet} \\
&= \sum_{k \geq 1} \sum_{\alpha \in \mathsf{LPBT}_N} \mathbb{X}_{ut}^{(k)}(\sharp Z_u^{\alpha} - R_{su}^{\alpha} \cdot \alpha)^{\bullet} - \sum_{\tau \in \mathsf{LPBT}_N} \mathbb{X}_{ut}^{(\|\tau\|)}(\sharp Z_u^{\tau})^{\bullet} \\
&= \sum_{\tau \in \mathsf{LPBT}_N} \mathbb{X}_{ut}^{(\|\tau\|)}(R_{su}^{\tau}) \in C_3^{\alpha(N-|\alpha|+1)+\alpha|\tau|}. \qquad (41)
\end{aligned}
$$

□

Theorem 4.15. *Let $0 < \alpha < 1$ be a real number and set $N = \lfloor \frac{1}{\gamma} \rfloor$. Let $\mathbb{X}$ be a truncated non-commutative rough path of order N. Pick Z a $\mathbb{X}$ controlled process, then*

1. *There exists an unique α-Hölder path $I^Z : [0,T] \to \mathcal{A}$ such that*
$$\|I_t^Z - I_s^Z - A_{st}^Z\| \preceq |t-s|^{(N+1)\alpha}, \ 0 \leq s,t \leq T.$$
Besides, the following estimate holds:
$$\|I^Z\|_{\mathcal{C}_1^{\gamma}} \leq \sum_{k=1}^{N} (T^{N\alpha}\|Z\|_{\mathcal{C}_2(\mathbb{X})} + T^{\alpha(k-1)}\|Z\|_{C_1^0})\|\mathbb{X}_{\cdot,\cdot}^{(k)}\|_{\mathcal{C}_2^{k\alpha}}$$
2. *The path $\bar{I}^Z = I^Z \bullet \oplus Z$ is a path controlled by $\mathbb{X}$, and*
$$\|\bar{I}^Z\|_{\mathcal{C}_1(\mathbb{X})} \leq T^{\alpha}\|Z\|_{\mathcal{C}_2(\mathbb{X})} + T^{\alpha}\|Z\|_{\mathcal{C}_2(\mathbb{X})} \sum_{k=1}^{N} \|\mathbb{X}_{\cdot,\cdot}^{\tau}\|_{\mathcal{C}_2^{k\alpha}} + \|Z_\cdot\|_{C_1^0} \sum_{k=1}^{N} \|\mathbb{X}_{\cdot,\cdot}^{(k)}\|_{C_2^{k\alpha}}$$

Proof. We prove the first estimate. We start with the following estimate, a consequence of the Sewing Lemma:
$$\|I_\cdot^Z - I_\cdot^Z - A_{\cdot,\cdot}^Z\|_{C_2^{(N+1)\alpha}} \leq \|\delta_{\cdot,\cdot,\cdot}A^Z\|_{C_3^{(N+1)\alpha}}.$$
From equation (41), we get
$$\|\delta_{\cdot,\cdot,\cdot}A^Z\|_{C_3^{(N+1)\alpha}} \leq \|Z\|_{C_2(\mathbb{X})} \sum_{k=1}^{N} \|\mathbb{X}_{\cdot,\cdot}^{(k)}\|_{C_2^{k\alpha}}.$$
Now, from the elementary identity $\|I_\cdot^Z - I_\cdot^Z - A_{\cdot,\cdot}^Z\|_{C_2^{\alpha}} \leq T^{N\alpha}\|I_\cdot^{\mathbb{X},Z} - I_\cdot^Z - A_{\cdot,\cdot}^Z\|_{C_2^{(N+1)\alpha}}$, we deduce the following one:
$$\|I_\cdot^Z - I_\cdot^Z - A_{\cdot,\cdot}^Z\|_{C_2^{\alpha}} \leq T^{N\alpha}\|Z\|_{C_2(\mathbb{X})} \sum_{k=0}^{N} \|\mathbb{X}_{\cdot,\cdot}^{(k)}\|_{C_2^{k\alpha}}.$$
From the very definition of the germ A^Z, we obtain:
$$\|A^Z\|_{C_2^{\alpha}} \leq \|Z_\cdot\|_{C_1^0} \sum_{k=1}^{N} T^{(k-1)\alpha}\|\mathbb{X}_{\cdot,\cdot}\|_{C_2^{k\alpha}}$$
and the first estimate follows. Let us turn our attention to the second assertion. First $\bar{I}_\cdot^Z$ is a 1-controlled path. In fact, owing to the sewing lemma,
$$\bar{I}_t^{Z\bullet} = \bar{I}_s^{Z\bullet} + \sum_{\tau \in \mathsf{LPBT}_N} \mathbb{X}_{st}^{\|\tau\|}(\sharp \bar{I}_s^{Z\tau} \cdot \tau)^{\bullet} + R_{st}^{\bullet}$$

with $R_{st} \in C_2^{N\alpha}$. Secondly, if τ is a tree with at least one generation, because Z is a controlled path

$$\begin{aligned}\bar{I}_t^{Z\tau} = Z_t^\tau &= \sum_{\substack{\tau':\tau\subset\tau'\\ \tau'\in\mathsf{LPBT}_{N-1}}} \mathbb{X}_{st}^{(\|\tau'\|-\|\tau\|)}(Z_s^{\tau'}) + \sum_{\substack{\tau':\tau\subset\tau'\\ \|\tau'\|=N}} \mathbb{X}_{st}^{(\|\tau'\|-\|\tau\|)}(Z_s^{\tau'}) + R_{st}^\tau \\ &= \sum_{\substack{\tau':\tau\subset\tau'\\ \tau'\in\mathsf{LPBT}_{N-1}}} \mathbb{X}_{st}^{(\|\tau'\|-\|\tau\|)}(\bar{I}_s^{Z\tau'}) + \sum_{\substack{\tau':\tau\subset\tau'\\ \|\tau'\|=N}} \mathbb{X}_{st}^{(\|\tau'\|-\|\tau\|)}(Z_s^{\tau'}) + R_{st}^\tau\end{aligned}$$

with $R_{st}^\tau \in C_2^{(N+1-\|\tau\|)\alpha}$ and the second sum is a sum of Hölder function in $C_2^{N\alpha}$. Thus $\bar{I}^Z$ is a $\mathbb{X}$-1-controlled path. The estimate on the norm of $\bar{I}^Z$ follows easily from the computations above.

□

We call I^Z *rough integral* of Z against X and denote it by

$$I_t^Z = \int_0^t \sharp Z(\mathrm{d}X_t).$$

4.5. **Loose ends.**

4.5.1. *Rough non-commutative differential equations.* In a following paper, we prove existence of a solution to equations (1) and continuity of the latter with respect to a topology on a space of n-c. rough path. However, we miss a transformation on the space controlled rough path, besides integration, that is composition with a smooth function of a n-c. controlled path.

4.5.2. *Existence of non-commutative rough paths.* We expect the method used in [] to apply in our case to state a n-c. version of the Lyons-Victoir extension theorem.

4.5.3. *Branched case.* In rough path theory, one distinguishes geometric rough paths to branched rough path (introdued by M. Gubinelli in [**?**]). For geometric rough paths, classical differential calculus holds and in particular if f is a $C^\infty(\mathbb{R}^2,\mathbb{R})$ function,

$$f(X_t) = \int_s^t f'\mathrm{d}X_t, \tag{42}$$

where X is a Hölder path that lifts to a geometric rough path $\mathbb{X}$ and the integral in the right hand side is a rough integral. There are situations for which the above equation does not hold ie Itö theory of integration. The above formula holds because $\mathbb{X}$ is an algebra morphism for the shuffle product, which makes possible to expand products of iterated integrals as sum over iterated integrals. It is however possible to build a complete theory of rough path in a non-geometric setting by collecting all iterated integrals and their products in a object (a representation) of an algebra of decorated trees (In fact a Hopf algebra). In our context it is not clear so far what are the good combinatorial objects replacing labeled trees indexing all product of the (full and partial) contraction operators.

4.5.4. *Probabilistic setting.* Let $\mathcal{A}$ be a $C^\star$-algebra accomodating a free Brownian motion $w=(w_t)_{t\geq 0}$ measurable with respect to a filtration $\mathcal{F}=(\mathcal{F}_t)_{t\geq 0}$. Using free theory of integration of Biane and Speicher, one can define a product Lévy area (see [5]) above w,

$$\mathcal{F}_s^{\times 3} \ni (A_0,A_1,A_2) \mapsto \int_{s<t_1<t_2<t} A_0\mathrm{d}w_{t_1}A_1\mathrm{d}w_{t_2}A_2. \tag{43}$$

with $\int_{s<t_1<t_2<t}\mathrm{d}w_{t_1}\otimes_\sigma \mathrm{d}w_{t_2}$ the spatial Lévy area introduced by C. Donati-Martin. In particular, this contraction operator is only partially defined, in contrast with our setting. Following the lines of reasoning exposed above, probabilistic n-c rough path should be defined as a 2-parameters family of partially defined operators on the model space $\mathsf{LPBT}(\mathsf{FC})$,

$$\mathbb{X}_{st} : \mathsf{LPBT}(\mathsf{FC})_s \to \mathsf{LPBT}(\mathsf{FC})_t$$

where $(\mathsf{LPBT}(\mathsf{FC})_s)_{s\geq 0}$ is a filtration of the model space induced by the filtration $\mathcal{F}$. The operator $\mathbb{X}_{st}$ is triangular but with a zero order component $\mathbb{X}_{st}^{(0)}$ acting as the injection $i_s : \mathsf{LPBT}(\mathsf{FC})_s \to \mathsf{LPBT}(\mathsf{FC})_t$.

These operators are not invertible, discarding the method exposed in [] based on the Baker–Campbell–Hausdorff formula to prove existence of a n-c. rough path above a n-c. stochastic process. However, we can state formally what are we are looking for. Besides compatibility with the shuffle product, we require for the operators $\mathbb{X}_{st}$ to be compatible with the involution. Remember that a stochastic process is a collection of self-adjoint operators, thus with $A_0, \ldots, A_n \in \mathcal{A}$ and X a stochastic process with bounded variations, for any leveled forest $f \in \mathsf{LPBF}$

$$(\mathbb{X}^f_{st})^\star(A_0 \otimes \cdots \otimes A_n) = \mathbb{X}^f_{st}((A_0 \otimes \cdots \otimes A_n)^\star)^\star = \mathbb{X}^{f^\bullet}_{st}(A_0 \otimes \cdots \otimes A_n). \tag{44}$$

with $f^\bullet$ is the horizontal mirror symmetric of the forest. The faces contraction map $\sharp$ is compatible with the involution on the model space $\mathsf{LPBT}(\mathcal{A})$ defined by $(A_{(1)} \otimes \cdots \otimes A_{(|\tau|)} \cdot \tau)^\star = A^\star_{(|f|)} \otimes \cdots \otimes A^\star_{(1)} \cdot \tau^\star$,

$$\sharp((A_{(1)} \cdots A_{(|\tau|)} \cdot \tau)^\star) = (\sharp(A_{(1)} \otimes \cdots \otimes A_{(|\tau|)}))^\star \tag{45}$$

with $A_1 \otimes \cdots \otimes A_{(|\tau|)} \cdot \tau \in \mathsf{LPBT}(\mathcal{A})$. With these definitions, the two models $\bar{\mathbb{X}}_{st}$ and $\tilde{\mathbb{X}}_{st}$ are $\star$ -homomorphisms. The filtration $\mathsf{LPBT}(\mathsf{FC})$ at time s induced by the filtration $\mathcal{F}$ is the closure (weak or strong) of the faces contractions operators

$$\sharp(A_1 \otimes \cdots \otimes A_{(|\tau|)} \cdot \tau),\ A_1, \ldots, A_{(|\tau|)} \in \mathcal{F}_s. \tag{46}$$

We can now give a definition of a probabilitstic n-c. rough path.

Definition 4.16. A *probabilistic n-c. rough path* above a path X of Hölder regularity α of self-adjoint operators progressively measurable with respect to a filtration $\mathcal{F}$ is the data of a two parameters family of $\star$ algebra endomorphisms $\{\mathbb{X}_{st}, 0 < s < t < T\}$ with

$$\mathbb{X}_{st} : \mathsf{LPBT}(\mathsf{FC})_s \to \mathsf{LPBT}(\mathsf{FC})_t \tag{47}$$

satisfying the following properties:

1. $\mathbb{X}_{st}(m_{\vee} \cdot \vee) = m_{\vee}(X_{st})$,
2. $\mathbb{X}_{ut} \circ \mathbb{X}_{su} = \mathbb{X}_{st}$,
3. $\mathbb{X}_{st} \prec |t-s|^{k\alpha}$,
4. $\mathbb{X}_{st}(\mathsf{FC}(\tau)_s - \mathrm{id}_{\mathcal{F}_s}) \subset \bigoplus_{\tau' \subsetneq \tau} \mathsf{FC}(\tau')$, $\tau \in \mathsf{LPBT}$.

We can go through the whole theory exposed above, in particular, provided that controlled n-c are measureable, the rough integral is well defined.

5. Annexe : Operads and pros

We start we the basic definitions of collections, non-symmetric operads, bi-collections and PROS and shuffle algebras.

Definition 5.1 (Collection)**.** A collection P is a sequence of vector spaces $(P(n))_{n\geq 1}$. A morphism between two collections is a sequence of linear morphisms $(\phi(n))_{n\geq 1}$ with $\phi_n : P(n) \to P(n)$, $n \geq 1$. The category of all collections is denoted Coll.

The tensor product $\bullet$ on the category Coll is the 2-functor from Coll $\times$ Coll to Coll defined by:

$$(P \bullet Q)(n) = \bigoplus_{\substack{k\geq 1\\ n_1+\cdots+n_k=n}} P(k) \otimes Q(n_1) \otimes \cdots \otimes Q(n_k),\ (f \bullet g)(n) = \bigoplus_{\substack{k\geq 1\\ n_1+\cdots+n_k=n}} f(k) \otimes g(n_1) \otimes \cdots \otimes g(n_k).$$

where $\otimes$ is the standard monoidal structure on the category $\mathrm{Vect}_{\mathbb{C}}$ of complex vector spaces. The unit element for the tensor product $\bullet$ is the collection denoted by $\mathbb{C}_\bullet$ with $\mathbb{C}_\bullet(n) = \delta_{n=1}\mathbb{C}$.

Definition 5.2 (Non-symmetric operad)**.** An non-symmetric operad $\mathcal{P}$ (or simply an operad) is a monoid in the monoidal category (Coll, $\bullet$, $\mathbb{C}$), i.e., a triple $\mathcal{P} = (P, \rho, \eta_P)$ with

$$P \in \text{Coll},\ \rho : P \bullet P \to P,\ \eta_P : \mathbb{C} \to P,$$

satisfying $(\rho \bullet \mathrm{id}_P) \circ \rho = (\mathrm{id}_P \bullet \rho) \circ \rho$ and $(\eta_P \bullet \mathrm{id}_P) \circ \rho = (\mathrm{id}_P \bullet \eta_P) \circ \rho = \mathrm{id}_P$.

We use the notation $\bullet$ for the tensor product on collection to not confuse it with composition of functions. It is common to use the notation $\circ$ for an operadic composition:

$$\rho(p \otimes (q_1 \otimes \cdots \otimes q_{|p|})) = p \circ (q_1 \otimes \cdots \otimes q_{|p|}) \tag{48}$$

Accordingly, the notations $\circ_i$ for partial compositions:

$$p \circ_i q = p \circ (1^{\otimes k-1} \otimes q \otimes \ldots 1^{|p|-k}), \ 1 \leq i \leq |p|. \tag{49}$$

We use these notations if there are no risks of confusion.

5.1. **Bi-collections.** In this section, we formalize the idea of composing operators with multiple in- and outputs (many-to-many operators). Branching outputs of an operator to the inputs of another one defines a product on a space of many-to-many operators. We refer to this product by the terminology *vertical.* There is another way to compose such operators: concatenating the outputs (resp. the inputs) of two operators. This is the *horizontal* product. We give a definition of a PROS in the category of bicollections using the language of 2-monoidal categories (or duoidal categories).

Definition 5.3 (Bicollection)**.** A bicollection is a two parameters family of vector spaces

$$P = (P(n,m))_{n,m\geq 0}.$$

A morphism between two bicollections P and Q is a family of linear maps $\phi(n,m) : P(n,m) \to Q(n,m)$. The category of all bicollections is denoted Coll_2.

Definition 5.4 (Horizontal tensor product)**.** The horizontal tensor product $\otimes$ is the functor $\otimes : \mathrm{Coll}_2 \times \mathrm{Coll}_2 \to \mathrm{Coll}_2$ defined by:

$$(P \otimes Q)(n,m) = \bigoplus_{\substack{n_1+n_2=n \\ m_1+m_2=m}} P(n_1,m_1) \otimes Q(n_2,m_2), \ (f \otimes g)(n,m) = \bigoplus_{\substack{n_1+n_2=n \\ m_1+m_2=m}} f(n_1,m_1) \otimes g(n_2,m_2).$$

The identity element for the horizontal tensor product $\otimes$ is the bicollection $\mathbb{C}_{\otimes}(n,m) = \delta_{n,m=0}\mathbb{C}$.

Definition 5.5 (Vertical tensor product)**.** The tensor product $\boxtimes$ on the category Coll_2 is defined by:

$$(P \boxtimes Q)(n,m) = \bigoplus_{k\geq 0} P(n,k) \otimes Q(k,m), \ (f \boxtimes g)(n,m) = \bigoplus_{k\geq 0} f(n,k) \otimes g(k,m).$$

The identity element for the tensor product $\boxtimes$ is the bicollection $\mathbb{C}_{\boxtimes}(n,m) = \delta_{n=m}\mathbb{C}$.

Fundamental examples of bicollections are obtained by taking polynomials on operators in a given collection. Pick $P = (P_n)_{n\geq 1}$ a collection, and define a bicollection P by

$$P(m,n) = \bigoplus_{k_1+\cdots+k_n=m} P_{k_1} \otimes \cdots \otimes P_{k_n} \text{ and set } T_{\otimes}(P) = \mathbb{C}1 \oplus P. \tag{50}$$

with 1 being an element with 0 inputs and zero 0 outputs. All bicollections we work with are of the form (50). For example, considering $P = (\mathrm{Hom}(B^{\otimes n}, B))_{n\geq 0}$ the bicollection $T_{\otimes}(\mathrm{Hom}(B))$ plays a prominent role in the sequel. We have already encountered another example of bicollection whose homogeneous components are spanned by forests with a certain number of trees and leaves.

In Figure 10 the reader will find a pictorial description of elements in the horizontal and vertical tensor products. In the vertical tensor product, the number of inputs of the operator on the lower level matches the number of outputs of the operator on the upper level. In comparison with the vertical tensor product introduced in [12], the tensor product $P \boxtimes Q$ we introduce here is a sum over planar 2-level diagrams with only one vertex on each level (see Fig. 10). In [12], the author considers bisymmetric sequences of vector spaces, and the monoidal structure involves either a sum over 2−level connected graphs for properads or on connected graphs for props.

It is easy to design a generalization of the vertical tensor product: we sum over connected planar diagrams connecting vertices placed on the integer points of the lines $\mathbb{R} \times \{0\}$ to vertices placed on the line $\mathbb{R} \times \{1\}$.

Let us mention that the vertical tensor $\boxtimes$ has also been considered by Bultel and Giraudo in [4], in which the authors define Hopf algebraic type structures on PROS. The vertical tensor product for a pair of bicollections of the form (50), can also be depicted as a sum over (not-necessarily connected) two level planar graphs, obtained as concatenation of corollas.

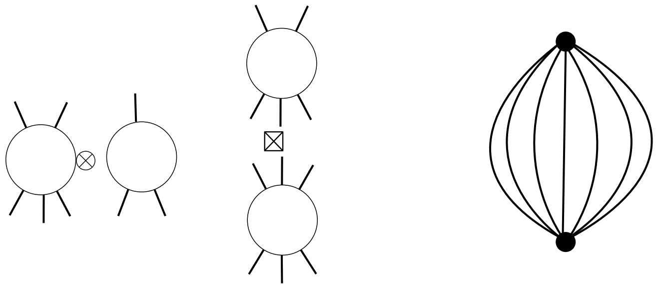

FIGURE 10. On the left, we have elements in the horizontal $\otimes$ and vertical $\boxtimes$ tensor product s. On the right, we have a bundle.

Remark 5.6. The tensor product $\otimes$ is a symmetric one, whereas $\boxtimes$ is not. Neither the horizontal nor the vertical tensor product come with injections and the units for these two tensor products are not initial objects.

In the sequel, to distinguish elements in the tensor products $A \boxtimes B$ or $A \otimes B$, we use the notation $a \boxtimes b$, respectively $a \otimes b$. In the first case, the notation emphasizes that fact that the number of inputs of a matches the number of outputs of b. The standard monoidal tensor product on the category $\mathrm{Vect}_{\mathbb{C}}$ of vector spaces is denotes $\otimes$.

Proposition 5.7. *Let C_i, $1 \leq i \leq 4$ be four bicollections, then*

$$(C_1 \boxtimes C_2) \otimes (C_3 \boxtimes C_4) \hookrightarrow (C_1 \otimes C_3) \boxtimes (C_2 \otimes C_4)\,. \tag{51}$$

The morphism is denoted by R_{C_1,C_2,C_3,C_4}. With C a collection, one has:

$$(C_1 \boxtimes T_{\otimes}(C)) \otimes (C_2 \boxtimes T_{\otimes}(C)) \simeq (C_1 \otimes C_2) \boxtimes T_{\otimes}(C). \tag{52}$$

Proof. Let C_1, C_2, C_3 and C_4 be four bicollections. Let p^1, p^2, p^3, p^4 be elements of respectively, C_1, C_2, C_3 and C_4 with the number of outputs of p^2 matching the number of inputs of p^1 and the same for p^3 and p^4. We denote by S the braiding of the symmetric monoidal category $(\otimes, \mathrm{Vect}_{\mathbb{C}})$ Next, we define

$$R_{C_1,C_2,C_3,C_4} : C_1 \boxtimes C_2 \otimes C_3 \boxtimes C_4 \to C_1 \boxtimes C_3 \otimes C_2 \boxtimes C_4$$

by

$$\begin{aligned} R_{C_1,C_2,C_3,C_4}\left((p^1 \boxtimes p^2) \otimes (p^3 \boxtimes p^4)\right) &= R_{C_1,C_2,C_3,C_4}\left((p^1 \otimes p^2) \otimes (p^3 \otimes p^4)\right) \\ &= (\mathrm{id} \otimes S \otimes \mathrm{id})(p^1 \otimes p^2 \otimes p^3 \otimes p^4) = p^1 \otimes p^3 \otimes p^2 \otimes p^4 = (p^1 \otimes p^3) \boxtimes (p^2 \otimes p^4). \end{aligned}$$

First, it is easy to see that R_{C_1,C_2,C_3,C_4}, is well defined, and if extended linearly it becomes a morphism of bicollections. Moreover it is injective. However, it is not surjective. In particular, the image of R is the span of the elements $(p^1 \otimes p^3) \boxtimes (p^2 \otimes p^4)$ with a perfect match between the inputs of p^3 and the outputs of p^4 on one hand, the inputs of p^1 and the outputs of p^2 on the other hand.

To prove the second assertion, we first notice that $T_{\otimes}(C)$ is endowed with an unital algebraic structure, given by the concatenation of words, for which $1 \in T_{\otimes}(C)$ is the unit. We denote by $m : T_{\otimes}(C) \otimes T_{\otimes}(C) \to T_{\otimes}(C)$ the algebra map. We denote by $q_1 \cdots q_s$ the product of operators $q_1, \ldots, q_s$ in $T_{\otimes}(C)$. For brevity, we also use the notation $|p|$ for the number of inputs of an operator p in a bicollection. Define the map

$$\tilde{R}_{C_1,T_{\otimes}(C),C_2,T_{\otimes}(C)} : (C_1 \otimes C_2) \boxtimes T_{\otimes}(C) \to (C_1 \boxtimes T_{\otimes}(C)) \otimes (C_2 \boxtimes T_{\otimes}(C))$$

by:

$$\tilde{R}_{C_1,T_{\otimes}(C),C_2,T_{\otimes}(C)}((p^1 \otimes p^2) \boxtimes (p_1 \otimes \cdots \otimes p_{|p^1|+|p^2|})) = (p^1 \boxtimes (p_1 \cdots p_{|p^1|})) \otimes (p^2 \boxtimes (p_{|p^1|+1} \cdots p_{|p^1|+|p^2|})),$$

with the convention that if $|p^1| = 0$ or $|p^2| = 0$, then we set $p_1 \cdots p_{|p_1|} = 1$, and, respectively, $p_{|p_1|+1} \cdots p_{|p_1|+|p_2|} = 1$. We should prove first that

$$\tilde{R}_{C_1,T_\otimes(C),C_2,T_\otimes(C)} \circ ((\mathrm{id}_{C_1} \otimes \mathrm{id}_{C_2}) \boxtimes m) \circ R_{C_1,T_\otimes(C),C_2,T_{\otimes(C)}} = \mathrm{id}. \tag{53}$$

Notice that $1 \in T_\otimes(C)$ is the unique element in $T_\otimes(C)$ with zero outputs (also the unique one with zero inputs). Assume first that $|p^1|, |p^2| > 0$. The left hand side of (53) applied to

$$p^1 \boxtimes (p_1 \otimes \cdots \otimes p_{|p^1|}) \otimes (p^2 \boxtimes (q_1 \otimes \cdots \otimes q_{|p^2|}))$$

gives:

$$\begin{aligned}\tilde{R}_{C_1,T_\otimes(C),C_2,T_\otimes(C)} &\Big((p^1 \otimes p^2) \boxtimes (p_1 \otimes \cdots \otimes p_{|p^1|} \otimes q_1 \otimes \cdots \otimes q_{|p^2|})\Big) \\ &= p^1 \boxtimes (p_1 \otimes \cdots \otimes p_{|p^1|}) \otimes (p^2 \boxtimes (q_1 \otimes \cdots \otimes q_{|p^2|})).\end{aligned}$$

Now assume that $|p^1| = 0$. Then, the left hand side of (53) applied to $(p^1 \boxtimes 1) \otimes (p^2 \boxtimes (q_1 \otimes \cdots \otimes q_{|p^2|}))$ gives:

$$\tilde{R}_{C_1,T_\otimes(C),C_2,T_\otimes(C)} \Big((p^1 \otimes p^2) \boxtimes (q_1 \cdots q_{|p^2|})\Big) = (p^1 \boxtimes 1) \otimes (p^2 \boxtimes (q_1 \cdots q_{|p^2|})).$$

Finally, the same line of thoughts applies to prove that

$$((\mathrm{id}_{C_1} \otimes \mathrm{id}_{C_2}) \boxtimes m) \circ R_{C_1,T_\otimes(C),C_2,T_{\otimes(C)}} \circ \tilde{R}_{C_1,T_\otimes(C),C_2,T_\otimes(C)} = \mathrm{id}. \tag{54}$$

□

The natural transformation R is sometimes called *exchange law* and the relation (51) is called *middle-four interchange.*

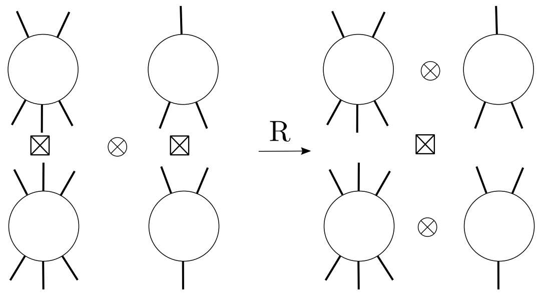

FIGURE 11. Drawing of the action of the natural functor R. On the left hand side, the vertical products are taken first between vertically arranged pairs, then we take the horizontal product. On the right hand side, we do the opposite.

A remark on the graphical presentation of the exchange law in Figure 11. In [1], the authors rather than using the symbols $\otimes$ and $\boxtimes$ replace them a by a simple straight line to indicate the operation that precede. Other authors follow a different convention and choose to represent by a straight line the last operation. In that case, on the left hand side in Figure 11, the horizontal line of symbol is replaced by a vertical line following that convention, and correspondingly for the right hand side.

The family of morphisms $\{R_{C_1,C_2,C_3,C_4}, C_i \in \mathrm{Coll}_2\}$ define a natural transformation between the two functors $\otimes \circ \boxtimes \times \boxtimes$ and $\boxtimes \circ \otimes \times \otimes$. In fact, pick four morphisms $f_i : C_i \to D_i$, $1 \leq i \leq 4$, the diagram in Figure 12 is a commutative diagram.

Proposition 5.8 (Proposition 6.3.5 in [1]). *The category* $(\mathrm{Alg}_\otimes, \boxtimes, \mathbb{C}_\boxtimes)$ *is a monoidal category. If* $(A, m^A_\otimes, \eta_A)$ *and* $(B, m^B_\otimes, \eta_B)$ *are horizontal algebras, then the horizontal product* $m^{A\boxtimes B}_\otimes$ *on* $A \boxtimes B$ *is defined by :*

$$m^{A\boxtimes B}_\otimes = (m^A_\otimes \boxtimes m^B_\otimes) \circ R_{A,B,A,B}. \tag{55}$$

The category $(\mathrm{coAlg}_\boxtimes, \otimes, \mathbb{C}_\otimes)$ *is a monoidal category. If* $(A, \Delta^\boxtimes_A)$ *and* $(B, \Delta^\boxtimes_B)$ *are two vertical co-algebras, then*

$$\Delta^\boxtimes_{A\otimes B} = R_{A,B,A,B} \circ (\Delta^\boxtimes_A \otimes \Delta^\boxtimes_B)$$

$$\begin{tikzcd}
(C_1 \boxtimes C_2) \otimes (C_3 \boxtimes C_4) \arrow[r, "(f_1\boxtimes f_2)\otimes(f_3\boxtimes f_4)"] \arrow[d, "R_{C_1,C_2,C_3,C_4}"] & (D_1 \boxtimes D_2) \otimes (D_3 \boxtimes D_4) \arrow[d, "R_{D_1,D_2,D_3,D_4}"] \\
(C_1 \otimes C_3) \boxtimes (C_2 \otimes C_4) \arrow[r, "(f_1\otimes f_3)\boxtimes(f_2\otimes f_4)"'] & (D_1 \otimes D_3) \boxtimes (D_2 \otimes D_4)
\end{tikzcd}$$

FIGURE 12. Naturality of R.

defines a coproduct on $A \otimes B$*.*

Following [1], a coalgebra in the category $(\mathrm{Alg}_{\otimes}, \boxtimes, \mathbb{C}_{\boxtimes})$ is called a *bimonoid*, while an algebra in the same monoidal category is called a *double monoid*.

Definition 5.9. (PROS) We call a PROS an algebra in the monoidal category $(\mathrm{Alg}_{\otimes}, \boxtimes, \mathbb{C}_{\boxtimes})$. Otherwise stated a PROS is a tuple $(C, \nabla, m^C_{\otimes}, \eta^C_{\otimes}, \eta^C_{\boxtimes})$ with

$$C \in \mathrm{Coll}_2, \quad \nabla : C \boxtimes C \to C, \quad m^C_{\otimes} : C \otimes C \to C, \quad \eta^C_{\otimes} : \mathbb{C}_{\otimes} \to C, \quad \eta^C_{\boxtimes} : \mathbb{C}_{\boxtimes} \to C$$

with ∇ and $\eta^C_{\boxtimes}$ two horizontal algebra morphisms and $\eta^C_{\otimes} = \eta^C_{\boxtimes} \circ \eta^{\mathbb{C}_{\boxtimes}}_{\otimes}$.

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