A SHUFFLE ALGEBRA POINT OF VIEW ON OPERATOR-VALUED PROBABILITY THEORY

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Abstract. We extend the shuffle algebra perspective on scalar-valued non-commutative probability theory to the operator-valued case. We start by associating to the operator-valued distribution and free cumulants of a random variable elements of the homomorphism operad corresponding to the algebra acting on the operator-valued probability space. Using notions coming from higher category theory, we are able to define an unshuffle Hopf algebra like structure on a properad of non-crossing partitions. We do the exact same construction for a properad of word insertions and construct an unshuffle morphism, the splitting map, between the two unshuffle Hopf algebra. We obtain two half-shuffle fixed point equations corresponding to, respectively, free and boolean moment-cumulants relations.

Contents

1. Introduction

1.1. **Motivation and overview.** In classical probability theory, it is now well established that moment-cumulant relations are best understood in the context of Möbius inversion on the lattice of set partitions

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and its associated incidence co-algebra. The combinatorial side of Voiculescu's (scalar-valued) free probability theory finds its roots in the seminal work of Speicher [30], who developed Rota's work by showing that upon replacing set partitions by non-crossing set partitions, the same machinery could be used to define an equivalent notion of moment-cumulant relations in free probability. More precisely, in free probability moments and cumulants are seen as linear maps on the incidence coalgebra of the lattice of non-crossing partitions and the moment-cumulant relations are expressed in terms of the convolution product of the cumulant map with the zeta function. We refer the reader to [26, 27] for an introduction to the theory of free probability.

When considering operator-valued moments and cumulants, Speicher's results can be (partially) extended [31]. In fact, let $(\mathcal{A}, e, B)$ be an operator-valued probability space. Recall that B is an algebra acting on the right on the left on the involutive algebra $\mathcal{A}$ and $e : \mathcal{A} \to B$ is a B-B linear map [26]. As in the scalar-valued case ($B = \mathbb{C}$), the operator-valued expectation e can be extended to a multiplicative function $E = (e_\pi)_{\pi \in \mathrm{NC}}$ on the lattice NC of non-crossing partitions. In comparison to the scalar-valued case, E does also depend on the *nesting* of the blocks in non-crossing partitions. Operator-valued cumulants depend as well on the nesting of the blocks. Still, the convolution of E as well as operator-valued cumulants with a *scalar-valued* function makes sense, giving rise to operator-valued moment-cumulant relations. Extracting algebraic structures encoding the nesting of blocks is then primordial to a better understanding of the properties of free cumulants. This may also participate in a concise description of relations with their boolean and monotone counterparts.

Recently, Ebrahimi-Fard and Patras proposed a rather different perspective on moment-cumulant relations in the scalar-valued case [12, 13, 14]. It does not involve Möbius inversion on lattices of set partitions. Instead, it is based on combinatorial Hopf algebra. More precisely, by describing a genuine shuffle algebra on words, (cumulants) moments are encoded as values taken by some Hopf algebra (infinitesimal) characters. This setting allows for a unified picture of the three different types of cumulants in non-commutative probability, i.e., free, monotone and boolean, as three faces of a single object, the unshuffle coproduct. This approach naturally gives rise to a (pre-)Lie theoretic description of the relations between the different cumulants in terms of shuffle adjoint transformations. It is critical to emphasize that the shuffle algebra setting does not involve at any point non-crossing partitions and that it has recently been successfully applied in the context of infinitesimal probability, provided that the base field of complex numbers is replaced by a nilpotent algebra of strictly triangular matrices, see [6]. In the case of present interest, the target algebra of the morphisms we consider is non-commutative. As a result, the (pre-)Lie theoretic machinery developed in [14] fails to work in the context of operator-valued probability spaces.

Until recently, it was unclear how the two perspectives, i.e., Möbius inversion on the lattice of non-crossing set partitions on the one hand and shuffle algebra on words on the other, could be related. In [11], the authors started to address this question. They showed that both the lattice and shuffle algebra approaches are governed each by their respective operad of non-crossing partitions and the associated incidence co-algebras. The shuffle algebra approach is associated with the so-called gap-insertion operad of non-crossing partitions, which is going to be extensively used in this work, while the Möbius inversion formulation is encoded by the incidence coalgebra of a set partition-refinement operad. In particular, the incidence co-algebra of the gap-insertion operad bears an unshuffle algebraic structure. The moment and free cumulant morphisms, M respectively K, are solution of left half-shuffle fixed point equations

$$M = \varepsilon + m \prec M, \quad K = \varepsilon + k \prec K$$

The two infinitesimal morphisms m and k encode moments and cumulants of all orders. The two morphisms M and K encode the functions on the lattice of non-cossing partitions extending moments and cumulants.

The main objective of the present work is to extend the shuffle algebraic approach of free, boolean and monotone moment-cumulant relations to the setting of operator-valued probability theory. Our approach mainly relies on the first part of reference [11]. We explain how considering moments and free cumulants of an operator-valued probability space as multiplicative functions on the lattice of non-crossing partitions, naturally leads to an operadic perspective. Such a point of view encompasses the boolean cumulants as well seen as "almost" operadic morphisms on the word insertion operad. This result extends the picture developed in [11] to the operator-valued case.

As already mentioned, in this context both the nesting and the linear ordering of the blocks of a non-crossing partition are essential, since the moments associated with each block do not commute with each other, even if we consider moments of a single random variable. To accommodate this fact, elements of the incidence coalgebra should be considered as operators with multiple outputs and the co-algebraic structure should be replaced by a co-properadic structure. To be more precise, a word build from non-crossing set partitions (including the partition of the empty set) is an operator with as many inputs as gaps between the elements of the partitioned sets. A single output is associated with each partition in the word. The co-properadic structure (which is actually a simpler version of a plain non-symmetric co-properadic structure) is then dual to the gap-insertion operad. A word on non-crossing set partitions should be seen as "a horizontal object" and applying the coproduct map on such a word results in two words that are vertically stacked.

We show that this new insight finds a transparent description by means of a so-called duoidal structure on bicollections (graded vector spaces with two gradings standing for the number of inputs and the outputs of an operator). A *duoidal category* is endowed with two tensor products (we use the symbols $\boxtimes$ and $\otimes$ throughout the article) satisfying a Lax property. The compatibility means essentially that composing horizontally and then vertically, or the other way around, results in the same object drawn with the bicollections at stake on the vertices. We shall use the terminology *vertically* and *horizontally* for the sub-categories of the categories of bicollections, or objects related to one or the other monoidal structure.

After having expounded the duoidal structure of the category of bicollections, we proceed to define the equivalent notion of a $\boxtimes\otimes$-Hopf algebra. The latter has both a vertical product and coproduct which are compatible through a horizontal algebraic structure. The associated convolution monoid of horizontal algebra morphisms valued in a properad of endomorphisms provides a unifying description of the gap-insertion and word insertion operadic morphisms. Free and boolean cumulants are implemented as (horizontal) algebra morphisms for the concatenation product on the space of words on non-crossing partitions. In the free case, this morphism is also an (extension of an) operadic morphism. However, this does not hold in the boolean case.

We enrich the structure of $\boxtimes\otimes$-Hopf algebra by introducing the notion of unshuffle Hopf algebra in a duoidal category. Once again, we show that the $\boxtimes\otimes$-Hopf algebra of words on non-crossing partitions can be endowed with such a structure. The dual of this unshuffle structure gives rise to a shuffle algebraic structure on the class of bicollection homomorphisms from the properad of words on non-crossing partitions to the properad of endomorphisms of B. The operator-valued moments and free cumulants implemented as operadic morphisms on the gap-insertion operad satisfy, separately, left half-shuffle fixed point equations. The horizontal morphism implementing boolean cumulants is the solution of right half-shuffle fixed point equations.

The introduction of a second monoidal structure is supported by the fact that the Lie theoretic perspective on moments and cumulants is restored. In particular, the notion of infinitesimal character makes sense in this setting and requires horizontal composition of partitions (words), while the vertical direction (operadic composition of non-crossing partitions) is used to define the monoid of which the moments as well as the free and boolean cumulants are elements of.

The free and boolean moment-cumulant relations are then retrieved as fixed point equations in a shuffle algebra of bicollection morphisms on a properad of words on random variables. This second shuffle algebra relates to the first one by mean of a shuffle algebra morphism, the so-called splitting map. The aforementioned fixed point equations are then retrieved by pulling back the half-shuffle fixed point equations satisfied by the boolean and free cumulants.

In the context of non-commutative probability theory, various authors have used Hopf algebras and operads from different perspectives. We mention the work of Friedrich–McKay [18], Hasebe–Lehner, [20], Mastnak–Nica [25] as well as the work of Gabriel [19]. In the latter, the author defines Hopf algebraic structures related to additive and multiplicative convolutions by using a geometric perspective on the space of (non-crossing) partitions. Operadic approaches to moment-cumulant relations have already been exploited by Joshuat-Vergès, Menous, Thibon and Novelli in [21] to obtain an operadic version of the shuffle point of view developed by Ebrahimi-Fard and Patras. Another perspective on moment-cumulant relations in an operadic framework was developed by Drummond-Cole in [7] and [8]. We end our (non-exhaustive) summary about previous works related to operator-valued probability theory with the two papers [9, 10] by Dykema, together with the following remark. In these two papers the point of view adopted by the author is fundamentally analytical. This translates in the way non-crossing partitions are considered as operators. It is radically different from our approach. For instance, in Dykema's work a partition of a set S has $|S|-1$ inputs, while in our case such a partition has $|S|+1$ inputs.

1.2. **Operator-valued probability theory.** We start with a small (historical) account on free probability theory and its operator-valued version. Free probability theory was created in 1985 by Dan Voiculescu to understand free factors of von Neumann algebras. Originally developed in the vicinity of the theory of algebras of operators, *freeness* drew probabilists' attention as the right algebraic framework to compute the asymptotic distribution of large random matrices. Creating a common notion encompassing (finite dimensional) distribution of random matrices and their asymptotic requires a further step in the abstraction, notably about what we understand as a *probability space*. In a nutshell, such a space allows for computation of *means* and has a notion of *positivity*, which is so important in classical probability. The very first example of a probability space is the algebra of essentially bounded random variables endowed with the usual expectation.

Conditional expectation is a map acting on a space of random variables measurable with respect to a σ-field $\mathcal{F}_1$ valued in a smaller algebra of random variables measurable with respect to a sub σ-field $\mathcal{F}_2 \subset \mathcal{F}_1$. The conditional mean of a random variable with respect to the sigma field $\mathcal{F}_2$ is not scalar-valued but algebra valued. Still, it enjoys the same positivity property as the usual mean does. Besides, it is linear with respect to left and right multiplication by random variables measurable with respect to the smaller sigma field. These properties in the settings of non-commutative probability translate as follows.

An operator-valued probability space $(\mathcal{A}, E, B)$ is a bi-module involutive complex unital algebra $\mathcal{A}$ over an unital involutive algebra B together with a B-bimodule positive unital morphism, $E : \mathcal{A} \to B$. In symbols, with $a \in \mathcal{A},\ b_1, b_2 \in B$

1. $(b_1 \cdot a) \cdot b_2 = b_1 \cdot (a \cdot b_2),\ (ba)^\star = a^\star b^\star$,
2. $E(b_1 a b_2) = b_1 E(a) b_2,\ E(aa^\star) \in BB^\star$

We define boolean, free and monotone conditional cumulants using Möbius inversion. As for the scalar-valued case, conditional free, boolean and monotone independence is characterized by the vanishing of mixed cumulants. The reader is directed to the monograph [31] for a detailed introduction on the combinatorial aspect of operator-valued probability theory. We denote by $\mathrm{NC}(n)$ the set of all non-crossing partitions of $[\![1, n]\!]$ and by 1_n the unique partition of $\mathrm{NC}(n)$ with only one block.

For simplicity, we pick a single random variable $a \in \mathcal{A}$. The B-valued distribution of the random variable a is the collection of elements in B:

$$E(b_0 a b_1 a \cdots a b_n), \quad b_0, \ldots, b_n \in B,\ n \geq 1. \tag{1}$$

Recall that $\mathbf{1}_n$ denotes the partition with a single block. Let $a \in \mathcal{A}$ be a random variable, we denote by $B[a]$ the smallest B-B bi-module algebra containing a. Speicher's original recursive definition of $e_\pi : B[a]^{\otimes_B |\pi|} \to B,\ \pi \in \mathrm{NC}$ is as follows:

$$e_{\mathbf{1}_n}(a_1 \otimes \cdots \otimes a_n) = E(a_1 \cdots a_{n+1}),$$

$$e_\pi(a_1 \otimes \cdots \otimes a_n) = e_{\pi_1}(a_1 \otimes \cdots \otimes a_{k-1} \otimes E(a_k \otimes \cdots \otimes a_l)a_{l+1} \otimes \cdots \otimes a_n). \tag{2}$$

Here $[\![k,l]\!]$ is an interval in π and π_1 is the restriction of π to $[\![1,n]\!]\backslash[\![k,l]\!]$. From this perspective, e_π is a map that takes random variables as inputs, which can be pictured as sitting on the legs of the partition π. These random variables are considered as fixed parameters or as defining a colourization of the partition. We remark that the operadic perspective developed in this work starts with a different point of view on the distribution of the random variable a. In fact, we see it as a collection of homomorphisms in the endomorphisms operad of B:

$$E(a^{\otimes n})(b_0, \ldots, b_n) = E(b_0 a b_1 a b_2 \cdots a b_n). \tag{3}$$

For each $n \geq 1$, $E(a^{\otimes n})$ is an element of $\mathrm{Hom}(B^{\otimes n+1}, B)$. We explain in the forthcoming sections how to construct for each partition π a map E_π from $B^{\otimes|\pi|}$ to B that if considered altogether comprises the distribution of a and is in fact an operadic morphism on a genuinely defined operadic structure on the set of non-crossing partitions. Denote by $\mathrm{Int}(n)$ the of all interval partitions in $\mathrm{NC}(n)$. Let us recall the free and boolean moment-cumulant relations for operator-valued cumulants:

$$E(a_1 \cdots a_n) = \sum_{\pi \in \mathrm{NC}(n)} \kappa_\pi(a_1, \ldots, a_n) = \sum_{I \in \mathrm{Int}(n)} \beta_I(a_1, \ldots, a_n),\ a_1, \ldots, a_2 \in \mathcal{A}. \tag{MC}$$

In the last equations, the definitions of κ_π and β_I follow from equations (2) with $\kappa_{1_{l-k+1}}(a_k \otimes \ldots \otimes a_l)$ (respectively $\beta_{1_{l-k+1}}(a_k \otimes a_l)$) in place of $E(a_k \otimes \cdots \otimes a_l)$. Since κ_{1_n} does not enter in the definition of κ_π with $\pi \neq 1_n$, the first relation in (MC) yields an inductive definition of the maps $\kappa_{1_n},\ n \geq 1$:

$$\kappa_{1_n}(a_1 \otimes \cdots \otimes a_n) = E(a_1 \cdots a_n) - \sum_{\substack{\pi \in \mathrm{NC}(n) \\ \pi \neq 1_n}} \kappa_\pi(a_1 \otimes \cdots \otimes a_n) \tag{4}$$

The inductive definition of the boolean cumulants $\beta_{1_n},\ n \geq 1$ proceeds from:

$$\beta_{1_n}(a_1, \ldots, a_n) = E(a_1 \cdots a_n) - \sum_{\substack{I \in \mathrm{Int}(n) \\ I \neq 1_n}} \beta_I(a_1 \ldots a_n). \tag{5}$$

1.3. **Outline.** We now outline the details of the relations (MC). First, we construct the operad $\mathcal{NC}$ of non-crossing partitions in Section 2. We then construct operadic morphisms E and K from this operad to the operad of homomorphisms on B implementing the set of cumulants κ_π and moments $E_\pi,\ \pi \in \mathcal{NC}$. We then address the problem of constructing a (convolution) monoid containing those two morphisms. To that aim, we introduce in Section 3 the notion of duoidal category as well as a notion of Hopf algebra in this context. In a duoidal category, objects can be composed in two different –but compatible– ways, either *horizontally*, either *vertically*. Pursuant to this are two notions of algebras and co-algebras, one for each tensor product. All of this is explained in Section 3.

The central result in Section 3 is Lemma 11. We show in Proposition 16 that the space of non-commutative polynomials on non-crossing partitions can be endowed with such a structure. As a consequence, its class of so-called horizontal algebra morphisms with values in the properad of endomorphisms of B is a monoid, containing both the maps E and K standing for the distribution and the free cumulants of a random variable. The main result of Section 4 is the following one.

Proposition (Proposition 27)**.** $(\mathrm{Hom}_{\mathrm{Coll}_2}(T_\otimes(\mathcal{NC}), T_\otimes(\mathrm{Hom}(B))), \prec, \succ, \star)$ *is a shuffle algebra.*

Thanks to the compatibility between the horizontal and vertical monoidal products, we can raise the notion of infinitesimal morphism in this context. In Section 4.3, we compute explicitly the left and right half-shuffle exponentials. In particular, we show that both K and E are solutions of left half-shuffle fixed point equations. See Proposition 28 as well as the Proposition 29:

$$K = \eta \circ \varepsilon + k \prec K, \quad E = \eta \circ \varepsilon + e \succ E. \tag{6}$$

Each summand on the righthand side of equation (MC) is interpreted as a value of solutions of half-shuffle fixed point equations.

Next, we define a structure for unshuffle Hopf algebra, similar to that of the operad of non-crossing partitions (adapted to the duoidal setting) on an operad of words insertions in Section 4.4. We prove the following proposition.

Proposition (Proposition 34)**.** $(\mathrm{Hom}_{\mathrm{Coll}_2}(\mathcal{W}, T_{\otimes}(\mathrm{Hom}(B)), \prec, \succ, \star)$ *is a shuffle algebra.*

In addition, we define a map $Sp : \mathcal{W} \to T_{\otimes}(\mathcal{NC})$, the splitting map, induces a morphisms between the two shuffle algebras constructed previously, see proposition Proposition 35 it reads:

$$Sp(a_1 \dots a_n) = \sum_{\pi \in \mathrm{NC}(n)} \pi \otimes a_1 \dots a_n. \tag{7}$$

Then, by pulling-back on the word insertion operad the first equation in (6), we finall arrive at our main result, Proposition 38 stating that (28) is equivalent to the fixed point equation in $\mathcal{W}$:

$$E = \eta \circ \varepsilon + k \prec E. \tag{8}$$

2. The gap-insertion operad of non-crossing partitions

In this section we settle the algebraic structure on non-crossing partitions used throughout this work. We start with a short reminder on collections and operads (both set and linear). Then, we formalize in this framework the idea of inserting a partition into the gaps of another partition. The reader is directed to [11] for a detailed exposition on this so-called gap-insertion operad and related structures. For general background on algebraic operads, both planar and symmetric as well as related concepts, we refer the reader to the monograph [24].

2.1. **Set partitions.** Let X be a finite, linearly ordered set. A *partition* of X into disjoint sets (called blocks), $\pi_i, 1 \leq i \leq k$, is denoted $\pi = \{\pi_1, \dots, \pi_k\}$. An isomorphism between two set partitions is a monotone bijection of the underlying linearly ordered sets compatible with the block structure. Then, any partition is equivalent to a partition of the linearly ordered set $[\![1, n]\!]$ for some $n \in \mathbb{N}$, which we call standard partition. It is convenient to work with the standard representative of each class.

For $k, n \in \mathbb{N}^\star$, we denote by $\mathrm{SP}(k, n)$ the set of iso-classes of partitions of sets of n elements into k blocks. The set $\mathrm{SP}(0, 0)$ contains only the empty partition. We put

$$\mathrm{SP} = \bigsqcup_{1 \leq k \leq n} \mathrm{SP}(k, n),\ \mathrm{SP}(n) = \bigsqcup_{k \leq n} \mathrm{SP}(k, n),\ \mathrm{SP}_0 = \mathrm{SP}(0, 0) \sqcup \mathrm{SP}.$$

Given a monotone inclusion of linearly ordered sets $X \subset Y$ and given a partition π of Y, we write $\pi_{|X}$ for the trace of the partition of π on X.

Definition 1. Let X be a non-empty finite subset of $\mathbb{N}$ (or a linearly ordered set Y). The *convex hull* of X is by definition $\mathrm{Conv}(X)=[\![\min(X), \max(X)]\!]$. We shall say that X is convex if $\mathrm{Conv}(X) = X$. Any finite subset $X \subset \mathbb{N}$ decomposes uniquely as

$$X = X_1 \sqcup \dots \sqcup X_k$$

with each X_i convex and each $X_i \sqcup X_j$ *not* convex for $i \neq j$. The X_i are called the convex components of X.

Definition 2 (Non-crossing partitions)**.** A partition $\pi = \{\pi_1, \dots, \pi_k\}$ is non-crossing if there are no $a, b \in \pi_i$ and no $c, d \in \pi_j$ with $i \neq j$ such that $a < c < b < d$.

See Figure 1 for examples of partitions. For a detailed overview of the algebraic structures of the set of non-crossing partitions, as well as an historical account, see [29]. The notion of non-crossing partitions has first been introduced by Kreweras in the seminal article [23].

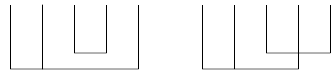

FIGURE 1. Example of a non-crossing partition on the left, and a partition with a crossing on the right.

Definition 3 (Interval partitions)**.** We say that a non-crossing partition $\pi \in \mathrm{NC}$ is an *interval partition* if and only if all the blocks of π are convex sets.

For each integer $n \geq 1$, let $\mathrm{Int}(n)$ be the set of all interval partitions of $[\![1, n-1]\!]$. Set $\mathrm{Int} = \bigcup_{n\geq 0} \mathrm{Int}(n)$, then Int is a sub-collection of $\mathcal{NC}$.

2.2. **Algebraic planar operads.** A collection P is a sequence of vector spaces $(P(n))_{n\geq 1}$. A morphism between two collections is a sequence of linear morphisms $(\phi(n))_{n\geq 1}$ with $\phi_n : P(n) \to P(n)$, $n \geq 1$. The category of all collections is denoted Coll. The tensor product $\bullet$ on the category Coll is the 2-functor from Coll $\times$ Coll to Coll defined by:

$$(P \bullet Q)(n) = \bigoplus_{\substack{k\geq 1 \\ n_1+\cdots+n_k=n}} P(k) \otimes Q(n_1) \otimes \cdots \otimes Q(n_k),\ (f \bullet g)(n) = \bigoplus_{\substack{k\geq 1 \\ n_1+\cdots+n_k=n}} f(k) \otimes g(n_1) \otimes \cdots \otimes g(n_k).$$

The unit element for the tensor product $\bullet$ is the collection denoted by $\mathbb{C}_\bullet$ such that $\mathbb{C}_\bullet(n) = \delta_{n=1}\mathbb{C}$. An operad $\mathcal{P}$ is a monoid in the monoidal category $(\mathrm{Coll}, \bullet, \mathbb{C})$, i.e., a triple (P, ρ, η_P) with

$$P \in \mathrm{Coll},\ \rho : P \bullet P \to P,\ \eta_P : \mathbb{C} \to P,$$

satisfying $(\rho \bullet \mathrm{id}_P) \circ \rho = (\mathrm{id}_P \bullet \rho) \circ \rho$ and $(\eta_P \bullet \mathrm{id}_P) \circ \rho = (\mathrm{id}_P \bullet \eta_P) \circ \rho = \mathrm{id}_P$. We use the notation $\bullet$ for the tensor product on collection to not confuse it with composition of functions. It is common to use the notation $\circ$ for an operadic composition:

$$\rho(p \otimes (q_1 \otimes \cdots \otimes q_{|p|})) = p \circ (q_1 \otimes \cdots \otimes q_{|p|}) \tag{9}$$

Accordingly, the notations $\circ_i$ for partial compositions:

$$p \circ_i q = p \circ (1^{\otimes k-1} \otimes q \otimes \ldots 1^{|p|-k}),\ 1 \leq i \leq |p|. \tag{10}$$

We should use these notations if there are no risks of confusion.

2.3. **Operad of partitions.** A partition $\pi \in \mathrm{SP}(n)$ is viewed as an operator with $n+1$ inputs. These inputs are the gaps between the elements of the partitioned set, including the front gap before 1 and the back gap after n. We can insert $n+1$ partitions inside these gaps. It is clear that if π is a non-crossing partition and we insert non-crossing partitions into the gaps of π then the resulting partition is again non-crossing.

Definition 4. We set $\mathcal{SP}(n) := \mathrm{SP}(n-1)$. In particular, we have $\mathcal{SP}(0) = \emptyset$ and $\mathcal{SP}(1) = \{\emptyset\}$. The empty partition is the operad unit. Let π be a partition and $(\alpha_1, \ldots, \alpha_{|\pi|})$ a sequence of set partitions. The composition $\rho_{\mathcal{SP}}(\pi \otimes \alpha_1 \otimes \cdots \otimes \alpha_{|\pi|})$ is obtained by inserting each partition α_i in between the two integers i and $i+1$, $i \leq 1$. In symbols:

$$\rho_{\mathcal{SP}}(\pi \otimes \alpha_1 \otimes \cdots \otimes \alpha_{|\pi|}) = \bigcup_{i=1}^{|\pi|} \{i - 1 + b,\ b \in \pi_i\} \cup \tilde{\pi}$$

where $\tilde{\pi}$ is the partition of $\{|\pi_1|, |\pi_1| + |\pi_2|, \ldots, |\pi_1| + \cdots + |\pi_n|\}$ induced by π.

Lemma 5. *The sequence $\mathcal{NC} = (\mathcal{NC}(n))_{n\geq 1}$ with $\mathcal{NC}(n-1) = \mathrm{NC}(n)$ defines a set operad called the non-crossing gap-insertion operad when equipped with the composition law $\rho_{\mathcal{NC}} = \rho_{\mathcal{SP}|\mathcal{NC}\bullet\mathcal{NC}}$.*

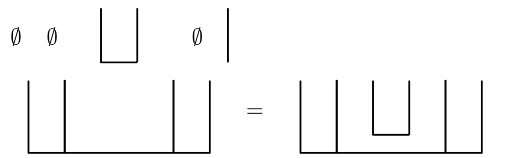

FIGURE 2. Example of a composition in the gap-insertion operad $\mathcal{NC}$.

The two set operadic structures $\rho_{\mathcal{SP}}$ and $\rho_{\mathcal{NC}}$ induce linear operadic structures on the free vector spaces spanned by SP, respectively NC. In the following, we shall not distinguish between them. The gap-insertion operad of non-crossing partitions admits the following presentation in terms of generators and relations.

Lemma 6 (Proposition 3.1.4 in [11])**.** *For any $n \geq 1$, we put $1_{n+1} = \{[\![1, n]\!]\}$. Then the operad $(\mathcal{NC}, \rho_{\mathcal{NC}})$ is generated by the elements 1_n, $n \geq 1$ with the relation:*

$$\forall m, n \geq 1, \quad 1_m \circ_m 1_n = 1_n \circ_1 1_m.$$

Let $a \in \mathcal{A}$ a random variable, we defined in the introduction for each $n \geq 1$ the map from $B^{\otimes n}$ to B:

$$E_{n+1}(b_0, \ldots, b_n) = E(a^{\otimes n})(b_0, \ldots, b_n) = E(b_0 a b_1 a \cdots a b_n) \tag{11}$$

with $b_0, \ldots, b_n \in B$. Now since E is B-B bimodule map, we get

$$E_1(b_0) = b_0 \tag{12}$$

$$E_n \circ_n E_m = E_m \circ_1 E_n. \tag{13}$$

In fact, we have, for $b_1, \ldots, b_{n+m} \in B$

$$\begin{aligned}(E_n \circ_n E_m)(b_0, \ldots, b_{n+m}) &= E(b_0 a b_1 \cdots a E_m(b_n, \ldots, b_{n+m}) = E(b_0 a b_1 \cdots a) E(b_n a \cdots b_{n+m}) \\ &= E(E(b_0 a b_1 \cdots a) b_n a \cdots b_{n+m}) = E(E(b_0 a b_1 \cdots a b_n) a \cdots b_{n+m}) \\ &= (E_m \circ_1 E_n)(b_0, \ldots, b_{n+m}).\end{aligned}$$

Hence, there exists an unique operadic morphism $E : \mathcal{NC} \to \mathrm{Hom}(B)$ such that $E(1_n) = E_n$. The free cumulants of E enjoy the same property: there exists an unique operadic morphism $K : \mathcal{NC} \to \mathrm{Hom}(B)$ such that

$$K(1_n)(b_0, \ldots, b_n) = k_n(b_0 a b_1, \ldots, a b_n). \tag{14}$$

Non-crossing partitions are central to free probability theory. In boolean probability theory, interval partitions are fundamental. Unfortunately, the collection of interval partitions is not a sub-operad in $\mathcal{NC}$. Boolean cumulants are rather multiplicative for the canonical algebra structure on interval partitions. Hence, we have to handle objects coming from free and boolean operator-valued probability theories, with very different algebraic properties. To implement the shuffle point of view for operator-valued probability theory, we will follow [11]. It starts with the definition of a coalgebraic structure on the vector space N of non-commutative polynomials in non-crossing partitions. On N, the operadic composition leads to a bialgebraic structure with coproduct $\Delta : N \to N \otimes N$:

$$\Delta(\pi) = \sum_{\alpha \circ \beta_1, \ldots, \beta_{|\alpha|}} \alpha \otimes \beta_1 \cdots \beta_{|\alpha|}. \tag{15}$$

Note that if $B = \mathbb{C}$, associated with the moment and the free cumulants of the random variable a are two algebra morphisms implementing the extensions of the moments and cumulants of a to the poset of non-crossing partitions. These morphisms are elements of the convolution algebra of characters of N. In our case, rather than having characters on N, we have maps $T_{\otimes}(E)$ and $T_{\otimes}(K)$ from N to a space of words on elements of the endomorphism operad of P, extending to N the maps E and K constructed previously.

In addition, since K and E are operadic morphisms, we see that two natural compositions of words on non-crossing partitions should be considered: a concatenation and a composition extending the operadic structure on non-crossing partitions. The following section formalizes this idea using the notion of duoidal category.

Notice that we considered the case of a single random variable, but the construction of the operadic morphism K and E extends readily to the multivariate case by considering coloured partitions.

3. The duoidal category of bicollections

Elements of a collection are seen as operations with many inputs and a single output. The operadic structure models compositions between these operations. In many branches of mathematics, ranging from probability theory, both classical and non-commutative, to gauge theory and quantum groups, algebraic structures with products and co-products that stand for merging respectively cutting processes have become popular. The framework of operads is however too narrow to treat such structures completely. Indeed, it turns out to be important to be able to handle operations with multiple in- and outputs. After the work of JP. Serre [28] and B. Vallette [32], the right algebraic framework appears to be the one of properads (props). The construction we expose in the section is reminiscent of the props setting, but is in fact much simpler.

We introduce now the pro-eminent algebraic structure to the present work, i.e., the category of bicollections endowed with two balanced monoidal structures. In the literature, such a category is called a *duoidal*[1] category. This section focuses on the so-called *laxity property* stated in (18). It is beyond the scope for the present work to provide the reader with a detailed account on the notion of duoidal category. Nevertheless, for the sake of completeness, we will give the definition of such a category, without fully commenting on it.

Definition 7 (Duoidal category)**.** *A duoidal category*, or 2-monoidal category, is a category endowed with a monoidal structure $(\otimes, E_{\otimes})$, together with an additional monoidal structure $(\boxtimes, E_{\boxtimes})$ such that $\boxtimes : C \times C \to C$ and $E_{\boxtimes} : 1 \to C$ are lax monoidal functors with respect to $(\otimes, E_{\otimes})$ and the coherence axioms of $(\boxtimes, E_{\boxtimes})$ are monoidal natural transformation with respect to $(\otimes, E_{\otimes})$. The laxity of $(\boxtimes, E_{\boxtimes})$ consists of natural transformations

$$(C_1 \boxtimes C_2) \otimes (C_3 \boxtimes C_4) \overset{R_{C_1,C_2,C_3,C_4}}{\longrightarrow} (C_1 \otimes C_3) \boxtimes (C_2 \otimes C_4),$$

together with morphisms $E_{\otimes} \to E_{\otimes} \boxtimes E_{\otimes}$, $\quad E_{\boxtimes} \otimes E_{\boxtimes} \to E_{\boxtimes}$, $\quad E_{\otimes} \to E_{\boxtimes}$.

To get acquainted with these two operations, the vertical and the horizontal composition, we draw an example. Denote by $\mathcal{F}$ the vector space generated by all planar rooted forests (words on planar rooted trees). There are two ways to compose forests, either we concatenate them or we stack them vertically. For the latter operation to be well defined, the number of leaves of the bottom forest should match the number of trees of the top forest, see Figure 3.

The main result of this section is the following one.

Proposition. *The category of bicollections is a duoidal category.*

For a nice exposition of duoidal categories, we refer to [1, Chap. 6]. Here duoidal categories are called 2-monoidal categories. From a more historical viewpoint, there are at least two other notions similar to that of a duoidal category, introduced in earlier work. The first one is the notion of *two fold monoidal categories* of Baltenau and Fiedorowcz [3, 22]. In such a category, the two monoidal structures are required to be *strict* (this property holds for the category of bicollections, see below) but also to share a common unit object (which is not the case for the category of bicollections). Later Forcey, Siehler and Sowers [17] improved upon this notion by removing the strictness assumption, allowing the unit objects to be different, but requiring stronger assumptions on the units. This fails for the duoidal category of bicollections.

[1] https://ncatlab.org/nlab/show/duoidal+category

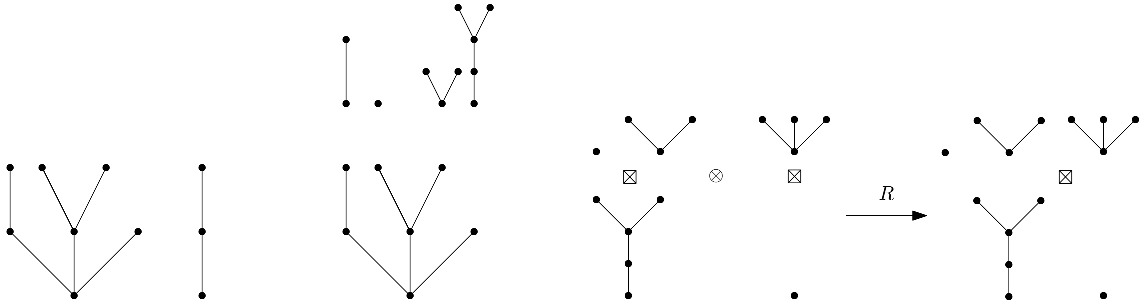

FIGURE 3. On the left, we stack horizontally two trees. In the middle, with stack vertically two forests. On the right we draw an example of the action of the natural transformation R on forests.

3.1. **The horizontal and vertical tensor product.** We formalize the idea of considering operators with multiple in- and outputs in the following definition.

Definition 8 (Bicollection)**.** A bicollection is a two parameter family of vector spaces $\boldsymbol{P} = (\boldsymbol{P}(n,m))_{n,m\geq 0}$. A morphism between two bicollections $\boldsymbol{P}$ and $\boldsymbol{Q}$ is a sequence of linear morphisms $\phi(n,m) : \boldsymbol{P}(n,m) \to \boldsymbol{Q}(n,m)$. The category of all bicollections is denoted Coll_2.

Example. *Let $Pl(n,m)$ be the linear span of planar forests (made of planar trees) with n trees and m leaves. Then $Pl = (Pl(n,m))_{n,m\geq 1}$ is a bicollection.*

Definition 9 (Horizontal tensor products)**.** The horizontal tensor product $\otimes$ is the functor $\otimes$: $\mathrm{Coll}_2 \times \mathrm{Coll}_2 \to \mathrm{Coll}_2$ defined by:

$$(\boldsymbol{P} \otimes \boldsymbol{Q})(n,m) = \bigoplus_{\substack{n_1+n_2=n \\ m_1+m_2=m}} \boldsymbol{P}(n_1,m_1) \otimes \boldsymbol{Q}(n_2,m_2),\ (f \otimes g)(n,m) = \bigoplus_{\substack{n_1+n_2=n \\ m_1+m_2=m}} f(n_1,m_1) \otimes g(n_2,m_2).$$

The identity element for the horizontal tensor product is the bicollection $\mathbb{C}_{\otimes}(n,m) = \delta_{n,m=0}\mathbb{C}$.

Definition 10 (Tensor product of bicollections)**.** The tensor product $\boxtimes$ on the category Coll_2 is defined by:

$$(\boldsymbol{P} \boxtimes \boldsymbol{Q})(n,m) = \bigoplus_{k\geq 0} \boldsymbol{P}(n,k) \otimes \boldsymbol{Q}(k,m),\ (f \boxtimes g)(n,m) = \bigoplus_{k\geq 0} f(n,k) \otimes g(k,m).$$

The identity element for the tensor product $\boxtimes$ is the bicollection $\mathbb{C}_{\boxtimes}(n,m) = \delta_{n=m}\mathbb{C}$.

Fundamental examples of bicollections are given by polynomials on operators in a collection. If $P = (P_n)_{n\geq 0}$ is a collection, define a bicollection $\boldsymbol{P}$ by its homogeneous component of degree m,n:

$$\boldsymbol{P}(m,n) = \bigoplus_{\substack{n\geq 1 \\ k_1+\cdots+k_n=m}} P_{k_1} \otimes \cdots \otimes P_{k_n}. \tag{16}$$

We set also:

$$T_{\otimes}(C) = \mathbb{C}1 \oplus \boldsymbol{P}, \tag{17}$$

with 1 being an element of 0 inputs and zeor 0 outputs.

All the bicollections appearing in this work are of the form (17). For example, considering $P = (\mathrm{Hom}(B^{\otimes n}, B))_{n\geq 0}$ the bicollection $T_{\otimes}(\mathrm{Hom}(B))$ plays a proeminent role in the sequel. We have already encountered another example of bicollection whose homogeneous components are generated by forests with a certain number of trees and leaves.

In Figure 4 the reader will find a pictorial description of elements in the horizontal and vertical tensor products. In the vertical tensor product, the number of inputs of the operator on the lower level matches the number of outputs of the operator on the upper level. In comparison with the vertical tensor product introduced in [32], the tensor product $\boldsymbol{P} \boxtimes \boldsymbol{Q}$ we introduce here is a sum over planar 2-level diagrams with only one vertex on each level (see Fig. 4). In [32], the author considers bisymmetric sequences of vector spaces, and the monoidal structure involves either a sum over 2−level connected graphs for properads or on connected graphs for props.

It is easy to design a generalization of the vertical tensor product: we sum over planar diagrams (connected or not) that connect vertices placed on integer points of the lines $\mathbb{R} \times \{0\}$ to vertices placed on the line $\mathbb{R} \times \{1\}$. Let us mention that the vertical tensor $\boxtimes$ has also been considered by Bultel and Giraudo in [5], in which the authors define Hopf algebraic type structures on pros. The vertical tensor product for a pair of bicollections of the form (17), can also be depicted as a sum over (not-necessarily connected) two level planar graphs, obtained as concatenation of corollas.

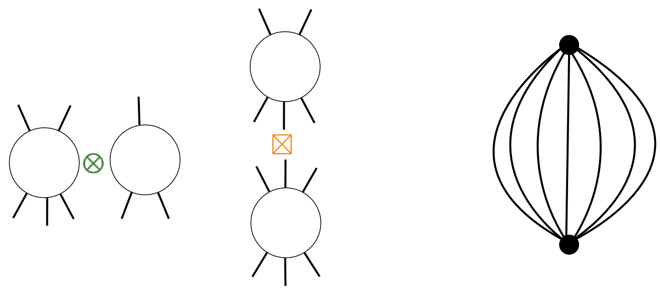

FIGURE 4. On the left, we have elements in the horizontal $\otimes$ and vertical $\boxtimes$ tensor product s. On the right, we have a bundle.

Remark 1. The tensor product $\otimes$ is a symmetric one, whereas $\boxtimes$ is not. Neither the horizontal nor the vertical tensor product come with injections and the units for these two tensor products are not initial objects.

The following lemma is crucial and is the cornerstone for definition of the notion Hopf-like structure in the context of duoidal category.

In the sequel, to distinguish elements in the tensor products $A \boxtimes B$ or $A \otimes B$, we use the notation $a \boxtimes b$, respectively $a \otimes b$. In the first case, the notation emphasizes that fact that the number of inputs of a matches the number of outputs of b. The standard monoidal tensor product on the category $\mathrm{Vect}_{\mathbb{C}}$ of vector spaces is denotes $\otimes$.

Proposition 11. *Let C_i, $1 \leq i \leq 4$ be four bicollections, then*

$$(C_1 \boxtimes C_2) \otimes (C_3 \boxtimes C_4) \hookrightarrow (C_1 \otimes C_3) \boxtimes (C_2 \otimes C_4)\,. \tag{18}$$

The morphism is denoted by R_{C_1,C_2,C_3,C_4}. If C is a collection

$$(C_1 \boxtimes T_{\otimes}(C)) \otimes (C_2 \boxtimes T_{\otimes}(C)) \simeq (C_1 \otimes C_2) \boxtimes T_{\otimes}(C). \tag{19}$$

Proof. Let C_1, C_2, C_3 and C_4 be four bicollections. Let p^1, p^2, p^3, p^4 be elements of respectively, C_1, C_2, C_3 and C_4 with the number of outputs of p^2 matching the number of inputs of p^1 and the same for p^3 and p^4. We denote by S the braiding of the symmetric monoidal category $(\otimes, \mathrm{Vect}_{\mathbb{C}})$ Next, we define

$$R_{C_1,C_2,C_3,C_4} : C_1 \boxtimes C_2 \otimes C_3 \boxtimes C_4 \to C_1 \boxtimes C_3 \otimes C_2 \boxtimes C_4$$

by

$$\begin{aligned} R_{C_1,C_2,C_3,C_4}\left((p^1 \boxtimes p^2) \otimes (p^3 \boxtimes p^4)\right) &= R_{C_1,C_2,C_3,C_4}\left((p^1 \otimes p^2) \otimes (p^3 \otimes p^4)\right) \\ &= (\mathrm{id} \otimes S \otimes \mathrm{id})(p^1 \otimes p^2 \otimes p^3 \otimes p^4) = p^1 \otimes p^3 \otimes p^2 \otimes p^4 = (p^1 \otimes p^3) \boxtimes (p^2 \otimes p^4). \end{aligned}$$

First, it is easy to see that R_{C_1,C_2,C_3,C_4}, is well defined, and if extended linearly it becomes a morphism of bicollections. Moreover it is injective. However, it is not surjective. In particular, the image of R is the span of the elements $(p^1 \otimes p^3) \boxtimes (p^2 \otimes p^4)$ with a perfect match between the inputs of p^3 and the outputs of p^4 on one hand, the inputs of p^1 and the outputs of p^2 on the other hand.

To prove the second assertion, we first notice that $T_{\otimes}(C)$ is endowed with an unital algebraic structure, given by the concatenation of words, for which $1 \in T_{\otimes}(C)$ is the unit. We denote by $m : T_{\otimes}(C) \otimes T_{\otimes}(C) \to T_{\otimes}(C)$ the algebra map. We denote by $q_1 \cdots q_s$ the product of operators $q_1, \ldots, q_s$ in $T_{\otimes}(C)$. For brevity, we also use the notation $|p|$ for the number of inputs of an operator p in a bicollection. Define the map

$$\tilde{R}_{C_1, T_{\otimes}(C), C_2, T_{\otimes}(C)} : (C_1 \otimes C_2) \boxtimes T_{\otimes}(C) \to (C_1 \boxtimes T_{\otimes}(C)) \otimes (C_2 \boxtimes T_{\otimes}(C))$$

by:

$$\tilde{R}_{C_1, T_{\otimes}(C), C_2, T_{\otimes}(C)}((p^1 \otimes p^2) \boxtimes (p_1 \otimes \cdots \otimes p_{|p^1|+|p^2|})) = (p^1 \boxtimes (p_1 \cdots p_{|p^1|})) \otimes (p^2 \boxtimes (p_{|p^1|+1} \cdots p_{|p^1|+|p^2|})),$$

with the convention that if $|p^1| = 0$ or $|p^2| = 0$, then we set $p_1 \cdots p_{|p_1|} = 1$, and, respectively, $p_{|p_1|+1} \cdots p_{|p_1|+|p_2|} = 1$. We should prove first that

$$\tilde{R}_{C_1, T_{\otimes}(C), C_2, T_{\otimes}(C)} \circ ((\mathrm{id}_{C_1} \otimes \mathrm{id}_{C_2}) \boxtimes m) \circ R_{C_1, T_{\otimes}(C), C_2, T_{\otimes}(C)} = \mathrm{id}. \tag{20}$$

Notice that $1 \in T_{\otimes}(C)$ is the unique element in $T_{\otimes}(C)$ with zero outputs (also the unique one with zero inputs). Assume first that $|p^1|, |p^2| > 0$. The left hand side of (20) applied to

$$p^1 \boxtimes (p_1 \otimes \cdots \otimes p_{|p^1|}) \otimes (p^2 \boxtimes (q_1 \otimes \cdots \otimes q_{|p^2|}))$$

gives:

$$\begin{aligned} &\tilde{R}_{C_1, T_{\otimes}(C), C_2, T_{\otimes}(C)} \left((p^1 \otimes p^2) \boxtimes (p_1 \otimes \cdots \otimes p_{|p^1|} \otimes q_1 \otimes \cdots \otimes q_{|p^2|}) \right) \\ &\qquad\qquad = p^1 \boxtimes (p_1 \otimes \cdots \otimes p_{|p^1|}) \otimes (p^2 \boxtimes (q_1 \otimes \cdots \otimes q_{|p^2|})). \end{aligned}$$

Now assume that $|p^1| = 0$. Then, the left hand side of (20) applied to $(p^1 \boxtimes 1) \otimes (p^2 \boxtimes (q_1 \otimes \cdots \otimes q_{|p^2|}))$ gives:

$$\tilde{R}_{C_1, T_{\otimes}(C), C_2, T_{\otimes}(C)} \left((p^1 \otimes p^2) \boxtimes (q_1 \cdots q_{|p^2|}) \right) = (p^1 \boxtimes 1) \otimes (p^2 \boxtimes (q_1 \cdots q_{|p^2|})).$$

Finally, the same line of thoughts applies to prove that

$$((\mathrm{id}_{C_1} \otimes \mathrm{id}_{C_2}) \boxtimes m) \circ R_{C_1, T_{\otimes}(C), C_2, T_{\otimes}(C)} \circ \tilde{R}_{C_1, T_{\otimes}(C), C_2, T_{\otimes}(C)} = \mathrm{id}. \tag{21}$$

$\square$

The natural transformation R is sometimes called *exchange law* and the relation (18) is called *middle-four interchange*.

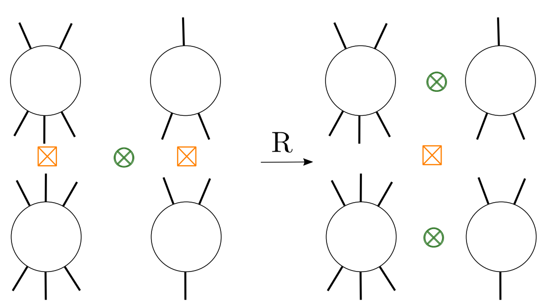

FIGURE 5. Drawing of the action of the natural functor R. On the left hand side, the vertical product are taken first between vertically arranged pairs, then we take the horizontal product. On the right hand side, we do the opposite.

A remark on the graphical presentation of the exchange law in Figure 5. In [1], the authors rather than using the symbols $\otimes$ and $\boxtimes$ replace them a by a simple straight line to indicate the operation that precede. Other authors follow a different convention and choose to represent by a straight line the last operation. In that case, on the left hand side in Figure 5, the horizontal line of symbol is replaced by a vertical line following that convention, and correspondingly for the right hand side.

The family of morphisms $\{R_{C_1,C_2,C_3,C_4}, C_i \in \mathrm{Coll}_2\}$ define a natural transformation between the two functors $\otimes \circ \boxtimes \times \boxtimes$ and $\boxtimes \circ \otimes \times \otimes$. In fact, pick four morphisms $f_i : C_i \to D_i$, $1 \leq i \leq 4$, the diagram in Figure 6 is a commutative diagram.

$$\begin{tikzcd}
(C_1 \boxtimes C_2) \otimes (C_3 \boxtimes C_4) \arrow[r, "(f_1 \boxtimes f_2) \otimes (f_3 \boxtimes f_4)"] \arrow[d, "R_{C_1,C_2,C_3,C_4}"] & (D_1 \boxtimes D_2) \otimes (D_3 \boxtimes D_4) \arrow[d, "R_{D_1,D_2,D_3,D_4}"] \\
(C_1 \otimes C_3) \boxtimes (C_2 \otimes C_4) \arrow[r, "(f_1 \otimes f_3) \boxtimes (f_2 \otimes f_4)"'] & (D_1 \otimes D_3) \boxtimes (D_2 \otimes D_4)
\end{tikzcd}$$

Figure 6. Naturality for R.

3.2. Monoids and comonoids.

We present here general results for duoidal categories, all the proofs can be found in the monograph [1, Chapt. 6, Sects. 6.5, 6.6]. We end this subsection with the definition of a $\boxtimes \otimes$-bialgebra. In the next section we will see how to introduce unshuffling in this context, which ultimately leads to the shuffle perspective on operator-valued probability.

We denote by $\mathrm{Alg}_{\boxtimes}$ the category of unital complex associative algebras in the monoidal category $(\mathrm{Coll}_2, \boxtimes, \mathbb{C}_{\boxtimes})$ (see Definition 10) and $\mathrm{Alg}_{\otimes}$ the category of complex associative algebras in the monoidal category $(\mathrm{Coll}_2, \otimes, \mathbb{C}_{\otimes})$. We write both horizontal and vertical algebras as triplets $(A, m^A_{\otimes}, \eta_{\otimes})$ respectively $(A, m^A_{\boxtimes}, \eta_{\boxtimes})$ with:

$$m^A_{\otimes} : A \otimes A \to A, \quad m^A_{\boxtimes} : A \boxtimes A \to A.$$

The unit $\mathbb{C}_{\boxtimes}$ of the vertical tensor product $\boxtimes$ is an algebra in the monoidal category $(\mathrm{Coll}_2, \otimes, \mathbb{C}_{\otimes})$:

$$m_{\mathbb{C}_{\boxtimes}} : \mathbb{C}_{\boxtimes} \otimes \mathbb{C}_{\boxtimes} \to \mathbb{C}_{\boxtimes}, \ 1_n \otimes 1_m \mapsto 1_{n+m}. \tag{22}$$

Proposition 12 (Proposition 6.3.5 in [1]). *The category $(\mathrm{Alg}_{\otimes}, \boxtimes, \mathbb{C}_{\boxtimes})$ is a monoidal category. If $(A, m^A_{\otimes}, \eta_A)$ and $(B, m^B_{\otimes}, \eta_B)$ are horizontal algebras, then the horizontal product $m^{A \boxtimes B}_{\otimes} : (A \boxtimes B) \otimes (A \boxtimes B)$ on $A \boxtimes B$ is defined by :*

$$m^{A \boxtimes B}_{\otimes} = (m^A_{\otimes} \boxtimes m^B_{\otimes}) \circ R_{A,B,A,B}. \tag{23}$$

The category $(\mathrm{coAlg}_{\boxtimes}, \otimes, \mathbb{C}_{\otimes})$ is a monoidal category. If $(A, \Delta^{\boxtimes}_A)$ and $(B, \Delta^{\boxtimes}_B)$ are two vertical coalgebras, then

$$\Delta^{\boxtimes}_{A \otimes B} = R_{A,B,A,B} \circ (\Delta^{\boxtimes}_A \otimes \Delta^{\boxtimes}_B)$$

defines a coproduct on $A \otimes B$.

We proceed with a fundamental example of a set of bicollections endowed with an horizontal and a vertical product. Let P be a collection. First, there exists a canonical isomorphism of bicollections

$$\phi : T_{\otimes}(C \bullet C) \longrightarrow T_{\otimes}(C) \boxtimes T_{\otimes}(C),$$

defined by

$$\phi\Big(\big[p^1 \bullet (q^1_1 \otimes \cdots \otimes q^1_{|p^1|})\big] \otimes \cdots \otimes \big[p^n \bullet (q^n_1 \otimes \cdots \otimes q^n_{|p^n|})\big] \Big) = (p^1 \otimes \cdots \otimes p^n) \boxtimes (q^1_1 \otimes \cdots \otimes q^n_{|p^n|})$$
$$\phi(1) = 1 \boxtimes 1.$$

In fact, from the second assertion of Lemma 11 with $C_1 = C_2 = C$, there exists an isomorphism of bicollections:

$$\bigoplus_{n\geq 1} (C \boxtimes T_{\otimes}(C))^{\otimes n} \simeq \bigoplus_{n\geq 1} C^{\otimes n} \boxtimes T_{\otimes}(C)$$

with $T^{+}_{\otimes}(C)(n,m) = T_{\otimes}(C)$, $n,m \geq 1$ and $T^{+}(C)(0,0) = \{0\}$, since $C_0 = \{0\}$, we have in fact

$$\bigoplus_{n\geq 1} \left(C \boxtimes T^{+}_{\otimes}(C)\right)^{\otimes n} \simeq \bigoplus_{n\geq 1} C^{\otimes n} \boxtimes T^{+}_{\otimes}(C)$$

which implies in turn:

$$T_{\otimes}(C \bullet C) = \mathbb{C}1 \oplus \bigoplus_{n\geq 1}(C \boxtimes T^{+}_{\otimes}(C)) \simeq \mathbb{C}(1 \boxtimes 1) \oplus \bigoplus_{n\geq 1} C^{\otimes n} \boxtimes T^{+}_{\otimes}(C) \simeq T_{\otimes}(C) \boxtimes T_{\otimes}(C).$$

As a consequence, the tensor product $T_{\otimes}(C) \boxtimes T_{\otimes}(C)$ is endowed with an algebra product obtained by pushing forward using ϕ the concatenation product on $T_{\otimes}(P)$. Assume next that $\mathcal{P}$ is endowed with an operadic composition $\rho : P \bullet P \to P$. Then ρ induces an *horizontal morphism* (for the concatenation) denoted $T_{\otimes}(\rho) : T_{\otimes}(P \bullet P) \to T_{\otimes}(P)$ which equals ρ on $P \bullet P \subset T_{\otimes}(\mathcal{P}) \boxtimes T_{\otimes}(\mathcal{P})$.

In addition, the two notions are self-dual.

Proposition 13. *A bicollection C is a cogebra in the monoidal category $(\mathrm{Alg}_{\otimes}, \boxtimes, \mathbb{C}_{\boxtimes})$ if and only if it is an algebra in the monoidal category $(\mathrm{coAlg}_{\boxtimes}, \otimes, \mathbb{C}_{\otimes})$.*

Proof. The two diagrams expressing compatibility between the multiplication map $m^{\otimes}$ and the coproduct $\Delta^{\boxtimes}$ (saying either that $m^{\otimes}$ is $\Delta^{\boxtimes}$ coproduct morphism or that $\Delta^{\boxtimes}$ is a $m^{\otimes}$ algebra map) are both equal to the diagram in Fig. 7. ☐

$$\begin{tikzcd}
C \otimes C \arrow[r, "\Delta \otimes \Delta"] \arrow[dd, "m^{\otimes}"] & (C \boxtimes C) \otimes (C \boxtimes C) \arrow[d, "R_{C,C,C,C}"] \\
 & (C \otimes C) \boxtimes (C \otimes C) \arrow[d, "m^{\otimes} \boxtimes m^{\otimes}"] \\
C \arrow[r, "\Delta"] & C \boxtimes C
\end{tikzcd}$$

FIGURE 7. Compatibility between the multiplication and comultiplication for $\boxtimes\otimes$-bialgebras.

We use the terminology $\boxtimes\otimes$-bialgebras for coalgebra in the monoidal category $(\mathrm{Alg}_{\otimes}, \boxtimes, \mathbb{C}_{\boxtimes})$ or for algebra in the category $(\mathrm{coAlg}_{\boxtimes}, \otimes, \mathbb{C}_{\otimes})$.

Definition 14 (Co-nilpotent $\boxtimes\otimes$-bialgebras)**.** A $\boxtimes\otimes$-bialgebras $(C, \bar{\Delta}, \varepsilon)$ is said *co-nilpotent* if

1. $C(n,n) = \mathbb{C}(n,n) = \mathbb{C}1_n,\ n \geq 0$ and $C(0,m) = C(m,0) = 0,\ m \geq 1$.
2. $\bar{\Delta}(c) = \Delta(c) + 1_m \boxtimes c + c \boxtimes 1_n,\ c \in C(n,m),\ n \neq m$, with

$$\bar{\Delta} : C \to C \boxtimes C,\ \Delta(C(n,m)) \subset \bigoplus_{\substack{k\geq 0\\ k\neq n,m}} C(n,k) \otimes C(k,m),$$

   and $\Delta(1_n) = 1_n \boxtimes 1_n, n \geq 0$.
3. Δ is point-wise nilpotent: for each $c \in C$, there exists an integer $n \geq 0$ such that $\Delta^n(c) = 0$.

3.3. $\boxtimes\otimes$-Hopf algebras and the monoid of horizontal morphisms.

Definition 15 ($\boxtimes\otimes$-Hopf algebras)**.** A bicollection $\boxtimes\otimes$-Hopf algebra is a tuple $(C, \Delta, \nabla, \varepsilon, S, \eta)$ of objects and morphisms in the category $\mathrm{Alg}_{\otimes}$ such that

(1) (C, Δ, ε) is a $\boxtimes\otimes$-bialgebra,
(2) (C, ∇, η) is an unital algebra in $(\mathrm{Alg}_{\otimes}, \boxtimes, \mathbb{C}_{\boxtimes})$,
(3) A morphism $S : C \to C$ of horizontal algebras such that

$$\nabla \circ (S \boxtimes \mathrm{id}_C) \circ \Delta = \nabla \circ (\mathrm{id}_C \boxtimes S) \circ \Delta = \eta \circ \varepsilon. \tag{24}$$

The $\boxtimes\otimes$-Hopf algebra is said *connected* if $\bigoplus_{n\geq 0} C(n,n) \overset{\eta}{\simeq} \mathbb{C}_{\boxtimes}$.

Remark 2. The map S is called an *antipode*. In the definition of a $\boxtimes\otimes$-Hopf algebra we do not assume any compatibility conditions between Δ and ∇. These two morphisms are algebra morphisms with respect to the horizontal algebraic product structure we have on the underlying bicollection, but nothing more. In particular, we can not require for Δ to be ∇ morphisms, this stems from the fact that $C \boxtimes C$ is *not* an $\boxtimes$-algebra, even if C is.

The map S does not enjoy the same properties as the antipodal map of a plain usual commutative Hopf algebra. In particular, it is not a morphism with respect to the product ∇, nor an anti-comorphism with respect to Δ nor an unipotent morphism ($S^2 = \mathrm{id}$). We shall see later that in the case of the $\boxtimes\otimes$-Hopf algebra canonically associated with the gap-insertion operad, the square of the antipode is in fact a projector.

Here again a remark on the terminology we use is in order. According to [1], a $\boxtimes\otimes$-Hopf algebra is a bimonoid and a dimonoid endowed with an extra map S. Defining the notion of Hopf monoid in a duoidal category is an highly non-trivial task and the various –equivalent– definitions of an Hopf algebra can lead to different notions of Hopf monoids in a duoidal category. Often, since for a bimonoid (a $\boxtimes\otimes$-bialgebra) the comonoidal and the monoidal structures are in different monoidal categories the notion of convolution monoid associated with a bimonoid is meaningless. Therefore, the notion of Hopf monoids in a duoidal category can not be defined using (24). See [4] for a detailed discussion on the different possibilities to define Hopf monoids in a duoidal category.

We now show that to the gap-insertion operad $\mathcal{NC}$ is associated a $\boxtimes\otimes$-Hopf algebra. As previously explained, the map $\rho_{\mathcal{NC}}$ extends to an horizontal morphism $T_{\otimes}(\mathcal{NC})$ defining a properadic structure, denoted $\nabla_{T_{\otimes}(\mathcal{NC})}$ on $T_{\otimes}(\mathcal{NC})$. The graded dual $\Delta_{T_{\otimes}(\mathcal{NC})}$ of $\nabla_{T_{\otimes}(\mathcal{NC})}$ which reads on a non-crossing partition π:

$$\Delta_{T_{\otimes}(\mathcal{NC})}(\pi) = \sum_{\substack{\alpha,\beta \in T_{\otimes}(\mathcal{NC}) \\ \nabla_{T_{\otimes}(\mathcal{NC})}(\alpha \boxtimes \beta) = \pi}} \alpha \boxtimes \beta, \tag{25}$$

is an horizontal algebra morphism. If π is a non-crossing partition, we denote by $\sharp\pi$ the number of non-empty blocks of π. Define then the algebra morphism $S : T_{\otimes}(\mathcal{NC}) \to T_{\otimes}(\mathcal{NC})$ by

$$S(I) = (-1)^{\sharp I} I, \text{ if } I \in \mathrm{Int} \text{ and } S(\pi) = 0 \text{ otherwise }. \tag{26}$$

Define the counit $\varepsilon : T_{\otimes}(\mathcal{NC}) \to \mathbb{C}_{\boxtimes}$ as the unique horizontal morphism such that $\varepsilon(\{\emptyset\}) = 1_1$ and $\varepsilon(\pi) = 0$ otherwise. Define also $\eta : \mathbb{C}_{\boxtimes} \to T_{\otimes}(\mathcal{NC})$ by $\eta(1_n) = \{\emptyset\}^n$ for each integer $n \geq 0$.

Proposition 16. *$(T_{\otimes}(\mathcal{NC}), \Delta, \nabla, S, \varepsilon, \eta)$ is a co-nilpotent $\boxtimes\otimes$-Hopf algebra.*

Proof. We check only that $\nabla_{T_{\otimes}(\mathcal{NC})} \circ S \boxtimes \mathrm{id} \circ \Delta_{T_{\otimes}(\mathcal{NC})} = \nabla_{T_{\otimes}(\mathcal{NC})} \circ \mathrm{id} \boxtimes S = \eta \circ \varepsilon$. Let π be a non-crossing partition. Let n be the number of intervals of π and let m be the number of blocks of π not contained in any other blocks. We have:

$$(\nabla_{T_{\otimes}(\mathcal{NC})} \circ S \boxtimes \mathrm{id} \circ \Delta_{T_{\otimes}(\mathcal{NC})})(\pi) = \sum_k \binom{n}{k} (-1)^k \pi = 0 = \sum_k \binom{m}{k} (-1)^k \pi = \nabla_{T_{\otimes}(\mathcal{NC})} \circ \mathrm{id} \boxtimes S \circ \Delta_{T_{\otimes}(\mathcal{NC})}.$$

$\square$

We now define the monoid containing the gap-insertion operadic morphism associated with the moments and cumulants of a random variable.

Definition 17 (Convolution product)**.** Let α, β be two morphisms of bicollection from $T_{\otimes}(\mathcal{NC})$ to the bicollection $T_{\otimes}(\mathrm{Hom}(B))$. We define the *convolution product* $\alpha \star \beta : T_{\otimes}(\mathcal{NC}) \to T_{\otimes}(\mathrm{Hom}(B))$ as the bicollection map:

$$\alpha \star \beta = \nabla_{\mathrm{Hom}(B)} \circ (\alpha \boxtimes \beta) \circ \Delta_{T_{\otimes}(\mathcal{NC})}.$$

Define $\eta_{\mathrm{Hom}(B)} : \mathbb{C}_{\boxtimes} \to T_{\otimes}(\mathcal{NC})$ by $\eta(1_m) = \mathrm{id}_B^m$, $m \geq 1$.

Proposition 18. *With the notation introduced so far, the product $\star$ defines a monoidal structure on* $\mathrm{Hom}_{\mathrm{Coll}_2}(T_{\otimes}(\mathcal{NC}), T_{\otimes}(\mathrm{Hom}(B)))$*.*

The unit element for the convolution product $\star$ is the morphism $\eta_{\mathrm{Hom}(B)} \circ \varepsilon$.

In addition, the class of horizontal algebra homomorphisms $(\mathrm{Hom}_{\mathrm{Alg}_{\otimes}}(T_{\otimes}(\mathcal{NC}), T_{\otimes}(\mathrm{Hom}(B)))$ *is also a monoid for $\star$.*

Finally, then an ($\boxtimes$-al)($\otimes$-al)gebra morphism α is invertible in the monoid of horizontal algebra morphisms $(\mathrm{Hom}_{\mathrm{Alg}_{\otimes}}(T_{\otimes}(\mathcal{NC}), T_{\otimes}(\mathrm{Hom}(B))), \star)$ *and* $\alpha^{-1} = \alpha \circ S$*.*

Proof. The fact that $\mathrm{Hom}_{\mathrm{Coll}_2}(B, A)$ is a monoid follows from coassociativity and associativity property of Δ and ∇. Let α and β be two horizontal morphisms. Since $(\mathrm{Alg}_{\otimes}, \boxtimes, \mathbb{C}_{\boxtimes})$ is a monoid implies that $\alpha \boxtimes \beta$ is a $\otimes$-algebra morphism. Then, $\nabla \circ (\alpha \boxtimes \beta) \circ \Delta$ is a $\otimes$-algebra morphism, as being a composition of such morphisms. Finally, if α is a vertical and a horizontal morphism, we get

$$\begin{aligned}\nabla_{\mathrm{Hom}(B)} \circ ((\alpha \circ S) \boxtimes \alpha) \circ \Delta_{T_{\otimes}(\mathrm{Hom}(B))} &= \nabla_{T_{\otimes}(\mathrm{Hom}(B))} \circ (\alpha \boxtimes \alpha) \circ (S \boxtimes \mathrm{id}) \circ \Delta_{T_{\otimes}(\mathrm{Hom}(B))} \\ &= \alpha \circ \nabla \circ (S \boxtimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon.\end{aligned}$$

□

4. Shuffle point of view on operator-valued probability theory

The main result of this section is Proposition 27. We then compute half-shuffle exponentials and show that any (extension as an horizontal morphism of an) operadic morphism on $\mathcal{NC}$ is a Left half-shuffle exponential. We compute the right half-shuffle exponential and the full shuffle exponential.

First, We recall classical results and definitions related to shuffle algebras. The terminology shuffle refers actually to different kind of objects. In the literature, the first meaning to shuffle arises from products of iterated integrals. As such it designates a commutative binary product. The second meaning refers to topological shuffles, the latter being non-commutative. These notions can be traced back at least to the 1950's, when these two notions were axiomatized in the work of Eilerberg–Maclane and Schützenberger. In this section, shuffle will always refer to the non-commutative case.

A *shuffle* (or *dendrimorphic*) algebra is a $\mathbb{K}$ vector space D together with two bilinear compositions $\prec$ and $\succ$ subject to the following three axioms

$$\begin{aligned}(a \prec b) \prec c &= a \prec (b \prec c + b \succ c), \\ (a \succ b) \prec c &= a \succ (b \prec c), \\ a \succ (b \succ c) &= (a \succ b + a \prec b) \succ c.\end{aligned}$$

These three relations yield the following associative shuffle algebra product $a \sqcup\!\sqcup b = a \prec b + a \succ b$ on D. The products $\prec$ and $\succ$ are called, respectively, *Left half-shuffle* and *right half-shuffle*. The standard example of a commutative shuffle algebra (meaning that $a \sqcup\!\sqcup b = b \sqcup\!\sqcup a$) is provided by the tensor algebra $\bar{T}(V)$ over a $\mathbb{K}$ vector space V endowed with a left half-shuffle recursively defined by

$$(x_1 \otimes \cdots \otimes x_n) \prec (y_1 \otimes \cdots \otimes y_n) = x_1 \otimes (x_2 \otimes \cdots \otimes x_n \sqcup\!\sqcup y_1 \otimes \cdots \otimes y_m).$$

Shuffle algebras are not naturally unital. This is because it is impossible to split the unit equation $1 \sqcup\!\sqcup a = a \sqcup\!\sqcup 1 = a$, into two equations involving the half-shuffles products $\succ$ and $\prec$. This issue is circumvented by using the "Schützenberger" trick, that is, for D a shuffle algebra, $\bar{D} = D \oplus \mathbb{K}1$. denotes the shuffle algebra augemented by a unit $\mathbf{1}$ such that

$$a \prec \mathbf{1} = a = \mathbf{1} \succ a,\ \mathbf{1} \prec a = 0 = a \succ \mathbf{1}$$

implying $\mathbf{1} \shuffle a = a \shuffle \mathbf{1} = a$. By convention, $\mathbf{1} \shuffle \mathbf{1} = \mathbf{1}$, but $\mathbf{1} \prec \mathbf{1}$ and $\mathbf{1} \succ \mathbf{1} = 0$ cannot be defined consistently. The following set of left- and right half-shuffle words in $\bar{D}$ are defined recursively for fixed elements $(x_1, \dots, x_n) \in D$, $n \in \mathbb{N}$

$$\begin{aligned}
w_{\prec}^{(0)}(x_1, \dots, x_n) &= 1 = w^{(0)}(x_1, \dots, x_n)\\
w_{\prec}^{(n)}(x_1, \dots, x_n) &= x_1 \prec w^{(n-1)}(x_2, \dots, x_n)\\
w_{\succ}^{(n)}(x_1, \dots, x_n) &= w^{(n-1)}(x_1, \dots, x_{n-1}) \succ x_n.
\end{aligned}$$

In the case $x_1 = \cdots = x_n = x$, we simply write $x^{\prec n} = w_{\prec}^{(n)}(x, \dots, x)$ and $x^{\succ n} = w_{\succ}^{(n)}(x, \dots, x)$. In the unital algebra $\bar{D}$, both the exponential and logarithm maps are defined in terms of the associative product $\shuffle$:

$$\exp_{\shuffle}(x) = 1 + \sum_{n \geq 1} \frac{x^{\shuffle n}}{n!},\ \log(1 + x) = -\sum_{n \geq 1} (-1)^n \frac{x^{\shuffle n}}{n!}.$$

In general, the two sums in the last equation are formal sums. However, in many cases of interest, we are able to identify a subset of elements of D for which these two sums are finite sums. The half-shuffle exponentials also called "time-ordered" exponentials and are defined by mean of the two shuffles $\prec$ and $\succ$:

$$\exp_{\prec}(x) = \mathbf{1} + \sum_{n \geq 1} x^{\prec n},\ \exp_{\succ}(x) = \mathbf{1} + \sum_{n \geq 1} x^{n \succ}.$$

Notice that the two *half-shuffle* exponentials are solution of the following fixed point equations:

$$X = \mathbf{1} + x \prec X, \quad X = \mathbf{1} + X \succ x.$$

These two time-ordered exponentials and the shuffle exponential are the key ingredients to the Hopf algebraic approach of moment-cumulant relations in non-commutative probability theory.

Lemma 19 (Lemma 2 in [12])**.** *Let A be a shuffle algebra, and $\bar{A}$ its augmentation by a unit 1. For $x \in A$, we have*

$$\exp_{\succ}(-x) \shuffle \exp_{\prec}(x) = \exp_{\succ}(-x) \shuffle \exp_{\prec}(x) = 1.$$

We proceed with a small overview on the shuffle approach on (scalar-valued) non-commutative probability theory. The core of this approach is developed in [12, 13, 14]. Let $(\mathcal{A}, E)$ be a scalar-valued non-commutative probability space. Consider the space $H = \bar{T}(T(\mathcal{A}))$ defined as the linear span of all words on words on elements in $\mathcal{A}$ including the empty word. Then H can be endowed with the unshuffle bialgebra structure $(\Delta, \varepsilon, \Delta_{\prec}, \Delta_{\succ})$, see for example Definition 3 in [12]. Because of the relations satisfied by the half unsuffle coproducts $\Delta_{\prec}$ and $\Delta_{\succ}$, the vector space of all linear forms on H is a shuffle algebra if endowed with the half-shuffles dual to the two unshuffle coproducts. The authors in [12] define a moment morphism $\Phi : H \to \mathbb{C}$, which is a morphism for the concatenation product on H whose value on a word $a_1 \otimes \cdots \otimes a_n$ (a "letter" in H) is

$$\Phi(a_1 \otimes \cdots \otimes a_n) = E(a_1 \cdot_{\mathcal{A}} \cdots \cdot_{\mathcal{A}} a_n). \tag{27}$$

Then, Φ is an element of the monoid $\mathrm{Hom}_{\mathrm{Alg}}(H, \mathbb{C})$ of characters of the algebra H, endowed with the shuffle product dual to Δ. Since H is connected and nilpotent, H is a Hopf algebra. Therefore $G = \mathrm{Hom}_{\mathrm{Alg}}(H, \mathbb{C})$ is a group and the two half-shuffle exponentials together with the shuffle exponential define three maps from the Lie algebra $Lie(G)$ to G. Thus, there exist three linear maps $k, b, m : H \to \mathbb{C}$ such that:

$$\Phi = \varepsilon + k \prec \Phi = \varepsilon + \Phi \succ b = \exp_{\shuffle}(m). \tag{28}$$

The three maps k, b and m can be identified with, respectively, the free, boolean and monotone cumulants in the following way. As elements of the Lie algebra $Lie(G)$ they are equal to zero on non-trivial products of words in H. On word $w \in T(\mathcal{A})$ they coincide each with one of the tree cumulant functions. Notice that equation (28) is equivalent to the free, boolean and monotone moment-cumulant relations. From this perspective cumulants and moments are not on the same footing. Indeed, cumulants are considered infinitesimal objects while moments are encoded by an algebra morphism. Later on, the authors in [11] linked the shuffle approach to a particular operad on non-crossing partitions and the Möbius inversion to another operad on non-crossing partitions. In this settings, the free cumulants of a random variable become an algebra morphism on the space N of words on non-crossing partitions, seen as solution of a Left half-shuffle fixed point equation and the moment-cumulant relations are retrieved through an action (compatible with the convolution coproduct on the dual $N^{\star}$) of an element of the monoid of morphisms on a coalgebra associated with the second operad. To retrieve the moments-cumulants relations for operator-valued probability spaces we will define an operator-valued counterpart of the splitting map defined in [13].

4.1. **Unshuffle $\boxtimes \otimes$-bialgebras.** The dual notion of unshuffle algebra appeared after the notion of shuffle algebra in the literature. It has first been considered by L. Foissy, in its seminal work [16] on the Duchamp–Hivert–Thibon "free Lie algebra" conjecture. We introduce a notion of unshuffle bialgebra adapted to our settings and show that the dual, in a certain sense, of such a bialgebra is a plain shuffle algebra.

Definition 20. An unshuffle co-algebra in Coll_2 is a coaugmented coassociative coalgebra

$$(\bar{C} = C \oplus \mathbb{C}_{\boxtimes}, \Delta),\ C(n,n) = 0,\ n \geq 0$$

in the monoidal category $(\mathrm{Coll}_2, \boxtimes, \mathbb{C}_{\boxtimes})$ with coproduct

$$\bar{\Delta} : \bar{C} \to \bar{C} \boxtimes \bar{C},\ \Delta \in \mathrm{Hom}_{\mathrm{Coll}_2}(\bar{C}, \bar{C} \boxtimes \bar{C})$$

such that for any $c \in C$, $\bar{\Delta}(c) = \Delta(c) + c \boxtimes 1_m + 1_n \boxtimes c$. The reduced coproduct Δ splits into two half unshuffle coproducts $\Delta_{\prec}$ and $\Delta_{\succ}$ such that

$$\Delta = \Delta_{\prec} + \Delta_{\succ}$$

and they satisfy the three following equations:

$$\begin{aligned}(\Delta_{\prec} \boxtimes I) \circ \Delta_{\prec} = (I \boxtimes \Delta) \circ \Delta_{\succ},\ (\Delta \boxtimes I) \circ \Delta_{\succ} = (I \boxtimes \Delta_{\succ}) \circ \Delta_{\succ} \\ (\Delta_{\succ} \boxtimes I) \circ \Delta_{\prec} = (I \boxtimes \Delta_{\prec}) \circ \Delta_{\succ}.\end{aligned} \tag{29}$$

In the following definition, we use the shorter notation $\rho^{\boxtimes}$ for the horizontal algebra product on $C \boxtimes C$ if (C, ρ) is an horizontal algebra in Coll_2.

Definition 21. *An unshuffle $\boxtimes \otimes$-bialgebra* is a conilpotent $\boxtimes \otimes$-bialgebra $(\bar{C} = C \oplus \mathbb{C}_{\boxtimes}, \Delta, \rho)$ with

$$\bar{\Delta}(c) = \Delta(c) + c \boxtimes 1_m + 1_n \boxtimes c,\ c \in C(n,m),$$

and $\bar{\Delta} = \Delta_{\prec} + \Delta_{\succ}$ is an unshuffle coproduct (see Definition 20), satisfying the following compatibility conditions:

$$\Delta \circ \rho = \rho^{\boxtimes} \circ (\Delta \otimes \Delta) \tag{30}$$

$$\begin{gathered}(\Delta^{+}_{\prec} \circ \rho)(p \otimes q) = \rho^{\boxtimes} \circ (\Delta^{+}_{\prec} \otimes \Delta)(p \otimes q),\ (\Delta^{+}_{\succ} \circ \rho)(p \otimes q) = \rho^{\boxtimes} \circ (\Delta^{+}_{\succ} \boxtimes \Delta)(p \otimes q) \\ p \notin \mathbb{C}_{\boxtimes},\ q \in C,\end{gathered} \tag{31}$$

$$\Delta^{+}_{\prec}(\rho(1_m \otimes q)) = \rho^{\boxtimes}((1_m \boxtimes 1_m) \otimes \Delta^{+}_{\prec}(q)), \Delta^{+}_{\succ}(\rho(1_m \otimes q)) = \rho^{\boxtimes}((1_m \boxtimes 1_m) \otimes \Delta^{+}_{\succ}(q)) \tag{32}$$

with $\Delta^{+}_{\prec}(c) = \Delta_{\prec}(c) + c \boxtimes 1_n$, $\Delta^{+}_{\succ}(c) = \Delta_{\succ}(c) + 1_m \boxtimes c$, $c \in C(m,n)$.

4.2. **The $\boxtimes\otimes$-unshuffle bialgebra of the gap-insertion operad of non-crossing partitions.** In this section, we focus on non-crossing partitions and define the $\boxtimes\otimes$-shuffle Hopf algebra relevant for the application to operator-valued non-commutative probability theory we expose in the next section.

Let π be a non-crossing partition of a linearly ordered set X. The set of blocks of π carries a pre-order defined by declaring for two blocks V_1 and V_2 of π that $V_1 \to_\pi V_2$ to mean that $\mathrm{Conv}(V_2) \cap V_1 \neq \emptyset$. In plain words, $V_1 \to_\pi V_2$ means that V_2 is nested in V_1.

Definition 22 (Upperset and lowerset)**.** A *lowerset* L of π is a set (which may be empty) of blocks of π such that if $V \in L$ and $V \to_\pi W$ in π then also $W \in L$. In plain words, if a block V is in L then all englobing blocks of V are also in L and L is a non-crossing partition.

An *upperset* of a non-crossing partition $\pi \in \mathrm{NC}(p)$ is a word $U_0 \otimes \cdots \otimes U_p$ of length $p+1$ in $T_\otimes(\mathcal{NC})$ on non-crossing partitions such that there exists a lowerset $L \in \mathrm{NC}(p)$ with

$$\pi = \nabla_{T_\otimes(\mathcal{NC})} L \boxtimes (U_1 \otimes \cdots \otimes U_p).$$

The notion of upperset and lowerset of a non-crossing partition (and for partitions) can be found in [11]. We denote by $\mathsf{Lo}(\pi)$ (respectively $\mathsf{Up}(\pi)$) the set of all lowersets (respectively uppersets) of a non-crossing partition π.

Let π be a non-empty non-crossing partition. Then a lowerset $L \in \mathrm{NC}(p)$ of π defines an upperset $U_0 \otimes \cdots \otimes U_p$. Each of the partitions U_i is either equal to the empty partition or is a subset of the partition π such that if $V \in U_i$ then all blocks $W \in \pi$ such that $V \to W$ are also in U_i. Given a lowerset L, we denote by $L^{\otimes}$ the associated upperset, by definition we have:

$$\pi = \nabla_{T_\otimes(\mathcal{NC})} \left(L \boxtimes L^{\otimes} \right). \tag{33}$$

Notice that the lowerset L in the definition of an upperset $U_1 \otimes \cdots \otimes U_p$ is unique, the blocks of L are the blocks of π not in any of the $U_i's$ and we denote it $U^{\cdot}$. A cut of π is then the data of a lowerset L and an upperset U such that $\pi = \nabla_{T_\otimes(\mathcal{NC})} L \boxtimes U$. Notice that in that case, $L = U^{\cdot}$ and $U = L^{\otimes}$.

Proposition 23. *Let π be a non-empty partition, then*

$$\Delta_{T_\otimes(\mathcal{NC})}(\pi) = \sum_{(L,U) \in cut(\pi)} L \boxtimes U. \tag{34}$$

In the following we denote by $T^+_\otimes(\mathcal{NC})$ the subspace of $T_\otimes(\mathcal{NC})$ generated by words on non-empty partitions. Notice that the horizontal morphism Δ splits as

$$\Delta_{T_\otimes(\mathcal{NC})}(w) = \bar{\Delta}(w) + \{\emptyset\}^m \boxtimes w + w \boxtimes \{\emptyset\}^n,\ w \in T_\otimes(\mathcal{NC})(m,n), m \neq n.$$

In the following definition, we write $1 \in L$ if the block of π that contain 1 is in the lowerset L.

To an upperset of a partition corresponds a subset of blocks of π. Hence, given a cut (L,U) of π we write $1 \in U$ (respectively, $1 \in L$) if the blocks of π that contains 1 is in U (in L).

Definition 24 (Half-unshuffles on $T_\otimes(\mathcal{NC})$)**.** We define two bicollection maps $\Delta^+_\prec : T^+_\otimes(\mathcal{NC}) \to T_\otimes(\mathcal{NC})^{\boxtimes 2}$, $\Delta^+_\prec : T^+_\otimes(\mathcal{NC}) \to T_\otimes(\mathcal{NC})^{\boxtimes 2}$. Let $\pi \in \mathcal{NC}$ be a non-empty partition and set

$$\Delta^+_\prec(\pi) = \sum_{\substack{(L,U) \in cut(\pi) \\ 1 \in L}} L \boxtimes U, \quad \Delta^+_\succ(\pi) = \sum_{\substack{(L,U) \in cut(\pi) \\ 1 \in U}} L \boxtimes U \tag{35}$$

We extend $\Delta^+_\prec$ and $\Delta^+_\succ$ by setting for a word $w \in T^+_\otimes(\mathcal{NC})$ and a partition $p \in \mathcal{NC}$ and integer $q \geq 0$:

$$\Delta_\prec(\{\emptyset\}^q p_1 w) = (\{\emptyset\}^q \boxtimes \{\emptyset\}^q)\Delta_\prec(p)\Delta(w),\ \Delta_\succ(\{\emptyset\}^q p w) = (\{\emptyset\}^q \boxtimes \{\emptyset\}^q)\Delta_\succ(p)\Delta(w). \tag{36}$$

From the very definition of the two left/right half-shuffles $\Delta^+_\prec$ and $\Delta^+_\succ$, its holds that $\Delta = \Delta_\prec + \Delta_\succ$.

Proposition 25. $(T_\otimes(\mathcal{NC}), \Delta, \Delta_\prec, \Delta_\succ)$ *is an unshuffle bialgebra in* Coll_2.

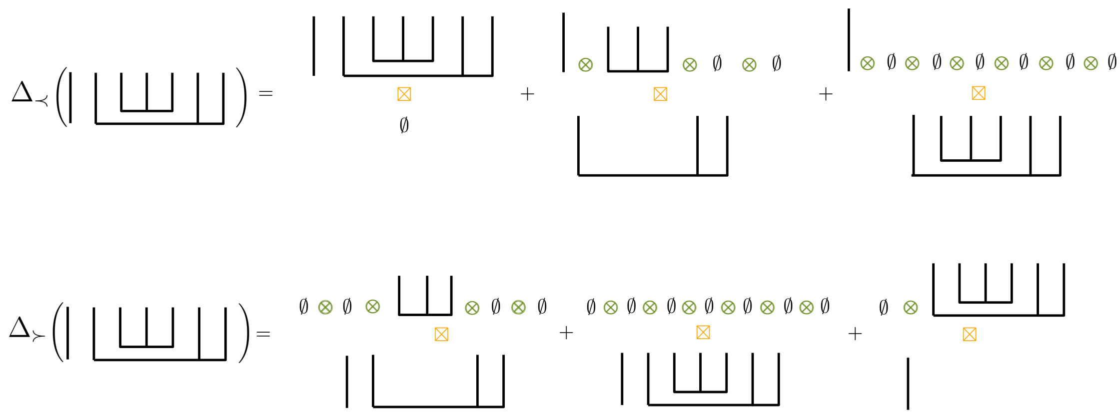

FIGURE 8. The two half unshuffle coproducts acting on a non-crossing partition.

Proof. For the sake of completeness, we present briefly the arguments given in [11], Proposition 3.4.3. Let π be a partition. It is sufficient to check the relations (29) for a single partition, because of the equations (36). Even so, we need to define lowersets, uppersets and cuts for words on (possibly empty) partitions. By convention, the only cut (L,U) of the empty partition is $(\{\emptyset\},\{\emptyset\})$. Notice that this convention is compatible with Proposition 23 since $\Delta(\{\emptyset\}) = \{\emptyset\} \boxtimes \{\emptyset\}$

Let $w = p_1 \cdots p_s$ be a word on partitions with $p_i \in \mathcal{NC}(k_i)$ with $k_i \geq 0$. A lowerset of w is a word $L_1 \cdots L_p$ with L_i a lowerset of the partition p_i. The notion of an upperset for w is defined similarly, an upperset of W is a word on uppersets one for each of the partition p_i. The notion of cut for partitions is then downwardly transferred to words on partitions. Then we have the formulas:

$$\Delta^{+}_{\prec}(w) = \sum_{\substack{(L,U)\in cut(w)\\ 1\in L}} L \boxtimes U, \quad \Delta^{+}_{\succ}(w) = \sum_{\substack{(L,U)\in cut(w)\\ 1\in U}} L \boxtimes U \tag{37}$$

for a word $w \in T^{+}_{\otimes}(\mathcal{NC})$. We say that (L,M,U) is a compatible pair of cuts of w if L is a lowerset of w, U is an upperset of w with $L^{\otimes} = \nabla_{T_{\otimes}(\mathcal{NC})} U \boxtimes M$ and $U^{\cdot} = \nabla_{T_{\otimes}(\mathcal{NC})}(L \boxtimes M)$ (because $\Delta_{T_{\otimes}(\mathcal{NC})}$ is coassociative these two conditions are equivalent) with $L, M, U \notin \mathbb{C}_{\boxtimes}$. We denote by $cut_2(w)$ the set of compatible pairs of cuts of a words in $T_{\otimes}(\mathcal{NC})$. Let π be a non-crossing partition, we have:

$$\begin{aligned}
(\Delta_{\prec} \boxtimes \mathrm{id}) \circ \Delta_{\prec}(\pi) &= \sum_{\substack{(U,M,L)\in cut_2(\pi)\\ 1\in L}} L \boxtimes M \boxtimes U = \mathrm{id} \boxtimes \Delta \circ \Delta_{\prec}(\pi),\\
(\Delta_{\succ} \boxtimes \mathrm{id}) \circ \Delta_{\prec}(\pi) &= \sum_{\substack{(U,M,L)\in cut_2(\pi)\\ 1\in M}} L \boxtimes M \boxtimes U = \mathrm{id} \boxtimes \Delta_{\prec} \circ \Delta_{\succ}(\pi)\\
(\Delta \boxtimes \mathrm{id}) \circ \Delta_{\succ}(\pi) &= \sum_{\substack{(U,M,L)\in cut_2(\pi)\\ 1\in U}} L \boxtimes M \boxtimes U = \mathrm{id} \boxtimes \Delta_{\succ} \circ \Delta_{\prec}(\pi).
\end{aligned}$$

□

Thanks to $\boxtimes\otimes$-bialgebra $T_{\otimes}(\mathcal{NC})$ being conilpotent, the following proposition holds. In the following, we focus on non-crossing partitions.

Proposition 26. *The $\boxtimes\otimes$-bialgebra $(T_{\otimes}(\mathcal{NC}), \Delta, \Delta_{\prec}, \Delta_{\succ})$ endowed with the vertical product $\nabla_{T_{\otimes}(\mathcal{NC})}$ is a unshuffle Hopf algebra.*

The splitting of the horizontal morphism $\Delta_{T_{\otimes}(\mathcal{NC})}$ into the two half-unshuffle $\Delta_{\prec}$ and $\Delta_{\succ}$ induces two bilinear (non-associative) composition on the vector space of bicollection morpisms from $T^{+}_{\otimes}(\mathcal{NC})$

to $T^{+}_{\otimes}(\mathrm{Hom}(B))$ (with obvious notations):

$$f \prec g = \nabla_{\mathrm{Hom}(B)} \circ (f \boxtimes g) \circ \Delta_{\prec},\ f \succ g = \nabla_{\mathrm{Hom}(B)} \circ (f \boxtimes g) \circ \Delta_{\succ},\ f,g \in \mathrm{Hom}_{\mathrm{Coll}_2}(T_{\otimes}(\mathcal{NC}),\ T_{\otimes}(\mathrm{Hom}(V))).$$

Recall that we wenote by $\eta_{\mathrm{Hom}(B)} : \mathbb{C}_{\boxtimes} \to T_{\otimes}(\mathrm{Hom}(V))$ the unique horizontal morphism such that $\eta_{\mathrm{Hom}(V)}(1_1) = \mathrm{id}_V$. We set

$$f \prec \Big(\eta_{\mathrm{Hom}(B)} \circ \varepsilon\Big) = \Big(\eta_{\mathrm{Hom}(V)} \circ \varepsilon\Big) \succ f = f$$

and

$$\Big(\eta_{\mathrm{Hom}(B)} \circ \varepsilon\Big) \prec f = f \succ \Big(\eta_{\mathrm{Hom}(B)} \circ \varepsilon\Big) = 0.$$

The following proposition is a corollary of Proposition 25 and equations (29). With the notation $\overline{\mathrm{Hom}}(T_{\otimes}(\mathcal{NC}), T^{+}_{\otimes}(\mathrm{Hom}(B)) = \mathbb{C}(\eta \circ \varepsilon_{\mathrm{Hom}(B)}) \oplus \mathrm{Hom}(T^{+}_{\otimes}(\mathcal{NC}), T_{\otimes}(\mathrm{Hom}(B))$, the following proposition is a direct corollary of the last proposition.

Proposition 27. $(\overline{\mathrm{Hom}}(T_{\otimes}(\mathcal{NC}), T_{\otimes}(\mathrm{Hom}(B))), \prec, \succ, \star)$ *is a shuffle algebra.*

4.3. **Half-shuffle and shuffle exponentials.** In this section we compute the half-shuffle and shuffle exponentials of infinitesimal morphisms. Those exponentials are always horizontal algebra morphisms and are compatible with the gap-insertion composition under some hypothesis. The three main results of this section are contained in Proposition 28, 29 and 33.

4.3.1. *Left half-shuffle.* Given an infinitesimal character k, we define the half-shuffle exponential $\exp_{\prec}(k)$ by:

$$\exp_{\prec}(k)(\pi) = \eta_{\mathrm{Hom}(B)} \circ \varepsilon + \sum_{p \geq 1} k^{\prec p},$$

with $k^{\prec p} = k \prec k^{\prec (p-1)}$ and $k^{\prec 1} = k$. It is easily seen that $\exp_{\prec}(k)$ is the unique solution of the following fixed point equation:

$$K = \eta_{\mathrm{Hom}(B)} \circ \varepsilon + k \prec K. \tag{38}$$

We denote by $\phi_{|1} : \mathcal{NC} \to \mathrm{Hom}(B)$ the restriction of a bicollection map $\phi : T_{\otimes}(\mathcal{NC}) \to T_{\otimes}(\mathrm{Hom}(B))$. Recall that we put 1_n for the partition in $\mathrm{NC}(n-1)$ with only one block, $n \geq 2$. If π is partition, recall that $\sharp\pi$ denotes the number of blocks of π.

Proposition 28. *With the notation introduced so far, the collection map K, being the solution of the fixed point equation* (38)*, is an horizontal morphisms. Beside the map $K_{|1}$ is an operadic morphism if and only if*

$$k(1_n) \circ_n k(1_m) = k(1_m) \circ_1 k(1_n) \tag{39}$$

and $k(\pi) = 0$ if $\sharp\pi > 1$.

Proof. We show that the solution K of (38) is an horizontal morphism. We do it recursively. Let $\tilde{K}$ be the horizontal morphism extending the values of $K_{|1}$. The two maps K and $\tilde{K}$ agree on words on partitions with no non-empty blocks, since in that case $K(\{\emptyset\}^q) = \tilde{K}(\{\emptyset\}^q) = \Big(T_{\otimes}(\eta_{\mathrm{Hom}(b)}) \circ \varepsilon\Big)(\{\emptyset^q\})$. Assume next that K and $\tilde{K}$ agree on words of partitions with a total number of non-empty blocks at most equal to $N \geq 1$. Pick a word on partitions with $N+1$ blocks and write $w = \emptyset^p \pi \tilde{w}$, with $\pi \neq \{\emptyset\}$ and $|\tilde{w}|$ a word of length s. Let V be the block of the partition associated with π that contains 1. Then by definition of an infinitesimal character, we get

$$K(w) = (k \prec K)(w) = \sum_{\substack{(L,U) \in cut(\pi) \\ 1 \in L}} \nabla_{\mathrm{Hom}(B)} \Big(\{\emptyset\}^p k(L) \{\emptyset\}^s \boxtimes K(\{\emptyset\}^p U \tilde{w})\Big).$$

Since the number of non-empty blocks of U and $U\tilde{w}$ is less than the number of non-empty blocks of w, we get

$$K(w) = \sum_{\substack{(L,U)\in Cut(\pi)\\ 1\in L}} \mathrm{id}_B \, \nabla_{\mathrm{Hom}(B)}\Big(k(L) \boxtimes \tilde{K}(U)\Big)\, \tilde{K}(\tilde{w}) = \mathrm{id}_B (k \prec K)(\pi)\tilde{K}(\tilde{w}) = \tilde{K}(w).$$

Next, Assume $k(\pi) = 0$ if $\sharp\pi > 1$ and $k(1_n) \circ_n k(1_m) = k(1_m) \circ_1 k(1_n)$. Let $\phi : \mathcal{NC} \to \mathrm{Hom}(B)$ be the operadic morphism extending the values $k(1_n)$, $n \geq 1$. If π is a partition with only one block, then

$$K(\pi) = (\eta_{\mathrm{Hom}(B)} \circ \varepsilon)(\pi) + (k \prec K)(\pi) = 0 + k(\pi) \circ (K(\emptyset^{|\pi|})) = k(\pi) = \phi(\pi). \tag{40}$$

Assume that the result holds for words on partitions with at most N blocks, $K(\pi_1 \cdots \pi_p) = \phi(\pi_1 \cdots \pi_p)$ for every element $\pi_1 \cdots \pi_p \in T_{\otimes}(\mathcal{NC})$ with $\sharp\pi_1 + \cdots + \sharp\pi_p \leq N$. Let π be a partition with $N+1$ blocks. We denote by V the block of π that contains 1. With this notation, we have

$$\begin{aligned} K(\pi) &= (\eta_{\mathrm{Hom}(B)} \circ \varepsilon)(\pi) + k \prec K(\pi) \\ &= \sum_{(L,U)\in Cut(\pi)} \nabla_{\mathrm{Hom}(B)}(k(L) \boxtimes K(U)) \\ &= \nabla_{\mathrm{Hom}(B)}(k(1_{\sharp V}) \boxtimes \phi(U)) = \phi(\pi). \end{aligned}$$

The last equality follows by application of the recursive hypothesis since $\sharp U \leq N$. Now assume that the solution $K_{|1}$ is an operadic morphism. Let $\pi \neq \emptyset$ be a non-crossing partition.

$$\begin{aligned} K(\pi) &= K(1_{\sharp V}) \circ (K(\pi_0), \ldots, K(\pi_{|V|})) = (k \prec K)(\pi) \\ &= \sum_{(L,U)\in Cut(\pi)} \nabla_{\mathrm{Hom}(B)}(k(L) \boxtimes K(U))(\pi) \\ &= \nabla_{\mathrm{Hom}(B)}(k(1_{\sharp V}) \boxtimes (K(\pi_1) \otimes \cdots \otimes K(\pi_{|V|}))) + \sum_{L\neq\{V\}} \nabla_{\mathrm{Hom}(B)}(k(L) \boxtimes K(U))(\pi). \end{aligned}$$

This last equality implies $\sum_{L\neq\{V\}} \nabla_{\mathrm{Hom}(B)}(k(L) \boxtimes K(U))(\pi) = 0$. A simple recursive argument on the number of blocks ends the proof. $\square$

4.3.2. *Right half-shuffle exponential.* Given an infinitesimal character $b : T_{\otimes}(\mathcal{NC}) \to T_{\otimes}(\mathrm{Hom}(B))$, we define the right half-shuffle exponential $\exp_{\succ}(b)$ by:

$$\exp_{\succ}(b)(\pi) = \eta_{\mathrm{Hom}(B)} \circ \varepsilon + \sum_{p\geq 1} b^{\succ p},$$

with $b^{\succ p} = b \succ b^{\succ(p-1)}$ and $b^{\succ 1} = b$. It is easily seen that $\exp_{\succ}(b)$ is the unique solution of the following fixed point equation:

$$B = \eta_{\mathrm{Hom}(B)} \circ \varepsilon + B \succ b. \tag{41}$$

Let π be a non-crossing partition. The *adjacency forest* $\tau(\pi)$ of π encodes nesting of the blocks of π. To each block of π we associate a vertex. Two blocks are connected in $\tau(\pi)$ if the convex hull of one of the block contains the other block. The root of each tree in $\tau(\pi)$ is a block not contained in any other block. In particular, the adjacencey forest of an irreductible partition (see [2]) is a tree.

We say that an horizontal morphism $B : T_{\otimes}(\mathcal{NC}) \to T_{\otimes}(\mathrm{Hom}(B))$ is *boolean* if it is equal to zero on any non-crossing partitions with at least two nested blocks. Those partitions have an adjacency forest with at least one tree containing two vertices. In addition, if $I_1 \cdots I_p$ is an interval partition, we require that:

$$B(I_1 \cdots I_p) = \Big(\cdots \Big(B(I_p) \circ_1 \cdots\Big) \circ_1 B(I_2)\Big) \circ_1 B(I_1)\Big). \tag{42}$$

Proposition 29. *With the notation introduced so far, the bicollection morphism B solution of the fixed point equation (41) is a horizontal morphism. Besides, B is boolean if and only if*

$$b(1_n) \circ_n b(1_m) = b(1_m) \circ_1 b(1_n)$$

and $b(\pi) = 0$ if $\sharp\pi > 1$.

Proof. The proof is very similar to the free case. Let $\tilde{B}$ be the horizontal morphism extending the values of $\bar{b}$. The two maps B and $\tilde{B}$ agree on words on partitions with at most 1 non-empty block. Assume that B and $\tilde{B}$ agree on words on partitions with at most N non-empty blocks.

Pick a word on partitions with $N+1$ blocks and write $w = \emptyset^p \pi \tilde{w}$, with $\pi \neq \{\emptyset\}$ and $\tilde{w}$ a word of length s. Let V be the block of the partition associated with π that contains 1. Then by definition of an infinitesimal morphism, we get

$$B(w) = (\underline{k} \prec K)(w) = \sum_{\substack{(L,U) \in Cut(\pi) \\ 1 \notin L}} \nabla_{\mathrm{Hom}(b)} \Big(B(\{\emptyset\}^p L \tilde{w}) \boxtimes b(\{\emptyset\}^p U \{\emptyset\}^{|\tilde{w}|}) \Big).$$

Since the number of non-empty blocks of $L\tilde{w}$ and L is less than the number of non-empty blocks in w, we get:

$$B(w) = \sum_{\substack{(L,U) \in Cut(\pi) \\ 1 \notin L}} \mathrm{id}_B^p \nabla_{\mathrm{Hom}(b)} \Big(\tilde{B}(L) \boxtimes b(U) \Big) \tilde{B}(\tilde{w}) = \mathrm{id}_B^p (B \succ b)(\pi) \tilde{B}(\tilde{w}) = \tilde{B}(w).$$

We assume that $b(\pi) = 0$ if $\sharp\pi > 1$. Let ϕ be the boolean morphism that extends the values $b(1_n)$, $n \geq 1$. We show recursively on the total number of non-empty blocks of word on partitions in $T_{\otimes}(\mathcal{NC})$ that $B = T_{\otimes}(\phi)$. First, the two maps coincide on words on partitions with a total number of non-empty blocks less than one. Let $N \geq 1$ and assume that $T_{\otimes}(\phi)$ and B are equal on multi-partition with at most N blocks. Pick π a partition with $N+1$ blocks. Assume first that the *adjacency forest* of π contains at least one tree not equal to the root.

$$B(\pi) = (B \succ b)(\pi) = \sum_{1 \notin L \in \mathsf{Lo}(\pi)} \nabla_{\mathrm{Hom}(b)} (B(L) \boxtimes b(U)) \tag{43}$$

A cut of the partition π corresponds to an admissible cut of its adjacency tree. Since $\bar{b}(U) = 0$ if U is a word on partitions either containing at least two non-empty partitions or equal to some $\emptyset^p, p \geq 1$, the cuts that contribute to the sum on the righthand side of (43) extract one and only one leaf of the adjacency forest. Hence, if the block V of π that contains 1 contains at least another block in its convex hull, $B(\pi) = 0$. Assume the opposite. It implies that the partition $\pi \backslash V$ is not an interval partition (and is not empty). Besides,

$$B(\pi) = \nabla_{\mathrm{Hom}(b)} (b(V) \boxtimes B(\emptyset \otimes \pi \backslash V))$$

The induction hypothesis implies $B(\emptyset \otimes \pi \backslash V)) = 0$. Now suppose that $\pi = I_1 \cdots I_p$ is an interval partition.

$$B(\pi) = \nabla_{\mathrm{Hom}(b)} (b(I_1) \boxtimes B(I_2 \cdots I_p)).$$

We apply the recursive hypothesis on $B(I_2 \cdots I_p)$ to end the proof. □

4.3.3. *Shuffle exponential.* In this section we compute the shuffle exponential (44). The restriction of this horizontal morphism to non-crossing partitions (operators with one output) is not compatible in any way, to the extend of our knowledge with the operation of gap-insertion. This boils down to the fact that the tree factorial defined hereafter is not multiplicative.

Definition 30 (Monotone partition)**.** Let π a partition with k blocks. An admissible labelling of the blocks by integers in $[\![1, k]\!]$ is an injective labelling which is increasing with respect to the nesting preoder on the blocks: If a block $V \in \pi$ is contained in the convex hull of a block W in π then the label of V is less than the label of W. A partition with an admissible labelling of its blocks is called a monotone partition. The set of all monotone partitions is denoted NC_m.

Definition 31 (Tree factorial, [2], Definition 3.2)**.** The tree factorial $t!$ of a rooted tree t is recursively defined as follows. Let t be a rooted tree with $n > 0$ vertices. It t consists of a single vertex, set $t! = 1$. Otherwise t can be decomposed into its root vertex and branches $t_1, \ldots, t_r$ and we defined recursively the number

$$t! = n \cdot t_1! \cdots t_k!$$

The tree factorial of a forest is the product of the factorials of the constituting trees.

Proposition 32 ([2], Proposition 3.3)**.** *The number $m(\pi)$ of monotone labellings of a non-crossing partition depends only on its adjacency forest $\tau(\pi)$ and is given by $m(\pi) = \frac{\sharp\pi!}{\tau(\pi)!}$*

Let $m : T_{\otimes}(\mathcal{NC}) \to T_{\otimes}(\mathrm{Hom}(B))$ be an infinitesimal morphism and define the shuffle exponential by

$$\exp_{\star}(m) = \eta_{\mathrm{Hom}(B)} \circ \varepsilon + \sum_{p \geq 1} \frac{1}{p!} m^{\star p}. \tag{44}$$

Proposition 33. *Pick $m : T_{\otimes}(\mathcal{NC}) \to T_{\otimes}(\mathcal{NC})$ an infinitesimal morphism such that:*

$$m(1_n) \circ_1 m(1_m) = m(1_m) \circ_{m+1} m(1_n) \tag{45}$$

with $m(\pi) = 0$ if $\sharp\pi > 1$. Then, $\exp_{\star}$ is an horizontal morphism and

$$\exp_{\star}(m)(\pi) = \frac{1}{\tau(\pi)!} \exp_{\prec}(m)(\pi),\ \pi \in \mathcal{NC}.$$

Proof. Let π be a non-crossing partition with k blocks. The number of admissible labelings of the partition π is equal to $\frac{k!}{\tau(\pi)!}$. Hence, to prove the statement, it is sufficient to show that

$$\exp_{\star}(m)(\pi) = \frac{1}{k!} \sum_{\pi \in \mathcal{NC}_m} \exp_{\prec}(m)(\pi).$$

To that end, we show first that there exists a natural embedding of the set of admissible labelings of a partition into the set of multiple admissible cuts of a partition. A multiple cuts of a partition π is a sequence $(L_1, \ldots, L_s)$ of (possibly empty) subsets of blocks of π such that L_i is a lower cut of L_{i-1} with the convention $L_0 = \pi$. For such a multiple cut of π, we denote by $L_i \backslash L_{i-1}$ the words on partition in $T_{\otimes}(\mathcal{NC})$ such that

$$\nabla_{T_{\otimes}(\mathcal{NC})}(L_i \boxtimes (L_{i-1} \backslash L_i)) = L_{i-1}.$$

Let (π, ℓ) be a monotone partition. We associate to the labelling ℓ of the block a multiple cut $\mathsf{L}(\pi, \ell)$ of π as follows. For each integer $i \in [\![1, k]\!]$, we denote by V_i the block of π labelled with the integer i. We define recursively $\mathsf{L}(\pi, \ell)$ by the following rule:

$$\mathsf{L}(\pi, \ell)_0 = \pi,\ \mathsf{L}(\pi, \ell)_i = \mathsf{L}(\pi, \ell)_{i-1} \backslash V_i.$$

Because the labelling ℓ is monotone, we obtain indeed a multiple cut of π. Next, from the definition of the coproduct Δ, we see that:

$$\exp_{\star}(m)(\pi) = \sum_{s \geq 1} \sum_{(L_1, \ldots, L_s)} \frac{1}{s!} \nabla^{\boxtimes s}_{\mathrm{Hom}(B)}(m(L_{s-1} \backslash L_s) \boxtimes \cdots \boxtimes m(L_0 \backslash L_1)),$$

with $\nabla^{\boxtimes s}_{\mathrm{Hom}(B)}$ defined recursively by $\nabla^{\boxtimes 1}_{\mathrm{Hom}(B)} = \nabla_{\mathrm{Hom}(B)}$ and $\nabla^{\boxtimes (s+1)}_{\mathrm{Hom}(B)} = \nabla^{\boxtimes s}_{\mathrm{Hom}(B)} \boxtimes \mathrm{id} \circ \nabla_{\mathrm{Hom}(B)}$. From the definition of an infinitesimal character, the sum on the right hand side of the last equation reduces to

$$\exp_{\star}(m)(\pi) = \sum_{(\pi, \ell) \in \mathcal{NC}_m} \frac{1}{k!} \nabla^{\boxtimes s}_{\mathrm{Hom}(B)}(m(\mathsf{L}(\pi, \ell)_{s-1} \backslash \mathsf{L}(\pi, \ell)_s) \boxtimes \cdots \boxtimes m(\mathsf{L}(\pi, \ell)_0 \backslash \mathsf{L}(\pi, \ell)_1)).$$

The result follows from the last equation. $\square$

4.4. **Moment-cumulant relations, operads of word insertions and the splitting map.** In this section, we introduce an $\boxtimes\otimes$ unshuffle bialgebra, see Definition 21 starting with an operad of words on random variables (defined hereafter). We proceed with the definition of a splitting map from this unshuffle bialgebra to the unshuffle bialgebra of words on non-crossing partitions we defined in the previous section. We prove finally that the moment-cumulant relations for free and boolean cumulants are equivalent to two half-shuffle fixed point equations, see Proposition 38.

In this section, all non-crossing partitions have their legs coloured with elements in the algebra $\mathcal{A}$. We use the same notation NC for the set of all coloured non-crossing partitions. The material exposed in the previous sections extends readily to coloured non-crossing partitions. A generic coloured non-crossing partition is written

$$\pi \otimes a_1 \otimes \cdots \otimes a_p,\ \pi \in NC(p),\ a_i \in \mathcal{A}.$$

We give only sketches of the proofs, if any, and the reader is directed to [13] where he or she will find detailed proofs readily adapted to our settings using line of thoughts we exposed in the previous sections for the operad of non-crossing partitions $\mathcal{NC}$. For the remainder of the sections we use the (heavier) notations $\Delta^{T_\otimes(\mathcal{NC})}$, $\Delta_\prec^{T_\otimes(\mathcal{NC})}$, $\Delta_\succ^{T_\otimes(\mathcal{NC})}$ and $\varepsilon^{T_\otimes(\mathcal{NC})}$ for the unshuffle structure on $T_\otimes(\mathcal{NC})$ we defined previously.

We start with the definition of the operad of words insertions. We denote by $T(\mathcal{A})$ the vector space of all non-commutative polynomials on elements in the algebra $\mathcal{A}$. We augment this space with the empty word $\emptyset$ and set $\bar{T}(\mathcal{A}) = \mathbb{C}\emptyset \oplus T(\mathcal{A})$. Each word $w_1 \cdots w_p$ is graded by its length plus one:

$$\mathsf{i}(w_1 \cdots w_p) = p + 1.$$

The empty word has length 0. The collection $\bar{T}(\mathcal{A})$ (for the graduation $|w| = \mathsf{i}(w)$) is an operad:

$$\rho_{\mathcal{WI}}: \begin{array}{ccc} \bar{T}(\mathcal{A}) \bullet \bar{T}(\mathcal{A}) & \longrightarrow & \bar{T}(\mathcal{A}) \\ x_1 \cdots x_p \otimes (y_1 \otimes \ldots \otimes y_{p+1}) & \mapsto & y_1 x_1 y_2 x_2 \cdots x_p y_{p+1} \end{array} \tag{46}$$

The empty word $\emptyset$ acts as the unit for the *word insertion operad* $(\bar{T}(\mathcal{A}), \rho_{\mathcal{WI}})$. We denote by $\mathcal{W} = T_\otimes(\bar{T}(\mathcal{A}))$ the space of all words on elements of $\bar{T}(\mathcal{A})$, augmented with an element 1 with 0 inputs and outputs. Recall that W is a bicollection $\mathcal{W}(n,m)$ stands for the vector space generated by elements in $\mathcal{W}$ with n outputs and m inputs. The horizontal morphism on $\mathcal{W}$ induced by $\rho_{\mathcal{WI}}$ is denoted $\nabla_{\mathcal{W}}$. We set $\Delta^{\mathcal{W}} : \mathcal{W} \to \mathcal{W} \boxtimes \mathcal{W}$ the unique horizontal algebra morphism such that:

$$\Delta^{\mathcal{W}}(w) = \sum_{\substack{\alpha,\beta \in \mathcal{W},\\ w = \nabla_{\mathcal{W}}(\alpha \boxtimes \beta)}} \alpha \boxtimes \beta, \quad w \in \bar{T}(\mathcal{A}). \tag{47}$$

The map $\Delta^{\mathcal{W}}$ is a vertical coproduct, the counit $\epsilon^{\mathcal{W}} : \mathcal{W} \to \mathbb{C}_\boxtimes$ is: $\epsilon^{\mathcal{W}}(w) = \delta_{w=\emptyset^n} 1_n, \quad w \in \mathcal{W}(n,m)$. We now proceed with a similar construction we gave for the operad of non-crossing partitions. If w is a word in $\mathcal{W}$, we denote by w^1 the first letter of the first non-empty word in w. Then the morphism $\Delta^{\mathcal{W}}$ splits as

$$\Delta^{\mathcal{W}}(w) = \{\emptyset\}^n \boxtimes w + w \boxtimes \{\emptyset\}^m + \bar{\Delta}^{\mathcal{W}}(w),\ w \in \mathcal{W}(n,m),\ n \neq m \tag{48}$$

and we have $\bar{\Delta}(w) = \Delta_\prec(w) + \Delta_\succ(w)$ with

$$\Delta_\prec^{+,\mathcal{W}}(w) = \sum_{\substack{\alpha,\beta \in \mathcal{W},\\ \nabla_{\mathcal{W}}(\alpha \boxtimes \beta),\\ w^1 \in \alpha}} \alpha \boxtimes \beta, \quad \Delta_\succ^{+,\mathcal{W}} = \sum_{\substack{\alpha,\beta \in \mathcal{W},\\ \nabla_{\mathcal{W}}(\alpha \boxtimes \beta),\\ w^1 \in \beta}} \alpha \boxtimes \beta. \tag{49}$$

Finally, define $S^{\mathcal{W}} : \mathcal{W} \to \mathcal{W}$ as the unique horizontal morphism such that:

$$S^{\mathcal{W}}(a_1 \cdots a_p) = (-1)^p a_1 \cdots a_p, \quad a_1 \cdots a_p \in \bar{T}(\mathcal{A}). \tag{50}$$

Proposition 34. $(\mathcal{W}, \Delta_\prec^{\mathcal{W}}, \Delta_\succ^{\mathcal{W}}, \rho, S^{\mathcal{W}})$ *is an unshuffle* $\boxtimes\otimes$*-Hopf algebra.*

Proof. We only sketch the proof, the same machinery of cuts and admissible cuts expounded for the gap-insertion operad applies here. Let w be a word in $\mathcal{W}$ containing at least one non-empty word. By definition, such a word can be written $w = \emptyset^q|x|w'$, with x a word in $T(\mathcal{A})$ not equal to the empty word. We call a lowerset of x a subset S of letters of x. Then a lowerset determines a sequence of words $S^{|} = U_0|\ldots|U_{|S|}$, each of the U_i being either an empty words or a connected component of the complementary set of S in x. We have:

$$\Delta^{\mathcal{W}}(x) = \sum_{S\subset x} S \boxtimes S^{|}. \tag{51}$$

The unique lowerset of the empty word is the empty word itself and $\emptyset^{|} = \emptyset$. The notion of is readily extended to words on words. An upperset of x is a sequence $U_0|\ldots|U_s$ such that each of the U_i is either the empty word of a subword of x, with the condition that there exists a subword $L \in x$ (a lowerset) of length s such that $x = \nabla_{\mathcal{WI}}(L \boxtimes U_0|\ldots|U_s)$. Notice that the only upperset of the empty word is the empty word itself. The notion of lowerset is then canonically extended to words on words. We denote by $U^{\cdot}$ the lowerset associated with an upperset of U.

A triple cut of w is a triplet (L, M, U) such that L is a lowerset of w, U is an upperset of w, $L^{|} = \nabla M \boxtimes U$ and $U^{\cdot} = \nabla L \boxtimes U$. In that case, M is a lowerset of $L^{|}$ and $U = M^{|}$. We denote by $\mathrm{Cut}_2(w)$ the set of triple cuts of w such that L, M, U are not in $\mathbb{C}_{\boxtimes}$. The following relations hold:

$$(\Delta^{\mathcal{W}}_{\prec} \boxtimes \mathrm{id}) \circ \Delta^{\mathcal{W}}_{\prec}(w) = \sum_{\substack{(L,M,U)\in \mathrm{Cut}_2(w)\\ w^1\in L}} L \boxtimes M \boxtimes U = \mathrm{id} \boxtimes \Delta^{\mathcal{W}} \circ \Delta^{\mathcal{W}}_{\prec}(w),$$

$$(\Delta^{\mathcal{W}}_{\succ} \boxtimes \mathrm{id}) \circ \Delta^{\mathcal{W}}_{\prec}(w) = \sum_{\substack{(L,M,U)\in \mathrm{Cut}_2(w)\\ w^1\in M}} L \boxtimes M \boxtimes U = \mathrm{id} \boxtimes \Delta^{\mathcal{W}}_{\prec} \circ \Delta^{\mathcal{W}}_{\succ}(w)$$

$$(\Delta^{\mathcal{W}} \boxtimes \mathrm{id}) \circ \Delta^{\mathcal{W}}_{\succ}(w) = \sum_{\substack{(L,M,U)\in \mathrm{Cut}_2(w)\\ w^1\in U}} L \boxtimes M \boxtimes U = \mathrm{id} \boxtimes \Delta^{\mathcal{W}}_{\succ} \circ \Delta^{\mathcal{W}}_{\prec}(w)$$

□

Now we define $Sp : \mathcal{W} \to T_{\otimes}(\mathcal{NC})$ the *splitting map* in our settings, following [13]. It is an horizontal morphism defined by:

$$Sp(a_1\cdots a_p) = \sum_{\pi\in \mathrm{NC}(p)} \pi \otimes (a_1\cdots a_p), \quad a_1\cdots a_p \in \bar{T}(\mathcal{A}),$$

Proposition 35. *The horizontal algebra morphism Sp is an unshuffle morphism, which means:*

$$(Sp \boxtimes Sp) \circ \Delta^{\mathcal{W}}_{\prec,\succ} = \Delta^{T_{\otimes}(\mathcal{NC})}_{\prec,\succ} \circ Sp,\ \varepsilon^{T_{\otimes}(\mathcal{NC})} \circ Sp = \varepsilon^{\mathcal{W}}.$$

Proof. The arguments exposed in [13] can be used verbatim to prove the result. Let us prove the statement involving the two coproducts $\Delta^{\mathcal{W}}$ and $\Delta^{T_{\otimes}(\mathcal{NC})}$. It is enough to show that

$$(Sp \boxtimes Sp) \circ \Delta^{\mathcal{W}}(a_1\cdots a_n) = \Delta^{T_{\otimes}(\mathcal{NC})}(Sp(a_1\cdots a_n)),\ a_1\cdots a_n \in \mathcal{A}^{\otimes n}. \tag{52}$$

$$\Delta^{T_{\otimes}(\mathcal{NC})}(Sp(a_1\cdots a_n)) = \sum_{\pi\in \mathrm{NC}(p)} \sum_{\nabla_{T_{\otimes}(\mathcal{NC})}(\alpha \boxtimes (\beta_1,\ldots \beta_{|\alpha|}))=\pi} (\alpha \otimes a_{\alpha}) \boxtimes (\beta_1 \otimes a_{\beta_1}) \otimes \cdots \otimes (\beta_{|\alpha|} \otimes a_{\beta_{|\alpha|}})$$

In the last equation, the second sum runs over non-crossing partitions α, and $\beta_1, \ldots, \beta_{\alpha}$ seen as subsets of π, with the condition that the operadic composition (in the operad $\mathcal{NC}$) $\nabla_{T_{\otimes}(\mathcal{NC})}(\alpha \boxtimes (\beta_1, \ldots \beta_{|\alpha|}))$ of their *standard representatives* is equal to π. The notation a_{α} is the word in $T(\mathcal{A})$ obtained from a by concatenation of the linearly order set of letters partitioned by α, by convention $a_{\{\emptyset\}} = \emptyset$. In the vein of the proof of the preceding proposition, if $S \subset [\![1,n]\!]$ is a (possibly empty) set, we denote by $U_0,\dots,U_{|S|}$ the words in $T(\mathcal{A})$ such that $a = \nabla_{\mathcal{W}}(a_S \boxtimes U_0 \otimes \cdots \otimes U_{|S|})$. Then, we see that:

$$\Delta^{T_{\otimes}(\mathcal{NC})}(Sp(a_1\cdots a_n)) = \sum_{S\subset[\![1,n]\!]} \sum_{\substack{\alpha\in \mathrm{NC}(S),\\ \beta_1\in\mathrm{NC}(U_0),\dots,\beta_{|\alpha|}\in\mathrm{NC}(U_{|S|})}} (\alpha\otimes a_S)\boxtimes(\beta_0\otimes U_0)\otimes\cdots\otimes(\beta_{|\alpha|}\otimes U_{|S|})$$
$$= (Sp\boxtimes Sp)\circ\Delta^{\mathcal{W}}(a_1\cdots a_n).$$

□

Remark 3. The map $\Delta^{\mathcal{W}}$ is an horizontal morphism (by definition) but is not a properadic morphism: $\nabla_{T_{\otimes}(\mathcal{NC})}\circ(Sp\boxtimes Sp)\neq Sp\circ\nabla_{\mathcal{W}}$. As a consequence, the splitting morphism Sp is not a $\boxtimes\otimes$-Hopf algebra morphism, in particular:

$$(S^{T_{\otimes}(\mathcal{NC})}\circ Sp)(w) = \sum_{\pi\in Int(p)}(-1)^{\sharp\pi}\otimes w \neq (Sp\circ S^{\mathcal{W}})(w) = \sum_{\pi\in\mathrm{NC}(p)}(-1)^{|w|}\pi\otimes w.$$

Definition 36. An *infinitesimal morphism* $k:\mathcal{W}\to T_{\otimes}(\mathrm{Hom}(B))$ is a bicollection map equal to zero on every word in $\mathcal{W}$ except that

$$k(\emptyset^p|w|\emptyset^q) = id^p|k(w)|id^q,\quad w\in T(\mathcal{A}),\ w\neq\emptyset. \tag{53}$$

Let $\pi\otimes a_1\otimes\cdots\otimes a_p\in\mathrm{NC}(p)$ be a coloured non-crossing partition and k and infinitesimal morphism satisfying the property:

$$k(a_1\otimes\cdots\otimes a_p)\circ_1 k(a_1'\otimes\cdots\otimes a_q') = k(a_1'\otimes\cdots\otimes a_q')\circ_{q+1} k(a_1\otimes\cdots\otimes a_p),\ a_i,a_i'\in\mathcal{A} \tag{54}$$

We define inductively the collection of maps $\{k_\pi : B^{\otimes|\pi|}\to B,\ \pi\in\mathrm{NC}\}$ by

$$k_\pi(b_0,\dots,b_p) = \Big[k_{\sharp V}(a_V)\circ(k_{\pi_1},\dots,k_{\pi_{|V|}})\Big](b_0,\dots,b_p),\quad k_{\{\emptyset\}} = \mathrm{id}_B. \tag{55}$$

with V the block of π containing 1 and $\pi_0,\dots,\pi_{\sharp V}$ the sequence of non-crossing partitions such that $\nabla_{T_{\otimes}(\mathcal{NC})}(V\boxtimes\pi_0\otimes\cdots\otimes\pi_{\sharp V}) = \pi$.

We set $\mathcal{W}^+ = \sum_{n\neq m}\mathcal{W}(n,m)$. Proposition 34 implies that the class of bicollection homomorphisms $\mathrm{Hom}_{\mathrm{Coll}_2}(\mathcal{W}^+,T_{\otimes}(\mathrm{Hom}(B)))$ is a shuffle algebra. We set

$$\overline{\mathrm{Hom}}_{\mathrm{Coll}_2}(\mathcal{W},T_{\otimes}(\mathrm{Hom}(B))) = \mathbb{C}\eta_{\mathrm{Hom}(B)}\circ\varepsilon^{\mathcal{W}}\oplus\mathrm{Hom}_{\mathrm{Coll}_2}(\mathcal{W}^+,T_{\otimes}(\mathrm{Hom}(B))) \tag{56}$$

The following equations endow $\overline{\mathrm{Hom}}_{\mathrm{Coll}_2}(\mathcal{W},T_{\otimes}(\mathrm{Hom}(B)))$ with the structure of an augmented shuffle algebra:

$$\eta_{\mathrm{Hom}(B)}\circ\varepsilon^{\mathcal{W}}\prec\alpha = \alpha\succ\eta_{\mathrm{Hom}(B)}\circ\varepsilon^{\mathcal{W}} = 0,\quad \eta_{\mathrm{Hom}(B)}\circ\varepsilon^{\mathcal{W}}\succ\alpha = \alpha\prec\eta_{\mathrm{Hom}(B)}\circ\varepsilon^{\mathcal{W}} = \alpha$$

Now set $\alpha^{\prec p} = \alpha^{\prec p-1}\prec\alpha$ and $\alpha^{p\succ} = \alpha\succ\alpha^{p-1\succ}$ with $\alpha^{\prec 1} = \alpha^{1\succ} = \alpha$. The following lemma is a corollary of Proposition 35 and the computations of the shuffle exponentials (of infinitesimal morphisms from the gap-insertion properad to the endomorphism properad of B) of the previous sections.

Lemma 37. *Let $k:\mathcal{W}\to T_{\otimes}(\mathrm{Hom}(B))$ be an infinitesimal morphism satisfying (54), with*

$$\exp_{\prec}(k) = \eta\circ\varepsilon^{\mathcal{W}} + \sum_{p\geq 1}k^{\prec p},\ \exp_{\succ}(k) = \eta\circ\varepsilon^{\mathcal{W}} + \sum_{p\geq 1}k^{p\succ} \tag{57}$$

the following formulas hold

$$\exp_{\prec}(k)(w)(b_0\otimes\cdots\otimes b_p) = \sum_{\pi\in\mathrm{NC}(p)}k_{\pi\otimes w}(b_0,\dots,b_p),\quad w\in\mathcal{A}^{\otimes p}, \tag{58}$$
$$\exp_{\succ}(k)(w)(b_0\otimes\cdots\otimes b_p) = \sum_{\pi\in\mathrm{Int}(p)}k_{\pi\otimes w}(b_0,\dots,b_p),\quad w\in\mathcal{A}^{\otimes p}$$

Denote by $\underline{k}$ and $\underline{b}$ the infinitesimal morphisms from $T_{\otimes}(\mathcal{NC})$ to $T_{\otimes}(\mathrm{Hom}(B))$ defined by:

$$\underline{k}(\pi \otimes a_1 \otimes \cdots \otimes a_n)(b_0, \ldots, b_n) = \delta_{\pi = \mathbf{1}_n} \kappa_n(b_0 a_1, \ldots, a_n b_n),$$
$$\underline{b}(\pi \otimes a_1 \otimes \ldots \otimes a_n)(b_0, \ldots, b_n) = \delta_{\pi = \mathbf{1}_n} \beta_n(a_1, \ldots, a_n)(b_0, \ldots, b_n),$$

where $\kappa_p(a_1, \ldots, a_n)$ respectively $\beta_n(a_1, \ldots, a_n)$ are the operator-valued free cumulant respectively boolean cumulants of the random variables $a_1, \ldots, a_p$. Then the maps $k, b : \mathcal{W} \to T_{\otimes}(\mathrm{Hom}(B))$ defined by

$$k(a_1 \cdots a_p) = (\underline{k} \circ Sp)(a_1 \cdots a_p), \; b(a_1 \cdots a_p) = (\underline{b} \circ Sp)(a_1 \cdots a_p) \tag{59}$$

are infinitesimal morphisms on $\mathcal{W}$. Let K and B be the horizontal morphisms from $T_{\otimes}(\mathcal{NC})$ to $T_{\otimes}(\mathrm{Hom}(B))$ solutions of the fixed point equations

$$K = \eta_{\mathrm{Hom}(B)} \circ \varepsilon^{T_{\otimes}(\mathcal{NC})} + \underline{k} \prec K, \; B = \eta_{\mathrm{Hom}(B)} \circ \varepsilon^{T_{\otimes}(\mathcal{NC})} + B \succ \underline{b}. \tag{60}$$

Owing to Proposition 35, the morphisms $K \circ Sp$ and $E \circ Sp$ are solutions of the following fixed point equations:

$$K \circ Sp = \eta \circ \varepsilon^{\mathcal{W}} + k \prec (K \circ Sp), \; B \circ Sp = \eta \circ \varepsilon^{\mathcal{W}} + (B \circ Sp) \succ b. \tag{61}$$

Now owing to Lemma 37 and definitions of the free and boolean cumulants, we have

$$(K \circ Sp)(a_1 \cdots a_n)(b_0, \ldots, b_n) = (B \circ Sp)(a_1 \cdots a_n)(b_0, \ldots, b_n) = E(b_0 a_1 \cdot_{\mathcal{A}} \cdots \cdot_{\mathcal{A}} a_n b_n), \; a_1, \ldots, a_n \in \mathcal{A}.$$

Proposition 38 (Operator-valued moment-cumulant relations)**.** *With the notation introduced so far, let* $k : \mathcal{W} \to T_{\otimes}(\mathrm{Hom}(B))$ *and* $b : \mathcal{W} \to T_{\otimes}(\mathrm{Hom}(B))$ *be the infinitesimal morphisms on* $\mathcal{W}$ *such that:*

$$k(a_1 \cdots a_n)(b_0, \ldots, b_n) = \kappa_n(b_0 a_1, \ldots, a_n b_n), \; b(a_1 \cdots a_n)(b_0, \ldots, b_n) = \beta_n(b_0 a_1, \ldots, a_n b_n). \tag{62}$$

Besides denotes by E *the horizontal morphism on* $\mathcal{W}$ *with values in* $T_{\otimes}(\mathrm{Hom}(B))$ *such that:*

$$E(a_1 \ldots a_n)(b_0, \ldots, b_n) = E(b_0 \cdot_{\mathcal{A}} a_1 \cdots a_n \cdot_{\mathcal{A}} b_n).$$

Then:

$$E = \eta_{\mathrm{Hom}(B)} \circ \varepsilon^{\mathcal{W}} + k \prec E = \eta_{\mathrm{Hom}(B)} \circ \varepsilon^{\mathcal{W}} + E \succ b. \tag{63}$$

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