ON THE TWISTED FACTORIZATION OF THE T-TRANSFORM

KURUSCH EBRAHIMI-FARD, NICOLAS GILLIERS

ABSTRACT. The amalgamated T-transform of a non-commutative distribution was introduced by K. Dykema and provides a fundamental tool for computing distributions of random variables in Voiculescu's free probability theory. The T-transform factorizes in a rather non-trivial way over a product of free random variables. In this article, we present a simple graphical proof of this property, followed by a more conceptual one, using the abstract setting of an operad with multiplication.

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1. INTRODUCTION

In reference [4], K. Dykema introduced and studied two central objects in free probability theory, i.e., the operator-valued R-transform, more precisely, the *unsymmetrised* R-transform, as well as the – interrelated – operator-valued *unsymmetrised* S- and T-transforms. Those transforms play a fundamental role in both scalar- as well as operator-valued free probability theory [10, 11] as they allow for the effective –algorithmic– calculation of the distribution of a sum respectively product of free random variables. In the scalar-valued case, they can be traced back to the seminal works by Voiculescu [13, 14]. Here, the R- and T-transforms with respect to a random variable a in a non-commutative probability space $(\mathcal{A}, \phi)$ are formal power series in one variable, $R_a(z), T_a(z) \in \mathbb{K}\langle\langle z\rangle\rangle$. If a and b are two free random variables in $\mathcal{A}$, with $\phi(a) = \phi(b) = 1$, then these transforms are linear

$$R_{a+b}(z) = R_a(z) + R_b(z) \tag{1}$$

respectively multiplicative

$$T_{ab}(z) = T_a(z) \cdot T_b(z). \tag{2}$$

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The product on the right-hand side of (2) is the Cauchy product, defined for two series $f(z), g(z) \in \mathbb{K}\langle\langle z\rangle\rangle$ by:

$$(f \cdot g)_n = \sum_{\substack{k,q\geq 0,\\ k+q=n}} f_k g_q.$$

The S-transform is defined as the inverse (with respect to the Cauchy product) of the T-transform, $S_a(z) = T_a^{-1}(z)$. All three transforms are related through the *distribution series* $\Phi_a(z) \in \mathbb{K}\langle\langle z\rangle\rangle$, $a \in \mathcal{A}$,

$$\Phi_a(z) = \sum_{n\geq 1} \phi(a^n) z^{n-1}, \tag{3}$$

namely [4],

$$\begin{aligned} (z + z^2\Phi_a(z))^{\langle -1\rangle} &= z(1 + zR_a(z))^{-1} \\ (z\Phi_a(z))^{\langle -1\rangle} &= z(1+z)^{-1} S_a(z). \end{aligned} \tag{4}$$

Inverses of formal series in the left-hand sides of the above equations are computed with respect to *composition* of formal power series, denoted by $\circ$ and defined on $z\mathbb{K}\langle\langle z\rangle\rangle$ by

$$(f \circ g)_n = \sum_{\substack{k,n_1,\ldots,n_k\geq 1\\ n_1+\cdots+n_k=n}} f_k g_{n_1} \cdots g_{n_k}. \tag{5}$$

The so-called free additive and multiplicative convolution problems have been shown by Voiculescu to admit solutions by constructing canonical random variables. Dykema verified in [4] that in the operator-valued case, these results admit counterparts, where *multilinear function series* play the role of formal power series.

More specifically, let $(\mathcal{A}, \phi, B)$ be an operator-valued probability space, that is, $\mathcal{A}$ is an unital algebra with unital subalgebra $B \subset \mathcal{A}$ and conditional expectation $\phi : \mathcal{A} \to B$, $\phi(b) = b$ and $\phi(b_1 a b_2) = b_1\phi(a)b_2$ for all $b, b_1, b_2 \in B$, $a \in \mathcal{A}$ [10]. Note that further below (Definition 1.1) we will work with a slightly more general notion of operator-valued probability space. K. Dykema introduced the notion of a multilinear function series, that is, the data of a sequence $(\alpha_n : B^{\otimes n} \to B)_{n\geq 0}$ of multilinear maps on the algebra B. We adopt the convention that $B^{\otimes 0} = \mathbb{C}$. This implies that α_0 can be considered as an element of B. Let $a \in \mathcal{A}$ be a random variable with $\phi(a) = 1_B (= 1_{\mathcal{A}})$. The multilinear function series $\Phi_a = (\Phi_{a,0}, \Phi_{a,1}, \ldots) \in \mathrm{Mult}[[B]]$ that replaces the scalar-valued distribution series (3) is defined through $\Phi_{a,0} = \phi(a)$ and for $n > 0$

$$\Phi_{a,n}(b_1, \ldots, b_n) = \phi(ab_1ab_2 \cdots ab_na). \tag{6}$$

Given two multilinear function series $\alpha, \beta \in \mathrm{Mult}[[B]]$, we define their formal product by

$$(\alpha \centerdot \beta)_n(b_1, \ldots, b_n) = \sum_{k=0}^{n} \alpha_k(b_1, \ldots, b_k)\beta_{n-k}(b_{k+1}, \ldots, b_n).$$

This product turns the space $\mathrm{Mult}[[B]]$ into an unital algebra, with unit $1 = (1_B, 0, \ldots)$. We define a second product on $\mathrm{Mult}[[B]]$ given by *composition*, denoted $\alpha \circ \beta$, if $\beta_0 = 0$, by $(\alpha \circ \beta)_0 = \alpha_0$, and

$$(\alpha \circ \beta)_n(b_1, \ldots, b_n) = \sum_{k=1}^{n} \sum_{\substack{p_1,\ldots,p_k\geq 1\\ p_1+\cdots+p_k=n}} \alpha_k\Big(\beta_{p_1}(b_1, \ldots, b_{p_1}), \ldots, \beta_{p_k}(b_{q_k+1}, \ldots, b_{q_k+p_k})\Big),$$

where $q_j := p_1 + \cdots + p_{j-1}$. Notice that composition is linear only in the left argument. The unit for this product is the multilinear function series

$$I = (0, \mathrm{id}_B, 0, \ldots). \tag{7}$$

Following an analogue approach as in (4) using Φ_a defined by (6), we obtain operator-valued counterparts of the S-, R- and T-transforms that are now multilinear function series. By constructing random variables with prescribed T-transform, Dykema showed in [4] that the multiplicativity of the T-transform (2) in the scalar-valued case generalises to the following so-called twisted factorization in the operator-valued case:

$$T_{ab} = (T_a \circ (T_b \centerdot I \centerdot T_b^{-1})) \centerdot T_b. \tag{8}$$

R. Speicher showed that non-crossing set partitions underlie the combinatorics of moment-cumulant relations. To relate free cumulants to the coefficients of the T-transform, the same role is played by so-called *non-crossing linked set partitions*. In particular, Möbius inversion on a certain poset $\mathrm{NCL}_D^{(1)}$ of *connected non-crossing linked set partitions* relates coefficients of the T-transform to free cumulants,

$$\kappa_n(ab_1, \ldots, ab_{n-1}, a) = \sum_{\pi \in \mathrm{NCL}_D^{(1)}(n)} t_a(\pi)(b_1, \ldots, b_{n-1}). \tag{9}$$

We refer to reference [4] for the definition of the multilinear map $t_a(\pi)$. Let us just mention that $t_a(\pi)$ factorises over the blocks of a non-crossing linked partition. In this article, this factorization is expressed by saying that a certain morphism, whose range comprises the values $t_a(\pi), \pi \in \mathrm{NCL}_D^{(1)}(n)$. For small values of n, one has

$$\begin{aligned}
\kappa_2(ab_1, a) &= t_a(1_1)(b_1)\\
\kappa_3(ab_1, ab_2, a) &= t_a(1_2)(b_1, b_2) + t_a(1_1)(b_1 t_a(1_1)(b_2))\\
\kappa_4(ab_1, ab_2, ab_3, a) &= t_a(1_3)(b_1, b_2, b_3) + t_a(1_2)(b_1 t_a(1_1)(b_2), b_3) +\\
&\quad t_a(1_2)(b_1, b_2 t_a(1_1)(b_3)) + t_a(1_1)(b_1 t_a(1_2)(b_2, b_3)).
\end{aligned}$$

The above relations (one for each integer $n \geq 1$) can be cast into a fixed point equation on multilinear function series, with

$$K_a = 1 + \sum_{n \geq 1} K_a^{(n)}, \quad K_a^{(n)}(b_1, \ldots, b_n) = \kappa_{n+1}(ab_1, ab_2, \ldots, ab_n, a).$$

The relations displayed in (9) are equivalent to

$$K_a = T_a \circ (I \centerdot K_a),$$

where I is the identity for composition (7) in $\mathrm{Mult}[[B]]$.

In the first part of this work we give a different (and shorter) proof of equation (8) which is operadic in nature. It does not involve the construction of canonical random variables and therefore yields to a conceptual understanding of equation (8) as resulting from distributivity of the composition product $\circ$ over the coproduct $\centerdot$.

This distributivity together with the specific form of the composition product $\circ$ are constitutive to the notion of Gerstenhaber algebra. Examples of such type of algebras are obtained by considering operads equipped with a distinguished operator of arity two that we call *multiplication* (for reasons explained below). This multiplication endows the set of operators with a monoidal law, over which the operadic product distributes. The operad with multiplication used in our work is the endomorphism operad of B spanned by multilinear maps over the algebra B. The multiplication coincides with the algebra product on B.

It is worth noticing that these algebras admit a homotopical version [6]. In this context, the operadic multiplication can be used to define a differential on the underlying collection of the operad. This construction yields for example the Hochschild cohomology of B, if one starts with the endomorphism operad of B.

In the second part of this work, we take a leap in abstraction and introduce in the setting of an operad $\mathcal{P}$ with multiplication m a free product on the set G^{inv} of formal series of operators – with non-zero constant coefficient. The latter is supporting a monoidal structure stemming from the multiplication m. We define in this context the T-transform.

This permits us to give another, in some sense more fundamental proof of the twisted factorization of the abstract T-transform. In particular, we highlight the role of left and right translations by the identity of the operad. These maps provide two different injections of the set G^{inv} into the *diffeomorphism group* G of the operad $\mathcal{P}$ (certain series of operators on $\mathcal{P}$) denoted ρ and λ.

These translations are not group morphisms. However, they are injective and their inverses are *cocycles with respect to a right action* $\curvearrowright$ *of the diffeomorphism group* G *on* G^{inv}. This algebraic setup is well understood. See for example [**?**]. We consider in addition a left action $\curvearrowleft$ defined by conjugation of the group $(G^{inv}, \centerdot)$ on G. This permits to relate the translations ρ to λ. Ultimately, the twisted factorization of the abstract T-transform is implied by

(1) Distributivity of the right action $\curvearrowright$ of the group G over the product $\centerdot$,
(2) The cocycle property of ρ with respect to the left action $\curvearrowleft$,
(3) Compatibility of the left action $\curvearrowleft$ with the product of the diffeomorphism group G.

1.1. **Outline.** • After the introduction, the article is divided into two parts. In the first part (Sections 2 and 3) we provide the reader with the necessary background on operads and brace algebras. We then define the two core operads for the present work, which are the operad of non-crossing partitions (see Definition 2.5) and the operad of non-crossing linked partitions (see Definition 2.9). In Subsection 2.3, we define operads with multiplication as well as brace algebras.

• In Section 3, we first address the problem of computing operator-valued free cumulants with products as entries and give a fixed point equation for computing the free cumulants of the product of two operator-valued free random variables, see Subsection 2.3. To the extend of our knowledge, this fixed point equation is new.

• Given this fixed point equation, we deduce a short proof of Theorem 3.3, pointing out the relation in Proposition 3.2 as the key property of the concatenation and composition products for the twisted factorization to hold.

• In Subsection 3.3, we define the free product and the T-transform in this abstract setting.

1.2. **Basic notions and notations.** We recall the definition of an operator-valued non-commutative probability space [10].

Definition 1.1 (Operator-valued probability space). *An operator-valued probability space is a triple* $(\mathcal{A}, \phi, B)$ *such that*

1. $\mathcal{A}$ *is a von-Neumann algebra*[1]*,*
2. B *is a* $C^\star$ *algebra,*
3. $\mathcal{A}$ *is a* B*-*B*-bimodule, meaning that* B *acts on the left as well as right on* $\mathcal{A}$

$$b_1 \cdot (a \cdot b_2) = (b_1 \cdot a) \cdot b_2, \quad (b \cdot a)^\star = a^\star \cdot b^\star \qquad b, b_1, b_2 \in B, a \in \mathcal{A}.$$

4. The state, $\phi : \mathcal{A} \to B$*, is a* B*-*B *bimodule morphism*[2] *which is positive:*

$$\phi(aa^\star) \in BB^\star \qquad a \in \mathcal{A}.$$

In the present work, functionals of random variables are restricted to polynomial ones. As a consequence, we can drop all topological assumptions in the above definition. In particular, $\mathcal{A}$ and B are only assumed to be involutive algebras.

2. Operads and brace algebras

In this section, we recall the definitions of operads and brace algebras. The reader is directed to the monograph [9] for a detailed introduction.

[1] $\mathcal{A}$ can also be $C^\star$ algebra, depending on the type of functional calculus on $\mathcal{A}$ we want to have access to, either borelian, continuous or polynomial.

[2] However, it is *not* an algebra morphism.

2.1. **Algebraic planar operads.** Operads are models for composing operators with multiple inputs and single output. They have been introduced by Boardman and Vogt in the 1970's. There are multiple equivalent definitions of an operad. In this section, we adopt a rather algebraic point of view by defining an operad first as a monoid in a certain monoidal category that we introduce. It should be understood that what we call an operad in this article is also called a planar operad; the set of operators we consider are vector spaces and are not assumed to be endowed with an action of the symmetric group.

Operators are organized in a *collection* C, that is a sequence of vector spaces $(C(n))_{n\geq 1}$. The vector space $C(n)$ comprises all operators with n inputs. We assume this collection to be reduced which means that all operators have at least one input. The number of inputs of an operator p is denoted $|p|$ and is called its *arity*.

A morphism between two collections C and D is a sequence of linear morphisms $(\phi(n))_{n\geq 1}$ with $\phi(n) : C(n) \to D(n)$, $n \geq 1$. We denote by Coll the category of all collections. We remark that it is an abelian category in an obvious way, the sum $C \oplus D$ of two collections is the collection defined by

$$(C \oplus D)(n) = C(n) \oplus D(n). \tag{10}$$

The category Coll can be endowed with a monoidal structure, where the tensor product, here denoted $\bullet$, is the 2-functor from Coll $\times$ Coll to Coll defined by:

$$(C \bullet D)(n) = \bigoplus_{\substack{k\geq 1\\ n_1+\cdots+n_k=n}} C(k) \otimes D(n_1) \otimes \cdots \otimes D(n_k),$$

$$(f \bullet g)(n) = \bigoplus_{\substack{k\geq 1\\ n_1+\cdots+n_k=n}} f(k) \otimes g(n_1) \otimes \cdots \otimes g(n_k).$$

The unit element for this product on Coll is the collection denoted by $\mathbb{C}_\bullet$ such that $\mathbb{C}_\bullet(n) = \delta_{n=1}\mathbb{C}$, this means

$$C \bullet \mathbb{C}_\bullet \simeq C \text{ and } \mathbb{C}_\bullet \bullet C \simeq C \tag{11}$$

Given two collections C and D, the set $\mathrm{Hom}_{\mathrm{Coll}}(C, D)$ of collection homomorphisms from C to D is a vector space. However, the tensor product of two collection morphisms is linear only on its left argument, in particular $f \bullet \lambda g \neq \lambda f \bullet g,\ f, g \in \mathrm{Hom}_{\mathrm{Coll}}(C, D)$.

Definition 2.1 (Operad). *An* operad $\mathcal{C}$ *is a monoid in the monoidal category* (Coll, $\bullet$, $\mathbb{C}$)*, i.e., a triple* $(C, \gamma_{\mathcal{C}}, \eta_{\mathcal{C}})$*,* $C \in$ *Coll, with*

$$\gamma_{\mathcal{C}} : C \bullet C \to C,\ \eta_{\mathcal{C}} : \mathbb{C} \to C,$$

satisfying associativity and unitality constraints, namely

$$\gamma_{\mathcal{C}} \circ (\gamma_{\mathcal{C}} \bullet id_{\mathcal{C}}) = \gamma_{\mathcal{C}} \circ (id_{\mathcal{C}} \bullet \gamma_{\mathcal{C}})$$

$$\gamma_{\mathcal{C}} \circ (\eta_{\mathcal{C}} \bullet id_{\mathcal{C}}) = \gamma_{\mathcal{C}} \circ (id_{\mathcal{C}} \bullet \eta_{\mathcal{C}}) = id_{\mathcal{C}}$$

It is common to use the symbol $\circ$ to denote the composition $\gamma_{\mathcal{C}}$, $\gamma_{\mathcal{C}}(p \otimes (q_1 \otimes \cdots \otimes q_{|p|})) = p \circ (q_1 \otimes \cdots \otimes q_{|p|})$. In the remaining part of this article we follow this convention. More practically, one can exploit the unital and associativity constraints to associate *partial compositions* to any operad $(C, \gamma_{\mathcal{C}}, \eta_{\mathcal{C}})$, which compose an operator p in C with another operator q at a certain input of p,

$$p \circ_k q = p \circ (\mathrm{id}^{\otimes k-1} \otimes q \otimes \mathrm{id}^{\otimes |p|-k}) \qquad 1 \leq k \leq |p|.$$

Associativity of the product γ_c translates as follows for the partial compositions:

$$(p \circ_i q) \circ_j r = (p \circ_j r) \circ_{i+|r|-1} q \qquad 1 \leq j < i \leq |p|$$

$$p \circ_i (q \circ_j r) = (p \circ_i q) \circ_{j+i-1} r \qquad 1 \leq i \leq |p|,\ 1 \leq j \leq |q|.$$

In the next sections, we define two operads on non-crossing set partitions and on non-crossing linked set partitions. These two operads are in fact set-operads that we regard as operads (in the category of vector spaces) with a preferred basis that behave well under the operadic product. For the time being, let us give two examples of operads.

Example 2.2. *1. The forgetful functor from the category of operads to the category of collections admits a left adjoint that we call the free functor $\mathcal{F}$. Given a collection C, the free operad $\mathcal{F}(C)$ over C, is spanned by* planar rooted trees *with internal vertices decorated with elements in the collection C. Recall that an internal vertex of a tree is a vertex with at least one input. The degree of a decoration of an internal vertex matches the number of its inputs. In a linear setting, that is if C is a collection of vector spaces, we identify a tree τ having a vertex decorated by a sum $a+b$ of elements $a, b \in C$ with the sum of two trees, each obtained by replacing the decoration $a+b$ with a or b. A leaf is a vertex of a tree with no inputs. The number of leaves of a tree is the number of its inputs. The collection C corresponds to decorated corollas, trees with only one internal vertex. Composition of a tree τ with another tree τ' at the i^{th} input of τ (the leaves of τ are ordred from the leftmost to the righmost leaf) is obtained by* grafting *the root of τ' to the leaf of τ'. The identity is the tree with a single vertex.*

2. Given a vector space V, we denote by End_V the operad whose underlying collection consists of all multilinear maps on V,

$$\mathrm{End}_V(n) = \mathrm{Hom}_{\mathrm{Vect}}(V^{\otimes n}, V),$$

and the composition is induced by composition of functions.

3. Operad $\mathcal{W}$ *of word-insertions. Let $\mathcal{A}$ be an alphabet. Set $T(\mathcal{A}) := \bigoplus_{n\geq 1} \mathcal{A}^n$, the space of all non-commutative polynomials in elements from $\mathcal{A}$, and $\bar{T}(\mathcal{A}) := \mathbb{C}\cdot\emptyset \oplus T(\mathcal{A})$. Then $\bar{T}(\mathcal{A})$ is an unital algebra for the concatenation product, denoted $\cdot$, and with unit $\emptyset$. The degree $|w|$ of an element $w = w_1 \cdots w_n \in \mathcal{A}^{\otimes n}$ is $|w| = n+1$ and $|\emptyset| = 1$. We turn $\bar{T}(\mathcal{A})$ into the operad $\mathcal{W}$ by defining the operadic composition $\gamma_{\mathcal{W}}$ for a word $w = w_1 \cdots w_n$:*

$$\gamma_{\mathcal{W}}(w \otimes (u_0 \otimes \cdots \otimes u_n)) = u_0 \cdot w_1 \cdot u_1 \cdots u_{n-1} \cdot w_n \cdot u_n.$$

The definition of a planar operad uses the monoidal structure of the category $\mathrm{Vect}_{\mathbb{C}}$ of all vector spaces. Replacing the monoidal category ($\mathrm{Vect}_{\mathbb{C}}$) by another yields the notion of an operad in a symmetric monoidal category. For example one can replace $\mathrm{Vect}_{\mathbb{C}}$ by the category Set of sets with bijections. It is a monoidal category for the cartesian product of sets. A monoid in the category of all set collections is called a set operad.

We recall here two different tensor products on the category of operads, that will be briefly used in the forthcoming sections.

Definition 2.3 (Hadamard product). *Let $\mathcal{P} = (P, \gamma_{\mathcal{P}})$ and $\mathcal{Q} = (Q, \gamma_{\mathcal{Q}})$ be two operads. The Hadamard product of $\mathcal{P}$ and $\mathcal{Q}$ is the operad $\mathcal{P} \otimes_H \mathcal{Q} = (P \otimes_H Q, \gamma_{\mathcal{P}\otimes_H\mathcal{Q}})$ defined by*

$$(\mathcal{P} \otimes_H \mathcal{Q})(n) = \mathcal{P}(n) \otimes \mathcal{Q}(n)$$

$$\gamma_{\mathcal{P}\otimes_H\mathcal{Q}}(p \otimes q \,\otimes\, (p_1 \otimes q_1 \otimes \cdots \otimes p_n \otimes q_n)) = \gamma_{\mathcal{P}}(p \otimes p_1 \otimes \cdots \otimes p_n) \otimes \gamma_{\mathcal{Q}}(q \otimes q_1 \otimes \cdots \otimes q_n).$$

Definition 2.4 (Free product). *Let $\mathcal{P} = (P, \gamma_{\mathcal{P}})$ and $\mathcal{Q} = (Q, \gamma_{\mathcal{Q}})$ be two operads. The* free product *of $\mathcal{P}$ and $\mathcal{Q}$ is the operad $\mathcal{P} \sqcup \mathcal{Q}$ obtained by quotiening the free operad on the collection $\mathcal{P} \oplus \mathcal{Q}$ by relations in the operad $\mathcal{P}$ as well as relations in the operad $\mathcal{Q}$, but no other.*

It may happen that operators which we want to compose have different input and output ranges. A model for composing such operators is called a *coloured operad.* Collections are replaced by coloured collections, each vector space $C(n)$ of operators with n inputs is split into a direct sum of spaces $C^{c'}_{c_1,\ldots,c_n}$, $c_1, \ldots, c_n, c' \in \mathsf{C}$ comprising all operators with source spaces labeled $c_1, \ldots, c_n$ and target labeled c' for some set of colors C. Formal composition of collections (the monoidal product $\bullet$) admits a coloured version, for which operators are formally composed provided that colorations of inputs and outputs match.

2.2. **Operads of non-crossing partitions.**

2.2.1. *The gap insertion operad.* In this section we recall the definition of the *gap insertion operad* first introduced in [5] on non-crossing partitions. We define a new operad on *non-crossing linked partitions.* We remark that we adopt a more general definition of the latter, which are in fact coverings (blocks may intersect).

To fix notations, we denote by $\mathrm{NC}(n)$ the set of all non-crossing partitions of the interval $[\![1,\ldots,n]\!]$ (with its natural order). Recall that $\pi \in \mathrm{NC}(n)$ if π is a partition of $[\![1,\ldots,n]\!]$ and no blocks of π cross, which means that for any sequence of integers $a<b<c<d$ one has for two blocks $V, W \in \pi$

$$\text{if } a,c \in V,\ b,d \in W, \quad \text{then } V = W. \tag{12}$$

A partition $\pi \in \mathrm{NC}(n)$ is viewed as an operator with $n+1$ inputs ($|\pi| = n+1$). An input corresponds to a gap between two consecutive elements of the partitioned set, including the front gap before the element 1 and the back gap after the element n. Hence, we may therefore insert $n+1$ non-crossing partitions into a non-crossing partition by stuffing them into the gaps of the latter.

Definition 2.5 (Operad $\mathcal{NC}$ of non-crossing partitions). *We set $\mathcal{NC}(n) := \mathbb{C}\,[\mathrm{NC}(n-1)]$. In particular, we have $\mathcal{NC}(1) = \mathbb{C}\{\{\emptyset\}\}$. The unique partition of the empty set acts as the operad unit. Let π be a non-crossing partition and $(\alpha_1,\ldots,\alpha_{|\pi|})$ a sequence of non-crossing partitions, we define*

$$\gamma_{\mathcal{NC}}(\pi \otimes \alpha_1 \otimes \cdots \otimes \alpha_{|\pi|}) = \bigcup_{i=1}^{|\pi|} \{i-1+b,\ b \in \pi_i\} \cup \tilde{\pi}$$

where $\tilde{\pi}$ is the non-crossing partition of $\{|\pi_1|, |\pi_1|+|\pi_2|, \ldots, |\pi_1|+\cdots+|\pi_n|\}$ induced by π.

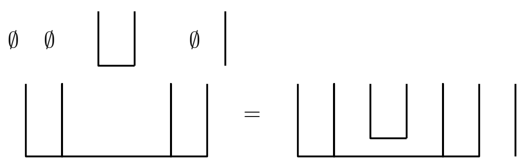

FIGURE 1. Example of a composition in the gap-insertion operad $\mathcal{NC}$.

Definition 2.6 (Coloured non-crossing partitions). *A coloured non-crossing partition is an element $\pi \otimes w$, $|w| = |\pi|$ in the Hadamard tensor product $\mathcal{NC} \otimes_H \mathcal{W}$ (see Example 2.2, item 3 for the definition of the word-insertions operad $\mathcal{W}$.).*

The gap-insertion operad of non-crossing partitions admits the following presentation in terms of generators and relations.

Lemma 2.7 (Proposition 3.1.4 in [5]). *For any $n \geq 1$, we put $\mathbb{1}_{n+1} = \{[\![1,n]\!]\}$. Then the operad $(\mathcal{NC}, \rho_{\mathcal{NC}})$ is generated by the elements 1_n, $n \geq 1$ with the relation:*

$$\forall m,n \geq 1, \quad \mathbb{1}_m \circ_m \mathbb{1}_n = \mathbb{1}_n \circ_1 \mathbb{1}_m.$$

Recall from [7] that the distribution of a random variable $a \in \mathcal{A}$ in an operator-valued probability space $(\mathcal{A}, \phi, B)$ yields an operadic morphism $\hat{\Phi}^a$ on the gap-insertion operad with values in the operad of endomorphisms End_B of B, prescribed by

$$\hat{\Phi}_a(1_n)(b_0,\ldots,b_n) := \phi(b_0ab_1a\cdots ab_n) \tag{13}$$

with $b_0,\ldots,b_n \in B$. Because ϕ is B-B bimodule map the morphism $\hat{\Phi}^a$ is well-defined.

The free cumulants $\{\kappa_n(a)\}_{n>0}$, $a \in \mathcal{A}$, correspond to another operadic morphism $\hat{\mathsf{K}}_a : \mathcal{NC} \to \mathrm{End}_B$ such that

$$\hat{\mathsf{K}}_a(1_n)(b_0,\ldots,b_n) = \kappa_n(b_0ab_1,\ldots,ab_n). \tag{14}$$

2.2.2. *Nesting-or-linking operad.* We now give the definition of a so-called non-crossing linked partition. As said, our definition is more general than what is usually given in the literature.

Definition 2.8 (Non-crossing linked partitions). *Let n be a positive integer. A* non-crossing linked (ncl) partition *is a collection π of subsets (blocks) of $[\![1,n]\!]$ such that:*

1. $\bigcup_{V\in\pi} V = [\![1,n]\!]$
2. *for U and V two blocks of π, if $a<c<b<d$ with $a,b\in U$ and $c,d\in V$, then $V=U$,*
3. *if $U\neq V$ and $|U|,|V|>1$, $|U\cap V|\geq 1$ then $U\cap V=\{x\}$ and $x=\min U$ or $x=\min V$.*

For $n\geq 1$, we denote by $\mathrm{NCL}(n)$ *the set of non-crossing linked partitions of $[\![1,n]\!]$.*

We refer the reader to Figure 2 for examples of non-crossing linked partitions.

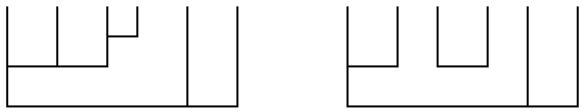

FIGURE 2. On the left, a connected ncl partition and on the right a ncl partition.

Two blocks of a ncl partition are allowed to meet at their minimal elements, this is the main difference between our definition of ncl partitions and the one given by K. Dykema in [4]. It allows us to define a natural operadic structure on ncl partitions. We denote NCL_D the subset of NCL comprising all non-crossing linked partitions with no pairs of blocks intersecting at their minimal elements.

Definition 2.9 (Nesting-or-linking operad). *Define the degree $|\pi|$ of a ncl partition by $|\pi|=n$ if $\pi\in\mathrm{NCL}(n)$. We denote by $\mathcal{NCL}$ the collection $(\mathrm{NCL}(n))_{n\geq 1}$. Given a ncl partition α in $\mathrm{NCL}(n)$ and $\beta_1,\ldots,\beta_n$ a sequence of ncl partitions, we define:*

$$\gamma_{\mathcal{NCL}}(\alpha\otimes\beta_1\otimes\cdots\otimes\beta_{|\alpha|}) = \bigcup_{i=1,\ldots,n}\{|\beta_{i-1}|+V,\ V\in\beta_i\}\cup\tilde{\pi} \tag{15}$$

where $\tilde{\pi}$ is the ncl partition of the set $\{1,1+|\beta_1|,|\beta_1|+|\beta_2|+1,\ldots,|\beta_1|+\cdots+|\beta_{n-1}|+1\}$ induced by π. We denote by $|$ the unique element in $\mathrm{NCL}(1)$*, which play the role of the unit of $\gamma_{\mathcal{NCL}}$.*

Blocks of non-crossing linked partitions can be either nested or linked. The former refers to the case where a block V of a ncl partition π is *nested* into another block W if it is contained in the convex hull of W and disjoint from W. This transitive closure of this elementary relation on π yields a partial order on π denoted $\leftarrow$. Linking refers to the case where a block V of a ncl partition π is *linked* to another block W of π if the minimum of V is contained in W and if $\min(V)=\min(W)$ then V is nested in W. Again, taking the transitive closure of this elementary relation yields an order on π, that we denote $\leftsquigarrow$.

One can represent a ncl partition as a *graph* on its blocks with two types of edges: $\leftsquigarrow$ and $\leftarrow$. Two blocks are connected if they are linked or nested. This graph has a lot of cycles: a block linked to another one is nested in any block the former is nested in, see Fig. **??**.

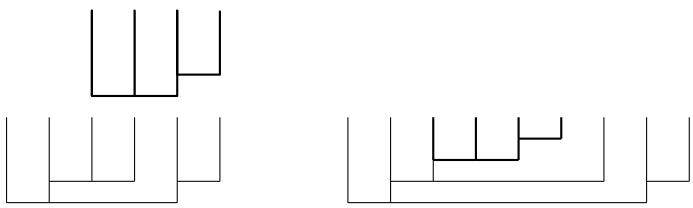

FIGURE 3. Example of a composition in the operad $\mathcal{NCL}$. We compose the ncl partition drawn in bold with the one below it at its third input.

In general, the relation $\Leftarrow$ on π defined as the transitive closure of the union of the two elementary orders, for $x, y \in \pi$,

$$x \Leftarrow y \Leftrightarrow x \leftarrow y \text{ or } x \leftarrow\!\!\!\sim y \tag{16}$$

is not antisymmetric. Remarkably, in our case, $\Leftarrow$ is antisymmetric. In details, if $x \Leftarrow y$ and $y \Leftarrow x$, the non-trivial cases are the following ones

(1) $x \leftarrow y$ and $y \leftarrow\!\!\!\sim x$,
(2) $x \leftarrow\!\!\!\sim y$ and $y \leftarrow x$

In the first case, the block y meets the block x as its minimum and x is included in the convex hull of y, which entails $x = y$ since π is non-crossing. The second case is similar.

As it will be clear after the proofs of Propositions, the Hasse diagram of $\Leftarrow$ is reminiscent of a tree monomial representing a ncl partition in the operad $\mathcal{NCL}$.

In [4] K. Dykema introduced two "projections" from the subset of non-crossing linked partitions NCL^D to non-crossing partitions to define a partial order on NCL^D. We define these two projections in our setting.

Definition 2.10 (Block merging). *Given a ncl partition π, we denote by $\hat{\pi}$ the non-crossing partition obtained by merging all blocks of π with a non-empty intersection.*

Example 2.11. *For example, if $\pi = \{\{1,5\},\{1,3\},\{2,4\}$, $\hat{\pi} = \{\{1,3,5\},\{2,4\}\}$.*

Pick a ncl partition π. A *connected component* of π is a subset of blocks that once merge all together for a block of $\hat{\pi}$.

Proposition 2.12 (Connected non-crossing linked partitions). *Denote by $\mathcal{NCL}^{(1)}$ the collection of connected ncl partitions, Let $n \geq 1$ be an integer and define the following subset of $\mathrm{NCL}(n)$ of connected ncl partitions:*

$$\mathrm{NCL}^{(1)}(n) = \{\pi \in \mathrm{NCL}(n) | \hat{\pi} = 1_n\}.$$

Then, the operadic composition $\gamma_{\mathcal{NCL}}$ restricts to $\mathcal{NCL}^{(1)}$. In addition, $(\mathcal{NCL}^{(1)}, \gamma_{\mathcal{NCL}})$ is isomorphic to the free operad on the collection of single block non-crossing linked partitions.

Notice that if π is a connected ncl partition, the Hasse diagram of π is a tree.

Proof. Let α and β be two connected ncl partitions and pick an integer $1 \leq i \leq |\alpha|$. From the very definition of the operadic composition on $\mathcal{NCL}$ and as illustrated in Figure 3, the block of α containing 1 intersects with the block of β containing i in the ncl partition $\beta \circ_i \alpha$.

Hence, $\beta \circ_i \alpha$ is connected if α and β are.

Then we construct a tree $\tau_{\leftarrow\!\sim}(\alpha)$ which is the Hasse diagram of $\leftarrow\!\!\!\sim$ augmented with leaves in order to interpret it as a tree monomial on (one block) ncl partitions. A vertex of $\tau_{\leftarrow\!\sim}(\alpha)$ corresponds to a block V of α and has $|V|$ incoming edges. We connect the output of a vertex V to an input of another block $W \in \pi$ if $V \leftarrow\!\!\!\sim W$ and if there is no other block U such that $V \leftarrow\!\!\!\sim U \leftarrow\!\!\!\sim W$. Since π is connected, there an unique minimal block for $\leftarrow\!\!\!\sim$ in π, the root of $\tau_{\leftarrow\!\sim}(\alpha)$.

Denote by $\mathcal{F}$ the free operad on the collection of one block ncl partitions and p the canonical projection $p : \mathcal{F} \to \mathcal{NCL}$.

Then $\tau_{\leftarrow\!\sim}(\alpha) \in \mathcal{F}$ and, clearly, $p(\tau_{\leftarrow\!\sim}(\alpha)) = \alpha$.

We show next that any tree τ in $\mathcal{F}$ such that $p(\tau) = \alpha$ is equal to $\tau_{\to}(\alpha)$. We prove this fact by induction on the number of block of a ncl partition π. This is obvious if α has only one block.

Assume that the result holds for ncl partitions with at most N blocks and pick a ncl partition α with $N+1$ blocks and $\tau \in \mathcal{F}$ such that $p(\tau) = \alpha$. Then τ has $N+1$ internal nodes. Write $\tau = V \circ (\tau_1, \ldots, \tau_{|V|})$. Notice that $V \leftarrow\!\!\!\sim W$ for any block W in $p(\tau_i)$, $1 \leq i \leq$. Thus V is the minimal block of α for the order $\leftarrow\!\!\!\sim$. We end the proof by applying the inductive hypothesis to the trees τ_i and ncl partitions $p(\tau_i)$. $\square$

Denote by I the collection of one blocks ncl partitions and by II the collection of ncl partitions defined by

$$\mathrm{II}_n = \{\, \{\{1\}, \{2, \dots, n-1\}\}\,\},\ n \geq 2. \tag{17}$$

We denote by θ_n the element of II_n See Fig. **??** for examples. Obvious relations among ncl partitions in $I \cup II$ are, for $m, n \geq 2$,

$$\theta_n \circ_1 1_m = 1_m \circ_m \theta_n, \quad \theta_n \circ_1 \theta_m = \theta_m \circ_m \theta_n \tag{18}$$

Proposition 2.13. *The operad $\mathcal{NCL}$ is generated* $\mathrm{I} \cup \mathrm{II}$ *with the relations* (18)

Proof. Denote by $\mathcal{F}$ the free operad on the collection $\mathrm{I} \cup \mathrm{II}$. The quotient of the free operad $\mathcal{F}$ by the relations (18) is isomorphic to the collection $\tilde{\mathcal{F}}$ of trees with $\mathrm{I} \cup \mathrm{II}$ decorated internal vertices meeting the following constraint. If v is an internal vertex of such a tree τ decorated with a ncl partition in the set II then its leftmost input is a leaf of τ. We show then that any ncl partition can uniquely be written as a monomial in $\tilde{\mathcal{F}}$. The proof is done by induction on the number of blocks of a ncl partition. First, the result is trivial for one block ncl partitions. We assume the result to hold for any ncl partition with at most N blocks and pick a ncl partition $\pi \in \mathrm{NCL}(p)$, $p \geq 2$ with $N+1$ blocks. Assume first there exists a tree $\tau \in \tilde{\mathcal{F}}$ such that $p(\tau) = \pi$ and write $\tau = V \circ (\tau_1, \tau_2, \dots, \tau_{|V|})$, with V a ncl partition in $I \sqcup II$. Follow two cases,

(1) If $V \in \mathrm{II}$, then τ_1 is the root tree and $\{1\} \in \pi$. In that case, V is equal to θ_n where n is the cardinal of the block $\{2 < i_2 < \ldots < i_n - 1\}$ containing 2. If we let π_j be the restriction of π to the interval $[\![i_j + 1, i_{j+1} - 1]\!]$ with the convention that $\pi_j = \{1\}$ if $i_j + 1 = i_{j+1}$ and $i_n = p$ then $p(\tau_j) = \pi_j$. The proof follows by applying the induction hypothesis to the ncl partitions π_j.
(2) If $V \in I$ then V is equal to the block that contain 1. The proof follows using the same line of arguments exposed in the previous case.

□

To construct a tree monomial on ncl partitions in $\mathrm{I} \cup \mathrm{II}$ representing a ncl partitions π, we can use the Hasse diagram $\tau_{\Leftarrow}(\pi)$ of the order $\Leftarrow$. First augment $\tau_{\Leftarrow}$ with as many leaves as needed for the degree of each corolla in $\tau_{\Leftarrow}$ to match the degree of the block of π it is decorated with. Place the additional leaves so that if $W \in \pi$ meets V at its i^{th} element, the corolla W is connected to the i^{th} input of the corolla V. Do the same if W is nested in V. From this process results a forest of blocks of π. Then, follow this rules. Firstly,

(1) If $\{1\} \in \pi$, erase the corolla representing this block and decorate the corolla representing the block of π containing 2 by θ_n where n is degree of this block minus one.
(2) If $\{1\} \not\in \pi$, decorate the corolla representing the block containing one with the corresponding block in I.

Secondly, if $V \leftsquigarrow W$ decorate the corolla representing W with the corresponding element in I and if $V \leftarrow W$, decorate the corresponding corolla with $\theta_{|W|+1}$ and a leftmost leaf. Finally, connect the root of each trees to the rightmost leaf of the previous one (if the forest is read from left to right).

Corollary 1. *The morphism of collections*

$$\begin{array}{rccc} j: & \mathbb{1} & \to & \mathrm{I} \\ & \mathbb{1}_n & \mapsto & \theta_n \end{array} \tag{19}$$

extends to an operadic morphism between the gap insertion operad and the linking-and-nesting operad.

The T-transform of a random variable a (in an operator-valued probability space $(\mathcal{A}, \phi, B)$) with $\phi(a) = 1_B$ is a sequence of multilinear maps on B,

$$t_a(n) : B^{\otimes n} \to B \qquad n \geq 0 \tag{20}$$

This sequence can be inductively defined by the following formula. Set $t_a(0) = 1 \in B$ and

$$\phi(b_1 a b_2 \cdots a b_n) = \sum_{\pi \in \mathrm{NCL}_D(n)} t_a(\pi)(b_1, \ldots, b_n), \quad b_1, \ldots, b_n \in B. \tag{21}$$

We refer the reader to [4], equations (67)-(70) for the definition of the map $\pi \mapsto t_a(\pi)$. Alternatively, by using the operadic structure on NCL we defined, $t_a(\pi)(b_0, \ldots, b_{n-1}$ can be seen to match the value

$$\hat{\mathsf{T}}_a(\pi)(b_0, b_1, \ldots, b_{n-1})$$

of the operadic morphism $\hat{T}_a : \mathrm{NCL} \to \mathrm{End}_B$ prescribed by the following equations

$$\hat{\mathsf{T}}_a(1_n)(b_0, \ldots, b_{n-1}) = b_0 \cdot t_a(n-1)(b_1, \ldots, b_{n-1}) \qquad n \geq 2. \tag{22}$$

We finish this section we additional remarks on relations between ncl partitions and non-crossing partitions. K. Dykema defined a second projection that associates to a non-crossing linked partition and non-crossing partition. This projection is only well defined on NCL^D. Pick a ncl partition $\pi \in \mathrm{NCL}^D$ and define the non-crossing partition $\check{\pi}$ by first, cutting out the minimum of a block if it belongs also to another block of π, and cutting out 1. For example, if $\pi = \{\{1,2,5\},\{2,3\}\}$ one has $\check{\pi} = \{\{1,4\},\{2\}\}$. Then, $\pi \mapsto (\hat{\pi}, \check{\pi})$ is a bijection from NCL^D to $\mathrm{NC} \times \mathrm{NC}$. A natural question concerns existence of an operadic structure on $\mathcal{NC} \times \mathcal{NC}$, pushforward of the nesting-or-linking operad. This question will be investigate in a further work. We can already make two remarks.

First, given two partitions $\alpha, \beta \in \mathrm{NCL}^D$, then certainly $\alpha \check{\circ}_i \beta = \check{\alpha} \circ_i \check{\beta}$. Secondly, if an operadic structure $\circ$ on NC (actually, on a collection of non-crossing partitions) makes $\hat{\pi}$ an operadic morphism, then the two non-crossing partitions $/ = \{\{1,2\}\}$ and $\backslash = \{\{1\},\{2\}\}$ have arity 2, and

$$/ \circ_1 / = / \circ_2 /, \ \backslash \circ_1 \backslash = \backslash \circ_2 \backslash, \ \backslash \circ_1 / = / \circ_2 \backslash. \tag{23}$$

The above relations are satisfied in the duplicial operad.

2.3. **Operads with multiplication.** In this section we introduce the notion of multiplication in an operad, which is a distinguished operator of arity 2. All operads we introduced so far admit a multiplication. We begin with the definition of *brace algebras* and refer to [1], [12], and [2] for details. Recall that $T(A)$ denotes the vector space of all non-commutative polynomials with entries in A. We shall use word notation, $a_1 \cdots a_n = a_1 \otimes \cdots \otimes a_n$, $a_i \in A$.

Definition 2.14 (Brace algebras). *A brace algebra is a tuple $(A, \{-;-\})$, where A is a vector space and $\{-;-\}$ is a linear map from $A \otimes T(A)$ to A such that*

$$\begin{aligned} &\{\{x; y_1 \cdots y_n\}; z_1 \cdots z_p\} \\ &\qquad = \sum \{x; z_1 \cdots z_{i_1}\{y_1; z_{i_1+1} \cdots z_{j_1}\} z_{j_1+1} \cdots \{y_n; z_{i_n+1} \cdots z_{j_n}\} z_{j_n+1} \cdots z_p\} \end{aligned} \tag{24}$$

with $x, y_1, \ldots, y_n, z_1, \ldots, z_p \in A$. The above sum runs over tuples $(i_1, \ldots, i_n)$ and $(j_1, \ldots, j_n)$ with

$$0 \leq i_1 \leq j_1 \leq i_2 \leq j_2 \leq \cdots \leq i_n \leq j_n \leq p.$$

Let $\mathcal{P}$ be an operad with $\mathcal{P}(1) = \mathbb{C} \cdot \mathrm{id}$ and operadic composition γ. It naturally yields a brace algebra structure on the collection $\mathcal{P}$ (more precisely on the direct sum of the vector spaces $\mathcal{P}(n), n \geq 1$),

$$\{x; y_1 \cdots y_n\} = \sum \gamma(x \otimes \mathrm{id} \cdots \otimes y_1 \otimes \mathrm{id} \otimes \cdots y_2 \otimes \mathrm{id} \cdots \otimes y_{n-1} \otimes \mathrm{id} \cdots \otimes y_n \otimes \mathrm{id} \cdots \mathrm{id}), \tag{25}$$

where the sum runs over all possible ways to branch the operators $y_1, \ldots, y_n$ to x while maintaining their linear order. Equation (24) follows from associativity of the product γ.

Definition 2.15 (Multiplication in an operad). *An operator $m \in \mathcal{P}(2)$ of arity 2 satisfying*

$$\gamma(m \otimes id \otimes m) = \gamma(m \otimes m \otimes id). \tag{26}$$

is called a multiplication*.*

We now browse through some examples, found among the operads we introduced in the previous sections.

Example 2.16. *1. The operad $\mathcal{NC}$ of non-crossing partitions is an operad with multiplication $m = |$. In fact, $\gamma(| \otimes \{\emptyset\} \otimes |)$ and $\gamma(| \otimes | \otimes \{\emptyset\})$ are both equal to the partition $|\,|$.*

2. The operad End_B of multilinear maps on an algebra (B, μ_B) (with multiplication μ_B), is an operad with multiplication with $m = \mu_B$.

3. In the nesting-or-linking operad of non-crossing linked partitions we have two operators of arity two, the partitions $|\,| = \{\{1\}, \{2\}\}$ and the partition $\sqcup = \{\{1, 2\}\}$. Only $||$ is a multiplication,since $|| \circ_1 || = || \circ_2 || = |||$. For $\sqcup$, $\sqcup \circ_1 \sqcup = \{\{1, 2\}, \{1, 3\}\}$ and $\sqcup \circ_2 \sqcup = \{\{1, 2\}, \{2, 3\}\}$.

4. In the word-insertions operad $\mathcal{W}$ any letter $a \in \mathcal{A}$ is an operator of arity two and a multiplication, since $a \circ_1 a = aa = a \circ_2 a$.

Remark 2.17. *In a graded context, that is, if the general term in the summation on the right hand side of (25) is multiplied by a sign, existence of a multiplication in an operad provides a rich structure on the collection $\mathcal{P}$ as observed by Gerstenhaber and Voronov in [6]. In particular, the multiplication m together with a certain* graded pre-lie product *yields a differential complex $(\mathcal{P}, d)$.*

We now assume that $\mathcal{P}$ is an operad with multiplication m. Following the above remark, the non-graded pre-lie product, denoted $\circ$, is defined by:

$$x \circ y = \{x; y\} \qquad x, y \in \mathcal{P}.$$

In fact, $x \circ (y \circ z) - (x \circ y) \circ z$ is symmetric in y and z. We denote by $[-, -]$ the bracket induced by $\circ$:

$$[x, y] = x \circ y - y \circ x \qquad x, y \in \mathcal{P},$$

which satisfies the Jacobi identity. In [3], the authors define a product, denoted $\times$, on the vector space $\mathbb{C}[[\mathcal{P}]]$ of formal series on operators in $\mathcal{P}$,

$$\mathbb{C}[[\mathcal{P}]] = \prod_{n \geq 1} \mathcal{P}(n), \tag{27}$$

defined for two series $a, b \in \mathbb{C}[[\mathcal{P}]]$ by

$$a \times b = \sum_{n \geq 1} \{a_n; b_{m_1} \cdots b_{m_n}\}.$$

Proposition 2.18 (see Proposition 4.1 in [3]). $(\mathbb{C}[[\mathcal{P}]], \times, \mathrm{id})$ *is an associative monoid. Besides, $a \in \mathbb{C}[[\mathcal{P}]]$ is an invertible element if and only if $a_1 \neq 0$.*

The above proposition implies that the subset $G \subset \mathbb{C}[[\mathcal{P}]]$ defined by

$$G = \Big\{ x \in \mathbb{C}[[\mathcal{P}]] : x_1 = \mathrm{id} \Big\}$$

endowed with the composition $\times$ is a group. The multiplication m yields another group product that we define in Section 3.3. But first, the multiplication m yields on the collection $\mathcal{P}$ a bilinear non-unital associative product $\cdot$ defined by

$$x \cdot y = \{m; xy\} \qquad x, y \in \hat{\mathsf{A}}_{\mathcal{P}}. \tag{28}$$

Proposition 2.19. $(\mathcal{P}, \{-; -\}, \cdot)$ *is a Gerstenhaber–Voronov algebra, which means that*

$$\{x \cdot y; z_1 \cdots z_p\} = \sum_{k=0}^{p} \{x; z_1 \cdots z_k\} \cdot \{y; z_{k+1} \cdots z_p\} \qquad x, y, z_1, \ldots, z_p \in \hat{\mathsf{A}}_{\mathcal{P}} \tag{29}$$

Equation (29) is key to the twisted factorization of the T-transform as explained below in Subsection 3.3.

3. Twisted factorization of the T-transform

We give a concise graphical proof of Theorem 7.18 in reference [4]. The starting point is a formula,for operator-valued free cumulants with the product of two free random variables as entries written using the language of operads. We then show how non-crossing linked partitions are naturally brought up by degree reduction, i.e., by filling inputs of a multilinear map with the unit of the algebra B.

3.1. **Free cumulants of products of random variables.** In this subsection, we explain how to compute the multilinear function series corresponding to free cumulants of the product of two free random variables as the solution of a certain fixed point equation. The proof of this fixed point equation (33) is sketched below. This formula is well known in the scalar-valued case. The authors have not been able to locate the operator-valued case in the literature.

Let us fix once and for all two free random variables a and b in the operator-valued probability space $(\mathcal{A}, \phi, B)$ with $\phi(a) = \phi(b) = 1_B$, and recall that we denote by K_x, $x \in \{a, b\}$, the multilinear function series

$$K_x = 1 + \sum_{n\geq 1} K_x^{(n)}, \quad K_x^{(n)}(b_1, \ldots, b_n) = \kappa_n(xb_1, xb_2, \ldots, xb_n, a),\ b_1, \ldots, b_n \in B. \tag{30}$$

Recall that in the scalar-valued case, that is, when $B = \mathbb{C}$, one has the beautiful formula [11]

$$\kappa_n(ab, \ldots, ab) = \sum_{\pi\in \mathrm{NC}(n)} \kappa_\pi(a)\kappa_{\mathrm{Kr}(\pi)}(b). \tag{31}$$

Here, the non-crossing partition $\mathrm{Kr}(\pi)$ is the Kreweras complement of $\pi \in \mathrm{NC}(n)$, first introduced in [8]. For two non-crossing partitions α and β in $\mathrm{NC}(n)$, one denotes by $\alpha \cup \beta$ the partition of $[\![1, 2n]\!]$ whose restriction to the odd integers, respectively to the even integers, coincides with α, respectively β. By definition, $Kr(\pi)$ is the maximal non-crossing partition (for the refinement order) such that $\pi \cup Kr(\pi)$ is a non-crossing partition of $\mathrm{NC}(2n)$.

In the operator-valued case, since the cumulants of a and b do not commute with each other (they are elements of the non-commutative algebra B), the right-hand side of equation (31) does not factorise over π and its Kreweras complement $Kr(\pi)$. In fact, we should maintain the linear order between random variables in the word $a \otimes b \otimes \cdots a \otimes b$.

We denote by $\tilde{\pi}$ the non-crossing partition $\pi \cup Kr(\pi)$. Each block of $\tilde{\pi}$ is coloured with 0 or 1, according to the parity of the elements in the block yielding an element of the free product $\mathcal{NC} \sqcup \mathcal{NC}$.

Recall that the operadic morphisms $\hat{\mathsf{K}}_a$ and $\hat{\mathsf{K}}_b$ are defined in equation (14). The free product $\mathsf{K}^a \sqcup \mathsf{K}^b$ is the unique operadic morphism on $\mathcal{NC} \sqcup \mathcal{NC}$ such that $\mathsf{K}^a \sqcup \mathsf{K}^b(\pi) = \mathsf{K}^a(\pi)$ if all blocks of π are coloured with 0 and $\mathsf{K}^a \sqcup \mathsf{K}^b(\pi) = \mathsf{K}^b(\pi)$ if all blocks of π are coloured with 1.

With these definitions, we *claim* that the operator-valued counterpart of formula (31) reads

$$x_0\kappa_n(ay_1bx_1, ay_2bx_2, \ldots, ay_nb)x_n = \sum_{\pi\in \mathrm{NC}(n)} (\mathsf{K}^a \sqcup \mathsf{K}^b)(\tilde{\pi})(x_0, y_0, \ldots, y_n, x_n), \tag{32}$$

with $x_0, \ldots, x_n \in B$ and $y_1, \ldots, y_n \in B$. We do not give a direct proof of this formula. To recover the free cumulants of ab, we set the $y's$ equal to $1 \in B$ in (32). We explain how this degree reduction (setting the $y's$ to 1_B) yields a sum over non-crossing *linked* partitions in place of a sum over non-crossing partitions.

Pick a non-crossing partition $\pi \in \mathrm{NC}(n)$. In Fig. 4, we pictured the partition $\tilde{\pi}$ with the blocks coloured according to the parity of the elements. We symbolize evaluation to 1 of the $y's$ by crosses. It is clear from the drawing in Fig. 4 that each block sees a cross either in its back or front gap.

A (odd) *black block* has its *front gap* marked with a cross whereas a (even) *blue block* – except the outer blue block – has its *back gap* marked with a cross. The blue outer block has both its front and back gap marked with a cross.

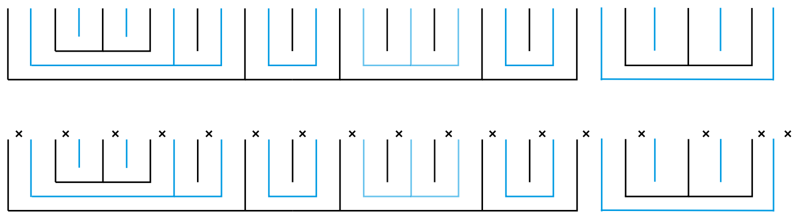

FIGURE 4. On the upper line, a partition $\tilde{\pi}$, the non-crossing partition π is drawn in black and its Kreweras complement is drawn in blue. On the second line, we symbolized with a cross evaluation to 1 of a variable in B (that fall within a gap)

The multilinear map $B^{\otimes(n+1)} \ni (x_0, \ldots, x_n) \mapsto (\hat{\mathsf{K}}_a \sqcup \hat{\mathsf{K}}_b)(\tilde{\pi})(x_0, 1, \ldots, x_n, 1)$ associated to the partition $\tilde{\pi}$ after evaluation of the $y's$ to 1 can thus be obtained by composing in the operad End_B the following multilinear maps

$$k^a_L(n)(b_1, \ldots, b_n) = \kappa_n(b_1 a, \ldots, b_n a),$$
$$k^b_R(n)(b_1, \ldots, b_n) = \kappa_n(b b_1, \ldots, b) b_n, \qquad k^a_R(n)(b_1, \ldots, b_n) = \kappa_n(a b_1, \ldots, a) b_n$$

with $b_1, \ldots, b_n \in B$. To explain how these multilinear maps are composed together, we associate to $\tilde{\pi}$ a non-crossing linked partition $\tilde{\pi}_\ell$. First, we choose a tree monomial $\tau(\tilde{\pi})$ representing $\tilde{\pi}$. We explained that $\tilde{\pi}$, with the blocks coloured, should be seen as an element of the free product $\mathcal{NC} \sqcup \mathcal{NC}$. Hence, the tree monomial $\tau(\tilde{\pi})$ representing $\tilde{\pi}$ as coloured corollas too. We can impose on $\tau(\tilde{\pi})$ the following:

1. the root corollas is either blue and its rightmost input is a leaf of $\tau(\tilde{\pi})$, either black and its rightmost input is leaf of $\tau(\tilde{\pi})$.
2. the leftmost and rightmost inputs of a corolla are leaves of $\tau(\tilde{\pi})$.

Next we erase leaves from the tree $\tau(\tilde{\pi})$ according the following rules:

1. we erase the rightmost leaf of a black corolla and of the corolla if the last is blue,
2. we erase the leftmost leaf of a blue corolla different from the root.

We obtain a tree $\tau(\tilde{\pi}_\ell)$ which, by making the substitution $\mathbb{1}_n \to 1_n$, $n \geq 2$ on each corolla, can be seen as a tree monomial on one-block ncl partitions representing a ncl partition $\tilde{\pi}_\ell$ in the nesting-or-linking operad. In Fig. 5, we have represented the partition $\tilde{\pi}_\ell$ resulting from the process described above, starting with the partition pictured in Fig. 4. Notice that all singletons in π are killed by this process.

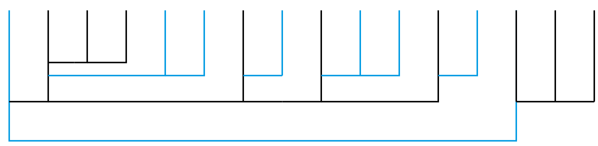

FIGURE 5. The resulting non-crossing linked partition.

To a blue root corolla of $\tau(\pi_\ell)$ (representing the outer blue block in Fig. 5) correspond a left B linear map from the sequence k^b_ℓ, whereas to the other blue blocks correspond a right B linear map from the sequence k^b_r. This entails that $\tilde{\pi}_\ell$ should in fact be seen as an element of the triple free product $\mathrm{NCL}^{\sqcup 3}$, with the blue outer block seen as element of the third tensor product and the other block distributed into the two remaining copies depending on their colour. Since monomials $\tau(\pi_\ell)$ with a blue root corolla can in fact be obtained by grafting such monomials but with black coloured root corolla to a blue corolla, we will assume first that $\tilde{\pi}$ is an *irreducible* non-crossing partition. In this case, the root corolla of $\tau(\pi_\ell)$ is black and $\tilde{\pi}_\ell$ has an outer blue block.

We call $\hat{K}^\ell_a$, $\hat{K}^r_b$ the operadic morphism on NCL with values in End_B that evaluates one-bock ncl partitions as k^ℓ_a and k^r_b respectively. Finally, set

$$\mathsf{K}_{a,b}(\tilde{\pi}_\ell) := \mathsf{K}^\ell_a \sqcup \mathsf{K}^r_b$$

Notice that the non-crossing linked partition $\tilde{\pi}_\ell$ does not meet the requirement of Dykema to have no blocks sharing their minimum. This is the reason for we gave a more general definition of non-crossing linked partitions.

Define the multilinear function series $V_{a,b}$ whose homogeneous component of order n is the sum of the multilinear maps $\mathsf{K}^{a,b}(\pi_\ell)$ when π ranges the set of all irreducible non-crossing partitions,

$$V_{a,b} = \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}} \mathsf{K}_{a,b}(\tilde{\pi}_\ell)$$

The next proposition follows from the discussion above.

Proposition 3.1. *Let a and b be two free random variables in $\mathcal{A}$, then*

$$V_{a,b} = K_a \times [(K_b \times [I \cdot V_{a,b}]) \cdot I)] \tag{33}$$

and K_{ab} is given by

$$K_{ab} = V_{a,b} \cdot (K_b \times [I \times V_{a,b}]). \tag{34}$$

3.2. **Short proof of the twisted factorization of the T-transform.** In this section, we give a short graphical proof for the twisted factorization of the T-transform. Following the work of K. Dykema, we define two subsets of multilinear function series,

$$\mathrm{Mult}[[B]]_0 = \{A \in \mathrm{Mult}[[B]] : A_0 = 0\}, \quad G = \mathrm{Mult}[[B]]_1 = \{A \in \mathrm{Mult}[[B]] : A_0 = 1\}.$$

We represent as rooted planar trees the operations of *concatenations* (the product $\cdot$) and *compositions* (the product $\times$) of multilinear function series. The composition $A \times B$ of two series $A \in \mathrm{Mult}[[B]]$ and $B \in \mathrm{Mult}[[B]]_0$ by a two nodes graph with a single vertical edge (see Fig.).

Doing so, we associate to the multilinear function series A an operator with a single input acting on $\mathrm{Mult}[[B]]_1$,

$$\mathrm{Mult}[[B]]_1 \ni B \mapsto A \times B. \tag{35}$$

The above operator is a set operator, in particular it is not linear. The outputed series belongs either to $\mathrm{Mult}[[B]]_0$ if $A \in \mathrm{Mult}[[B]]_0$ either to $\mathrm{Mult}[[B]]_1$ if $A \in \mathrm{Mult}[[B]]_1$.

More generally, if $W = E_1E_2 \ldots E_n$ is a *word* on multilinear function series, we associate to W set operators, with multiple inputs, acting on $\mathrm{Mult}[[B]]_1$. Each of these operators is drawn as a corolla decorated with W and with at most n leaves. Each leaf corresponds to composition of the inputed multilinear function series with one of the letter E_i, followed by concatenation of the resulting multilinear function series. For example, in the case $W = E_1E_2$, we have drawn in Fig. 6 the associated operators.

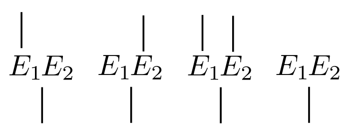

FIGURE 6. Operators associated with the word E_1E_2, from left to right, $A \mapsto (E_1 \circ 1)E_2$, $A \mapsto E_1(E_2 \circ A), (A, B) \mapsto (E_1 \circ A)(E_2 \circ B)$ and $E_1 \cdot E_2$.

Notice that the edges of the corollas drawn in Fig. 6 should be coloured, with 1 for the inputs and with 0 for the output, respectively 1, if ${E_1 \cdot \dots \cdot E_n}^0 = 0$, respectively, ${E_1 \cdot \dots \cdot E_n}^1 = 1$. We omit these colourizations to lighten notations.

In Fig. 7, we represented graphically the defining relation of the T-transform and in Fig. 8 the two equations (33) and (34).

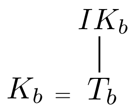

FIGURE 7. Equation satisfied by the T-transform and the free cumulants.

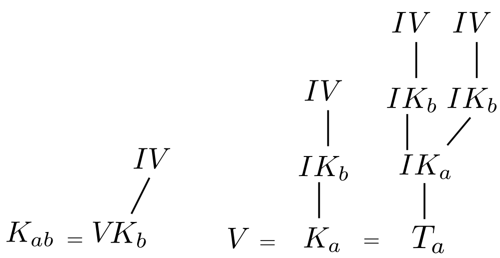

FIGURE 8. Graphical representation of equations (33) and (34).

Proposition 3.2 (Proposition 2.3. in [4]). *Let A, B and C be three*

$$[A \cdot B] \times C = (A \times C) \cdot (B \times C), \;\; C_0 = 0.$$

Notice that since $1 = 1 \circ E$, one has $(A \circ E)^{-1} = A^{-1} \circ E$.

Theorem 3.3 (Theorem 7.18 in [4]). *Let a, b be two free random variables, then*

$$T_{ab} = (T_a \times [T_b \centerdot I \centerdot T_b^{-1}]) \centerdot T_b.$$

Proof. The proof of the statement is contained in Fig. 9 and Fig. 11.

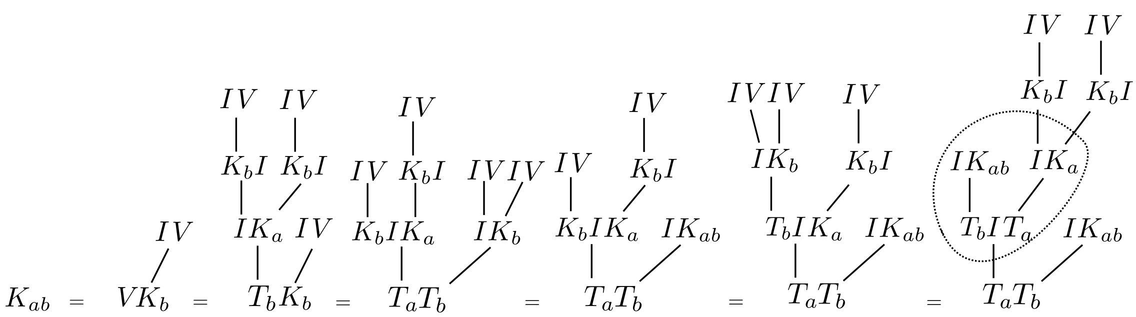

FIGURE 9. Proof of the multiplicativity property of the T-transform with respect to multiplication of free random variables

We detail the computations of Fig 9. For the first equality, we use equation (33) and for the second one equation (34). The third one follows from inserting the defining equation for the T-transform of b (see Fig. 7). We then recognize the equation (33) in the leftmost tree attached to the node T_aT_b. The fourth and fifth equalities proceed from the same computations. To continue, we use the relation drawn in Fig. 10 for the expression circled with a dotted line.

∎

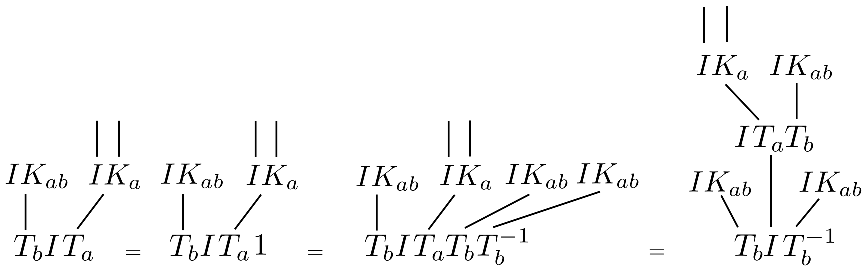

FIGURE 10. Direct corollary of the relation $1 = 1 \times E = (A \times E)(A^{-1} \times E)$.

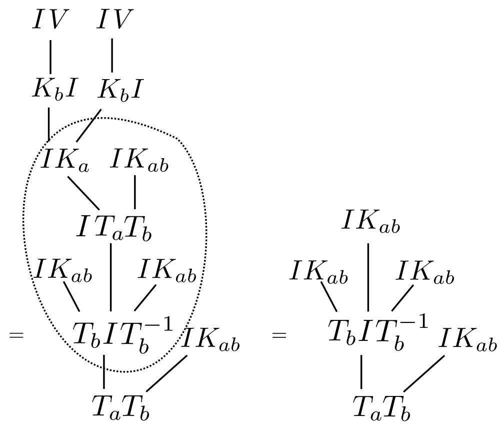

FIGURE 11. End of the proof of the multiplicativity property of the T-transform with respect to multiplication of free random variables

3.3. **Free product in an operad with multiplication.** In this section, we give a more conceptual proof of the twisted factorization for a T transform that we define in the abstract setting of an operad with multiplication $(\mathcal{P}, \gamma, m)$. This serves two objectives, the first one aims at emphasizing the key role played by distributivity of the product $\times$ with respect to the product $\cdot$ (see Section 2.3). And the second one is to draw connection with the algebraic setup of Hopf remormalization of quantum field theories. Pick a and b two formal series in $\mathbb{C}[[\mathcal{P}]]$ and define their concatenation product $a \centerdot b \in \mathbb{C}[[\mathcal{P}]]$ by

$$a \centerdot b = \sum_{n\geq 1} \sum_{\substack{k,q\geq 0\\ k+q=n}} a_k \cdot b_q = \sum_{n\geq 1} \sum_{\substack{k,q\geq 0\\ k+q=n}} \{m; a_k \otimes b_q\}. \tag{36}$$

The following proposition is a direct consequence of associativity of m and of the operadic product γ.

Proposition 3.4. $(\mathbb{C}[[\mathcal{P}]], \centerdot)$ *is an associative graded algebra.*

Set $\mathbb{C}[[\mathcal{P}]]_0$ equal to the unitization (for the product $\centerdot$) of $\mathbb{C}[[\mathcal{P}]]$,

$$\mathbb{C}[[\mathcal{P}]]_0 = \mathbb{C}1 \oplus \mathbb{C}[[\mathcal{P}]],\ \deg 1 = 0$$

and put

$$G^{inv} = \{x \in \mathbb{C}[[\mathcal{P}]]_0 : x_0 = 1\}.$$

The set G^{inv} if endowed with the concatenation product $\centerdot$ is a group that we denote $G^{inv}_{\centerdot}$. We have introduced so far two formal groups, G and $G^{inv}_{\centerdot}$. The group G is sometimes called the diffeomorphisms group of the operad $\mathcal{P}$. Regarding notations, a generic element of the group G will be denoted g and a generic element of G^{inv} will be denoted h.

We define next two actions, a right action $\curvearrowleft$ of the group G on G^{inv}, compatible with the concatenation product $\centerdot$, and a left action by conjugation $\curvearrowright$ of $G^{inv}_{\centerdot}$ on G. We begin with the former.

Definition 3.5 (Right action $\curvearrowleft$ of G on G^{inv}). *Pick an element $h \in G^{inv}$ and $g \in G$ and define $h \curvearrowleft g$ by*

$$h \curvearrowleft g = h \times g,\ 1 \curvearrowleft g = 1. \tag{37}$$

Owing to associativity of $\times$, $\curvearrowleft$ is a right action of G. Besides, it is compatible with $\centerdot$ in the following sense.

Proposition 3.6. *Pick $h, h' \in G^{inv}_{\centerdot}$ and $g \in G$, then*

$$(h \centerdot h') \curvearrowleft g = (h \curvearrowleft g) \centerdot (h' \curvearrowleft g),\ (h \curvearrowleft g)^{-1} = h^{-1} \curvearrowleft g. \tag{38}$$

Proof. Let $h, h' \in H$ and $g \in G$. Owing to the definition of the concatenation product $\centerdot$ and the action $\curvearrowleft$,

$$\begin{aligned}
(h \centerdot h') \curvearrowleft g &= 1 + \sum_{\substack{k,q\geq 0\\ k+q\geq 1}} \gamma(\{m; h_k h'_q\} \otimes g \otimes \ldots \otimes g) \\
&= 1 + \sum_{\substack{k,q\geq 0\\ k+q\geq 1}} \gamma(\gamma(m \otimes (h_k \otimes h'_q)) \otimes (g \otimes \cdots \otimes g)) \\
&= 1 + \sum_{\substack{k,q\geq 0\\ k+q\geq 1}} \gamma(m \otimes (\gamma(h_k \otimes g \otimes \cdots \otimes g) \otimes \gamma(h_q \otimes g \otimes \cdots \otimes g))) \\
&= (h \curvearrowleft g) \centerdot (h' \curvearrowleft g).
\end{aligned}$$

This concludes the proof. □

Definition 3.7 (Left action $\curvearrowright$ of G^{inv} on G). *The group $G^{inv}_{\centerdot}$ acts on the left of an element of G by conjugation,*

$$h \curvearrowright g = h \centerdot g \centerdot h^{-1}, \quad h \in G^{inv}_{\centerdot},\ g \in G \tag{39}$$

This last proposition immediately implies that H is a stable set for the action of G. The two actions $\curvearrowleft$ and $\curvearrowright$ are compatible in the following sense.

Proposition 3.8. *Let $h \in G^{inv}_{\centerdot}$, $g, g' \in G$, then*

$$a)\ (h \curvearrowright g') \times g = (h \curvearrowleft g) \curvearrowright (g' \times g) \quad b)\ . \tag{40}$$

Proof. Let $h \in H$ and $g, g' \in G$,

$$\begin{aligned}
(h \curvearrowright g') \times g &= 1 + h \centerdot g + g \centerdot h^{-1} \\
&\quad + \sum_{k,l,m\geq 1} \gamma(\gamma(m \otimes \gamma(m \otimes (h_k \otimes g_l)) \otimes h^{-1}_m) \otimes (g \otimes \cdots \otimes g)) \\
&= 1 + h \centerdot g + g \centerdot h^{-1} \\
&+ \sum_{k,l,m\geq 1} \gamma\Big(m^{(2)} \otimes \Big[\gamma(h_k \otimes (g \otimes \cdots \otimes g)) \otimes \gamma(g_l \otimes (g \otimes \cdots \otimes g)) \otimes \gamma(h^{-1}_l \otimes (g \otimes \cdots \otimes g))\Big]\Big) \\
&= (h \curvearrowleft g) \centerdot (g \times g') \centerdot (h^{-1} \curvearrowleft g) \\
&= (h \curvearrowleft g) \centerdot (g \times g') \centerdot (h \curvearrowleft g)^{-1} \\
&= (h \curvearrowleft g) \curvearrowright (g' \times g)
\end{aligned}$$

□

The identity I of the operad $(\mathcal{P}, \gamma)$ induces two injections of the set G^{inv} into G, obtained by left translation and right translation by I,

$$\begin{array}{cccc} \lambda: & G^{inv} & \to & G, \\ & h & \mapsto & I \centerdot h \end{array}, \qquad \begin{array}{cccc} \rho: & G^{inv} & \to & G \\ & h & \mapsto & h \centerdot I \end{array}$$

The maps ρ and λ are not group morphisms. However, they satisfie a number of interesting properties listed below,

1. both maps are injective,
2. their ranges is stable by the product $\times$,
3. they extend to co-algebras morphisms between the bialgebras of polynomial functions on G^{inv} and G,
$$(\lambda \otimes \lambda) \circ \Delta^H = \Delta^G \circ \lambda, \; (\rho \otimes \rho) \circ \Delta^H = \Delta^G \circ \rho$$
with $\Delta^G(g) = g \otimes g$ and $\Delta^H(h) = h \otimes h$.

The range of ρ, respectively λ is a subgroup of G that is usually denoted G^ρ, respectively G^λ in the literature, see [**?**]. Next, owing to Proposition 3.8 and item 3, G^{inv} can be endowed with two additional products,
$$h \star_l h' = h' \centerdot (h \curvearrowleft \lambda(h')), \; h \star_r h' = (h \curvearrowleft \rho(h')) \centerdot h'$$

Proposition 3.9. *Let $h, h' \in H$ and $g \in G$,*
$$a)\; \lambda(h) \times \lambda(h') = \lambda(h \star_l h'), \quad b)\; \rho(h) \times \rho(h') = \rho(h \star_r h') \tag{41}$$
$$\rho(h) = h \curvearrowright \lambda(h) \tag{42}$$

Proof. With $h, h' \in H$, one has
$$\lambda(h) \times \lambda(h') = (I \centerdot h) \times (I \centerdot h') = (I \centerdot h') \centerdot (h \times (I \centerdot h')) = I \centerdot (h' \centerdot (h \times (I \centerdot h')))).$$
The computations with ρ in place of λ are similar. The second statement is obvious. □

The above proposition can be restated by saying that λ^{-1} and ρ^{-1} are two cocycles with respect to the right action $\curvearrowleft$ and the group product on G^λ, respectively G^ρ,
$$\lambda^{-1}(g \times g') = \lambda^{-1}(g) \centerdot \lambda^{-1}(g') \curvearrowleft g'.$$

Proposition 3.10. *Pick h and h' in G^{inv}, then*
$$a)\; \rho(h) \times \lambda(h') = (h')^{-1} \curvearrowright \rho(h \star_l h'), \quad b)\; \lambda(h) \times \rho(h') = h' \curvearrowright \lambda(h \star_r h'), \tag{43}$$

Proof. We relie on the very definition of the left action $\curvearrowleft$ and of the maps ρ and λ. Let $h, h' \in G^{inv}$, then
$$\lambda(h) \times \rho(h') = h' \centerdot I \centerdot (h \times (h' \centerdot I)) = h' \centerdot (I \centerdot (h \times (h' \centerdot I)) \centerdot h') \centerdot (h')^{-1} \tag{44}$$
□

Relation $b)$ in the above proposition can be seen to be implied by Proposition 3.8 and the cocycle property for ρ. However, it does not seem to be a consequence of the cocycle property for λ together with Proposition 3.8 and 42.

Next, we define the free product of two elements in H in the abstract setting of an operad with multiplication.

Definition 3.11 (Free product). *Pick k_a and k_b in H and define the free product $k_a \boxtimes k_b \in H$ as*
$$v = k_a \curvearrowleft \rho(k_b \curvearrowleft \lambda(v))$$
$$k_a \boxtimes k_b = k_b \star_l v$$

Theorem 3.12. *Pick k_a and k_b in H and assume the following fixed point equations in H to hold*
$$k_a = t_a \curvearrowleft \lambda(k_a), \quad k_b = t_b \curvearrowleft \lambda(k_b), \tag{45}$$
with t_a and t_b in H. One has
$$k_a \boxtimes k_b = \big([t_a \curvearrowleft (t_b \curvearrowright I)] \centerdot t_b\big) \curvearrowleft \lambda(k_a \boxtimes k_b).$$

Proof. Set $k_{ab} = k_a \boxtimes k_b$. First, we use the fact that $\curvearrowleft$ is a right action of the group $(G, \times)$ to write

$$k_{ab} = v \centerdot (k_b \curvearrowleft \lambda(v)) = v \centerdot (t_b \curvearrowleft (\lambda(k_b) \times \lambda(v))) = v \centerdot (t_b \curvearrowleft \lambda(k_b \star_l v)) = v \centerdot (t_b \curvearrowleft \lambda(k_{ab}))$$

We then use the fixed point equation satisfied by k_a and k_b,

$$\begin{aligned} v = t_a \curvearrowleft (\lambda(k_a) \times \rho(k_b \curvearrowleft \lambda(v))) &= t_a \curvearrowleft (\lambda(k_a) \times \rho(t_b \curvearrowleft (\lambda(k_b) \times \lambda(v)))) \\ &= t_a \curvearrowleft (\lambda(k_a) \times \rho(t_b \curvearrowleft \lambda(k_b \star_l v))) \\ &= t_a \curvearrowleft (\lambda(k_a) \times \rho(t_b \curvearrowleft \lambda(k_{ab}))). \end{aligned}$$

By using equation (43) with $h' = t_b \curvearrowleft \lambda(k_{ab})$ and $h = k_a$ we obtain

$$\lambda(k_a) \times \rho(t_b \curvearrowleft \lambda(k_{ab})) = (t_b \curvearrowleft \lambda(k_{ab})) \curvearrowright \lambda(k_a \star_r (t_b \curvearrowleft \lambda(k_{ab})).$$

Coming back to the fixed point equation satisfied by v and inserting the fixed point equation satisfied by k_b, we get also

$$v = k_a \curvearrowleft \rho(t_b \curvearrowleft \lambda(k_b \star_l v)) = k_a \curvearrowleft \rho(t_b \curvearrowleft \lambda(k_{ab})).$$

This last equation implies

$$\begin{aligned} \lambda(k_a) \times \rho(t_b \curvearrowleft \lambda(k_{ab})) &= (t_b \curvearrowleft \lambda(k_{ab})) \curvearrowright (v \centerdot (t_b \curvearrowleft \lambda(k_{ab}))) \\ &= (t_b \curvearrowleft \lambda(k_{ab})) \curvearrowright \lambda(k_{ab}) \end{aligned}$$

Thus we obtain, for v and k_{ab},

$$v = t_a \curvearrowleft \Big((t_b \curvearrowleft \lambda(k_{ab})) \curvearrowright \lambda(k_{ab}) \Big),$$

$$k_{ab} = \Big[t_a \curvearrowleft \Big((t_b \curvearrowleft \lambda(k_{ab})) \curvearrowright \lambda(k_{ab}) \Big) \Big] \centerdot \Big[t_b \curvearrowleft \lambda(k_{ab}) \Big]$$

It follows from equations (40) and (38) that

$$\begin{aligned} k_{ab} = \Big[t_a \curvearrowleft \Big((t_b \curvearrowright I) \times \lambda(k_{ab}) \Big) \Big] \centerdot \Big[t_b \curvearrowleft \lambda(k_{ab}) \Big] &= \Big[\Big(t_a \curvearrowleft (t_b \curvearrowright I) \Big) \curvearrowleft \lambda(k_{ab}) \Big] \centerdot \Big[t_b \curvearrowleft \lambda(k_{ab}) \Big] \\ &= \Big((t_a \curvearrowleft (t_b \curvearrowright I)) \centerdot t_b \Big) \curvearrowleft \lambda(k_{ab}) \end{aligned}$$

□

If one chooses for the the operad with multiplication $\mathcal{P}$ the endomorphism operad of B, with m equal to the product in B, the above proof gives a third proof of the twisted multiplicativity for the T-transform in operator-valued free probability.

As a final remark, let us point at the following fact. First, the proof of Theorem 3.12 relies only on the cocycle property for λ, Proposition 3.8 and the fact that $h'^{-1} \curvearrowright \lambda(h) \times \rho(h')$ is in the range of λ, the explicit form of the antecedent is not crucial. This brings a generalization of the above formula, if left and right translation by the identity are not performed with respect to the same product. In [**?**], the author considers two multiplication on the Duplicial operad, $/$ and $\backslash$ satisfying the duplicial relations mentioned earlier. In this setting, the left translation λ and the right translation ρ are defined as

$$\lambda(h) = I \backslash h, \quad \rho(h) = h/I, \;\; h \in G^{inv} \tag{46}$$

With the right action by conjugation $\curvearrowright$ of G^{inv} on G defined by

$$h \curvearrowright g = h/g \backslash h^{\backslash -1}, \quad h \in G^{inv}, g \in G, \tag{47}$$

Proposition 3.8 holds. Moreover, by defining for $h, h' \in G^{inv}$,

$$h \star_r h' = [h \curvearrowleft (h'/I)] \backslash h', \quad h \star_l h' = h' \backslash h \curvearrowleft (I \backslash h') \tag{48}$$

then it can be checked by simple computations that

$$\lambda(h) \times \rho(h') = h' \curvearrowright \lambda(h \star_r h'), \lambda(h) \times \lambda(h') = \lambda(h \star_l h') \tag{49}$$

Notice that $\rho(h) \times \rho(h') \neq \rho(h \star_r h')$. As a matter of fact, Theorem 3.12 holds in this setting.

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