Non-commutative gauge symmetries

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07.06.2020
Free probability online seminar

Yang–Mills fields and master fields

$\boxed{\text{Symmetries} \longrightarrow \text{Interactions}}$

(circle) $U(1)$ $\longrightarrow$ Maxwell electromagn.

(quaternion) $SU(2)$ $\longrightarrow$ electroweak interaction.

$SU(3)$ $\longrightarrow$ strong interaction (quark)

(classical) Yang-Mills theory

Space time geometry (Lorentzian geometry)

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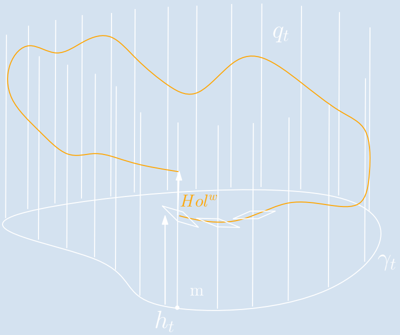
Kobayashi–Nomizu
Bleecker

Holonomy



Loop on the plane $\mapsto$ An element of the group (Holonomy)

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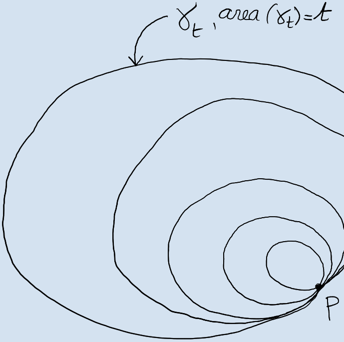
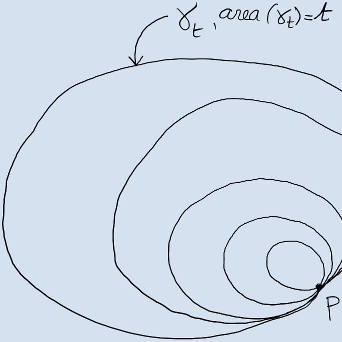
$$\text{Curvature :} \qquad \Omega^{\omega} = \mathrm{d}\omega + \frac{1}{2}[\omega \wedge \omega] \in \Lambda^2(P, \mathfrak{g})$$



γ_t, area$(\gamma_t) = t$
P



- The process $\big(\mathrm{Hol}(\gamma_t)\big)_{t>0}$ is a Brownian motion on the group, whose law is conjugacy-invariant.
- If $\overset{\circ}{\ell}_1 \cap \overset{\circ}{\ell}_2 = \emptyset$, $\mathrm{Hol}(\ell_1)$ are $\mathrm{Hol}(\ell_2)$ independent.

Holonomy fields

An holonomy field with structure group G is a collection $\{h_\ell, \ell \in \mathit{Loops}\}$ of G valued random variables.

- *Multiplicativity* : $h_{\ell_1 \ell_2} = h_{\ell_1} h_{\ell_2}$

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- *Holonomy of simple loops with non-intersecting domains are independent.*

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⟪ Lévy process indexed by loops ⟫

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Which generalization ?

Lévy (2010)

Brownian motion $\longrightarrow$ Lévy process $\quad F(G)$

$N \to +\infty$

Free Brownian motion $\longrightarrow$ Lévy process $\quad$ free. $\quad G\langle 1 \rangle$

Lévy 2011

Cébron, Dahlqvist, Gabriel 2016

quantum Lévy process. $\quad H$

Arrow reversing

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$$f(Zh_{\ell_1}Z^{-1},\ldots,Zh_{\ell_n}Z^{-1}) \overset{(d)}{=} f(h_{\ell_1},\ldots,h_{\ell_n})$$

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Gauge symmetry and Zhang algebras

Zhang algebra vs Hopf algebra

$\mathcal{F}(G)$

Commutative Hopf algebra

commutative Zhang algebra

Woronowicz compact quantum groups

- Alexeev, Gross & Schomerus (1993)
- Buffenoir, Roche (1995)
- Meusburger (2015)

Algebraic Category

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- Set with disjoint union, $k = \varnothing$,
- ComAlg with (the usual) tensor product, $k = \mathbb{C}$,
- Alg with the free product.
- $\mathrm{Alg}(B)$ with the free product over Bn the unit is B.

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» A commutative Hopf algebra is a Zhang algebra in the category ComAlg.

- Algebra of functions on a group in $\mathrm{ComAlg}^\star$

$$f_1 f_2(g) = f_1(g) f_2(g),\ \Delta(f)(g_1, g_2) = f(g_1 g_2),\ S(f)(g) = f(g^{-1}).$$

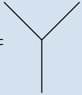
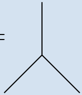
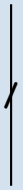
- Voiculescu's dual groups, in $\mathrm{Alg}^\star$,

$$\mathcal{O}\langle n\rangle = \langle u_{ij}, u_{ij}^\star \mid \sum_k u_{ik} u_{jk}^\star = 1\rangle \quad \Delta(u_{ij}) = u_{ik} u_{ki},\ S(u_{ij}) = u_{ji}^\star.$$

- Set $B = \langle\, p_1, \ldots, p_n \mid p_i p_j = p_j p_i = 0, p_i^2 = p_i, i \neq j \,\rangle$ in $\mathrm{Alg}(B)$.

$$\mathcal{O}_B\langle n\rangle = \langle u, u^\star \mid uu^\star = 1\rangle_B \quad \Delta(u) = u_{|1} u_{|2}, S(u) = u^\star$$

Graphical calculus à la Hopf

$\Delta =$  $\mu =$  $S =$  $\varepsilon =$  $\eta =$ 

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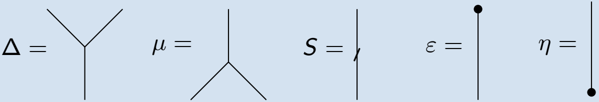

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J. Zhang

For any functor $F : \mathrm{Alg}(\mathcal{C}) \to \mathcal{G}rp$, there exists an Zhang algebra

$$F(\cdot) = \mathrm{Hom}_{\mathcal{C}}(H, \cdot)$$

J. Zhang, H-algebra, 1991.

Comodules and Conjugation coaction

Let $(\mathcal{C}, \sqcup, k)$ be an algebraic category A comodule (M, ρ) over a Zhang algebra $(H, \Delta, \varepsilon, S)$ is :

- An object in $\mathcal{C}$
- A morphism $\rho \in \mathrm{Hom}_{\mathcal{C}}(M, M \sqcup H)$,

$$(\rho \sqcup \mathrm{id}_H) \circ \rho = (\mathrm{id}_M \dot{\sqcup} \Delta) \circ \rho, \quad (\mathrm{id}_M \dot{\sqcup} \varepsilon) \circ \rho = \mathrm{id}_M.$$

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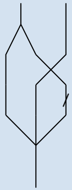
Proposition

The category *ICoModC*(H) is an algebraic category.

Proposition (G. 2019)

A Zhang algebra H is a left comodule over H :



$$\Omega_c = (\iota_1 \dot{\sqcup} \iota_2 \dot{\sqcup} \iota_1) \circ (\mathrm{id}_{H \sqcup H} \sqcup S) \circ (\Delta \sqcup \mathrm{id}_H) \circ \Delta.$$

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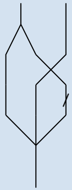

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Conjugation coaction on the algebra of functions on a group

$$\Omega^c(f)(h, g) = f(hgh^{-1}), \; f \in \mathcal{F}(G), \; g, h \in G.$$

...on the dual Voiculescu group $\mathcal{O}\langle n \rangle$:

$$\Omega^c(o_{ij}) = \sum_{k,q=1}^{n} o_{ik}|_1 o_{kq}|_2 o^{\star}_{kq}|_1$$

Categorical independence and monoidal structures

Classical independence and commutative diagrams

X, Y two random variables. They are independent if and only if :

$$\begin{tikzcd}
& (L^\infty(\Omega,\mathcal{F}),\mathbb{E}) & \\
(L^\infty(S,\mathcal{S}),\tau_X) \arrow[ur, "j_X"] & & (L^\infty(T,\mathcal{T}),\tau_Y) \arrow[ul, "j_Y"']
\end{tikzcd}$$

avec :

$$j_X : \begin{array}{ccc} L^\infty(S,\mathcal{S}) & \rightarrow & L(\Omega,\mathcal{F},\mathbb{P}) \\ f & \mapsto & f(X) \end{array},$$

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Monoidal structures

- tensor product of vector spaces, $V, W \in \mathrm{Vect}$,

$$V \otimes W = \{vw, v \in V,\ w \in W\},\ v(\lambda w) = (\lambda v)w = \lambda(vw).$$

- tensor product of unital algebras, $A, B \in \mathrm{ComAlg}$

$$a_1 \otimes b_1 \cdot a_2 \otimes b_2 = a_1 a_2 \otimes b_1 b_2$$

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$\otimes$ is...

- ...associative :

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \xrightarrow{\cong} (A \otimes B) \otimes C,$$

- ...has a unit object $E \in \mathrm{Obj}(\mathcal{C})$:

$$\ell_A : E \otimes A \xrightarrow{\cong} A,\ r_A : A \otimes E \xrightarrow{\cong} A$$

Categorical independence

$$\begin{tikzcd}
 & A & \\
C_1 \arrow[ur, "f_1"] & & C_2 \arrow[ul, "f_2"']
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Independence in Prob($\mathbb{C}$)

Pick $(\mathcal{A}_1, \phi_{\mathcal{A}_1})$ and $(\mathcal{A}_2, \phi_{\mathcal{A}_2})$ two probability spaces.

Universal classical tensor product $\hat{\otimes}$

$$\mathcal{A}_1 \hat{\otimes} \mathcal{A}_2 = \mathcal{A}_1 \sqcup \mathcal{A}_2,$$

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Non-commutative holonomy fields

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$$\mathcal{C} = \mathrm{Alg}^\star, (\mathrm{Prob}, \otimes) \ni (\mathcal{A}, \tau)$$

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- $\tau \circ j_{s,t} = \tau \circ j_{t-s}$

Holonomy fields

- $\mathsf{H}_\ell, \ell \in \mathsf{Loops}_0(\mathbb{R}^2)$, $\mathsf{H}_\ell \in \mathrm{Hom}_{\mathrm{Alg}^\star}(H, (\mathcal{A}, \tau))$

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∘ Si $(\ell_1, \ldots, \ell_n)$ and $(\ell'_1, \ldots, \ell'_n)$ are two family of loops contained in two simple loops with non-intersecting domains,

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$\circ$Invariance by area preserving morphisms :

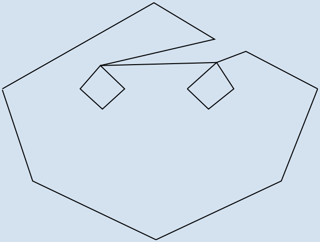
$$\phi_{\mathcal{A}} \circ \mathsf{H}_{\ell_1} \dot{\sqcup} \ldots \dot{\sqcup} \mathsf{H}_{\ell_n} = \phi_{\mathcal{A}} \circ \mathsf{H}_{\phi(\ell_1)} \dot{\sqcup} \ldots \dot{\sqcup} \mathsf{H}_{\phi(\ell_n)}.$$

Theorem

To any quantum Lévy process satisfying two properties one can associate an (algebraic) non-commutative holonomy field.

How to construct such a field ?

Affine loops, direct limit



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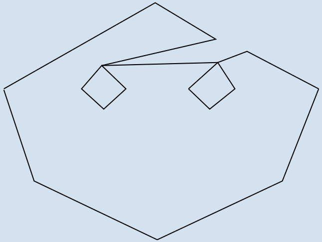

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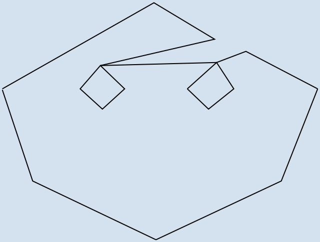

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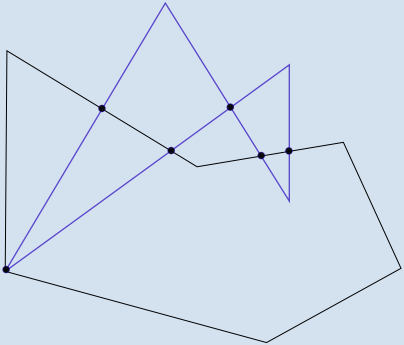
To a sequence of loops there is an associated graph :



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We construct $(\mathcal{A}, \tau)$ as a direct limit,

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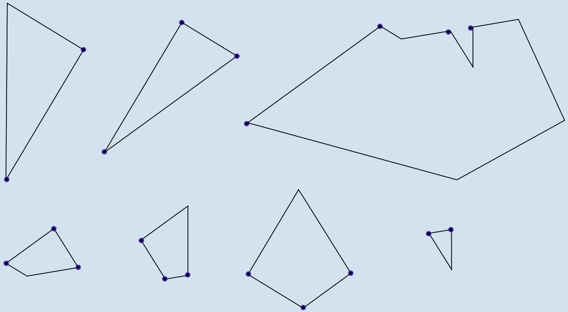

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We can find a basis of lassos $(c_1, \ldots, c_{|\mathbb{F}|})$ for the previous graph...

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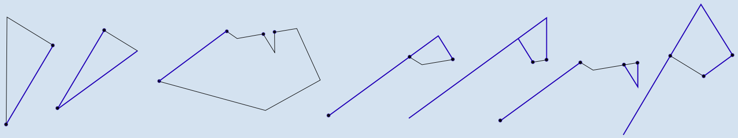

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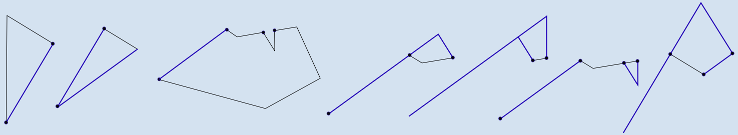

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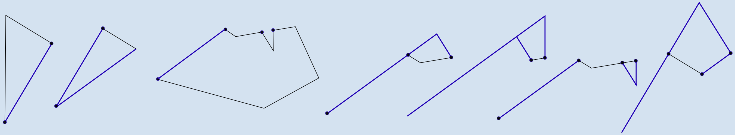

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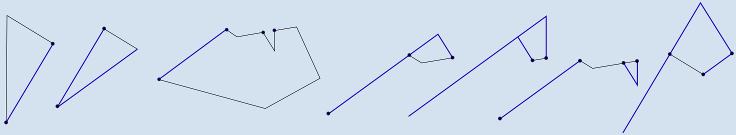

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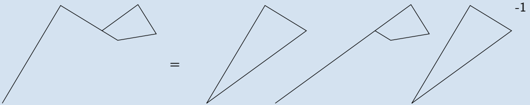
Braids and Artin's theorem

... but wait, there are plenty of them !



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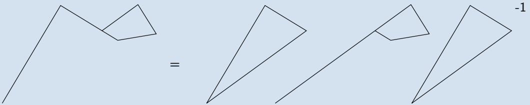

Proposition

Two basis of counter clockwise lassos are related by mean of the action of a *braid*.

Braids & Lassos

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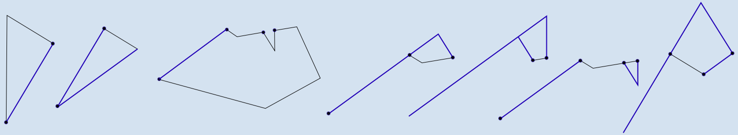

What is a non-commutative Hopf algebras ?

Commutative Hopf algebras represent certain functors

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» Zhang algebra are also useful for the shuffle approach to (operator-valued) non-commutative proba !

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$$R : (A \boxtimes B) \otimes (C \boxtimes D) \to (A \otimes C) \boxtimes (B \otimes D)$$

PROS & Hopf PROS

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$$\Delta : (A, m^1) \to (A \boxtimes A, (m^1 \boxtimes m^1) \circ R),\ \ \varepsilon : A \to \mathbb{C}_{\boxtimes}.$$

» Applications to pre-Lie theoretical perspective on amalgamated non-commutative probability.