École doctorale de Sciences Mathématiques de Paris Centre
(ED 386)

Thèse de doctorat de Sorbonne Université

Discipline : Mathématiques

présentée par

Nicolas GILLIERS

Non-commutative gauge symmetry and pseudo-unitary diffusions

---

dirigée par Thierry LÉVY

Rapporteurs de la thèse:

Mr Uwe Franz	Institut mathématiques de Besançon
Mr Todd Kemp	Université de San Diego, Californie

Soutenue le 10 Juillet 2019 devant le jury composé de :

Mr Philippe Biane	Institut Gaspard Monge (Paris Est)	Examinateur
Mme Manon Defosseux	MAP 5 (Paris Descartes)	Examinatrice
Mme Catherine Donati-Martin	Université de Versailles - St Quentin	Examinatrice
Mr Uwe Franz	Institut mathématiques de Besançon	Rapporteur
Mr Thierry Lévy	LPSM (Sorbonne Université)	Directeur de thèse
Mr Lorenzo Zambotti	LPSM (Sorbonne Université)	Examinateur

Sorbonne Université
Laboratoire de Probabilités, Statistiques et Modélisation
LPSM, UMR 8001
Case courrier 158
4 place Jussieu
75252 Paris Cedex 05

Sorbonne Université
Campus Pierre et Marie Curie
École doctorale de Sciences Mathématiques de Paris Centre
Case courrier 290
4 place Jussieu
75252 Paris Cedex 05

Abstract. This thesis is devoted to the study of two quite different questions, which are related by the tools that we use to study them. The first question is that of the definition of lattice gauge theories with a non-commutative structure group. Here, by non-commutative, we do not mean non-Abelian, but instead non-commutative in the general sense of non-commutative geometry. The second question is that of the behaviour of Brownian diffusions on non-compact matrix groups of a specific kind, namely groups of pseudo-orthogonal, pseudo-unitary or pseudo-symplectic matrices.

In the first chapter, we investigate lattice and continuous quantum gauge theories on the Euclidean plane with a structure group that is replaced by a Zhang algebra. Zhang algebras are non-commutative analogues of groups and contain the class of Voiculescu's dual groups. We are interested in non-commutative analogues of random gauge fields, which we describe through the random holonomy that they induce. We propose a general definition of a holonomy field with Zhang gauge symmetry, and construct such a field starting from a quantum Lévy process on a Zhang algebra. As an application, we define higher dimensional generalizations of the so-called master field.

In the second chapter, we study matricial approximations of higher dimensional master fields constructed in the previous chapter. These approximations (in non-commutative distribution) are obtained by extracting blocks of a Brownian unitary diffusion (with entries in the algebras of real, complex or quaternionic numbers) and letting the dimension of these blocks tend to infinity. We divide our study into two parts: in the first one, we extract square blocks while in the second one we allow rectangular blocks. In both cases, free probability theory appears as the natural framework in which the limiting distributions are most accurately described.

In the last two chapters, we use tools introduced (Zhang algebras and coloured Brauer diagrams) in the first two ones to study Brownian motion on pseudo-unitary matrices in high dimensions. We prove convergence in non-commutative distribution of the pseudo-unitary Brownian motions we consider to free with amalgamation semi-groups under the hypothesis of convergence of the normalized signature of the metric. In the split case, meaning that at least asymptotically the metric has as much negative directions as positive ones, the limiting distribution is that of a free Lévy process, which is a solution of a free stochastic differential equation. We leave open the question of such a realization of the limiting distribution in the general case. In addition we provide (intriguing) numerical evidences for the convergence of the spectral distribution of such random matrices and make two conjectures. At the end of the thesis, we prove asymptotic normality for the fluctuations.

Keywords. Brauer diagrams, holonomy fields, master fields, Yang–Mills fields, gauge theories, Hopf algebras, dual Voiculescu groups, non-commutative probability theory, free probability theory, amalgamated freeness, Brownian motion, pseudo-unitary matrices.

Résumé. Cette thèse est consacrée à l'étude de deux questions très différentes, reliées par les outils que nous utilisons pour les étudier. La première question est celle de la définition des théories de jauge sur un réseau avec un groupe de structure non commutatif. Ici, non commutatif ne signifie pas non Abelian, mais plutôt non commutatif au sens général de la géométrie non commutative. La deuxième question est celle du comportement des diffusions Browniennes sur des groupes matriciels non compacts d'un type spécifique, à savoir des groupes de matrices pseudo-orthogonales, pseudo-unitaires ou pseudo-symplectiques.

Dans le premier chapitre, nous étudions des théories de jauge quantiques sur un réseau et leur limite continue sur le plan euclidien ayant une algèbre de Zhang pour groupe de stucture. Les algèbres de Zhang sont des analogues non commutatifs des groupes et contiennent la classe des groupes duaux de Voiculescu. Nous nous intéressons donc aux analogues non commutatifs des champs de jauges quantiques, que nous décrivons par l'holonomie aléatoire qu'ils induisent. Nous proposons une définition générale d'un champ d'holonomies ayant une symétrie de jauge présentant la structure d'une algèbre de Zhang, et construisons un tel champ à partir d'un processus quantique de Lévy sur une algèbre de Zhang.

Dans le deuxième chapitre, nous étudions les approximations matricielles des champs maîtres en dimensions supérieures construits dans le chapitre précédent. Ces approximations (en distribution non commutative) sont obtenues en extrayant des blocs d'une diffusion unitaire Brownienne (à coefficients dans les algèbres de nombres réels, complexes ou quaternioniques) et en laissant la dimension de ces blocs tendre vers l'infini. Nous divisons notre étude en deux parties : dans la première, nous extrayons des blocs carrés tandis que dans la seconde, nous autorisons des blocs rectangulaires.

Dans les deux derniers chapitres, nous utilisons les outils introduits (algèbres de Zhang et diagrammes de Brauer colorés) dans les deux premiers pour étudier des diffusions sur des groupes de matrices pseudo-unitaires. Nous prouvons la convergence non commutative des mouvements Browniens pseudo-unitaires que nous considérons vers des semi-groupes libres avec amalgamation sous l'hypothèse de convergence de la signature normalisée de la métrique de l'espace sous-jacent. Dans le cas déployé, c'est-à-dire, qu'au moins asymptotiquement, la métrique a autant de directions négatives que de directions positives, la distribution limite est la distribution d'un processus de Lévy, solution d'une équation différentielle stochastique libre. Nous laissons ouverte la question d'une telle réalisation de la distribution limite dans le cas général. De plus, nous présentons des résultats numériques sur la convergence de la distribution spectrale de ces matrices aléatoires et faisons deux conjectures. Dans le dernier chapitre, nous prouvons la normalité asymptotique des fluctuations.

Mots clés. diagrammes de Brauer, champs d'holonomie, champ maître, champs de Yang–Mills, théories de jauge, algèbre de Hopf, groups duaux de Voisculescu, probabilités non-commutatives, probabilités libres, liberté amalgamée, mouvement Brownien, matrices pseudo-unitaires

Remerciements. Tout d'abord, je souhaite exprimer ma reconnaissance aux deux rapporteurs, Messieurs Uwe Franz et Todd Kemp, pour avoir accepté de réviser mon travail, ainsi qu'aux membres du jury: Philippe Biane, Manon Defosseux, Catherine Donati-Martin, Uwe Franz, Thierry Lévy, Lorenzo Zambotti.

Une immense gratitude envers mes parents pour leur soutien inconditionnel, tant pratique qu'affectif, et leur patience inépuisable.

Thierry, je suis très heureux d'avoir découvert la recherche en mathématiques en travaillant sous ta direction. Toutes nos discussions ont orienté et m'ont appris à organiser mon travail de thèse, m'ont fait progresser dans ma manière de communiquer mes recherches. Tes anciens étudiants, Antoine, Franck, Guillaume m'ont transmis leur enthousiasme pour les théories de jauge et les probabilités non commutatives. Je suis très heureux de pouvoir partager cet intérêt avec vous.

Les discussions avec Damien Simon ont certainement été, au cours des premières années de ma thèse, une raison importante de ma présence quotidienne au labo. Au cours de ces discussions, j'ai découvert un vaste panorama de la physique mathématiques, les chemins rugueux, les groupes quantiques. Je suis très admiratif de l'énergie qu'il déploie à accompagner tous ses étudiants dans leur travail. De la même manière, j'ai tiré grand profit de mes discussions avec Manon Defosseux, dont la patience, la bienveillance et les connaissances en font une interlocutrice privilégiée.

Merci à toute l'équipe des doctorants du LPSM, anciennement LPMA: Babacar, Carlo, Eric, Flaminia, Franck, Henry, Isao, Nikolas, Nicolas, Olga, Paul, Rancy, Sarah, Thibault, Thibault... Et à celle du MAP5 pour m'avoir accueilli la dernière année de ma thèse. Je remercie le LPSM de m'avoir accueilli ces dernières années et toute l'équipe administrative, pour leur soutien logistique, entre autres.

Merci à tous mes amis, pour lesquels j'ai beaucoup d'affection.

Enfin, merci à Nathalie pour son soutien et ses éclats qui brisent la trivialité du quotidien. C'est un immense bonheur de compagnonner ensemble. C'est plus ça que j'dis.

Contents

CHAPTER 0

Introduction

Contents

This thesis is devoted to the study of two quite different questions, which are related by the tools that we use to study them.

The first question is that of the definition of lattice gauge theories with a non-commutative structure group. Here, by non-commutative, we do not mean non-abelian, but instead non-commutative in the general sense of non-commutative geometry. For instance, several authors investigated lattice gauge theories with a quantum group or another kind of Hopf algebra in place of the structure group. We propose an alternative, and in a sense simpler, approach using another kind of non-commutative analogues of groups, namely Zhang algebras. We give a definition, and several examples, of lattice gauge theories based on a Zhang algebra.

The second question is that of the behaviour of Brownian diffusions on non-compact matrix groups of a specific kind, namely groups of pseudo-orthogonal, pseudo-unitary or pseudo-symplectic matrices. The convergence results are also expressed in terms of Zhang algebras. Their proofs involve a version of Schur–Weyl duality based on the combinatorics of coloured and oriented Brauer diagrams. They are the same Brauer diagrams that we use in relation with the first question, to produce matricial approximations of the Zhang algebra based gauge theories.

Before describing the content of this thesis in more detail, we will give a brief overview of non-commutative probability (see Section 0.1) and two-dimensional gauge theories (see Section 0.2).

Then, in Section 0.3, we describe more specifically the problems that we address in this thesis, and discuss at a general level the way in which we solve them.

Finally, Section 0.4 presents a commented selection of our main results.

0.1. Non-commutative probability theory

0.1.1. Non-commutative probability spaces. Since Kolmogorov's seminal work in the early 1930's, the probability triple $(\Omega,\mathcal{F},\mathbb{P})$ has been almost universally considered as the fundamental object from which to develop any mathematical model incorporating a notion of randomness. Non-commutative probability arose from a shift from this perspective analogous to that which gave rise to non-commutative geometry, and which brings together two essential ideas:

1. the idea, familiar to algebraic geometers, that a good algebra of functions on a space, or a good sheaf of functions, allow one to describe this space just as efficiently as (or perhaps more efficiently than) the points of this space;

2. the idea, originating in quantum mechanics, that non-commutative algebras, in the guise of algebras of observables, are as meaningful as (and perhaps more meaningful than) commutative ones.

One way to define a non-commutative probability theory is the following: start with a classical probability triple $(\Omega,\mathcal{F},\mathbb{P})$ and associate to it the commutative algebra $L^\infty(\Omega,\mathcal{F},\mathbb{P})$ of essentially bounded complex valued random variables. Endowed with the sup norm, it is a unital Banach algebra, that is, a normed unital algebra in which the product is continuous, and which is complete as a normed vector space. It is moreover an *involutive* algebra: the map $\star$ that sends a function to its pointwise complex conjugate is involutive and an anti-morphism of algebra (and, in this case, a morphism, since the algebra $L^\infty(\Omega,\mathcal{F},\mathbb{P})$ is commutative). The norm and the involution on $L^\infty(\Omega,\mathcal{F},\mathbb{P})$ are related by the relation

$$\|ff^\star\| = \|f\|^2, \tag{1}$$

which turns out to be an extremely rigid one. For instance, no other norm on $L^\infty(\Omega,\mathcal{F},\mathbb{P})$ makes it a Banach algebra and satisfies this relation. An involutive unital Banach algebra satisfying (1) is called a $C^\star$-algebra.

With the previous observations, we have almost completely extracted from the algebra $L^\infty(\Omega,\mathcal{F},\mathbb{P})$ what makes it the L^∞ algebra of a probability space. The first and most famous theorem in this respect is the following.

Theorem 0.1 (Gelfand). *Any commutative unital $C^\star$ algebra is isomorphic to the algebra of continuous functions on a compact space endowed with the sup norm.*

The algebra of continuous functions on a compact space is not necessarily isomorphic to the algebra of bounded random variables on a measured space. Indeed, an L^∞ space is the dual of the corresponding L^1 space, whereas the space of continuous functions on a compact space is not, in general, the dual of a Banach space.

Theorem 0.2. *Any commutative unital $C^\star$ algebra which is, as a Banach space, the dual of a Banach space, is isomorphic with the algebra of essentially bounded random variables on a measured spaces.*

A unital $C^\star$ algebra which is the dual of a Banach space is called a von Neumann algebra. Anticipating on what we are going to say in the next few lines, one can say that von Neumann algebras are the non-commutative measurable spaces.

Finally, we want to select amongst all measured spaces and to single out the *probability* spaces. For this, we take into account the linear form given by the integration against the measure. Indeed, the algebra $L^\infty(\Omega,\mathcal{F},\mathbb{P})$ is endowed with the linear form $\mathbb{E}$ that allows for computation of the distribution of random variables. This linear form $\mathbb{E}$ enjoys the following properties:

$$\mathbb{E}(\mathbf{1}) = 1,\ \mathbb{E}(ff^\star) \geq 0,\ \mathbb{E}(f^\star) = \overline{\mathbb{E}(f)},\ f \in L^\infty(\Omega,\mathcal{F},\mathbb{P}), \tag{2}$$

where $\mathbf{1}$ denotes the constant random variable equal to $1 \in \mathbb{C}$.

Having reached this point, one defines a non-commutative probability space by erasing the commutativity from the list of properties of the pair $(L^\infty(\Omega,\mathcal{F},\mathbb{P}),\mathbb{E})$.

Here is a possible definition, or rather a possible variety of definition from one is free to choose the one best suited to any particular problem under consideration.

Definition 0.3. A non-commutative probability space is a pair (A,τ) where A is a complex unital algebra (resp. a unital involutive algebra, resp. a unital C^*-algebra, resp. a unital von Neumann algebra) and τ is a linear form on A satisfying $\tau(1) = 1$ (resp. $\tau(a^\star) = \overline{\tau(a)}$ and $\tau(aa^\star) \geq 0$, resp. appropriate continuity properties).

0.1.2. Non-commutative random variables. A classical random variable is a map from a classical probability space to a measurable space. This map sends — what else ? — a point to a point. However, a distinctive feature of the non-commutative perspective is to never consider points, which are regarded at best as meaningless, and at worst as non existing. Thus, in a non-commutative world, we must take another definition of a random variable. For this, we use the concepts introduced in the previous section, namely the analogy between measurable spaces and von Neumann algebras, and the principle of reversal of arrows[1].

Definition 0.4. Let (A,τ) be a non-commutative probability space (possibly involutive, a $C^\star$ algebra, a von Neumann algebra). Let B be a unital algebra with a structure corresponding to that of A (involutive, $C^\star$, von Neumann). A B-random variable on A is a morphism of unital algebras (with the corresponding structure) from B to A.

For instance, if $B = L^\infty(\mathbb{R}^n,\mathcal{B}_{\mathbb{R}^n},\mathrm{Leb})$ and $\phi : B \to A$ is a B-random variable, then the range of ϕ is a commutative sub-algebra of A which, ignoring analytical issues, can be identified with some $L^\infty(\Omega,\mathcal{F},\mathbb{P})$, with τ being the expectation with respect to $\mathbb{P}$. Then, the morphism of algebras ϕ corresponds, more or less heuristically, to a map $(\Omega,\mathcal{F},\mathbb{P}) \to \mathbb{R}^n$, that is, to a classical random vector.

This extended definition of a random variable allows us to define a non-commutative analogue of a group-valued random variable, an object which will play a central role in this thesis. A semi-commutative definition of a random variable with values in a topological group G would be: a morphism of algebras $\phi : C(G) \to A$, where (A,τ) is a non-commutative probability space and $C(G)$ is the algebra of continuous functions on G. Let us spell out again the meaning of this ϕ in the case where A is commutative, of the form $L^\infty(\Omega,\mathcal{F},\mathbb{P})$: it is then determined by a classical G-valued random $X : \Omega \to G$ and it sends a continuous function f to the random variable $\phi(f) = f \circ X$.

[1] This principle is based on the fact that a map $f : X \to Y$ gives rise, by composition on the right, to a morphism of algebras $\mathcal{F}(Y) \to \mathcal{F}(X)$ between the spaces of functions on X and Y.

We want to consider a more non-commutative picture where the commutative algebra $C(G)$ is replaced by a possibly non-commutative algebra sharing some of the properties of an algebra of the form $C(G)$. Such an algebra can be called a non-commutative group. There exist many classes of algebras that could play this role, for instance quantum groups, quantum compact groups, Hopf algebras. We will be interested by another class of algebras, called Zhang algebras and which we will present extensively in Chapter 1. We will also explain later in this introduction why we choose to consider this class of algebras and not the more widely considered Hopf algebras, or quantum groups. For the time being, we want to discuss another point, which is the notion of stochastic process in this fully non-commutative framework.

Let us fix, again, a non-commutative probability space (A,τ) and an algebra H, of which we think as some non-commutative analogue of a group. The naive picture that a stochastic process should be a 1-parameter family $\phi_t : H \to A$ of morphisms of algebras is satisfactory as long as one is interested in 1-dimensional marginals of the process. Indeed, the distributions of the morphisms ϕ_t are defined without ambiguity. However, it is natural to expect to be able to define *increments* of a process, for instance to define the notion of Lévy process. A first step in this direction would be the possibility of multiplying two group-valued random variables.

Any reasonable algebra of functions on a group G inherits from the group structure a coproduct, that is, a map

$$\begin{aligned}\Delta : \mathcal{F}(G) &\longrightarrow \mathcal{F}(G)\otimes\mathcal{F}(G) \simeq \mathcal{F}(G\times G)\\ f &\longmapsto \big(\Delta f : (g,h)\mapsto f(gh)\big).\end{aligned}$$

Using this coproduct, one can form the convolution product on the space of linear maps $\mathcal{L}(H,A)$: given two linear maps $\phi,\psi : H \to A$, we define

$$\phi * \psi = m\circ(\phi\otimes\psi)\circ\Delta, \tag{3}$$

where $m : A\otimes A \to A$ denotes the multiplication of A.

However, we want to multiply maps that are not only linear, but also morphisms of algebras. And the problem is that, *unless A is commutative, the convolution product of two morphisms of algebras is not, in general, a morphism of algebras*. Indeed, using Sweedler's notation, we have, for all $h,k\in H$,

$$(\phi*\psi)(hk) = \phi(h_{(1)})\phi(h_{(2)})\psi(k_{(1)})\psi(k_{(2)}) \neq \phi(h_{(1)})\psi(k_{(1)})\phi(h_{(2)})\psi(k_{(2)}) = (\phi*\psi)(h)(\phi*\psi)(k).$$

The reason for this failure to produce a morphism of algebras when A is not commutative is ultimately the fact that, in the algebra $A\times A$, the commutation relation

$$(a\otimes 1)(1\otimes b) = (1\otimes b)(a\otimes 1)$$

holds.

This discussion shows that the tensor product is, in a sense, too commutative for our purposes. This is why we consider algebras endowed with a coproduct which takes its values not in the tensor product, but rather in the free product of the algebra with itself. We insist, moreover, that this coproduct be a morphism of algebras.

Let us summarise : we consider a non-commutative probability space (A,τ) and an algebra H endowed with a free coproduct $\Delta : H \to H \sqcup H$. In this setting, the equation (3), in which Δ is read as the free coproduct, yields a morphism of algebra $\phi*\psi$ from two morphisms of algebras.

The reader may wonder why he was able to add or multiply non-commutative random variables before reading this discussion about free coproducts. The reason is that a usual non-commutative random variable a can be understood as the morphism of algebras $\phi : \mathbb{C}[X] \to A$ sending a polynomial P to $P(a)$. Given a second random variable b and the corresponding morphism $\psi : \mathbb{C}[X] \to A$, it is well known that there is no general and canonical way of using a and b to form a morphism $\mathbb{C}[X]\otimes\mathbb{C}[X] \simeq \mathbb{C}[X,Y] \to A$. On the other hand, we have an isomorphism $\mathbb{C}[X]\sqcup\mathbb{C}[X] \simeq \mathbb{C}\langle X,Y\rangle$ and the coproducts $\Delta_+,\Delta_\times : \mathbb{C}[X] \to \mathbb{C}\langle X,Y\rangle$ given respectively by $\Delta_+(X^n) = (X+Y)^n$ and $\Delta_\times(X^n) = (XY)^n$ yield the usual sum and product of non-commutative random variables.

We want to define an H-stochastic process on A, and in particular the increments of such a process. There are at least two ways to do this. The first is to postulate the increments and to define the process as a family $(\phi_{s,t})_{0\leq s\leq t}$ of morphisms of algebras from H to A satisfying the compatibility condition

$$\phi_{s,t} * \phi_{t,u} = \phi_{s,u},\ 0 \leq s \leq t \leq u.$$

This is not the way that we choose. Instead, we impose more structure on H in order to be able to define the inverse of a random variable. For this, we need an antipode, that is, an involutive morphism of algebras $S : H \to H$ which will be described in detail in due time, and which is of course inspired by the map $f \mapsto (g \mapsto f(g^{-1}))$ on $C(G)$. Using this antipode, we define the *inverse* of an H-random variable ϕ as the random variable $\phi \circ S : H \to A$.

With this rich structure, it is now possible to define an H-stochastic process on A as a 1-parameter of random variables $(\phi_t)_{t\geq 0}$ and to define, for all $0 \leq s \leq t$, the right and left increments between s and t as the random variables

$$(\phi_s \circ S) * \phi_t \ \text{ and } \ \phi_t * (\phi_s \circ S).$$

0.1.3. Independence and freeness. In this work, we will use various non-commutative analogues of the classical notion of independence. We will give a general discussion of what can be understood by a notion of independence in Section 1.3.1. In this introduction, we will only describe the notion that will be for us the most important, namely that of freeness.

Freeness, or free independence, was introduced in the 1980's by Voiculescu for the purposes of the study of von Neumann algebras of free groups [**50**]. Voiculescu soon realised that large random matrices gave finite dimensional approximations of free random variables, and this opened a vast and to this day extremely active field of research. Let us recall the main definition.

Definition 0.5. Let (A,τ) be a non-commutative probability space. A collection $(A_i)_{i\in I}$ of unital sub-algebras of A is said to be *freely independent* if for all $n \geq 2$, all $i_1,\ldots,i_n \in I$ with $i_1 \neq \ldots \neq i_n$, all $a_1 \in A_{i_1},\ldots,a_n \in A_{i_n}$ such that $\tau(a_1) = \ldots = \tau(a_n) = 0$, the equality $\tau(a_1 \ldots a_n) = 0$ holds.

Ruling out the trivial case of constant random variables, neither classical random variables nor random variables taken in a finite-dimensional non-commutative probability space can be freely independent in the sense that the $\star$-algebras generated by the random variables are not mutually free. However, as pointed out by one of the reviewer of the thesis, any collection of nilpotent matrices in $\left(\mathcal{M}_N(\mathbb{C}), \mathrm{tr}_N\right)$ is freely independent, but not $\star$-free.

This makes it the more important that large random matrices that are rotationally invariant in distribution are almost free. There exist dozens of statements of this fact, and we give one of them, which is by far not the most general. In this theorem, $N\times N$ matrices are seen as elements of the non-commutative probability space $(M_N(\mathbb{C}), \frac{1}{N}\mathrm{Tr})$, and random $N\times N$ matrices as elements of $(L^\infty(\Omega,\mathcal{F},\mathbb{P})\otimes M_N(\mathbb{C}), \mathbb{E}\otimes\frac{1}{N}\mathrm{Tr})$. Moreover, $\mathbb{C}^N$ is endowed with its usual Hermitian structure.

Theorem 0.6. *Let $n \geq 2$ be an integer. For each $i \in \{1,\ldots,n\}$, let $(A_{N,i})_{N\geq 1}$ be a sequence of $N\times N$ Hermitian matrices with uniformly bounded operator norm, and convergent in non-commutative distribution. For each $N \geq 1$, let $U_{N,1},\ldots,U_{N,n}$ be n independent Haar distributed $N\times N$ unitary matrices. Then the sequence $(U_{N,1}A_{N,1}U_{N,1}^{-1},\ldots,U_{N,n}A_{N,n}U_{N,n}^{-1})_{n\geq 1}$ converges in non-commutative distribution to a family of freely independent random variables.*

There is a theory of the addition of freely independent random variables wich parallels closely that of classically independent random variables. One can in particular define infinitely divisible non-commutative distributions and non-commutative Lévy processes, and the Lévy-Khinchine formula and Lévy measure have a non-commutative analogue called the Schürmann triple (see Section 2.3.1 and Definition 3.7.3).

A free Lévy process that will feature prominently in our work is the free unitary Brownian motion. It was first described by Biane in and it is the limit as N tends to infinity of the appropriately normalised Brownian motion on the unitary group $\mathbb{U}(N)$. It is also the limit of the Brownian motion on the orthogonal and symplectic group. These convergences can be expressed in one stroke, as follows. Let us choose a division algebra $\mathbb{K}$ to be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Let us define the parameter β to be equal to 1, 2 or 4 accordingly. Let $\mathbb{U}(N,\mathbb{K})$ denote the group of $N \times N$ unitary matrices with entries in $\mathbb{K}$, that is, the orthogonal, unitary or symplectic group. Let $\mathfrak{u}(N,\mathbb{K}) \subset M_N(\mathbb{K})$ be its Lie algebra, endowed with the scalar product

$$\langle X, Y\rangle = \frac{N\beta}{2}\mathrm{Tr}(X^*Y) \text{ if } \mathbb{K}=\mathbb{R} \text{ or } \mathbb{K}=\mathbb{C} \text{ and } \langle X, Y\rangle = \frac{N\beta}{2}\mathfrak{Re}\mathrm{Tr}(X^*Y) \text{ if } \mathbb{K}=\mathbb{H}.$$

Let $(K_t)_{t\geq 0}$ be a standard linear Brownian motion in the Euclidean space $(\mathfrak{u}(N,\mathbb{K}),\langle\cdot,\cdot\rangle)$. Set $c = 1 - \frac{2-\beta}{\beta N}$. Let $(U_t)_{t\geq 0}$ be the solution of the stochastic differential equation

$$dU_t = U_t\,\mathrm{d}K_t - \frac{c}{2}U_t\,\mathrm{d}t.$$

Then the trajectories of the process $(U_t)_{t\geq 0}$ belong almost surely to $\mathbb{U}(N,\mathbb{K})$ and the following convergence holds.

Theorem 0.7. *As N tends to infinity, the process $(U_t)_{t\geq 0}$ converges in non-commutative distribution to the free unitary Brownian motion.*

Another convergence result from which the present work is inspired is the recent result of Cébron and Kemp [?] about the convergence in non-commutative distribution of the Brownian motion on $\mathrm{GL}_N(\mathbb{C})$ towards a free Lévy process. It is instructive to compare this work and ours. A common first problem is the fact that, on non-compact groups, there is no canonical definition of a Brownian motion. Indeed, the Lie algebra of a non-compact group does not carry a bi-invariant scalar product, and the very definition of a Brownian motion requires choices that are more than just a normalisation of speed.

There are more subtle points of comparison between $\mathrm{GL}_N(\mathbb{C})$ and the pseudo-unitary groups $\mathbb{U}(p,q,\mathbb{K})$ that we will study. In the pseudo-unitary case, the compact subgroup $\mathbb{U}(p,\mathbb{K})\times\mathbb{U}(q,\mathbb{K})$ plays an important role and replaces the compact subgroup $\mathbb{U}(N,\mathbb{C})$ of $\mathrm{GL}_N(\mathbb{C})$. Moreover, the group $\mathrm{GL}_N(\mathbb{C})$ comes with a complex structure, opening the possibility of using tools such as the Segal-Bargmann transform. It is not clear if the analogue of this structure exists in the pseudo-unitary case.

0.1.4. Spectrum and non-commutative distribution. One of the distinctive features of the present work is that we will deal with *non-normal* random matrices, in this instance with pseudo-orthogonal, pseudo-unitary and pseudo-symplectic matrices. In comparison with normal random matrices, their study is relatively recent, and still much less developed. One fundamental aspect in which the study of non-normal matrices differs from that of normal matrices is in the correspondence between non-commutative distribution and spectral distribution.

For a sequence of Hermtian matrices such as those considered in Theorem 0.6 above, the convergence in non-commutative distribution is equivalent to the convergence of the empirical spectral measures, both being equivalent to the convergence of the sequence of moments.

In contrast, for a sequence of non-normal random matrices, the convergence in non-commutative distribution cannot be the same thing as the convergence of the empirical spectral measure. One problem is, for example, that a non-normal non-commutative random variable does not have a classical distribution in the form of a probability measure on $\mathbb{C}$. The Brown measure is a substitute for this non-existing distribution, but its study is much more delicate (see for instance [11]).

Our results will not prove convergence of empirical spectral distributions towards Brown measures, but instead convergences of non-commutative distributions, and of the empirical spectral distributions towards a measure of which we characterise a variant of the Stieltjes transform (see [11]).

Convergence towards Brown measure seems to be a more difficult, but of course very interesting, question, which we only scratched from the numerical side. We show below the results of some of our simulations (see Fig. 1). A guide towards rigorous results in this direction is the recent work of Hall, Driver and Kemp [25].

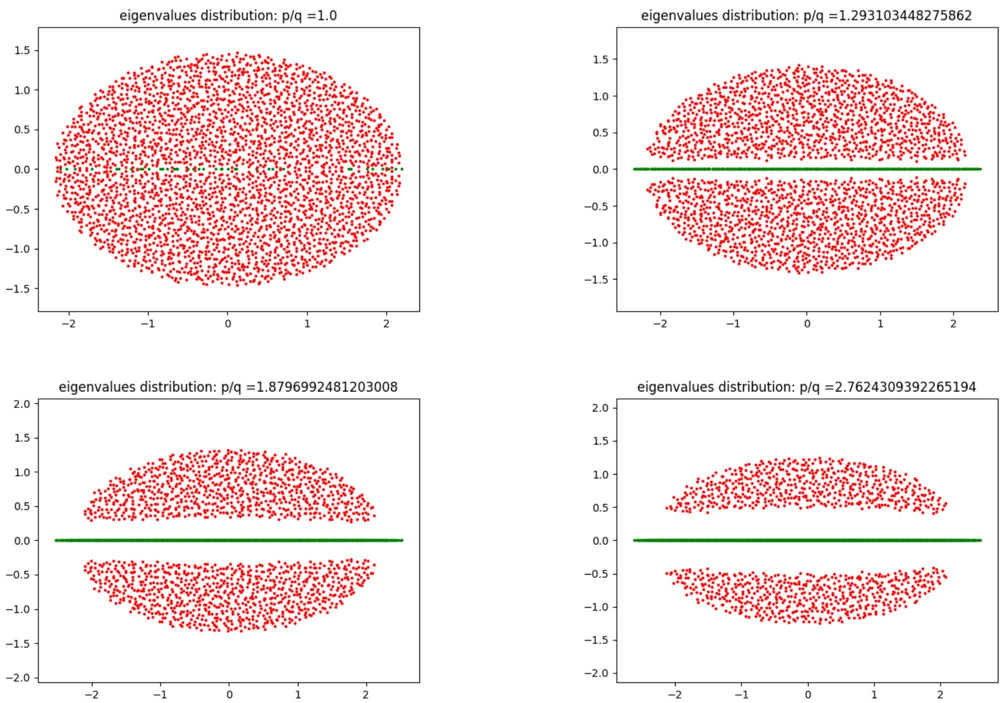


Figure 1. Spectrum of a Gaussian pseudo-Hermitian matrix. The real part of the spectrum is displayed in green.

0.1.5. Brauer diagrams. We conclude the part of this introduction devoted to non-commutative probability theory to a review of one of the combinatorial tools of which we will make extensive use, after mildly generalising it, namely Brauer diagrams.

There are deep connections between unitary and symmetric groups, the study of which is a classical subject, called invariant theory. These connections can be put to good use in the study of stochastic processes taking their values in the unitary group, or more generally in questions of probability theory or integration on the unitary group.

The core result in this respect is the so-called Schur–Weyl duality theorem. It is concerned with two group actions on $(\mathbb{C}^N)^{\otimes n}$, namely the action ρ of the unitary group $\mathbb{U}(N)$ and the action π of the symmetric group $\mathfrak{S}_n$ given by

$$\rho(U)(v_1 \otimes \ldots \otimes v_n) = Uv_1 \otimes \ldots \otimes Uv_n \quad \text{and} \quad \pi(\sigma)(v_1 \otimes \ldots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)}.$$

These actions induce linear maps $\mathbb{C}[\mathbb{U}(N)] \to \mathrm{End}((\mathbb{C}^N)^{\otimes n})$ and $\mathbb{C}[\mathfrak{S}_n] \to \mathrm{End}((\mathbb{C}^N)^{\otimes n})$. Let us denote respectively by $\mathcal{U}$ and $\mathcal{S}$ the ranges of these linear maps. The main theorem is the following.

Theorem 0.8 (Schur–Weyl duality). *The two sub-algebras $\mathcal{U}$ and $\mathcal{S}$ of* $\mathrm{End}((\mathbb{C}^N)^{\otimes n})$ *are each other's commutant.*

From the point of view of the study of random unitary matrices with a distribution that is invariant under conjugation, the content of this theorem is that any conjugation-invariant unitary quantity can be expressed in terms of permutations, that is, in combinatorial terms. The following result is an example of this principle.

Proposition 0.9. *Let $n \geq 1$ be an integer and $t \geq 0$ be a real. Let U_t be the value at time t of the Brownian motion on $\mathbb{U}(N)$. Then the following equality holds in* $\mathrm{End}((\mathbb{C}^N)^{\otimes n})$*:*

$$\mathbb{E}\Big[U_t^{\otimes n}\Big] = \exp\left(-\frac{nt}{2} + \frac{1}{N} \sum_{1 \leq i < j \leq n} \pi\big((i\,j)\big)\right).$$

Theorem 0.8 is called the first fundamental theorem of the theory of invariants of the unitary groups. Its orthogonal and symplectic versions exist, and are almost as classical as the unitary version. They involve the actions on $(\mathbb{R}^N)^{\otimes n}$ and $(\mathbb{H}^N)^{\otimes n}$ of an algebra that contains the group algebra of the symmetric group, but that is not itself the algebra of a group. It is the Brauer algebra $\mathcal{B}_n$, which as a vector space admits a basis indexed by pairings of points drawn on the long edges of a horizontal rectangular box and in which multiplication is given by the superposition of boxes. A much more detailed presentation of Brauer diagrams will be given in Section 2.4. In particular, we will explain how to decorate the links of a Brauer diagram with orientations and colours, and how this allows us to deal with block decompositions of matrices, with blocks that are of differenti sizes, square or rectangular.

0.2. Yang–Mills theory and planar quantum field theory

One of the original motivations of the present work is to investigate non-commutative generalisations of the two-dimensional Yang–Mills holonomy field. This is why we devote a part of this introduction to a brief overview of this theory, despite the fact that a large part of the present thesis can be read without any particular familiarity with gauge theories.

Let us emphasize that the non-commutative generalisation that we have in mind is one where classical probability spaces and random variables are replaced by non-commutative ones, and the structure group, abelian or not, is replaced by a non-commutative variant of a group, namely a Hopf algebra, or indeed a Zhang algebra, as we will explain.

0.2.1. Yang–Mills theory and gauge symmetry. Yang–Mills theory, and more generally gauge theories, are physical theories in which the description of the fields that fill space-time, be they matter fields or interaction fields, is subject to the action, at each point of space-time, of a compact symmetry group called the structure group. This *gauge symmetry* determines the whole structure of the theory: the spaces in which the fields take their values, the algebra of observable quantities, and the set of possible dynamics.

Mathematically, the theory is built around a principal fibre bundle above space-time with structure group dictated by the kind of interaction that is being studied. In a classical picture, interaction fields are connections on this prinpal bundle, and matter fields are sections of vector bundles associated with this principal bundle. Interaction fields can be used, through covariant differentiation, to produce differential operators on the spaces of matter fields. These differential operators are the building blocks of the Lagrangians which describe the details of the physical theory.

The gauge group, in this differential geometric picture, is the group of automorphisms of the principal bundle. It acts on every object of the theory, and a fundamental principle is that only gauge-invariant quantities have a physical meaning.

Let us describe a bit more specifically the part of Yang–Mills theory that is concerned with interaction fields, without matter fields. We will also work in the Euclidean setting, where the Lorentzian structure of space-time is replaced by a Riemannian structure.

Let us consider a compact Riemannian manifold M. Let us choose a compact Lie group G, which will be our structure group. Let $P \to M$ be a principal G-bundle over M. The configuration space of the pure Euclidean Yang–Mills theory in this setting is the infinite dimensional affine space $\mathcal{A}(P)$ of all connections on P. These connections are differential 1-forms on P with values in the Lie algebra $\mathfrak{g}$ of G. Without entering the details (for which we refer the reader to [38]), each connection ω admits a curvature Ω, which is a differential 2-forms on M with values in a Euclidean fibre bundle with fibre isomorphic to $\mathfrak{g}$. The Riemannian structure of M allows one to define a Hodge operator $\star$ on differential forms, and to define the Yang–Mills Lagrangian

$$S_{\mathrm{YM}}(\omega) = \frac{1}{2}\int_M \langle \Omega \wedge \star\Omega\rangle.$$

One of the many questions that can be asked about this Lagrangian is whether it can be used to produce a probability measure on the space of connections on P, according to the heuristic formula

$$\mathrm{d}\mu_{\mathrm{YM}}(\omega) = \frac{1}{Z} e^{-S_{\mathrm{YM}}(\omega)}\, \mathrm{d}\omega.$$

We will not enter the details of the reasons why this is difficult question, and will content ourselves with a brief description of the way in which it can be answered in the case where M is a two-dimensional manifold.

0.2.2. The two-dimensional Yang–Mills measure. The construction of the two-dimensional Yang–Mills measure that is now fairly classical is based on the fact that the algebra of observables of Yang–Mills theory is generated by a class of very special functionals called Wilson loops. This fact is true on a manifold of arbitrary dimension, but it is only on surfaces that it has, so far, led to a construction of the measure.

Wilson loops are constructed using the holonomy induced by a connection. A connection ω, being a differential 1-form with values in $\mathfrak{g}$, can be integrated in a multiplicative way along a closed curve ℓ in M to produce an element of G, called the holonomy of the connection along ℓ and denoted by $\mathrm{hol}(\omega,\ell)$. The holonomy along a loop is not intrinsically defined as an element of G, but is really a G-equivariant automorphism of the fibre above the starting point of the loop. Expressing this automorphism as an element of G requires a choice, very much in the same way that expressing an automorphism of a vector space as a matrix requires the choice of a basis of this vector space. Different choices lead to different elements of G that lie, fortunately, in the same conjugacy class. It is thus the conjugacy class of $\mathrm{hol}(\omega,\ell)$ that is well defined. A Wilson loop is a functional of the connection of the form $\omega \mapsto \chi(\mathrm{hol}(\omega,\ell))$, where $\chi : G \to \mathbb{C}$ is a central function.

Wilson loops generate an algebra of functions on the space $\mathcal{A}(P)$ of all connections on P and in most cases of interest, for instance if the structure group is a unitary group, or an orthogonal group, or a symplectic group, or a product thereof, this algebra separates points. Whether this is true for any compact structure group does not seem to be known.

However, all we need to know is that a connection is characterised by its holonomies along loops. We must add to the previous discussion the fact that the holonomies along several loops based at the same point of M are defined up to conjugation by the same element of G. The important fact, which holds for any compact Lie group G, is the following : for any base point $m \in M$ and any reasonable class $\mathsf{L}_m(M)$ of loops on M based at m, the map

$$\begin{array}{rccc} \mathrm{hol}: & \mathcal{A}(P)/\mathrm{Aut}(P) & \longrightarrow & G^{\mathsf{L}_m(M)}/G \\ & \omega & \longmapsto & (\mathrm{hol}(\omega,\ell) : \ell \in \mathsf{L}_m(M)) \ (\mathrm{mod}\ G) \end{array}$$

is injective.

Therefore, one interprets the problem of constructing a probability measure on $\mathcal{A}(P)$ invariant under the action of the gauge group as the problem of constructing a collection of G-valued random variables indexed by the set of loops based at an arbitrary base point $m \in M$, with the requirement that the distribution of this family should be invariant under global conjugation. Each of these random variables is seen as the random holonomy determined along a particular loop by a random connection distributed according to the Yang–Mills measure.

The details of the way in which such a collection can be constructed when M is a compact surface are not really relevant to the present work and we spare them to the reader, who can instead consult [41]. Let us mention that in the two-dimensional setting, the Riemannian structure of M appears in the construction only through the Riemannian volume, that is, the Riemannian area that it induces. Let us also indicate that when this construction is performed on a disk, or on the Euclidean plane, the random holonomies along two loops based at the same point but surrounding disjoint domains of the plane are independent. Finally, it is an elementary property of the holonomy that it is multiplicative, in the sense that concatenation of loops results in the multiplication of holonomies.

These properties lead us, hopefully fairly naturally, to the following definition of a classical holonomy field on the Euclidean plane. We denote by $\mathsf{L}_0(\mathbb{R}^2)$ a nice class of loops based at the origin on $\mathbb{R}^2$, for instance smooth paths, or piecewise affine paths.

Definition 0.10. Let G be a compact Lie group. A G-valued holonomy field on the Euclidean plane is a collection $(H_\ell)_{\ell \in \mathsf{L}_0(\mathbb{R}^2)}$ of G-valued random variables that satisfies the following four properties.

(1) (Multiplicativity) For all $\ell_1, \ell_2 \in \mathsf{L}_0(\mathbb{R}^2)$, we have almost surely $H_{\ell_1\ell_2} = H_{\ell_1}H_{\ell_2}$ and $H_{\ell_1^{-1}} = H_{\ell_1}^{-1}$.

(2) (Gauge invariance) For all $\ell_1, \dots, \ell_n \in \mathsf{L}_0(\mathbb{R}^2)$ and all $g \in G$, we have the equality in law

$$(H_{\ell_1}, \dots, H_{\ell_n}) \overset{(d)}{=} (gH_{\ell_1}g^{-1}, \dots, gH_{\ell_n}g^{-1}).$$

(3) (Independence) If $(\ell_1, \dots, \ell_n)$ and $(\ell'_1, \dots, \ell'_m)$ are two finite sequences of loops such that $\bigcup_{i=1}^n \mathrm{Int}(\ell_i)$ and $\bigcup_{j=1}^m \mathrm{Int}(\ell'_j)$ are disjoint, then $(\mathsf{H}_{\ell_1}, \dots, \mathsf{H}_{\ell_n})$ and $(\mathsf{H}_{\ell'_1}, \dots, \mathsf{H}_{\ell'_m})$ are independent.

(4) (Invariance by area-preserving diffeomorphisms) For all area-preserving diffeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ and all loops $\ell_1, \dots, \ell_n$ such that $\phi(\ell_1), \dots, \phi(\ell_n)$ belong to $\mathsf{L}_0(\mathbb{R}^2)$, we have the equality $(\mathsf{H}_{\ell_1}, \dots, \mathsf{H}_{\ell_n}) = (\mathsf{H}_{\phi(\ell_1)}, \dots, \mathsf{H}_{\phi(\ell_n)})$.

0.2.3. The master field. One of the directions in which the study of the two-dimensional Yang–Mills measure has made progress during the recent years is the study of the large N limit of the theory with structure group $G = \mathbb{U}(N)$. The central result of this study is the existence of a deterministic limit to the to the normalised traces of the holonomies along loops. This was proved by Lévy in the case of the plane [41], and by Dahlqvist and Norris in the case of the sphere [19].

The limiting object is thus a deterministic function $\Phi : \mathsf{L}_m(M) \to \mathbb{R}$, which satisfies, at least when M is the plane $\mathbb{R}^2$ or the shpere S^2, and for all $\ell \in \mathsf{L}_m(M)$,

$$\Phi(\ell) = \lim_{N\to\infty} \frac{1}{N}\mathrm{Tr}\Big[\mathsf{H}_\ell^{\mathbb{U}(N)}\Big],$$

where the $\mathbb{U}(N)$ in exponent indicates that we consider, for each N, the theory with structure group $\mathbb{U}(N)$, and where the convergence is a convergence in probability.

The function Φ can be seen as a state on the group algebra of the group of reduced loops $\mathsf{RL}_m(M)$ (see Section for the definition of reduced loops). It can be computed recursively, for loops with a finite number of self-intersections, thanks to the Makeenko–Migdal equations, which express the variation of $\Phi(\ell)$ when ℓ is modified by increasing or decreasing the areas of some of the faces that it delineates.

0.3. Overview of the thesis

The material of the present thesis consists of contributions in three main directions:

1. the definition of a non-commutative lattice gauge theory which incorporates gauge invariance at a fundamental level and is well suited to the probabilistic approach to quantum field theories,
2. the construction of a family of examples of such non-commutative lattice gauge theories which contains and generalises widely both the existing classical gauge theories and their large N limits, the so-called master fields,
3. the study of Brownian diffusions on pseudo-orthogonal, pseudo-unitary and pseudo-symplectic groups, their large N limits, and their fluctuations as N tends to infinity.

It is clear that the first two directions are related. It is perhaps less clear that the third is also related to the first two, and one of the purposes of this general introduction is to explain how.

Most of the mathematical literature about the two-dimensional Yang–Mills theory is devoted to the study of the case where the structure group is a compact Lie group, typically $\mathbb{U}(N)$. This is what could be called the '$\mathbb{U}(N)$ theory'. In the last ten years, the study of the large N limit of this theory has made substantial progress, and the limiting object, called the master field, is on the way of being well understood. For reasons that will become clear later, we would like to call this branch of the study of Yang–Mills theory the '$\mathcal{O}\langle 1\rangle$ theory'.

The construction and study of non-commutative analogues of the Yang–Mills theory is a problem that has been addressed both by physicists and by mathematicians, at various levels of rigour and within diverse non-commutative frameworks [12], [2] for example. From a probabilistic point of view, the existing constructions share the frustrating property that they do not have, or at least do not seem to have the involutive structure that allows one to incorporate the amount of positivity required to do non-commutative probability. One of the reasons for this lack of positivity is probably that these theories are set in the framework of Hopf algebras, which is notoriously difficult to handle probabilistically. One of our first motivations for the work presented in this thesis was to find a way around this difficulty.

Another source of inspiration for our work is the recent work of Cébron, Dahlqvist and Gabriel, where they extend the $\mathcal{O}\langle 1\rangle$ theory from the Brownian case to the Lévy case: they consider the large N limits of random holonomy fields that are not Brownian.

Our first goal was thus to propose a definition of a non-commutative gauge field that is amenable to a probabilistic treatment and is, in an essential way, invariant under a gauge group that can be taken in a category of quantum-like, or non-commutative, groups.

In order to achieve this goal, we decided to place ourselves in a maximally non-commutative setting. Indeed, we came to realise that one of the obstacles to the construction of useful involutive structures on a Hopf algebra based lattice gauge theory is related to the presence of *too many relations*. In a sense, these theories are still too commutative, and it is too constraining to accomodate this remnant of commutativity. Just as the study of Voiculescu's dual groups is, in some respects, less difficult than that of quantum groups, it turns out to be easier to construct a lattice gauge theory in a much more radically non-commutative setting than that of Hopf algebras.

The category of 'maximally non-commutative analogues of Hopf algebras' that we choose to consider is that of Zhang algebras. The definition of Zhang algebras (also cometimes called H algebras in the literature) mimics closely that of Hopf algebras, with the essential difference that the coproduct, instead of taking its value in the tensor product of the algebra with itself, takes its values in the free product of the algebra with itself. The definition of these algebras that we give in Chapter 1 is made slightly more complicated by the necessity of incorporating the gauge action into the structure, and of defining coactions of a Zhang algebra. Nevertheless, once Zhang algebras and their coactions are defined, it is not very difficult to write a sensible definition of a lattice gauge field with a Zhang algebra as its structure group. Then, going from the lattice to the continuum is, at least algebraically, a matter of taking an inverse limit of the lattice theories associated with a family of graphs.

It turns out that the $\mathbb{U}(N)$ theory and the $\mathcal{O}\langle 1\rangle$ theory are special case of lattice gauge theories with Zhang symmetries. Our second contribution is to construct a family of such lattice theories which extend the $\mathcal{O}\langle 1\rangle$ theory, and that we call the $\mathcal{O}\langle n\rangle$ theories. The structure group of these non-commutative gauge theories is indeed Voiculescu's dual group $\mathcal{O}\langle n\rangle$, which is obtained from the algebra of coordinate functions on the unitary group $\mathbb{U}(n)$, writing the most natural presentation of this algebra, and erasing all commutation relations:

$$\mathcal{O}\langle n\rangle = \mathbb{R}\langle u_{ij}, u_{ij}^* : i,j \in \{1,\dots,n\}\rangle \Big/ \Big(\sum_{k=1}^{n} u_{ik}u_{kj}^* = \delta_{ij} \Big).$$

Elaborating on Ulrich's work, we realise this $\mathcal{O}\langle n\rangle$ theory as the large N limit of a theory with $\mathbb{U}(nN)$ symmetry in which the unitary matrices are looked at as block matrices, with n^2 blocks of size N.

The family of gauge theories that we consider is in fact slightly more general than this $\mathcal{O}\langle n\rangle$ theory, in that we consider the large N limits of unitary theories looked at as block matrices with *rectangular* blocks. These theories enjoy a property of asymptotic freeness that is not the most usual one, but an amalgamated variant of it. It is to be able to express this amalgamted freeness that we define Zhang algebras not only in the category of usual algebras, but on an arbitrary algebraic category.

On the combinatorial side, our study of block matrices and their large N limits relies on Schur-Weyl duality and involves Brauer diagrams. In order to deal with block matrices, these Brauer diagrams need to be augmented with colourings, orientations, and floating loops, that is, appropriate central extensions of the Brauer algebra.

The tools that we develop in the course of this study, namely Zhang algebras and coloured Brauer diagrams, turn out to be just those needed to study a problem that is, at first sight, rather different: the large N limit of Brownian diffusions on groups of pseudo-unitary (or pseudo-orthogonal, or pseudo-symplectic) matrices. These matrices, although they are not block matrices in nature, are indeed adapted to a certain decomposition of the ambient space according to the sign of the indefinite quadratic form. This is the third direction of research of which the results are presented in this thesis.

The cases where the sizes of the positive and negative subspaces of the quadradic form are asymptotically of equal dimension, and the case where they are different, demand distinct treatments, and give rise to two chapters of the thesis, which are completed by a study of the fluctuations associated to the large N convergences that we establish.

We will now describe more linearly the content and main results of the thesis.

0.4. Description of the main results

0.4.1. Chapter 1. This first chapter contains the main definitions and fundamental results about Zhang algebras that we use throughout our work. As we explained above, a Zhang algebra is, informally, a Hopf algebra in which the coproduct takes its values in the free product of the algebras with itself. Let us give a list of the first definitions and results, with a dictionary explaining how they relate to the classical setting.

- Algebraic category (Def. 1.1). An element of an algebraic category plays the role of an algebra of random variables. Our construction is based on the choice, at the very beginning, of a particular algebraic category.
- Zhang algebra (Def. 1.2). This plays the role of the algebra of functions on the structure group of a gauge theory. After choosing the algebraic category in which we work, we choose one particular Zhang algebra H on this category. An important point is that H is itself an object of this algebraic category.
- Comodule algebras (Def. 1.4). A comodule algebra over H is an element of our algebraic category that is coacted on by H. It corresponds to an algebra of random variables endowed with an action of the structure group.
- H is a comodule algebra over itself in a natural way (Lemma 1.6), the category of comodule algebras over H is algebraic (Lemma 1.5), and H is a Zhang algebra over this category (Lemma 1.7). We consider only algebras of random variables that are acted on by the structure group, and this category behaves well with respect to the action of H.

In the following table, we summarise the operations available in the structure of a Zhang algebra. Let us emphasize that a Zhang algebra, say H, is not in general an algebra in the usual sense. It is an object of an algebraic category, and this endows it with a map

$$H \sqcup H \longrightarrow H$$

which is a morphism in this category and plays the role of the product. In particular, it admits a unit in a natural sense. On the other hand, asking whether there is a natural map $H \otimes H \to H$ does not make sense, because $H \otimes H$ is not defined in the first place.

Zhang algebra	Functions on a group $\mathcal{F}(G)$
product	$f \otimes g \mapsto fg$
coproduct	$f \mapsto ((g,h) \mapsto f(gh))$
unit	$g \mapsto 1$
counit	$f \mapsto f(1)$
antipode	$f \mapsto (g \mapsto f(g^{-1}))$

After discussing independence and inductive limits, we give the definition of a holonomy field over a Zhang algebra (Def. 1.18). This definition comes in two variants (see also Def. 1.19), depending on whether the algebraic category on which we work is a ‘concrete’ category of algebras, or an abstract algebraic category.

We then prove the main theorem of this first chapter.

Theorem 0.11 (Thm. 1.25). *Every (appropriate) quantum Lévy process gives rise to a holonomy field over a Zhang algebra in the sense of Def. 1.18.*

0.4.2. Chapter 2. In this second chapter, we construct a matricial approximation of the higher dimensional version of the master field that we constructed in Chapter 1. This approximation is obtained, informally, as follows. Let $N \geq 1$ an integer and consider the $\mathbb{U}(N)$ based Yang–Mills theory on the plane. Each (piecewise affine) loop on the plane has a holonomy that is a unitary random matrix in $\mathbb{U}(N)$. In its usual formulation, Yang–Mills theory considers a class of observables given by traces of these holonomies, also called Wilson loops. As explained in Section 0.2.3, the distribution of the one-dimensional master field is approximated by these Wilson loops by letting the dimension N of the structure group $\mathbb{U}(N)$ tend to infinity.

We propose to consider a larger class of observables given by traces of product of submatrices of the loop holonomy. This leads us to study the convergence in high dimensions of the non-commutative distribution of square blocks extracted from a unitary Brownian motion. In [49], Ulrich studies the convergence of such extractions for complex unitary matrices. Using a Schur–Weyl duality, we simplify significantly the computations and study the convergence of the non-commutative distribution of square blocks extracted from complex, real and quaternionic Brownian unitary matrices.

More precisely, let $n \geq 1$ and $d \geq 1$ be two integers and set $N = nd$. Let $\mathsf{U}_N^{\mathbb{K}}$ be a Brownian motion on $\mathbb{U}(nd, \mathbb{K})$. The process $U^{\langle n \rangle}$ is the quantum Lévy process on the dual Voiculescu group $\mathcal{O}\langle n \rangle$ that extracts blocks of size $d \times d$ in the matrix $\mathsf{U}_N^{\mathbb{K}}$, see equation 1.18. The free n-dimensional Brownian motion is defined in Section 2.3.3, Chapter 2.

Theorem (Theorem 2.28). *Let $n \geq 1$ an integer. Each of the three processes $U_{n,d}^{\mathbb{K}}$, where $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, converges in non-commutative distribution as $d \to +\infty$ to the free n-dimensional Brownian motion $U^{\langle n \rangle}$.*

To state this Schur–Weyl duality, we define a coloured version of the algebra of Brauer diagrams and a central extension of this algebra that is obtained by adding loops to a Brauer diagram. We also consider oriented Brauer diagrams, and use them to tackle a generalization of the initial problem of approximation of the higher dimensional master field. In fact, in the second part of this chapter, we allow the blocks extracted from the unitary Brownian motion to be rectangular. Under the assumption that the ratios of the dimensions of each block, relatively to the dimension of the Brownian motion, converge, we are able to prove the convergence of the joint non-commutative distribution of the blocks. We use the theory of non-commutative amalgamated probability and the notion of amalgamated freeness to describe the limiting distribution. The free semi-group that is obtained in this way enjoys the invariance properties required for the construction of a holonomy field.

Let d a sequence of positive numbers of length n. The rectangular extraction process $U_{\mathsf{d}}^{\mathbb{K}}$ and the cluster process $U_{\{\mathsf{d}\},\pi}^{\mathbb{K}}$ are defined in Section 2.6.2, Chapter 2.

Theorem (Theorem 2.37). *Let $n \geq 1$ an integer. For each integer $N \geq 1$, pick a partition d_N of N into n parts. Let $\mathbb{K}$ be one of the three divisions algebras $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Assume that as N tends to infinity, there exists positive real numbers $r_i \in (0,1]$, $1 \leq i \leq n$, such that*

$$\frac{d_N(i)}{N} \underset{N \to +\infty}{\longrightarrow} r_i,\ 1 \leq i \leq n.$$

As the dimension N tends to infinity, the non-commutative distribution of $U_{d_N}^{\mathbb{K}}$ converges to a $\mathcal{D}_n$-amalgamated free semi-group.

Theorem (Theorem 2.39). *Let $n \geq 1$ an integer and for each integer $N \geq 1$, let d_N be a partition of N.*

We assume that for all integer $i \leq n$, the ratio $r_N(i) = \frac{d_N(i)}{N}$ converges as N tends to infinity to a positive value r_i less than one. We assume further that the kernel of the partition d_1 is finer than the kernels of the partitions d_N, $N \geq 1$.

Let $\mathbb{K}$ be one the three divisions algebras $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. As the dimension N tends to infinity, the non-commutative distribution of $U_{\{d_N\},\mathsf{ker}(d_1)}^{\mathbb{K}}$ converges to a $\mathcal{D}_{d_1}$-amalgamated free semi-group.

0.4.3. Chapters 3 to 4. In Chapter 3, we use again the algebra of coloured Brauer diagrams to study the convergence of Brownian pseudo-unitary matrices.

In chapter 3 to 4, We explore the asymptotic behaviour of Brownian diffusions on not necessarily compact unitary group, namely, unitary groups associated with a quadratic form that is not necessarily definite. There are actually more than one asymptotic regimes to consider, depending on the relative growth of the dimensions of the spaces on which the metric is of constant sign.

There are physical motivations for the investigation of the properties of such random matrices, and of similarly defined pseudo-hermitian matrices: during the last thirty years, the theory of PT-symmetric quantum mechanics that deals with non-hermitian matrices (with respect to a definite metric) has become increasingly popular, see [7].

Before studying the convergence in non-commutative distribution of the pseudo-unitary diffusion, and following the historical development, it is necessary to study the convergence of the driving noise of the stochastic differential equation of which the pseudo-unitary diffusion is the solution. This is what we do in chapter 3 in case metric is split, by proving that this process converges under the same asymptotic regime.

But first, we adress the problem of defining Brownian motions on the Lie-algebras of pseudo-Hermitian matrices. Let $\mathbb{K}$ be one of the three division algebras $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, in addition let $p,q \geq 1$ two integers. There are properties that can be required to be satisfied for the law of a pseudo-Hermitian Gaussian matrix. First, we impose invariance by conjugation by a maximal compact subgroup of $\mathbb{U}(p,q,\mathbb{K})$, to such a choice corresponds a Cartan decomposition of the Lie algebra we use to construct a two parameters family of scalar products. The Gaussian laws we obtain in this way lack of invariance by conjugation by any matrix of $\mathbb{U}(p,q,\mathbb{K})$. Conjugation by a pseudo-unitary matrix corresponds to a change of pseudo-orthonormal basis, since there is no preferred pseudo-orthonormal basis, any distribution obtained by conjugation of the previously constructed laws can serve as a good notion of a pseudo-Hermitian gaussian law. Thus to any maximally compact subgroup of $\mathbb{U}(p,q,\mathbb{K})$ corresponds a two parameters family of Gaussian pseudo-Hermitian law.

We argued that there is no preferred pseudo-orthogonal basis, which leads to multiple definition of Gaussian pseudo-Hermitian laws. But there is an other set of basis that enjoys good properties which we could have used also. To a Witt basis corresponds a Witt decomposition of the ambient space that is a splitting into two orthogonal summands, one is an hyperbolic space and the other is an anisotropic space. In the present work, we do not adress Gaussian laws associated with such decompositions.

The pseudo-unitary Brownian motion are defined as a two-parameters family of stochastic processes, each being a solution of a stochastic differential equation with two parameters being the speeds of the diffusion in the compact and non-compact directions of the pseudo-Hermitian Brownian motion driving the equation.

In the split case, studied in Chapter 3, where the metric has as many positive and negative directions, and corresponding to a maximally non-compact group in the real case, the pseudo-unitary Brownian motion converges in non-commutative distribution to a two-parameters family of processes that can be constructed as the solution of a free stochastic differential equation. It is interesting to observe that free probability can be used to describe the asymptotic regimes of Brownian diffusions on both compact and maximally non compact real Lie groups.

Here is one of the results that we obtain.

Theorem (Theorem 3.43). *For each $\mathbb{K} \in \{\mathbb{R},\mathbb{C},\mathbb{H}\}$, the diffusion $V_p^{\mathbb{K}}$ in the split unitary group $\mathbb{U}(p,p,\mathbb{K})$ converges in non-commutative distribution, as $p \to +\infty$, to the free split unitary Brownian motion V.*

The theorem expressing the same convergence in the case where the dimensions of the positive and negative spaces of the quadratic form are different is the following.

Theorem (Theorem 3.49). *Let $l \geq 1$ an integer and let $(s_1,t_1,\ldots,s_l,t_l) \in \mathbb{R}_+^l$ be a tuple of times. The rectangular probability space*

$$\left(\mathcal{RO}(-,+)^{\sqcup_{\mathcal{D}} l}, \phi_+^{p,q}(s_1,t_1,\ldots,s_l,t_l), \phi_-^{p,q}(s_1 t_1,\ldots,s_l,t_l), \frac{p}{p+q}, \frac{q}{p+q}\right)$$

converges under hypothesis (H). In addition, if we denote by $\mathbb{E}_\lambda(s_1,t_1,\ldots,s_l,t_l)$ the limit conditional expectation,

$$\mathbb{E}_\lambda(s_1,t_1,\ldots,s_l,t_l) = \mathbb{E}_\lambda(t_1-s_1) \,\dot{\sqcup} \cdots \dot{\sqcup}\, \mathbb{E}_\lambda(t_1-s_1).$$

In Chapter 4, we prove a central limit theorem for the convergence of the non-commutative distribution of the Brownian pseudo-unitary diffusions by making use of a combinatorial method that founds its roots in the theory of second order freeness; the fluctuations scale as the dimension and are asymptotically Gaussian. The method we use in this chapter can be applied to study fluctuations for the convergence of rectangular and square extractions of an unitary Brownian motion. In the following Theorem, $\mathfrak{m}_{p,q}^{\mathbb{K}}(\mathrm{t}) \circ X_{p,q}^{\mathbb{K}} \circ (p+q)^N$ is convenient way to express mean of product of traces of products of blocks extracted from the pseudo-unitary Brownian motions.

THEOREM (Theorem 4.7). *The state* $\mathbb{m}^{\mathbb{K}}_{p,q}(\mathrm{t}) \circ X^{\mathbb{K}}_{p,q} \circ (p+q)^N$ *on* $\mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_{irr,\infty}\right]$ *converges to a Gaussian state as* $p,q \to +\infty$ *with* $\frac{p}{q} \to \lambda > 0$*. In addition, with respect to the limiting state, the covariance between two Brauer diagrams* b, b' *is given by*

$$\sigma_{\lambda,\mathrm{t}}(b,b') = \left[\int_0^1 2e^{u\mathcal{D}^{\mathbb{K}}_\lambda}\Gamma_{\mathcal{S}^{\mathbb{K}}_\lambda}\left(e^{(1-u)\mathcal{D}^{\mathbb{K}}_\lambda}b, e^{(1-u)\mathcal{D}^{\mathbb{K}}_\lambda}b'\right)\mathrm{d}u\right],\ b,b' \in \mathcal{B}^{\mathbb{K}}.$$

0.5. Introduction en français

0.5.1. Probabilités non commutatives. Depuis les années 30 et le travail de Kolomogorov, le triplet probabiliste $\Omega, \mathcal{F}, \mathbb{P}$ est considéré par une très large partie de la communauté mathématiques comme l'objet fondamental de tout modèle stochastique. Abandonnant ce paradigme, les probabilités non commutatives naissent de l'adoption d'un nouveau point de vue, analogue à celui donnant naissance à la géométrie non commutative:

1. l'idée familière à tout géomètre, qu'un espace n'est pas moins bien perçu au travers de son faisceau de fonctions, que comme une collection de points,
2. l'idée que les observables de la mécanique quantique forment des algèbres non commutatives.

On commence par décrire les probabilités classiques comme une théorie non-commutative des probabilités. Au triplet probabiliste classique $(\Omega, \mathcal{F}, \mathbb{P})$, on associe l'algèbre $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ commutative des variables aléatoires complexes essentiellement bornée. Equipées de la norme infinie, cette algèbre devient une algèbre de Banach unifère, ce qui veut dire que le produit de cette algèbre est une opération continue et elle est complète en tant qu'espace vectoriel topologique. C'est également une algèbre involutive pour l'anti-morphisme $*$ égal à la conjugaison complexe ponctuelle d'une fonction. Cette involution et la norme infinie sur $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ satisfont à la relation:

$$\|ff^*\| = \|f\|^2. \tag{4}$$

Une algèbre involutive satisfaisant (4) est appelée C^**-algèbre*. Avec cette dernière relation, on a une liste presque exhaustive des propriétés de $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ la caractérisant comme algèbre de variables aléatoires bornées.

THEOREM 0.12 (Gelfand). *Toute* C^* *algèbre commutative est isomorphe à une algèbre de fonctions continues sur un espace compact équipés de la norme infinie.*

Un autre théorème caractérise les C^* algèbre duales.

THEOREM 0.13. *Toute* C^* *algèbre commutative duale d'une algèbre de Banach, est isomorphe à une algèbre de variables aléatoires essentiellement bornées.*

Une C^* algèbre duale d'une algèbre de Banach est appelée algèbre de von Neumann.

DEFINITION 0.14. Un espace de probabilité non commutatif est une paire (A, τ) avec A une algèbre unifère (resp. une algèbre unifère involutive, resp. une C^* algèbre unifère, resp. une algèbre de von Neumann) et τ est une forme linéaire sur A satisfaisant $\tau(1) = 1$ (resp. $\tau(a^*) = \overline{\tau(a)}$ et $\tau(a^*a) \geq 0$, resp. des propriétés de continuité).

L'équivalent non commutatif de la notion de variable aléatoire est obtenue en utilisant le principe de renversement.

DEFINITION 0.15. Soit (A, τ) un espace de probabilité non commutatif, et B une algèbre unifère ayant une structure algébrique analogue à celle de A. Une variable aléatoire de B dans A est un morphisme de B vers A.

Cette notion de variable aléatoire non commutative est suffisement générique pour nous permettre de donner un analogue non commutatif d'une variable aléatoire prenant ses valeurs dans un groupe. Si G est un groupe, une variable aléatoire $X : Omega \to G$ induit un morphisme $\phi_X : \mathcal{F}(G) \to Ł^\infty(\Omega, \mathcal{F}, \mathbb{P})$. La structure de groupe de G induit une structure d'algèbre de Hopf commutative sur $\mathcal{F}(G)$:

$$\Delta : \begin{array}{ccc} \mathcal{F}(G) & \to & \mathcal{F}(G) \otimes \mathcal{F}(G) \\ f & \mapsto & \Delta f : (g,h) \to f(gh) \end{array}, \ \varepsilon(f) = f(e).$$

Ce coproduit permet de définir le produit de convolution de deux morphismes ϕ_X, et ϕ_Y via:

$$\phi_X * \phi_Y(f) = m \circ (\phi_X \otimes \phi_Y)(\Delta f) = \phi_{XY}(f)$$

A la multiplication ponctuelle de deux variables aléatoires X and Y prenant leurs valeurs dans G correspond un produit de convolution sur une algèbre de Hopf. Il est important que l'algèbre but des variables aléatoires ϕ_X et ϕ_Y soit commutative, dans le cas contraire le produit de convolution de deux morphismes n'est pas nécessairement un morphisme, cela est dû principalement à la commutation des deux copies de l'algèbre $\mathcal{F}(G)$ dans $\mathcal{F}(G) \otimes \mathcal{F}(G)$.

Dans la suite non considérerons donc des bigèbres pour lesquelles le coproduit prend ses valeurs dans le produit libre d'algèbres, c'est ce que l'on appelle des bigèbres libres. Pour cette structure, le produit de convolution de deux morphismes (prenant leurs valeurs dans une algèbre éventuellement non commutative) est un morphisme.

Grâce à cette structure, nous pouvons définir la notion de processus stochastique non commutatif comme une collection à deux paramètres de variables aléatoires $\phi_{s,t} : (B, \Delta, \varepsilon) \to (A, \tau)$ vérifiant:

$$\phi_{s,t} * \phi_{t,u} = \phi_{s,u}, \ 0 \leq s \leq t \leq u.$$

Si la bigèbre libre est équipée d'une antipode S vérifiant les même propriétés que l'antipode $S : f \mapsto g \mapsto f(g^{-1})$ de $\mathcal{F}(G)$. Un processus stochastique non commutatif peut être défini à partir d'une famille à un paramètre $(\phi_t)_{t>0}$ de morphismes en posant:

$$\phi_{s,t} = (\phi_S \circ S) * \phi_t \text{ ou } \phi_{s,t} = \phi t * (\phi_s \circ S).$$

0.5.2. Indépendance et liberté. Dans le premier chapitre de cette thèse, on donne une définition très générale (catégorique) de la notion d'indépendance. Dans cette introduction, on décrit la plus importante, celle de liberté. La liberté a été introduite par Voiculescu dans les années 80, dans le but d'étudier les algèbres de von Neumann associées à des groupes libres. On en rappelle maintenant la définition.

Soit (A, τ) un espace de probabilité non commutatif. Une collection $(A_i)_{i \in I}$ de sous-algèbres unifères de A est dite *libre* si pour tout entier $n \geq 2$, et tout $i_1, \ldots, i_n \in I$ avec $i_1 \neq \ldots \neq i_n$, tout $a_1 \in A_{i_1}, \ldots, a_n \in A_{i_n}$ tel que $\tau(a_1) = \ldots = \tau(a_n) = 0$, on a l'égalité $\tau(a_1 \cdots a_n) = 0$. Des variables aléatoires classiques ou des variables aléatoires non commutatives d'un espace de probabilité de dimension fini ne peuvent être libres entre elles. Voisculescu a montré que des matrices aléatoires de grandes dimensions et indépendantes ayant des lois invariantes par conjugaison unitaire sont presque libres. Voir le théorème 0.6.

Ayant défini une autre notion d'indépendance, on peut remplacer dans la définition classique d'un processus de Lévy la propriété d'indépendance (tensorielle, classique) des incréments du processus par leur liberté pour obtenir des processus de Lévy libres. La formule de Lévy Khinchine et la mesure de Lévy ont des équivalent libres que l'on appelle triplets de Schürmann.

Un processus joue un rôle prédominant dans notre travail, c'est le mouvement Brownien libre unitaire obtenu comme processus limite en grande dimensions du mouvement Brownien $(U = U_t)_{t>0}$ sur le groupe des matrices unitaires à entrées complexes, réelles ou quaternioniques, solution de l'équation différentielle stochastique:

$$dU_t = U_t dK_t - \frac{c}{2} U_t dt,$$

où c est une constante et K un mouvement Brownien Hermitien de variance inversement proportionelle à la dimension. On envisage également dans la deuxième partie de cette thèse des mouvements Browniens sur des groupes de matrices pseudo-unitaires, qui ne sont pas compacts. Il n'y a pas de définition naturelle d'un tel processus car les algèbres de Lie de ces groupes ne sont pas canoniquement équipées d'un produit scalaire invariant.

On s'inspire du travail de Kemp sur la convergence du mouvement Brownien sur le groupe des matrices inversibles ([35]) pour étudier en grande dimensions ces processus. Cependant, certains outils tels que la transformée de Segal–Bargmann construite en utilisant la structure complexe de $\mathcal{M}_n(\mathbb{C})$ n'ont pas d'analogues immédiats pour les groupes pseudo-unitaires.

0.5.3. La théorie de Yang Mills et les champs planaires. La motivation principale de cette thèse est l'étude d'analogues non commutatifs de théorie de Yang Mills bidimensionnelles. C'est pour cette raison qu'une partie de cette introduction est consacrée aux théories de Yang–Mills, même si la thèse peut être lue sans connaissances approfondies sur ces théories. Précisons que les analogues non commutatifs que l'on cherche a construire sont ceux où les notions classiques d'espace de probabilité, de variables aléatoires, d'indépendance sont remplacées par leurs équivalents non commutatifs et où le groupe de structure (abélien ou non) est remplacé par (une structure ressemblant à) un groupe quantique.

0.5.3.1. *La théorie de Yang Mills et les symétries de jauge.* Les théories de Yang–Mills et plus généralement les théories de jauges sont des théories physiques dans lesquels les champs sont soumis à l' action d'un groupe (de Lie, compact) en chaque point de l'espace-temps. On appelle ce groupe le groupe de structure. Cette *symétrie de jauge détermine entièrement la théorie*: l'espace dans lequel ces champs prennent leur valeurs, la dynamique de la théorie, les observables... On décrit, avec un tout petit peu plus de détails, la théorie traitant des champs d'interaction, sans champs de matières. On considère une variété Riemannienne M et un group de Lie compact G, qui jouera le rôle du groupe de structure. Soit $P \to M$ un G fibré principal. L'espace des configurations est l'espace affine de dimension infinie des connexions sur le fibré P. Ces connexions sont des 1-formes différentielles à valeurs dans l'algèbre de Lie de G. Chacune des ces connexions admet une courbure Ω, qui est une une 2-forme différentielle prenant ses valeurs dans un fibré associé à P dont la fibre typique est $\mathfrak{g}$. La structure Riemannienne de M permet de définir une étoile de Hodge $\star$ sur les formes différentielles et de définir l'action de yang Mills par:

$$S_{YM}(w) = \frac{1}{2}\int_M \langle \Omega \wedge \star\Omega \rangle \tag{5}$$

Heuristiquement, l'amplitude associée a chaque connexion w, dans le cadre d'une quantification à la Feynmann serait:

$$\mathrm{d}\mu_{YM}(w) = \frac{1}{Z}e^{-S_{YM}(w)}dw$$

avec dw la mesure de Lebesgues sur l'espace des connexions (qui n'existe pas). Nous ne rentrerons pas plus dans les détails concernant les difficultés d'une telle approche, nous nous contentons de donner une brève description dans la section suivante d'une quantification des théories de Yang–Mills en dimension deux.

0.5.3.2. *La mesure de Yang–Mills en dimension deux.* La construction de la mesure de Yang–Mills en dimension deux est maintenant un sujet très classique et se base sur la génération de l'algèbre des observables par les boucles de Wilson. Ces observables particulières sont construites via l'holonomie induite par une connexion. En effet, une connexion peut être intégrée le long d'une courbe fermée de M en une trajectoire sur le groupe de structure. L'holonomie le long de cette boucle est le point final de cette trajectoire. Cette construction est valide dans pour n'importe quelle choix de jauge, des choix différents entraînent des holonomies différentes mais qui sont toutes conjuguées. Ainsi, la nature intrinsèque de l'holonomie d'une connexion le long d'une courbe serait plutôt une classe de conjugaison qu'un élément du groupe.

Si le groupe de structure est compact, l'holonomie d'une connexion le long de toutes les boucles basées en un même point de la surface caractérise cette connexion. Ainsi, on peut interpréter le problème de la construction d'une mesure sur l'espace de toutes les connexions, comme le problème de la construction d'une famille de variables aléatoires indexée par les boucles basées en un point de la surface ayant une distribution jointe invariante par conjugaison par un même élément du groupe de structure. Le lecteur pourra consulter [41] pour la construction d'une telle famille de variables aléatoires. Sur le plan, ces holonomies associées à des boucles englobant des domaines disjoints sont indépendantes. Aussi, cette famille est fermée pour la concaténation de boucles et produit dans le groupe : l'holonomie associée à la concaténation de deux boucles est le produit des holonomies de chacune de ces boucles.

Ces propriétés mènent à la définition suivante d'un champ d'holonomies planaires.

Soit G un groupe de Lie compact. Un champ d'holonomie sur le plan est une collection $(H_\ell)_{\ell\in \mathsf{L}_0(\mathbb{R}^2)}$ de variables aléatoires à valeurs dans G satisfaisant aux propriétés suivantes:

1. *(Multiplicativité)* Pour toute paire de boucles (ℓ_1,ℓ_2), presque sûrement: $H_{\ell_1\ell_2} = H_{\ell_1}H_{\ell_2}$ and $H_{\ell^{-1}} = H_\ell^{-1}$.
2. *(Invariance de jauge)* Pour tout famille de boucles $\ell_1,\dots,\ell_n$, et tout $g\in G$, on a l'égalité:
$$(H_{\ell_1},\dots,H_{\ell_n}) \overset{(d)}{=} (gH_{\ell_1}g^{-1},\dots,gH_{\ell_n}g^{-1}).$$
3. *(Indépendance)* Si $(\ell_1,\dots,\ell_n)$ et $(\ell'_1,\dots,\ell'_n)$ sont deux familles de boucles telles que: $\cup_{i=1}^n \mathrm{Int}(\ell_i)$ and $\cup_{i=1}^n \mathrm{Int}(\ell'_i)$ alors les familles $(H_{\ell_1},\dots,H_{\ell_n})$ et $(H_{\ell'_1},\dots,H_{\ell'_n})$ sont indépendantes l'une de l'autre
4. *(Invariance par homéomorphismes préservant l'aire)* Pour tout difféomorphisme $\phi:\mathbb{R}^2\to\mathbb{R}^2$ préservant l'aire, et boucles $\ell_1,\dots,\ell_n$ tel que $\phi(\ell_1),\dots,\phi(\ell_n)$ soit dans $\mathsf{L}_0(\mathbb{R}^2)$, on a $(H_{\ell_1},\dots,H_{\ell_n}) \overset{(d)}{=} (H_{\phi(\ell_1)},\dots,H_{\phi(\ell_n)})$

0.5.4. Résumé de la thèse. Les contributions de cette thèse peuvent être regroupées selon trois directions:

1. la définition d'analogues non commutatifs de théories de jauge en identifiant la structure algèbrique des symétries adéquates, du point de vue probabiliste sur la théorie quantique des champs,
2. la construction d'une famille d'exemples de telles théories incluant une généralisation d'un champ maître,
3. l'étude de la convergence d'ensembles de matrices aléatoires pseudo-unitaires, pseudo-orthogonales, pseudo-symplectiques et pseudo-hermitiennes en grande dimensions, ainsi que leurs fluctuations.

Il est clair que les deux premières directions sont apparentées. L'objet de cette introduction est d'expliquer comme le troisième point est relié aux deux premiers.

La littérature traite généralement de théories de Yang–Mills associées à des groupes de structure compacts, ce que l'on pourrait appeler des "théories U(N)". Ces dix dernières années, l'étude de ces théories dans la limite où N tend vers l'infini a beaucoup progressé, et l'objet limite que l'on appelle le champ maître est maintenant bien compris. Pour des raisons qui deviendront claires par la suite, l'étude en grande dimensions de ces théories sera appelée "théorie $\mathcal{O}\langle 1\rangle$".

La construction et l'étude d'analogues non commutatifs des théories de Yang–Mills a été entreprise à différents niveaux de rigueur et dans des contextes différents par des physiciens et des mathématiciens. Cependant, ces constructions résistent (ou semblent résister) à une interprétation probabiliste, la principale raison étant, à notre avis, l'emploi d'algèbres de Hopf comme structure pour les symétries, sur lesquels il est (plus) difficile de définir des analogues non commutatifs de notions probabilistes classiques. Une autre source d'inspiration à notre travail est le récent article de Cébron, Dahlqvist et Gabriel, dans lequel ils étendent la théorie $\mathcal{O}\langle 1\rangle$ à des processus de Lévy libres généraux.

Notre premier objectif a été de proposer une définition d'une théorie de jauge non commutative qui soit formulable avec des concepts de la théorie non commutative des probabilités et qui présente des symétries de jauge ayant une structure algébrique proche des groupes quantiques (de Woronowicz). Cela nous a conduit à considérer des analogues maximalement non commutatifs des algèbres de Hopf, que l'on appelle algèbres de Zhang. On donne une définition, très proche de celle des algèbres de Zhang, dans le premier chapitre. On définit également la notion de comodule à gauche et à droite sur une algèbre de Zhang ainsi que la coaction par conjugaison d'une algèbre de Hopf sur elle-même.

Une fois toutes ces définitions posées, il n'est pas difficile de définir la notion de champs de jauge présentant des symétries ayant une structure d'algèbre de Zhang. Les théories $\mathbb{U}(N)$ et

$\mathcal{O}\langle 1\rangle$ apparaissent alors comme des théories particulières de théorie de jauges sur des algèbres de Zhang. Nous construisons dans un deuxième temps la théorie $\mathcal{O}\langle n\rangle$ pour tout entier n, l'algèbre de structure étant le groupe dual de Voisculescu $\mathcal{O}\langle n\rangle$:

$$\mathcal{O}\langle n\rangle = \mathcal{R}\langle u_{ij}, u_{ij}^* : i,j \in \{1,\dots,n\}\rangle \Big/ \left(\sum_{k=1}^{n} u_{ik}u_{kj}^* \right)$$

En poursuivant le travail d'Ulrich, on réalise cette théorie $\mathcal{O}\langle n\rangle$ comme limite de théories finies dimensionelles dans lesquelles une matrice unitaire est vue comme une matrice par blocs, en nombre fixé, mais dont les tailles tendent vers l'infini. On construit également des champs de jauge ayant des symétries présentant une structure de bimodule sur une algèbre de dimension finie que l'on appellera champs maitres amalgamés. Ces algèbres de symétries sont obtenues à partir de $\mathcal{O}\langle 1\rangle$ en ajoutant un système complet de projecteurs, les traces ces derniers servant de paramètres à cet ensemble de champs de jauge amalgamés. C'est pour considérer ces théories que l'on a adopté dans le chapitre 1 une définition très générale d'algèbre de Zhang.

D'un point de vue combinatoire, l'étude de la convergence de blocs extraits de matrices aléatoires unitaires en grandes dimensions repose sur une dualité de Schur–Weyl exprimée à l'aide d'une version colorée de l'algèbre de diagrammes de Brauer, agrémentés d'une orientation et de boucles constituant une extension centrale de cette dernière.

Les outils développés dans les deux premiers chapitres (les algèbres de Zhang et les diagrammes de Brauer colorés) se revèlent également utiles dans les deux derniers chapitres pour une étude approfondie, et apparemment sans rapport avec les deux premiers chapitres, de la convergence en grande dimensions de matrices aléatoires pseudo-unitaires, pseudo-orthogonales et pseudo-symplectiques. Ces matrices, bien que n'étant pas naturellement par blocs, sont néanmoins adaptées à une décomposition de l'espace en deux sous-espaces sur lesquels la forme quadratique est de signe défini. C'est la troisième direction de recherche mentionnée plus tôt.

Pour une description détaillée (en anglais) des principaux résultats de la thèse, on consultera la section 0.4.

CHAPTER 1

Gauge symmetry, Zhang algebras and holonomy fields

We investigate lattice and continuous quantum gauge theories on the Euclidean plane with a structure group that is replaced by a Zhang algebra. Zhang algebras are non-commutative analogues of groups and contain the class of Voiculescu's dual groups (omitting topology). We are interested in non-commutative analogues of random gauge fields, which we describe through the random holonomy that they induce. We propose a general definition of a holonomy field with Zhang gauge symmetry, and construct such a field starting from a quantum Lévy process on a Zhang algebra. As an application, we define higher dimensional generalizations of the so-called master field.

Contents

1.1. Introduction

The present work deals with the question of defining gauge theories on non-commutative analogue (or deformation) of spaces of functions on a group. In 1954, Yang and Mills exposed the theory at the origin of the Standard Model, which is now known as the Yang–Mills theory. This theory is the culminating point of a process started decades beforehand. Physicists do not longer consider symmetries as a tool to reduce computations but rather as generators of a dynamic. In fact, Yang–Mills theory produces a Lagrangian, a particle, the input being, essentially, a group of symmetries and a space on which the particles should evolve.

The two dimensional $\mathbb{U}(N)$–Yang–Mills theory has a long story and has been studied both by physicists and mathematicians who raised incredibly difficult questions, e.g quantization of the Yang–Mills theory on a four dimensional Lorentzian space. In two dimensions, the problem of defining what a quantum YM theory is, given a surface and a compact Lie group of symmetries, has been rigorously addressed by Lévy in [41]. From the point of view adopted by T. Lévy, the problem lies, to put it in few words, in defining what a Gaussian measure on the infinite dimensional space of connections on a fiber bundle with a compact Lie group as a structure group is. Because a translation invariant measure on such a space does not exists, it is meaningless to define such a measure by means of its density.

Let Σ be a riemannian surface and P a $G-$ principal bundle with G a compact Lie group. Let $p \in \sigma$ be a point. A connection A on P is most accurately described through its holonomy morphism from the group of loops based at p, and drawn on σ. By fixing a gauge, the holonomy associated with a loop becomes an element of the group G, an other choice of gauge induces a conjugation of this element. Under the non-existing Yang–Mills measure on the space of all connections on P, the process that associates to a growing family of loops the holonomy of a random connection along these loops is a Brownian motion on the group which law is invariant by the conjugation action on the group. A random connection induced a loop indexed process on the gauge group that share features with time parametrized classical Levy processes (independence, stationarity, see below), this is what we call a Yang–Mills field.

In this chapter, we focus on the case for which the underlying surface of the theory is the plan. The master field is the limit, in a vague sense, as N tends to infinity of the $\mathbb{U}(N)$–Yang–Mills field. The master field is most accurately understood and described with the help of free probability theory. In particular, it admits a description as an holonomy field that is the free counterpart of the Yang–Mills given above in which the unitary Brownian motion is replaced by its free version. Generalizations of these fields (YM and free) have been studied. For example, the authors in [13] considered the holonomy fields associated with (free) infinite divisible semi-group.

The present work includes these generalized YM and master fields into a larger class of fields having a Zhang algebra as set of symmetries by performing what is called *arrow reversing*. As such, free generalized master fields defined in [13] are seen as being *one dimensional* (sic). In fact, our main concern is to produce higher dimensional counterparts of the master field by increasing the algebra of symmetries of that field, which is slightly different from the point of view adopted in [13].

This work is organized as follows. In Section 1.2.2 we define what Zhang algebras are, make a short review on algebraic categories, left and right comodules. In Section 1.8, we define the notion of monoidal structure on a category, give examples and introduce the notion of categorical independence as defined by Franz in [29] and Schürmann in [31]. The main definitions and results are located in Section 1.4. In particular, the definition of a categorical algebraic holonomy field can be found in Definition 1.19. Theorem 1.25 is the central result of this work. Applications of Theorem 1.25 can be found in Section 1.4.

1.2. Zhang algebras

In this section, we review Zhang algebras, starting with a definition. Recall that all our algebras are over the field of complex numbers, unital and associative.

The definition of Zhang algebras is very similar to that of Hopf algebras, the main difference being that the coproduct takes its values not in the tensor product, but in a free product of the algebra with itself. There is therefore a variety of notions of Zhang algebras, corresponding to various notions of free products, or more properly speaking of categorical coproducts, of algebras. Since we will use several notions of categorical coproduct, we prefer not to specify a particular one at this point, and we choose instead to adopt the more abstract point of view of *algebraic categories*. The main point of this section and the following one is to recall some basics fact on algebraic categories, monoidal categories and Zhang algebras. We use the language of category theory, functors the reader is directed to the monograph [1] for a detailed exposition. Equalities between morphisms are expressed as a commutativity property of some *diagrams*. A *diagram* is a directed graph with object labelled vertices and morphism labelled edges. We say that a diagram is commutative if the composition of morphisms along any two directed paths with same source and target yield the same result. We will frequently drop labels of edges, if it is clear from the context how edges of a diagram are morphisms tagged to increased readability.

1.2.1. Algebraic categories. Recall that in a category $\mathcal{C}$, an object k is called an initial object if for every object A, there is exactly one object in $\mathrm{Hom}(k,A)$. Recall also that a coproduct of two objects A and B is the data of an object C and two morphisms $\iota_A : A \to C$ and $\iota_B : B \to C$ such that for any object D and any two morphisms $f : A \to D$ and $g : B \to D$, there exists a unique morphism $h : C \to D$ such that $f = h \circ \iota_A$ and $g = h \circ \iota_B$. If a coproduct of two objects A and B exists, it is unique up to isomorphism and it is denoted by $A \sqcup B$. Moreover, with the current notation, the morphism h is denoted by $f \,\dot\sqcup\, g : A \sqcup B \to D$. The dot in the symbol $\dot\sqcup$ indicates that the elements of D obtained by applying f and g to the elements of A and B are multiplied in D. There is also a natural way of combining two morphisms $f : A \to D_1$ and $g : B \to D_2$ into a morphism $f \sqcup g : A \sqcup B \to D_1 \sqcup D_2$, by first forming $\iota_{D_1} \circ f : A \to D_1 \sqcup D_2$ and $\iota_{D_2} \circ g : B \to D_1 \sqcup D_2$ and then setting $f \sqcup g = (\iota_{D_1} \circ f) \,\dot\sqcup\, (\iota_{D_2} \circ g)$.

Definition 1.1 (Algebraic category). An algebraic category is a category with an initial object and in which any two objects admit a coproduct.

Let, for example, Alg be the category of unital complex associative algebras. The algebra $\mathbb{C}$ is an initial object of this category. Moreover, given two algebras A and B, we can form the algebra $A \sqcup B$ freely generated by A and B, the units of A and B being identified with the unit of $A \sqcup B$. This algebra can be described as a quotient of the tensor algebra:

$$A \sqcup B = T(A \oplus B)/(a \otimes a' - aa', b \otimes b' - bb', 1_A - 1, 1_B - 1 : a, a' \in A, b, b' \in B)$$

Concretely, $A \sqcup B$ is the vector space of all formal linear combinations of alternating words in elements of A and B. Any occurrence of the units of A or B in one of these words can be ignored, and multiplication of words is given by concatenation followed, in the case where they belong to the same algebra, by the multiplication of the last letter of the first word with the first letter of the second. Then $\sqcup$ is a coproduct in the category Alg, so that Alg is an example of an algebraic category.

We will consider the following other examples of algebraic categories.

1. The category $\mathrm{Alg}^\star$ of involutive (complex unital associative) algebras. The initial object of this category is still $\mathbb{C}$. The coproduct of two objects A and B of $\mathrm{Alg}^\star$ is, as an algebra, their coproduct in Alg. Moreover, $A \sqcup B$ is endowed with the unique antimultiplicative involution which extends those of A and B. Concretely, the involution of $A \sqcup B$ reverses the order of the letters in a word, and transforms each letter according to the involutions of A and B.
2. The category $\mathrm{Alg}(R)$ of (complex unital associative) algebras endowed with a structure of bimodule over a fixed unital algebra R. The initial object in this category is R. The coproduct of two objects A and B is the coproduct of the category Alg with amalgamation over R. It can be described as

$$\begin{aligned} A \sqcup_R B = (R \oplus \bigoplus_{n \geq 1} T^n(A \oplus B))/(&ar \otimes r'a' - arr'a', br \otimes r'b' - brr'b', ar \otimes b - a \otimes rb, \\ &br \otimes a - b \otimes ra, r1_A r' - rr', r1_B r' - rr' : a, a' \in A, b, b' \in B, r, r' \in R). \end{aligned}$$

In English, it is the free product of A and B in which multiplication by elements of R can circulate between neighbouring factors.

3. The category $\mathbb{Z}_2$-Alg of complex unital associative $\mathbb{Z}_2$-graded algebras is algebraic. A $\mathbb{Z}_2$-graded algebra is the data of a complex unital algebra and an unipotent morphism on that algebra. Morphisms are unital morphisms of algebras that preserve the grading. The free product of two graded algebras (A, D_A) and (B, D_B), is as an algebra $A \sqcup B$ and the grading is defined through $D_{A \sqcup B} = D_A \sqcup D_B$. It means that a word in $A \sqcup B$ is even if and only if it contains an odd number of odd letters. The initial object of the category is $\mathbb{C}$ endowed with the trivial grading.
4. The category $\mathcal{G}rp$ of groups. The coproduct is the free product of groups and the initial object is the group having only one element. If G and H are groups, a word in G and H is a product of the form $s_1 s_2 \cdots s_n$, where each s_i, $i \leq n$ is either an element of the group G or an element of the group H. Such a word may be reduced using the following operations:
   1. Remove an instance of the identity element (of either G or H).
   2. Replace a pair of the form $g_1 g_2$ by its product in G, or a pair $h_1 h_2$ by its product in H, with obvious notation.

   Every reduced word is an alternating product of elements of G and elements of H. The free product $G \sqcup H$ is the group whose elements are the reduced words in G an H, under the operation of concatenation followed by reduction.
5. The category biMod(R) of bimodules over a fixed unital algebra R can be endowed with two coproduct with injections. Let A and B two R-bimodules. The first product $A \sqcup_1 B$ in biMod(R) is, as a vector space, isomorphic to the sum of vector spaces $A \oplus B$. The R bimodule structure on $A \oplus B$ is the sum of the two structures:
$$r(a \oplus b)r' = rar' \oplus rbr', a \in A,\ b \in B,\ r, r' \in R.$$
   The initial object is again R.
6. The category coModAlg(H) of comodule algebras over a Zhang algebra H, which we will described later (see Section 1.2.3).

1.2.2. Zhang algebras. We can now give the definition of a Zhang algebra of an algebraic category.

Definition 1.2 (Zhang algebra). Let $\mathcal{C}$ be an algebraic category with initial object k and coproduct $\sqcup$. A Zhang algebra of $\mathcal{C}$ is a quadruplet (H, Δ, ϵ, S) where

1. H is an object of $\mathcal{C}$,
2. $\Delta : H \to H \sqcup H$ is a morphism of $\mathcal{C}$ such that $(\Delta \sqcup \mathrm{id}_H) \circ \Delta = (\mathrm{id}_H \sqcup \Delta) \circ \Delta$,
3. $\epsilon : H \to k$ is a morphism of $\mathcal{C}$ such that $(\epsilon \dot\sqcup \mathrm{id}_H) \circ \Delta = \mathrm{id}_H = (\mathrm{id}_H \dot\sqcup \epsilon) \circ \Delta$,
4. $S : H \to H$ is a morphism of $\mathcal{C}$ such that $(S \dot\sqcup \mathrm{id}_H) \circ \Delta = \eta \circ \epsilon = (\mathrm{id}_H \dot\sqcup S) \circ \Delta$, where η is the unique morphism from k to H.

Definition 1.2 is formally very similar to the definition of Hopf algebras. More succisely, replacing $\mathcal{C}$ by the category of (complex unital associative) algebras, k by $\mathbb{C}$ and $\sqcup$ by the tensor product in Definition 1.2 yields the exact definition of a Hopf algebra. However, the tensor product is not a coproduct in the category of algebras, and Hopf algebras are not a special case of Zhang algebras. It is instructive to understand the reason why the tensor product is not a coproduct. Consider indeed two algebras A and B and two morphisms $f : A \to D$ and $g : B \to D$. Let $\iota_A : A \to A \otimes B$ and $\iota_B : B \to A \otimes B$ be the natural morphisms. Should there exist a morphism $h : A \otimes B \to D$ such that $f = h \circ \iota_A$ and $g = h \circ \iota_B$, the relation
$$\iota_A(a)\iota_B(b) = (a \otimes 1_B)(1_A \otimes b) = a \otimes b = (1_A \otimes b)(a \otimes 1_B) = \iota_B(b)\iota_A(a)$$
would impose the equalities $h(a \otimes b) = f(a)g(b) = g(b)f(a)$, the second of which has no reason of being satisfied unless, of course, D is commutative.

The fact that the definition of Zhang algebras involves a coproduct is crucial for us, in particular because it has the consequence that we now explain. Consider a Zhang algebra H on an algebraic category $\mathcal{C}$. Then for every object A of $\mathcal{C}$, the set $\mathrm{Hom}(H, A)$ is endowed with a group structure by the formula
$$f * g = (f \dot\sqcup g) \circ \Delta.$$

The unit element of this group is $\eta_A \circ \epsilon_H$, where η_A is the unique morphism from k to A, and the inverse of an element f of $\mathrm{Hom}(H,A)$ is $f \circ S$.

In contrast with this situation, if H is a Hopf algebra and A is an algebra, the convolution product of two morphisms of algebras needs not be a morphism of algebras, unless A is commutative.

The reason why this fact is so important for us is that $\mathrm{Hom}(H,A)$ plays the role of a set of group-valued random variables, and it is extremely natural for us to be able to take the inverse of such a random variable, or two multiply two of them.

Let us conclude this discussion with the following theorem, which shows that Zhang algebras are in a sense exactly the right class of objects for our purposes.

Theorem 1.3 ([52], theorem 3.2). *Let $\mathcal{C}$ be an algebraic category. Let H be an object of $\mathcal{C}$. Then* $\mathsf{Hom}(H,\cdot)$ *a functor from $\mathcal{C}$ to the category of groups if and only if there exists Δ, ϵ, S, such that (H,Δ,ϵ,S) is a Zhang algebra of $\mathcal{C}$.*

We end this section with examples of Zhang algebras. Some of the following Zhang algebras are defined and used in Chapter 2 to study asymptotic of observables of large unitary matrices.

1. Let V be a complex vector space. We claim that V is a Zhang algebra in the algebraic category $(\mathrm{Vect}_{\mathbb{C}}, \oplus, \{0\})$. In fact, define:
$$\Delta(x) = x \oplus x \in V \oplus V,\ S(x) = -x,\ \varepsilon(x) = 0.$$
It is easy to check that, with these definitions, (V,S,ε) is a Zhang algebra.
2. Let $n \geq 1$ an integer. The Dual Voiculescu group $\mathcal{O}\langle n\rangle$ is the involutive unital associative algebra generated by $2n^2$ variables; $u_{ij},\ u_{ij}^{\star}\ i, 1 \leq j \leq n$ subject to the relations:
$$\sum_{k=1}^{n} u_{ik}u_{jk}^{\star} = \delta_{ij},\ \sum_{k=1}^{n} u_{ki}^{\star}u_{kj} = \delta_{ij},\ 1 \leq i,j \leq n.1 \leq i,j \leq n.$$
The dual voiculescu group is a turned into a Zhang algebra $(\mathcal{O}\langle n\rangle, \Delta, \varepsilon, S)$ if we define the structural morphisms by:
$$S(u_{ij}) = u_{ji}^{\star},\ \Delta_{u_{ij}} = \sum_{k=1}^{n} u_{ik}|_1 u_{kj}|_2,\ \varepsilon(u_{ij}) = \delta_{ij},\ 1 \leq i,j \leq n.$$
3. The rectangular unitary algebra $\mathcal{RO}\langle n\rangle$ is the involutive unital associative algebra generated by one unitary element and a set of mutually autoadjoint orthogonal projectors $\{p_i,\ i \leq n\}$. Set $\mathcal{R} = \{p_i,\ 1 \leq i \leq n\}$. The rectangular unitary algebra is a bimodule algebra over $\mathcal{R}$. The algebra $\mathcal{RO}\langle n\rangle$ is a Zhang algebra in the algebraic category $\mathrm{Alg}^{\star}(\mathcal{R})$ with structural morphisms:
$$S(u) = u^{\star} = u^{-1},\ \Delta(u) = u|_1 u|_2,\ \varepsilon(u) = 1 \in R.$$
4. Any commutative Hopf algebra is a Zhang algebra, thus if G is a group then its space of polynomial functions $\mathcal{F}(G)$ is a Zhang algebra with structure morphisms given by:
$$\Delta(f)(g,h) = f(gh),\ S(f)(g) = f(g^{-1}),\ \varepsilon(f) = f(e).$$

1.2.3. Comodule algebras over Zhang algebras. In this section, we define the category of comodule algebras over a Zhang algebra of an algebraic category $\mathcal{C}$ with initial object k and coproduct $\sqcup$. In this definition, and for every object B of $\mathcal{C}$, we will identify without further mention the objects B, $B \sqcup k$ and $k \sqcup B$. They are indeed isomorphic by the maps $\mathrm{id}_B \,\dot{\sqcup}\, \eta : B \sqcup k \to B$ and $\eta \,\dot{\sqcup}\, \mathrm{id}_B : k \sqcup B \to B$, where $\eta : k \to B$ is the unique element of $\mathrm{Hom}(k,B)$.

For this section, let us fix an algebraic category $\mathcal{C}$ with initial object k and coproduct $\sqcup$, and a Zhang algebra (H,Δ,ϵ,S) of this category.

Definition 1.4 (Comodule algebras). A right H-comodule algebra of $\mathcal{C}$ is a pair (M,Ω), where M is an object of $\mathcal{C}$ and $\Omega : M \to M \sqcup H$ is a morphism such that

1. M is an object of $\mathcal{C}$,

2. $\Omega : M \to M \sqcup H$ is a morphism of $\mathcal{C}$ satisfying the following two conditions:

$$(1_R) \qquad (\Omega \sqcup \mathrm{id}_H) \circ \Omega = (\mathrm{id}_M \sqcup \Delta) \circ \Omega \ \text{ and } \ (\mathrm{id}_M \sqcup \epsilon) \circ \Omega = \mathrm{id}_M.$$

The definition of a left comodule is deduced from the definition of a right comodule by replacing (1_R) by

$$(1_L) \qquad (\mathrm{id}_H \sqcup \Omega) \circ \Omega = (\Delta \sqcup \mathrm{id}_M) \circ \Omega \ \text{ and } \ (\epsilon \sqcup \mathrm{id}_M) \circ \Omega = \mathrm{id}_M.$$

A morphism between two right H-comodule algebras (M, Ω_M) and (N, Ω_N) is, by definition, an element f of $\mathrm{Hom}_{\mathcal{C}}(M, N)$ which respects the structure of H-comodule in the sense that

$$(1.2) \qquad \Omega_N \circ f = (f \sqcup \mathrm{id}_H) \circ \Omega_M.$$

We denote respectively by $\mathrm{rCoMod}\mathcal{C}(H)$ and $\mathrm{lCoMod}\mathcal{C}(H)$ the categories of right and left H-comodule algebras of $\mathcal{C}$. We will now state, and prove, that they are algebraic categories.

Lemma 1.5. *The category* $\mathrm{rCoMod}\mathcal{C}(H)$ *is an algebraic category.*

Of course, an analogous statement holds for the category $\mathrm{lCoMod}\mathcal{C}(H)$.

Proof. It is a simple verification that k is a right H-comodule algebra and that for every H-comodule algebra M, the unique morphism in $\mathcal{C}$ from k to $\mathcal{C}$ satisfies (1.2), hence is a morphism in the category of right H-comodule algebras. This shows that k is an initial element of $\mathrm{rCoMod}\mathcal{C}(H)$.

We must now define a coproduct in $\mathrm{rCoMod}\mathcal{C}(H)$. Let (M, Ω_M) and (N, Ω_N) be two right H-comodules. We will endow the object $M \sqcup N$ of $\mathcal{C}$ with a coaction of H. For this, we start from the map $\Omega_M \sqcup \Omega_N : M \sqcup N \to M \sqcup H \sqcup N \sqcup H$. We will compose this map with a map which, informally, forgets the origin of the factors belonging to H. Pictorially, we want a morphism $M \sqcup H_{|1} \sqcup N \sqcup H_{|2} \to M \sqcup N \sqcup H$ which sends, for instance, $nh_{|2}n'mh'_{|1}m'n''h''_{|1}h'''_{|2}$ to $nhn'mh'm'n''(h''h''')$. This map is built from the canonical maps $\iota_M : M \to M \sqcup N \sqcup H$, $\iota_N : N \to M \sqcup N \sqcup H$, $\iota_H : H \to M \sqcup N \sqcup H$ by the formula

$$\Omega_{M \sqcup N} = (\iota_M \mathbin{\dot\sqcup} \iota_H \mathbin{\dot\sqcup} \iota_N \mathbin{\dot\sqcup} \iota_H) \circ (\Omega_M \sqcup \Omega_N).$$

We claim that $(M \sqcup N, \Omega_{M \sqcup N})$ is a coproduct of (M, Ω_M) and (N, Ω_N). The fact that $\Omega_{M \sqcup N}$ is a morphism in $\mathcal{C}$ follows from its very definition. There remains to prove that it satisfies the equalities (1_R). Let us treat the first equality in detail.

We begin by drawing (see Fig. 1) the diagram associated with the universal problem of which the pair $(M \sqcup N, \Omega_{M \sqcup N})$ is the solution.

$$\begin{tikzcd}
N \arrow[r, "\Omega_M"] \arrow[dr] & N \sqcup H \arrow[dr] & \\
& M \sqcup N \arrow[r, "\Omega_{M \sqcup N}"] & M \sqcup N \sqcup H \\
M \arrow[r, "\Omega_N"'] \arrow[ur] & M \sqcup H \arrow[ur] &
\end{tikzcd}$$

Figure 1. The universal problem solved by $(M \sqcup N, \Omega_{M \sqcup N})$.

Using this universal property and the fact that Ω_M and Ω_N are coactions, we draw a second diagram (see Fig. 2) in which a map $f : M \sqcup N \to M \sqcup N \sqcup H \sqcup H$ appears. We claim that the two maps of which we want to prove the equality, namely $(\Omega_{M \sqcup N} \sqcup \mathrm{id}_H) \circ \Omega_{M \sqcup N}$ and $(\mathrm{id}_{M \sqcup N} \sqcup \Delta) \circ \Omega_{M \sqcup N}$, are equal to this map.

This is done by two more diagrams. The first (Fig. 3) shows the map $(\Omega_{M \sqcup N} \sqcup \mathrm{id}_H) \circ \Omega_{M \sqcup N}$. The commutativity of this diagram and its comparison with Fig. 2 shows that this map is indeed equal to the map f.

For the second map, the diagram that we draw (Fig. 4) has four squares and the commutativity of the rightmost two needs to be checked. This is a simple verification that we leave

$$\begin{tikzcd}
N \arrow[r] \arrow[dr] & N \sqcup H \arrow[r, shift left, "\Omega"] \arrow[r, shift right, "\Delta"'] & N \sqcup \tilde{H} \sqcup H \arrow[dr] & \\
& M \sqcup N \arrow[rr, "f"] & & M \sqcup N \sqcup \tilde{H} \sqcup H \\
M \arrow[r] \arrow[ur] & M \sqcup H \arrow[r, shift left] \arrow[r, shift right, "\Delta"'] & M \sqcup \tilde{H} \sqcup H \arrow[ur] &
\end{tikzcd}$$

FIGURE 2. In this diagram, blue arrows indicate a coaction of H and orange arrows a coproduct of H. We use the notation $\tilde{H}$ for the sake of clarity.

$$\begin{tikzcd}
N \arrow[r] \arrow[dr] & N \sqcup H \arrow[r] \arrow[dr] & N \sqcup \tilde{H} \sqcup H \arrow[dr] & \\
& M \sqcup N \arrow[r] & M \sqcup N \sqcup H \arrow[r] & M \sqcup N \sqcup \tilde{H} \sqcup H \\
M \arrow[r] \arrow[ur] & M \sqcup H \arrow[r] \arrow[ur] & M \sqcup \tilde{H} \sqcup H \arrow[ur] &
\end{tikzcd}$$

FIGURE 3. This diagram, in which blue arrows correspond to the coaction of H, shows the map $(\Omega_{M\sqcup N} \sqcup \mathrm{id}_H) \circ \Omega_{M\sqcup N}$.

to the reader. The equality of $(\mathrm{id}_{M\sqcup N} \sqcup \Delta) \circ \Omega_{M\sqcup N}$ with the map f, and the fact that $\Omega_{M\sqcup N}$ satisfies the first equality of (1_R), follows immediately.

$$\begin{tikzcd}
N \arrow[r] \arrow[dr] & N \sqcup H \arrow[r] \arrow[dr] & N \sqcup \tilde{H} \sqcup H \arrow[dr] & \\
& M \sqcup N \arrow[r] & M \sqcup N \sqcup H \arrow[r] & M \sqcup N \sqcup \tilde{H} \sqcup H \\
M \arrow[r] \arrow[ur] & M \sqcup H \arrow[r] \arrow[ur] & M \sqcup \tilde{H} \sqcup H \arrow[ur] &
\end{tikzcd}$$

FIGURE 4. In this diagram, orange arrows correspond to the coproduct of H. We see the map $(\mathrm{id}_{M\sqcup N} \sqcup \Delta) \circ \Omega_{M\sqcup N}$ in the middle line.

The proof of the second equality of (1_R) is similar and simpler that the proof of the first, and we leave it to the reader.

We are not only stating that the free product of two comodules is a comodule, but also that the category of all comodules over the Zhang algebra H is algebraic. Three points remain to be proved. The first is the equivariance of the coproduct of two equivariant morphisms. The second is the fact that k, the initial object of $\mathcal{C}$, can be endowed with a coaction of H. The third is that, with respect to this coaction, the unique morphism from k to any comodule algebra M is equivariant.

Let us discuss the first point. Let $(M,\Omega_M),(N,\Omega_N),(C,\Omega_C)$ be three right comodules algebras. Let $f : M \to C$ and $g : N \to C$ be two morphisms of the category $\mathrm{rCoMod}\mathcal{C}(H)$. We claim that $f \dot{\sqcup} g$ is equivariant with respect to coactions $\Omega_{M\sqcup N}$ and Ω_C, which means that the equality $\Omega_C \circ (f \dot{\sqcup} g) = (\mathrm{id}_H \sqcup (f \dot{\sqcup} g)) \circ \Omega_{M\sqcup N}$ holds. Fig. 5 shows a diagram in which, as before, blue arrows indicate the coaction of H. The equivariance of $f \dot{\sqcup} g$ is equivalent to the commutativity, in this diagram, of the face delimited by the gray bended arrow and the horizontal symmetry axis of the diagram. This commutativity property will be implied by commutativity of the two outer faces bounded by the violet and gray arrows. This commutativity, in turn, is implied by the associativity of the free product, drawn in Fig. 6.

The second and third points concern the initial object k. There exists an unique morphism $\eta_{k\sqcup H} : k \to k \sqcup H$ and we prove that $(k, \eta_{k\sqcup H})$ is an object of $\mathrm{rCoMod}\mathcal{C}(H)$. The two morphisms

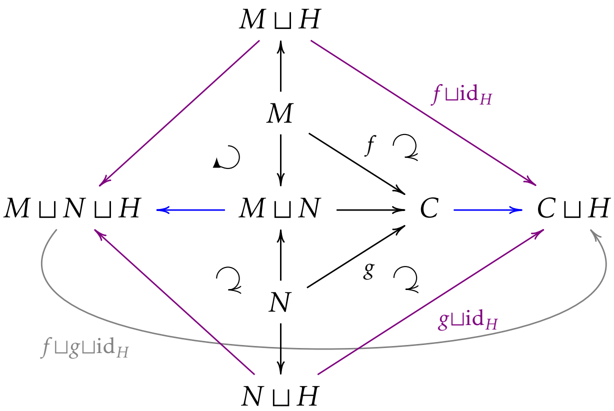


FIGURE 5. Equivariance of the map $f \dot{\sqcup} g$. The morphisms drawn in violet and gray are equal to the morphisms drawn with the same colour in Figure 6.

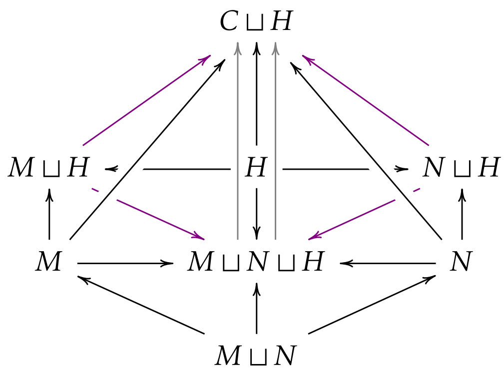


FIGURE 6. Associativity of the free product.

$(\mathrm{id}_H \sqcup \eta_{k\sqcup H}) \circ \eta_{k\sqcup H}$ and $(\Delta \sqcup \mathrm{id}_H) \circ \eta_{k\sqcup H}$ are equal because there is a unique morphism from k to $k \sqcup H \sqcup H$. For the same reason, $(\varepsilon \sqcup \mathrm{id}_k) \circ \eta_{k\sqcup H} = \mathrm{id}_k$. The third point, that the unique map from k to any comodule algebra M is equivariant, also follows from the same argument. The proof is complete. □

Let us denote by $\iota_1 : H \to H \sqcup H$ and $\iota_2 : H \to H \sqcup H$ the canonical maps. We define the morphism $\Omega_c : H \to H \sqcup H$ by the following formula:

$$\Omega_c = (\iota_1 \dot{\sqcup} \iota_2 \dot{\sqcup} \iota_1) \circ (\mathrm{id}_{H\sqcup H} \sqcup S) \circ (\Delta \sqcup \mathrm{id}_H) \circ \Delta.$$

For an integer $n \geq 1$, we denote by Ω_c^n the induced coaction on $H^{\sqcup n}$. It is convenient to introduce a graphical calculus to perform computations involving free products. All morphisms we handle act on free products of H with itself and are valued in the same type of objects. We do not forget that co-product on $\mathcal{C}$ comes with morphisms. Let $n \geq 1$ an integer, the morphisms from H to $H^{\sqcup n}$ are denoted $\iota_1, \ldots, \iota_n$. Let $f = (f_1, \ldots, f_n)$ a finite sequence of morphisms from H to itself. To each permutation σ of $[1, \ldots, n]$ is attached a morphism from $H^{\sqcup n}$ to $H^{\sqcup n}$, denoted f_σ and defined as the unique morphism satisfying the property: $f_\sigma \circ \iota_i = i_{\sigma(i)} \circ f_i$. Such morphisms are depicted as follows: we draw n vertical lines, labelled with the symbols $f_1, \ldots, f_n$ from left to right. We add at the beginning of the i^{th} vertical line the integer i and the integer $\sigma(i)$ at the end of the line. We draw examples in Figure 7 in case $n = 3$.

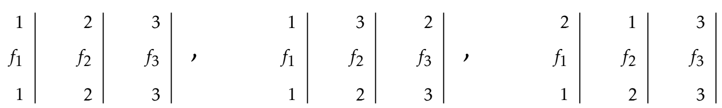


FIGURE 7. Morphisms f_{id}, $f_{(2,3)}$, $f_{(1,2)}$.

Of primary importance is the case where all the f_i $'s$ are equal to the identity of H. In that case, we use the notation τ_σ for the morphism f_σ. In words, τ_σ relabel the letters by substituting the label σ_i to i. The Figure 8 shows the graphical representation of the morphism τ_{12}. To draw the graphical representations of the morphisms at stake, we make the convention that a sequence of edges starting and ending on same levels have ends labelled with increasing integers from left to right, the ends being labelled with the same integer. Using graphical

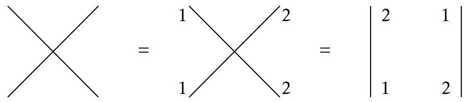


FIGURE 8. The permutation τ_{12} of labels.

calculus, the structural morphisms S, Δ, ϵ and $\mu = id_H \dot{\sqcup} \mathrm{id}_H$ are pictured as in Fig. 9 and the relations they are subject to are drawn in Fig. 10.

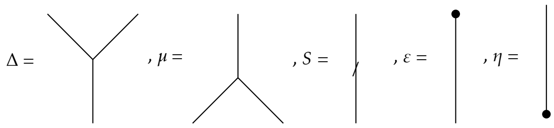


FIGURE 9. Drawings of the structural morphisms of a Zhang algebra (H,Δ,ε,S) and $\mu = \mathrm{id}_H \dot{\sqcup} \mathrm{id}_H$.

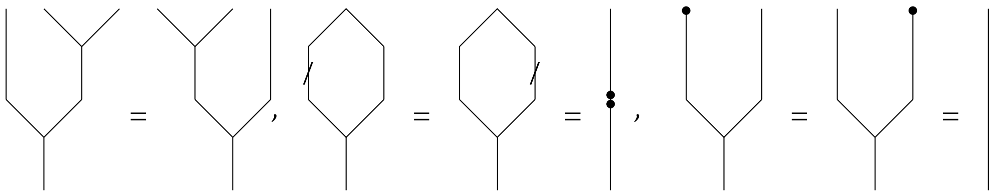


FIGURE 10. Relations amongst the structural morphisms of a Zhang algebra, from left to right: $\Delta \sqcup \mathrm{id}_H \circ \Delta = \mathrm{id}_H \sqcup \Delta \circ \Delta$, $S \sqcup \mathrm{id}_H \circ \Delta = \mathrm{id}_H \sqcup S \circ \Delta = \epsilon \circ \eta, \epsilon \sqcup \mathrm{id}_H = \mathrm{id}_H \sqcup S = \mathrm{id}_H$.

On a Zhang algebra, the antipode S is not a comorphism, however a simple relation between $S \sqcup S \circ \Delta$ and $\Delta \circ S$ can be deduced from the three structural relations drawn in Fig. 10 which is reminiscent from the fact that the antipode S of an Hopf algebra is an anti-co-morphism:

$$\tau_{(12)} \circ (S \sqcup S) \circ \Delta = \Delta \circ S$$

This last relation is pictured in Fig 11 and its proof can be found in the seminal article of Zhang [52]. Let us, however, illustrate how the graphical calculus we introduce work on this first example. The morphism $\Delta \circ S$ is the inverse of Δ in the group $\mathrm{Hom}_{\mathcal{C}}(H, H \sqcup H)$. We have to show that $((\tau_{12} \circ (S \sqcup S) \circ \Delta) \dot{\sqcup} \Delta) \circ \Delta = \eta_{H \sqcup H} \circ \epsilon$, where $\eta_{H \sqcup H}$ is the unique morphism from the initial object to $H \sqcup H$, graphical computations are performed in Fig. 12. By using the same method, it can be proved that $S^2 = \mathrm{id}_H$. The main goal of the two following lemmas (Lemma

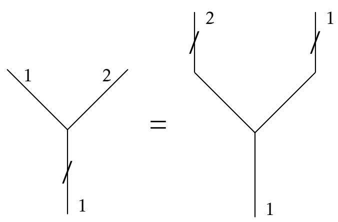


FIGURE 11. Diagram corresponding to the relation $\tau_{(12)} \circ (S \sqcup S) \circ \Delta = \Delta \circ S$.

1.7 and Lemma 1.6) is to prove that a Zhang algebra in algebraic category $\mathcal{C}$ is also a Zhang algebra in the category $rCoModC(H)$ of left comodules over H in $\mathcal{C}$.

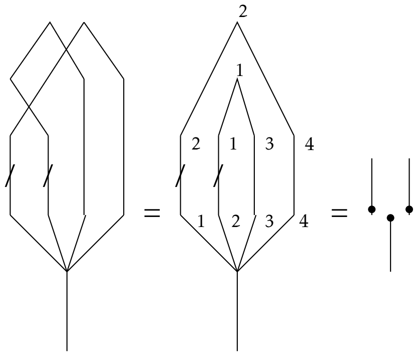


FIGURE 12. The graphical proof of the relation $\tau_{12} \circ (S \sqcup S) \circ \Delta = \Delta \circ S$.

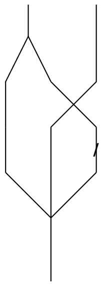


FIGURE 13. Diagram representing the coaction Ω_c.

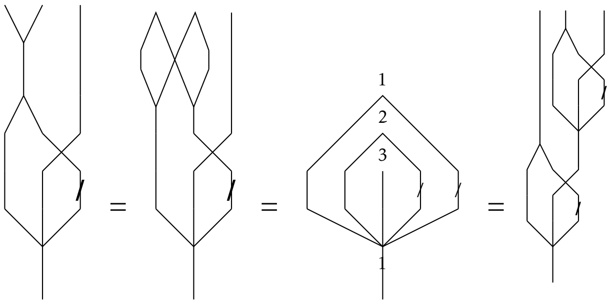


FIGURE 14. Graphical proof of the relation $(\Delta \dot{\sqcup} 1) \circ \Omega_c = (1 \dot{\sqcup} \Omega_c) \circ \Omega_c$.

LEMMA 1.6. *The pair (H, Ω_c) is a right H-comodule algebra of $\mathcal{C}$.*

PROOF. The equality $(\mathrm{id}_H \sqcup \Omega_c) \circ \Omega_c = (\Delta \sqcup \mathrm{id}_M) \circ \Omega_c$ follows from the fact that both sides are equal to

$$(\iota_1 \dot{\sqcup} \iota_2 \dot{\sqcup} \iota_3 \dot{\sqcup} \iota_2 \dot{\sqcup} \iota_1) \circ (\mathrm{id}_{H^{\sqcup 4}} \sqcup S) \circ \Delta^4,$$

as one checks using the coassociativity of Δ and the fact that it is a morphism. Let us write more details to convince the reader with the efficiency of the graphical calculus we introduced. The co-action Ω_c has the graphical presentation showed in Fig. 13. We begin with the first relation of (1_L). In figure 14, we drew the sequence of diagram that proves the equality $(\Delta \sqcup \mathrm{id}_H) \circ \Omega_c = (\iota_1 \dot{\sqcup} \iota_2 \dot{\sqcup} \iota_3 \dot{\sqcup} \iota_2 \dot{\sqcup} \iota_1) \circ (\mathrm{id}_{H^{\sqcup 4}} \sqcup S) \circ \Delta^4$.

The verification of the equality $(\epsilon \sqcup \mathrm{id}_H) \circ \Omega_c = \mathrm{id}_H$ is simpler.

□

Furthermore, we claim that Δ, ϵ, and S, which are defined as morphisms in the category $\mathcal{C}$, are in fact morphisms in the category $\mathrm{rCoMod}\mathcal{C}(H)$. This means that they satisfy the equation (1.2).

LEMMA 1.7. *$((H, \Omega_c), \Delta, \epsilon, S)$ is a Zhang algebra of the category* $\mathrm{rCoMod}\mathcal{C}(H)$.

PROOF. We have to check that the three morphisms Δ, ϵ, S are co-module morphisms. To that end, we use the graphical calculus introduced previously. We begin with the antipode, compatibility between S and the right co-action Ω_c means $\Omega_c \circ S = (\mathrm{id}_H \sqcup S) \circ \Omega_c$. The computations are pictured in Fig. 15, to perform them we use the relation drawn in Fig. 11.

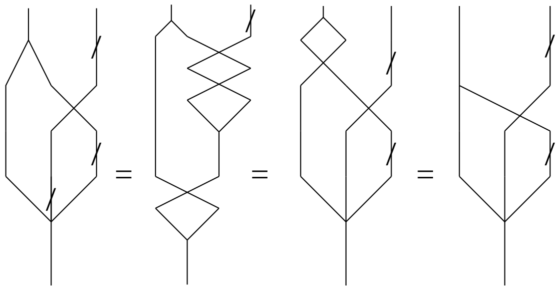


FIGURE 15. The relation $\Omega_c \circ S = (\mathrm{id}_H \sqcup S) \circ \Omega_c$.

We now turn our attention to the relation that needs to be satisfied by the counit ε. The computations are drawn in Fig. 16. We have to check that $\eta_{H \sqcup k} \circ \varepsilon = (\mathrm{id}_H \sqcup \varepsilon) \circ \Omega_c$, because we identify $H \sqcup k$ with H, the last relation is written as $\eta_H \circ \varepsilon = (\mathrm{id}_H \sqcup \varepsilon) \circ \Omega_c$. The final relation

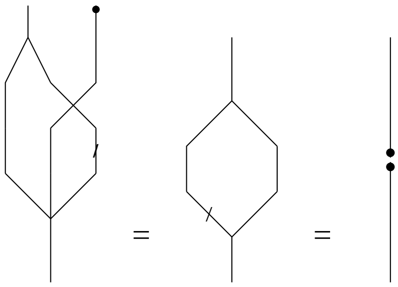


FIGURE 16. Diagrammatic proof of the relation $\eta_H \circ \varepsilon = (\mathrm{id}_H \sqcup \varepsilon) \circ \Omega_c$.

that needs to be checked implies that Δ is a comodule morphism with respect to the coactions Ω_c on H and Ω_c^2 on $H \sqcup H$:$\Omega_c^2 \circ \Delta = (\mathrm{id}_H \sqcup \Delta) \circ \Omega_c$. Once again, we perform diagrammatic computations that are pictured in Fig. 17.

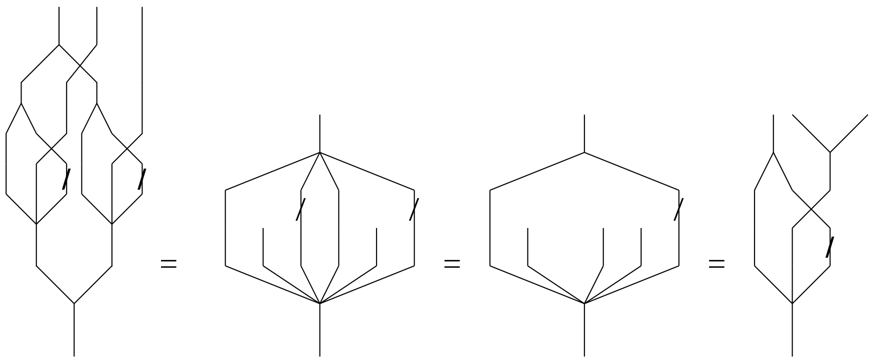


FIGURE 17. Proof of the relation $\Omega_c^2 \circ \Delta = (\mathrm{id}_H \sqcup \Delta) \circ \Omega_c$.

□

1.3. Monoidal structures and inductive limits

In this section, we recall basic facts on monoidal structures with inclusions and inductive (direct) limits. We define the notion of categorical independendence.

1.3.1. Monoidal structures and independence. In order to motivate our definitions, we start by reviewing the classical, commutative case. Let

$$X : (\Omega,\mathcal{F},\mathbb{P}) \to (S,\mathcal{S}) \text{ and } Y : (\Omega,\mathcal{F},\mathbb{P}) \to (T,\mathcal{T}) \tag{1.3}$$

be two essentially bounded random variables defined on the same classical probability space. Let τ_X and τ_Y denote the linear forms defined respectively on $L^\infty(S,\mathcal{S})$ and $L^\infty(T,\mathcal{T})$ by the distributions of X and Y.

The random variables X and Y also induce homomorphisms of the algebras of measurable functions

$$j_X : L^0(S,\mathcal{S}) \to L^0(\Omega,\mathcal{F}) \text{ and } j_Y : L^0(T,\mathcal{T}) \to L^0(\Omega,\mathcal{F}),$$

defined by $j_X(f) = f \circ X$ and $j_Y(g) = g \circ Y$.

The independence of the random variables X and Y is equivalent to the existence of a morphism j making the following diagram commutative:

$$\begin{tikzcd}
& (L^\infty(\Omega,\mathcal{F}),\mathbb{E}) & \\
(L^\infty(S,\mathcal{S}),\tau_X) \arrow[ur, "j_X"] \arrow[r, "\iota"] & (L^\infty(S,\mathcal{S}),\tau_X)\otimes(L^\infty(T,\mathcal{T}),\tau_Y) \arrow[u, "j"] & (L^\infty(T,\mathcal{T}),\tau_Y) \arrow[ul, "j_Y"'] \arrow[l, "\iota"']
\end{tikzcd}$$

In order to generalize the notion of independence from the category of commutative probability spaces to an arbitrary category of non-commutative probability spaces, it appears that a notion of tensor product is needed. A category in which the tensor product of two objects is defined is called a *monoidal category or a tensor category*. A monoidal category $\mathcal{C}$ is a category $\mathcal{C}$ together with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ with the following properties:

1. The bifunctor $\otimes$ is associative under a natural isomorphism with components

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \xrightarrow{\cong} (A \otimes B) \otimes C,\ A, B, C \in \mathcal{C}$$

   called *associativity constraints*.
2. The bifunctor $\otimes$ has a unit object $E \in \mathrm{Obj}(\mathcal{C})$ acting as left and right identity under natural isomorphisms with components:

$$\ell_A : E \otimes A \xrightarrow{\cong} A,\ r_A : A \otimes E \xrightarrow{\cong} A$$

   called *left unit constraint and right unit constraint* such that the *pentagon and triangle identities* hold, see Fig. 18 and Fig. 19

$$\begin{tikzcd}
& (A\otimes(B\otimes C))\otimes D \arrow[dr] & \\
((A\otimes B)\otimes C)\otimes D \arrow[ur] \arrow[d] & & A\otimes((B\otimes C)\otimes D) \arrow[d] \\
(A\otimes B)\otimes(C\otimes D) \arrow[rr] & & A\otimes(B\otimes(C\otimes D))
\end{tikzcd}$$

FIGURE 18. Pentagonal coherence axiom for monoidal categories

$$\begin{tikzcd}
(A\otimes E)\otimes B \arrow[rr] \arrow[dr] & & A\otimes(E\otimes B) \arrow[dl] \\
& A\otimes B &
\end{tikzcd}$$

FIGURE 19. Triangle coherence axiom for monoidal categories

The pentagon identities Fig.18 and triangle identities Fig.19 imply commutativity of all diagrams which contain the associativity constraint, the natural isomorphisms ℓ and r. The following definition is motivated by the fact that in a tensor category, there is in general no canonical morphism from an object to its tensor product with another object.

Definition 1.8 (Monoidal category with inclusions). A monoidal category with inclusions $(\mathcal{C}, \otimes, \iota)$ is a tensor category $(\mathcal{C}, \otimes)$ in which, for any two objects B_1 and B_2, there exist two morphisms $\iota_{B_1} : B_1 \to B_1 \otimes B_2$ and $\iota_{B_2} : B_2 \to B_1 \otimes B_2$ such that for any two objects A_1 and A_2 and any two morphisms $f_1 : A_1 \to B_1, f_2 : A_2 \to B_2$, the following diagram is commutative:

$$\begin{tikzcd} A_1 \arrow[r, "\iota_{A_1}"] \arrow[d, "f_1"] & A_1 \otimes A_2 \arrow[d, "f_1 \otimes f_2"] & A_2 \arrow[l, "\iota_{A_2}"'] \arrow[d, "f_2"] \\ B_1 \arrow[r, "\iota_{B_1}"] & B_1 \otimes B_2 & B_2 \arrow[l, "\iota_{B_2}"'] \end{tikzcd}$$

Let $\mathcal{P}_i : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $i \in \{1,2\}$ be the projections functors on the first and second component. We can reformulate Definition 1.8, a monoidal category with inclusions if a monoidal category with two natural transformations $\iota_1 : P_1 \Rightarrow \otimes$ and $\iota_2 : P_2 \Rightarrow \otimes$. The natural transformation ι_1 is called a *left inclusion* and ι_2 is called a *right inclusion*. We can now give a general definition of independence of two morphisms.

Definition 1.9. Let $(\mathcal{C}, \otimes, \iota)$ be a tensor category with injections. Two morphisms $f_1 : C_1 \to A$ and $f_2 : C_2 \to A$ are said to be *independent* if there exists a third morphism $f : C_1 \otimes C_2 \to A$ such that the following diagram commutes:

$$\begin{tikzcd} & A & \\ C_1 \arrow[ur, "f_1"] \arrow[r, "\iota_{C_1}"'] & C_1 \otimes C_2 \arrow[u, "f"] & C_2 \arrow[ul, "f_2"'] \arrow[l, "\iota_{C_2}"] \end{tikzcd}$$

If we want to be explicit about the monoidal structure that is involved, we say that (f_1, f_2) is a $\otimes$-independent family of morphisms. Note that independence is not a symmetric notion (unless the monoidal structure is symmetric or braided): (f_2, f_1) may not be an independent family of morphisms. Definition 2 defines what it means for two morphisms to be independent but not what mutual independence of a finite set of morphisms is. To define mutual independence, the monoidal structural has to satisfy compatibility condition with the inclusions morphisms (see below). The morphism f of the last definition is called an *independence morphism*. In the examples that are treated below this morphism, if it exists, will be uniquely determined but this is not the case in general. In fact, if $\otimes$ is *braided*, which means that for any pair of objects (A, B) there exists a (necessarily involutive) morphism $\Psi_{A,B} : A \otimes B \to B \otimes A$, the injections ι_A, ι_B can be composed with the (torsion) morphism $T_{A,B} = \Psi_{B,A}\Psi_{A,B}$ to obtain injections ι'_A and ι'_B. All couples of independent morphisms for this new monoidal structure with injections $(\otimes, \iota')$ are also independent morphisms for the former ones, but the independence morphisms are different, with obvious notations : $f' = f \circ T^{-1}_{A,B}$ is $\otimes$ independence morphism if f is a $(\otimes, \iota')$ independence morphisms for a pair of $(\otimes, \iota)$-independent morphisms (f_1, f_2). Assume further that the monoidal structure is symmetric: with the notations introduced previously, $\Psi_{A,B} = \Psi^{-1}_{B,A}$. Symmetry of the tensor product is not sufficient for the associated notion of independence to be symmetric, we have to assume the compatibility conditions with the natural morphisms ι drawn in Fig. 20 In the sequel, any monoidal category we consider satisfies this assumption, that otherwise will be recalled below, when defining categorical holonomy fields on Zhang algebras. A second warning: a coproduct with injections is a tensor product with injections for which any two morphisms with the same target space are independent. A tensor coproduct is not a co-product.

We now give examples of monoidal categories. Let R an unital associative algebra and denote by $\mathrm{Prob}(R)$ the category of algebras endowed with a conditional expectation. We saw that the category $\mathrm{Alg}^\star(R)$ is algebraic, the coproduct being the free product of algebras amalgamated overs R. Pick a tensor product $\otimes$ on $\mathrm{Prob}(R)$. Let A and B two (R-valued) probability

$$\begin{tikzcd}
 & A \otimes B & \\
A \arrow[ur, "\iota_A^{A\otimes B}"] \arrow[dr, "\iota_A^{B\otimes A}"'] & & A \arrow[ul, "\iota_A^{A\otimes B}"'] \arrow[dl, "\iota_A^{B\otimes A}"] \\
 & B \otimes A \arrow[uu, "\Psi_{B,A}"'] &
\end{tikzcd}$$

Figure 20. Compatibility condition between the two natural transformations Ψ and ι.

spaces and write $A \otimes B = (C, \tau_{A\otimes B})$ with $C \in \mathrm{Alg}^\star$. From the universal property satisfied by the co-product, there exist a morphism of involutive algebras $\pi : A \sqcup B \to C$. Define the bi-functor $\otimes'$ on $\mathrm{Prob}(R)$ by, for two objects A and $B \in \mathrm{Prob}(R)$:

$$A \otimes' B = (A \sqcup B, \tau_{A\otimes B} \circ \pi) \tag{1.4}$$

and for two morphisms $f : A_1 \to A_2$, $g : B_1 \to B_2$, $\otimes'(f,g) = f \sqcup g$. We prove that $\otimes'$ is a tensor product with injections on Prob. One fact needs to be proved. Let $(A_i, \tau_{A_i}), (B_i, \tau_{B_i}) \in \{1,2\}$ objects of Prob, $f : (A_1, \tau_{A_1}) \to (B_1, \tau_{B_1})$ and $g : (A_2, \tau_{A_2}) \to (B_2, \tau_{B_2})$ two morphisms. First, we prove that $\tau_{A_2\otimes B_2} \circ (f \sqcup g) = \tau_{A_1\otimes B_1}$. To that end, we draw the commutative diagram in Fig. 21, blue arrows are morphisms of the category Prob, while black arrows are morphisms in the category $\mathrm{Alg}^\star$. In Fig. 21, the morphisms $\iota_{A_i} : A_i \to A_i \sqcup B_i$ and $\iota_{B_i} : B_i \to A_i \sqcup B_i$ are drawn

$$\begin{tikzcd}
 & (A_1 \otimes B_1, \tau_{A_1\otimes B_1}) \arrow[ddd, bend right=60] & \\
(A_1, \tau_{A_1}) \arrow[ur] \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowleft"] & (A_1 \sqcup B_1, \tau_{A_1\otimes B_1} \circ \pi) \arrow[u] \arrow[d, "f \sqcup g"] & (B_1, \tau_{B_1}) \arrow[ul] \arrow[l] \arrow[d, "g"] \arrow[dl, phantom, "\circlearrowleft"] \\
(A_2, \tau_{A_2}) \arrow[r] \arrow[dr] & (A_2 \sqcup B_2, \tau_{A_2\otimes B_2} \circ \pi) \arrow[d] & (B_2, \tau_{B_2}) \arrow[l] \arrow[dl] \\
 & (A_2 \otimes B_2, \tau_{A_2\otimes B_2}) &
\end{tikzcd}$$

Figure 21.

in blue. In fact, it is easily seen that preserving the trace for these morphisms is equivalent to commutativity of the four triangles in Fig. 21. To show that the free product $f \sqcup q$ preserves the trace, it is enough to shows the commutativity of the two outer faces, the ones bordered with blue arrows. The commutativity of theses two faces is implied by the universal property satisfied by $\otimes$. In conclusion, the arrow $f \sqcup g$ is equal to a composition of blue arrow and is thus trace preserving. Assume that $A_2 = B_2$. If the two morphisms f and g are $\otimes$-independent then they are also $f \otimes' f$-independent. Hence, in the sequel, there is no loss in replacing the tensor product $\otimes$ by $\otimes'$ and thus assuming that the underlying involutive algebra of the tensor product of two probability spaces is the free product of the algebras. We provide examples of monoidal structures with inclusions on the categories of probability spaces ($R = \mathbb{C}$) and amalgamated probability spaces, see [29] for a detailed overview.

The category $\mathrm{Prob}(\mathbb{C})$ of complex probability spaces can be endowed with several monoidal structures, there are three of them that will be interesting for the present work and are denoted by $\hat{\otimes}$, $*$ and $\diamond$. Let (A, E_A) and (B, E_B) two probability spaces; $E_A : A \to \mathbb{C}$ and $E_B : B \to \mathbb{C}$ are two complex positive unital linear forms.

a. The linear for $E_A \hat{\otimes} E_B$ is a linear form on the free product of algebras $A \sqcup B$ and is defined by, for an alternating word $s \in A \sqcup B$,by:

$$(E_A \hat{\otimes} E_B)(s) = E_A(a_1 \cdots a_p) E_B(b_1 \cdots b_m). \tag{1.5}$$

In the last equation, the elements $a_1, \ldots, a_p$ and $b_1, \ldots, b_m$ are indexed according to the order they appear in the word s. The bifunctor $\hat{\otimes}$ that send the pair of probability spaces $((A, E_A), (B, E_B))$ to $A \sqcup B, E_A \hat{\otimes} E_B)$ is a monoidal structure which is, in addition, symmetric. To this monoidal structure is related the notion of *universal tensor independence*. To define $\hat{\otimes}$ we use the free product of star algebras, we could have instead used the tensor product of algebras. In fact, the functor $\otimes$ that sends (A, E_A) and (B, E_B) to $(A \otimes B, E_A \otimes E_B)$ is, first, well defined and is a monoidal structure on $\mathrm{Prob}(\mathbb{C})$. With obvious notations, two morphisms $f : (A, E_A) \to (C, E_C)$ and $g : (B, E_B) \to (C, E_C)$ are $\otimes$ independent if and only if:
   1. $\forall a \in A,\ b \in B,\ [f(a), g(b)] = 0,$
   2. $E_C\Big(f(a_1)g(b_1) \cdots f(a_p)g(b_p)\Big) = E_A\Big(f(a_1 \cdots a_p)\Big) E_B\Big(g(b_1 \cdots b_p)\Big).$

The two morphisms f ang g are $\hat{\otimes}$ independent if only point $a1$ holds. From the explicit expression (1.5), positivity of E_A and E_B implies positivity of $E_A \hat{\otimes} E_B$. Of course, the map $\tau_{A \sqcup B} : A \sqcup B \to B \sqcup A$ equal to the identity on A and B is a state preserving morphism: $\tau_{A \sqcup B} \circ E_B \otimes E_A = E_A \otimes E_B$. In that case, we say that $\hat{\otimes}$ is *symmetric*.

b. The second monoidal structure is related to the notion of *boolean independence*. The state $E_A \diamond E_B$ is defined on the free product $A \sqcup B$ and satisfies:

$$(E_A \diamond E_B)(s_1 \cdots s_p) = E_{\varepsilon_1}(s_1) \cdots E_{\varepsilon_p}(s_p).$$

with $s_1, \ldots, s_p \in \{A, B\}$, $\varepsilon_i = A$ if $s_i \in A$ and $\varepsilon_i = B$ if $s_i \in B$. Since E_A and E_B are unital map, the boolean product is well defined. The bifunctor $\diamond$ that send the pair of probability spaces (A, E_A), (B, E_B) to $(A \sqcup B, E_A \diamond E_B)$ is a symmetric monoidal structural.

c. The third and last monoidal structural we define is related to the notion of *free independence*, at the origin of free probability theory.The free product of E_A and E_B is defined by the following requirement. For all alternating word $s \in A \sqcup B$,

$$(E_A \ \dot{\sqcup}\ E_B)(s) = 0,\ s_i \in \mathsf{ker}(E_{\varepsilon_i}),\ i \le p,\ j \le m. \tag{1.6}$$

A formula for $E_A \ \dot{\sqcup}\ E_B(s)$ can be computed inductively. Computation of the free product of E_A and E_B are simple for words with small lengths,

$$\begin{aligned}
&(E_A \ \dot{\sqcup}\ E_B)(ab) = E_A(a) E_B(b),\\
&(E_1 \ \dot{\sqcup}\ E_B)(a_1 b a_2) = E_A(a_1) E_B(b) E_A(a_2) + E_A(a_1 E_B(b) a_2) + E_A(a_1) E_B(b) E_A(a_2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + 2E_A(a_1) E_B(b) E_A(a_2)
\end{aligned}$$

See [47] and [44]. Let R be an unital associative algebra, we define two monoidal structures on the category of amalgamated probability spaces, $\mathrm{Prob}(R)$. In case R is commutative, we define a monoidal structure on the category $\mathrm{comProb}(R)$ of commutative bimodule algebras. Note that, by definition, the left and right action of R on an probability space in $\mathrm{comProb}(R)$ are equal. Let (A, E_A) and (B, E_B) two probability spaces in $\mathrm{Prob}(R)$.

a. We begin with the amalgamated free product $E_A \ \dot{\sqcup}_R\ E_B$ of E_A and E_B. It is defined by the equation (1.6), for all alternating word $s \in A \sqcup_R B$ in the amalgamated free product $A \sqcup_R B$, $E_A \ \dot{\sqcup}_R\ E_B$ is defined by requiring:

$$(E_A \ \dot{\sqcup}_R\ E_B)(s) = 0,\ s_i \in \mathsf{ker}(E_{\varepsilon_i}),\ i \le p,\ j \le m.$$

b. The amalgamated boolean product $E_A \diamond_R E_B$ of the two R bimodule maps E_A and E_B is a R bimodule map on the amalgamated free product $A \sqcup_R B$ and is defined by the equation:

$$(E_A \diamond_R E_B)(s_1 \cdots s_p) = E_{\varepsilon_1}(s_1) \cdots E_{\varepsilon_p}(s_p)$$

with $s_1, \ldots, s_p \in \{A, B\}$, $\varepsilon_i = A$ if $s_i \in A$ and $\varepsilon_i = B$ if $s_i \in B$.

We defined amalgamated versions of the notion of boolean and free independences, using essentially the same formulae as for the non-amalgamated case. We can not do so for tensor independance, at least for two reasons. First, the amalgamated tensor product $A \otimes_R B$ with bimodule structure given by

$$r(a \otimes_R b)r' = ra \otimes_R br', \ a \in A, \ b \in B, \ r, r' \in R,$$

is not naturally an algebra, meaning that the canonical product $A \otimes B$ does not descend to a product on $A \otimes_R B$. Secondly, formula 1.5 can not be used to define an amalgamated version of tensor independance, since for all a,b $E(a_1 r a_2)E(b_1) \neq E(a_1 a_2)E(rb_1)$, $a_1, a_2 \in A$, $b_1 \in B$ and $r \in R$. Finally, in the last section of the present work we use amagamated probability spaces over commutative algebras R. We mentioned earlier there might be an issue if we try to define independence of more than two morphisms. An additional constraint needs to be satisfied by the monoidal structure so between the natural morphisms ι^1, ι^2 and the unit E of the monoidal structure. This is the content of the next definition.

Definition 1.10. Inclusions ι^1 and ι^2 are called *compatible with unit constraints* if the diagram 22 is a commutative diagram.

$$\begin{tikzcd}
E \otimes A \arrow[dr] & A \arrow[l] \arrow[r] \arrow[d] & A \otimes E \arrow[dl] \\
 & A &
\end{tikzcd}$$

Figure 22. Compatibilty constraint between inclusions and unit

In english, the constraint of Definition 1.10 can be phrased by saying that the inclusions ι^1 and ι^2 and the left and right unit morphisms are, respectively, inverse from each other. The next theorem shows how to produce a monoidal category with inclusions starting from a monoidal category that has an initial object as unit.

Theorem 1.11 (Theorem 3.6 in [31]). *Let $(\mathcal{C}, \otimes, E, \alpha, \ell, r, \iota^1, \iota^2)$ be a monoidal category.*

1. *If ι^1 and ι^2 are inclusions which are compatible with unit constraints, the unit object E is initial, i.e there is an unique morphism $\eta_A : E \to A$ for every object $A \in \mathrm{Obj}(\mathcal{C})$. Furthermore,*

$$(\star) \qquad \iota^1_{A,B} = (\mathrm{id}_A \otimes \eta_B) \circ r_A^{-1}, \quad \iota^2_{A,B} = (\eta_A \otimes \mathrm{id}_B) \circ \ell_B^{-1}.$$

   holds for all objects $A, B, C \in \mathrm{Obj}(\mathcal{C})$.
2. *Suppose that the unit object E is an initial object. Then $\star$ read as a definition yields inclusions ι^1, ι^2 which are compatible with the unit constraints.*

As pointed out in [31], Maclane's coherence theorem can be extended to all diagrams built with the associativity constraints, left and right units, and the natural morphisms η. This extended Maclane coherence theorem implies, in particular, commutativity of the diagrams in Fig. 23 and Fig. 24 in case compatibility with unit of the inclusions is satisfied. In the

$$\begin{tikzcd}
(A \otimes C) \arrow[r, "\iota^1"] \arrow[d, "\mathrm{id}_1 \otimes \iota^2"] & ((A \otimes B) \otimes C) \\
(A \otimes B) \otimes C \arrow[ur, "\alpha_{A,B,C}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[d, "\iota^1"] & B \arrow[l, "\iota^2"] \arrow[r, "\iota^1"] & B \otimes C \arrow[d, "\iota^2"'] \\
(A \otimes B) \otimes C \arrow[rr, "\alpha"] & & A \otimes (B \otimes C)
\end{tikzcd}$$

Figure 23.

sequel, until the end of this chapter, monoidal structures with injections we consider satisfy the compatibility with unit constraints. If $1 \leq i_1 < \cdots < i_k \leq n$ is an increasing k tuple of integers, there exists morphisms $\iota^{i_1,\ldots,i_k;n}_{A_1,\ldots,A_n} : A_{i_1} \otimes \cdots \otimes A_{i_{i_k}} \to A_1 \otimes \cdots A_n$. Grouped, these morphisms induce natural transformations $\iota^{i_1,\ldots,i_k;n}$. We refer to these morphisms as inclusion morphisms.

In the sequel, the symbol $\mathcal{C}$ stands for a monoidal category such that the unit object is initial.

$$\begin{tikzcd} C \arrow[r, "\iota^2"] \arrow[d, "\iota^2"] & B\otimes C \arrow[d, "\iota^2"] \\ (A\otimes B)\otimes C \arrow[r, "\alpha"] & A\otimes(B\otimes C) \end{tikzcd}$$

Figure 24.

Definition 1.12 (Mutual independence, Definition 3.7 in [31]). Let $B_1,\ldots,B_n,A$ be objects in the category $\mathcal{C}$ and $f_i : B_i \to A$ morphisms. The family $(f_1,\ldots,f_n)$ is said to be independent if there exists a morphism $h : B_1 \ldots \cdots \otimes B_n \to A$ such that the following diagram (Fig. 25):

$$\begin{tikzcd} & A \\ B_i \arrow[ur, "f_i"] \arrow[r, "\iota^{i;n}"] & B_1\otimes\cdots\otimes B_n \arrow[u, "h"] \end{tikzcd}$$

Figure 25. Mutual independence of a sequence of morphisms.

As a consequence of the existence of the natural injections $\iota^{i_1,\ldots,i_k}_{A_1,\ldots,A_n}$, a subfamily $(f_{i_1},\ldots,f_{i_k})$ with $(1 \leq i_1 < \cdots < i_k \leq n)$ of a family of independent morphisms $(f_1,\ldots,f_n)$ is an independent family.

1.3.2. Directed set, poset and inductive limits of $C^\star$-algebra. In this section, we briefly recall definitions of a directed set and a poset. As before, R denotes an associative unital algebra. The construction of an inductive limits of R-bi-module involutive algebras play a prominent role for the present work, we thus offer to the reader details on its construction, which otherwise can be found in [48].

An upward directed set is a set S endowed with a pre-order $\leq$, (a reflexive and transitive binary relation) with the additional property that for any pair of elements $s_1 \in S$ and $s_2 \in S$ there exists a third element $s \in S$, called *an upper bound*, such that $s \geq s_1$ and $s \geq s_2$. The notion of downward directed set is obtained by substituting the existence of an upper bound with the existence a lower bound; a third element $s \in S$ such that $s \leq a$ and $s \leq b$.

An upward directed poset is an upward directed set $(S,\leq)$ with $\leq$ satisfying the extra property of being anti-symmetric $a \leq b,\ b \leq a \implies a = b$. As an example, the set of finite sequences of affine loops drawn on the plane equipped with the binary operation:

$$\left(\ell_1,\ldots,\ell_p\right) \leq \left(\ell'_1,\ldots,\ell'_p\right) \Leftrightarrow \{\ell'_1,\ldots,\ell'_p\} \subset \{\ell_1,\ldots,\ell_p\}. \tag{1.7}$$

is an upward directed set. Note that because we consider sequences of loops and not set of loops, $\leq$ is not anti-symmetric. The set of finite sets of loops drawn on the plane is a poset for the binary operation inducs by inclusion.

For the remaining of this section, we closely follow the exposition of Ziro Takeda in its seminal article [48] on inductive limit of $C^\star$-algebra.

Definition 1.13. Let Γ be an increasingly directed set. A direct system in a category $\mathcal{C}$ is the data of a family of objects $\{O_\gamma, \gamma \in \Gamma\}$ in $\mathcal{C}$ and morphisms $f_{\alpha,\beta}$ for all couples (α,β) with $\alpha \leq \beta$ such that:

1. $\forall \alpha \in \Gamma,\ f_{\alpha,\alpha} = \mathrm{id}_{O_\alpha}$
2. $\forall \alpha \leq \beta \leq \gamma,\ f_{\gamma,\beta} \circ f_{\beta,\alpha} = f_{\gamma,\alpha}$

A direct system can alternatively be seen as a functor. In fact, to the upward directed set Γ is associated a category, also denoted Γ, whose class of objects is the set Γ. The set of homomorphisms between two distinct elements $(\alpha \leq \beta)$ contains an unique element and we denote by (α,β) this morphism. With the notations of the last definition, the functor O associated with a direct system is defined by:

$$O(\gamma) = O_\gamma,\ O((\alpha,\beta)) = f_{\beta,\alpha}.$$

For the rest of this section, we fix an increasingly directed set Γ.

Definition 1.14. Let $\mathcal{C}$ a category and let O_γ, $\gamma \in \Gamma$, $\{f_{\alpha,\beta}, \alpha \leq \beta\}$ a direct system of $\mathcal{C}$. An inductive limit is the data of an object O of $\mathcal{C}$ and morphisms $\phi_\gamma : O_\gamma \to O$ such that

1. $\phi_\beta \circ f_{\beta,\alpha} = \phi_\alpha$
2. The following universal property holds. For all objects $Y \in \mathcal{C}$ and morphisms $g_\gamma : O_\gamma \to Y$ there exists a morphism $G : O \to Y$ such that the diagram in Fig. 26 is commutative for all pairs $\alpha \leq \beta$ in Γ.

$$\begin{tikzcd}
O_\alpha \arrow[rr, "f_{\beta,\alpha}"] \arrow[dr, "\phi_\alpha"] \arrow[ddr, "G_\alpha"'] & & O_\beta \arrow[dl, "\phi_\beta"'] \arrow[ddl, "G_\beta"] \\
& O \arrow[d, "G"] & \\
& Y &
\end{tikzcd}$$

Figure 26. Universal property of the inductive limit.

From the universal property satisfied by inductive limits, we see that an inductive limit of a family of objects $\{A_\gamma, \gamma \in \Gamma\}$ is unique up to isomorphism. If any direct family in a category $\mathcal{C}$ admits a direct limit, we say that $\mathcal{C}$ is closed for taking inductive limits, or using the language of category theory that $\mathcal{C}$ is *inductively complete*, see [1]

Let R an unital associative algebra. The following theorem states that the category of R-bimodule involutive algebras is closed for taking inductive limits.

Theorem 1.15. *The categories* $\mathrm{Prob}(R)$, $\mathrm{Alg}^\star(R)$ *and* $\mathrm{biMod}(R)$ *are inductively complete.*

Proof. We prove only that $\mathrm{Prob}(R)$ is closed for taking inductive limits. Let $\mathcal{A}$ be the set of equivalence classes $\{[\big(\gamma, a_\gamma\big)],\ a_\gamma \in A_\gamma,\ \gamma \in \Gamma\}$, with

$$a_\alpha \sim a_\beta \Leftrightarrow \exists \delta \geq \alpha, \beta \text{ such that } f_{\delta,\alpha}(a_\alpha) = f_{\delta,\beta}(a_\beta).$$

Since the maps f_γ $(\gamma \in \Gamma)$ are trace preserving, one has

$$\tau_\delta(f_{\delta,\alpha}(a_\alpha)) = \tau_\delta\left(f_{\delta,\beta}(a_\beta)\right) = \tau_F\left(a_F\right) \text{ for all pairs } (a_\alpha, a_\beta) \text{ with } a_\alpha \sim a_\beta$$

hence, the function $(\gamma, a_\gamma) \to \phi_\gamma(a_\gamma)$ is constant on the classes for $\sim$ and thus descends to a linear form τ on the quotient space $\bigsqcup_F \mathcal{A}_F/\sim$. The algebraic operations on $\mathcal{A}$ are defined as follows

1. Addition : $[x_\alpha] + [x_\beta] = [x_\delta + y_\delta]$, with $\delta \geq \alpha,\ \delta \geq \beta,\ x_\delta = f_{\delta,\alpha}(x_{\alpha)}$ and $y_\delta = f_{\delta,\beta}(x_\beta)$.
2. Multiplication : $[x_\alpha] \cdot [y_\beta] = [x_\delta \cdot y_\delta]$,
3. Star operation : $[x_\alpha]^\star = [x_\alpha^\star]$.
4. To define the R-bimodule structure on $\mathcal{A}$, we simply set:
$$r[x_\alpha]r' = [rx_\alpha r'],\ r, r' \in R.$$
In fact, if $[x_\alpha] = [x_\beta]$ with $\beta \geq \alpha$, then $x_\beta = f_{\beta,\alpha}(x_\alpha)$ and $rx_\beta r' = rf_{\beta,\alpha}(x_\alpha)r' = f(rx_\alpha r')$ for $r, r' \in R$

□

1.4. Zhang algebra holonomy fields

1.4.1. Classical lattice holonomy fields. Before giving the main definition of this paper, (Definition 1.18), we review briefly the classical notion of a holonomy field on a lattice. Part of what we will explain in this section, in particular the content of Section 1.4.2, makes sense on an arbitrary surface, or even on an arbitrary graph, but for the sake of simplicity and concision, we will restrict ourselves to the framework of graphs on the Euclidean plane $\mathbb{R}^2$.

In this paper, all the paths that we will consider will be piecewise affine continuous paths on the Euclidean plane $\mathbb{R}^2$. We denote by $\mathsf{P}(\mathbb{R}^2)$ the set of all these paths. Each path c has a starting point $\underline{c}$, an end point $\overline{c}$, an orientation, but it has no preferred parametrisation. Constant paths are allowed. Given two paths c_1 and c_2 such that c_1 finishes at the starting point of c_2, the concatenation of c_1 and c_2 is defined in the most natural way and denoted by c_1c_2. Reversing the orientation of a path c results in a new path denoted by c^{-1}. A path which finishes at its starting point is called a *loop*. A loop which is as injective as possible, that is, a loop which visits twice its starting point and once every other point of its image, is called a *simple loop*. A path of the form $c\ell c^{-1}$, where c is a path and ℓ is a simple loop, is called a *lasso* (see Fig. 27).

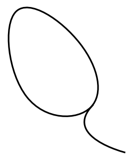


FIGURE 27. A lasso drawn on the plane.

Let us call *edge* a path that is either an injective path (thus with distinct endpoints) or a simple loop. By a *graph* on $\mathbb{R}^2$, we mean a finite set $\mathbb{E}$ of edges with the following properties:

1. for all edge e of $\mathbb{E}$, the edge e^{-1} also belongs to $\mathbb{E}$,
2. any two edges of $\mathbb{E}$ which are not each other's inverse meet, if at all, at some of their endpoints,
3. the union of the ranges of the elements of $\mathbb{E}$ is a connected subset of $\mathbb{R}^2$.

From the set $\mathbb{E}$, we can form the set $\mathbb{V}$ of *vertices* of the graph, which are the endpoints of the elements of $\mathbb{E}$, and the set $\mathbb{F}$ of *faces* of the graph, which are the bounded connected components of the complement of the union of the range of the edges of the graph. Although it is entirely determined by $\mathbb{E}$, it is the triple $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$ that we regard as the graph.

From the graph $\mathbb{G}$, and given a vertex v, we form the set $\mathsf{L}_v(\mathbb{G})$ of all loops based at v that can be obtained by concatenating edges of $\mathbb{G}$. The operation of concatenation makes $\mathsf{L}_v(\mathbb{G})$ a monoid, with the constant loop as unit element. In order to make a group out of this monoid, one introduces the backtracking equivalence of loops and the notion of reduced loops. A loop is *reduced* if in its expression as the concatenation of a sequence of edges (which is unique) one does not find any two consecutive edges of the form ee^{-1}. We denote by $\mathsf{RL}_v(\mathbb{G})$ the subset of $\mathsf{L}_v(\mathbb{G})$ formed by reduced loops. It is however not a submonoid of $\mathsf{L}_v(\mathbb{G})$, for the concatenation of two reduced loops needs not be reduced. The appropriate operation on $\mathsf{RL}_v(\mathbb{G})$ is that of concatenation followed by reduction where, as the name indicates, one concatenates two loops, and then erases sub-loops of the form ee^{-1} until no such loops remain. It is true, although perhaps not entirely obvious, that this operation is well defined, in the sense that the order in which one erases backtracking sub-loops does not affect the final reduced loop.

From the graph $\mathbb{G}$ and the vertex v, we thus built a group $\mathsf{RL}_v(\mathbb{G})$. This group is in fact isomorphic in a very natural way with the fundamental group based at v of the subset of $\mathbb{R}^2$ formed by the union of the edges of $\mathbb{G}$. An important property of the group $\mathsf{RL}_v(\mathbb{G})$ is that it is generated by lassos (see Fig. 28)

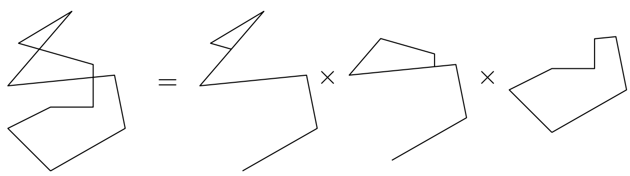


FIGURE 28. Decomposition of a loop into a product of lassos.

In fact, $\mathsf{RL}_v(\mathbb{G})$ is a free group, with rank equal to the number of faces of $\mathbb{G}$, and it admits bases formed by lassos. More precisely, we will use the following description of a basis of this group. Let us say that a lasso $c\ell c^{-1}$ surrounds a face F of the graph if the loop ℓ traces the boundary of F.

PROPOSITION 1.16. *The group $\mathsf{RL}_v(\mathbb{G})$ admits a basis formed by a collection of lassos, each of which surrounds a distinct face of $\mathbb{G}$.*

In fact, the group $\mathsf{RL}_v(\mathbb{G})$ admits many such bases, but it will be enough for us to know that there exists one. A proof of this proposition, and more details about graphs in general, can be found in [40].

A classical lattice gauge field on $\mathbb{G}$ with structure group G is usually described as an element of the configuration space

$$\mathcal{C}_{\mathbb{G}} = \{g = (g_e)_{e\in\mathbb{E}} \in G^{\mathbb{E}} : \forall e \in \mathbb{E}, g_{e^{-1}} = g_e^{-1}\}.$$

This configuration space is acted on by the lattice gauge group $\mathcal{J}_{\mathbb{G}} = G^{\mathbb{V}}$, according to the formula

$$(j \cdot g)_e = j_{\overline{e}}^{-1} g_e j_{\underline{e}}.$$

Let us fix a vertex v of our graph. Any element g of the configuration space $\mathcal{C}_{\mathbb{G}}$ induces a holonomy map $\mathsf{L}_v(\mathbb{G}) \to G$, which to a loop ℓ written as a concatenation of edges $e_1 \dots e_n$ associates the element $g_{e_n} \cdots g_{e_1}$ of G. This map descends to the quotient by the backtracking equivalence relation, and induces a morphism of groups $\mathsf{RL}_v(\mathbb{G}) \to G$. The action of a gauge transformation $j \in \mathcal{J}_{\mathbb{G}}$ on $\mathcal{C}_{\mathbb{G}}$ modifies this morphism by conjugating it by the element j_v. These observations can be turned in the following convenient description of the quotient $\mathcal{C}_{\mathbb{G}}/\mathcal{J}_{\mathbb{G}}$.

Proposition 1.17. *For all $v \in \mathbb{V}$, the holonomy map induces a bijection*

$$\mathcal{C}_{\mathbb{G}}/\mathcal{J}_{\mathbb{G}} \longrightarrow \mathrm{Hom}(\mathsf{RL}_v(\mathbb{G}), G)/G.$$

It follows from this proposition that describing a probability measure on the quotient space $\mathcal{C}_{\mathbb{G}}/\mathcal{J}_{\mathbb{G}}$ is equivalent to describing the distribution of a random group homomorphism from $\mathsf{RL}_v(\mathbb{G})$ to G, provided this random homomorphism invariant under conjugation. Combining this observation with the fact that $\mathsf{RL}_v(\mathbb{G})$ is a free group, and choosing for instance a basis $l_1, \dots, l_n$ formed by lassos surrounding the n faces of $\mathbb{G}$, we see that a probability measure on $\mathcal{C}_{\mathbb{G}}/\mathcal{J}_{\mathbb{G}}$ is the same thing as a G^n-valued random variable $(H_{l_1}, \dots, H_{l_n})$, invariant in distribution under the action of G on G^n by simultaneous conjugation on each factor.

Let us finally introduce some further notation. We denote by $\mathsf{L}_0(\mathbb{R}^2)$ the set of loops on $\mathbb{R}^2$ based at the origin and by $\mathsf{RL}_0(\mathbb{R}^2)$ the group of reduced loops. From now on, we will always assume that 0 is a vertex of all the graphs that we consider.

It is important to observe that any reduced loop on $\mathbb{R}^2$ belongs to $\mathsf{RL}_v(\mathbb{G})$ for some graph $\mathbb{G}$. Thus,

$$\mathsf{RL}_0(\mathbb{R}^2) = \bigcup_{\mathbb{G}\text{ graph}} \mathsf{RL}_0(\mathbb{G})$$

and accordingly,

$$\mathrm{Hom}(\mathsf{RL}_0(\mathbb{R}^2), G) = \varprojlim \mathrm{Hom}(\mathsf{RL}_0(\mathbb{G}), G).$$

1.4.2. Holonomy field on a Zhang algebra. In the following, we set $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let B be a unital algebra over $\mathbb{K}$. Recall from Example 2 page 37 that the category of involutive bimodule algebras over B is denoted $\mathrm{Alg}^\star(R)$.

We fix a R-probability space, that is, a pair $(\mathcal{A}, E)$ where $\mathcal{A}$ is an object of $\mathrm{lCoModAlg}^\star(R)$ and $E : \mathcal{A} \to B$ is a morphism of bimodules. We choose a monoidal structure $\otimes$ on the category Prob(R). At that point, we should make two restrictive assumptions. First, the underlying algebra $(A, \phi_A) \otimes (B, \phi_B)$ is supposed to be the free product $A \sqcup B$ with $A, B \in \mathrm{Prob}(R)$ and secondly, $\otimes$ is assumed to be symmetric with commutativity constraint maps given by $\tau_{A\otimes B} : A \otimes B \to B \otimes A$ (the canonical maps equal to identity on both A and B). As a result of these last two assumptions, any permutation of a sequence of mutually independent morphisms is a sequence of mutually independent morphisms. Also, the independence morphism associated with a sequence of independent morphisms is given by the free product of these morphisms. Finally, we choose a Zhang algebra H of $\mathrm{Alg}^\star(B)$.

Definition 1.18. (Probabilistic holonomy fields on a Zhang algebra) An H-algebraic holonomy field is a group homomorphism $\mathsf{H} : \mathsf{RL}_{\mathrm{Aff},0}(\mathbb{R}^2) \to \mathrm{Hom}_{\mathrm{lCoModAlg}^\star(H)}((H, \Omega_c), \mathcal{A})$ satisfying the following three properties:

(1) *(Gauge invariance)* Let $l_1,\ldots,l_n \in \mathsf{L}_{\mathrm{Aff},0}(\mathbb{R}^2)$ a finite sequence of loops. For all morphism $\phi_H : H \to k$ of $\mathrm{Hom}(\mathcal{D})$:

$$(\phi_H \otimes \phi_{\mathcal{A}}) \circ (\mathrm{id}_B \sqcup \mathsf{H}_{\ell_1,\ldots,\ell_1})) \circ \Omega_c^n = \phi_{\mathcal{A}} \circ \mathsf{H}_{\ell_1,\ldots,\ell_n}.$$

(2) *(Independence)* If $(l_1,\ldots,l_n)$ and $(l'_1,\ldots,l'_m)$ are two finite sequences of loops such that $\bigcup_{i=1}^n \mathrm{Int}(l_i)$ and $\bigcup_{j=1}^m \mathrm{Int}(l'_j)$ are disjoint, then $\mathsf{H}_{l_1} \dot{\sqcup} \ldots \dot{\sqcup} \mathsf{H}_{l_n}$ and $\mathsf{H}_{l'_1} \dot{\sqcup} \ldots \dot{\sqcup} \mathsf{H}_{l'_m}$ are $\otimes$-independent.

(3) *(Invariance by area-preserving homeomorphisms)* For all area-preserving diffeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ and all loops $(l_1,\ldots,l_n)$, we have the equality $E \circ (\mathsf{H}_{l_1} \dot{\sqcup} \ldots \dot{\sqcup} \mathsf{H}_{l_n}) = E \circ (\mathsf{H}_{\phi(l_1)} \dot{\sqcup} \ldots \dot{\sqcup} \mathsf{H}_{\phi(l_n)})$.

This generalisation of the usual notion of a gauge field, although already wide enough to encompass classical gauge fields with compact structure groups as well as their large N limits — the so-called master fields — is set in a context that is less general than our definition of a Zhang algebra. Indeed, in this definition, we work on the particular algebraic category $\mathrm{Alg}^\star(B)$. It is not much more difficult to extend the definition to a setting where the algebraic category in which we take our probability spaces is almost arbitrary.

Let $(\mathcal{C}, \sqcup_{\mathcal{C}}, k)$ be an algebraic category with initial object k and $(\mathcal{D}, \sqcup_{\mathcal{D}}, k)$ a second category and $F : \mathcal{C} \to \mathcal{D}$ a faithfull (injective on the homorphisms classes)and wide (surjective on the class of objects) functor of categories. To put it in words, $\mathcal{D}$ is an enlargement of the category $\mathcal{D}$: the classes of objects are the same whereas the homomorphisms classes between O and O' are larger if these objects belong to $\mathrm{Obj}(\mathcal{D})$. Morphisms of the category $\mathcal{D}$ with target spaces the initial object k are to be seen as distributions or states. In fact, define $\mathcal{B}$ as the category which objects are pairs (O, ϕ_O) with O an object of $\mathcal{C}$ and ϕ a morphism of $\mathcal{D}$ from $F(O)$ to the initial object $k \in \mathcal{D}$. A morphism between two objects (O, ϕ_O) and $(O', \phi_{O'})$ of $\mathcal{B}$ is a morphism $f : O \to O'$ of $\mathcal{C}$ such that $\phi_{O'} \circ F(f) = \phi_O$.

Let $(\mathcal{A}, \phi_A)$ an object of $\mathcal{B}$. Let $B \in \mathcal{B}$. A morphism $f : A \to B$ of the category $\mathcal{C}$ defines a morphism, denoted $\hat{f}$, of the category $\mathcal{B}$ with source $(A, \phi_B \circ F(f))$ and target space (B, ϕ_B).

We assume the category $\mathcal{B}$ to be endowed with a symmetric monoidal structure with injections $(\otimes, E)$ with E an initial object of $\mathcal{B}$. Denote by P_1 the obvious functor from $\mathcal{B}$ to $\mathcal{C}$. We assume this functor to conserve the monoidal structure of $\mathcal{B}$ and the product structure of $\mathcal{C}$:

$$P_1((A, \phi_A) \otimes (B, \phi_B)) = A \sqcup B, A, B \in \mathrm{Obj}(\mathcal{C}).$$

Let $\mathcal{A} = (A, \phi_A) \in \mathcal{B}$ with $\mathcal{A}$ a H-comodule of $\mathcal{C}$, $\mathcal{A} \in \mathrm{lCoMod}\mathcal{C}(H)$ and denote by $\overline{\Omega}$ the coaction. Furthermore, let H a Zhang algebra of $\mathcal{C}$. In the following definition, we use the shorter notation $A_{\ell_1,\ldots,\ell_n}$ for the free products of a sequence $(A_{\ell_1},\ldots,A_{\ell_n})$ of morphisms from H to $\mathcal{A}$.

Definition 1.19. (Categorical holonomy field on a Zhang algebra) An H-categorical holonomy field is a group homomorphism $\mathsf{H} : \mathsf{RL}_{\mathrm{Aff},0}(\mathbb{R}^2) \to \mathrm{Hom}_{\mathrm{lCoMod}\mathcal{C}(H)}((H, \Omega_c), \mathcal{A})$ that satisfies the following three properties:

1. *(Gauge invariance)* Let $\ell_1,\ldots,\ell_n \in \mathsf{L}_{\mathrm{Aff},0}(\mathbb{R}^2)$ a finite sequence of loops. For all morphism $\phi_H : H \to k$ of $\mathrm{Hom}(\mathcal{D})$:

$$(\phi_H \otimes \phi_{\mathcal{A}}) \circ (\mathrm{id}_B \sqcup \mathsf{H}_{\ell_1,\ldots,\ell_1})) \circ \Omega_c^n = \phi_{\mathcal{A}} \circ \mathsf{H}_{\ell_1,\ldots,\ell_n}.$$

2. *(Independence)* If $(\ell_1,\ldots,\ell_n)$ and $(\ell'_1,\ldots,\ell'_m)$ are two finite sequences of loops such that $\bigcup_{i=1}^n \mathrm{Int}(l_i)$ and $\bigcup_{j=1}^m \mathrm{Int}(\ell'_j)$ are disjoint, then $\mathsf{H}_{l_1} \dot{\sqcup} \ldots \dot{\sqcup} \mathsf{H}_{\ell_n}$ and $\mathsf{H}_{\ell'_1} \dot{\sqcup} \ldots \dot{\sqcup} \mathsf{H}_{\ell'_m}$ are $\otimes$-independent.
3. *(Invariance by area-preserving homeomorphisms)* For all area-preserving diffeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ and all loops $(\ell_1,\ldots,\ell_n)$, we have the equality $\phi_{\mathcal{A}} \circ F(\mathsf{H}_{\ell_1} \dot{\sqcup} \ldots \dot{\sqcup} \mathsf{H}_{\ell_n}) = \phi_{\mathcal{A}} \circ F(\mathsf{H}_{\phi(\ell_1)} \dot{\sqcup} \ldots \dot{\sqcup} \mathsf{H}_{\phi(\ell_n)})$.

In the next section we construct a categorical holonomy field that satisfies a strengthened gauge invariance property; in fact the left comodule $\mathcal{A}$ is built as an inductive limit and the morphism $\phi_{\mathcal{A}}$ is gauge invariant, which means

$$(\phi_H \otimes \phi_{\mathcal{A}}) \circ \overline{\Omega}_c = \phi_{\mathcal{A}}, \text{ for all } \phi_H \in \mathrm{Hom}_D(H, k).$$

This last equation trivially implies gauge invariance property 1.

In Definition 1.19, we use the co-product structure $\sqcup$ of the category $\mathcal{C}$ to state what gauge invariance and invariance by area preserving homeomorphisms of the plane means. This is not compulsory as we should see now and further in the next section on the course of defining a categorical holonomy field.

Let $\ell_1,\ldots,\ell_p$ a finite sequence of loops and $c_1,\ldots,c_n$ a family of lassos that have disjoint bulks and such that $\ell_1,\ldots,\ell_n \in \mathsf{RL}_{\mathrm{Aff},0}(c_1,\ldots,c_n)$, we will see in the next section how such family is obtained. We claim that property 1 for the sequence $(\ell_1,\ldots,\ell_n)$ can be obtained from the fact that **??** holds for the sequence of lassos $(c_1,\ldots,c_n)$.

In fact, let w a word in M_n. We denote by m^w the unique morphism from the free product $H_w = H_{w_1} \sqcup \cdots \sqcup H_{w_n}$ to $H_1 \sqcup \cdots \sqcup H_n$ such that $m^w \circ (\iota^{H_w}_{H_{w_k}} = \iota^{H_1\cdots H_n}_{H_{w_k}})$.

$$\begin{aligned}(\phi_H \otimes \phi_{\mathcal{A}}) &\circ (\mathrm{id}_H \sqcup (\mathsf{H}_{w_1(c)} \dot\sqcup \mathsf{H}_{w_2(c)} \dot\sqcup \cdots \dot\sqcup \mathsf{H}_{w_p(c)})) \circ \Omega \\ &= (\phi_H \otimes \phi_{\mathcal{A}}) \circ (\mathrm{id}_H \sqcup (\mathsf{H}_{c_1} \dot\sqcup \cdots \dot\sqcup \mathsf{H}_{c_n})) \circ (\mathrm{id}_H \sqcup m^{w_1\cdots w_p}) \circ \Omega_c^{p_1+\cdots+p_n} \circ \Delta^{p_1+\cdots+p_n-1} \\ &= (\phi_H \otimes \phi_{\mathcal{A}}) \circ (\mathrm{id}_H \sqcup (\mathsf{H}_{c_1} \dot\sqcup \cdots \dot\sqcup \mathsf{H}_{c_n})) \circ \Omega_c^{p_1+\cdots+p_n} \circ m^{w_1\cdots w_p} \circ \Delta^{p_1+\cdots+p_n-1} \\ &= \phi_{\mathcal{A}} \circ \mathsf{H}_{c_1} \dot\sqcup \cdots \dot\sqcup \mathsf{H}_{c_n} \circ m^{w_1\cdots w_p} \circ \Delta^{p_1+\cdots+p_n-1} = \phi_{\mathcal{A}} \circ (\mathsf{H}_{\ell_1} \dot\sqcup \cdots \dot\sqcup \mathsf{H}_{\ell_p}).\end{aligned}$$

We leave to the reader the verification that invariance by are-preserving homeomorphisms property 3 and independence property 2 hold for any sequence of affine loops if it holds for all sequence of affine lassos. In short, properties 1. – 3. of Definition 1.19 are equivalent to:

1′. (Independence) If $(c_1,\ldots,c_n)$ is a finite sequences of lassos with two by two disjoint bulks, $\mathsf{H}_{c_1},\ldots,\mathsf{H}_{c_n}$ is a mutually independent family of morphisms.

2′. (Gauge invariance) Let $c_1,\ldots,c_n \in \mathsf{L}_{\mathrm{Aff},0}(\mathbb{R}^2)$ a finite sequence of lassos with disjoint bulks. For all morphism $\phi_H : H \to k$ of $\mathrm{Hom}(\mathcal{D})$:

$$(\phi_H \otimes \phi_{\mathcal{A}}) \circ (\mathrm{id}_B \otimes \mathsf{H}_{c_1} \otimes \ldots \otimes \mathsf{H}_{c_n}) \circ \Omega_c^n = \phi_{\mathcal{A}} \circ (\mathsf{H}_{c_1} \otimes \ldots \otimes H_{c_n}).$$

3′. (Invariance by area-preserving homeomorphisms) For all area-preserving diffeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ and all lassos with disjoint interiors $(c_1,\ldots,c_n)$, we have the equality $\phi_{\mathcal{A}} \circ F(\mathsf{H}_{c_1} \otimes \ldots \otimes \mathsf{H}_{c_n}) = \phi_{\mathcal{A}} \circ F(\mathsf{H}_{\phi(c_1)} \otimes \ldots \otimes \mathsf{H}_{\phi(c_n)})$.

The distribution of a categorical holonomy field H is the collection $\{\Phi^{\mathsf{H}}_\ell,\ \ell \in \mathsf{RL}_{\mathrm{Aff},0}(\mathbb{R}^2)\}$ of morphisms from H to k defined by:

$$\Phi^{\mathsf{H}}_\ell = \tau \circ \mathsf{H}_\ell,\ \ell \in \mathsf{RL}_{\mathrm{Aff},0}(\mathbb{R}^2). \tag{1.8}$$

We denote by $\star_\otimes$ the product on the space homomorphisms from H to k in the category $\mathcal{D}$ an defined by:

$$\alpha \star_\otimes \beta = (\alpha \otimes \beta) \circ \Delta,\ \alpha, \beta \in \mathrm{Hom}_{\mathcal{D}}(H,k).$$

Properties 1.–3. of Definition 1.19 for the categorical holonomy field implies the following ones for its distribution:

1. $\Phi_{\ell_1\ell_2\ell_1^{-1}} = \Phi_{\ell_2},\ \Phi_{\ell_1^{-1}} = \Phi_{\ell_1} \circ S$ for all $\ell_1,\ell_2 \in \mathsf{RL}_{\mathrm{Aff},0}(\mathbb{R}^2)$,
2. $\Phi_{\ell_1\ell_2} = \Phi_{\ell_1} \star_\otimes \Phi_{\ell_2}$, for all simple loops $\ell_1,\ell_2 \in \mathsf{RL}_{\mathrm{Aff},0}(\mathbb{R}^2)$ with disjoint interiors,
3. $\Phi_\ell = \Phi_{\psi(\ell)}$ for all area preserving homeomorphisms ψ of $\mathbb{R}^2$.

Let us now draw a comparison between the definition of a categorical Lévy process as given in [**31**] and Definition 1.19. Let γ and γ_1 be two reduced loops. We write $\gamma \prec \gamma_1$ if $\gamma_1 = \gamma\ell$ for a certain loop ℓ such that closure of interiors of the loops ℓ and γ meet only at the origin. The relation $\prec$ is a preorder:

1. It is transitive: if $\gamma \prec \gamma_1,\ \gamma_1 \prec \gamma_2$ then $\gamma_2 = \gamma_1\ell = \gamma\ell\tilde{\ell}$ and $\overline{\mathrm{Int}}(\ell) \cap \overline{\mathrm{Int}}(\gamma_1) = \emptyset$, $\overline{\mathrm{Int}}(\tilde{\ell}) \cap \overline{\mathrm{Int}}(\gamma) = \emptyset$, we obtain $\overline{\mathrm{Int}}(\gamma) \cap \overline{\mathrm{Int}}(\ell\tilde{\ell}) = \emptyset$,
2. It is reflexive.

Let H an categorical holonomy field. For all loop $\gamma \in \mathsf{RL}_{\mathrm{Aff},0}$, set $H_\gamma = (H, \phi_A \circ \mathsf{Hol}_\gamma)$. The group $\mathsf{RL}_{\mathrm{Aff},0}$ is seen as category with class of objects the set of points $\mathsf{RL}_{\mathrm{Aff},0}$ and empty morphisms sets between two different loops. The system formed by the functor $\mathsf{L} : \mathsf{RL}_{\mathrm{Aff},0} \to \mathcal{B}$, $\ell \mapsto H_\ell$ and the maps:

$$\Delta : H_{\gamma\gamma^1} \to H_\gamma \otimes H_{\gamma^1},\ h \mapsto \Delta(h),\ \delta : H \to k,\ h \mapsto \varepsilon(h)$$

is a comonoidal system. In addition, if we define $\mathsf{j}^{\mathsf{H}}_{\gamma,\gamma_1} : H_{\gamma^{-1}\gamma_1} \to (\mathcal{A}, \phi_{\mathcal{A}})$ with $\gamma \prec \gamma_1$ by $\mathsf{j}^{\mathsf{Hol}}_{\gamma,\gamma_1} = \mathsf{H}_{\gamma^{-1}\gamma_1}$, then $\mathsf{j}^{\mathsf{Hol}}$ is a categorical Lévy process:

1. $j_{\gamma,\gamma} = \eta_H \circ \delta$,
2. $\left(j_{\alpha_1,\beta_1}, \dots, j_{\alpha_p,\beta_p}\right)$, $\alpha_p \leq \beta_p$
3. $j^{\mathsf{Hol}}_{\gamma,\gamma_1} \otimes j^{\mathsf{Hol}}_{\gamma_1,\gamma_2} \circ \Delta_{\gamma^{-1}\gamma_1,\ \gamma_1^{-1}\gamma_2} = j^{\mathsf{Hol}}_{\gamma,\gamma_2}$.

1.4.3. Construction of a categorical holonomy field from a direct system. In that section, we explain how to construct a categorical holonomy field. Of course, we need initial datas that will be of two kinds. Our exposition is divided into three steps. First, we show how to construct a morphism H from the group of reduced loops $\mathsf{RL}_{\mathrm{Aff},0}$ into the group of homomorphisms from a Zhang algebra in $\mathcal{C}$ to a certain object of the category $\mathrm{lCoMod}\mathcal{C}(H)$, starting from a direct system of objects and a projective system of morphisms. We continue our exposition by showing how to construct a categorical holonomy field that have property 2, 1 and 3 of Definition 1.19.

Our settings is the one of Definition 1.19. In addition to the hypothesis on the three categories $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ we made, we add two more of them:

1. The categories $\mathcal{C}$ and $\mathcal{D}$ are inductively complete.
2. The tensor product $\otimes$ on $\mathcal{B}$ is symmetric and we denote by τ the commutativity constraint:

$$\tau_{A,B} : (A \otimes B, \tau_{A\otimes B}) \to (B \otimes A, \tau_{B\otimes A}),\ \tau_{B,A} \circ \tau_{A,B} = \mathrm{id}_{A\otimes B},\ \tau_{A,B} \circ \tau_{B,A} = \mathrm{id}_{B\otimes A}.$$

The set of finite sequences of loops is an upward directed set as we saw in the previous section. We recall that $\mathcal{P}\left(\mathsf{L}_{\mathrm{Aff},0}\left(\mathbb{R}^2\right)\right)$ denotes the set of finite sequences of affine loops. For a finite sequence of loops $F = \left(l_1, \dots, l_p\right)$, we use the short notation $\{F\}$ for the set $\{l_1, \dots, l_p\}$.

We focus now on the construction of the morphism H of definition 1.19, regardless of the property $1-3$. We fix a direct system A of the category $\mathcal{B}$. It means that

1. $\mathsf{A}((L', L)) : \mathsf{A}(L) \to \mathsf{A}(L')$ satisfies $E_{\mathsf{A}(L')} = E_{\mathsf{A}(L)} \circ \mathsf{A}((L', L))$ for all finite sequences of loops $L' \succ L$,
2. $\mathsf{A}((L'', L')) \circ \mathsf{A}((L', L)) = A((L'', L))$ for all finite sequences of loops L, L', L'',
3. $\mathsf{A}((L, L)) = \mathrm{id}_L$ for all finite sequence $L \in \mathcal{P}\left(\mathsf{L}_{\mathrm{Aff},0}\left((\mathbb{R}^2\right)\right)$.

Below, we explain how to construct a direct system $\mathsf{A} : \mathcal{P}\left(\mathsf{L}_{\mathrm{Aff},0}\left(\mathbb{R}^2\right)\right) \to \mathcal{B}$ starting from a direct system from a subcategory $S \subset \mathcal{P}\left(\mathsf{L}_{\mathrm{Aff},0}\left(\mathbb{R}^2\right)\right)$ to $\mathcal{B}$ by using the monoidal structure of $\mathcal{B}$.

We denote by $((\mathcal{A}, E), j)$ the inductive limit of the direct system A in the category $\mathcal{B}$.

Let L and L' two finite sequences of affine loops of length $p \geq 1$ such that $\{L\} = \{L'\}$. The group of permutations of $[1, \dots, p]$ is denoted $\mathcal{S}_p$. We recall that a group G defines a category whose objects are the points of G and the space of morphisms $\mathsf{Hom}(g, h)$ between two different elements g, h in G has an unique element: $L_{gh^{-1}}$ with $L_{gh^{-1}} : G \to G$ the left multiplication by gh^{-1}. In the following definition, $\mathcal{S}_p$ is seen as a category.

Definition 1.20. We call *commutativity constraint* of the direct system A the functors γ^L_{A}, $L \in \mathcal{P}\left(\mathsf{L}_{\mathrm{Aff},0}\left(\mathbb{R}^2\right)\right)$ defined by, for all permutations $\sigma, \tau \in \mathcal{S}_p$ and $L \in \mathsf{L}_{\mathrm{Aff},0}$:

$$\gamma^L_{\mathsf{A}}(\sigma) = \left(\mathsf{A}(\sigma \cdot L), E_{\mathsf{A}(L)}\right),\quad \sigma \cdot \tau^{-1} \in \mathsf{Hom}(\tau, \sigma),\ \gamma^F_{\mathsf{A}}(\sigma \cdot \tau^{-1}) = \mathsf{A}((\sigma \cdot L, \tau \cdot L)). \tag{1.9}$$

Owing to properties 2. and 3. satisfied by the family of morphisms $\{\mathsf{A}((L, L')), L \prec L' \in \mathcal{P}\left(\mathsf{L}_{\mathrm{Aff},0}\left(\mathbb{R}^2\right)\right)\}$, the equation (1.9) does define a co-variant functor from S_p to $\mathcal{B}$.

For a finite sequence of loops L, we denote by $\mathsf{RL}\langle L \rangle \subset \mathsf{RL}(\mathbb{R}^2)$ the subgroups of reduced loops that are concatenation (and reduction) of loops in L. For two finite sequences of loops

$L < L'$, we define the map $\phi_{L'L}$ as:

$$\phi_{L'L}: \begin{array}{ccc} \mathsf{RL}\langle L\rangle & \to & \mathsf{RL}\langle L'\rangle \\ \ell & \mapsto & \ell \end{array}.$$

The functor $\mathsf{L} : \mathcal{P}\Big(\mathsf{L}_{\mathrm{Aff},0}(\mathbb{R}^2)\Big) \to \mathcal{G}rp$ defined by: $\mathsf{L}(L) = \mathsf{RL}\langle L\rangle$ and $\mathsf{L}((L,L')) = \phi_{L',L}$ is a direct system. The family of morphisms $(\phi_{L',L})_{L<L'\in\mathcal{P}_{\mathrm{Aff},0}(\mathbb{R}^2)}$ enjoys the trivials, yet important, two following properties. Let L, L' be two finite sequences of affine loops with $L < L'$, then:

$$\phi_{L,L'} = \phi_{\alpha\cdot L,\beta\cdot L'},\ \alpha \in \mathcal{S}_{\sharp L},\ \beta \in S_{\sharp L'}.$$

Also, if $L_1 < M_1$ and $L_2 < M_2$ are four finite sequences of affine loops, one has:

$$\phi_{M_1,L_1}(\ell) = \phi_{M_2,L_2}(\ell),\ \ell \in \mathsf{RL}\langle L_1\rangle \cap \mathsf{RL}\langle L_2\rangle.$$

We remind the reader with the fact that the set of affine loops drawn on the plane is the direct limit of the direct functor L:

$$\mathsf{RL}_{\mathrm{Aff},0}\left(\mathbb{R}^2\right) = \overleftarrow{\lim}\ \mathsf{L}.$$

We explain how to construct the morphism H from definition 1.19 starting from a family $\{\mathsf{H}_L, L \in \mathcal{P}\Big(\mathsf{L}_{\mathrm{Aff},0}\left(\mathbb{R}^2\right)\Big)\}$ of homomorphisms of $\mathcal{C}$ with, for each finite sequence of loops $L \in \mathcal{P}\Big(\mathsf{L}_{\mathrm{Aff},0}\left(\mathbb{R}^2\right)\Big)$, $\mathsf{H}_L \in \mathrm{Hom}_{\mathcal{C}}(\mathsf{RL}\langle L\rangle, \mathrm{Hom}_{\mathcal{C}}(H, \mathsf{A}(L)))$.

Set $\overline{\mathsf{H}}_L = j_L \circ \mathsf{H}_L \ \Big(L \in \mathcal{P}\Big(\mathsf{L}_{\mathrm{Aff},0}\left(\mathbb{R}^2\right)\Big)\Big)$ and assume that the following compatibility relation holds:

$$\overline{\mathsf{H}}_L = \overline{\mathsf{H}}_{L'} \circ \phi_{L',L} \tag{1.10}$$

Note that this last relation is implied by the following one on the family $\{\mathsf{H}_L, L \in \mathcal{P}\Big(\mathsf{L}_{\mathrm{Aff},0}\left(\mathbb{R}^2\right)\Big)\}$:

$$\mathsf{A}((L',L)) \circ \mathsf{H}_L = \mathsf{H}_{L'} \circ \mathsf{L}((L',L)),\ L, L' \in \mathcal{P}\Big(\mathsf{L}_{\mathrm{Aff},0}(\mathbb{R}^2)\Big).$$

From the universal property of the direct limit of L and equation (1.10), there exists a morphism H from $\mathsf{RL}(\mathbb{R}^2)$ into $\mathrm{Hom}_{\mathcal{C}}(H,\mathcal{A})$ such that the diagram in Fig. 29 is commutative. That it: from the two intial datas of the direct system A and projective family H we constructed

$$\begin{tikzcd}
\mathsf{RL}(L) \arrow[rr] \arrow[dr] \arrow[ddr, "\overline{\mathsf{H}}_L"'] & & \mathsf{RL}(L') \arrow[dl] \arrow[ddl, "\overline{\mathsf{H}}_{L'}"] \\
& \mathsf{RL}(\mathbb{R}^2) \arrow[d, "\mathsf{H}"] & \\
& \mathrm{Hom}_{\mathcal{C}}(H,\mathcal{A}) &
\end{tikzcd}$$

FIGURE 29. The holonomy H obtained as a solution of an universal problem.

a morphism $\mathsf{H} \in \mathrm{Hom}_{\mathcal{G}rp}(\mathsf{RL}(\mathbb{R}^2), \mathrm{Hom}(H,\mathcal{A}))$. Now, we focus on the question of constructing a direct system A such that property 2 of Definition 1.19 holds for the morphism H constructed as above.

The set, denoted lassos, of anti-clockwise oriented lassos drawn on the plane has the property that for a finite sequence F of loops drawn on the plane, there exists a set $S_F \subset \mathsf{lassos}$ such that $\mathsf{RL}(S_F) = \mathsf{RL}(F)$. In fact, two loops in F have finite self intersections, one can thus built a graph out of the set of loops F : the vertices are the intersection points and the oriented edges are the segments of the loops in F that connect two intersection points. We denote by $\mathbb{G}_F$ that graph. One can pick for S_F a set of lassos starting at the origin and surrounding, one time and in anticlockwise manner, a face of that graph and each face is the bulk of a lasso in S_F. Note that lassos in S_F have disjoint bulks.

For each anti-clockwise oriented lasso c, let E_c a morphism in the category $\mathcal{D}$ from $F(H)$ to the initial object k and set $H_c = (H, E_c) \in \mathcal{B}$. Furthermore, let $\left(c_1, \ldots, c_p\right)$ a finite sequence of lassos in $\overset{\circ}{\mathcal{P}}(\text{lassos})$ and define an object $\left(H_{(c_1,\ldots,c_p)}, E_{(c_1,\ldots,c_p)}\right)$ of the category $\mathcal{B}$ by:

$$(H_{(c_1,\ldots,c_p)}, E_{(c_1,\ldots,c_p)}) = (H_{c_1}, E_{c_1}) \otimes \cdots \otimes (H_{c_p}, E_{c_p}) = (H_{c_1} \sqcup \cdots \sqcup H_{c_p}, E_{c_1} \otimes \cdots \otimes E_{c_p}). \tag{1.11}$$

For each integer $1 \leq i \leq p$, denote by $\iota_{c_i}^{(c_1,\ldots,c_p)}$ the canonical morphism from c_i to the tensor product $\otimes$ on $\mathcal{B}$ (we use the same style of notation, the greek letter ι for these morphisms because they are equal to the canonical injections associated with the co-product on $\mathcal{C}$) and define the morphism $\mathsf{H}_{(c_1,\ldots,c_p)} : \mathsf{RL}\langle(c_1, \ldots, c_p)\rangle \to \mathrm{Hom}_{\mathcal{C}}(H, H_{(c_1,\ldots,c_p)})$ by:

$$\mathsf{H}_{(c_1,\ldots,c_p)} = i_{c_1}^{(c_1,\ldots,c_p)} \times \cdots \times i_{c_p}^{(c_1,\ldots,c_p)}. \tag{1.12}$$

In the last equation, the symbol $\times$ denotes products of morphisms in the group $H_{(c_1,\ldots,c_p)}$,

$$A \times B = A \mathbin{\dot\sqcup} B \circ \Delta, \; A, B \in \mathrm{Hom}_{\mathcal{G}rp}(H, H_{(c_1,\ldots,c_p)}).$$

In the sequel, if $(l_1, \ldots, l_q)$ is a finite sequence of loops in $\mathsf{RL}\langle(c_1, \ldots, c_p)\rangle$, we use the shorter notation $\mathsf{H}_{c_1,\ldots,c_p}(\ell_1, \ldots, \ell_p) : H_{\ell_1} \sqcup \cdots \sqcup H_{\ell_p} \to H_{(c_1,\ldots,c_p)}$ with

$$\mathsf{H}_{c_1,\ldots,c_p}(\ell_1, \ldots, \ell_p) = \mathsf{H}_{c_1,\ldots,c_p}(\ell_1) \mathbin{\dot\sqcup} \cdots \mathbin{\dot\sqcup} \mathsf{H}_{c_1,\cdots,c_p}(\ell_p).$$

The morphism H is the holonomy morphism of the family $\ell_1, \ldots, \ell_p$ in $(c_1, \ldots, c_p)$.

At that point, starting from a family of morphisms E indexed by anticlockwise lassos drawn on the plane, we constructed a family of objects in $\mathcal{B}$ indexed by finite sequences of lassos and a family H of morphisms.

We set $E^{c_1,\ldots,c_p}_{(\ell_1,\ldots,\ell_q)} = E_{(c_1,\ldots,c_p)} \circ \mathsf{H}_{(c_1,\ldots,c_p)}(\ell_1, \ldots, \ell_p)$. In the classical case, the morphism $E^{(c_1,\ldots,c_p)}_{\ell_1,\ldots,\ell_p}$ is the distribution of the holonomies of loops $\ell_1, \ldots, \ell_p$ in $(c_1, \ldots, c_p)$. We emphasized the dependence of $E^{(c_1,\ldots,c_p)}_{(\ell_1,\ldots,\ell_q)}$ toward the set of lassos we picked initially, otherwise stated, toward the basis of the group of reduced loops $\mathsf{RL}\langle(c_1, \ldots, c_p)\rangle$.

In the next part, we are concerned with making the morphisms$E^{(c_1,\ldots,c_p)}_{(\ell_1,\ldots,\ell_p)}$ independent from $(c_1, \ldots, c_p)$, with obvious notations. To that end, we need to assume two more properties on the morphisms E_c, $c \in \text{lassos}$.

To construct, for each finite sequence $F \in \mathcal{P}\left(\mathsf{L}_{\mathrm{Aff},0}\left(\mathbb{R}^2\right)\right)$, an object H_F of $\mathcal{B}$, we pick first an enumeration $(f_1, \ldots, f_p)$ of the faces of the graph $\mathbb{G}_F$ and anticlockwise oriented lassos $(c_1, \ldots, c_p)$ based at 0 that surround the faces $f_1, \cdots, f_p$. It would be natural to define:

$$(H_F, H_F) = \left(H_{\ell_1} \sqcup \cdots \sqcup H_{\ell_p},\; E_F^{c_1,\ldots,c_p}\right)$$

In order for the state $E_F^{(c_1,\ldots,c_p)}$ to not depend on a choice of a basis for $\mathsf{RL}(\mathbb{G}_F)$, the morphisms $(E_c)_{c \in \text{lassos}(\mathbb{R}^2)}$ has to satisfy invariance properties that are stated in the next definition. We should first, motivate further their introduction. Two sequences of lassos with disjoint interiors that generate the same group of reduced loops can be characterized as follows. To a braid $\beta \in \mathcal{B}_n$, is associated a permutation σ_β, it is defined on an elementary braid $\langle i,j\rangle$ as $\sigma_{\langle i,j\rangle} = (i,j)$ for all $i < j \leq n$ and extended to $\mathcal{B}_n$ as a morphism from $\mathcal{B}_n$ to $\mathcal{S}_n$.

Proposition 1.21. *Let $(c_1, \ldots, c_n)$ and $\left(c'_1, \ldots, c'_p\right)$ be two families of lassos. For each family, we assume that the interiors of the bulks are pairwise distinct and that $\mathsf{RL}\langle c_1, \ldots, c_n\rangle = \mathsf{RL}\langle c'_1, \ldots, c'_p\rangle$. Then $n = p$ and it exists a braid $\beta \in \mathcal{B}_n$ such that*

$$\sigma_\beta \cdot (c'_1, \cdots, c'_n) = \beta \cdot (c_1, \cdots, c_n).$$

Proof. The proposition is a consequence of a theorem of Artin, see for example, [4]. □

This last proposition motivates the first point of Definition 1.22. For the second point, we consider a simple situation. Let c be a lasso that can be written as the product of two lassos $c = c_1 \cdot c_p$. It is natural to ask for the morphism E_c to be equal to the holonomie's distribution of

c in c_1, c_p (once again in comparison with the classical case), i.e $E_c = E^{(c)}_{(c_1,c_p)}$. This is the meaning of equation (1.14) in definition 1.22.

Definition 1.22. (1) The family $(H_c, E_c)_{c\in \mathsf{lassos}(\mathbb{R}^2)}$ is said to be *braid invariant* if for any sequence $\left(c_1,\ldots,c_p\right)$ of lassos with disjoint interiors, one has

(1.13) $$E_{\beta\cdot c} = E_{c_{\beta(1)},\ldots,c_{\beta(n)}} \circ \mathsf{H}_{\sigma_\beta\cdot c}(\beta\cdot c), \quad \text{for any braid } \beta\in\mathcal{B}_n.$$

(2) With the notations above, a family of morphisms $(E_c)_{c\in\mathsf{lassos}(\mathbb{R}^2)}$ on H is *infinitely divisible* if for any pair of lassos (c_1,c_2) with disjoint interiors, we have:

(1.14) $$E_{c_1\cdot c_2} = E_{c_1,c_2}\circ \mathsf{H}_{(c_1,c_2)}(c_1c_2) \quad \text{if} \quad c_1c_2 \quad \text{is a lasso.}$$

From now on, we assume the family $(H_c,E_c)_{c\in\mathsf{lassos}}$ to be braid invariant and infinitely divisible.

Proposition 1.23. *With the notations introduced so far, let $L=(\ell_1,\ldots,\ell_q)$ a finite sequence of loops. Let $(c_1,\ldots,c_p)$ and $(c'_1,\ldots,c'_q)$ two sequences of lassos in $\mathring{\mathcal{P}}(\mathsf{lassos})$ such that $L\subset \mathsf{RL}\langle(c_1,\ldots,c_p)\rangle\cap \mathsf{RL}\langle(c'_1,\ldots,c'_q)$, then $E_L^{(c_1,\ldots,c_p)} = E_L^{(c'_1,\ldots,c'_q)}$.*

In addition, if $L' > L$ is a second finite sequence of loops, we have $\mathsf{H}_{\ell'_1,\ldots,\ell'_q}(\ell_1,\ldots,\ell_p) : (H_L,E_L)\to (H_{L'},E_{L'})$.

Proof. Let L and L' be two finite sequences of loops, with $L\subset L'$, $L=(l_1,\cdots,l_p)$ and $L' = \left(\ell'_1,\ell'_2,\cdots,\ell'_{p'}\right)$. Let $(c_1,\ldots,c_q)$ be a sequence of lassos in $\mathring{\mathcal{P}}(\mathsf{lassos})$ that is a basis of $\mathsf{RL}(\mathbb{G}_{L'})$. Assume that the first point of proposition 1.23 holds, $E_L = E_L^{(c_1,\ldots,c_q)}$ and $E_{L'} = E_{L'}^{(c_1,\ldots,c_q)}$. We aim at proving the equality:

$$(\star)\qquad E_{L'}^{(c_1,\ldots,c_p)}\circ \mathsf{H}_{L'}(L) = E_L^{(c_1,\ldots,c_p)}.$$

Because of functorial properties of the tensor product on $\mathcal{B}$, $\mathsf{H}_{c_1,\ldots,c_p}(L')\circ\mathsf{H}_{L'}(L) = \mathsf{H}_{c_1,\ldots,c_p}(L)$. Hence, from the definition of the morphism $E_{L'}^{(c_1,\ldots,c_p)}$, equation ($\star$) is equivalent to:

$$E_{c_1,\ldots,c_p}\circ \mathsf{H}_{(c_1,\ldots,c_p)}(L) = E_L^{(c_1,\ldots,c_p)}.$$

We prove now that the morphism $E_L^{c_1,\ldots,c_p}$ does not depends on the sequences of lassos $(c_1,\ldots,c_p)$ we pick, provided we have $L\in \mathsf{RL}\langle(c_1,\ldots,c_q)\rangle$. Let $f_1,\ldots,f_q$ be an enumeration of the faces of $\mathbb{G}_L$ and pick lassos $(c_1,\ldots,c_p)$ that surrounds (resp.) the faces $f_1,\ldots,f_p$. First, since we assumed that the tensor product is symmetric, $E_{c_1,\ldots,c_p}$ does not depends on the enumeration of the faces we chose. Then, if β is a braid,

$$\begin{aligned}
E_L^{\beta\cdot(c_1,\ldots,c_p)} &= E_{\beta\cdot(c_1,\ldots,c_p)}\circ \mathsf{H}_{\beta\cdot(c_1,\ldots,c_p)}(\ell_1,\ldots,\ell_q)\\
&= E_{c_{\beta(1)},\ldots c_{\beta(q)}}\circ \mathsf{H}_{\sigma_\beta\cdot c}(\beta\cdot c)\circ \mathsf{H}_{\beta\cdot c}(\ell_1,\ldots,\ell_p) = E_{c_{\beta(1)},\ldots,c_{\beta(n)}}\circ\mathsf{H}_{\sigma_\beta\cdot c}(\ell_1,\ldots,\ell_q)\\
&= E_L^{c_{\beta(1)},\ldots,c_{\beta(n)}} = E_L^{c_{\beta(1)},\ldots,c_{\beta(n)}} = E_L^{c_1,\ldots,c_n}.
\end{aligned}$$

In conclusion, we can pick any enumeration and basis of $\mathsf{RL}(\mathbb{G}_F)$ to compute $E_L^{(c_1,\ldots,c_p)}$. Let $C' = (c'_1,\ldots,c'_p)$ a finite sequence such that $L\subset \mathsf{RL}\langle(c'_1,\ldots,c'_p)\rangle$.

The graph $\mathbb{G}_{C'}$ is finer than the graph $\mathbb{G}_L$ and can thus be obtained by iterative application of two transformations, starting from the graph $\mathbb{G}_L$:

1. adding a vertex on an edge,
2. connecting two vertices.

Let $\mathbb{G}_L \prec \mathbb{G}_1\prec\cdots\prec\mathbb{G}_n\prec\mathbb{G}_{C'}$ be a sequence of graphs obtained by successive applications of the transformations 1 and 2: $\mathbb{G}_{i+1}$ is obtained from $\mathbb{G}_i$ by one of the transformation 1, 2. Next, we define inductively a sequence of objects $(H^{(i)})_{0\leq i\leq n+1}$ in the category $\mathcal{B}$. We first define inductively a sequence $(c^{(1)},\ldots,c^{(n+1)})$ with $c^{(i)}\in\mathring{\mathcal{P}}(\mathsf{lassos})$ such that for each $i\leq n+1$, $c^{(i)}$ is a basis of $\mathsf{RL}\langle\mathbb{G}_i\rangle$. Put $c^{(n)} = (c'_1,\ldots,c'_p)$. If $\mathbb{G}_{i+1}$ is obtained by adding a vertex to $\mathbb{G}_i$, the groups of loops drawn on $\mathbb{G}^{i+1}$ is equal to the group of loops drawn on $\mathbb{G}^{(i)}$. In that case we set $c^i = c^{i+1}$. If a face f of G_i is cut into two faces $f_1 f_2$ to obtain the graph $\mathbb{G}_{i+1}$, we obtain a basis $c^{(i)}$ of $\mathsf{RL}(\mathbb{G}_i)$ by extracting the lassos in the basis $c^{(i+1)}$ that surround nor the face f_1 nor the face f_2 and doing the product of the two lassos that surround the faces f_1, f_2 (in any order). Next we define, inductively, the sequence of objects in $\mathcal{B}$ by the equations:

$$uH^{(0)} = (H_L, E_L),\ H^{(i+1)} = \left(H_L,\ E_{c^{(i)}} \circ \mathsf{H}_{c^{(i)}}\left(\ell_1,\ldots,\ell_p\right)\right). \tag{1.15}$$

Let $i \leq n$ an integer such that $\mathbb{G}^{(i+1)}$ is obtained by cutting a face of $G^{(i)}$ in two. We claim that the diagram in Fig. 30 is commutative diagram of morphisms in $\mathcal{C}$. In Fig. 30, the blue arrows are morphism of $\mathcal{B}$. The upper arrow in Fig. 30 is thus painted in blue owing to the infinite divisibility of the morphisms E_c, $c \in \mathsf{lassos}$.

$$\begin{tikzcd}
(H_{c^{(i)}}, E_{c^{(i)}}) \arrow[r, "\mathsf{H}_{c^{(i+1)}}(c^{(i)})"] & (H_{c^{(i+1)}}, E_{c^{(i+1)}}) \\
H^{(i)} \arrow[u, "\mathsf{H}_{c^{(i)}}(l_1,\ldots,l_p)"] \arrow[r, "\mathrm{id}"] & H^{(i+1)} \arrow[u, "\mathsf{H}_{c^{(i+1)}}(\ell_1,\ldots,\ell_p)"']
\end{tikzcd}$$

FIGURE 30.

From Fig. 30, the sequence of morphism $(E_i, i \leq n)$ is thus constant which leads readily to the conclusion, since $E_L^{(c_1,\ldots,c_p)} = E_0 = E_{n+1} = E_L^{(c'_1,\ldots,c'_p)}$. $\square$

Proposition 1.23 shows that the functor:

$$\mathsf{A}(L) = (H_L, E_L),\ \mathsf{A}(L', L) = \mathsf{H}_{L'}(L),\ L \prec L' \in \mathcal{P}\left(\mathsf{L}_{\mathrm{Aff},0}\left(\mathbb{R}^2\right)\right)$$

is well defined. We denote by $(\mathcal{A}, E)$ the inductive limit of A in $\mathcal{B}$. Properties satisfied by the tensor product $\otimes$ implies that $\left(\overline{\mathsf{H}}_L, \phi_{L,L'}\right)$ is a projective system. By construction, the property 2 holds for the morphism H constructed as above from A and $\left(\overline{\mathsf{H}}_L, \phi_{L,L'}\right)$. Assume further that each morphism E_c, $c \in \mathsf{lassos}$ is gauge invariant:

$$E_c \circ ((\mathsf{K} \,\dot{\sqcup}\, \mathrm{id}_H) \circ \Omega_c) = E_c,\ c \in \mathsf{lassos}.$$

for all morphism $K : H \to H_c$ such that $\hat{K}$ is independent from id_{H_c}. We prove first that the inductive limit $\mathcal{A}$ can be endowed with a co-action $\overline{\Omega}_c$ of the Zhang algebra H that makes the diagram in Fig. 31 commutative.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\overline{\Omega}_c"] & H \sqcup \mathcal{A} \\
H_L \arrow[u, "j^{\mathcal{A}}_{H_L}"] \arrow[r, "\Omega^L_c"] & H \sqcup H_L \arrow[u, "\mathrm{id}_H \sqcup j^{\mathcal{A}}_{H_L}"']
\end{tikzcd}$$

FIGURE 31.

For all pairs of loops $L \prec L'$, we saw that the holonomy $\mathsf{H}_{L'}(L)$ is gauge covariant, $\Omega^{L'}_c \circ \mathsf{H}_{L'}(L) = \mathrm{id}_H \,\dot{\sqcup}\, \mathsf{H}_{L'}(L) = \Omega^L_c$. Hence, the coaction $\overline{\Omega}_c : \mathcal{A} \to H \sqcup \mathcal{A}$ is defined by setting:

$$\overline{\Omega}_c([X_L]) = ((\mathrm{id}_H \sqcup j^{\mathcal{A}}_{H_L}) \circ \Omega_c)(X_L),\ [X_L] \in \mathcal{A}.$$

The map $\overline{\Omega}_c$ is well defined:

$$\begin{aligned}
((\mathrm{id}_H \sqcup j^{\mathcal{A}}_{H_{L'}}) \circ \Omega^{L'}_c \circ \mathsf{H}_{L'}(L))(X_L) &= (\mathrm{id}_H \sqcup (j_{H_{L'}} \circ \mathsf{H}_{L'}(L))) \circ \Omega^L_c)(E_L) \\
&= ((\mathrm{id}_H \sqcup \iota_{H_L}) \circ \Omega^L_c)(E_L).
\end{aligned}$$

Property 1 is satisfied by any morphism E_L, $L \in \overset{\circ}{\mathcal{P}}(\mathsf{lassos})$, in fact, let $\phi_H : H \to k$ a morphism of $\mathcal{D}$. For all integer $1 \leq k \leq n$,

$$(\phi_H \otimes (E_{c_1} \otimes \cdots \otimes E_{c_n})) \circ (\mathrm{id}_H \sqcup i_k) \circ \Omega_c = E_{c_k} \tag{1.16}$$

Since the morphisms i_k are mutually independent, the morphisms $\iota_{c_i} \circ \Omega_{c_i} : H_{c_i} \to (H, \phi_H) \otimes H_{c_1} \otimes \cdots \otimes H_{c_n}$ are also mutually independent. Hence, owing to equation (1.16),

$$\Omega_c^{(n)} = \iota_{c_1} \circ \Omega_{c_1} \dot\sqcup \cdots \dot\sqcup \iota_{c_n} \circ \Omega_{c_n} \in \mathrm{Hom}_{\mathcal{B}}(H_{c_1} \otimes \cdots \otimes H_{c_n}, (H, \phi_H) \otimes H_{c_1} \cdots \otimes H_{c_n}). \tag{1.17}$$

Gauge invariance of E_L implies gauge invariance of the morphism E with respect to $\overline{\Omega}_c$ in a straightforward manner.

Invariance by are preserving homomorphisms of $\mathbb{R}^2$ is implied by invariance by area preserving homomorphisms of the family $\{E_c,\ c \in \mathsf{lassos}\}$: For each homomorphism ψ of $\mathbb{R}^2$, we have the equality $E_c = E_{\Psi(c)}$.

In the next section, we expose how to construct a family (H, E_c), $c \in \mathsf{lassos}$ that is braid invariant, infinitely divisible, gauge invariant and invariant by area preserving homomorphisms starting from a Lévy process on a Zhang algebra H.

1.4.4. Categorical master field field from a Lévy process. In this section, we use the same algebraic settings of the previous section. In particular, H denotes a Zhang algebra in a category $\mathcal{C}$ and $\mathcal{A} = (A, \phi_{\mathcal{A}})$ is an object of the category $\mathcal{B}$. We define the notion of Lévy process on a bialgebra B in the category $\mathcal{C}$ (that is a Zhang algebra without antipode). The tensor product $\otimes$ on $\mathcal{B}$ is supposed to satisfy the conditions stated at the beginning of the last section (symmetry, compatibility with unit constraints,...).

Definition 1.24 (Lévy process). Let (B, Δ, ε) a bialgebra in the category $\mathcal{C}$. For all pair of times $s \leq t$, let $j_{s,t} : \mathcal{H} \to \mathcal{A}$ be a morphism of $\mathcal{C}$. We say that $j = (j_{s,t})_{s \leq t}$ is a *Lévy process* if

1. *(Increments 1)* for all triple of times $u < s < t$, $j_{u,s} \dot\sqcup j_{s,t} = j_{u,t}$,
2. *(Increments 2)* for all time $s \geq 0$ and $b \in B$, $j_{s,s}(b) = \varepsilon(b)$,
3. *(Independence)* for any tuple $(s_1 < t_1 \leq s_2 < t_2 \ldots \leq s_p < t_p)$,
$$\tau \circ j_{s_{1,1}} \dot\sqcup \cdots \dot\sqcup j_{s_p, t_p} = \tau \circ j_{s_1, t_1} \otimes \cdots \otimes \tau \circ j_{s_p, t_p}$$
4. *(Stationnarity)* $\tau \circ j_{s,t} = \tau \circ j_{t-s}$.

In the case $\mathcal{C} = \mathrm{Alg}^\star$ of involutive algebras over the field of complex numbers and $\mathcal{B}$ the category of non-commutative probability spaces (that is the usual settings of non-commutative probability theory), since the initial object is $\mathbb{C}$, it is endowed with a natural norm and we require also the following compatiblity condition that is necessary the existence of a generators (see Chapter 2),

$$\lim_{t \to s^+} \tau \circ j_{s,t} = \varepsilon.$$

If considering amalgamated probability spaces on an algbra R, the last condition is also required if B is endowed with a natural norm, that will be the case in the applications of the main Theorem 1.25 that are adressed in Section 1.4.5. A Lévy process on a bi-algebra is at start, a two parameters family of morphisms of the category $\mathcal{C}$ that, owing to conditions 1 and **??** are interpreted as increments. If $j = (j_t)_{t \geq 0} : H \to \mathcal{A}$ is a one parameter family of morphisms on a Zhang algebra A, by mean of the antipode, we can associate to j a set of increments

$$j_{s,t} = \Big((j_s \circ S) \dot\sqcup j_t\Big) \circ \Delta,\ s < t.$$

The family j is said to be a Lévy process if its increments satisfy the two last conditions of Definition 1.24. The continuity condition 1.4.4 is satisfied for j if $\lim_{s \to 0^+} j_s(h) = \varepsilon(h)$ for any $h \in H$. We let $j = (j_{s,t})_{s \leq t}$, $j_{s,t} : H \to (\mathcal{A}, E)$, $s \leq t$ be a $\otimes$-Lévy process on the Zhang Algebra H, valued in an object $(\mathcal{A}, E)$ of $\mathcal{B}$. Let c be a lasso drawn on the plane and denote by $|c|$ the area enclosed by the bulk of c. For each anticlockwise oriented lasso, we define the object H_c in the category $\mathcal{B}$ by $H_c = (H, E_c) = (H, E \circ j_{|c|})$.

The infinite divisible property of the family H_c, $c \in \mathsf{lassos}$ is implied by the fact that j is a Lévy process. In fact, let c_1 and c_2 two lassos such that $c_1 c_2$ is also a lasso. In that case, the area enclosed by the bulk of c is the sum of the two areas enclosed by c_1 and by c_2. Hence, $E_{c_1c_2} = E \circ j_{|c_1|+|c_2|} = E \circ (j_{0,c_1} \times j_{|c_1|,|c_1|+|c_2|}) = (E_{c_1} \otimes E_{c_2}) \circ (\iota_{c_1} \times \iota_{c_2})$. The last equality from from $\otimes$ independence of the two increments $j_{0,|c_1|}$ and $j_{|c_1|,|c_1|+|c_2|}$. braid and gauge variance should be required for the Lévy process j in order for the objects $(H_c,\ c \in \text{lasso})$ to be braid and gauge invariant. Invariance by area preserving homomorphisms is implied by the definition of the state E_c: it does only depends on the area of the bulk. The following theorem is a formal statement of the discussion concerning the construction of a categorical master field from a direct system (see Section 1.4.3).

Theorem 1.25. *Let $(\mathcal{C}, \sqcup, k)$ an algebraic category. Let $F : \mathcal{C} \to \mathcal{D}$ be a wide and faithful functor from $\mathcal{C}$ to $\mathcal{D}$. We assume that $\mathcal{C}$ and $\mathcal{D}$ are inductively complete. Define the category $\mathcal{B}$ as previously and pick a symmetric monoidal structure $\otimes$ on $\mathcal{B}$ such that P_1 is a monoidal functor. Finally, let $j = (j_{s,t})_{s\leq t}$ be braid and gauge invariant Lévy process, meaning that:*

1. *for all integer $n \geq 1$ and braid $\beta \in \mathcal{B}_n$, $\beta \cdot (j_{s_1,t_1}, \ldots, j_{s_n,t_n}) = (j_{s_1,t_1}, \ldots, j_{s_n,t_n})$ for tuples of times $s_1 < t_1 \leq s_2 < t_2 \leq \ldots \leq s_n < t_n$,*
2. *$(\phi_H \otimes \phi_{\mathcal{A}}) \circ (\mathrm{id}_H \sqcup j_{s_1,t_1}) \circ \Omega_c$.*

There exists a categorical master field H, in the sense of Definition 1.19 satisfying the following property. For all one parameter family of simple loops $\gamma = (\gamma_t)_{t\geq 0}$ based at O with

1. *for all time $t \geq 0$, $|\mathrm{Int}(\gamma_t)| = t$, and*
2. *for all times $s \leq t$, $\mathrm{Int}(\gamma_s) \subset \mathrm{Int}(\gamma_t)$.*

The process $(H_{\gamma_t})_{t\geq 0}$ has the same non-commutative distribution as the initial Lévy process j.

1.4.5. Examples of non-commutative holonomy fields.

1.4.5.1. *Classical algebraic master fields.* In this section, we specialize our construction of a probabilistic algebraic master field to classical Lévy processes (increments are tensor independent) values in a compact Lie group. Let $N \geq 1$ an integer. Let $\mathbb{K}$ be one of the three division algebras $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$, the quaternions. In Chapter 2, we define a Brownian diffusion on the group $\mathbb{U}(N,\mathbb{K})$ of unitary matrices of size $N \times N$ with entries in $\mathbb{K}$, see Chapter 2, Section 2.2. The Brownian diffusion $\mathrm{U}_N^{\mathbb{K}}$ is the solution of a classical stochastic differential equation:

$$\begin{aligned} &\mathrm{dU}_N^{\mathbb{K}}(t) = \mathrm{dW}_N^{\mathbb{K}}(t)\mathrm{U}_N^{\mathbb{K}}(t) + \frac{1}{2}c_N^{\mathbb{K}}\mathrm{U}_N^{\mathbb{K}}(t)\\ &\mathrm{U}_N^{\mathbb{K}}(0) = I_N, \end{aligned}$$

with $\mathrm{W}_N^{\mathbb{K}}$ a linear Brownian motion on the Lie algebra of antihermitian matrices with entries in $\mathbb{K}$ and with respect to a conjugation invariant scalar product. The entries of $\mathrm{W}_N^{\mathbb{K}}$ are, up to symmetries, independent Brownian motions which variance scale as the inverse of the dimension N. To this diffusion are associated four Lévy processes that are of interest for the present work that are defined in the next chapter. Recall that we denote by $\mathcal{F}(\mathbb{U}(N,\mathbb{K}))$ the algebra of polynomial functions on $\mathbb{U}(N,\mathbb{K})$. First, we define a classical Lévy process j_N, (the increments are tensor independent), by setting for all time $s \geq 0$:

$$\begin{array}{rccc} j_N^{\mathbb{K}}(s): & \mathcal{F}(\mathbb{U}(N,\mathbb{K})) & \to & (L^\infty(\Omega,\mathcal{F},\mathbb{P}),\mathbb{E}) \\ & f & \mapsto & f(U_N^{\mathbb{K}}(s)). \end{array}$$

We recall that $\mathcal{F}(U(N,\mathbb{K})$ is an involutive Zhang algebra, being a commutative Hopf algebra with structure morphisms:

$$\Delta(f)(U,V) = f(UV),\ S(f)(U) = f(U^{-1}),\ \varepsilon(f) = f(I_N),\ \star(f) = \bar{f}.$$

The law of $j_N^{\mathbb{K}}$ is invariant by conjugation by any Unitary matrices in $\mathbb{U}(N,\mathbb{K})$ since this property holds for the driving noise $\mathrm{W}_N^{\mathbb{K}}$. To prove braid invariance for $j_N^{\mathbb{K}}$, it is sufficient to prove :

$$\left(j_N^{\mathbb{K}}(t) \times j_N^{\mathbb{K}}(s,t) \times \left[j_N^{\mathbb{K}}\right]^{-1}(t),\ j_N(t)\right) \overset{\text{distrib.}}{=} \left(j_N^{\mathbb{K}}(t),\ j_N^{\mathbb{K}}(s,t)\right).$$

This last equation is readily implied by gauge invariance and independence increments throught Fubini's Theorem.

We apply Theorem 1.25 to obtain a Holonomy field associated with $j_N^{\mathbb{K}}$. This field is the $U(N)$-*Yang–Mills field* on the plane with structure group $U(N,\mathbb{K})$.

There are two other gauge and braid invariant processes associated with the unitary diffusion $\mathsf{U}_N^{\mathbb{K}}$. The first of these quantum processes depends on two integers $n,d \geq 1$, it extracts $d\times d$ square blocks from the matrix $\mathsf{U}_N^{\mathbb{K}}$ with $N = nd$ and is defined by:

$$\begin{array}{rcl} U_{n,d}^{\mathbb{K}}: \ \mathcal{O}\langle n\rangle & \to & \left(\mathcal{M}_d(L^\infty(\Omega,\mathcal{F},\mathbb{P})),\ \mathbb{E}\otimes(\tfrac{1}{d}\mathsf{Tr})\right) \\ u_{ij} & \mapsto & \mathsf{U}_N^{\mathbb{K}}(i,j). \end{array} \tag{1.18}$$

where for $nd\times nd$ matrix A and integers $1\leq i,j\leq n$, $A(i,j)$ is the $d\times d$ sub-matrix at position (i,j) in A.

The third quantum process we consider extracts rectangular blocks from the matrix$\mathsf{U}_N^{\mathbb{K}}$. Let $n\geq 1$ an integer and $d_N = (d_N^1,\ldots,d_N^n)$ a partition of N, which means:

$$1\leq d_N^i,\ d_N^1+\ldots+d_N^n = N,\ \text{ for all } 1\leq i\leq n.$$

The Zhang algebra $\mathcal{RO}\langle n\rangle$ we call rectangular unitary algebra belongs to the category $\mathrm{Alg}^\star(R)$, we defined it in Section 1.2.2. Recall that R an involutive algebra generated by a complete set of n autoadjoint projectors. The R-amalgamated quantum stochastic process $j_{d_N}^{\mathbb{K}}$ is valued in the rectangular probability space $\mathcal{M}_{d_N}(L^{\infty-}(\Omega,\mathcal{F},\mathbb{P},\mathbb{K}))$ (see Chapter 2, Section 2.6),

$$\begin{array}{rcl} U_{d_N}^{\mathbb{K}}: \ \mathcal{RO}\langle n\rangle & \mapsto & \mathcal{M}_{d_N}(\mathbb{K}) \\ u & \to & \mathsf{U}_N^{\mathbb{K}} \\ p_i & \to & E_{d_n}^i. \end{array}$$

In Chapter 2, we prove for the process of square and rectangular extractions the convergence in non-commutative distributions. The limiting distributions are free (amalgamated) semi-groups, which are gauge and braid invariant. We use these semi-groups in the forthcoming section, Section 1.4.5.2, to construct higher dimensional counter part of the free master field.

1.4.5.2. *Higher dimensional probabilistic free master field.* In Chapter 2, Section 2.3.3 we define a higher dimensional counterpart of a free unitary Brownian motion. More precisely, pick an integer $n\geq 1$ and a von Neumann algebra $\mathcal{A}$ endowed with a tracial state τ, ($\tau(aba^{-1}) = \tau(a)$, $\tau(aa^\star)\leq 0$, $\tau(a^\star) = \overline{\tau(a)}$, $a,b\in A$). We define a free quantum Levy process $\mathsf{U}^{\langle n\rangle} = (\mathsf{U}^{\langle n\rangle}(t))_{t\geq 0}$, with $\mathsf{U}^{\langle n\rangle}(t)\in\mathcal{A}\otimes\mathcal{M}_n(\mathbb{C})$ for all time $t\geq 0$ which matricial elements are solution of a free stochastic differential system:

$$\mathrm{d}\mathsf{U}^{\langle n\rangle}(t) = \frac{\mathrm{i}}{\sqrt{n}}\mathrm{d}\mathsf{W}_t\mathsf{U}^{\langle n\rangle}(t) - \frac{1}{2}\mathsf{U}^{\langle n\rangle}(t)\mathrm{d}t,\ t\geq 0.$$

From $\mathsf{U}^{\langle n\rangle}$, one can built several Lévy processes, one of them is the free unitary Brownian motion of dimension n. The dual Voiculescu group is defined in Section 1.2.2. Define the Lévy process $U^{\langle n\rangle} = ({U^{\langle n\rangle}}_{s,t})_{s\leq t}$ with $U^{\langle n\rangle}:\mathcal{O}\langle n\rangle\to\mathcal{A}$, by setting:

$${U^{\langle n\rangle}}_t(u_{ij}) = \mathsf{U}^{\langle n\rangle}(t)(i,j),\ 1\leq i,j\leq n.$$

We saw that $\mathcal{O}\langle n\rangle$ is a Zhang algebra in the category $\mathrm{Alg}^\star$ of involutive algebras. Let $\mathcal{D}$ be the category with the same objects class as the category $\mathrm{Alg}^\star$ but with morphisms class between two objects given by the set of all complex linear involutive positive maps, if $A,B\in\mathrm{Obj}(\mathcal{C})$:

$$\mathrm{Hom}_{\mathcal{D}}(A,B) = \{\tau\in\mathrm{Hom}_{\mathrm{Vect}_{\mathbb{C}}}(A,B) : \tau(aa^\star)\geq 0,\ \tau(a^\star) = \tau(a)^\star, a\in A\}.$$

The category $\mathcal{B}$ is the usual category of (complex involutive) probability spaces, Prob. Let V be a unitary matrix in $\mathcal{M}_n(\mathbb{C})\otimes\mathcal{A}$ and set, for all times $t\geq 0$, ${\mathsf{U}^{\langle n\rangle}}^V(t) = V\mathsf{U}^{\langle n\rangle}(t)V^{-1}$. If u is an element of the Voiculescu dual group $\mathcal{O}\langle n\rangle$, $\mathsf{U}^{\langle n\rangle}(t)(u)$ and ${\mathsf{U}^{\langle n\rangle}}^V(t)(u)$ stand for the values on u of the morphisms on $\mathcal{O}\langle n\rangle$ induced by the two matrices $\mathsf{U}^{\langle n\rangle}(t)$ and ${\mathsf{U}^{\langle n\rangle}}^V(t)$. We assume further that the involutive subalgebra of $\mathcal{A}$ generated by the entries of V is free from the algebra generated by the entries of $\mathsf{U}^{\langle n\rangle}(t)$ for all times $t\geq 0$.

Lemma 1.26. *Let $t\geq 0$ a time, $u\in\mathcal{O}\langle n\rangle$, V an unitary element of $\mathcal{M}_n(\mathbb{C})\otimes\mathcal{A}$, with the notations introduced so far,*

$$\tau\circ{\mathsf{U}^{\langle n\rangle}}^V(t)(u) = \tau\circ\mathsf{U}^{\langle n\rangle}(t)(u),\ t\geq, u\in\mathcal{O}\langle n\rangle. \tag{1.19}$$

Proof. The process U^V is solution of the following free stochastic differential system, with obvious notation,

$$\mathrm{d}\mathsf{U}^{\langle n\rangle V}(t) = \frac{\mathrm{i}}{\sqrt{n}}\mathrm{d}\mathsf{W}^V_t\mathsf{U}^{\langle n\rangle V}(t) - \frac{1}{2}\mathsf{U}^{\langle n\rangle V}(t),\ t\geq 0.$$

Hence, gauge invariance of the driving noise W implies (1.19), property that is proved now. For an integer $m\geq 1$, we denote by NC_{2m} the set of matchings of the intervale $[\![1,2m]\!]$. A matching $\mathsf{m}\in\mathsf{NC}^{(2)}_{2m}$ is alternatively seen as a non-crossing partition or as an involution of $[\![1,2m]\!]$ verifying:

$$\text{for all } k<l\in[\![1,2m]\!],\ \mathsf{m}(k)<\mathsf{m}(l).$$

We compute the cumulants of the family $\{\mathsf{W}^V(\alpha,\beta),\ 1\leq\alpha,\beta\leq n\}$ and prove that:

$$\begin{aligned}(1.20)\quad & k_{2m+1}\left(\mathsf{W}^V(\alpha_1,\beta_1),\ldots,\mathsf{W}^V(\alpha_{2m+1},\beta_{\beta_{2m+1}})\right)=0,\ m\geq 1, \text{ for all } \alpha,\beta\in\{1,\ldots,n\}^{2m+1},\\ & k_{2m}\left(\mathsf{W}^V(\alpha_1,\beta_1),\ldots,\mathsf{W}^V(\alpha_{2m+1},\beta_{\beta_{2m}})\right)=k_{2m}\left(\mathsf{W}(\alpha_1,\beta_1),\ldots,\mathsf{W}(\alpha_{2m},\beta_{\beta_{2m}})\right)\\ &\qquad = \sum_{\mathsf{m}\in\mathsf{NC}_{2m}}\prod_{i=1}^{2m}\delta_{\alpha_i,\beta_{\mathsf{m}(i)}}\delta_{\alpha_{\mathsf{m}(i)},\beta_i},\ \alpha,\beta\in\{1,\ldots,n\}^{2m}.\end{aligned}$$

To compute these cumulants, we use the following fondamental formula relating cumulants with products of elements of $\mathcal{A}$ as entries to the cumulants of these elements. Let $p\geq 1$ an integer and $k_p\geq 1$ an other one. We define the interval partition $\sigma=\{\{1,\ldots,k_1\},\{k_1+1,\ldots,k_2\},\ldots,\{k_{p-1}+1,\ldots,k_p\}\}$ and denote by 1_{k_p} the partition of $[\![1,k_p]\!]$ with only one block. Let $a_1,\ldots,a_{k_p}\in\mathcal{A}$, then:

$$(1.21)\qquad k_{k_p}(a_1\cdots a_{k_1},a_{k_1+1}\cdots a_{k_2},\ldots,a_{k_{p-1}+1}\cdots a_{k_p}) = \sum_{\substack{\pi\in\mathsf{NC}_{k_p}\\ \pi\vee\sigma=1_{k_p}}} k_\pi(a_1,\ldots,a_{k_p}).$$

By using formula (1.21) and freeness of the matricial entries of V with W_t, it is easy to prove that the odd cumulants in (1.20) are equal to zero. Let $p\geq 1$ and $\alpha_1,\ldots,\alpha_{2p}\in\{1,\ldots,n\}^{2p}$. Consider a word $u=v_1w_1\tilde{v}_1\cdots v_1\tilde{w}\tilde{v}_1)$ with the $v^{'s}$ and the $\tilde{v}^s$ in the algebra generated by the matrix coefficients of V and the $w^{'s}$ in the algebra generated by the matrix coefficients of W_t. We say that an integer $i\leq 3p$ is white coloured if u_i is equal to v_i or v_i' and black coloured if u_i is equal to u_i.

Let σ be the interval partition $\sigma=\{\{1,2,3\},\ldots,\{6p-2,6p-1,6p\}\}$. Let $\pi\in\mathsf{NC}_{6p}$ such that $\sigma\vee\pi=1_{6p}$. By using nullity of mixed cumulants having components of V and W_t as entries, we prove that $k_\pi(V(\alpha_1,k_1),\mathsf{W}(k_1,q_1),V^\star(q_1,\beta_1),\ldots,V(\alpha_{2p},k_{2p}),\mathsf{W}(k_{2p},q_{2p}),V^\star(q_{2p},\beta_{2p}))$ is equal to zero if a block of π is white a black coloured. Also, the trace of π on the set of black coloured integers is a matching m, in that case,

$$\begin{aligned}(1.22)\quad & k_\pi(V(\alpha_1,k_1),\mathsf{W}(k_1,q_1),V^\star(q_1,\beta_1),\ldots,V(\alpha_{2p},k_{2p}),\mathsf{W}(k_{2p},q_{2p}),V^\star(q_{2p},\beta_{2p}))=\\ &\qquad k_{\mathsf{m}}(W(k_1,q_1),\ldots,W(k_{2p},q_{2p}))k_{\mathsf{K}(\mathsf{m},\pi)}(V(\alpha_1,k_1),V^\star(q_1,\beta_1),\ldots,V^\star(q_{2p},\beta_{2p})).\end{aligned}$$

The non crossing partition denoted $\mathsf{K}(\mathsf{m},\pi)$ is a partition of the white coloured integers of $[\![1,6p]\!]$ and equal to the complement of m in the partition π: $\mathsf{m}\cup\mathsf{K}(\mathsf{m},\pi)=\pi$. We sum (1.22) over non crossing partitions having the same trace $\mathsf{m}\in\mathsf{NC}^2_{2p}$ over black coloured integers and over integers $k_1,\ldots,k_{2p}$,$q_1,\ldots,q_{2p}$ in $\{1,\ldots,n\}^2$. By using the moments-cumulants formula, we obtain:

$$\sum_{\substack{1\leq k_1,\ldots,k_{2p}\leq n,\\ 1\leq q_1,\ldots,q_{2p}\leq n}} \sum_{\substack{\pi\in \mathsf{NC}_{6p}\\ \pi_\bullet=\mathsf{m}}} k_\pi(V(\alpha_1,k_1),\mathsf{W}(k_1,q_1),V^\star(q_1,\beta_1),\ldots,V(\alpha_{2p},k_{2p}),\mathsf{W}(k_{2p},q_{2p}),V^\star(q_{2p},\beta_{2p}))$$

$$= \sum_{\substack{1\leq k_1,\ldots,k_{2p}\leq n,\\ 1\leq q_1,\ldots,q_{2p}\leq n}} k_{\mathsf{m}}(W(k_1,q_1),\ldots,W(k_{2p},q_{2p}))\tau_{\mathsf{K}(\mathsf{m},1_{6p})}(V(\alpha_1,k_1),V^\star(q_1,\beta_1),\ldots,V^\star(q_{2p},\beta_{2p}))$$

$$= \sum_{\substack{1\leq k_1,\ldots,k_{2p}\leq n,\\ 1\leq q_1,\ldots,q_{2p}\leq n}} \prod_{i=1}^{2p} \delta_{k_i,q_{\mathsf{m}(i)}}\delta_{q_i,k_{\mathsf{m}(i)}}\tau_{\mathsf{K}(\mathsf{m},1_{6p})}(V(\alpha_1,k_1),V^\star(q_1,\beta_1),\ldots,V^\star(q_{2p},\beta_{2p}))$$

To compute right hand side of the last equation, pick a block V of the partition $\mathsf{K}(\mathsf{m},1_{6p})$, then by using cyclicity of τ, we have:

(1.23)
$$\sum_{\substack{1\leq k_1,\ldots,k_{2p}\leq n,\\ 1\leq q_1,\ldots,q_{2p}\leq n}} \tau_V(V^\star(q_{i_1},\beta_{i_1})V(\alpha_{i_2},k_{i_2})\cdots V(\alpha_{i_l},k_{\mathsf{m}(i_l)})\prod_l \delta_{k_{i_l},q_{\mathsf{m}(i_l)}}\delta_{q_{i_l},k_{\mathsf{m}(i_l)}} = \prod_l \delta_{\alpha_l,\beta_{\mathsf{m}(l)}}\delta_{\beta_l,\alpha_{\mathsf{m}(l)}}.$$

The proof of the formulas (1.20) is now complete. □

We saw that for a classical Levy process on the Zhang algebra of function on a group, gauge invariance and independence of increments implies braid invariance. It seems difficult to prove braid invariance of $U^{\langle n\rangle}$ with the same arguments. However, we prove in Chapter 2 that $U^{\langle n\rangle}$ is a limit, in non-commutative distribution of a braid invariant process: the process $U^{\mathbb{C}}_{n,d}$ (which is not a Lévy process) that is defined in Chapter 2. Braid invariance of the finite dimensional approximations of $U^{\langle n\rangle}$ implies braid invariance of the latter. It is interesting to notice that Braid invariance of the finite dimensional Levy processes $U^{\mathbb{C}}_{n,d}$ is implied by conjugation invariance of its one dimensional marginals and the Fubini theorem.

We can apply Theorem 1.25: there exists an algebraic generalized master field associated with the higher dimensional counterpart of the free unitary Brownian motion $U^{\langle n\rangle}$.

Proposition 1.27. *Let $n\geq 1$ an integer. There exists a probabilistic generalized master field (see Definition 1.19), denoted $\Phi^{\langle n\rangle}$, such that for any one parameter family of simple loops γ with $|\gamma_t|=t$ and $\mathrm{Int}(\gamma_s)\subset\mathrm{Int}(\gamma_t)$ for all times $0\leq s\leq t$, the process $\Phi^{\langle n\rangle}(\gamma_t)$ is a free unitary Brownian motion of dimension n.*

1.4.5.3. *Amalgamated probabilistic algebraic master fields.* Let $n\geq 1$ an integer. In Section 1.2.2, we defined the Zhang algebra $\mathcal{RO}\langle n\rangle$ as being the involutive algebra generated by one unitary element u and a complete set of mutually orthogonal projectors $\{p_i,\ 1\leq i\leq n\}$. Set $\mathcal{R}=\langle p_i, 1\leq i\leq n\rangle$. The algebra $\mathcal{RO}\langle n\rangle$ belongs to the category $\mathrm{lCoModAlg}(\mathcal{R})$. Let $\mathcal{D}$ be the category having the same objects class as $\mathrm{lCoModAlg}(\mathcal{R})$ and the class of morphisms between two objets being given by the positive complex $\mathcal{R}$-bimodule maps, for all $A,B\in\mathrm{Obj}(\mathcal{D})$:

$$\mathrm{Hom}_{\mathcal{D}}(A,B)=\{\phi\in\mathrm{Vect}_{\mathbb{C}}(A,B):\phi(a^\star)=\phi(a)^\star,\ \phi(aa^\star)\geq 0,\ \phi(rar')=r\phi(a)r'\} \tag{1.24}$$

With the notations of Definition 1.19, the category $\mathcal{B}$ is the usual category of $\mathcal{R}$-amalgamated probability spaces, also known as rectangular probability spaces. Let $r=(r_1,\ldots,r_n)$ be positive real numbers such that $r_1+\cdots+r_n=1$. In Chapter 2, Section 2.6, we define the semi-group E_r on the Zhang algebra $\mathcal{RO}\langle n\rangle$ and prove that E_r is a free, with amalgamation, semi-group. We explain briefly how this semi-group is obtained.

We fix a integer $n\geq 1$. For each integer $N\geq 1$, we pick $(d_N^1,\ldots,d_N^n)$ a partition of N into n parts and we assume that the ratio $\frac{d_N^i}{N}$ converges to r_i for each integer $1\leq i\leq n$ as $N\to+\infty$. Let $\mathbb{K}$ be one of the three division algebras $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. In Chapter 2, Section 2.6, we define for a partition d of N of length n the quantum process $U^{\mathbb{K}}_{\mathsf{d}}$ on the Zhang algebra $\mathcal{RO}\langle n\rangle$ extracting rectangular blocks an unitary Brownian motion of dimension N. This process is valued into a rectangular probability space, which we denote by $\mathcal{M}_{d_N}$. As an algebra, $\mathcal{M}_{d_N}$ is isomorphic to the space of matrices with dimension $N \times N$ and entries in the algebra of bounded variables (or random variables having moments of all order) with values in $\mathbb{K}$. The algebra $\mathcal{M}_{d_N}$ is also a bimodule algebra over the commutative unital algebra generated by the projectors:

$$p_i = \sum_{j=d_1+\ldots+d_{i-1}}^{d_1+\ldots+d_i} e^j \otimes e_j,\ 1 \leq i \leq n,$$

where $(e_1, \ldots, e_n)$ denotes the canonical basis of $\mathbb{K}^N$ (as a left $\mathbb{K}$-module and $(e^1, \ldots, e^n)$ is its dual basis. If $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the conditional expectation $\mathbb{E}_{d_N}$ on $\mathcal{M}_{d_N}$ is the mean of the trace of the diagonal blocks:

$$\mathbb{E}_{d_N}(A) = \sum_{i=1} \frac{1}{d_N(i)} \mathbb{E}[\mathsf{Tr}(p_i A p_i)]\, p_i,\ A \in \mathcal{M}_{d_N}.$$

In Chapter 2, we define the semi-group E_r as the limiting distribution of the non commutative distribution of the rectangular extraction process:

$$\mathsf{E}_{d_N} = \mathbb{E}_{d_N} \circ U^{\mathbb{K}}_{d_N}(t) \xrightarrow{N \to +\infty} \mathsf{E}_{r_1,\ldots,r_n}(t), \text{ for all time } t \geq 0.$$

We apply our main Theorem 1.25 to obtain a probabilistic generalized master field $\Phi^{\langle r \rangle}$ associated with the free semi-group E_r that we name *amalgamated higher dimensional master field* with parameter r. The following proposition sums up this last discussion.

Proposition 1.28. *Let $r_1, \ldots, r_n \geq 1$ positive real numbers. There exists a probabilistic generalized master field (see Definition 1.18), which we $\Phi^{\langle r_1,\ldots,r_n \rangle}$, such that for all one-parameter family of simple loops γ with $|\gamma_t| = t$ and $\mathrm{Int}(\gamma_s) \subset \mathrm{Int}(\gamma_t)$ for all times $0 \leq s \leq t$, the process $\left(\Phi^{\langle r_1,\ldots,r_n\rangle}(\gamma_t)\right)_{t\geq 1}$ is a free with amalgamation quantum Lévy process which non-commutative distribution at time $t \geq 0$ is $\mathsf{E}_{r_1,\ldots,r_n}(t)$.*

CHAPTER 2

Matricial approximations of higher dimensional master fields

We study matricial approximations of master fields constructed in Chapter 1. These approximations (in non-commutative distribution) are obtained by extracting blocks of a Brownian unitary diffusion and letting the dimension of these blocks tend to infinity. We divide our study in two parts: in the first one square blocks are extracted while in the second one we allow rectangular blocks. In both cases, free probability theory appears as the natural framework in which the limiting distributions are most accurately described. However, a generalization called amalgamated freeness is needed in the second case.

Contents

2.1. Introduction

In this work, we study convergence in non-commutative distribution of random matrices extracted from a unitary Brownian motion in high dimensions. We consider three cases: Brownian motions with real, complex and quaternionic entries and denote by $\mathbb{U}(\mathbb{K},N)$ the group of unitary matrices with entries in the division algebra $\mathbb{K}$. Brownian motion on $U(\mathbb{K},N)$, as a non-commutative process, is studied for quite a long time. We make a short and non-exhaustive list of available results.

The story begins with the work of Wigner on Hermitian random matrices having, up to symmetries, independent and identically distributed entries. Under mild assumptions (satisfied for Gaussian distributed entries with a variance that scales as the square root inverse of the dimension), Wigner proved the convergence of this random matrix's moments (the non-commutative distribution) as the dimension tends to infinity. Later, the question of the convergence in high dimensions of not only one random matrix, but of a process of Hermitian random matrices, the Hermitian Brownian motion was addressed. The result of Wigner implies convergence of the one-dimensional marginals of this process. The convergence of multi-dimensional marginals is most easily expressed and understood using non-commutative probability theory and a new notion of independence between random variables, which is Voiculescu's freeness. The notion of freeness (within the framework of operators algebras) was introduced by Voiculescu, but aroused probabilists' interest with the work of Speicher. Freeness is a non-commutative counterpart of classical independence between two random variables; it is a property of two non-commutative random variables that allows computation of the joint distribution from the individual distribution of the random variables. One significant result appearing in the work of Voiculescu (see [51]) is the asymptotic freeness of two (classically) independent random matrices. This theorem implies asymptotic freeness of a Hermitian Brownian motion's increments, which leads in turn to the convergence of multidimensional marginals of this process. The limiting process was named semi-circular Brownian motion. In the 1980's, Biane got interested in the stochastic exponential of a Hermitian Brownian motion, the unitary Brownian motion. He proved a similar result for this integrated version of Hermitian Brownian motion, namely the asymptotic freeness of the increments and convergence of the one-dimensional marginals. The limiting non-commutative process was named free unitary Brownian motion and is a solution of a free stochastic differential equation.

In the sequel, we will see that the processes mentioned above can be considered as being one dimensional, meaning that they can be seen to be members of a family of processes indexed by integers that can be approximated, in distribution, by random matrices. These processes are called higher dimensional version of the free unitary Brownian motion. Let us explain, with more details, this point. An approach for the construction of these higher dimensional free unitary Brownian motions is to extract square matrices and let the dimension tend to infinity while maintaining the number of such extractions constant. In other words, the ratio between the dimension of a block and the total dimension is kept constant. Our first result can be informally stated as follows.

Theorem 2.1. *As the dimension tends to infinity, the non-commutative distribution of square blocks extracted from a unitary Brownian motion converges to the distribution of a free process.*

See Theorem 2.28 for a more precise statement. The second part deals with a generalisation the readers may have already guessed. Why settle for square extractions? The answer may be that the framework of free non-commutative probability is not the good one to study asymptotics of products of rectangular blocks. Going from the square case to the rectangular case corresponds to an algebraic move, from the category of stellar algebras to the category of bi-module stellar algebras. We will not develop this point at the moment; we only underline that for rectangular extractions processes, the right framework for studying asymptotic as the dimension of the blocks tend to infinity is amalgamated free probability theory or rectangular free probability theory.

Theorem 2.2. *Under the assumption that the ratios between the dimensions of the extracted blocks and the total dimension of the matrix tend to positive real numbers, we prove that normalised* *traces of product of rectangular extractions from an Unitary Brownian motion at a fixed time converge. In addition, the time parametrized family of distributions is an amalgamated free semi-group.*

We have been vague on a point : which products between these rectangular blocks are allowed? Do we consider all products that have a meaning, regarding the dimensions of the blocks? At this point, our work splits in two and the last theorem holds for the two possibilities.

We mention that the question related to the convergence of square extractions of a Brownian motion has already been addressed by Michael Ulrich in [49]. The present work extends this initial investigation in two directions. The first concerns the division algebras the matricial coefficients belong to, considering the three cases: complex, real and quaternionic, while the author in [49] focuses only on the complex case. The second direction of generalisation we explored concerns convergence of rectangular extractions.

We end this introduction with an indication on the method we used. It a variation of the one developed by Levy in [41] to study Browian motion on the orthogonal, unitary, and symplectic compact group. We start from the algebra of Brauer diagrams and add colors to the vertices of a diagram. We obtain a coloured Brauer algebra that proves helpful for the first part of our work concerning square extractions. This coloured Brauer algebra is however too small for the investigation we conduct in the second part: we need a central extension of this algebra.

2.1.1. Outline. In the first part of this work (Section 2.2), we make a brief reminder on Unitary groups $\mathbb{U}(N,\mathbb{K})$ with $\mathbb{K}$ a division algebra equal either to the field of real numbers, complex numbers or to algebra of quaternions. We then introduce Brownian diffusions on such groups. A multidimensional counterpart of the free Brownian motion is introduced in Section 2.3.3. Our main combinatoric tool, the algebra of coloured Brauer diagram is introduced in Section 2.4. Convergence in non-commutative distribution of square blocks extractions of a unitary, symmetric and symplectic Brownian motion is studied in Section 2.5. The case were rectangular blocks are allowed for extractions is exposed in Section 2.6.

2.2. Brownian diffusions on unitary matrices

2.2.1. Unitary matrices over the three finite dimensional division algebras. We let $\mathbb{K}$ be one of the three associative algebras $\mathbb{C},\mathbb{H}$ and $\mathbb{R}$. We denote by i,j and k the linear real basis of $\mathbb{H}$:

$$\mathsf{i}^2=\mathsf{j}^2=\mathsf{k}^2=-1,\quad \mathsf{ij}=\mathsf{k},\mathsf{jk}=\mathsf{i},\mathsf{ki}=\mathsf{j}.$$

The adjoint of element $x\in\mathbb{K}$ is denoted $x^\star$ and the adjoint of a matrix $M=\left(M_{ij}\right)_{1\leq i,j\leq N}\in\mathcal{M}_N(\mathbb{K})$, $N\geq 1$ is $M^\star=\left(M^\star_{ij}\right)_{1\leq i,j\leq N}=\left(M^\star_{ji}\right)_{1\leq i,j\leq N}$. The group of unitary matrices with entries in $\mathbb{K}$ is the connected subgroup of $\mathcal{M}_N(\mathbb{K})$, which depends on an integer $N\geq 1$ and defined by

$$\mathbb{U}(N,\mathbb{K})=\{M\in\mathcal{M}_N(\mathbb{K}),\ MM^\star=M^\star M=1\}^0.$$

where the exponent ^{0} means that we take the connected component of the identity (it is needed for the real case). If $K=\mathbb{R}$ the group $U(N,\mathbb{R})$ is the group of special orthogonal matrices $SO(N,\mathbb{R})$ and for $K=\mathbb{C}$, $U(N,\mathbb{C})$ is the group of unitary matrices. The Lie algebra $\mathfrak{u}(N,\mathbb{K})$ is given by

$$\mathfrak{u}(N,\mathbb{K})=\{H\in\mathcal{M}_N(\mathbb{K}):H^\star+H=0\}.$$

The real Lie algebra of skew-symmetric matrices of size $N\times N$ is denoted a_N and the vector space of symmetric matrices of size $N\times N$ is denoted s_N. As real Lie algebras, one has the decompositions:

$$\mathfrak{so}_N=\mathfrak{a}_N,\quad \mathfrak{u}_N=\mathfrak{a}_N+\mathsf{i}\mathfrak{s}_N,\quad \mathfrak{sp}_N=\mathfrak{a}_N+\mathsf{i}\mathfrak{s}_N+\mathsf{j}\mathfrak{s}_N+\mathsf{k}\mathfrak{s}_N,\ N\geq 1. \tag{2.1}$$

It follows that, with $\beta=\dim_{\mathbb{R}}(\mathbb{K})$,

$$\dim(\mathfrak{u}_N(\mathbb{K}))=\frac{N(N-1)}{2}+(\beta-1)\frac{N(N+1)}{2},N\geq 1.$$

Amongst the groups $\mathbb{U}(N,\mathbb{K})$, $\mathbb{K}=\mathbb{R},\mathbb{C}$ or $\mathbb{H}$, only $U(N,\mathbb{C})$ has a non trivial center and is thus non simple. We shall add to the list the group of special unitary matrices $SU(N)$ defined as the subgroup of unitary matrices with complex entries with trace equal to one. The Lie algebra

$\mathfrak{su}(N,\mathbb{C})$ is the subalgebra of $\mathcal{M}_N(\mathbb{C})$ of anti-Hermitian matrices with null trace. Let $N \geq 1$. To define a Brownian motion on the group $\mathbb{U}(N,\mathbb{K})$ one needs to pick first a scalar product on the Lie algebra $\mathfrak{u}(N,\mathbb{K})$. Since $U(N,\mathbb{K})$ is simple, there exists up to a multiplication by a positive scalar only one scalar product on $\mathfrak{u}(N,\mathbb{K})$ which is invariant by the adjoint action of $\mathbb{U}(N,\mathbb{K})$ on its Lie algebra

As the group $\mathbb{U}(N,\mathbb{K})$ is compact, the negative of the Killing form is an invariant scalar product. Since we are going to let the dimension N tend to $+\infty$, we care about the normalization of the Killing form. Let $\langle\cdot,\cdot\rangle_N$ be the scalar product

$$\langle X,Y\rangle_N = \frac{\beta N}{2}\mathcal{R}e(\mathrm{Tr}(X^\star Y)),\ X,Y \in \mathfrak{u}(N,\mathbb{K})$$

The direct sums in the equations (2.1) are decompositions into mutually orthogonal summands for $\langle\cdot,\cdot\rangle_N$. Let $\{H_k^N\}$ be an orthonormal basis of $\mathfrak{u}(N,\mathbb{K})$, the Casimir element $C_{\mathfrak{u}_N(\mathbb{K})}$ is a bivector in $\mathfrak{u}_N(\mathbb{K})\otimes_{\mathbb{R}}\mathfrak{u}_N(\mathbb{K})$ defined by the formula:

$$C_{\mathfrak{u}_N(\mathbb{K})} = \sum_{k=1}^{\beta} H_k \otimes H_k.$$

We can cast the last formula for the Casimir element into a more concrete form by setting first

$$\mathsf{P} = \sum_{ab} E_{ab}\otimes E_{ab} \in \mathcal{M}_N(\mathbb{K})\otimes\mathcal{M}_N(\mathbb{K}) \quad \mathsf{T} = \sum_{a,b} E_{ab}\otimes E_{ba} \in \mathcal{M}_N(\mathbb{K})\otimes\mathcal{M}_N(\mathbb{K}),$$

then a simple calculation we shall not reproduce here for brevity shows that:

$$C_{\mathfrak{a}_N} = -\mathsf{T}+\mathsf{P} \quad C_{\mathfrak{s}_N} = \mathsf{T}+\mathsf{P}.$$

The letters T and P stand for *transposition* and *projection*. Eventually, put $\mathsf{I}(\mathbb{K}) = \{1,\mathsf{i},\mathsf{j},\mathsf{k}\}\cap\mathbb{K}$ and for the needs of the quaternionic case, define

$$\mathsf{Re}^{\mathbb{K}} = \sum_{\gamma\in\mathsf{I}(\mathbb{K})} \gamma\otimes\gamma^{-1} \in \mathbb{K}\otimes_{\mathbb{R}}\mathbb{K},\ \mathsf{Co}^{\mathbb{K}} = \sum_{\gamma\in\mathsf{I}(\mathbb{K})}\gamma\otimes\gamma \in \mathbb{K}\otimes_{\mathbb{R}}\mathbb{K}.$$

For the complex case, formulae for $\mathsf{Re}^{\mathbb{C}}$ and $\mathsf{Co}^{\mathbb{C}}$ are given below if these quantities are seen in the tensor product over the complex field of $\mathcal{M}_N(\mathbb{C})$ with itself, not over the real field as stated in the last equation.

Lemma 2.3. *The Casimir element of the real Lie algebra $\mathfrak{u}_N(\mathbb{K})$ is given by*

$$C_{\mathfrak{u}_N(\mathbb{K})} = \frac{1}{\beta N}\left(-\mathsf{T}\otimes_{\mathbb{R}}\mathsf{Re}^{\mathbb{K}} + \mathsf{P}\otimes_{\mathbb{R}}\mathsf{Co}^{\mathbb{K}}\right)$$

We agree with the author in [41], for the complex case it is more natural to take tensor products over the complex field, not over the real field. In the sequel, the symbol $\otimes$ stands for the symbol $\otimes_{\mathbb{C}}$ if taking the tensor product of complex vector spaces and $\otimes_{\mathbb{R}}$ otherwise. With this convention, we can give simple formulae for $C_{\mathfrak{u}(N,\mathbb{C})}$ and $C_{\mathfrak{u}(N,\mathbb{R})}$:

$$C_{\mathfrak{u}_N} = -\frac{1}{N}T \in \mathcal{M}_N(\mathbb{C})\otimes\mathcal{M}_N(\mathbb{C}), \quad C_{\mathfrak{so}_N} = -\frac{1}{N}(T-P) \in \mathcal{M}_N(\mathbb{R})\otimes\mathcal{M}_N(\mathbb{R}).$$

Let $m : \mathcal{M}_N(\mathbb{K})\otimes\mathcal{M}_N(\mathbb{K}) \to \mathcal{M}_N(\mathbb{K})$ be the multiplication map and let $c_{\mathfrak{u}_N(\mathbb{K})} = m(C_{\mathfrak{u}_N(\mathbb{K})})$.

Lemma 2.4. *If $\mathfrak{g}$ is one of the three Lie algebras at hand, then:*

$$c_{\mathfrak{u}_N(\mathbb{K})} = -1 + \frac{2-\beta}{\beta N} I_N.$$

2.2.2. Brownian motion on unitary groups. Let $N \geq 1$ be an integer and $\mathbb{K}$ one of the three division algebra $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$. Let $(B_k)_{k \leq \dim(\mathfrak{u}_N(\mathbb{K}))}$ be a $\dim_{\mathbb{R}}(\mathfrak{u}_N(\mathbb{K}))$ dimensional Brownian motion and let $\left(H_k^N\right)_{1 \leq k \leq N}$ an orthonormal basis for $\mathfrak{u}(N, \mathbb{K})$, a Brownian motion $K = (K(t))_{t \geq 0}$ with values in the Lie algebra $\mathfrak{u}(N, \mathbb{K})$ is

$$K(t) = \sum_{k=1}^{\dim_{\mathbb{R}}(\mathfrak{u}(N,\mathbb{K}))} B_k(t) H_k^N, \; t \geq 0. \tag{2.2}$$

If $d \leq 1$ and $n \geq 1$ are two integers such that $N = nd$, a matrix $A \in \mathcal{M}_N(\mathbb{K})$ is seen as an element of $\mathcal{M}_N(\mathbb{K}) \otimes_{\mathbb{R}} (\mathbb{K})$ through the identification $M \mapsto E_j^i \otimes M(i,j)$ if $M(i,j)$ is the matrix of size $d \times d$ in place i,j in M, $1 \leq i,j \leq N$. The Brownian motion $\mathsf{U}_N^{\mathbb{K}} = (\mathsf{U}_N^{\mathbb{K}}(t))_{t \geq 0}$ on the unitary group $\mathbb{U}(N, \mathbb{K})$ is the solution of the following stochastic differential equation with values in the tracial algebra $(\mathcal{M}_N(\mathbb{K}), \mathrm{tr})$:

$$\begin{cases} \mathrm{d}\mathsf{U}_N^{\mathbb{K}}(t) = \mathsf{U}_N^{\mathbb{K}}(t)\mathrm{d}K(t) + \frac{c_{\mathfrak{u}_N(\mathbb{K})}}{2} \mathsf{U}_N^{\mathbb{K}}(t)\mathrm{dt} \\ \mathsf{U}_N^{\mathbb{K}}(0) = I_N. \end{cases} \tag{2.3}$$

For all $t \geq 0$, $\mathsf{U}_N^{\mathbb{K}}(t)$ is an unitary matrix, a random variable with values in the dual Voiculescu group $\mathcal{O}\langle n \rangle$ is defined by:

$$U_{n,d}^{\mathbb{K}}(t) : \begin{cases} \mathcal{O}\langle nd \rangle & \rightarrow & L^{\infty}(\Omega, \mathcal{A}, \mathcal{M}_d(\mathbb{C}), \mathbb{P} \otimes \mathrm{tr}) \\ u_{ij} & \mapsto & \mathsf{U}_N^{\mathbb{K}}(t)(i,j) \\ u_{ij}^{\star} & \mapsto & \left(\mathsf{U}_N^{\mathbb{K}}(t)(i,j)\right)^{\star}. \end{cases}$$

In Section 2.5, we study the convergence in non-commutative distribution of $U_{n,d}^{\mathbb{K}}$ as the dimension $d \rightarrow +\infty$ to the free unitary Brownian motion. A crucial step toward this goal is to give formulae for mean of polynomials in the matrix $\mathsf{U}_N^{\mathbb{K}}(t)$, $t \geq 0$.

In the following proposition, let i,j,k be three integers such that $i,j \leq k$, to a tensor $A \in \mathcal{M}_N(\mathbb{K}) \otimes \mathcal{M}_N(\mathbb{K})$ we associate the endomorphism $\iota_{ij}(A) \in \mathcal{M}_N(\mathbb{K})^{\otimes k}$ that acts as: $\iota_{ij}(A)(v_1 \otimes v_i \otimes \cdots \otimes v_j \otimes v_k) = v_1 \otimes \cdots A^{(1)}(v_i) \otimes \cdots \otimes A^{(2)}(v_j) \otimes \cdots \otimes v_k$, $v_1 \cdots \otimes v_k \in (\mathbb{R}^N)^{\otimes k}$ if we use the Sweedler notation. For the complex case, mean of tensor product of $\mathsf{U}_N^{\mathbb{C}}$ and its conjugate are also needed. We denote by $\bar{\mathsf{M}}_2$ the free monoid generated by x_1 and $\bar{x}_1$. If $A \in \mathcal{M}_N(\mathbb{C})$ and $w \in \bar{\mathsf{M}}_2$, then $w^{\otimes}(A)$ denotes the monomial in $\mathcal{M}_N(\mathbb{C})^{\otimes k}$ obtained via the substitution $x_1 \rightarrow A$ and $\bar{x}_1 \rightarrow \bar{A}$.

Proposition 2.5 ([41]). *Let $\mathbb{K}$ be one the three division algebra $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Let $k \geq 1$ be an integer and $t \geq 0$ a time. We have*

$$\mathbb{E}\left[\left(\mathsf{U}_N^{\mathbb{K}}(t)\right)^{\otimes k}\right] = \exp\left(kt\frac{c_{\mathfrak{g}}}{2} + t\sum_{1 \leq i < j \leq n} \iota_{ij}(C_{\mathfrak{g}})\right).$$

For the complex case, let $w \in \overline{\mathsf{M}}_2$, then:

$$\mathbb{E}\left[w^{\otimes}\left(\mathsf{U}_N^{\mathbb{C}}\right)\right] = \exp\left(-\frac{kt}{2} + \sum_{\substack{1 \leq i,j \leq k, \\ w_i \neq w_j}} \iota_{i,j}(\mathsf{P}) - \sum_{\substack{1 \leq i,j \leq k, \\ w_i = w_j}} \iota_{i,j}(\mathsf{T})\right). \tag{2.4}$$

2.3. Higher dimensional free Brownian motion

Let n be an integer greater than one. Let $(w_{ij}^{\mathsf{i}})_{1 \leq i < j \leq n}, \{w_i,\ i \leq n\}$ with $\mathsf{i} \in \{1,2\}$ be three mutually free families of free Brownian motions on a tracial von Neumann algebra $(\mathcal{A}, \tau)$. We define the algebra $\mathcal{H}\langle n \rangle$ as the real unital algebra freely generated by $n(n+1)$ elements $(h_{ij})_{1 \leq i \leq j \leq n}$ and $(h_{ij}^{\star})_{1 \leq i \leq j \leq n}$. We turn $\mathcal{H}\langle n \rangle$ into a $\star$ algebra by defining the involutive antimorphism $\star$ as $(h_{ij})^{\star} = h_{ij}^{\star}$. The complexification of $\mathcal{H}\langle n \rangle$ is denoted $\mathcal{H}^{\mathbb{C}}\langle n \rangle$. The involution $\star$ is extended as an anti-linear anti-morphism of $\mathcal{H}^{\mathbb{C}}\langle n \rangle$. We prefer to work with the real algebra $\mathcal{H}\langle n \rangle$ since the random variables we are interested in are valued into real algebras (and we do not want to complexify those algebras). For each time $t \geq 0$, we define a free noise process, that is a quantum process W by setting for each time $t \geq 0$, the random variable $\mathsf{W}(t) : \mathcal{H}\langle n\rangle \to (\mathcal{A}, \tau)$ equal to:

$$\mathsf{W}_{ii}(t) = w_i(t), \mathsf{W}_{ij}(t) = \frac{1}{\sqrt{2}}\left(w^1_{ij}(t) + \mathrm{i}w^2_{ij}(t)\right), \mathsf{W}_{ji}(t) = \left(\mathsf{W}_{ij}(t)\right)^{\star} \quad i < j,$$

where $\mathsf{W}_{ij}(t) = \mathsf{W}(t)(h_{ij})$. We refer to the matrix with non-commutative entries W as the *freen-dimensional Hermitian Brownian motion*. We define further the dual unitary group $\mathcal{O}\langle n\rangle$, in the sense of Voiculescu. As a real algebra $\mathcal{O}\langle n\rangle$ is generated by $2n^2$ variables $(o_{ij})_{1\leq i,j\leq n}$ and $(o^{\star}_{ij})_{1\leq i,j\leq N}$ subject to the relations:

$$\sum_{k=1}^{n} o_{ik}o^{\star}_{jk} = \sum_{k=1}^{n} o^{\star}_{ki}o_{kj} = \delta_{ij},\ 1 \leq i,j \leq n. \tag{2.5}$$

To define morphisms on $\mathcal{O}\langle n\rangle$, it is convenient to introduce the matrix O with entries in $\mathcal{O}\langle n\rangle$ and defined by $O_{ij} = o_{ij}$, $i,j \leq n$. With the help of these notations, the dual Voiculescu group is turned into a free bialgebra by defining a coproduct Δ that takes values into the free product $\mathcal{O}\langle n\rangle \sqcup \mathcal{O}\langle n\rangle$ and satisfies the equation:

$$\Delta(O) = O_{|1}O_{|2}.$$

The associated counit $\varepsilon : \mathcal{O}\langle n\rangle \to \mathbb{R}$ is subsequently defined by $\varepsilon(O) = I_n \in \mathcal{O}\langle n\rangle \otimes \mathcal{M}_n(\mathbb{R})$. In addition, the morphism S of $\mathcal{O}$ that takes the values $S(o_{ij}) = o^{\star}_{ji}$ on the generators of $\mathcal{O}\langle n\rangle$ is an antipode for the free bi-algebra $\mathcal{O}\langle n\rangle$: $(S \sqcup 1)(o_{ij}) = (1 \sqcup S)(o_{ij}) = o_{ij}$ for all $i,j \leq n$. The tuple $(\mathcal{O}\langle n\rangle, \Delta, \varepsilon, S)$ is a Zhang algebra in the category of involutive algebra.

Our goal now is to define the higher dimensional analog of the free unitary Brownian motion. This process is a one parameter family of random variables on the Voiculescu dual unitary group denoted by $U^{\langle n\rangle}$ which values $U_{ij}(t) = U(t)(o_{ij})$ $i,j \leq n$ on the generators of $\mathcal{O}\langle n\rangle$ satisfy the following free stochastic differential system:

$$\left\{\begin{array}{l} \mathrm{d}\mathsf{U}^{\langle n\rangle}(t)(i,j) = \frac{\mathrm{i}}{\sqrt{n}}\sum_{k=1}^{n}(\mathrm{d}\mathsf{W}_t(i,k))\mathsf{U}^{\langle n\rangle}(t)(k,j) - \frac{1}{2}\mathsf{U}^{\langle n\rangle}(t)(i,j)\mathrm{d}t \\ \mathsf{U}^{\langle n\rangle}(0) = I_n \end{array}\right. \tag{2.6}$$

The following lemma states that for each time $t \geq 0$, the entries of the matrix $\mathsf{U}^{\langle n\rangle}$ satisfy the defining relations of $\mathcal{O}\langle n\rangle$. In the next lemma, we use the fact the algebra $\mathcal{A} \otimes \mathcal{M}_n(\mathbb{R})$ is an involutive algebra if endowed with the tensor product of the star anti-morphisms of $\mathcal{A}$ and $\mathcal{M}_n(\mathbb{R})$ (the transposition).

Lemma 2.6. *For each time $t \geq 0$, the matrix $\mathsf{U}^{\langle n\rangle} = \left((\mathsf{U}^{\langle n\rangle}(t))(i,j)\right)_{1\leq i,j\leq n}$ is an unitary element of $\mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})$.*

Proof. The defining relations of the algebra $\mathcal{O}\langle n\rangle$ admit the following compact form : $OO^{\star} = O^{\star}O = I_n$. We compute the derivative of $t \mapsto \mathsf{U}^{\langle n\rangle}\mathsf{U}^{\langle n\rangle\star}(t)$. Let $t \geq 0$ be a time and $i,j \leq n$ integers.

$(\star)$

$$\begin{aligned}\mathrm{d}\left(\mathsf{U}^{\langle n\rangle}(t)\mathsf{U}^{\langle n\rangle}(t)^{\star}\right)(i,j) &= -\left(\mathrm{d}\mathsf{U}^{\langle n\rangle}(t)\mathsf{U}^{\langle n\rangle}(t)^{\star}\right)(i,j) + \left(\mathsf{U}^{\langle n\rangle}(t)\mathrm{d}\mathsf{U}^{\langle n\rangle}(t)^{\star}\right)(i,j)\\ &= -\left(\mathsf{U}^{\langle n\rangle}(t)\mathsf{U}^{\langle n\rangle}(t)^{\star}\right)(i,j)\mathrm{d}t\\ &\qquad + \frac{1}{n}\sum_{k,l,q}\mathrm{d}\mathsf{W}_t(i,k)\mathsf{U}^{\langle n\rangle}(t)(k,q)\mathsf{U}^{\langle n\rangle}(t)^{\star}(q,l)\mathrm{d}\mathsf{W}_t(l,j)\\ &= -\mathsf{U}^{\langle n\rangle}(t)\mathsf{U}^{\langle n\rangle}(t)^{\star} + \mathrm{tr}(\mathsf{U}^{\langle n\rangle}(t)^{\star}\mathsf{U}^{\langle n\rangle}(t))I_n.\end{aligned}$$

By applying the linear form tr on the left and right hand sides of equation $(\star)$,we prove that $\mathrm{tr}(\mathsf{U}^{\langle n\rangle}(t)^{\star}\mathsf{U}^{\langle n\rangle}(t))$
$= 1$ for all time $t \geq 0$. Inserting this last relation in equation $(\star)$ gives

$$\mathrm{d}\mathsf{U}^{\langle n\rangle}(t)\mathsf{U}^{\langle n\rangle}(t)^{\star} = -\mathsf{U}^{\langle n\rangle}(t)^{\star}\mathsf{U}^{\langle n\rangle}(t) + I_n,\ \mathsf{U}^{\langle n\rangle}{}_0 = 1.$$

By using unicity of the solution of equation ($\star$), we prove that $\mathsf{U}^{\langle n\rangle}(t)\mathsf{U}^{\langle n\rangle\star}_t = 1$. A similar argument shows that the relation $\mathsf{U}^{\langle n\rangle\star}_t\mathsf{U}^{\langle n\rangle}(t) = I_n$ also holds. □

We use the symbol $U^{\langle n\rangle}$ for the process on the dual Voiculescu group which values on the generators of $\mathcal{O}\langle n\rangle$ are given by $\mathsf{U}^{\langle n\rangle}$. It is not gard to prove, by using equation (2.6), that $U^{\langle n\rangle}$ is actually a free Levy process on $\mathcal{O}\langle n\rangle$. We compute the derivative of the non cmmutative-distribution at time $t=0$ of $U^{\langle n\rangle}$. We define an operator L_n, that will be called th generator of $U^{\langle n\rangle}$, such that

$$\frac{d}{dt}\Big|_{t=0} \tau\circ U^{\langle n\rangle}(t)(u) = L_n(u).$$

We define in the next section what a generator on a bi-algebra is in the next Section and compute an associated Schürmann triple. We will not say that much on what a Schürmann triple is, we only indicate that it is a non-commutative analogue of the famous Lévy triple that is associated to every classical Lévy process. As such, Schürmann triples classify laws of non-commutative Lévy processes and allow for the definition of what Gaussian, Poisson or pure drift quantum processes are. We provide, to put it informally, a combinatorial formula for L_n by using the algebra of coloured Brauer diagrams we introduced in the last Section. This formula is important because it will allow for a comparison between the limiting distribution of Brownian motions on unitary groups in high dimension and the free process $U^{\langle n\rangle}$. We fix an element $u = u_{i_1,j_1}\cdots u_{i_p,j_p}\in\mathcal{O}\langle n\rangle$ in the Voiculescu dual group and compute the derivative of $t\mapsto \tau(U^{\langle n\rangle}{}_t(u))$ at time $t=0$. By using the free Itô formula, we find

$$\begin{aligned}\mathrm{d}\Big(U_t(u^{\varepsilon(1)}_{i_1j_1})\cdots U_t(u^{\varepsilon(p)}_{i_pj_p})\Big) &= \sum_{k=1}^{p} U_t(u^{\varepsilon(1)}_{i_1j_1})\cdots \mathrm{d}(U_t(u^{\varepsilon(k)}_{i_kj_k}))U_t(u^{\varepsilon(p)}_{i_pj_p})\\ &\quad+\sum_{1\leq k<l\leq p} U_t(u^{\varepsilon(1)}_{i_1j_1})\cdots \mathrm{d}U_t(u^{\varepsilon(k)}_{i_kj_k})\cdots \mathrm{d}U_t(u^{\varepsilon(l)}_{i_lj_l})\cdots U_t(u^{\varepsilon(p)}_{i_pj_p})\end{aligned}\tag{2.7}$$

We insert in the last equation the stochastic differential equation 2.6 that is satisfied by the matrix $\mathsf{U}^{\langle n\rangle}$ and $\mathsf{U}^{\langle n\rangle\star}$ and apply the trace τ to both side of the resulting equation, note that:

$$\mathrm{d}\mathsf{U}^{\langle n\rangle\star}_t = \frac{\text{-i}}{\sqrt{n}}\mathrm{d}\mathsf{W}_t\,\mathsf{U}^{\langle n\rangle\star}_t - \frac{1}{2}\mathsf{U}^{\langle n\rangle\star}_t.$$

We obtain for the derivative of $t\mapsto\tau(U^{\langle n\rangle}{}_t(u))$:

$$\begin{aligned}\mathrm{d}\tau\Big(U_t(u^{\varepsilon(1)}_{i_1j_1})\cdots U_t(u^{\varepsilon(p)}_{i_pj_p})\Big) &= -\frac{p}{2}\tau(U_t(u^{\varepsilon(1)}_{i_1j_1})\cdots U_t(u^{\varepsilon(k)}_{i_kj_k})U_t(u^{\varepsilon(p)}_{i_pj_p}))\mathrm{dt}\\ &\quad+\sum_{1\leq k<l\leq p}\tau\Big(U_t(u^{\varepsilon(1)}_{i_1j_1})\cdots \mathrm{d}U_t(u^{\varepsilon(k)}_{i_kj_k})\cdots \mathrm{d}U_t(u^{\varepsilon(l)}_{i_lj_l})\cdots U_t(u^{\varepsilon(p)}_{i_pj_p})\Big).\end{aligned}\tag{2.8}$$

We divide the second sum in (2.7) according to the values of $(\varepsilon(k),\varepsilon(l))$, $k,l\leq n$. First, if $\varepsilon(k)=\varepsilon(l)$ we obtain:

$$\begin{aligned}&\tau\Big(U_t(u^{\varepsilon(1)}_{i_1j_1})\cdots \mathrm{d}U_t(u_{i_kj_k})\cdots \mathrm{d}U_t(u_{i_lj_l})\cdots U_t(u^{\varepsilon(p)}_{i_pj_p})\Big)\\ &\qquad= -\frac{1}{n}\tau\Big(U_t(u^{\varepsilon(1)}_{i_1j_1})\cdots\tau(U_t(u_{i_lj_k})\cdots)U_t(u_{i_kj_l})\cdots U_t(u^{\varepsilon(p)}_{i_pj_p})\Big)\end{aligned}\tag{9_1}$$

$$\begin{aligned}&\tau\Big(U_t(u^{\varepsilon(1)}_{i_1j_1})\cdots \mathrm{d}U_t(u^{\star}_{i_kj_k})\cdots \mathrm{d}U_t(u^{\star}_{i_lj_l})\cdots U_t(u^{\varepsilon(p)}_{i_pj_p})\Big)\\ &\qquad= -\frac{1}{n}\tau\Big(U_t(u^{\varepsilon(1)}_{i_1j_1})\cdots U_t(u^{\star}_{i_kj_l})\tau(\cdots U_t(u^{\star}_{i_lj_k}))\cdots U_t(u^{\varepsilon(p)}_{i_pj_p})\Big)\end{aligned}\tag{9_2}$$

Secondly, if $\varepsilon(k)\neq\varepsilon(l)$:

$$\begin{aligned}&\tau\Big(U_t(u^{\varepsilon(1)}_{i_1j_1})\cdots \mathrm{d}U_t(u_{i_kj_k})\cdots \mathrm{d}U_t(u^{\star}_{i_lj_l})\cdots U_t(u^{\varepsilon(p)}_{i_pj_p})\Big)\\ &\qquad= \frac{1}{n}\tau\Big(\delta_{i_kj_l}U_t(u^{\varepsilon(1)}_{i_1j_1})\cdots\tau(U_t(u_{i_ki_l})\cdots U_t(u^{\star}_{j_kj_l}))\cdots U_t(u^{\varepsilon(p)}_{i_pj_p})\Big)\end{aligned}\tag{9_3}$$

$$(9_4)\qquad \tau\Big(U_t(u^{\varepsilon(1)}_{i_1j_1})\cdots \mathrm{d}U_t(u_{i_kj_k})\cdots \mathrm{d}U_t(u^{\star}_{i_lj_l})\cdots U_t(u^{\varepsilon(p)}_{i_pj_p})\Big) = \frac{1}{n}\tau\Big(\delta_{j_lj_k}U_t(u^{\varepsilon(1)}_{i_1j_1})\cdots U_t(u^{\star}_{i_kj_k})\tau(\cdots)U_t(u_{i_li_k})\cdots U_t(u^{\varepsilon(p)}_{i_pj_p})\Big)$$

In the sequel, we use the coloured Brauer algebra $\mathcal{B}_k(\underbrace{1,\ldots,1}_{n})$ and its representation $\rho_{(1,\ldots,1)}$ to write formulae (9$_1$)-(9$_4$). We set $c_p=(1,\ldots,p)$ and consider c_p alternatively as a permutation or as a non-coloured Brauer diagram. The non-coloured diagram $b^{\bullet}_{\varepsilon}$ is obtained by twisting c_p at positions i's such that $\varepsilon(i)=\star$:

$$b^{\bullet}_{\varepsilon}=\Big(\prod_{i:\varepsilon(i)=\star}^{n}\mathsf{Tw}_i\Big)(c_p).$$

We colourize $b^{\bullet}_{\varepsilon}$ with the colourization c defined by : $c(k)=i_k$, $c(k')=j_k$ if $\varepsilon(k)=1$ and $c(k)=j_k$, $c(k')=i_k$ if $\varepsilon(k)=\star$ to obtain a coloured Brauer diagram denoted $b^{\bullet}_{\varepsilon}$. We claim that each of the equations (9$_1$) − (9$_4$) can be put in the following form:

$$\tau\Big(U_t(u^{\varepsilon(1)}_{i_1j_1})\cdots \mathrm{d}U_t(u^{\varepsilon(k)}_{i_kj_k})\cdots \mathrm{d}U_t(u^{\varepsilon(l)}_{i_lj_l})\cdots U_t(u^{\varepsilon(p)}_{i_pj_p})\Big) = \frac{-1}{n}\Big(\tau\otimes\mathrm{Tr}^{\otimes n}\Big)\Big[\rho_{(1\ldots,1)}(\tau^{\bullet}_{kl}\circ b_{\varepsilon,\mathrm{i},\mathrm{j}})\circ \mathsf{U}^{\langle n\rangle}(t)^{\otimes k}\Big]\mathrm{d}t,\ \text{if } \varepsilon(k)=\varepsilon(l)$$

$$\tau\Big(U_t(u^{\varepsilon(1)}_{i_1j_1})\cdots \mathrm{d}U_t(u^{\varepsilon(k)}_{i_kj_k})\cdots \mathrm{d}U_t(u^{\varepsilon(l)}_{i_lj_l})\cdots U_t(u^{\varepsilon(p)}_{i_pj_p})\Big) = \frac{1}{n}\Big(\tau\otimes\mathrm{Tr}^{\otimes n}\Big)\Big[\rho_{(1,\ldots,1)}(e^{\bullet}_{kl}\circ b_{\varepsilon,\mathrm{i},\mathrm{j}})\circ \mathsf{U}^{\langle n\rangle}(t)^{\otimes k}\Big]\mathrm{d}t,\ \text{if } \varepsilon(k)\neq\varepsilon(l).$$

A combinatorial formula for the generator of the process $U^{\langle n\rangle}$ follows readily from this last four formulae. In fact, by using equation (2.8) and the characterisation of the sets $\mathsf{T}^{+}_{k}(b^{\epsilon})$ and $\mathsf{W}^{+}_{k}(b^{\epsilon})$ we gave in Section 2.4, we get:

$$\frac{d}{dt}\Big|_{t=0}\tau\Big(U_t(u^{\varepsilon(1)}_{i_1j_1})\cdots U_t(u^{\varepsilon(p)}_{i_pj_p})\Big) = -\frac{p}{2}\delta_{\mathrm{i},\mathrm{j}}-\frac{1}{n}\sum_{\tau^{\bullet}\in\mathsf{T}^{+}_{k}(b_{\varepsilon,\mathrm{i},\mathrm{j}})}\delta_{\Delta}(\tau^{\bullet}\circ b_{\varepsilon,\mathrm{i},\mathrm{j}})+\frac{1}{n}\sum_{e^{\bullet}\in\mathsf{W}^{+}_{k}(b_{\varepsilon,\mathrm{i},\mathrm{j}})}\delta_{\Delta}(e^{\bullet}\circ b_{\varepsilon,\mathrm{i},\mathrm{j}})$$

$$(2.10)\qquad = \mathcal{L}_n(u^{\varepsilon(1)}_{i_1j_1}\cdots u^{\varepsilon(p)}_{i_pj_p}),$$

2.3.1. Schürmann triple for the higher dimensional free unitary Brownian motion. For a detailed introduction to the theory of quantum stochastic calculus and Schürmann triple, the reader is directed to [29] and [46].

Definition 2.7. Let B be a unital associative complex free bi-algebra. A *generator* is a complex linear functional $L:B\to\mathbb{C}$ satisfying the properties:

1. $L(1_B)=0$, $L(b^{\star})=\overline{L(b)}$, for all $b\in B$,
2. $L(b^{\star}b)\geq 0$ for all $b\in B$ with $\varepsilon(b)=0$.

Generators show up naturally if differentiating a convolution semi-group of states. In the present work, we are interested in two types of (convolution) semi-groups, which are tensor semi-groups and free semi-groups (see [29]):

$$(2.11)\qquad \alpha_{s+t}=\alpha_s\hat{\otimes}\alpha_t=(\alpha_s\otimes\alpha_t)\circ\Delta\ (\text{tensor}),\ \alpha_{s+t}(\alpha_s\hat{\sqcup}\alpha_t)=(\alpha_s\sqcup\alpha_t)\circ\Delta\ (\text{free})\,,$$

The reader should not be confused by the notation $\alpha_s\sqcup\alpha_t$ use in the last equation. In fact, $\alpha_s\sqcup\alpha_t$ refers to the free product of states, defined by: $\alpha_s\sqcup\alpha_t(w)=0$ with w an alternating words in letters belonging to the kernels of α_s and α_t. We assume also the continuity condition,

$$\lim_{s\to 0^{+}}\alpha_s=\varepsilon$$

where $s,t\geq 0$ are two times. The word convolution is used to indicate the similitude of the products $\hat{\otimes}$ and $\hat{\sqcup}$ with the usual convolution product of functions on a group. From the definition 1.24, Chapter 1, if j is a free Lévy process on a bi-algebra B, its one dimensional marginals, $(\tau \circ j_t)_{t\geq 0}$ constitute a free semi-group. Let B be a free bi-algebra. Below we state the Schoenberg correspondence, that relates precisely some type semi-groups to generators. In Proposition 2.8, we use the symbol $\exp_\star$ to denote the exponentiation with respect to a convolution product on $B^\star$ denoted $\star$, equal either to the tensor convolution product or the free convolution product? The correspondence holds also for what is called the boolean convolution product, obtained by replacing the free product of states in equation (2.11) by boolean product of states (see Chapter 1).

Proposition 2.8 (Schoenberg correspondence). *Let $\star$ be a convolution product on $B^\star$.*

(1) Let $\psi : B \to \mathbb{C}$ be a linear functional on B, then the series:

$$\exp_\star(\psi)(b) = \sum_{k=1}^{\infty} \frac{\psi^{\star n}}{n!}(b)$$

converges for all $b \in B$.

(2) Let $(\phi_t)_{t\geq 0}$ be a convolution semi-groups, with respect to the product $\star$ on $B^\star$, then

$$L(b) = \lim_{t\to 0} \frac{1}{t}\left(\phi_t(b) - \varepsilon(b)\right) \text{ exists}$$

for all $b \in B$. Furthermore, $\exp_\star(tL)(b) = \phi_t(b)$ and the two following statements are equivalent:

a. L is a generator,

b. ϕ_t is a state for all $t \geq 0$: $\phi_t(bb^\star) \geq 0$ and $\phi_t(1) = 1$.

In the next definition, we introduce the central object of this section, the Schürmann triple. Let $(D, \langle \cdot \mid \cdot \rangle)$ be a pre-Hilbert space, we denote by $\mathcal{L}(D)$ the vector space of all linear operators on D that have an adjoint defined everywhere on D:

$$\mathcal{L}(D) = \left\{A : D \to D : \exists A^\star : D \to D,\ \langle A(v), w\rangle = \langle v, A^\star(w)\rangle,\ v, w \in D\right\}.$$

Definition 2.9 (Schürmann triple). A Schürmann triple on (B, Δ, ε) is triple (π, η, L) with

1. a unital $\star$-representation $\pi : \mathcal{B} \to \mathcal{L}(D)$ on a pre-Hilbert space D,
2. a linear map $\eta : B \to D$ verifying

$$\eta(ab) = \pi(a)\eta(b) + \eta(a)\varepsilon(b), \tag{1CC}$$

3. a generator L such that:

$$-\langle \eta(a^\star), \eta(b)\rangle = \varepsilon(a)L(b) - L(ab) + L(a)\varepsilon(b). \tag{2CB}$$

A map $\eta : B \to D$ satisfying (1CC) is called a π-ϵ cocyle and a map $L : B \to \mathbb{C}$ satisfying condition (2CB) is called a ε-ε coboundary. A schurman triple is said to be *surjective* if the cocycle η is a surjective map.

Proposition 2.10. *With the notation introduced so far, there is a one-to-one correspondence between surjective Schürmann triples, generators and convolution semi-groups $\{\phi_t, t \geq 0\}$.*

Gaussian and drift generators can be classified using Schürmann triples. In particular, the next definition introduces the notion of Gaussian processes on $\mathcal{O}\langle n\rangle$

Definition 2.11. Let $(j_t)_{t\geq 0}$ be a quantum Lévy process on $\mathcal{O}\langle n\rangle$. Let (π, η, L) be a Schürmann triple associated with j. The process j is said to be *Gaussian* if one of the following equivalent conditions hold :

1. For each $a, b, c \in \mathrm{Ker}(\varepsilon)$, we have $L(abc) = 0$,
2. For each $a, b, c \in \mathrm{Ker}(\varepsilon)$, we have $L(b^\star a^\star ab) = 0$,
3. For each $a, b, c \in \mathcal{O}\langle n\rangle$, the following formula holds

$$\begin{aligned} L(abc) = {} & L(ab)\varepsilon(c) + L(ac)\varepsilon(b) + \varepsilon(a)L(bc) - \varepsilon(a)\varepsilon(b)L(c) \\ & - \varepsilon(a)\varepsilon(c)L(b) - L(a)\varepsilon(b)\varepsilon(c) \end{aligned}$$

4. The representation π is zero on $\mathrm{Ker}(\varepsilon)$,

5. For each $a \in \mathcal{O}\langle n\rangle, \pi(a) = \varepsilon(a)1$
6. For each $a, b \in \mathsf{Ker}(\varepsilon)$, we have $\eta(ab) = 0$
7. For each $a, b \in \mathcal{O}\langle n\rangle$, $\eta(ab) = \varepsilon(a)\eta(b) + \eta(a)\varepsilon(b)$

Proposition 2.12 ([49]). *Take $D = \mathcal{M}_N(\mathbb{C})$. We define a Schürmann triple for $U^{\langle n\rangle}$ by setting*

$$\begin{aligned} &\eta(u_{ij}) = \varepsilon_{ij},\ \eta(u^{\star}_{ij}) = \varepsilon_{ij}, \\ &\pi(u_{ij}) = \delta_{ij}1, \\ &\mathcal{L}_n(u_{ij}) = -\frac{1}{2}\delta_{ij} = -\frac{1}{2}\sum_{r=1}^{n}\langle \eta(u^{\star}_{ir}), \eta(u_{rj})\rangle \end{aligned} \tag{2.12}$$

In the last equation, $\langle\,\cdot\,,\,\cdot\,\rangle = \mathsf{tr}(\,\cdot^{\star}\,\cdot\,)$

Proof. Let $n \geq 1$ an integer. The operator $\mathcal{L}_n$ is denoted L. First we prove that the three operators (π, η, L) defined by their values given in equation (2.12) on the generators of $\mathcal{O}\langle n\rangle$ exist. This is trivial for the representation π. For η and L we have to check that (2.12) is compatible with the defining relations of the algebra $\mathcal{O}\langle n\rangle$. Denote by $\mathcal{F}(n^2)$ the free algebra with n^2 generators. Let $\eta : \mathcal{F}(n^2) \to D$ be the extension of the values (2.12) by using the cocycle property (1CC). Denote also by $L : \mathcal{F}(n^2) \to \mathbb{C}$ the operator extending the values (2.12) by using point (2CB) in definition 3.7.3. For the maps η and L to descend to the quotient of the free algebras $\mathcal{F}(n^2)$ by the defining relations of $\mathcal{O}\langle n\rangle$, we have to check

$$\eta(\sum_r u^{\star}_{ri}u_{rj}) = 0,\ L(\sum_r u^{\star}_{ri}u_{rj}) = 0.$$

First, let $1 \leq i, j, r \leq n$,

$$\begin{aligned} L(u^{\star}_{ri}u_{rj}) &= \langle \eta(u_{ri}), \eta(u_{rj})\rangle + \varepsilon(u^{\star}_{ri})L(u_{rj}) + L(u^{\star}_{rj})\varepsilon(u_{rj}) \\ &= \langle \varepsilon(ri), \varepsilon(rj)\rangle + \varepsilon(u^{\star}_{ri})L(u_{rj}) + L(u^{\star}_{ri})\varepsilon(u_{rj}) \\ &= \frac{1}{n}\delta_{ir}\delta_{rj} - \frac{1}{2}\delta_{ri}\delta_{rj} - \frac{1}{2}\delta_{ri}\delta_{rj}. \end{aligned}$$

By summing the last equation over $1 \leq r \leq n$, we obtain $L(\sum_{r=1}^{n} u^{\star}_{ri}u_{rj}) = 0$. Also, using property (1CC),

$$\eta(\sum_{r=1}^{n} u^{\star}_{ri}u_{rj}) = -\sum_{r=1}^{n}\delta_{ri}\varepsilon_{rj} + \sum_{r}\varepsilon_{ir}\delta_{rj} = 0$$

By construction, (L, η, π) is a Schürmann triple. It is easy to show by induction the following formula for the cocycle η, $\varepsilon_i \in \{1, \star\}$, $1 \leq a_i, b_i \leq n,\ 1 \leq i \leq p$,

$$\eta\left(u^{\varepsilon_1}_{a_1,b_1}\cdots u^{\varepsilon_p}_{a_p,b_p}\right) = \sum_{k\leq p}\delta_{a_1,b_1}\cdots(-1)^{(\varepsilon_k\,=\,\star)}\varepsilon_{a_k b_k}\cdots\delta_{a_p,b_p} \tag{$\star$}$$

On the generators, the linear functional L and L_n agree: $L(u^{\varepsilon}_{ij}) = L_n(u^{\varepsilon}_{ij})$, $i, j \leq n$. We prove by induction on length of words on generators of $\mathcal{O}\langle n\rangle$, that L and L_n agree. Assume that L and L_n agree on words of length less than m and let w a word on the alphabet $\{u_{i,j}, u^{\star}_{i,j}\ 1 \leq i, j \leq n\}$ of length $m+1$, $w = \tilde{w}u^{\varepsilon_{m+1}}_{i_{m+1}j_{m+1}}$ with $\tilde{w}$ a word of length m.

$$L_n(w) = L(\tilde{w})\varepsilon(u^{\varepsilon_{m+1}}_{i_{m+1}j_{m+1}}) - \frac{1}{n}\sum_{\tau_{i,p+1}\in\mathsf{T}^{+}_{k}(b^{\varepsilon}_{\mathsf{i,j}})}\delta_{\Delta}(\tau_{i,p+1}\circ b^{\varepsilon}_{\mathsf{i,j}}) + \frac{1}{n}\sum_{e_{i,p+1}\in\mathsf{W}^{+}_{k}(b^{\varepsilon}_{\mathsf{i,j}}))}\delta_{\Delta}(e_{i,p+1}\circ b^{\varepsilon}_{\mathsf{i,j}}).$$

We compute next the remaining two sums. Let $1 \leq k \leq p$ be an integer, owing to:

$$\delta_{\Delta}(\tau_{k,p+1}\circ b^{\varepsilon}_{\mathsf{i,j}}) = \delta_{i_1,j_1}\cdots\delta_{i_k j_{p+1}}\cdots\delta_{i_{p+1},j_k},\ \delta_{\Delta}(e_{k,p+1}\circ b^{\varepsilon}_{\mathsf{i,j}}) = \delta_{i_1,j_1}\cdots\delta_{j_k j_{p+1}}\cdots\delta_{i_{p+1},i_k},$$

we can write the last formula for $L_n(w)$ in a more explicit form as:

$$L_n(w) = L(\tilde{w})\varepsilon(u^{\varepsilon_{m+1}}_{i_{m+1}j_{m+1}}) + \frac{1}{n}\sum_{k}(-1)^{1+(\varepsilon_{p+1}\,=\varepsilon_k}\delta_{\mathsf{i}_1,\mathsf{j}_1}\cdots\left\{\begin{array}{ll}\varepsilon_{p+1} = \varepsilon_k & \delta_{\mathsf{i}_{p+1},\mathsf{i}_k}\delta_{\mathsf{j}_{p+1},\mathsf{j}_k} \\ \varepsilon_{p+1} \neq \varepsilon_k & \delta_{\mathsf{i}_{p+1},\mathsf{j}_k}\delta_{\mathsf{j}_{p+1},\mathsf{i}_k}\end{array}\right\}\cdots\delta_{\mathsf{i}_p,\mathsf{j}_p}.$$

On the other hand, formula ($\star$) implies:

$$\langle \eta(u^{\neg\varepsilon_{p+1}}_{\mathrm{i}_{p+1},\mathrm{j}_{p+1}}), \eta(u^{\varepsilon_1}_{\mathrm{i}_1,\mathrm{j}_1}\cdots u^{\varepsilon_p}_{\mathrm{i}_p,\mathrm{j}_p})\rangle = \frac{1}{n}\sum_{k=1}^{p}(-1)^{1+(\varepsilon_{p+1}=\varepsilon_k)}\delta_{\mathrm{i}_1,\mathrm{j}_1}\cdots \mathrm{Tr}\left(\varepsilon^{\neg\varepsilon_{p+1}}_{\mathrm{i}_{p+1},\mathrm{j}_{p+1}}\varepsilon^{\varepsilon_k}_{\mathrm{i}_k\mathrm{j}_k}\right)\cdots\delta_{\mathrm{i}_p,\mathrm{j}_p}$$
$$= \frac{1}{n}\sum_{k=1}^{p}(-1)^{1+(\varepsilon_{p+1}=\varepsilon_k)}\delta_{\mathrm{i}_1,\mathrm{j}_1}\cdots\left\{\begin{array}{ll}\varepsilon_{p+1}=\varepsilon_k & \delta_{\mathrm{i}_{p+1},\mathrm{i}_k}\delta_{\mathrm{j}_{p+1},\mathrm{j}_k}\\ \varepsilon_{p+1}\neq\varepsilon_k & \delta_{\mathrm{i}_{p+1},\mathrm{j}_k}\delta_{\mathrm{j}_{p+1},\mathrm{i}_k}\end{array}\right\}\delta_{\mathrm{i}_p,\mathrm{j}_p}$$

This achieves the proof of Proposition 2.12. □

2.3.2. Some cumulants of higher dimensional free Brownian motions. In this section we compute some cumulants functions of higher dimensional Brownian motion's distribution. Our main result is contained in Proposition 2.15. Computing mixed cumulants of $\{U^{\langle n\rangle}, U^{\langle n\rangle^\star}\}$ is rather difficult task, see for example [21] in which formulae for only two types of mixed cumulants $\{U^{\langle n\rangle}, U^{\langle n\rangle^\star}\}$ are proved. We will not address this question for the process $U^{\langle n\rangle}$, although it would be interesting to.

2.3.2.1. *Free cumulants.* Let $n\geq 1$, we consider the set $\mathsf{NC}(\{1,\ldots,n\}) = \mathsf{NC}(n)$ of all non-crossing partitions of $\{1,\ldots,n\}$. A generic partition in $\mathsf{NC}(n)$ will be denoted π (sometimes ρ). A notation for $\pi = \{V_1,\ldots,V_k\}$ where $V_1,\ldots,V_k$ are called the blocks of π.

On NC(n), we consider the partial order given by reversed refinement, where for $\pi,\rho\in\mathsf{NC}(n)$ we have $\pi\leq\rho$ if and only if every blocks of ρ is a union of blocks of π. The minimal partition for the order $\leq$ is the non crossing partitions having n blocks (denoted $\mathbf{0}_n$ and the maximal element is the partition having only one block (denoted $\mathbf{1}_n$). The Möbius function on $\mathsf{NC}(n)$ will be denoted μ. This function is defined on $\{(\pi,\rho)\mid \pi,\rho\in\mathsf{NC}(n),\pi\leq\rho\}$. We will use only the Moebius function restricted to set of pairs $(\mathbf{0}_n,\pi)$ with $\pi\in\mathsf{NC}(n)$ for which we have

$$\mu(\mathbf{0}_n,\pi) = \pi_{W\in\pi}(-1)^{\sharp W}C_{\sharp W-1}$$

where for $k\in\mathbb{N}$,

$$C_k = \frac{(2k)!}{k!(k+1)!}$$

is the k^{th} Catalan number.

Let (A,ϕ) be a non commutative probability space. The n^{th} moment functional of (A,ϕ) is the multilinear functional $\phi_n : A^n\to\mathbb{C}$ defined by $\phi_n(a_1,\ldots,a_n) = \phi(a_1\cdots a_n)$, $a_1,\ldots,a_n\in A$.

The n^{rh} cumulant functional of (A,ϕ) is the multilinear functional $k_n : A^n\to\mathbb{C}$ defined by

$$k_n(a_1,\ldots,a_n) = \sum_{\pi\in\mathsf{NC}(n)}\mu(\pi,\mathbb{1}_n)\cdot\pi_{\{i_1<\ldots<ik\}\in\pi}\phi_k(a_{i_1},\ldots,a_{i_k}) \tag{2.13}$$

We refer to the equation (2.13) as the *moment-cumulant formula*. The cumulants functionals satisfy some important properties:

1. Invariance under cyclic permutations of the entries
$$k_n(a_1,\ldots,a_n) = k_n(a_m,\ldots a_n,a_1,\ldots,a_1),\ \forall 1\leq m\leq n \text{ and } a_1,\ldots,a_n\in A,$$
2. If $\mathcal{C}\subset A$ is a commutative algebra, then
$$k_n(c_1,\ldots,c_n) = k_n(c_n,\ldots,c_2,c_1),\ \forall n\geq 1 \text{ and } c_1,\ldots,c_n\in\mathcal{C}.$$
3. $k_n(a_1,\ldots,a_n) = \overline{k_n(a_n^\star,\ldots,a_1^\star)}$.

2.3.3. Computations of cumulants of the higher dimensional free Brownian motion. Let $p,n\geq 1$ two integers. We use the symbol $\mathcal{C}_{2p}$ for the set comprising all sequences $((i_1,j_1),\ldots,(i_p,j_p))$ with $i_l,j_l\in\{1,\ldots,n\}$ and refer to an element of $\mathcal{C}_{2p}$ as a colourization. For each time $t\geq 0$, we denote by $\phi_t\in\mathcal{O}\langle n\rangle^\star$ the distribution of the process $U^{\langle n\rangle}$. Let $p\geq 1$ an integer, π a non-crossing partition in NC_p and define $\phi_t(\pi):\mathcal{O}\langle n\rangle^p\to\mathbb{C}$, $\phi_t(\pi):\mathcal{C}_{2p}\to\mathbb{C}$ by

$$\phi_t(\pi)(a_1,\ldots,a_p) = \prod_{b\in\pi}\phi_t\Big(\overrightarrow{\prod_{k\in b}}a_k\Big),\ \phi_t(\pi)(\mathrm{i},\mathrm{j}) = \phi_t^\pi(u_{i_1,j_1},\ldots,u_{i_p,j_p}). \tag{2.14}$$

The function $\phi(pi)$ taking as argument a colourization is introduced for easing exposition. The group $\mathfrak{S}_p$ of permutations acts on a finite sequence $\mathsf{i} \in \{1,\ldots,n\}^p$ in a canonical way:

$$s \cdot \sigma = (s_{\sigma(1)}, \ldots, s_{\sigma(p)}),\ s \in \{1,\ldots,n\}^p,\ \sigma \in \mathfrak{S}_p.$$

The following lemma is a downward consequence of equation (2.10) for the derivative of ϕ.

Lemma 2.13. *With the notation above, for each time $t \geq 0$, non crossing partition π and colourization $(\mathsf{i},\mathsf{j}) \in \mathcal{C}_{2p}$*

$$\frac{d}{dt}\phi_t^{\pi}(\mathsf{i},\mathsf{j}) = -\frac{p}{2}\phi_t^{\pi}(\mathsf{i},\mathsf{j}) - \frac{1}{n}\sum_{\tau \in \mathsf{T}^+(\sigma_\pi)} \phi_t^{\pi_{\sigma_\pi \circ \tau}}(\mathsf{i}\cdot\tau,\mathsf{j}). \tag{2.15}$$

We fix once for all a non-crossing partition π and a colourization $(\mathsf{i},\mathsf{j}) \in \mathcal{C}$. We solve the differential equation (2.15). We introduce the normalized function:

$$L(\mathsf{i},\mathsf{j},\pi)(s) = e^{\frac{ps}{2}}\phi_s^{\pi}(\mathsf{i},\mathsf{j}),\ s \in \mathbb{R}_+.$$

For each integer $k \in \{0,\ldots,p-1\}$, we denote by $P^k(\mathsf{i},\mathsf{j},\pi) \in \mathbb{C}$ k^{th} coefficient of the Taylor expansion of $\mathbb{R}^+ \ni s \mapsto L(\mathsf{i},\mathsf{j},\pi)(s)$ (we prove in a moment this function is polynomial in its time variable). Owing to formula (2.15),

$$\frac{d}{ds}L(\mathsf{i},\mathsf{j},\pi)(s) = -\frac{1}{n}\sum_{\tau \in \mathsf{T}_k^+(\sigma_\pi)} L(\tau\cdot\mathsf{i},\mathsf{j},\pi_{\sigma_\pi\circ\tau})(s),\ s \in \mathbb{R}_+. \tag{2.16}$$

Frm the definition of $\phi(\pi,\mathsf{i},\mathsf{j})(s)$ as a product over the blocks of π, we prove that:

$$L_{\mathsf{i},\mathsf{j}}^{\pi}(s) = \prod_{V \in \pi} L_{\mathsf{i}_{|V},\mathsf{j}_{|V}}^{1_{\sharp V}}(s),\ s \geq 0,\ (\mathsf{i},\mathsf{j}) \in \mathcal{C}_p,\ \pi \in \mathsf{NC}(p) \tag{2.17}$$

We gave now the argument to prove that $s \mapsto L_{\mathsf{i},\mathsf{j}}^{\pi}(s)$ is polynomial. Let N be the operator acting on functions of non-crossing partition and colourizations such that

$$\frac{d}{ds}L(\pi,\mathsf{i},\mathsf{j}) = N(L(s,\cdot,\cdot))(\pi,\mathsf{i},\mathsf{j}).$$

The operator $N$is a nilpotent operator of order p. In fact, if $\mathbf{1}$ is the constant function equals to 1 on $\mathsf{NC}(p) \times \mathcal{C}_{2p}$, then $\mathsf{L}^s(\mathbf{1})((p\ldots,1),\mathsf{i},\mathsf{j})$ is the number of minimal factorisations of the cycle $(p\ldots 1)$ of length s. Thus $\mathsf{L}^{p-1}(\mathbf{1}) \neq 0$ and for all $s \geq p$, $\mathsf{L}^s(\mathbf{1}) = 0$. Since $L(f) \leq L(\mathbf{1})\sup(f)$, we have $L^s(f) = 0$, $\forall s \geq p$. Hence L is nilpotent of order p which implies that the exponential of L is a finite sum and the function $s \mapsto L_{\mathsf{i},\mathsf{j}}^{\pi}(s)$ is indeed a polynomial of degree stricly less than p.

Owing to equation (2.16), the following inductive relationship for the Taylor coefficients holds:

$$\begin{aligned} P_{\mathsf{i},\mathsf{j}}^{\pi,k} &= \frac{1}{n}\sum_{\tau \in \mathsf{T}^+(\sigma)} P_{\tau\cdot\mathsf{i},\mathsf{j}}^{\pi_{\sigma_\pi\circ\tau},k-1}, \\ P_{\mathsf{i},\mathsf{j}}^{\pi,0} &= L_{\mathsf{i},\sigma}(0) = \prod_{i=1}^{n}\delta_{\mathsf{i}_i,\mathsf{j}_i}, \quad k \leq p-1,\ \pi \in \mathsf{NC}_p,\ (\mathsf{i},\mathsf{j}) \in \mathcal{C}_p. \end{aligned} \tag{R}$$

In particular, the coefficients $P_{\mathsf{i},\mathsf{j}}^{\pi,0}$ are independent of the non-crossing partition π. Recall that the type of a permutation $\sigma \in \mathfrak{S}_p$ is a sequence $\mathsf{t}(\sigma) = (n_i)_{1\leq i\leq p}$ of p integers with n_i the number of cycles of σ of length i, $1 \leq i \leq p$. The geodesic distance $\mathsf{d}(\sigma)$ of a permutation $\sigma \in \mathfrak{S}_p$ to the identity id $\in \mathfrak{S}_p$ is the minimal numbers of transpositions needed to write σ as a product of transpositions:

$$\mathsf{d}(\sigma) = \min\{\ell \geq 0 : \sigma = \tau_1\cdots\tau_\ell,\ \tau_i \in \mathsf{T}_p,\ 1 \leq i \leq p\}.$$

The geodesic distance can be computed as $\mathsf{d}(\sigma) = p - \sharp(\sigma)$ with $\sharp(\sigma)$ the number of cycles of σ, $\sigma \in \mathfrak{S}_p$.

LEMMA 2.14. *Let* $(\mathrm{i},\mathrm{j}) \in \mathcal{C}_{2p}$ *be a colourization and* $\pi \in \mathsf{NC}(p)$ *a non-crossing partition, it holds that:*

$$P^{\pi,k}_{\mathrm{i},\mathrm{j}} = \frac{1}{n^k} \sum_{\tau_1,\dots,\tau_k \in \mathsf{T}^k_+(\sigma_\pi)} \delta_{\mathrm{i}\cdot\tau_1\dots\tau_k,\mathrm{j}}, \quad \forall k \geq 1. \tag{2.18}$$

The set $\mathsf{T}^k_+(\sigma_\pi)$ *is defined by:*

$$\mathsf{T}^k_+(\sigma_\pi) = \left\{\tau_1,\dots,\tau_k \in \mathsf{T}_p^{\times k} : \sigma_\pi \circ \tau_1 \circ \cdots \circ \tau_k \text{ has exactly } k + \sharp\sigma_\pi + 1 \text{ cycles}\right\}.$$

In particular if the type $t(\sigma_\pi)$ *of* σ_π *is* $(n_1,\cdots,n_p)$, *then*

$$P^{\pi,\mathsf{d}(\sigma_\pi)}_{\mathrm{i},\mathrm{j}} = \left(\frac{p-\sum_{i=1}^p n_i}{0!^{n_1}\dots(p-1)^{n_p}} \prod_{i=1}^p i^{n_i(i-2)}\right) \delta_{\mathrm{i}_{\sigma_\pi^{-1}(1)},\mathrm{j}_1} \cdots \delta_{\mathrm{i}_{\sigma_\pi^{-1}(p)},\mathrm{j}_p}.$$

Before we prove the last proposition, a simple consequence of the relation (2.20) is the nullity of the Taylor coefficients $P^k(\pi,\mathrm{i},\mathrm{j}$ of order k larger than the geodesic distance of σ_π to the identity in $\mathfrak{S}_p$:

$$\forall k \in \mathbb{N}, \quad k > \mathsf{d}(\sigma_\pi) \Rightarrow P^{\pi,k}_{\mathrm{i},\mathrm{j}} = 0.$$

We can be more precise. If σ is a non-crossing permutation in $\mathfrak{S}_p$, define $\mathfrak{S}(\sigma,i,j) \subset \mathfrak{S}_p$ as the set comprising all permutations ρ lying on a geodesic between the identity permutation and σ and satisfying $\mathrm{i}\cdot\rho = \mathrm{j}$. The last proposition implies:

$$k > \max\{d(\rho), \rho \in S(\sigma_\pi,\mathrm{i},\mathrm{j})\} \implies P^{\pi,k}_{\mathrm{i},\mathrm{j}} = 0.$$

PROOF. Let $k \geq 1$ be an integer, let π a non-crossing partition of size $p \geq 1$, and $(\mathrm{i},\mathrm{j}) \in \mathcal{C}_{2p}$ a colourization. A simple application of the recurrence relation (R) shows that:

$$P^{\pi,k}_{\mathrm{i},\mathrm{j}} = \frac{1}{n^k} \sum_{\substack{\tau_1,\dots,\tau_k \in \mathsf{T} \\ \tau_i \in \mathsf{T}_+(\sigma_\pi \circ \tau_1 \dots \tau_{i-1})}} P^{\sigma_\pi\circ\tau_1\cdots\tau_k,0}_{\mathrm{i}\cdot\tau_1\dots\tau_k,\,\mathrm{j}}.$$

The sum runs over k-tuple $(\tau_1,\cdots,\tau_k)$ such that $\tau_i \in \mathsf{T}^+(\sigma_\pi \circ \tau_1 \cdots \circ \tau_{i-1})$ for all $i \leq k$. By definition such a k-tuple of transpositions belongs to $\mathsf{T}^k_+(\sigma_\pi)$. The first relation is proved. A minimal factorisation of a permutation π is a tuple $(\tau_1,\dots,\tau_q)$ such that $\pi = \tau_1 \cdots \tau_q$. Such a tuple is of length $p - \sharp\pi$. A result of Dénes (see [22]) on the number of minimal factorisations of a cycle of length s assesses that there are s^{s-2} such factorisations. Hence the number of minimal factorisations of a permutation σ of type $n_1,\cdots,n_p$ is

$$\frac{p-\sum_{i=1}^p n_i}{0!^{n_1}\cdots(p-1)^{n_p}} \prod_{i=1}^p i^{n_i(i-2)} \tag{2.19}$$

where the factor $\frac{p-\sum_{i=1}^p n_i}{0!^{n_1}\cdots(p-1)!^{n_p}}$ accounts for the number of shuffling of a given minimal factorisations of σ_π. □

We use the notation $\mathsf{mf}(\sigma)$ for the number of minimal factorisation of a non-crossing partition σ, see equation (2.19). Before going further into the analysis of the sum (2.18), in Lemma 2.14, there are other simple consequences of Lemma 2.14. First, for the coefficient $P^{k,\pi}_{\mathrm{i},\mathrm{j}}$, the sequences i and j must contain the same number of different colours with same number of occurences:

$$\phi_t((\mathrm{i},\mathrm{j}),\pi) \neq 0 \Rightarrow (\forall i \in \{1,\cdots,p\}, \sharp\{k \in \{1,\cdots,p\} : \mathrm{i}_k = i\} = \sharp\{k \in \{1,\cdots,p\} : \mathrm{i}_k = i\}) \tag{2.20}$$

Given two sequences i and j in $\{1,\dots,n\}^p$, it seems a rather difficult task to decide whether there exists a non-crossing permutation σ such that $\sigma(\mathrm{i}) = \mathrm{j}$. Related questions have drawn our interest but we did not succeed to make progress on them:

(1) Give sufficient conditions on the sequence j such that there exists a non-crossing permutation σ satisfying $\mathrm{i}\cdot\sigma = \mathrm{j}$,

(2) if such a permutation exists, give a way to construct all of them (or at least one), and finally
(3) compute the maximal distance and minimal distance to the identity of such permutations.

This last relation implies in turn that for all pairs of integers $1 \leq i \neq j \leq p$ and all integers $n \geq 1$ that $\phi_t(\big(u_{ij}\big)^n) = 0$. One should emphasize that the last relation does not imply $u_{ij} = 0$, because u_{ij} is not self-adjoint and thus $\phi_t(u_{ij}\big(u_{ij}\big)^\star) \neq \phi_t(u_{ij}^2)$. Let σ be a permutation of $\{1, \cdots n\}$. An other simple consequence of Lemma 2.14 is independence of the state ϕ_t with respect to permutations of blocks:

$$P^{\pi,k}_{\sigma\cdot \mathrm{i},\sigma\cdot\mathrm{j}} = P^{\pi,k}_{\mathrm{i,j}},\ \forall k \geq 0,\ \sigma \in \mathfrak{S}_p \text{ and } \phi_t(\mathrm{i},\mathrm{j},\pi) = \phi_t(\sigma\cdot\mathrm{i},\sigma\cdot\mathrm{j},\pi).$$

This last property is related to an invariance of the non-commutative distribution of the driving noise W of equation (2.6). In fact, the group of permutations $\mathfrak{S}$ is injected into the group of unitary elements of $\mathcal{A}\otimes\mathcal{M}_n(\mathbb{C})$ by setting for the matrix $[\sigma]$ corresponding to a permutation σ, $[\sigma]_{ij} = \delta_{i,\sigma(j)}$, $1 \leq i,j \leq n$. With this definition, because we chose for the entries of W circular brownian motion with same covariance:

$$[\sigma]\mathsf{W}(t)[\sigma]^{-1} \overset{\text{nc. dist.}}{=} \mathsf{W}(t),\ \text{ for all time } t \geq 0.$$

We reformulate equation (2.18) of Lemma 2.14 by rewriting the right hand side as a sum over non-crossing partitions. In fact, for a pair of non-crossing partitions ρ,π in NC_p, we denote by $[\rho,\ldots,\pi]$ the set of all non-crossing partitions that are greater than ρ and smaller than π. It is a simple fact that

$$\gamma \in [\hat{0}_{[1,\ldots,p]},\pi] \Leftrightarrow \exists k > 0, \exists \tau_1\cdots\tau_k \in \mathsf{T}^+(\sigma_\pi) : \sigma_\gamma = \tau_1\cdots\tau_k \circ \sigma_\pi,$$

Thus,

$$P^{\pi,k}_{\mathrm{i,j}} = \frac{1}{n^k}\sum_{\gamma\leq\pi}\sum_{\tau_1,\ldots,\tau_k=\gamma}\delta_{\gamma\cdot\mathrm{i,j}} = \frac{1}{n^k}\sum_{\gamma\leq\pi}\mathsf{mf}(\sigma_\gamma)\delta_{\sigma_\gamma\cdot\mathrm{i,j}},\quad \forall k \geq 1. \tag{2.21}$$

Let $(u_t)_t$ be a unitary Brownian motion in a tracial algebra $(\mathcal{A},\tau)$. For each time $t \geq 0$, let $\nu_t \in \mathbb{C}[u,u^\star]^\star$ be the non commutative distribution of u_t. A formula due to Nica (see for example [21]) for some of the cumulants $k_t^p, p \leq 1$ of the distribution ν_t is the following:

$$\kappa_t^q(u,\ldots,u) = e^{-\frac{qt}{2}}\frac{(-qt)^{q-1}}{q!},\quad q \geq 1,$$

We briefly recall how this formula can be obtained. It can be shown (see [8]) that :

$$\phi_t(u_t^p) = e^{-\frac{pt}{2}}\sum_{k=0}^{n-1}\frac{(-t)^k}{k!}P_k.$$

with $P_k = \sharp\{(\tau_1,\cdots,\tau_k) \in \mathsf{T}_p : \tau_1\cdots\tau_k \leq (1,\cdots,n),\ \sharp\pi_{\tau_1\ldots\tau_k} = p-k\}$. According to a result of Denes, there are $\ell^{\ell-2}$ minimal factorisations of a cycle of length ℓ. Thus, if the partitions $p_\pi \in \mathsf{NC}(p,p-k)$ of a non-crossing permutation define a partition $(s_1,\cdots,s_p)$ of the integer p, taking into account all the shuffles of a minimal factorisation leads to the following formula for the number $\mathsf{mf}(\pi)$ of minimal factorisations of π: This leads to the desired formula for the cumulants κ_t^q, $q \geq 1$. The next proposition is downright implication of equations (2.19) and (2.21).

Proposition 2.15. *Let $p \geq 1$ an integer and $t \geq 0$ a time. Denote by k_t^p the p^{th} cumulant function of the distribution of $U^{\langle n\rangle}(t)$. With the notation introduced so far, if $u_{i_1,j_1}\ldots,u_{i_p,j_p}$ are elements of $\mathcal{O}\langle n\rangle$ taken amongst the generators,*

$$\kappa_t^p(u_{\mathrm{i}_1,\mathrm{j}_1},\ldots,u_{\mathrm{i}_p,\mathrm{j}_p}) = e^{-\frac{pt}{2}}\left(\frac{-t}{n}\right)^{p-1}\frac{p^{p-2}}{(p-1)!}\delta_{c_p\cdot\mathrm{i,j}}.$$

2.4. Coloured Brauer diagrams and Schur–Weyl dualities

2.4.1. Coloured Brauer diagrams. In that section, we introduce the set of coloured Brauer diagrams and the algebra they generate. We take the opportunity to make a brief reminder on Brauer diagrams, see [6] for a detailed review on these combinatorial objects. We strive to motivate all definitions that are introduced. However, we are aware that the combinatoric developed here may seem to be quite raw but is absolutely fundamental for our work. Let $k \geq 1$ an integer. We use the notation $i' = 2k+i$ for $1 \leq i \leq k$ and denote by $\{1,\ldots,k,1',\ldots,k'\}$ the interval of integers $[\![1,2k]\!]$.

Definition 2.16. A Brauer diagram of size k is a fixed point free involution of the set $\{1,\ldots,k,1',\ldots,k'\}$.

We denote by $\mathcal{B}_k^\bullet$ the set of all Brauer diagrams. (The superscript $\bullet$ is used to make clear the difference between Brauer diagrams and the notion of coloured Brauer diagram we introduce below). A Brauer diagram of size k may alternatively be seen as a partition of the set $\{1,\ldots,k,1',\ldots,k'\}$: two integers i,j are related if and only if $i = \sigma(j)$. Such a partition associated with a Brauer diagram is depicted as follows. We draw first k vertices on a line labelled by the integers in $1',\ldots,k'$ from left to right and k other vertices on an other line under the first one and labelled by the integers in $\{1,\ldots,k\}$. We add strands that connect two integers if one is the other image by the Brauer diagram (in the same block for the associated partition). In the sequel, we make the identification without mentioning it between a Brauer diagram, a partition which has blocks of cardinal two, and the picture that depicts it. We perform operations on Brauer diagrams which are naturally defined on the set $\mathcal{P}_k$ of all partitions of the set $\{1,\ldots,k,1',\ldots,k'\}$. These operations are the following ones and are related to the lattice structure of the set $\mathcal{P}_k$. Let p_1 and p_2 be two partitions. We write $p_1 \prec p_2$ and say that p_1 is a refinement of p_2 if each block of p_1 is included in a block of p_2. We denote by $p_1 \vee p_2$ the smallest partition which is greater than p_1 and p_2:

$$p_1 \vee p_2 = \cup_{V \in p_1} \cup_{W \in p_2 : V \cap W \neq \emptyset} V \cup W. \tag{2.22}$$

The greatest partition which is smaller than p_1 and p_2 is denoted $p_1 \wedge p_2$:

$$p_1 \wedge p_2 = \cup_{V \in p_1, W \in p_2} V \cap W.$$

The block number of a partition p is denoted $\mathsf{nc}(p)$. Of course, the function nc is constant on the set $\mathcal{B}_k^\bullet$ and equal to k. We denote by the symbol $\mathbf{1}_k$ the Brauer diagram that is pictured as in Fig. 1. A *cycle* of a Brauer diagram $b^\bullet$ is a block of the partition $b^\bullet \vee \mathbf{1}$.

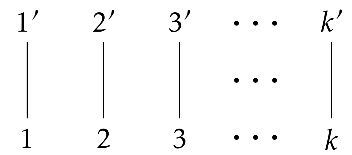


Figure 1. The identity element of the algebra of coloured Brauer diagram

Let $n \geq 1$ an integer and let $\mathsf{d} = (d_1,\ldots,d_n)$ a finite sequence of positive integers.

A *colouring* of $\{1,\ldots,k,1',\ldots,k'\}$ is a function $c : \{1,\ldots,k,1',\ldots,k'\} \to [\![1,n]\!]$. We use the symbol $\mathcal{C}_{2k}^n$ to denote the set of colourings. The *dimension* function d_c associated with a colouration c and the finite sequence d is defined by:

$$\begin{array}{rccc} \mathsf{d}_c : & \{1,\ldots,k,1',\ldots,k'\} & \longrightarrow & \{d_1,\ldots,d_n\} \\ & i & \mapsto & d_{c_i} \end{array}.$$

We define the object of interest for this section. For the rest of this section, we fix a dimension function d.

Definition 2.17 (Coloured Brauer diagrams). A *coloured Brauer diagram* is a pair $(b^\bullet, c)$ with $b^\bullet$ a Brauer diagram and c a colouring which dimension function d_c satisfies $d_c \circ b = d_c$.

A coloured Brauer diagram is conveniently depicted as in Fig. 2. The set of all coloured Brauer diagrams is denoted by $\mathcal{B}_k^{\mathsf{d}}$. If there is no risk of misunderstanding, we drop the superscript d, that indicates the dependence of the set of Brauer diagrams toward the sequence d. This sequence is also named *dimension function* in the following. The set of coloured Brauer diagram $\mathcal{B}_k^{\mathsf{d}}$ depends solely on the partition $\mathsf{Ker}(\mathsf{d})$ of $[\![1,n]\!]$ that is the set of all level sets of d. We would thus write for a partition π of $[\![1,n]\!]$ $\mathcal{B}_k^{\pi}$ for the set of Brauer diagrams which links are coloured by two integers in the same blocks of π. If π' is a partition finer than π then $\mathcal{B}_k^{\pi'} \subset \mathcal{B}_k^{\pi}$.

Let $\mathbb{K}$ the field of real numbers or the field of complex numbers. We use $\mathbb{K}[\mathcal{B}_k]$ for the $\mathbb{K}$ vector space with basis $\mathcal{B}_k^{\mathsf{d}}$. If b is a coloured Brauer diagram, $b^{\bullet}$ stands for its underlying Brauer diagram and c_b is the colouring.

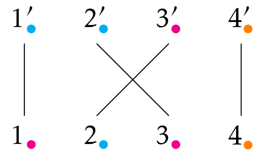


FIGURE 2. A coloured Brauer diagram with $n = 3$, and $d_{\bullet} = d_{\bullet} \neq d_{\bullet}$.

We define on the vector space $\mathbb{R}\left[\mathcal{B}_k^{\mathsf{d}}\right]$ an algebra structure. Let $b_1, b_2 \in \mathcal{B}_k$ two coloured Brauer diagrams.

We begin with reviewing the definition of a composition law on the real span of $\mathcal{B}_k^{\bullet}$, that depends on an integer $N \geq 1$. Let $b_1^{\bullet}$ and $b_2^{\bullet}$ be two Brauer diagrams. We stack $b_1^{\bullet}$ over $b_2^{\bullet}$ to obtain a third diagram that may contain closed connected components. If so, we remove these components to obtain the concatenation $b_1^{\bullet} \circ b_2^{\bullet}$ of $b_1^{\bullet}$ and $b_2^{\bullet}$. Let $\mathcal{K}(b_1^{\bullet}, b_2^{\bullet})$ be the number of components that were removed. The product $b_1^{\bullet} b_2^{\bullet}$ is defined by the formula:

$$b_1^{\bullet} b_2^{\bullet} = N^{\mathcal{K}(b_1^{\bullet}, b_2^{\bullet})} b_1^{\bullet} \circ b_2^{\bullet}.$$

We denote by $\mathcal{B}_k(N)$ this algebra. The unit of this algebra is the Brauer diagram $1_k = \{\{i, i', i \leq k\}$. We define now on the vector space $\mathbb{K}[\mathcal{B}_k]$ an algebra structure. Let $b_1, b_2 \in \mathcal{B}_k$ two Brauer diagrams. We stack b_1 over b_2, if we look at the non-coloured components of b_1 and b_2, we obtain a diagram $c^{\bullet}$ which contains, eventually, closed connected components. Now, if the colourization of the bottom line of b_1 matches the colourization of the upper line of b_2, we set $b_1 \circ b_2$ equal to the Brauer diagram obtained by colouring the bottom line (resp. upper line) of $c^{\bullet}$ with the colourization of the bottom line of b_1 (resp. the upper line of b_2) and removing all the closed components of $c^{\bullet}$. Otherwise, we set $b_1 \circ b_2 = 0$

Suppose $b_1 \circ b_2 \neq 0$. Since the colourizations of b_1 and b_2 match, the closed connected components that have been removed were eventually coloured with multiple integers lying in the same level set of the dimension function, so that we can associate to each loop a dimension in $\{\mathsf{d}\}$ For each $d \in \{\mathsf{d}\}$, we let $\mathcal{K}_d(b_1, b_2)$ be the number of closed connected components c which were erased to obtain $b_1 \circ b_2$. The product $b_1 b_2$ is subsequently defined by

$$b_1 b_2 = \prod_{d \in \{\mathsf{d}\}} d^{\mathcal{K}_d(b_1, b_2)} (b_1 \circ b_2).$$

We use the notation $\mathcal{B}_k(\mathsf{d})$ for the algebra we just defined. Endowed with this composition law, $\mathbb{R}\left[\mathcal{B}_k^{\mathsf{d}}\right]$ is an associative complex unital algebra with unit

$$e = \sum_{\substack{c \in \mathcal{C}_{2k}^n \\ c(i) = c(i')}} (\mathbf{1}_k, (c, c')). \tag{2.23}$$

The product we just defined on $\mathbb{R}\left[\mathcal{B}_k^{\mathsf{d}}\right]$ is relevant for studying distribution of square blocks extracted from an unitary Brownian motion in high dimension. However, in Section 2.6 we consider the more general problem of rectangular extractions. To tackle this question, we need a central extension of $\mathcal{R}\left[\mathcal{B}_k^{\mathsf{d}}\right]$ that allows us to track the loops, and the dimension of their colourings that are possibly created if two Brauer diagrams are multiplied together. In short,

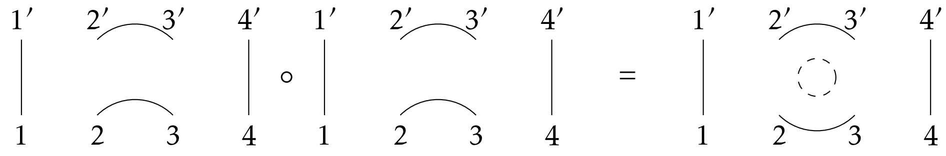


FIGURE 3. Concatenation of two non cloured Brauer diagrams.

this central extension is constructed by considering diagrams that may have closed connected components. We should make an intensive use of the following fundamental relation, which is easily proved by a drawing:

$$\mathsf{nc}((b_1^\bullet \circ b_2^\bullet) \vee 1) = \mathsf{nc}(b_1 \vee b_2) + \mathcal{K}(b_1^\bullet, b_2^\bullet). \tag{2.24}$$

If $b^\bullet$ is a non-coloured Brauer diagram, $C(b)$ is the set of all colourizations of $b^\bullet$ so as to $(b^\bullet, c)$ is a coloured Brauer diagram which each block is coloured with only one integer. We end this section by defining an injection of the algebra of non-coloured Brauer diagram $\mathcal{B}_k\left(\sum_{i=1}^n \mathsf{d}(i)\right)$ into the algebra of coloured Brauer diagram $\mathcal{B}_k(\mathsf{d})$ that will be used, without mentioning it, in computations,

$$\begin{array}{rccc} \Delta: & \mathcal{B}_k(\sum_{i=1}^n \mathsf{d}(i)) & \to & \mathcal{B}_k(\mathsf{d}) \\ & b^\bullet & \mapsto & \displaystyle\sum_{c \in C(b)} (b^\bullet, c). \end{array}$$

2.4.2. **Representation.** Let $k \geq 1$ an integer, if $\mathsf{i} = \left(i_j\right)_{1 \leq j \leq k}$ is a k-tuple of integers, we denote by $\mathsf{ker}(\mathsf{i})$ the partition of $\{1, \ldots, k\}$ equal to the set of all level sets of i. Also, if i, j are two integer sequences of length k, $\mathsf{ker}(\mathsf{i}, \mathsf{j})$ is the partition equal to the set of all level sets of the function defined on $\{1, \ldots, k, 1', \ldots, k'\}$ equal to i on $\{1, \ldots, k\}$ and j on $\{1', \ldots, k'\}$.

Let $N \geq 1$ an integer. A representation $\rho_N^\bullet$ of the algebra $\mathcal{B}_k^\bullet(N)$ is defined by setting:

$$\begin{array}{rccc} \rho_N^\bullet: & \mathcal{B}_k^\bullet(N) & \to & \mathsf{End}(\mathbb{R}^N) \\ & b^\bullet & \to & \displaystyle\sum_{\substack{\mathsf{i},\mathsf{j} \in \{1,\ldots,p+q\}^k, \\ \mathsf{ker}(\mathsf{i},\mathsf{j}) \geq b}} E_{i_1, j_1} \otimes \cdots \otimes E_{i_k, j_k}. \end{array}$$

If $N \geq 1$ is sufficiently large, it can be shown that $\rho_N^\bullet$ is injective.

Let $n \geq 1$ an integer and $\mathsf{d} = (d_1, \ldots, d_n)$ a sequence of positive integers of length n, set $N = d_1 + \ldots + d_n$. A representation ρ_{d} of the algebra $\mathbb{R}\left[\mathcal{B}_k^{\mathsf{d}}\right]$ on the k-fold tensor product $\left(\mathbb{R}^N\right)^{\otimes k}$ is defined by setting:

$$\begin{array}{rccc} \rho_{\mathsf{d}}: & \mathcal{B}_k(\mathsf{d}) & \to & \mathsf{End}(\mathbb{R}^N) \\ & (b^\bullet, c_b) & \mapsto & \displaystyle\sum_{\substack{\mathsf{i},\mathsf{j} \in \{1,\ldots,p+q\}^k, \\ \mathsf{ker}(\mathsf{i},\mathsf{j}) \geq b}} E_{i_1, j_1}^{c(1'), c(1)} \otimes \cdots \otimes E_{i_k, j_k}^{c(k'), c(k)} \end{array}.$$

With the definition of the injection Δ we gave in the previous section, simple computations show that $\rho_{\mathsf{d}} \circ \Delta = \rho_N^\bullet$ if $N = \sum_{i=1}^n d_i$.

We turn our attention to the definition of three real representations, $\rho_{\mathsf{d}}^{\mathbb{R}}, \rho_{\mathsf{d}}^{\mathbb{C}}$ and $\rho_{\mathsf{d}}^{\mathbb{H}}$ that will be used later to define statistics of the unitary Brownian motions. For the real and complex case, we set $\rho_{\mathsf{d}}^{\mathbb{R}} = \rho_{\mathsf{d}}^{\mathbb{C}} = \rho_{\mathsf{d}}$.

The representation $\rho_{p,q}^{\mathbb{C}}$ of the real algebra $\mathcal{B}_k(\mathsf{d})$ defines a representation, denoted by the same symbol, of the complex algebra $\mathcal{B}_k(\mathsf{d}) \otimes \mathbb{C}$. A real linear representation $\rho^{\mathbb{H}}$ of the algebra of Brauer diagrams $\mathcal{B}_k^\bullet(-2N)$ on $(\mathbb{H}^n)^{\otimes k}$ is defined in [41], equation (36) as a convolution of two representations : the representation $\rho_N^\bullet$ of $\mathcal{B}_k^\bullet(N)$ and a representation γ of $\mathcal{B}_k^\bullet(-2)$ which commutes with $\rho_N^\bullet$. Let us explain how a representation of the coloured Brauer algebra $\mathcal{B}_k(-2\mathsf{d})$ is defined similarly as a convolution product of two representations.

Let m be the multiplication map of endomorphisms in $\mathsf{End}(\mathbb{H}^n)^{\otimes k})$ and let $s, t > 0$ be two positive real numbers. The key observation is the existence of a morphism $\Delta_{st}^{s,t} : \mathcal{B}_k^\bullet(st) \to \mathcal{B}_k^\bullet(s) \times$

$\mathcal{B}_k^\bullet(t)$ which is the real linear extension of a function defined on the set of Brauer diagrams $\mathcal{B}_k^\bullet$ by $\Delta_{st}^{s,t}(b) = b \otimes b$, $b \in \mathcal{B}_k^\bullet$.

The representation $\rho_N^{\mathbb{H}}$ of the algebra $\mathcal{B}_k^\bullet(-2N)$ defined by Lévy in [41] is the convolution product:

$$\rho_N^{\mathbb{H}} = m \circ \left(\rho^{\mathbb{R}} \otimes \gamma\right) \circ \Delta_{-2N}^{N,-2}.$$

The definition of a coloured version of the representation $\rho_N^{\mathbb{H}}$ is ensured by the existence of coloured version $\Delta_{-2\mathsf{d}}^{\mathsf{d},-2} : \mathcal{B}_k(-2\mathsf{d}) \to \mathcal{B}_k(\mathsf{d}) \times \mathcal{B}_k^\bullet(-2)$ of $\Delta_{-2N}^{N,-2}$, namely, for $b \in \mathcal{B}_k$:

$$\Delta_{-2\mathsf{d}}^{\mathsf{d},-2}(b) = b \otimes b^\bullet.$$

There are no difficulties in checking that the map $\Delta_{-2\mathsf{d}}^{\mathsf{d},-2}$ is a morphism from $\mathcal{B}_k(-2\mathsf{d})$ to $\mathcal{B}_k(\mathsf{d}) \otimes \mathcal{B}_k^\bullet(-2)$. Finally, the representation $\rho_{p,q}^{\mathbb{H}}$ is defined by the equation:

$$\rho_{p,q}^{\mathbb{H}} = m \circ \left(\rho_{p,q}^{\mathbb{R}} \otimes \gamma\right) \circ \Delta_{-2(p+q)}^{-2p,-2q}. \tag{2.25}$$

2.4.3. Orienting and cutting a Brauer diagram. Let $b = (b^\bullet, c_b)$ a coloured Brauer diagram.

To the partition $b^\bullet$ we associate a graph Γ_b: the vertices are the points $\{1,\dots,k,1',\dots,k'\}$ and the edges are the links of the partition $b^\bullet$ together with the vertical edges $\{x,x'\}$, $x \leq k$. Each of the connected components of this graph is a loop and we pick an orientation of these loops. To that orientation of Γ_b, we associate a function $s : \{1,\dots,k\} \to \{-1,1\}$ defined as follows. Let $i \in \{1,\dots,k\}$ an integer, we set $s(i) = 1$ if the edge that belongs to b which contains i is incoming at i in the chosen orientation of Γ_b and -1 otherwise. Of course an orientation of Γ_b is completely known through its associated sign function s, thus we will in the sequel freely identify these last two objects.

We use the notation $b^s = (b,s)$ for an oriented Brauer diagram with sign function s and the set of oriented Brauer diagrams is denoted $\mathcal{OB}_k^{\mathsf{d}}$. To each oriented Brauer diagram (b,s) there are two associated permutations $\Sigma_{(b,s)}$ and $\sigma_{(b,s)}$ defined as follows. An oriented Brauer diagram (b,s) is naturally a permutation $\Sigma_{(b,s)}$ of the set $\{1,\dots,k,1',\dots,k'\}$. The cycles of the permutation $\sigma_{(b,s)}$ are the traces on $\{1,\dots,k\}$ of the cycles of $\Sigma_{(b,s)}$.

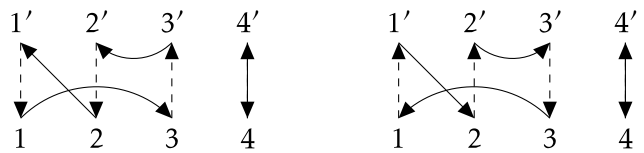


FIGURE 4. Two orientations of the same Brauer diagram.

We denote by $\mathcal{OB}_k$ the set of oriented uncoloured Brauer diagrams. We were not able to endow the real vector space with basis $\mathcal{OB}_k^{\mathsf{d}}$ with an algebra structure that would turn the canonical projection from $\mathbb{R}\left[\mathcal{OB}_k^{\mathsf{d}}\right]$ into a morphism. Also, we introduce Brauer algebras and related for two reasons : to represent quantities that are of interest for us in Section 2.6 and to define operators that will ease computations. As we should see, these operators act on the Brauer component of a oriented Brauer diagram by multiplication, hence we need, somehow, to associate to an unoriented coloured Brauer diagram and to an oriented Brauer diagram a third Brauer diagram. There is no canonical way in doing that. We may just simply pick a section $\mathcal{B}_k^{\mathsf{d}} \to \mathcal{OB}_k^{\mathsf{d}}$ and use it to give orientation to a Brauer diagram if needed. Such a section can be defined, for example, by choosing for the orientation of a diagram the sign function that is equal to *one* on the minimum of each cycle. We prefer, given an oriented Brauer (b_1,s) diagram and a Brauer diagram b, to define a orientation of $b \circ b_1$ in the following way. We choose for $s_{b_1 \circ b_2}$ the sign function that is equal to s_{b_1} on the minimum of each cycle of $b_1 \circ b$. We denote by $b \diamond b_1$ the oriented Brauer diagram obtained in this way.

The subset of permutations $\mathcal{S}_k \subset \mathcal{B}_k$ is defined as the subset of Brauer diagrams that represent a permutation. In symbols, the Brauer diagram b_σ associated with a permutation σ is equal to $\{\{i,\sigma(i)'\},\ i \leq k\}$. It is easily seen that a Brauer diagram b is a permutation diagram if and only if any orientation of b is constant on the cycles, meaning that the vertical edges of Σ_b belonging to the same cycle have the same orientation. We denote by $s_b^\bullet$ the orientation of a Brauer diagram that is equal to one on the minimum of each cycle of b.

For $i \in \{1,\ldots,k,1',\ldots,k'\}$, denote by i^k the integer $i+k$ if $i \leq k$ and $i-k$ if $i > k$. The *transposition* of a diagram $b = (b^\bullet, c_b)$ is the diagram $b^t = \left(b^{t\bullet}, c_{b^t}\right)$ defined by the equation

$$b^{t\bullet} = \cup_{l \in b^\bullet}\{i^\star, i \in l\},\ c_{b^t}(t) = c(i \bmod k).$$

See Figure 5.

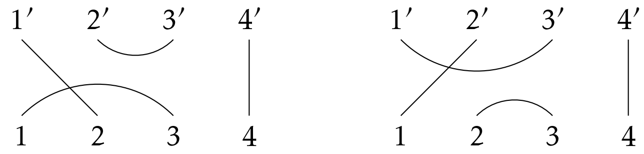


Figure 5. Transposition of a non-coloured Brauer diagram.

We define now the twist operators $\mathsf{Tw}_i^\bullet$, $i \leq k$ that act on (non-coloured) partitions of $\{1,\ldots,k,1',\ldots,k'\}$. The set of (non-coloured partitions) of $\{1,\ldots,k,1',\ldots,k'\}$ is denoted $\mathcal{P}_k$.

Definition 2.18 (Twist operators). Let $i \leq k$. The twist operator $\mathsf{Tw}_i^\bullet : \mathcal{P}_k \to \mathcal{P}_k$ is the complex linear extension to $\mathbb{C}[\mathcal{P}_k]$ of the set function defined on $\mathcal{P}_k$ by the equation

$$\mathsf{Tw}_i^\bullet(p) = \cup_{l \in p}\{i^\star, j, j \in l \backslash i\}$$

To put it in words, the twist operator $\mathsf{Tw}_i^\bullet$ exchanges the integer i and $i^\star$ in their own blocks. The subset of Brauer diagrams is stable by the Twists operators $\mathsf{Tw}_i^\bullet$, $i \leq k$.

Let us recall that the set $\mathcal{P}_k$ is a lattice. The minimum $p_1 \wedge p_2$ of two partitions p_1 and p_2 is the partition which blocks are the intersection of the blocks of p_1 and p_2. The maximum $p_1 \vee p_1$ of two partitions is the partition which blocks are union of blocks of p_1 and p_2 that have a non empty intersection.

Lemma 2.19. *The twist operator is a morphism of the lattice $(\mathcal{P}_k, \wedge, \vee)$. In addition,* $\mathsf{nc}(p) = \mathsf{nc}(\mathsf{Tw}^\bullet(p)), p \in \mathcal{P}_k$. *In particular, the number of cycles of a Brauer diagram is preserved by twisting.*

Proof. A simple drawing of the diagrams does the proof. Let's nevertheless do the proof. Let p_1, p_2 two partitions. Let S be a block of $\mathsf{Tw}_i^\bullet(p) \vee \mathsf{Tw}_i^\bullet(q)$. The set S enjoys the maximality property:

$$U \in \mathsf{Tw}_i^\bullet(p_1) \cup \mathsf{Tw}_i^\bullet(p_2),\ U \cap S \neq \emptyset \Rightarrow U \subset S. \tag{2.26}$$

Define the set $\tilde{S}$ by

- $\tilde{S} = S$ if $i, i' \in S$ or $i, i' \notin S$,
- $\tilde{S} = S \backslash \{i\} \cup \{i'\}$ if $i \in S, i' \notin S$,
- $\tilde{S} = S \backslash \{i'\} \cup \{i\}$ if $i' \in S, i \notin S$.

In order to prove that S is a block of the partition $\mathsf{Tw}_i^\bullet(p_1 \vee p_2)$, we prove that $\tilde{S}$ enjoys the maximality property:

$$U \in p_1 \cup p_2,\ U \cap \tilde{S} \neq \emptyset \Rightarrow U \subset \tilde{S}. \tag{2.27}$$

Assume that S does not contain nor i nor i' then $S = \tilde{S}$, S is an union of blocks of p_1 and p_2, and further $S \in \mathsf{Tw}_i^\bullet(p_1 \vee p_2)$.

Assume that $i \in S$, $i' \notin S$. Then $\tilde{S} = S \backslash \{i\} \cup \{i'\}$. Let $U \in p_1 \cup p_2$, $U \cap S \neq \emptyset$. We distinguish four cases:

- $i \in U, i' \notin U$, then $U \backslash \{i\} \cup \{i'\} \cap S \neq \emptyset$, $U \backslash \{i\} \cup \{i'\} \in \mathsf{Tw}_i^\bullet(p_1) \cup \mathsf{Tw}_i^\bullet(p_2)$ implies $U \backslash \{i\} \cup \{i'\} = S$ and then $\tilde{S} = U$.
- $i' \in U$, $i \in U$ then $U \in \mathsf{Tw}_i^\bullet(p_1) \cup \mathsf{Tw}_i^\bullet(p_2)$ thus $\tilde{S} = U$.
- $i \notin U, i' \notin U$ then $U \in \mathsf{Tw}_i^\bullet(p_1) \cup \mathsf{Tw}_i^\bullet(p_2)$ thus $\tilde{S} = U$.
- $i \in U$, $i' \in U$, then $U \backslash \{i'\} \cup \{i\} \cap S \neq \emptyset$, $U \backslash \{i'\} \cup \{i\} \in \mathsf{Tw}_i^\bullet(p_1) \cup \mathsf{Tw}_i^\bullet(p_2)$ implies $U \backslash \{i'\} \cup \{i\} = S$ and then $\tilde{S} = U$.

We conclude that $\tilde{S}$ has the maximal property (2.27), moreover $\tilde{S}$ is an union of blocks of p_1 and p_2, it follows that $S \in p_1 \vee p_2$.

Assume now that $i \in S$, $i' \in S$. Let $U \in p_1 \cup p_2$ such that $U \cap S \neq \emptyset$. We prove that S has the maximality property (2.27). If $U \in \mathsf{Tw}^{\bullet}(p_1) \cup \mathsf{Tw}^{\bullet}_i(p_2)$, then $U = S$ so let us assume that $U \notin \mathsf{Tw}^{\bullet}(p_1) \cup \mathsf{Tw}^{\bullet}_i(p_2)$. Either i, either i' belongs to U but not both, we can make the hypothesis that $i \in U$ and $i' \notin U$. We have $U \backslash \{i\} \cup \{i'\} \cap S \neq \emptyset$ and $U \backslash \{i\} \cup \{i'\} \in \mathsf{Tw}^{\bullet}_i(p_1) \cup \mathsf{Tw}^{\bullet}_i(p_2)$. It follows that $U \backslash \{i\} \cup \{i'\} = S$ and $U = S$. Since S is an union of sets in p_1 and p_2, one has $S \in p_1 \vee p_2$. □

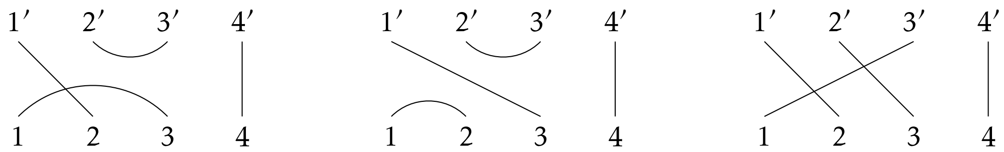


Figure 6. The second diagram is the twist at 1 of the first one. The third diagram is the twist at 2 of the first one.

Lemma 2.20. *Let $i \leq k$ an integer. Let (b,s) be an oriented Brauer diagram. For any orientation u of the diagram $\mathsf{Tw}^{\bullet}_i(b^{\bullet})$, we have*

$$u(i)u(j) = -u(i)u(j),\ i \neq j,\ i \sim_{b^{\bullet}\vee 1},\ \textit{and}\ u(i)u(j) = -u(i)u(j), i \neq j,\ i \nsim_{b^{\bullet}\vee 1} j$$

Proof. We use the notations introduced in Lemma 2.20. For any $k,l \leq k$ integers the products $u(k)u(l)$ does not depend on the orientation we pick to orient the twist at i of the diagram b. Define an orientation u of $\mathsf{Tw}^{\bullet}(b^{\bullet})$ by setting

$$u(j) = s(j) \text{ if } j \neq i, \text{ and } u(i) = -s(i).$$

To prove that u does indeed define an orientation of $b^{\bullet}$, a simple drawing is, once again, sufficient. In fact, twisting at i the diagram $b^{\bullet}$ reverses the orientation of the vertical edge that connects the vertices i, and $i+k$. □

In the previous proof, we defined an orientation of the twist of a non-coloured Brauer diagram given an orientation of the diagram. This suggests that the twist operator can be lifted to the set of non-coloured oriented Brauer diagrams.

Also, extending the twist operator to the set of coloured Brauer diagrams is straightforward; if a coloured Brauer diagram is twisted at a site i, the colours the integers i and i' are coloured with are exchanged. In the following lemma we denote by $\langle \mathsf{Tw}^{\bullet}_i,\ i \leq k\rangle$ the algebra generated by the twist operators $\{\mathsf{Tw}^{\bullet}_i,\ i \leq k\}$. For an ordered set $\{i_1 \leq i_2 \leq \ldots \leq i_q\}$, we use sometimes the notation $\mathsf{Tw}^{\bullet}_S = \mathsf{Tw}^{\bullet}_{s_1} \cdots \mathsf{Tw}^{\bullet}_{s_q}$.

Lemma 2.21. *Let $b \in \mathcal{B}^{\bullet}_k$ be an irreducible Brauer diagram ($\mathsf{nc}(b)) = 1$). There are exactly two permutation diagrams in the orbit $\{\mathsf{Tw}^{\bullet}(b),\ \mathsf{Tw}^{\bullet} \in \langle \mathsf{Tw}^{\bullet}_i,\ i \leq k\rangle\}$. In addition, two permutations in the orbit $\{\mathsf{Tw}^{\bullet}(b),\ \mathsf{Tw}^{\bullet} \in \langle \mathsf{Tw}^{\bullet}_i,\ i \leq k\rangle\}$ are related by transposition.*

Proof. There are exactly two possible orientations for an irreducible Brauer diagram. We pick one and denote it by s (the other is $-s$). Define $\mathsf{Tw}^{\bullet}_s = \prod_{i:s(i)=-1} \mathsf{Tw}^{\bullet}_i$. As in the proof of Lemma 2.20, an orientation u of $\mathsf{Tw}^{\bullet}_s(b^{\bullet})$ is defined by setting

$$u(i) = 1 \text{ if } s(i) = -1 \text{ and } u(i) = 1 \text{ if } s(i) = 1.$$

Hence, $u = 1$ and the diagram $\mathsf{Tw}^{\bullet}_s(b^{\bullet})$ is a permutation diagram. Let σ an other permutation diagram in the orbit of $b^{\bullet}$. There exists a set $S \subset \{1,\ldots,k\}$ such that $\mathsf{Tw}^{\bullet}_s(b^{\bullet}) = \left(\prod_{i\in S} \mathsf{Tw}^{\bullet}_i\right)(\sigma)$. Once again, the diagram $\left(\prod_{i\in S} \mathsf{Tw}^{\bullet}_i\right)(\sigma)$ is oriented by mean of the sign function v:

$$v(i) = -1 \text{ if } i \in S,\ v(i) = 1 \text{ if } i \notin S.$$

Unless $S = \{1,\ldots,k\}$, v is not constant. This achieves the proof. □

We defined the twist operators to prove the following Proposition, which is needed in Section 3.7 to prove that the free unitary Brownian of dimension n motion is the limit of the process extracting square block of a Brownian unitary matrix.. We recall the following formula which is used extensively in Section 2.5 and is needed for the proof of the forthcoming proposition:

$$\mathsf{nc}(b_1^\bullet \circ b_2^\bullet \vee 1) + \mathcal{K}(b_1^\bullet, b_2^\bullet) = \mathsf{nc}(b_1^\bullet \vee b_2^\bullet). \tag{2.28}$$

Proposition 2.22. *Let $b^\bullet \in \mathcal{B}_k^\bullet$ an irreducible Brauer diagram and $e^\bullet$ a projector. Then $\mathsf{nc}(e^\bullet \vee b^\bullet) - \mathsf{nc}(b^\bullet \vee 1) \in \{0,1\}$ and $\mathsf{nc}(e^\bullet \vee b^\bullet) = \mathsf{nc}(b^\bullet \vee 1) + 1$ if and only if $s(i)s(j) = -1$ for any orientation s of $b^\bullet$.*

Proof. First, pick $T \in \langle \mathsf{Tw}_i^\bullet,\ i \leq k\rangle$ such that $T(b)$ is a permutation diagram (see Lemma 2.21). Let $i,j \leq k$ be two integers. Then $\mathsf{nc}(e_{ij}^\bullet \vee b) = \mathsf{nc}(T(e_{ij}^\bullet \vee b)) = \mathsf{nc}(T(e_{ij}^\bullet) \vee T(b))$. The diagram $T(e_{ij}^\bullet)$ is equal either to the projector $e_{ij}^\bullet$ if $s(i)s(j) = 1$ either to the transposition τ_{ij} if $s(i)s(j) = -1$. It is easily checked that no loops nor cycles are created if multiplying an irreducible permutation diagram by a projector, thus from equation 2.28 $\mathsf{nc}(b^\bullet \vee e_{ij}^\bullet) = \mathsf{nc}(b^\bullet)$ if $s(i)s(j) = 1$. If $s(i)s(j) = -1$, we multiply by a transposition an irreducible permutation (a permutation with only one cycle). A direct calculation shows that $\mathsf{nc}\big(\tau_{ij}T(b^\bullet)\big) = \mathsf{nc}(T(b^\bullet)) + 1$, hence $\mathsf{nc}(b^\bullet \vee e_{ij}^\bullet) - \mathsf{nc}(b^\bullet \vee 1) = 1$. □

Proposition 2.23. *Let $k \geq 1$. Denote by c the cycle $(1,\ldots,k)$. Let $S \subset \{1,\ldots,k\}$ a set of integers. Define the Brauer diagram $b^\bullet$ as $b^\bullet = \prod_{s\in S}(\mathsf{Tw}_s^\bullet)(c)$. Let $i \neq j$ be two integers, then*

- *if $i \in S, j \in S$, $(\tau_{ij}^\bullet \circ b^\bullet \vee 1 = \{\{1,\ldots,i-1,j,\ldots,k\},\{i,i+1,\ldots,j-1\}\}$,*
- *if $i \notin S, j \notin S$, $(\tau_{ij}^\bullet \circ b^\bullet) \vee 1 = \{\{1,\ldots,i,j+1,\ldots,k\},\{i+1,i+1,\ldots,j\}\}$.*

In addition, for any orientation s of $b^\bullet$ and u of $\tau_{ij} \circ b$, we have $u(x)u(y) = s(x)s(y)$, $1 \leq x,y \leq k$, $x \sim_{\tau_{ij}\circ b} y$.

- *if $i \in S, j \notin S$, $(e_{ij}^\bullet \circ b^\bullet) \vee 1 = \{\{1,\ldots,i-1,j+1,\ldots,k\},\{i,i+2,\ldots,j\}\}$,*
- *if $i \notin S, j \in S$, $(e_{ij}^\bullet \circ b^\bullet) \vee 1 = \{\{1,\ldots,i-1,j,\ldots,k\},\{i+1,\ldots,j-1\}\}$.*

In addition, for any orientation s of $b^\bullet$ and u of $e_{ij} \circ b$, we have $u(x)u(y) = s(x)s(y)$, $1 \leq x,y \leq k$, $x \sim_{e_{ij}\circ b} y$.

Proof. Let S and b be as in Proposition 2.22. As shown in Lemma 2.20, an orientation s of b is defined by setting

$$s(i) = -1 \text{ if } i \in S,\ s(i) = 1 \text{ if } i \notin S.$$

Let $i,j \leq k$ be two integers. Assume that $i \in S$ and $j \in S$. Easy computations show that

$$(\tau_{ij} \circ b)(i') = b(j),\ (\tau_{ij} \circ b)(j') = b(i),\ (\tau_{ij} \circ b)(k) = b(k), k \neq i', j'.$$

By using Proposition 2.22, we prove that $\mathsf{nc}(\tau_{ij} \circ b \vee 1) = 2$. Recall that b is involution of $\{1,\ldots,k,1',\ldots,k'\}$, for each $i \in \{1,\ldots,k,1',\ldots,k'\}$, $b(i)$ is the integer that lies in the same block of b as i. Recall that $\star(x)$ denotes x' if $x \leq k$ and $i-k$ if $x > k$. As shown, the partition $\tau_{ij} \circ b \vee 1$ has two blocks and if x is in one of this block, so is $x^\star$. The block that contains $\{i,i'\}$ is equal to the set of alternate products of $b \circ \tau_{ij}$ and $\star$ applied to i:

$$\{i, \star(i), \big((b\circ\tau_{ij})\circ\star\big)(i), \big(\star\circ(b\circ\tau_{ij})\circ\star\big)(i), \big((b\circ\tau_{ij})\circ\star\circ(b\circ\tau_{ij})\circ\star\big)(i), \ldots\}$$

which is equal to $\{i, i', b(j), \star(b(j)), (b\circ\tau_{ij})(\star(b(j)), \ldots\}$. We have $\star(b(j))[k] = j-1$ (from the definition of b). Thus if $i \neq j-1$, we have $(\tau_{ij}\circ b)(\star(b(j))) = b(\star(b(j)))$. Continuing in the same manner, we find

$$\{i, i', b(j), \star(b(j)), b(\star(b(j)), \star(b(\star(b(j))), \star(b(\star(b(j)))\ldots i, (\star)i\} = \{i, i', j-1, j'-1, j-2, j'-2, \ldots, i, i'\}.$$

We do the same for the case $i \in S$ and $j \notin S$, the details are left to the reader. Assume now that $i \in S$ and $j \notin S$. The partition $e_{ij} \circ b \vee 1$ has two blocs (this follows from Proposition 2.22). Since

$(e_{ij}\circ b)(i')=j'$, the set $\{i',j'\}$ is contained within a block of $(e_{ij}\circ b)$. Once again to compute the blocks that contains the set $\{i',j'\}$, we have to compute the set

$$\{i',i,(e_{ij}\circ b)(i),(\star\circ(e_{ij}\circ b))(i),((e_{ij}\circ b)\circ\star\circ(e_{ij}\circ b))(i),\ldots,\} \tag{2.29}$$

We remark that for any integer x in the interval $[\![i,j-1]\!]$, $\{(e_{ij}\circ b)(x),\star((e_{ij}\circ b)(x)\}=\{x+1,x'+1\}$. We have $(e_{ij}\circ b)(i)[k]=i+1$, thus we find the set 2.29 is equal to:

$$\{i',i,i+1,i'+1,i'+2,i+2,\ldots,...j-1,j-1,j,j'\}$$

The case $i\notin S$ and $j\in S$ is left to the reader. □

Conjugation of a Brauer diagram by a permutation α results in a Brauer diagram that has the same number of cycles, and

$$\alpha\circ b\circ\alpha^{-1}=\bigcup_{l\in b}\{\alpha(i),\alpha(j),i,j\in l\}.$$

Hence, $\alpha\circ e_{ij}\circ\alpha^{-1}=e_{\alpha(i),\alpha(j)}$ and $\alpha\circ\tau_{ij}\circ\alpha^{-1}=\tau_{\alpha(i),\alpha(j)}$ with $i,j\le k$ two integers. In addition, orientation and conjugation enjoy a remarkable property. For any oriented Brauer diagram (b,s_b) and permutation α, the sign function $s_{\alpha\circ b\circ\alpha^{-1}}=s_b\circ\alpha^{-1}$ defines orientation of $\alpha\circ b\circ\alpha^{-1}$.
The twists operators are also equivariant with respect to conjugation action:

$$\mathsf{Tw}^\bullet_S\left(\alpha\circ b\circ\alpha^{-1}\right)=\alpha\circ\mathsf{Tw}^\bullet_{\alpha^{-1}(S)}(b)\circ\alpha^{-1},\ S\subset\{1,\ldots,k\},\ \mathsf{Tw}^\bullet_S=\prod_{s\in S}\mathsf{Tw}^\bullet_s,\ \alpha\in\mathcal{S}_k.$$

Let $S\subset\{1,\ldots,k\}$. Let α a permutation in $\mathcal{S}_k$. The proposition is easily generalised to twists of the cycle $\alpha\circ c\circ\alpha^{-1}=(\alpha(1),\ldots,\alpha(k))$. Define $b=\mathsf{Tw}^\bullet_S((\alpha(1),\ldots,\alpha(k)))$. In fact, $(\alpha(1),\ldots,\alpha(k))=\alpha\circ(1,\ldots,k)\circ\alpha^{-1}$ and

$$\left(e_{ij}\circ\mathsf{Tw}^\bullet_S(\alpha\circ c\circ\alpha^{-1})\right)\vee 1=\alpha\circ\left(\left(e_{\alpha^{-1}(i),\alpha^{-1}(j)}\circ\mathsf{Tw}^\bullet_{\alpha^{-1}(S)}(c)\right)\vee 1\right)\circ\alpha^{-1}$$

We apply Proposition 2.23 to find

- if $i\in S,j\in S$, $(\tau_{ij}\circ b)\vee 1=\{\{\alpha(1),\ldots,\alpha(\alpha^{-1}(i)-1),j,\alpha(\alpha^{-1}(j+1),\ldots,\alpha(k)\},$ $\{\alpha^{-1}(i),\alpha(\alpha^{-1}(i)+1),\ldots,\alpha(\alpha^{-1}(j)-1)\}\}$,
- if $i\notin S,j\notin S$, $(\tau_{ij}\circ b)\vee 1=\{\{\alpha(1),\ldots,\alpha(\alpha^{-1}(i)-1),\alpha(\alpha^{-1}(j)+1),\ldots,\alpha(k)\},$ $\{\alpha(\alpha^{-1}(i)),\alpha(\alpha^{-1}(i)+1),\ldots,\alpha(\alpha^{-1}(j)-1),j\}\}$.

In addition, or any orientation s of $b^\bullet$ and u of $\tau_{ij}\circ b$, we have $u(x)u(y)=s(x)s(y)$, $1\le x,y\le k$, $x\sim_{\tau_{ij}\circ b}y$.

- if $i\in S,j\notin S$, $(e_{ij}\circ b)\vee 1=\{\{\alpha(1),\ldots,\alpha(\alpha^{-1}(i)-1),\alpha(\alpha^{-1}(j)+1,\ldots,\alpha(k)\},$ $\{i,\alpha(\alpha^{-1}(i)+1),\ldots,j\}\}$,
- if $i\notin S,j\in S$, $(e_{ij}\circ b)\vee 1=\{\{\alpha(1),\ldots,\alpha(\alpha^{-1}(i)-1),j,\alpha(j+1),\ldots,\alpha(k)\},$ $\{\alpha(\alpha^{-1}(i)+1),\ldots,\alpha(\alpha^{-1}(j)-1)\}\}$.

In the same way, for any orientation s of $b^\bullet$ and u of $e_{ij}\circ b$, we have $u(x)u(y)=s(x)s(y)$, $1\le x,y\le k$, $x\sim_{e_{ij}\circ b}y$.

We have seen in Proposition 2.22 that multiplication of an irreducible Brauer diagram by a transposition or a projector produces at most one cycle. Given a non-necessarily irreducible Brauer diagram $b^\bullet$, the following proposition specifies how many cycles are deleted or created if we multiply $b^\bullet$ by a transposition or a projector.

Proposition 2.24. *Let b be a non-coloured Brauer diagram. Let $i\neq j\in\{1,\ldots,k\}$ two integers. If i,j do not lie in the same cycle of $b^\bullet$ ($i\nsim_{b^\bullet\vee 1}j$) then $\mathsf{nc}(e^\bullet_{ij}\vee b^\bullet)=\mathsf{nc}(e_{ij}\vee 1)-1$. If i and j are in the same cycle of $b^\bullet$, we have for any orientation s of $b^\bullet$:*

$$\begin{aligned}&\text{if } s(i)s(j)=-1,\ \mathsf{nc}(b^\bullet\vee e^\bullet_{ij})=\mathsf{nc}(b^\bullet\vee 1)+1,\\&\text{if } s(i)s(j)=1,\ \mathsf{nc}(b^\bullet\vee\tau^\bullet_{ij})=\mathsf{nc}(b^\bullet\vee 1)+1.\end{aligned}\tag{2.30}$$

2.4.4. Central extension of the algebra of coloured Brauer diagrams. As it will appear in Section 2.6, it will be necessary to keep track of the colouration of the loops that are created if two Brauer diagrams are multiplied together. In fact, as of now, it is not possible to do so : from the definition of the algebra structure on $\mathcal{B}_k(\mathsf{d})$, a loop that is produced by multiplication of two diagrams multiply by a positive scalar the concatenation of the two diagrams. It there are least two loops that are created, it not possible to find back the colourizations from this multiplication factor. That is the reason why we introduce a central extension.

The central extension $\mathring{\mathcal{B}}_k(\mathsf{d})$ is, as a vector space, equal to the direct sum of vector spaces $\mathbb{R}\left[\mathcal{B}_k^{\mathsf{d}}\right] \oplus \mathbb{R}[\{\mathsf{o}_d, d \in \{\mathsf{d}\}\}]$. The set $\mathsf{o} = \{\mathsf{o}_d, d \in \{\mathsf{d}\}\}$ of commuting variables is referred to as the set of *loops variables* or *ghost variables*. Two elements $b \oplus P(\mathsf{o})$ and $b' \oplus Q(\mathsf{o})$ in $\mathcal{B}_k^{\mathsf{d}} \oplus \mathbb{R}[\mathsf{o}]$ are multiplied as follows:

$$(b \oplus P(\mathsf{o})) \cdot (b' \oplus Q(\mathsf{o})) = \left(b \circ b', PQ \times \prod_d \mathsf{o}_d^{\mathcal{K}_d(b,b')}\right).$$

We indexed the loops variable by the set $\{\mathsf{d}\}$, we could have equivalently indexed it by the blocks of the partition $\ker(\mathsf{d})$. In the the sequel, we will mainly deal with operators that are defined on a subalgebra of $\mathring{\mathcal{B}}_k(\mathsf{d})$. If $(\alpha_i)_{i \in \mathsf{d}}$ is a multi-index, we denote by $\mathsf{o}_{\{\mathsf{d}\}}^{\alpha}$ the monomial $\mathsf{o}_{d_1}^{\alpha_{d_1}} \cdots \mathsf{o}_{d_p}^{\alpha_{d_p}}$ if $\{\mathsf{d}\} = \{d_1, \ldots, d_p\}$. We set $\mathring{\mathcal{B}}_k^{\mathsf{d}} = \{b \oplus \mathsf{o}_{\{\mathsf{d}\}}^{\alpha},\ b \in \mathcal{B}_k^{\mathsf{d}}, \alpha \in \mathbb{N}^{\{\mathsf{d}\}}\}$. An element of $\mathring{\mathcal{B}}_k^{\mathsf{d}}$ is named a diagram with loops and is pictured as in Fig. 7. Let π a partition of $[\![1,n]\!]$ that is

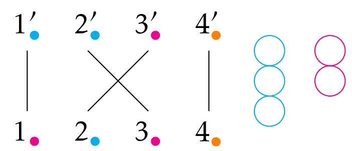


FIGURE 7. A diagram with loops.

greater than $\ker(\mathsf{d})$. As mentioned, the set $\mathcal{B}_k^{\mathsf{d}}$ of coloured Brauer diagram is injected into the set $\mathcal{B}_k^{\pi}$. There is no injection of the vector space $\mathbb{R}\left[\mathcal{B}_k^{\mathsf{d}}\right] \oplus \mathbb{R}[\{\mathsf{o}_d, d \in \{\mathsf{d}\}\}]$ into the vector space $\mathbb{R}\left[\mathcal{B}_k^{\pi}\right] \oplus \mathbb{R}[\{\mathsf{o}_V, V \in \{\pi\}\}]$. The only canonical map from $\mathbb{R}[\{\mathsf{o}_d, d \in \{\mathsf{d}\}\}]$ to $\mathbb{R}[\{\mathsf{o}_V, V \in \{\pi\}\}]$ is the projection induced by the change of variable $\mathsf{o}_V \to \mathsf{o}_W$, $V \in \ker(\mathsf{d})$, $W \in \pi$ with $V \subset W$.

The space $\mathring{\mathcal{B}}_k^{\mathsf{d}}$ projects onto the algebra of coloured Brauer diagrams. The projection $\mathring{\pi}$: $\mathring{\mathcal{B}}_k(\mathsf{d}) \to \mathcal{B}_k(\mathsf{d})$ specializes a loop variable o_d to the corresponding dimension d:

$$\pi((b, P(\mathsf{o}_d, d \in \{d\}))) = P(d, d \in \{d\})b.$$

We draw in Fig. 8 the short exact sequence.

$$0 \longrightarrow \mathbb{R}[\mathsf{o}] \longrightarrow \mathbb{R}\left[\mathring{\mathcal{B}}_k^{\mathsf{d}}\right] \xrightarrow{\ \pi\ } \mathbb{R}\left[\mathcal{B}_k^{\mathsf{d}}\right] \longrightarrow 0.$$

FIGURE 8. A central extension of the algebra of coloured Brauer diagrams.

If $b \in \mathcal{B}_k^{\mathsf{d}}$ is a Brauer diagram, we denote by $\mathring{b}$ the element $(b, 1)$ in the $\mathring{\mathcal{B}}_k^{\mathsf{d}}$. In this way we define a section from $\mathcal{B}_k^{\mathsf{d}}$ to $\mathring{\mathcal{B}}_k^{\mathsf{d}}$ which is not a algebra morphism.

We finish with the definition of the functions that justify alone the introduction of this central extension. In the last section, we gave orientation to Brauer diagrams. We will do the same for Brauer diagrams with loops in a consistent way. We recall that we denote by $\mathcal{OB}_k(\mathsf{d})$ the vector space with basis the set $\mathcal{OB}_k^{\mathsf{d}}$ of all oriented coloured Brauer diagrams. We denote by $\mathcal{O}\mathring{\mathcal{B}}_k(\mathsf{d})$ the vector space $\mathcal{OB}_k(\mathsf{d}) \oplus \mathbb{R}[\mathsf{o}_d, d \in \{\mathsf{d}\}]$.

Let $d \in \{\mathsf{d}\}$ and $\tilde{b} = ((b,s), \prod_{d\in\{d_N\}} \mathsf{o}_d^{n_d}) \in \mathcal{O}\mathring{\mathcal{B}}_k(\mathsf{d})$, the function fnc_d counts the number of loops variables o_d and the number of cycles of b whose minimum m or minimum m' is coloured with the dimension d, depending on the orientation of the cycle. More formally:

$$\mathsf{fnc}_d\left(\tilde{b}\right) = n_d + \sharp\left\{\{i_1 < \ldots < i_k\} \in b^\bullet \vee 1 : d_{c_b(i_1)} = d \text{ and } s(m) = 1\right\}$$
$$+ \sharp\left\{\{i_1 < \ldots < i_k\} \in b^\bullet \vee 1 : d_{c_b(i_1')} = d \text{ and } s(m) = -1\right\}.$$

In the last section, given an oriented Brauer diagram (b_1, s) and a Brauer diagram b, we defined the oriented Brauer diagram $b \diamond (b_1, s)$. This operation can be lifted to $\mathcal{O}\mathring{\mathcal{B}}_k(\mathsf{d})$:

$$((b,P) \diamond ((b_1,s),Q) = \left(b \diamond (b_1,s), PQ \times \prod_{d\in\{d\}} \mathsf{o}_d^{\mathcal{K}_d(b,b_1)} \right),\ (b,P) \in \mathring{\mathcal{B}}_k(\mathsf{d}),\ ((b_1,s),Q) \in \mathcal{O}\mathring{\mathcal{B}}_k(\mathsf{d}).$$

2.4.5. Special subsets of coloured Brauer diagrams. We introduce subsets of Brauer diagrams that will be used in Sections 2.5 and 2.6 to express generators of differential systems satisfied by statistics of the unitary Brownians motions.

The first of these sets is the set of non-mixing Brauer diagrams, which we denote $\mathcal{B}_{k,n}$, that is defined as a set of coloured Brauer diagrams which blocks are coloured by a single integer $1 \le i \le n$.

We denote by $\Delta_{k,n}$ the subset of $\mathcal{B}_{k,n}$ of diagonally coloured Brauer diagrams : diagrams b that have a colouring c_b satisfying $c_b(i) = c_b(i')$ for all integers $1 \le i \le n$.

Let $i, j \le k-1$. The projector $e_{ij}^\bullet \in \mathcal{B}_k^\bullet$ and the transposition $\tau_{ij}^\bullet$ are non-coloured Brauer diagrams that are defined by

$$\begin{aligned} e_{ij}^\bullet &= \{\{i,j\},\{i',j'\}\} \cup \{\{x,x'\},\ x \neq i,j,\ 1 \le x \le k\} \\ \tau_{ij}^\bullet &= \{\{i,j'\},\{j,i'\}\} \cup \{\{x,x'\},\ x \neq i,j,\ 1 \le x \le k\}. \end{aligned} \tag{2.31}$$

The set of non-coloured Brauer diagrams comprising all non-coloured transpositions, respectively all non-coloured projectors, is denoted $\mathsf{T}_k^\bullet$ and $\mathsf{W}_k^\bullet$. The subset of coloured Brauer diagrams in $\mathcal{B}_k^{\mathsf{d}}$ which non-coloured component is a projector, respectively a transposition, is denoted $\mathsf{W}_{k,\mathsf{d}}$, respectively $\mathsf{T}_{k,\mathsf{d}}$. The set of non-mixing transpositions, respectively projectors is denoted $\mathsf{T}_{k,n}$, respectively $\mathsf{W}_{k,n}$. We will frequently drop the subscript n if the numbers of colours is clear from the context or the subscript d if the partition is clear from the context. Elements of the set $\mathsf{T}_k \cup \mathsf{W}_k$ are called *elementary diagrams*.

We also define the sets of *exclusive transpositions* $\mathsf{T}_{k,\mathsf{d}}^{\neq}$ and the set of *exclusive projectors* $\mathsf{W}_{k,\mathsf{d}}^{\neq}$ by setting

$$\mathsf{T}_{k,\mathsf{d}}^{\neq} = \{(\tau_{ij}^\bullet, c_{\tau_{ij}}) \in \mathsf{T}_k : c(i) \neq c(j)\},\ \mathsf{W}_{k,\mathsf{d}}^{\neq} = \{(e_{ij}^\bullet, c_{e_{ij}}) \in \mathsf{W}_k : c(i) \neq c(i')\}.$$

The sets of *diagonal transpositions* and *diagonal projectors* are defined by

$$\mathsf{T}_{k,\mathsf{d}}^{=} = \{(\tau_{ij}^\bullet, c) \in \mathsf{T}_{k,\mathsf{d}} : c(i) = c(j)\},\ \mathsf{W}_{k,\mathsf{d}}^{=} = \{(e_{ij}^\bullet, c) \in \mathsf{W}_{k,\mathsf{d}} : c(i) = c(i')\}.$$

In Fig. 9, we draw examples of elements of the subsets defined above.

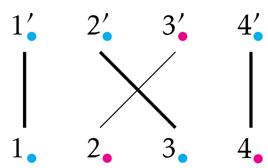


FIGURE 9. Example of an exclusive transposition, $n = 2$.

Let $b^\bullet$ be a non coloured Brauer diagram. The sets $\mathsf{T}_k^{+,\bullet}(b^\bullet)$ and $\mathsf{W}_k^{+,\bullet}(b^\bullet)$ of elementary non-coloured Brauer diagrams that create a cycle if concatenated with b are defined as

$$\mathsf{T}_k^{+,\bullet}(b^\bullet) = \{\tau^\bullet \in \mathsf{T}_k^\bullet :\ \mathsf{nc}(b^\bullet \vee \tau^\bullet) = \mathsf{nc}(b^\bullet \vee 1) + 1\},$$
$$\mathsf{W}_k^{+,\bullet}(b^\bullet) = \{e^\bullet \in \mathsf{W}_k^\bullet :\ \mathsf{nc}(b^\bullet \vee e^\bullet) = \mathsf{nc}(b^\bullet \vee 1) + 1\}.$$

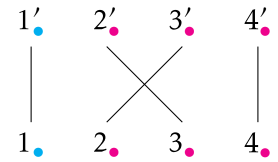


FIGURE 10. Example of a diagonal transposition, $n = 2$.

We end this section by defining four more subsets. Let $b \in \mathcal{B}_k$, we define the subsets

$$\mathsf{T}_k^{\neq,+}(b) = \{\tau \in \mathsf{T}_k^{\neq} : \tau b \neq 0,\ \mathsf{nc}(b^\bullet \vee \tau^\bullet) = \mathsf{nc}(b^\bullet \vee 1) + 1\},$$
$$\mathsf{W}_k^{\neq,+}(b) = \{e \in \mathsf{W}_k^{\neq} : eb \neq 0,\ \mathsf{nc}(b^\bullet \vee e^\bullet) = \mathsf{nc}(b^\bullet \vee 1) + 1\}.$$

and

$$\mathsf{T}_{k,2}^{=,+}(b) = \{\tau \in \mathsf{T}_k^{=} : \tau b \neq 0,\ \mathsf{nc}(b^\bullet \vee \tau^\bullet) = \mathsf{nc}(b^\bullet \vee 1) + 1\},$$
$$\mathsf{T}_{k,2}^{=,+}(b) = \{e \in \mathsf{W}_k^{=} : eb \neq 0,\ \mathsf{nc}(b^\bullet \vee e^\bullet) = \mathsf{nc}(b^\bullet \vee 1) + 1\}.$$

2.4.6. Invariant polynomials and Brauer diagrams. Let $k \geq 1$ and $\mathsf{d} = (d_1, \ldots, d_n)$ a partition of N and $\{\mathsf{d}\} = \{d^1, \ldots, d^n\}$. Let $\mathbb{K}$ be one ot the three division algebras $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. We introduce the two following compact Lie groups:

$$U(\mathsf{d},\mathbb{K}) = U_{d_1}(\mathbb{K}) \times \cdots \times U_{d_n}(\mathbb{K}) \text{ and } U^\sharp(\mathsf{d},\mathbb{K}) = U_{d^1}(\mathbb{K}) \times \ldots \times U_{d^p}(\mathbb{K}),$$

with $\{d_1, \ldots, d_n\} = \{d^1 < \ldots < d^p\}$. The group $U^\sharp(\mathsf{d},\mathbb{K})$ is injected into $U(\mathsf{d},\mathbb{K})$ using the diagonal injection of each factor $U(d^i)$ into the product $U(\mathsf{d},\mathbb{K})$.

An $U(\mathsf{d},\mathbb{K})$-invariant polynomial on $\mathcal{M}_N(\mathbb{K})^k$ is a polynomial function $p : \mathcal{M}_N(\mathbb{K})^k \to \mathbb{K}$ and invariant by the diagonal conjugacy action $\mathsf{conj}_{\mathsf{d}}^k$ of the group $U(\mathsf{d},\mathbb{K})$ on $\mathcal{M}_N(\mathbb{K})^{\times k}$,

$$p\left(\mathsf{conj}_{\mathsf{d}}^k(U)(A_1, \ldots, A_k)\right) = f(UA_1U^{-1}, \ldots, UA_kU^{-1}) = f(A_1, \ldots, A_k),\ \forall U \in U^\sharp(\mathsf{d},\mathbb{K}).$$

The $U^\sharp(\mathsf{d},\mathbb{K})$ invariant polynomial functions are defined similarly. We denote by $P(k,\mathbb{K},\mathsf{d})$ (resp. $P^\sharp(k,\mathbb{K},\mathsf{d})$) the set of all $U(\mathsf{d},\mathbb{K})$ (resp. $U^\sharp(\mathsf{d},\mathbb{K})$) (real if $\mathbb{K} = \mathbb{R}$ and complex if $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$) invariant polynomials on $\mathcal{M}_N(\mathbb{K})^{\times k}$. The set of invariant tensors $\mathrm{Inv}(k,\mathbb{K},\mathsf{d})$ is the set of all elements in $\mathcal{M}_N(\mathbb{K})^{\otimes k}$ fixed by the k folded conjugacy action $\mathsf{conj}_{\mathsf{d}}^k$ of $U(\mathsf{d},\mathbb{K})$ on $\mathcal{M}_N(\mathbb{K})^{\otimes k}$:

$$Z \in \mathsf{inv}_{\mathsf{d}}^k \Leftrightarrow \mathsf{conj}_{\mathsf{d}}^k(Z) = Z.$$

In the sequel, we denote by $\mathsf{Tr}_{\mathbb{K}}^{\otimes k}$ the k folded tensor product of the matricial trace on $\mathcal{M}_N(\mathbb{K})^{\otimes k}$ if $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\mathsf{Tr}_{\mathbb{H}}^{\otimes k} = \mathcal{R}e(\mathsf{Tr})^{\otimes k}$. The trace $\mathsf{Tr}^{\otimes k}$ defines a non degenerate bilinear pairing, $\langle A, B\rangle = \mathsf{Tr}^{\otimes k}(AB)$ for $A, B \in \mathcal{M}_N(\mathbb{K})$ which is an $U(\mathsf{d},\mathbb{K})$ invariant polynomial of $(\mathcal{M}_N(\mathbb{K})^{\otimes k} \otimes \mathcal{M}_N(\mathbb{K})^{\otimes k})^\star$. Hence, any $U(\mathsf{d},\mathbb{K})$-invariant tensor Z defines an invariant polynomial f_Z:

$$f_Z(A_1, \ldots, A_k) = \langle Z, A_1 \otimes \ldots \otimes A_k\rangle,\ A_1, \ldots, A_k \in \mathcal{M}_N(\mathbb{K}). \tag{2.32}$$

We denote by $\mathbb{K}\left[X_1, \ldots, X_k, X_1^\star, \ldots, X_k^\star\right]$ the algebra of polynomials in k non commutating indeterminates. Any invariant polynomial is of the form 2.32. A set of generators for the $U(d,\mathbb{K})$ invariant polynomials with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, can be set equal to

$$\mathsf{Tr}(P(A_1, \ldots, A_k)),\ \text{with } P \in \mathbb{K}[X_1, \cdots, X_k, X_1^\star, \ldots, X_k^\star]\}$$

with $A^\star$ equal to the transpose of A for the case $\mathbb{K} = \mathbb{R}$ and $A^\star$ equal to the Hermitian conjugated matrix A for the case $\mathbb{K} = \mathbb{C}$. For the compact symplectic group $U(N,\mathbb{H}) = Sp(N)$, the complex algebra generated by the set

$$\mathsf{Tr}(P(A, A^\star)), P \in \mathbb{C}[X_1, \cdots, X_k]\}$$

with $\star$ the quaternionic conjugation being a proper subalgebra of $P(\mathbb{H}, k, d)$. Simple results of invariant theory we recall below lead us to the determination of the $U^\sharp(\mathsf{d},\mathbb{K})$ and $U(\mathsf{d},\mathbb{K})$ tensor invariants in $\mathcal{M}_N(\mathbb{K})^{\otimes k}$.

In fact, let $d \geq 1$ an integer and ρ^i_d, $1 \leq i \leq 2$ two representations of $U(d,\mathbb{K})$ on $\mathbb{K}^d$. The space invariants of the sum $\rho^1 \oplus \rho^2$ of the two representations is the sum of the invariants of ρ_1 and ρ^2 since

$$\forall U \in U(d,\mathbb{K}),\ Z_1, Z_2 \in \mathbb{K}^d,\ \left(\rho^1 \oplus \rho^2\right)(U)(Z_1+Z_2) = Z_1 + Z_2, \Leftrightarrow \rho^1(U)(Z_1) = Z_1 \text{ and } \rho^2(Z_2) = Z_2.$$

Hence the space of invariants of $(\mathsf{nat}_d \oplus \mathsf{nat}_d) \otimes \left(\mathsf{nat}^\star_d \oplus \mathsf{nat}^\star_d\right)$ is the sum of four spaces. Each of these spaces is isomorphic to a space of endomorphisms $\mathsf{End}(V_1, V_2)$ invariant by $\mathsf{nat}_d \otimes \mathsf{nat}^\star_d$ with V_i, $i \leq 2$ one of the two copies of $\mathbb{K}^d$ in $\mathbb{K}^d \oplus \mathbb{K}^d$. A straightforward generalization of this argument for the k-folded action $(\mathsf{nat}_d \oplus \mathsf{nat}_d) \otimes \mathsf{nat}^\star_d \oplus \mathsf{nat}^\star_d$ proves that the polynomials invariants $P(k,\mathbb{K},\mathrm{d})$ with $d_1 = \ldots = d_n = d$ admit as a set of generators

$$\mathsf{Tr}\left(P\left(A^{i_1}_{j_1}, A^{i_2}_{j_2}, \cdots, A^{i_p}_{j_p}, \left(A^{i_1}_{j_1}\right)^t, \cdots, \left(A^{i_p}_{j_p}\right)^t\right)\right)$$

with P a non-commutative polynomial and A^i_j a square block in position (i,j) in the matrix A. For the group $U_{d_1} \times U_{d_2}$ that acts by $\rho_1 \times \rho_2$ on $\mathbb{K}^{d_1} \oplus \mathbb{K}^{d_2}$ defined as

$$(\rho_1 \times \rho_2)(U_1 \times U_2)(Z_1 + Z_2) = \rho_1(U_1)(Z_1) + \rho_2(U_2)(Z_2)$$

the space of invariant is also the sum of the spaces of invariants for ρ_1 and ρ_2. Using the expression

$$\mathsf{nat}_{d_1} \times \mathsf{nat}_{d_2} = \left(\mathsf{nat}_{d_1} \times 1\right) \oplus \left(1 \times \mathsf{nat}_{d_2}\right),$$

we find that the space of tensor invariants for the k-folded action $\left(\left(\mathsf{nat}_{d_1} \times \mathsf{nat}_{d_2}\right) \otimes \left(\mathsf{nat}^\star_{d_1} \times \mathsf{nat}^\star_{d_2}\right)\right)^{\otimes k}$ is a sum of spaces and each term is the space of invariants for a representation of the form

$$\mathsf{nat}_{d_{i_1}} \otimes \mathsf{nat}^\star_{d_{j_1}} \otimes \cdots \otimes \mathsf{nat}_{d_{i_k}} \otimes \mathsf{nat}^\star_{d_{j_k}}, \tag{2.33}$$

where we have written $\mathsf{nat}_{d_1} = \mathsf{nat}_{d_1} \times 1$ and $\mathsf{nat}_{d_2} = 1 \times \mathsf{nat}_{d_2}$ for brevity. We write a representation (2.33) as the tensor products of two representations a and b of respectively $U(d_1)$ and $U(d_2)$

$$\mathsf{nat}_{d_{i_1}} \otimes \mathsf{nat}^\star_{d_{j_1}} \otimes \cdots \otimes \mathsf{nat}_{d_{i_k}} \otimes \mathsf{nat}^\star_{d_{j_k}} = (a \otimes b),$$

by setting a to be equal to the tensor product (2.33) in which nat_{d_2} and $\mathsf{nat}^\star_{d_2}$ have been replaced by the trivial representation. The representation b is defined similarly. The space of invariant of (2.33) is the tensor product of the space of invariants of a and b. One must have an equal number of the representation nat_{d_i} and of its contragredient representation for the space of invariants of the representation (2.33) to be non trivial. Let $A \in \mathcal{M}_n$ and $1 \leq i,j \leq n$ two integers. We denote by $A(i,j)$ the block of A of dimension $d_i \times d_j$ in position (i,j). We denote by $\tilde{A}^i_j$ the matrix of dimension $(d^1 + d^2) \times (d^1 + d^2)$ having the block $A(i,j)$ in position (i,j) and the remaining coefficient equal to 0. A set of generators for the polynomials invariant for the representation $U(d_1) \times U(d_2)$ is given by

$$\mathsf{Tr}\left(P\left(\left(\tilde{A}^i_j,\ \left(\tilde{A}^k_l\right)^t,\ i,j,k,l \leq n\right)\right)\right) \tag{2.34}$$

with P a non commutative polynomial in $2n$ indeterminates. Having discussed what are the polynomials invariants of $\mathsf{nat}_{d_1} \times \mathsf{nat}_{d_2}$ and $\mathsf{nat}_{d_1} \oplus \mathsf{nat}_{d_2}$, it is now straightforward to prove that a set of generators for the polynomial invariants of $U(\mathrm{d})^\sharp$ is given by

$$\mathsf{Tr}\left(P\left(A^i_j,\ i,j,k,l \leq n\right)\right)$$

with P a polynomial in the non commutative indeterminates $X^i_j, i,j \leq n$ and $(X^t)^k_l, k,l \leq n$ that satisfies the following condition. A monomial X is in P is for two consecutive indeterminates in X, say X^i_j and X^l_k in this order, we have $d_j = d_l$. The same condition holds for two consecutives indeterminates X^i_j and ${X^t}^l_k$.

We state three lemmas that justify the introduction of the algebra of coloured Brauer diagrams. We use the following convention: for a matrix $A \in \mathcal{M}_N(\mathbb{R})$ or in $\mathcal{M}_N(\mathbb{C})$, we set the expression $[A]^{-1}$ equal to the transpose of A and we set $[A]^1$ equal to A. If A is a matrix with quaternionic entries, we set $[A]^{-1}$ equal to $A^\star$ (the quaternionic transpose of A).

Lemma 2.25 ([41]). *Let $N \geq 1$ an integer and $A_1,\ldots,A_k \in \mathcal{M}_n(\mathbb{R})$. Let b be a non-coloured oriented Brauer diagram, then*

$$\mathrm{Tr}^{\otimes k}((A_1 \otimes \ldots \otimes A_k) \circ \rho_n^\bullet(b)) = \prod_{(i_1,\ldots,i_k)\in\sigma_{(b,s)}} \mathrm{Tr}\left(\left[A_{i_1}\right]^{s(i_1)} \cdots \left[A_{i_k}\right]^{s(i_k)}\right),$$

where we have denoted by $(i_1,\ldots,i_k)$ a cycle of $\sigma_{(b,s)}$.

With the coloured version of the representation ρ_d we defined in the previous section, the following lemma is a simple consequence of Lemma 3.16.

Lemma 2.26. *Let $N \geq 1$ an integer, and $\mathsf{d} = (d_1,\ldots,d_n)$ a partition of N into n parts. Let $A_1,\ldots,A_k \in \mathcal{M}_n(\mathbb{R})$. Let b be a coloured oriented Brauer diagram, then*

$$\mathrm{Tr}^{\otimes k}((A_1 \otimes \ldots \otimes A_k) \circ \rho_{\mathsf{d}}((b,s))) = \prod_{\substack{(i_1,\ldots,i_k)\\ \in\sigma_{(b,s)}}} \mathrm{Tr}\left(\left[A_{i_1}(c_b(i_1),c_b(i_1'))\right]^{s(i_1)} \cdots \left[A_{i_k}(c_b(i_k),c_b(i_k')\right]^{s(i_k)}\right).$$

We also defined a coloured version of the representation $\rho^{\mathbb{H}}$ defined in [41] to study large Brownian quaternionic matrices. The following lemma is a straightforward corollary of the lemma 2.6 in [41].

Lemma 2.27. *Let $A_1,\ldots,A_n \in \mathcal{M}_n(\mathbb{K})$, (b,s) an oriented coloured Brauer diagram, then*

$$\mathcal{R}e(\mathrm{Tr}^{\otimes k})\left(\left(A_1 \otimes \cdots \otimes A_k \circ \rho^{\mathbb{H}}(b)\right)\right) = \prod_{\substack{(i_1,\ldots,i_k)\\ \in\sigma_{(b,s)}}} \mathcal{R}e\mathrm{Tr}\left(\left[A_{i_1}(c_b(i_1),c_b(i_1'))\right]^{s(i_1)} \cdots \left[A_{i_k}(c_b(i_k),c_b(i_k')\right]^{s(i_k)}\right)$$

2.5. Square extractions of an unitary Brownian motion

Let $\mathbb{K} = \mathbb{R},\mathbb{C}$ or $\mathbb{H}$. If $n,d \geq 1$ are two integers, we denote by $U_{n,d}^{\mathbb{K}}$ the quantum process on the dual Voiculescu group $\mathcal{O}\langle n\rangle$ extracting blocks of size $d\times d$ from a unitary Brownian motion of dimension nd:

$$U_{n,d}^{\mathbb{K}}(t): \begin{array}{ccc} \mathcal{O}\langle n\rangle & \to & \mathcal{M}_d(L^{\infty-}(\Omega,\mathcal{F},\mathbb{P})) \\ u_{ij} & \mapsto & \mathrm{U}_N^{\mathbb{K}}(t)(i,j) \end{array} \tag{2.35}$$

The quantum process $U_{n,d}^{\mathbb{C}}$ and $U_{n,d}^{\mathbb{R}}$ are seen as valued in the algebraic probability spaces $\mathcal{M}_N(\mathbb{R}),\mathbb{E}\otimes\frac{1}{d}\mathrm{Tr}$, respectively $\mathcal{M}_N(\mathbb{C}),\mathbb{E}\otimes\frac{1}{d}\mathrm{Tr}$ while for the quaternionic case, it is mandatory to take the real part of the trace, $U_{n,d}^{\mathbb{H}}$ is valued into the algebraic probabilit space $\mathcal{M}_N(\mathbb{H}),\mathbb{E}\otimes\frac{1}{d}\mathcal{R}e\circ\mathrm{Tr}$. This section is devoted to the proof of our main theorem stated below.

Theorem 2.28. *Let $n \geq 1$ an integer. Each of the three processes $U_{n,d}^{\mathbb{K}}$ $\mathbb{K} = \mathbb{R},\mathbb{C}$ or $\mathbb{H}$, converges in non-commutative distribution as $d \to +\infty$ to the free n-dimensional Brownian motion $U^{\langle n\rangle}$.*

Let $t \geq 0$, $\mathbb{K}$ be one of the three divisions algebra in 2.28 and $n \geq 1$ an integer, that stands for the number of blocks the random matrices will be cut into and $d \geq 1$ an other integer that is the dimension of each of these blocks. Set $N = nd$. The proof of 2.28 proceeds as follows. First, we focus on the one dimensional marginals. We begin by showing that a suitable set of statistics of the process which comprises its distribution satisfies a differential system. Secondly, the convergence of the generator as the dimension $d \to +\infty$ of that system is proved and we give a formula for the limit. Finally, we draw a comparison between the generator of the limit of the process $U_{n,d}^{\mathbb{K}}$ and the generator of the free n–dimensional Brownian motion $U^{\langle n\rangle}$. Our proof of the convergence of the multidimensional marginals of the process $U_{n,d}^{\mathbb{K}}$, heavily rely on Theorem 3.39 and conjugation invariance of the law of the Browian motions on unitary groups.

This method has already been applied by Levy in [41] to prove the convergence in non commutative distribution of the Unitary Brownian motion, this corresponds to the case $n = 1$

in our setting. Introducing of the algebra of coloured Brauer diagrams makes the computations for the case $n=1$ and $n>1$ very similar, which is obviously an argument in favor of the lengthy exposition made in Section 2.4.

Concerning the outline, the complex, real and quaternionic cases are treated separately to prove the convergence of the one dimensional marginals.

We use the shorter notation $\rho^{\mathbb{K}}$ for one the three representations of the algebra of coloured Brauer diagrams $\mathcal{B}_k((\underbrace{d,\ldots,d}_{n}))$ we defined in the last section. We use the injection Δ of the algebra $\mathcal{B}_k^{\bullet}(nd)$ of uncoloured Brauer diagrams without making mention of it.

2.5.1. Convergence of the one-dimensional marginals, the complex case. Let $t\geq 0$, the convergence in distribution of the one dimensional marginals of the quantum process $U^{\mathbb{K}}_{n,d}$ is implied by the following proposition.

Proposition 2.29. *Let $t\geq 0$, $r\geq 0$ and $\alpha,\beta\in\{1,\ldots,n\}^r$. The mixed moments of the family*

$$\left\{\mathsf{U}^{\mathbb{C}}_N(t)(\alpha_i,\beta_i),\left(\mathsf{U}^{\mathbb{C}}_N(t)(\alpha_i,\beta_i)\right)^t,\overline{\mathsf{U}^{\mathbb{C}}_N(t)(\alpha_i,\beta_i)},\left(\mathsf{U}^{\mathbb{C}}_N(t)(\alpha_i,\beta_i)\right)^{\star},i\leq r\right\}$$

in the tracial algebra $\mathcal{M}_d(L^\infty(\Omega,\mathcal{F},\mathbb{P},\mathbb{C}),\mathbb{E}\otimes\mathsf{Tr})$ converge as $d\to+\infty$.

Remark. *The last proposition does not only imply the convergence of the distribution of $U^{\mathbb{K}}_{n,d}(t)$. In fact, the convergence stated in Proposition 2.29 is much more general and holds also for words on the matrix transpose ${\mathsf{U}^{\mathbb{K}}_N}^t$ and on the complex conjugate $\overline{\mathsf{U}^{\mathbb{K}}_N}(t)$ (without transposition).*

Owing to equation 2.4 of Proposition 2.5, mean of polynomials of the matrix $\mathsf{U}^{\mathbb{C}}_N(t)$ admits the following combinatorial formula :

$$\mathbb{E}\left[w^{\otimes}(\mathsf{U}^{\mathbb{C}}_N(t))\right]=\exp\left(-\frac{kt}{2}-\frac{1}{N}\sum_{\substack{1\leq i,j\leq k,\\ w_i=w_j}}\rho^{\mathbb{C}}(\tau^{\bullet}_{ij})+\frac{1}{N}\sum_{\substack{1\leq i,j\leq k,\\ w_i\neq w_j}}\rho^{\mathbb{C}}(e^{\bullet}_{ij})\right),\ w\in\bar{\mathsf{M}}_2(k). \tag{2.36}$$

As explained previously, we are looking for a linear space of statistics that are polynomial functionals in the coefficients of a matrix (and its conjugate), which is manifestly invariant by the generator of the unitary Brownian motion and contains traces of matrix powers. To define such statistics, we use the algebra of coloured Brauer diagrams defined in Section 2.4. The symbol $\overline{\mathsf{M}}_1$ stands for the free monoids generated by two letters $\{x_1,\overline{x}_1\}$. If $w\in\overline{\mathsf{M}}_1$ is a word of length k and $A\in\mathcal{M}_{nd}(\mathbb{C})$ a complex matrix, $w^{\otimes}(A)$ stands for the monomial in $\mathcal{M}_{nd}(\mathbb{C})^{\otimes}$ that is obtained by the substitutions rules: $x_1\to A,\overline{x}_1\to\overline{A}$. Let $k\geq 1$ an integer. The subset of words in $\overline{M}_1$ of length k is denoted $\overline{\mathsf{M}}_1(k)$. Let $w\in\overline{\mathsf{M}}_1(k)$ a word and b a (coloured) Brauer diagram in $\mathcal{B}_k$. We set

$$\mathsf{m}^{\mathbb{C}}_d(w,b,A)=\mathsf{Tr}(\rho^{\mathbb{C}}(b)\circ w^{\otimes}(A)),\ A\in\mathcal{M}_N(\mathbb{C}). \tag{2.37}$$

We are now concerned with the derivative of the normalized statistic $\mathsf{m}^{\mathbb{R}}_s$ defined by

$$\mathbb{m}^{\mathbb{C}}_d(w,b,t)=d^{-\mathsf{nc}(b^{\bullet}\vee 1_k)}\mathsf{m}^{\mathbb{C}}_d(w,b,\mathbb{E}\left[\mathsf{U}^{\mathbb{C}}_N\right])$$

The statistic $\mathbb{m}^{\mathbb{C}}_d$ is extended as a linear function on the space $\mathcal{B}_k\otimes\overline{\mathsf{M}}_2(k)=\mathcal{B}_k(nd)\otimes\mathbb{R}\left[\overline{\mathsf{M}}_2(k)\right]$. Note that owing to 3.15, the range of $b\to\mathbb{m}^{\mathbb{C}}_d(b,t)$ comprises the distribution of $U^{\mathbb{C}}_{n,d}(t)$. By using formula (2.36), we prove the existence of an operator $L^{\mathbb{C}}_d:\mathcal{F}\left(\mathcal{B}_k\otimes\bar{\mathsf{M}}_2(k)\right)\to\mathcal{F}\left(\mathcal{B}_k\otimes\bar{\mathsf{M}}_2(k)\right)$ such that

$$\frac{d}{dt}\mathbb{m}^{\mathbb{C}}_d(b\otimes w,t)=L^{\mathbb{C}}_d\left(\mathbb{m}^{\mathbb{C}}_d(t)\right)(b\otimes w),\ \mathbb{m}^{\mathbb{C}}_d(b\otimes w,0)=\delta_{\Delta_k}(b),b\otimes w\in\mathcal{B}_k\times\overline{\mathsf{M}}_2(k).$$

In the last formula, we use the notation Δ_k for the support function of the set $\Delta_k\subset\mathcal{B}_{k,n}$ of Brauer diagrams that are diagonally coloured : $(b^{\bullet},c_b)\in\mathcal{B}_k\Leftrightarrow c_b(i)=c_b(i'),\ \forall 1\leq i\leq k$. An explicit expression for the operator $L_d^{\mathbb{C}}$ is given by

$$
\begin{aligned}
L_d^{\mathbb{C}}(g)(w,b) = \frac{k}{2}g(w,b) - \frac{1}{nd}\sum_{\substack{1\leq i,j\leq k\\ w_i=w_j}} d^{\mathsf{nc}(b^\bullet\vee\tau_{ij}^\bullet)-\mathsf{nc}(b^\bullet\vee 1)} g(w,\tau_{ij}^\bullet\circ b)\\
+\frac{1}{nd}\sum_{\substack{1\leq i,j\leq k\\ w_i\neq w_j}} d^{\mathsf{nc}(b^\bullet\vee e_{ij}^\bullet)-\mathsf{nc}(b^\bullet\vee 1)} g(w,e_{ij}^\bullet\circ b).
\end{aligned}
\tag{2.38}
$$

The coefficients $d^{\mathsf{nc}(b^\bullet\vee r_{ij}^\bullet)-\mathsf{nc}(b^\bullet\vee\mathbf{1}_k)}$ are obtained by using the fundamental equality $d^{\mathsf{nc}(b^\bullet\vee r^\bullet)}=\mathsf{nc}(rb^\bullet\vee\mathbf{1}_k)+\mathcal{K}_d(b,r)=\mathsf{nc}(b\vee r)$, with r an elementary diagram and b a coloured Brauer diagram. Note that the symbol $r^\bullet$ stands for two different objects in the last equation. If acting by multiplication on a coloured Brauer diagram, $r^\bullet$ is to be seen as an element of $\mathcal{B}_k$ as described in the introduction of the present section. On the other hand, in the expression $r^\bullet\vee b^\bullet$, the symbol $r^\bullet$ stands for an elementary uncoloured Brauer diagram in $\mathcal{B}_k^\bullet$.

Proposition 3.15 implies that $\mathsf{nc}(b^\bullet\vee\tau_{ij}^\bullet)-\mathsf{nc}(b^\bullet\vee 1)\in\{-1,0,1\}$ and $\mathsf{nc}(b^\bullet\vee e_{ij}^\bullet)-\mathsf{nc}(b^\bullet\vee 1)\in\{-1,0,1\}$. Hence, the two sums in the right hand side of equation (2.38), as $d\to+\infty$, converge to two sums over elementary diagrams r such that $r\circ b$ has one more cycle than b, or rb has a loop (which means $rb=d(r^\bullet\circ b)$). With the notation introduced in Section 2.4,

$$
\begin{aligned}
L_d^{\mathbb{C}}(g)(b\otimes w) &= \frac{k}{2}b\otimes w - \frac{1}{n}\sum_{\substack{1\leq i<j\leq k\\ w_i=w_j,\\ \tau_{ij}^\bullet\in\mathsf{T}_k^{+,\bullet}(b^\bullet)}} g(\tau_{ij}^\bullet\circ b\otimes w) + \frac{1}{n}\sum_{\substack{1\leq i<j\leq k\\ w_i\neq w_j\\ e_{ij}^\bullet\in\mathsf{W}_k^{+,\bullet}(b)}} g(e_{ij}^\bullet\circ b\otimes w) + o_{d,\infty}(\frac{1}{d})\\
&= \bar{L}_n(g)(b\otimes w) + o_{d,\infty}(\frac{1}{d})
\end{aligned}
\tag{2.39}
$$

for any function $g\in\mathcal{F}\left(\mathcal{B}_k\otimes\overline{\mathsf{M}}_2(k)\right)$. Note that the range of the two sums in the last equation does not exhaust the sets $\mathsf{T}_k^+(b)$ or $\mathsf{W}_k^+(b)$ since the elementary diagrams $r^\bullet, r\in\{e,\tau\}$ are, by definition, in the range of the injection Δ of the algebra of uncoloured Brauer diagrams, non mixing. The differential system satisfied by the functional $t\to\mathbb{m}_d^{\mathbb{C}}(t)$ is linear and finite dimensional. Its solutions can be expressed as the (matrix) exponential of its generator $L_d^{\mathbb{C}}$. Since the matrix exponential is a continuous map, convergence of the generator $L_d^{\mathbb{C}}$ implies the convergence of $\mathbb{m}_d^{\mathbb{C}}$ to the solution $\overline{\mathbb{m}}_n$ of:

$$
\frac{d}{dt}\overline{\mathbb{m}}_n(t)=\bar{L}_n(\overline{\mathbb{m}}_n(t)),\ \overline{\mathbb{m}}_n(0)=\delta_{\Delta_k}.
$$

We end this section by stating the convergence of the one dimensional marginal of the process $U_{n,d}^{\mathbb{K}}$. In fact, for each word u on the generators of the dual Voiculescu group, there exists a Brauer diagram b_u and a word w_u such that $\mathsf{m}_d^{\mathbb{C}}(b_w,u_w,t)=\frac{1}{d}\mathsf{Tr}(U_{n,d}^{\mathbb{K}}(t)(w))$. The point-wise convergence of $\mathsf{m}_d^{\mathbb{C}}$ implies the convergence of $\frac{1}{d}\mathsf{Tr}(U_{n,d}^{\mathbb{K}}(t)(w))$ as $d\to+\infty$. Later, we should exploit a property satisfied by the pair (b_u,w_u) to draw a comparison between $\overline{L}_n$ and the generator of the free n-dimensional unitary Brownian motion.

2.5.2. Convergence of the one-dimensional marginals, the real case. Let $t\geq 0$ a time and $k\geq 0$ an integer, that will be the size of the diagrams and length of words that are considered. To treat the real case, we define the real counterpart of the statistics $\mathbb{m}_d^{\mathbb{C}}$ that were defined in the previous section. We are more brief in this section to expose the convergence of the one dimensional marginals of $U_{n,d}^{\mathbb{R}}$ since the method used for the complex case is applied here too. The representation $\rho^{\mathbb{R}}$ is defined in Section 2.4.For each real matrix $A\in\mathcal{M}_N(\mathbb{R})$, we define the function $\mathsf{m}_d^{\mathbb{R}}(t)$ on the set of coloured brauer diagram $\mathcal{B}_k$ by the equation

$$
\mathsf{m}_d^{\mathbb{R}}(b,t)=\mathsf{Tr}\Big(\mathbb{E}\Big[A^{\otimes k}\Big]\circ\rho^{\mathbb{R}}(b)\Big),\ b\in\mathcal{B}_k
$$

and extend it linearly to $\mathbb{R}[\mathcal{B}_k]$. The normalized statistics $\mathbb{m}_d^{\mathbb{R}}(b,t)$ of the real unitary Brownian diffusion is

$$\mathbb{m}_d^{\mathbb{R}}(b,t) = d^{-\mathsf{nc}(b^\bullet \vee 1)}\mathsf{Tr}\Big(\mathbb{E}\Big[\mathsf{U}_N^{\mathbb{R}}(t)^{\otimes k}\Big] \circ \rho^{\mathbb{R}}(b)\Big),\ b \in \mathcal{B}_k.$$

By using the representation $\rho^{\mathbb{R}}$, the mean $\mathbb{E}\Big[\mathsf{U}_N^{\mathbb{R}}(t)^{\otimes k}\Big]$ can be expressed as:

$$\mathbb{E}\Big[\mathsf{U}_N^{\mathbb{R}}(t)^{\otimes k}\Big] = \exp\left(-\frac{kt(N-1)}{2N} + t\left(-\frac{1}{N}\sum_{1\leq i,j\leq k}\rho^{\mathbb{R}}(\tau_{ij}^\bullet) + \frac{1}{N}\sum_{1\leq i<j\leq k}\rho^{\mathbb{R}}(e_{ij}^\bullet)\right)\right). \tag{2.40}$$

Once again the range of $\mathbb{m}_d^{\mathbb{R}}(t)$ comprises the distribution of $U_{n,d}^{\mathbb{R}}$. We compute the derivative of $\mathbb{m}_d^{\mathbb{R}}$. We apply formula (2.40) to get:

$$\frac{d}{dt}\mathbb{m}_d^{\mathbb{R}}(b,t) = L_d^{\mathbb{R}}(\mathbb{m}_d^{\mathbb{R}}),\ \mathbb{m}_d^{\mathbb{R}}(0) = \delta_{\Delta_k}.$$

where the operator $L_d^{\mathbb{R}}$ acting on linear forms $\mathbb{R}[\mathcal{B}_k]$ is defined by the formula:

$$L_d^{\mathbb{R}}(g)(b) = -\tfrac{k(N-1)}{2N}g(b) - \tfrac{1}{nd}\left(\sum_{\tau^\bullet\in\mathsf{T}_k^\bullet} d^{\mathsf{nc}(\tau^\bullet\vee b^\bullet)-\mathsf{nc}(b^\bullet\vee 1)}g(\tau^\bullet\circ b) + \sum_{e^\bullet\in\mathsf{W}_k^\bullet} d^{\mathsf{nc}(e^\bullet\vee b^\bullet)-\mathsf{nc}(b^\bullet\vee 1)}g(e^\bullet\circ b)\right).$$

with $g \in \mathbb{R}[\mathcal{B}_k]^\star$. Again, as for the complex case, the last two sums localize to sums over diagrams in, respectively $\mathsf{T}_k^+(b)$ and $\mathsf{W}_k^+(b)$ if we let $d \to +\infty$. If we define the operator L_n acting on functions of Brauer diagrams, by

$$L_n(g) = -\frac{k}{2}g(b) - \frac{1}{n}\sum_{\tau^\bullet\in\mathsf{T}_k^{+,\bullet}(b)} g(\tau^\bullet\circ b) + \frac{1}{n}\sum_{e^\bullet\in\mathsf{W}_k^{+,\bullet}(b)} g(e^\bullet\circ b), \tag{2.41}$$

with $g \in \mathbb{R}[\mathcal{B}_k]^\star$, we get $L_d^{\mathbb{R}} = L_n + o_{d,\infty}(d)$. Again, as for the complex case, the convergence of the generator of the system satisfied by the statistics $\mathsf{m}_d^{\mathbb{R}}$ implies the convergence of the solution to the function $\mathbb{m}_n$ that satisfies the differential system:

$$\frac{d}{dt}\mathbb{m}_n(t) = L_n(\mathbb{m}_n(t)),\ \mathbb{m}_n(0) = \delta_{\Delta_k}.$$

2.5.3. Convergence of the one-dimensional marginals, the quaternionic case. Let $t \geq 0$ a time. As we did for the complex and real cases, we use the representation $\rho^{\mathbb{H}}$ to define, for any matrix of size $N\times N$ with quaternionic entries, a function on the set of coloured Brauer diagrams $\mathcal{B}_{kk}$ of size k (we recall that the dimension function used to defined $\mathcal{B}_{kk}$ is the constant dimension function on $[\![1,\ldots,n]\!]$ equal to d),

$$\mathsf{m}_d^{\mathbb{H}}(A,b) = -2\mathcal{R}e\mathsf{Tr}\Big(A\circ\rho^{\mathbb{H}}(b)\Big),\ A \in \mathcal{M}_N(\mathbb{H})$$

Further, we normalize these statistics using a slighlty different normalization factor that was used for the real and the complex case, the main reason being that $\rho^{\mathbb{H}}$ is a representation of the algebra $\mathcal{B}_k(-2N)$:

$$\mathbb{m}_d^{\mathbb{H}}(t,b) = \frac{1}{(-2d)^{\mathsf{nc}(b^\bullet\vee 1_k)}}\mathsf{m}_d^{\mathbb{H}}(b,\mathbb{E}\Big[\mathsf{U}_N^{\mathbb{H}}\Big]).$$

For the third time, the range of $\mathbb{m}_d^{\mathbb{H}}(t)$ comprises the distribution of $U_{n,d}^{\mathbb{H}}$. Owing to Proposition 2.5, the mean of tensor monomials of $U_{n,d}^{\mathbb{H}}(t)$ is expressed as:

$$\mathbb{E}\Big[\mathsf{U}_N^{\mathbb{H}}(t)^{\otimes k}\Big] = \exp\left(-\frac{tk}{2}\left(\frac{2N-1}{2N}\right) - \frac{1}{N}\sum_{e^\bullet\in\mathsf{W}_k^\bullet}\rho^{\mathbb{H}}(e^\bullet) + \frac{1}{N}\sum_{\tau^\bullet\in\mathsf{T}_k^\bullet}\rho^{\mathbb{H}}(\tau^\bullet)\right). \tag{2.42}$$

Similar computations as done for the real and complex case lead to a differential system satisfied by the statistics $\mathbb{m}_d^{\mathbb{H}}$. Note that, contrary to the complex case, we do not need an extra parameter (a word in M_2), even if the conjugation is not trivial on $\mathbb{H}$. Let $b \in \mathcal{B}_k$ a Brauer diagram, there exists an on operator $L_d^{\mathbb{H}}$ acting on the space of functions on $\mathcal{B}_k$ such that:

$$\frac{d}{dt}\mathbb{m}_d^{\mathbb{H}}(b,t) = L_d^{\mathbb{H}}(\mathbb{m}_d^{\mathbb{H}}(t))(b),\ \mathbb{m}_d^{\mathbb{H}}(0) = \delta_\Delta.$$

The operator $L_d^{\mathbb{H}}$ is given by the formula

$$L_d^{\mathbb{H}}(g)(b) = -\frac{k(2N+1)}{4N}g(b) - \frac{1}{n}\sum_{\tau^\bullet \in \mathsf{T}_k}(-2d)^{\mathsf{nc}(b^\bullet\vee\tau^\bullet)-\mathsf{nc}(b^\bullet\vee 1)-1}g(\tau^\bullet\circ b)$$
$$+\frac{1}{n}\sum_{e^\bullet\in\mathsf{W}_k^\bullet}(-2d)^{\mathsf{nc}(b^\bullet\vee e^\bullet)-\mathsf{nc}(b^\bullet\vee 1)-1}g(e^\bullet\circ b)$$

where $g \in \mathbb{R}[\mathcal{B}_k]^\star$. As d tends to infinity the generator $L_d^{\mathbb{H}}$ converges to L_n defined in equation (2.41). This is sufficient to prove the convergence of the one dimensional marginals of $U_{n,d}^{\mathbb{R}}$. In addition, $U_{n,d}^{\mathbb{R}}$ and $U_{n,d}^{\mathbb{H}}$ converge in non-commutative distribution to the same limit.

2.5.4. Convergence of the one-dimensional marginals : conclusion. Let $\mathbb{K}$ be one of the three division algebras $\mathbb{R},\mathbb{C}$ or $\mathbb{H}$. In the last section we proved the convergence of the one dimensional marginals of the process $U_{n,d}^{\mathbb{K}}$. We exhibit differential system the limiting distributions are solution of and saw that limiting non-commutative distributions of the one dimensional marginals of the real and quaternionic processes are equal, we prove now that it also holds for the complex one dimensional marginals.

To that end, we define first the notion of compatible pairs in $\mathcal{B}_k\times\overline{\mathsf{M}}_1(k)$. Let $b\in\mathcal{B}_k$ a Brauer diagram. Recall $s_b^\bullet$ denotes the orientation of b that is positive on the minimum of the cycles of b. Define the subset $\mathcal{S}(b)$ of $\{-1,1\}^k$ by $S(b) = \{s\in\{-1,1\}^k : s_b^\bullet(i)s_b^\bullet(j) = s_is_j\ \forall\, i,j\le k,\ i\sim_{b^\bullet\vee 1} j\}$ and the set of compatible words and diagrams by $\mathcal{C} = \{(b,w^s), b\in\mathcal{B}_k,\ s\in\mathcal{S}(b)\}$. If $w\in\overline{\mathsf{M}}_1(k)$ is a word, $\overline{w}$ is obtained by substituting x_1 in place of $\overline{x}_1$ and vice-versa. The operator L_n,respectively $\overline{L}_n$, acts on the space of linear forms on $\mathbb{R}[\mathcal{B}_k]$, respectively on the space of linear forms on $\mathbb{R}\left[\overline{\mathsf{M}}_1\right]\otimes\mathbb{R}[\mathcal{B}_k]$. To state the next proposition, we find convenient to consider the dual operators $L_n^\star$ and $\bar{L}_n^\star$ acting, respectively, on $\mathbb{R}[\mathcal{B}_k]$ and $\mathbb{R}\left[\overline{\mathsf{M}}_1(k)\right]\otimes\mathbb{R}[\mathcal{B}_k]$.

Lemma 2.30. *Let $b\in\mathcal{B}_k$ a Brauer diagram and $w\in\mathsf{M}_2(k)$ a word. For any time $t\ge 0$, one has $\overline{\mathbb{m}}_n(b,w,t) = \overline{\mathbb{m}}_n(b,\overline{w},t)$. The real vector space generated by tensors of compatible words and diagrams is stable by the action $\overline{L}_n^\star$. In addition, $\bar{L}_n^\star(b\otimes w) = (L_n^\star(b)\otimes w)$ for any pair $(b,w)\in\mathcal{C}$.*

Proof. The first assertion of Lemma 2.30 is trivial. Let $r\in\mathsf{W}_k^+(b)\cup\mathsf{T}_k^+(b)$, then $s_{r_{ij}\circ b}^\bullet(i)$ $s_{r_{ij}\circ b}^\bullet(j) = s_b^\bullet(i)s_b^\bullet(j)$ whenever $i\sim_{rb^\bullet\vee 1} j$. It follows that $\mathcal{C}$ is stable by $\bar{L}_n$. Let b a Brauer diagram. Let $(b,w^s)\in\mathcal{C}$ with $s\in S(b)$. From Proposition 3.15, Section 2.4, $e_{ij}^\bullet\in\mathsf{W}_k^+(b)$ if and only if $s_b^\bullet(i)s_b^\bullet(j) = -1 = s(i)s(j)$ and $\tau_{ij}^\bullet\in\mathsf{T}_k^+(b)$ if and only if $s_b^\bullet(i)s_b^\bullet(j) = 1 = s(i)s(j)$. Since $w_i^s = \overline{w^s}(j)\Leftrightarrow s(i)s(j) = -1$ and $w_i^s = w^s(j)\Leftrightarrow s(i)s(j) = 1$ we get $\bar{L}_n^\star(b,w) = (L_n^\star(b),w)$. $\square$

If b and w are compatible diagrams and word, with b having only one cycle, Lemma 2.30 implies the equalities:

$$\overline{\mathbb{m}}_n(b,w,t) = \overline{\mathbb{m}}_n(b,\overline{w},t),\text{ and }\overline{\mathbb{m}}_n(b,w) = \mathbb{m}_n(b),$$

if the diagram b has more than one cycle, the word w in the first of the last two inequalities can partially conjugated : we can swap exchange all letters x_i and $\overline{x_i}$ which positions in the word w are two integers belonging to the same cycle of b and leave the other letters untouched.

Now, to each word $u\in\mathcal{O}\langle n\rangle$ is associated a pair (b_u,w_u) of compatible word and diagram (we are simply requiring that transposition and conjugation occur both at a time on the matrix $\mathsf{U}_N^{\mathbb{C}}(t)$) such that $\frac{1}{d}\mathbb{E}\mathsf{Tr}(U_{n,d}^{\mathbb{C}}(t)(u)) = \mathbb{m}_d^{\mathbb{C}}(b_u,w_u,t)$. As the dimension tends to infinity, $\frac{1}{d}\mathbb{E}\left[\mathsf{Tr}(U_{n,d}^{\mathbb{C}}(t)(u))\right]\to\overline{\mathbb{m}}_n(b_u,w_u,t) = \mathbb{m}_n(b_u,t)$.

Now owing to the formula (2.10) for the generator of the pseudo-unitary diffusion, $\mathcal{L}_n(u) = \delta_{\Delta_k}(L_n(b_u))$. This last inequalities implies

$$U^{\mathbb{K}}_{n,d}(t) \overset{\text{dist.nc.}}{\to} U^{\langle n\rangle}(t). \tag{2.43}$$

2.5.5. Convergence of the multidimensional marginals. To finish the proof of Theorem 2.28, we prove convergence of the multidimensional marginals of $U^{\mathbb{K}}_{n,d}$ by using Theorem 3.39, which in turn relies on conjugation invariant property of the process's distribution; for any unitary matrix in $\mathbb{U}(d,\mathbb{K})$ and words $u_1,\dots,u_q \in \mathcal{O}\langle n\rangle$ in the dual Voiculescu group, the family $\{UU^{\mathbb{K}}_{n,d}(t_1)U^{-1},\dots,UU^{\mathbb{K}}_{n,d}(t_q)U^{-1}\}$ has the same non-commutative distribution as $\{U^{\mathbb{K}}_{n,d}(t_1),\dots,U^{\mathbb{K}}_{n,d}(t_q)\}$.

Theorem 2.31 (Voiculescu; Collins,Sniady). *Choose $\mathbb{K}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$. Let $(A_{N,1},\dots,A_{N,n})_{N\geq 1}$ and $(B_{N,1},\dots,B_{N,n})$ be two sequences of families of random matrices with coefficients in $\mathbb{K}$. Let $a_1,\dots,a_n$ and $b_1,\dots,b_n$ be two families of elements of a non commutative probability space $(\mathcal{A},\tau)$. Assume that the convergence in non-commutative distribution*

$$(A_{N,1},\dots,A_{N,n}) \to (a_1,\dots,a_n) \text{ and } (B_{1,N},\dots,B_{N,n}) \to (b_1,\dots,b_n)$$

hold. Assume also that for all N, given a random matrix U distributed according to the Haar measure on $U(N,\mathbb{K})$ and independent of $(A_{N,1},\dots,A_{N,n},B_{N,1},\dots,B_{N,n})$, the two families $(A_{N,1},\dots,A_{N,n},B_{N,1},\dots,B_{N,n})$ and $\left(UA_{N,1}U^{-1},\dots,UA_{N,n}U^{-1},B_{N,1},\dots,B_{N,n}\right)$ have the same distribution. Then the families $\{a_1,\dots,a_n\}$ and $\{b_1,\dots,b_n\}$ are free.

Recall that we endowed the Dual Voiculescu group $\mathcal{O}\langle n\rangle$ with a coproduct Δ with value in the free product $\mathcal{O}\langle n\rangle$, a counit ϵ and an antipode S that makes $\mathcal{O}\langle n\rangle$ into a Zhang's algebra (see Chapter 1). The two parameters family $\left(U^{\mathbb{K}}_{n,d}(s,t)\right)_{s,t\geq 0}$ of increments of the process $\left(U^{\mathbb{K}}_{n,d}(t)\right)_{t\geq 0}$ is defined as

$$U^{\mathbb{K}}_{n,d}(s,t) = \left(U^{\mathbb{K}}_{n,d}(t) \sqcup \left(U^{\mathbb{K}}_{n,d}(s)\circ S\right)\right)\circ\Delta.$$

and the increments $\left(U^{\langle n\rangle}(s,t)\right)_{0\leq s\leq t\leq+\infty}$ of the free n dimensional unitary Brownian motion satisfy

$$U^{\langle n\rangle}(s,t) = \left(U^{\langle n\rangle}(t) \sqcup \left(U^{\langle n\rangle}(t)\circ S\right)\right)\circ\Delta.$$

The $U(d)$ invariance of the finite dimensional marginals of $U^{\mathbb{K}}_{n,d}(s,t)$ combined with the following result of asymptotic freeness stated in Theorem 3.39 are the last two ingredients that end the proof of Theorem 2.28. In fact, we show by recurrence that as $d\to+\infty$, for any tuples $s_1 < t_1 \leq s_2 < t_2 \cdots s_p < t_p$, the random variables

$$U^{\mathbb{K}}_{n,d}(s_1,t_1),\dots,U^{\mathbb{K}}_{n,d}(s_p,t_p)$$

are asymptotically free. Let $s_0 < s_1$, then $U^{\mathbb{K}}_{n,d}(s_0,s_1)$ has the same non-commutative distribution as $U^{\mathbb{K}}_{n,d}(s_1-s_0)$. Thus, $U^{\mathbb{K}}_{n,d}(s_0,s_1)$ converges to $U_n(s_0,s_1)$. Pick a two tuples of time such that $0<s_0<s_2<\cdots<s_p$. Assume that the family $\{U^{\mathbb{K}}_{n,d}(s_0,s_1),\dots,U^{\mathbb{K}}_{n,d}(s_{p-2},s_{p-1})\}$ converges to $\{U^{\langle n\rangle}(s_0,s_1),\dots,U^{\langle n\rangle}(s_{p-2},s_{p-1})\}$ in non-commutative distribution. We proved that $U^{\mathbb{K}}_{n,d}(s_{p-1},s_p)$ converges to $U^{\langle n\rangle}(s_{p-1},s_p)$. Besides, the law of $U^{\mathbb{K}}_{n,d}(s_{p-1},s_p)$ is invariant by conjugation by any element of $U(d)$ and $U^{\mathbb{K}}_{n,d}(s_{p-1},s_p)$ is independent from the family $\{U^{\mathbb{K}}_{n,d}(s_0,s_1),\dots,U^{\mathbb{K}}_{n,d}(s_{p-2},s_{p-1})\}$ thus an application of Theorem 3.39 shows that

$$\{U^{\mathbb{K}}_{n,d}(s_0,s_1),\dots,U^{\mathbb{K}}_{n,d}(s_{p-1},s_p)\} \underset{d\to+\infty}{\overset{n.c}{\to}} \{\{U^{\langle n\rangle}(s_0,s_1),\dots,U^{\langle n\rangle}(s_{p-1},s_p)\}.$$

2.6. Rectangular extractions of an unitary Brownian motion

In that section, we extend the result we proved in the last section stating the convergence in non-commutative distribution of square blocks extracted from an unitary matrix by allowing these blocks to be rectangular. But first, we briefly expose amalgamated non-commutative probability theory.

2.6.1. Operator valued probability theory. In this section, we make an overview of operator valued probability theory. Categorical notions are used without recalling them, for brevity. The reader can refer to the first chapter in which he will find a detailed exposition on Zhang algebras, categorical independance, categorical coproduct and comodule algebras.

2.6.1.1. *Involutive bimodule Zhang algebras.* In the sequel, algebras are complex or real unital algebras. Let A and B two algebras. Let R be a third algebra and assume that A and B are R-bi-modules; there exists a left and a right action commuting which each other such that:

$$(rr')a = r(r'a),\; a(rr') = ar(r'),\; 1a = a,\; r(ar') = (ra)r',\; r \in R,\; a \in A$$

In this work, we mainly deal with involutive algebras. An involutive algebra C is endowed with an involutive anti-morphism $\star_A : C \to C$ that is linear if C is a real algebra, anti-linear if C is complex. We assume the three algebras A, B and R to be involutive and the following compatibility condition between the bi-module structure and the anti-morphisms $\star_R$ and $\star_A$:

$$\star_A(r \cdot a) = \star_A(a) \cdot \star_R(r),\; \star_A(a \cdot r) = \star_R(r) \cdot \star_A(a),\; r \in R,\; a \in A,$$

A significant construction is the amalgamated free product of A and B over R, denoted by $A \sqcup_R B$. This amalgamated free product turns the category of involutive bi-module algebras into an algebraic category, more on this is explained at the end of the paragraph and in Chapter 1.

The amalgamated free product is, to put it in words, the free product of A and B in which we forget from which algebra the letters that belong to R comes from. For instance, if $a \in A$, $b \in B$ and $r \in R$: $a(rb) = (ar)b \in A \sqcup_R B$. In symbols, the amalgamated free product is the quotient:

$$\begin{aligned} A \sqcup_R B = (R \oplus \bigoplus_{n \geq 1} T^n(A \oplus B))/(&ar \otimes r'a' - arr'a', br \otimes r'b' - brr'b', ar \otimes b - a \otimes rb, \\ &br \otimes a - b \otimes ra, r1_A r' - rr', r1_B r' - rr' : a, a' \in A, b, b' \in B, r, r' \in R). \end{aligned}$$

The free product $A \sqcup_R B$ is endowed in a canonical way with a R-bi-module structure, since $R \subset A \sqcup_R B$ so that R acts by left and right multiplication on $A \sqcup_R B$. In addition, the two star morphisms $\star_A$ and $\star_B$ induce a morphism $\star_{A \sqcup_R B}$ on the amalgamated free product $A \sqcup_R B$ which makes the diagram Fig.11 commutative.

$$\begin{tikzcd}
A \arrow[rr, "\star_A"] \arrow[dr] & & A \arrow[dr] & \\
& A \sqcup_R B \arrow[rr, "\star_{A \sqcup B}"] & & A \sqcup_R B \\
B \arrow[rr, "\star_B"'] \arrow[ur] & & B \arrow[ur] &
\end{tikzcd}$$

FIGURE 11. The amalgamated free product as an involutive algebra.

The category of involutive bi-modules algebras over R is algebraic, with R as initial object. It is therefore meaningful to define Zhang algebras in this category (see Chapter 1) and subsequently the notion of quantum processes in this category. Let us put this discussion into a more formal way by writing the definition of an involutive R-bi-module Zhang algebra.

DEFINITION 2.32. Let R be a unital algebra. A involutive B-bi-module Zhang algebra is a quadruplet (H, Δ, S, ϵ) with H an involutive unital algebra that is also an involutive R-bi-module andv three structural maps:

$$\Delta : H \to H \sqcup_R H,\; \epsilon : H \to B,\; S : H \to H.$$

Each of the maps Δ, S, ϵ is a morphism of unital algebra and R-bi-module maps. They are subject to the relations:

1. $\Delta(\Delta \sqcup \mathrm{id}_H) = (\mathrm{id}_H \sqcup \mathrm{id}_H)\Delta$,
2. $S \sqcup \mathrm{id}_H = \mathrm{id}_H \sqcup S = \eta \circ \epsilon$,
3. $(\epsilon \sqcup 1)\Delta = (\mathrm{id}_H \sqcup \epsilon)\Delta$.

We end this section with a remark on amalgamated tensor product $A \otimes_R B$ of two R bimodules. The space $A \otimes_R B$ is the quotient:

$$A \otimes_R B = (A \otimes B)/\{a \otimes (r \cdot b) = (a \cdot r) \otimes b, a \in A, b \in B, r \in R\}$$

The projection in $A \otimes_R B$ of a tensor $a \otimes b \in A \otimes_R B$ will be denoted $a \otimes_R b$. It is trivial to define a R-bi-module structure on $A \otimes_R B$, $r \cdot (a \otimes_R b) r' = (ra) \otimes_R (br')$. However, even if A and B are algebras R-bimodules, the product $(a_1 \otimes_R b_1)(a_2 \otimes_R b_2) = a_1 a_2 \otimes_R b_1 b_2$ is, in general, not associative. In fact, associativity would imply that for all $a_1, a_2 \in A$, $b \in B$ and $r \in R$:

$$((a_1(rb))a_2) = (((a_1 r)b)a_2)$$

On the other hand, we have:

$$((a_1(rb))a_2) = a_1 a_2 \otimes_R rb \neq a_1 r a_2 \otimes_R b = (((a_1 r)b)a_2),\ a_1, a_2 \in A,\ b \in B.$$

Since $A \otimes_R B$ is not a R bi-module algebra, we cannot define an R bi-module algebra morphism from the amalgamated free product $A \sqcup_R B$ to $A \otimes_R B$ equal to identity on A and on B as solution of an universal problem. Assume that R is commutative and denote by comBiModAlg(R) the category of commutative R-bimodule algebras which left and right module structures are equal. If $A, B \in$ comBiModAlg(R), the aforementioned issue disappears: the natural product on $A \otimes_R B$ turns this space into an R-bimodule structure.

A probabilistic implication of the last discussion needs to be clarified. Amalgamated tensor independence, as an amalgamated counterpart of classical (from the point of view of non-commutative probability with $R = 1$) tensor independence cannot be defined in the category of operator valued probability spaces (see below for the definition of such spaces).

2.6.1.2. *Operator valued probability spaces, rectangular probability spaces.* Let R be an unital involutive algebra. If A and B are two involutive R bimodule that are R-valued probability space, an R-valued random variable from A to B is a morphism of the category of bi-module involutive algebras. Having introduced the notion of amalgamated Zhang algebra in the last section, its now meaningful to talk about amalgamated quantum processes. However, to define the notion on non-commutative distribution, we need an appropriate definition of expectation on bimodule algebras.

Definition 2.33. Let A a bi-module algebra over R. A conditional expectation E on A is a positive involutive R-bi-module map $E : A \to R$:

1. $E(bab') = bE(a)b'$ (R-bi-module map),
2. $E(a^\star) = E(a)^\star$ (involutive map),
3. $E(aa^\star) \geq 0$ (positivity).

An R-valued probability space is the data of a bimodule algebra A and a conditional expectation E on A. We introduce as of now the class of R valued probability spaces we are interested in, the rectangular probability spaces. Let A a bi-module algebra and assume from now that A contains a set of complete, mutually orthogonal projectors $\{p_1, \ldots, p_n\}$:

$$\sum_{p \in \{p_1, \ldots, p_n\}} p = \mathrm{id}_A,\ p^\star = p,\ p^2 = p,\ pq = 0,\ p \neq q \in \{p_1, \ldots, p_n\}.$$

and that $R = \langle \{p_1, \ldots, p_n\} \rangle$, the commutative algebra generated by the projectors. Each element x of A can be written as a matrix, since $x = \sum_{i,j=1}^n p_i x p_j$, we adopt the notation: $\mathrm{x} = (p_i x p_j)_{i,j \leq n} \in \mathcal{M}_n(A)$. We use the terminology *compressed spaces* for the vector spaces $A_{ij} = p_i A p_j$, $1 \leq i, j \leq n$. To construct a conditional expectation on A, we assume further that each the diagonal compressed algebras A_{ii}, $i \leq n$ is a (usual) probability space and denote by ϕ_i the expectation on $\mathcal{A}_{ii}$. A conditional expectation E on A is defined by the formula:

$$E(x) = \sum_{i=1}^{n} \phi_i(\mathrm{x}_{ii}) p_i,\ x \in A.$$

Having introduced the analogue notion of mean for R-valued probability space, we focus now on cumulants. Recall that the set of non-crossing partitions of an interval $[\![1, \ldots, k]\!]$ is denoted $\mathsf{NC}(k)$. In the sequel, we are handling multilinear bimodule maps over A, these maps are defined naturally on tensor products of the algebra A with itself over R, for which we use the symbol $A \otimes_R \cdots \otimes_R A$. By definition : $a \otimes_R (ra') = (ar) \otimes_R a'$, $a, a' \in A$, $r \in R$. For $n \geq 1$ an integer, define the map $E_n : A^{\otimes_R n} \to A$, by $E_n(a_1 \otimes_R \cdots \otimes_R a_n) = E(a_1 \cdots a_n)$, $a_1, \ldots, a_n \in A^{\otimes_R n}$.

Let $\pi \in \mathsf{NC}(k)$ a non-crossing partition. The map E_π from $A \otimes_R \cdots \otimes_R A$ is defined recursively as follows. For two blocks V and W of π, we write $V < W$ if there exists two integers i, j in W such that $V \subset [i, j]$. Denote by $V_1, \ldots, V_p$ blocks of π that are maximal for π. Let $m \leq p$ an integer and write $V_m = \{i_1^m < \cdots < i_{k^m}^m\}$, the partition π restricts to a non-crossing partition π_l^m of the interval $\rrbracket i_l, i_{l+1} \llbracket$. The family of functions $(E_\pi)_{\pi \in \mathsf{NC}}$ is defined recursively by the equation

$$E_\pi(a_1 \cdots a_k) = E_{\sharp V_1}\left(a_{i_1^1} E_{\pi_1^1}\left(a_{\rrbracket i_1^1, i_{k^1}^1 \llbracket}\right) \cdots E_{\pi_{k^1-1}^1}\left(a_{\rrbracket i_{k^1-1}^1, i_{k^1}^1 \llbracket}\right) a_{i_{k^1}^1}\right) \cdots$$
$$E_{\sharp V_p}\left(a_{i_1^p} E_{\pi_1^p}\left(a_{\rrbracket i_1^p, i_2^p \llbracket}\right) \cdots E_{\pi_{k^p-1}^p}\left(a_{\rrbracket i_{k^p-1}^p, i_{k^p}^p \llbracket}\right) a_{i_{k^p}^p}\right)$$

and the initial condition $E_\emptyset = 1$. Recall that we use the notation μ for the Möebius function of $\mathsf{NC}(k)$. The R-valued cumulants $(c_\pi : A \otimes_R \cdots \otimes_R A \to R)_{\pi \in \mathsf{NC}(k)}$ are obtained by Möebius transformation:

$$c_\pi = \sum_{\gamma \leq \pi} \mu(\gamma, \pi) E_\gamma, \; E_\pi = \sum_{\gamma \leq \pi} c_\gamma.$$

As we shall see below, amalgamated freeness (defined below) is most efficiently seen on cumulants, which is the main reason for introducing them. Since cumulants and conditional expectations are obtained from each others by a linear transformation, they share a lot of properties.

First, for any non-crossing partition π, c_π is a R-bi-module map. Secondly, let $k \geq 1$ an integer and $i_1, \ldots, i_k$, $j_1, \ldots, j_k$ two k-tuples of integers in $\llbracket 1, n \rrbracket$. The conditional expectation E_k is equal to zero on the space $A_{i_1,j_1} \otimes_R \cdots \otimes_R A_{i_k,j_k}$ if there exists a pair (i_l, j_{l+1}), $l \leq n$ such that $i_l \neq j_{l+1}$. It can be proved, by induction, that for any non-crossing partition π, E_π evaluates to zero on $A_{i_1,j_1} \otimes_R \cdots \otimes_R A_{i_k,j_k}$ if there exists a block $\{l_1, \ldots, l_p\} \in \pi$ such that $i_{l_1 - 1} \neq i_{l_p}$. This property of the conditional expectation is shared with the cumulants.

Finally, $c_k(a_1 \otimes_R \cdots \otimes_R a_k) = 0$ if there exists an integer $i \leq k$ and $r \in R$ such that $a_i = r1$. Let us prove this property. To ease the exposition, we assume that $a_1 = r1$ for some $r \in R$. Define $\tilde{c}_\pi(a_1 \otimes_R \cdots \otimes_R a_k) = c_\pi(a_1 \otimes_R \cdots \otimes_R a_k)$ if $\{1\} \in \pi$ and set $\tilde{c}_\pi(a_1 \otimes_R \cdots \otimes_R a_k) = 0$ otherwise. We claim that

$$E_\pi(a_1 \otimes_R \cdots \otimes_R a_k) = \sum_{\gamma \leq \pi} \tilde{c}_\gamma(a_1 \otimes_R \cdots \otimes_R a_k)$$

This last relation, obviously, holds if $\{1\} \in \pi$. Let $V = \{1 < i_1 \cdots < i_p\} \in \pi$ the block of π containing 1. Using R linearity, $E_\pi(a_1 \otimes_R \cdots \otimes_R a_k) = E_{\tilde{\pi}}(a_1 \otimes_R \cdots \otimes_R a_k)$ with $\tilde{p}$ the partition obtained from π by splitting the block V of π into the two blocks $\{1\}, \{i_1 < \cdots < i_p\}$. Hence,

$$\begin{aligned} E_\pi(a_1 \otimes_R \cdots \otimes_R a_k) &= E_{\tilde{\pi}}(a_1 \otimes_R \cdots \otimes_R a_k) \\ &= \sum_{\gamma \leq \tilde{\pi}} c_\gamma(a_1 \otimes_R \cdots \otimes a_k) = \sum_{\gamma \leq \pi} \tilde{c}_\gamma(a_1 \otimes_R \cdots {}_R \otimes a_k). \end{aligned}$$

From which if follows that $\tilde{c}_\pi(a_1 \otimes_R \cdots \otimes_R a_k) = c_\pi(a_1 \otimes_R \cdots \otimes_R a_k)$ and finally $\tilde{c}_k(a_1 \otimes_R \cdots \otimes_R a_k) = 0$ if $k \geq 2$.

Proposition 2.34 (see [47]). *Let A be an involutive algebra endowed with a bimodule action of an algebra R. Let x_1 and x_2 two elements in A. The two R-bimodule algebras $R[a]$ and $R[b]$ are free from each other if and only if for all $n \geq 1$ the cumulants $c_n(x_{i_1}, \ldots, x_{i_n})$ are null if there exists two integers $1 \leq k, q \leq n$ with $i_k \neq i_q$.*

2.6.1.3. *Amalgamated semi-groups and Lévy processes.* Let R a unital associative algebra. Equivalent notions for tensor semi-groups and free semi-groups on involutive bi-algebras can be defined in the context of amalgamated bi-algebras. Let $\mathcal{C}$ be either the algebraic category of R-bimodule algebras, $(\mathrm{biModAlg}(R), \sqcup_R, R)$, either the algebraic category of commutative R-bimodule algebras $(\mathrm{comBiModAlg}(R), \otimes_R, B)$ and recall that these two categories are algebraic if endowed, respectively, with the amalgamated free product or the amalamated tensor product. The notion of bi-algebra in comBiModAlg(R) (in the case R is commutative) is obtained by replacing the free amalgamated product by the tensor product in the definition of a free amalamated Zhang algebra and removing the antipode S from the set of structural morphism.

Let (B,Δ,ε) be an associative bi-algebra in biModAlg(R), and $\alpha : B \to R$, $\beta : B \to R$ two R-bimodule linear maps. The free product $\alpha \sqcup_R \beta \in (B \sqcup_R B)^\star$ of α and β is the unique R-bimodule map satisfying the condition: for any alternating word $s_1 s_2 \cdots s_m \in B \sqcup_R B$, with:

$$\alpha(s_i) = 0 \text{ if } s_i \in B_{|1} \text{ or } \beta(s_i) = 0 \text{ if } s_i \in B_{|2}, \text{for all } 1 \leq i \leq m,$$

we have $(\alpha \sqcup_R \beta)(s_1 s_2 \cdots s_m) = 0$. On the commutative side, if B is an object of the category comBiModAlg(R), the tensor product of α and β is an R-bimodule map on the amalgamated tensor product $A \otimes_R B$ defined by:

$$(\alpha \hat{\otimes}_R \beta)(b_1 \otimes_R b_2) = \alpha(b_1)\beta(b_2),\ b_1 \otimes b_2 \in B \otimes_R B.$$

The free convolution product $\alpha \hat{\sqcup}_R \beta$ of α and β is an R-bimodule map on B and is defined by $(\alpha \hat{\sqcup}_R \beta) = (\alpha \sqcup_R \beta) \circ \Delta$. Again, in case B is an object of comBiModAlg(R), we can define the tensor convolution product $\alpha \hat{\otimes} \beta$ of α and β by setting: $\alpha \hat{\otimes}_R \beta = (\alpha \otimes_R \beta) \circ \Delta$.

An amalgamated free semi-group is a time parametrized family $(E_t)_{t\geq 0}$ of R-bimodule maps on B satisfying:

$$E_{t+s} = E_t \hat{\sqcup}_R E_s,\ t,s \geq 0.$$

The notion of amalgmated tensor semi-group is obtained be replacing the amalamated free convolution product in the last equation by the tensor amalgamated product. We now expose the amalmagmated counterpart of free independence. Let (A,E) a R valued probability space. Let B_1 and B_2 two R-bimodule sub-algebras of A. We say that B_1 and B_2 are freely independent with amalgamation if:

$$E(b_1^1 b_2^1 ... b_1^p b_2^p) = 0, \text{ with } b_1^i \in B_1,\ b_2^i \in B_2 \text{ and}, E(b_1^i) = E(b_2^i) = 0,\ 1 \leq i \leq p. \tag{2.44}$$

Working in the category *biModAlg*(R),we say that two sub-R-bimodule algebras B_1 and B_2 of commutative operator valued probability space (A,E) are *amalgamated tensor independent* if

$$E(b_1 b_2) = E(b_1)E(b_2).$$

Let $a \in A$ an element of A, we denote by $R[a]$ the R bimodule algebra generated by a. By definition $R[a]$ is the set of all linear combinations of monomials in the element a with coefficients in the algebra R:

$$R[a] = \mathbb{K}\Big[\{r_0 a^{n_1} r_1 a^{n_2} \cdots a^{n_p} r_p, r_0, \ldots, r_p \in R,\ p \geq 1\}\Big] \subset A.$$

We say that two elements $a \in A$ and $b \in B$ are free with amalgamation over R if $R[a]$ and $R[b]$ are free sub-modules of A.

The restriction of E to the algebra generated by two mutually free with amalgamation algebras B_1 and B_2 is entirely determined by the restriction of E to B_1 and to B_2 and is equal to the amalgamated free product of the restrictions of E to these algebras; with ι_1 and ι_2 the injections of, respectively, B_1 into B and B_2 into B:

$$\Big(B_1 \sqcup_R B_2, E_{|B_1} \hat{\sqcup}_R E_{|B_2}\Big) \xrightarrow{\iota_{B_1} \dot{\sqcup} \iota_{B_2}} (B,E).$$

We are now in position to give the definition of an *amalgamated free Lévy process*. Let (H,Δ,ε,S) be an R-amalgamated free Zhang algebra, that is, a Zhang algebra in biModAlg(R) and (A,E) an R-valued probability space. An amalamated free Lévy process $j = (j_t)_{t\geq 0}$ is a time parametrized collection of homomorphisms of biModAlg$R$ with values in A that satisfy the following three conditions, with $j_{st} = j_t \dot{\sqcup} j_s \circ s$:

1. for all times $s_1 < t_1 \leq s_2 < t_2 \leq \cdots \leq s_p < t_p$, $\{j_{s_1 t_1}, \ldots, j_{s_p t_p}\}$ is a mutually free with amalgamation family, meaning that the algebras $j_{s_1 t_1}(H), \ldots, j_{s_p t_p}(H)$ are mutually free in A.
2. For all times $t > s$, the distribution $E \circ j_{st}$ depends only on the difference $t - s$,

In addition, if R is naturally endowed with a norm, we require also the continuity condition:

$$\lim_{s\to t} j_{st}(h) = \varepsilon(h), h \in H.$$

To be complete, if considering the category comBiModAlg(R), the defintion of a tensor amalgamated Lévy process is obtained by requiring amalagmated tensor independence of the increments in the last definition. With the notation of the definition, the one dimensional marginals $t \mapsto E \circ j_t$ of a free amalgamated Lévy process is a free semi-group (the same holds if working with tensor amalgamated tensor Lévy process). We do not know if an amalgamated version of the Schoenberg correspondance for free or tensor amalgamated Lévy process holds.

2.6.2. Extraction processes and their statistics in high dimensions. In the Section 2.5, we proved the convergence in non-commutative distribution of the process on the dual Voiculescu group $\mathcal{O}\langle n\rangle$ that extracts square blocks from a unitary Brownian motion in the limit for which the dimensions of these blcks tend to infinity. A natural extension of this result is investigated here; square blocks are replaced by rectangular ones. Our main results are stated in Theorem 2.37 and Theorem 2.39. Given a partition d of the dimension, we construct for each time $t \geq 0$ two quantum processes on two amalgamated Zhang algebras (defined below in Section 2.6.2.1 and 2.6.2.2) that extract rectangular blocks in the matrix $\mathsf{U}_N^{\mathbb{K}}(t)$. For one of these processes, product of blocks, even if the dimensions match, may be equal to zero while it is never the case for the other process. More on this point is explained below. The method we use in this section for proving the convergence of the one dimensional marginals of these processes is similar to the one used in Section 2.5. However, to prove the convergence of the multi-dimensional marginals, Theorem 3.39 can not be applied. We fix an integer $n \geq 1$. Let $N \geq 1$ an integer and let d be a partition of N into n parts. The algebra $\mathcal{M}_N(\mathbb{R})$ can be endowed with a structure of operator valued probability space. In fact, denote by $\mathcal{D}_{\mathsf{d}}$ generated by the projectors:

$$p_i(k,l) = \begin{cases} \delta_{k,l} & \text{if } k,l \in [d_1 + \cdots + d_{i-1},\ d_1 + \cdots + d_i] \\ 0 & \text{otherwise} \end{cases},\ i \leq n.$$

and define the complex linear form $E_{\mathsf{d}} : \mathcal{M}_N(\mathbb{R}) \to \mathcal{D}_{\mathsf{d}}$ by

$$E_{\mathsf{d}}[A] = \sum_{i=1}^{n} \frac{1}{d_i} \mathsf{Tr}(p_i A p_i) p_i.$$

The algebra $\mathcal{M}_N(\mathbb{R})$ is a $\mathcal{D}_{\mathsf{d}}$ bimodule and E_{d} is a positive bimodule map : $(\mathcal{M}_N(\mathbb{R}), E_{\mathsf{d}})$ is an operator valued probability space.

If considering matrices with entries in $\mathbb{C}$, the conditional expectation we choose on $\mathcal{M}_N(\mathbb{C})$ is the same as for the real case. If we consider matrices with entries in the quaternionic division algebra, we choose for the conditional expectation $E_{\mathsf{d}}[A] = \sum_{i=1}^{n} \frac{1}{d_i} \mathcal{R}e\mathsf{Tr}(p_i A p_i) p_i$, $A \in \mathcal{M}_N(\mathbb{H})$. For $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, we denote by $\mathcal{M}_{\mathsf{d}}(\mathbb{K})$ the operator valued probability space we just defined. Considering matrices with random entries in a $L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P})$, a rectangular probability space amalgamated over $\mathcal{D}_{\mathsf{d}}$ is obtained by taking the mean of the conditional expectation E_{d}. Set $\mathbb{E}_{\mathsf{d}} = \mathbb{E} \circ E^{\mathsf{d}}$. The rectangular probability space $\left(\mathcal{M}_{\mathsf{d}}, \mathbb{E}_d, \frac{\mathsf{d}}{N}\right))$ is denoted $(\mathcal{M}_{\mathsf{d}}(L^{\infty-}(\mathbb{K}))$.

Put $n' = \sum_{d \in \mathsf{d}} d$. The two sections 2.6.2.1 and 2.6.2.2 are devoted to the definition of the two processes we are interested in and their structure algebras. These two processes depend on a partition d. The first one takes its values in $\mathcal{M}_{n'}(\mathbb{K})$, extracts blocks of dimensions prescribed by the partition d and puts blocks of same size at the same place into a matrix of dimension $n' \times n'$. The second one extracts blocks but holds them in place in the matrix, meaning that all the other blocks are set to zero.

2.6.2.1. *Dimension cluster algebra.* Let $N \geq 1$ an integer and pick a partition d of N into n parts. We use the short notation $\{\mathsf{d}\}$ for the set $\{d,\ d \in \mathsf{d}\}$. If $d \in \{\mathsf{d}\}$, we denote by n_d the number of occurrences of d in d and we set $n' = \sum_{d \in \{\mathsf{d}\}} d$. We define the first process which asymptotics in high dimensions is the object of study.

Pick two integers $d, d' \in \{\mathsf{d}\}$. To that pair of dimensions (d, d') we associate formal variables $x_{d,d'}(i,j),\ x^{\star}_{d',d}(j,\ i)\ i \leq n_d,\ j \leq n_{d'}$ and define the matrix X of size n with entries in the free algebra generated by the variables $x_{d,d'}(i,j)$ by

$$X(k,l) = x_{d_k,d_l}\left(\sharp\{1 \le i \le k : d_i = d_k\}, \sharp\{1 \le i \le l : d_i = d_l\}\right). \tag{2.45}$$

Definition 2.35 (Cluster Rectangular Unitary algebra). Cluster Rectangular Unitary algebra is the involutive unital associative algebra, denoted $\mathcal{CRO}\langle \mathsf{d}\rangle$, that is generated by all the variables $x_{dd'}(i,j)$ and projectors $p_d, d \in \{\mathsf{d}\}$ subject to the relations:

$$XX^\star = X^\star X = 1, p_d^\star = p_d,\ p_d p_{d'} = \delta_{dd'} p_d,$$
$$p_{d_1} x_{d_2 d_3}(i,j) p_{d_4} = \delta_{d_1 d_2}\delta_{d_3 d_4} x_{d_2 d_3}(i,j),\ p_{d_1} x^\star_{d_2 d_3}(i,j) p_{d_4} = \delta_{d_1 d_3}\delta_{d_4 d_2} x_{d_2 d_3}(i,j).$$

As a consequence of the set of relations 2.35, the family of projectors $\{p_d, d \in \{\mathsf{d}\}$ for a complete set of projectors:

$$\sum_{d \in \{\mathsf{d}\}} p_d = 1 \in \mathcal{CRO}\langle \mathsf{d}\rangle.$$

To a pair of dimensions (d,d') we associate the sub-algebra $\mathcal{CRO}\langle \mathsf{d}\rangle_{d,d'}$ generated by the set of variables $\{x_{d,d'}(i,j), x^\star_{d',d}(j,i),\ i \le n_d,\ j \le n_{d'}\}$. The unital algebra generated by the projectors $\{p_d, d \in \{\mathsf{d}\}\}$ is denoted $\mathcal{D}_{\mathsf{d}}$. Finally, we define a $\mathcal{D}_{\mathsf{d}}$-amalgamated Zhang algebra structure on $\mathcal{CRO}\langle \mathsf{d}\rangle$ by setting for the structural morphisms S, Δ, ε:

$$\begin{aligned}
&\Delta : \mathcal{CRO}\langle \mathsf{d}\rangle \to \mathcal{CRO}\langle \mathsf{d}\rangle \sqcup_{\mathcal{D}_{\mathsf{d}}} \mathcal{CRO}\langle \mathsf{d}\rangle, && \Delta(X) = X_{|1} X_{|2},\\
&S : \mathcal{CRO}\langle \mathsf{d}\rangle \to \mathcal{CRO}\langle \mathsf{d}\rangle, && S(X) = X^\star,\\
&\varepsilon : \mathcal{CRO}\langle \mathsf{d}\rangle \to D_{\mathsf{d}}, && \epsilon(X) = 1.
\end{aligned}$$

We denote by d' the partition of n' obtained by sorting $\{\mathsf{d}\} = \{d_1, \dots, d_p\}$ in ascending order. As we should see below, this choice for the partition d' is quite arbitrary and is, somehow, maximal.

To a matrix $A \in \mathcal{M}_n(\mathbb{K})$, we associate a random variable $j_A : \mathcal{CRO}\langle \mathsf{d}\rangle \to \mathcal{M}_{\mathsf{d}'}(\mathbb{K})$ defined as follows. For $d, d' \in \{\mathsf{d}\}, i \le n_d, j \le n_{d'}$: $j_A(x_{d,d'}(i,j))$ is the block of dimensions $d \times d'$ at position (i,j) in the matrix of size $n_d d \times n_{d'} d'$ obtained by extracting all blocks of size $d \times d'$ (and keeping them at their relative position) of A.

We make few remarks regarding the way we chose to define the algebra cluster and the random variables j_A^{d}, with obvious notations. We indexed the generators of $\mathcal{CRO}\langle \mathsf{d}\rangle$ by using, in particular, the set $\{\mathsf{d}\}$. It is important to notice that the algebra $\mathcal{CRO}\langle \mathsf{d}\rangle$ depends solely on the kernel of d, if another sequence d_2 of integers of length n has the same kernel as d then $\mathcal{CRO}\langle \mathsf{d}\rangle = \mathcal{CRO}\langle \mathsf{d}_2\rangle$. Only the random variabl j_A^{d} depends on the sequence d.

If $V = \{i_1 < \ldots < i_p\}$ and $V' = \{i'_1 < \ldots < i'_{p'}\}$ are two blocks of $\mathsf{Ker}(\mathsf{d})$, we denote by $x_{V,V'}(i_k, i'_l)$ the element $x_{\mathsf{d}(V)}(k,l)$, and accordingly $p_{d(V)} = p_V$. The constitutive relations of $\mathcal{CRO}\langle \mathsf{d}\rangle$ can thus be written:

$$XX^\star = X^\star X = 1, p_V p_{V'} = \delta_{V,V'} p_V, p_V^\star = p_V \tag{2.46}$$

$$p_{V_1} x_{V,V'}(i,j) p_{V_2} = \delta_{V_1,V}\delta_{V_2,V'} x_{V,V'}(i,j), i \in V, j \in V' \tag{2.47}$$

If π is a partition of $[\![1,n]\!]$, we denote by $\mathcal{CRO}\langle \pi\rangle$ the algebra generated by the random variables $\{x_{V,V'}(i,j), V, V' \in \pi,\ i \in V,\ j \in V'\}$ subject to the relations (2.46).

Let A be a matrix of size N and d a partition of N into n parts which kernel is coarser than π. Set p equal to the number of blocks of π and pick σ a permutation of $\mathfrak{S}_p$. Using the lexicographic order on the block of π, we write $\pi = \{V_1 < \ldots < V_p\}$ and define $j_A^{\mathsf{d},\pi,\sigma}$ the random variable that takes it values in the rectangular probability space $\mathcal{M}_{(d(V_{\sigma(1)}),\ldots d(V_{\sigma(p)}))}$ and defined in the same way as j_A^{d}. It is clear that the random variable j_A^{d} we defined previously corresponds to the choice $\pi = \mathsf{Ker}(\mathsf{d})$ for some permutation σ. We settle quite a level of generalities, in the sequel we use only the random varirable $j_A^{\pi,\mathsf{d},1_p}$ for which we use the shorter notation $j^{\pi,\mathsf{d}}$. After all all these defintions, we can now define the process which asymptotic asymptotic in high dimensions is studied:

$$U^{\mathbb{K}}_{\{\mathsf{d}\},\pi}(t) = j^{\mathsf{d},\pi}_{\mathsf{U}^{\mathbb{K}}_N(t)}, \text{ for all times } t \ge 0. \tag{2.48}$$

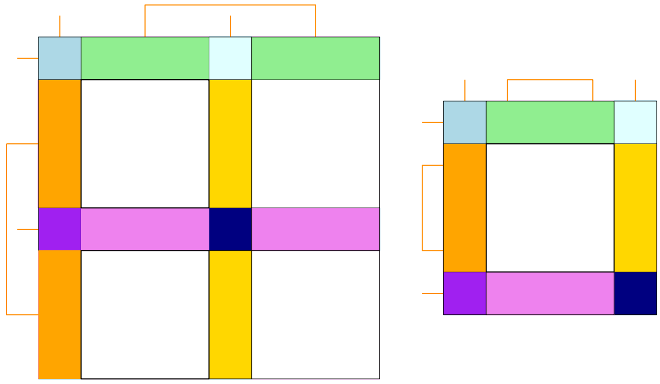


Figure 12. This figure pictures the action of the morphism $j_A^{\mathsf{d},\pi,\mathrm{id}_4}$, with $\mathsf{d} = (1,3,1,3)$ and $\pi = \{\{1\},\{2,4\},\{3\}\}$ on a matrix A, drawn on the left hand side of the figure. Blocks of A coloured in the same way are sent by $j_A^{\mathsf{d},\pi,\mathrm{id}_4}$ on the block in the matrix on the right hand side coloured with their common colour.

2.6.2.2. *Rectangular unitary algebra.* Let $N, n \geq 1$ integers and d a partition of N into n parts. In that section, we define the rectangular unitary algebra $\mathcal{RO}\langle n\rangle$ that is nothing more but the dual Voiculescu group augmented with auto-adjoint projectors. Hence, as an unital algebra, $\mathcal{RO}\langle n\rangle$ is generated by n auto-adjoint mutually orthogonal idempotent elements $(p_i, i \leq n)$ and an unitary element u subject to the relations:

$$p_k p_l = 0,\ k \neq l, \quad p_k^\star = p_k, \quad p_k^2 = p_k, \quad k,l \leq n, \text{ and } u^\star = u^{-1}. \tag{2.49}$$

Denote by $\mathcal{D}_n$ the algebra generated by the projectors p_i, $i \leq n$. The algebra $\mathcal{RO}\langle n\rangle$ is an involutive $\mathcal{D}_n$ bimodule algebra. Let π be a partition finer than $\mathsf{Ker}(\mathsf{d})$. At that point, let us draw comparisons between the algebra $\mathcal{CRO}(\pi)$ we introduced in the last section and the rectangular unitary algebra. We claim that there exists a surjective morphism $\phi : \mathcal{CRO}(\pi) \to \mathcal{RO}\langle n\rangle$ taking the following values on the generators:

$$\phi(x_{V,V'}(i,j)) = p_i u p_j,\ d,d' \in \{\mathsf{d}\}, \phi(p_V) = \sum_{q=1}^{\sharp V} p_q.$$

with $i \in V$, $j \in V', V, V' \in \pi$. For each time $t \geq 0$, the random variable $U_{\mathsf{d}}^{\mathbb{K}}(t) : \mathcal{RO}\langle n\rangle \to L^{\infty-}(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{R}) \otimes \mathcal{M}_N(\mathbb{K})$ that extract blocks from $\mathsf{U}_N^{\mathbb{K}}$ but hold them in place, is defined by:

$$\begin{array}{rccl} U_{\mathsf{d}}^{\mathbb{K}}(t): & \mathcal{RO}\langle n\rangle & \to & \mathcal{M}_N(L^\infty(\Omega,\mathcal{F},\mathbb{P},\mathbb{K})) \\ & u & \mapsto & \mathsf{U}_N^{\mathbb{K}}(t) \\ & p_i & \mapsto & p_i \end{array}. \tag{2.50}$$

The algebra $\mathcal{RO}\langle n\rangle$ is an involutive $\mathcal{D}_n$-bimodule and is endowed with a Zhang algebra structure. We define three bi-module morphisms Δ, ϵ, and S by specifying their values on the generators of $\mathcal{RO}\langle n\rangle$ and show that $(\mathcal{RO}\langle n\rangle, \Delta, \epsilon, S)$ is a Zhang algebra. We claim that there exist involutive algebra morphisms S,Δ, ϵ, defined on the free real algebra generated by $\{u, u^\star, p_1, \ldots, p_n\}$ satisfying:

$$\begin{aligned} &\Delta : \mathbb{R}\Big[u,u^\star,p_1,\ldots,p_n\Big] \to \mathcal{RO}\langle n\rangle \sqcup_R \mathcal{RO}\langle n\rangle, && \Delta(u) = u_{|1}u_{|2},\ \Delta(p_i) = p_i,\ 1 \leq i \leq n \\ &S : \mathbb{R}\Big[u,u^\star,p_1,\ldots,p_n\Big] \to \mathcal{RO}\langle n\rangle, && S(u) = u^\star,\ S(p_i) = p_i,\ 1 \leq i \leq n \\ &\epsilon : \mathbb{R}\Big[u,u^\star,p_1,\ldots,p_n\Big] \to \mathcal{B}, && \epsilon(u) = 1,\ \epsilon(p_i) = p_i, i \leq n \end{aligned} \tag{2.51}$$

The algebra $\mathcal{RO}\langle n\rangle$ is a quotient of $\mathbb{R}\Big[u,u^\star,p_1,\ldots,p_n\Big]$ by the relations (2.49). Hence, the three maps in (2.51) descend to morphisms on $\mathcal{RO}\langle n\rangle$ if the images of the generator $\{u,u^\star,p_1,\ldots,p_n\}$ by these maps satisfy the same relations (2.49). This verification shows no difficulties and we omit it for brevity. With this definition, $U_{\mathsf{d}}^{\mathbb{K}}$ is non-commutative process on the Zhang algebra H taking its values in the rectangular probability space $\mathcal{M}_{\mathsf{d}}(L^{\infty-}(\Omega,\mathcal{F},\mathbb{P},\mathbb{K}))$. The next sections are devoted to the investigation of the convergence in non commutative distribution of $U^{\mathbb{K}}_{d_N}$ as the dimension N tends to infinity, with $(d_N)_{N\geq 1}$ a sequence of partitions of N into a fixed number n of parts.

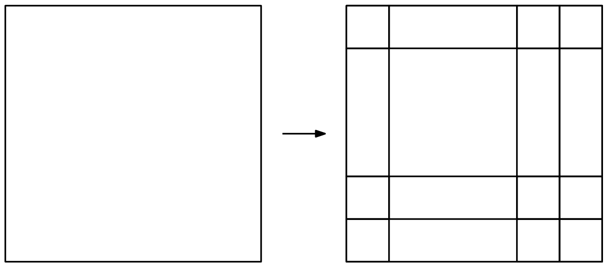


FIGURE 13. A picture standing for the action of the morphism j_A that cuts a matrix A(on the left) into 16 blocks which sizes exhaust the set $\{1,3\}\times\{1,3\}$.

2.6.2.3. *Statistics of the extractions processes.* Let $N\geq 1$ an integer and pick a partition d_N into n parts of N. The purpose of the forthcoming sections are to prove the convergence in high dimensions, in non-commutative distribution, of the two processes $U^{\mathbb{K}}_{\{d_N\}}$ and $U^{\mathbb{K}}_{d_N}$. The method we use has been expounded in Section 2.5 to prove the convergence in non-commutative distribution of the process square blocks extraction from an unitary Brownian motion. We recall some features of that method. If $N=nd$ and $d_N=(d,\ldots,d)$, we defined for each time $t\geq 0$ a statistic $\mathbbm{m}^{\mathbb{K}}_d(t)$ which is a function on the set of Brauer diagrams which range comprises the distribution of $U^{\mathbb{K}}_{d_N}$. We proved next the convergence of this statistic by exhibiting a differential system $\mathbbm{m}^{\mathbb{K}}_{d_N}$ is solution of. Hence, in order to apply this method, we need first to define the statistic $\mathsf{m}^{\mathbb{K}}_{\mathsf{d}}$ that is the rectangular counterpart of $\mathsf{m}^{\mathbb{K}}_{d}$. After that, to normalize this statistic we use the functions fnc_d, $d\in\{d_N\}$ we introduced in Section 2.4. To explain why the use of these functions is needed, recall that for a matrix $A\in\mathcal{M}_N(\mathbb{R})$ and $(b,s)\in\mathcal{B}_k$ an oriented irreducible Brauer diagram,

$$(\star)\qquad \mathrm{Tr}^{\otimes k}\left(\rho_{d_N}(b)\circ(A\otimes\cdots\otimes A)\right)=\mathrm{Tr}^{\otimes k}\left(A\left(c_b(i_1')^{s_b(1)},c_b(i_1)\right)\cdots A\left(c_b(i_k'),c_b(i_k)^{s_b(k)}\right)\right)$$

with $\sigma_{b^\bullet}=(i_1,\ldots,i_k)$. Of course,the right hand side of ($\star$) does only depend on the cyclic order induced by $\sigma_{b^\bullet}$ and truly independent of the orientation s. However, to normalize these quantities (and retrieve the distribution of $U^{\mathbb{K}}_{d_N}$) we need first to choose which block sits at front, this amongst to pick a linear order on $(1',\ldots,k')$. By doing this, we are able to associate a word on the blocks of A to each Brauer diagram, not only a cyclic word. A second step is to choose the dimension we use to normalize, either the number of lines, either the number of columns of the blocks that sits at front. If this block is transposed, we normalized by the number of columns, and it is not we normalize by the number of lines. By doing this we break an other symmetry of the right hand side of ($\star$), which is the invariance by transposition. Let's draw an example with three blocks and $b=((1,2,3),(1,2)(1,1)(1,2))$, one has

$$\mathrm{Tr}(\rho_{\mathsf{d}}\circ A\otimes A\otimes A)=\mathrm{Tr}\left(A^2_1A^1_1A^1_2\right)=\mathrm{Tr}\left(A^1_1A^1_2A^2_1\right)=\mathrm{Tr}\left(A^2_1A^1_1A^1_2\right).$$

For each of three linear orders on $(1,2,3)$, we get the normalizations

$$\frac{1}{d_2}\mathrm{Tr}\left(A^2_1A^1_1A^1_2\right)\ 1<2<3,\ \frac{1}{d_1}\mathrm{Tr}\left(A^1_1A^1_2A^2_1\right)\ 1<3<2,\ \frac{1}{d_2}\mathrm{Tr}\left(A^1_1A^1_2A^2_1\right)\ 3<2<1.$$

The support of a cycle c of a Brauer diagram $b^\bullet\in\mathcal{B}^\bullet_k$ is seen as being endowed with the linear order that is left by putting the minimum, for the natural order, of the support of c out of the cyclic order induced by c.

After this discussin, the definition of the rectangular extractions' statistic will seem natural for the reader. The function $\mathsf{m}^{\mathbb{R}}_{d_N}$ on the set of oriented Brauer diagrams $\mathcal{OB}_k$ and valued in the space of linear forms on $\mathcal{M}_N(\mathbb{R})^{\otimes k}$ is defined by, for matrices $A_1,\ldots,A_k\in\mathcal{M}_N(\mathbb{K})$, and an oriented Brauer diagram $(b,s) \in \mathcal{OB}_k$:

$$\mathsf{m}^{\mathbb{K}}_{d_N}((b,s))(A_1 \otimes \cdots \otimes A_k) = \left(\prod_{d \in \{d_N\}} d^{-\mathsf{fnc}_{\mathsf{d}}((\mathsf{b},\mathsf{s}))} \right) \mathrm{Tr}_{\mathbb{K}}(\rho^{\mathbb{R}}_{\mathsf{d}}(b) \circ A_1 \otimes \cdots \otimes A_k)), \tag{2.52}$$

where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and $\mathrm{Tr}_{\mathbb{R}} = \mathrm{Tr}_{\mathbb{C}} = \mathrm{Tr}$, $\mathrm{Tr}_{\mathbb{H}} = \mathcal{R}e \circ \mathrm{Tr}$. In the following section, we are making some hypothesis on the sequence of partitions $(d_N)_{N \geq 1}$ and study the convergence of:

$$\mathsf{m}^{\mathbb{K}}_{d_N}\left(\mathbb{E}\left[\mathsf{m}^{\mathbb{K}}_{d_N}(\mathsf{U}^{\mathbb{K}}_N(s_1,t_1) \otimes \cdots \otimes \mathsf{U}^{\mathbb{K}}_N(s_q,t_q)\right)\right]$$

as the dimension N tends to infinity (see 2.36). To study the aforementioned convergence, we let $q \geq 1$ an integer and pick $\left[\mathsf{U}^{\mathbb{K}}_N\right]^{(1)}, \ldots, \left[\mathsf{U}^{\mathbb{K}}_N\right]^{(q)}$ independent unitary Brownian motions. We denote by M_q, respectively $\overline{M}_q$ the free monoid generated by q letters $\{x_1, \ldots, x_q\}$, respectively $2q$ letters $\{x_1, \ldots, x_q, \bar{x}_1, \ldots, \bar{x}_q\}$ and the identity element $\emptyset$. If $k \geq 1$ is an integer, we denote by $\mathsf{M}_q(k)$, respectively $\overline{\mathsf{M}}_q(k)$ the subset of M_q , respectively of $\overline{\mathsf{M}}_q$, comprising all words of length k.

If $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$, we are now defining for a tuple of times $\mathsf{t} = (t_1, \ldots, t_q)$, a word $w \in \mathsf{M}_q(k)$ and an oriented Brauer diagram (b,s):

$$\mathfrak{m}^{\mathbb{K}}_{d_N}((b,s),w,\mathsf{t})(u) = \mathsf{m}^{\mathbb{K}}_{d_N}((b,s)\left(\mathbb{E}\left[w^{\otimes}\left(\left[\mathsf{U}^{\mathbb{K}}_N(ut_1)\right]^{(1)} \otimes \cdots \otimes \left[\mathsf{U}^{\mathbb{K}}_N(ut_q)\right]^{(q)}\right)\right]\right), u \in [0,1].$$

In the complex case, we need also to take component wise conjugation of the Brownian diffusion in order for its non-commutative distribution to be in the range of the statistic $\mathfrak{m}^{\mathbb{C}}(\mathsf{t})$, with w a word in $\overline{\mathsf{M}}_q(k)$ and an oriented Brauer diagram (b,s), we set:

$$\mathfrak{m}^{\mathbb{C}}_{d_N}((b,s),w,\mathsf{t})(u) = \mathsf{m}^{\mathbb{C}}_{d_N}((b,s))\left(\mathbb{E}\left[w^{\otimes}\left(\left[\mathsf{U}^{\mathbb{C}}_N(ut_1)\right]^{(1)} \otimes \cdots \otimes \left[\mathsf{U}^{\mathbb{C}}_N(ut_q)\right]^{(q)}\right)\right]\right), u \in [0,1].$$

For each time $u \in [0,1]$, the statistic $\mathfrak{m}^{\mathbb{K}}_{d_N}(\mathsf{t})(u)$ extends linearly to the tensor product $\mathbb{R}[\mathcal{OB}_k] \otimes \mathbb{R}\left[\mathsf{M}_q(k)\right]$ (resp. to $\mathbb{R}[\mathcal{OB}_k] \otimes \mathbb{R}\left[\overline{\mathsf{M}}_q(k)\right]$), if $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$ (resp. if $\mathbb{K} = \mathbb{C}$).

2.6.3. Convergence of the extraction processes' statistics. We assume in this section that as N tends to infinity, the ratio $\frac{d_N(i)}{N}$ converges for each integer $1 \leq i \leq n$:

$$\frac{d_N(i)}{N} \underset{N \to +\infty}{\longrightarrow} r_i \in]0,1], \text{ for all } 1 \leq i \leq n. \tag{Δ}$$

Recall that we denote by $\mathsf{Ker}(d_N)$ the partition of $[\![1,n]\!]$ of all level sets of the function $[\![1,n]\!] \ni i \mapsto \frac{d_N(i)}{N}$. As noticed in Section 2.4, if two dimensions functions f and f' satisfy $\mathsf{Ker}(f) = \mathsf{Ker}(f')$ then $\mathcal{B}^f_k = \mathcal{B}^{f'}_k$. This section is devoted to the proof of the following proposition, which main corollary is Theorem 2.37. In the sequel we set $\mathsf{r} = (r_1, \ldots, r_n)$. The sequence r is a sequence of positive integers, we means that the dimensions of the blocks that are extracted grow linearly compared to the total dimension N.

Proposition 2.36. *As $N \to +\infty$, for each non-mixing oriented Brauer diagram (b,s) and word w in $\mathsf{M}_q(k)$, $\mathfrak{m}^{\mathbb{R}}_{d_N}((b,s),w,\mathsf{t})$ and $\mathfrak{m}^{\mathbb{H}}_{d_N}((b,s),w,\mathsf{t})$ converge to the same limit.*

As $N \to +\infty$, for each non-mixing Brauer diagram (b,s) and word w in $\overline{\mathsf{M}}_q(k)$, $\mathfrak{m}^{\mathbb{C}}_{d_N}((b,s),w,\mathsf{t})$ converges.

In addition, if we assume that the sequence of Kernels $(\mathsf{ker}(d_N))_{N \geq 1}$ is bounded from below by $\mathsf{Ker}(d_1)$, $\mathsf{Ker}(d_1) \leq \mathsf{Ker}(d_N)$, $\forall N \geq 1$, the above convergence is extended to the whole set of Brauer diagrams $\mathcal{B}^{d_1}_k$.

Let us explain with more details the hypothesis we made on the sequence of kernels, requiring for each element of this sequence to be larger than d_1 is the same as cutting a matrix into blocks, letting each block growing while maintaining the dimensions of blocks that were equal initially, equal. The kernel $\mathsf{ker}(\mathsf{r})$ is, in general, greater than the kernels $\mathsf{ker}(d_N)$, $N \geq 1$. Note that the sequence of algebras $\mathcal{B}_k(d_N)$ is a sequence of *linearly* isomorphic spaces, and each algebra $\mathcal{B}_k(d_N)$ is injected canonically into the limit algebra $\mathcal{B}_k(\mathsf{r})$.

We denote by $\mathfrak{m}_{\mathsf{r}}(\mathsf{t})$ and $\overline{\mathfrak{m}}_{\mathsf{r}}(\mathsf{t})$ the limit of $\mathfrak{m}^{\mathbb{R}}_{d_N}(\mathsf{t})$, respectively, $\mathfrak{m}^{\mathbb{C}}_{d_N}(\mathsf{t})$. In the course of proving Proposition 2.36, we find two differential systems the functions $\mathfrak{m}_{\mathsf{r}}(\mathsf{t})$ and $\overline{\mathfrak{m}}_{\mathsf{r}}$ are solutions of. The generators of these systems are denoted L_{r} and $\bar{L}_{\mathsf{r}}$ and are defined below as operators acting on $\mathbb{R}[\mathcal{OB}_k]\otimes\mathbb{R}\left[\mathsf{M}_q(k)\right]$ for L_{r} and $\mathbb{R}[\mathcal{OB}_k]\otimes\mathbb{R}\left[\overline{\mathsf{M}}_q(k)\right]$ for $\bar{L}_{\mathsf{r}}$. To provide tractable formulae for these operators, with a slight abuse of notation, we introduce for each positive dimension function f, the function f on the set product $\overset{\circ}{\mathcal{OB}}{}^f_k\times\overset{\circ}{\mathcal{OB}}{}^f_k$ by the equation:

$$\mathsf{f}(((b,s),\mathsf{o}),((b',s'),\mathsf{o}')) = \prod_{d\in\{\mathsf{f}\}} d^{\mathsf{fnc}_d((b,s),\mathsf{o})-\mathsf{fnc}_d((b',s'),\mathsf{o}')}, \tag{2.53}$$

with $((b,s),\mathsf{o}),((b',s'),\mathsf{o}')\in\overset{\circ}{\mathcal{OB}}{}^f_k\times\overset{\circ}{\mathcal{OB}}{}^f_k$ and remark that f is well defined owing to the positivity of f. We set

$$c^{\mathbb{R}}_N = -\frac{1}{2}\frac{N-1}{N},\ c^{\mathbb{H}}_N = -\frac{1}{2}\frac{N-3}{N},\ c^{\mathbb{C}}_N = -\frac{1}{2}.$$

We are now ready to compute the derivatives of the statistics $\mathfrak{m}^{\mathbb{K}}_{d_N}$. By using formulae (2.36), (2.40) and (2.42) of Section 2.5 for the mean $\mathbb{E}\left[\mathsf{U}^{\mathbb{K}}_N(t_i)\right]$, $i\leq q$ and obtain for each integer $1\leq i\leq q$ existence of an operator $L^{\mathbb{K}}_{i,N}$ acting on the space $\mathbb{R}[\mathcal{OB}_k]\otimes\mathbb{R}\left[\mathsf{M}_q(k)\right]$ for $\mathbb{K}=\mathbb{R}$ or $\mathbb{H}$ and on $\mathbb{R}[\mathcal{OB}_k]\otimes\mathbb{R}\left[\overline{\mathsf{M}}_q(k)\right]$ if $\mathbb{K}=\mathbb{C}$ such that:

$$\frac{d}{du}\mathfrak{m}^{\mathbb{K}}_{d_N}(\mathsf{t})(u) = \sum_{i=1}^{q}\mathfrak{m}^{\mathbb{K}}_{d_N}(\mathsf{t})(u)\circ L^{\mathbb{K}}_{i,N} = \mathfrak{m}^{\mathbb{K}}_{d_N}(\mathsf{t})(u)\circ L^{\mathbb{K}}_{N},\ \text{for all } u\in[0,1], \tag{2.54}$$

with $L^{\mathbb{K}}_N = \sum_{i=1}^q t_i L^{\mathbb{K}}_{i,N}$.

The two sets of coloured Brauer diagrams $\mathcal{B}^{d_N}_k$ and $\mathcal{B}^{r_N}_k$ are equal, the operators $L^{\mathbb{K}}_{i,N}$ may thus be seen as acting on the linear span of the tensor product of $\mathbb{R}\left[\mathcal{OB}^{r_N}_k\right]$ with $\mathbb{R}\left[\overline{\mathsf{M}}_q(k)\mathbb{R}\right]$(or $\mathbb{R}\left[\mathsf{M}_q(k)\right]$ in the complex case). For $1\leq i\leq n$ and a word in $w\in\mathsf{M}_q$ (resp. in $\overline{\mathsf{M}_q}$), we denote by $\mathsf{n}_i(w)$ the number of letters in the word w equal to x_i (resp. to x_i or $\overline{x}_i$). We recall that the sets of non mixing coloured Brauer diagrams which underlying component is a transposition of a projection is denoted, respectively, by $\mathsf{T}_{k,n}$ and $\mathsf{W}_{k,n}$. Let $b\in\mathcal{OB}^{d_N}_k$ a Brauer diagram and $w\in\mathsf{M}_q(k)$, for $\mathbb{K}=\mathbb{R}$ or $\mathbb{H}$,

$$\begin{aligned} L^{\mathbb{K}}_{i,N}(b\otimes w) = c^{\mathbb{K}}_N\mathsf{n}_i(w)(b\otimes w) &+ \sum_{\substack{e_{ij}\in\mathsf{W}_{k,n},\\ w_i=w_j=x_i}} N^{\mathsf{nc}(b^\bullet\vee e^\bullet_{ij})-\mathsf{nc}(b^\bullet\vee 1)-1}\mathsf{r}_N\left(\overset{\circ}{e}_{ij}\diamond\overset{\circ}{b},\overset{\circ}{b}\right)(e_{ij}\diamond b\otimes w)\\ &- \sum_{\substack{\tau_{ij}\in\mathsf{T}_{k,n},\\ w_i=w_j=x_i}} N^{\mathsf{nc}(b^\bullet\vee\tau^\bullet_{ij})-\mathsf{nc}(b^\bullet\vee 1)-1}\mathsf{r}_N\left(\overset{\circ}{\tau}_{ij}\diamond\overset{\circ}{b},\overset{\circ}{b}\right)(\tau_{ij}\diamond b\otimes w). \end{aligned} \tag{2.55}$$

with the finite sequence r_N defined by $\mathsf{r}_N=\left(\frac{d_N(1)}{N},\ldots,\frac{d_N(n)}{N}\right)$. For the the complex case, we let w be a word in the monoid $\overline{\mathsf{M}}_q(k)$,

$$\begin{aligned} L^{\mathbb{C}}_{i,N}(b\otimes w) = c^{\mathbb{C}}_N\mathsf{n}_i(w)(b\otimes w) &+ \sum_{\substack{e_{ij}\in\mathsf{W}_{k,n},\\ w_i,w_j\in\{x_i,\overline{x}_i\}}} N^{\mathsf{nc}(b^\bullet\vee e^\bullet_{ij})-\mathsf{nc}(b^\bullet\vee 1)-1}\mathsf{r}_N\left(\overset{\circ}{e}_{ij}\diamond\overset{\circ}{b},\overset{\circ}{b}\right)(e_{ij}\diamond b\otimes w)\\ &- \sum_{\substack{\tau_{ij}\in\mathsf{T}_{k,n},\\ w_i=w_j\in\{x_i,\overline{x}_i\}}} N^{\mathsf{nc}(b^\bullet\vee\tau^\bullet_{ij})-\mathsf{nc}(b^\bullet\vee 1)-1}\mathsf{r}_N\left(\overset{\circ}{\tau}_{ij}\diamond\overset{\circ}{b},\overset{\circ}{b}\right)(\tau_{ij}\diamond b\otimes w). \end{aligned} \tag{2.56}$$

Let us detail the computations for the real case. For an oriented Brauer diagram (b,s), set $D_N(b,s) = \prod_{d\in\{d_N\}} d^{-\mathsf{fnc}_d(b)}$. By definition, we have

$$\frac{d}{du}\mathbb{m}^{\mathbb{R}}_{d_N}(\mathsf{t},w,b)(u) = D_N(b,s)\mathsf{Tr}\left(\frac{d}{du}\mathbb{E}\left[w^{\otimes}([\mathsf{U}^{\mathbb{R}}_N]^1(ut_q)\otimes\cdots\otimes[\mathsf{U}^{\mathbb{R}}_N(ut_N))]^q\right]\circ\rho^{\mathbb{R}}_{\mathsf{d}_N}(b)\right). \tag{2.57}$$

Owing to formulae in Section 2.5 we proved for the mean of tensor monomials of the Unitary Brownian diffusion and mutual independence of the family $\left\{[\mathsf{U}^{\mathbb{K}}_N]^i,\ 1\leq i\leq q\right\}$,

$$\begin{aligned}\frac{d}{du}\mathbb{E}\left[w^{\otimes}([\mathsf{U}^{\mathbb{R}}_N]^1(ut_q)\otimes\cdots\otimes[\mathsf{U}^{\mathbb{R}}_N(ut_N))]^q\right] &= \mathbb{E}\left[w^{\otimes}([\mathsf{U}^{\mathbb{R}}_N]^1(ut_q)\otimes\cdots\otimes[\mathsf{U}^{\mathbb{R}}_N(ut_q))]^q\right]\\ &\times\sum_{i=1}^q t_i\left(c^{\mathbb{R}}_N\mathsf{n}_i(w)+\sum_{\substack{e_{ij}\in\mathsf{W}_{k,n},\\ w_i=w_j=x_i}}\frac{1}{N}e_{ij}-\sum_{\substack{\tau_{ij}\in\mathsf{T}_{k,n},\\ w_i=w_j=x_i}}\frac{1}{N}\tau_{ij}\right)\end{aligned}$$

We insert this last equation into formula (2.57) to obtain:

$$\begin{aligned}&\frac{d}{du}\mathbb{m}^{\mathbb{R}}_{d_N}(\mathsf{t},w,b)(u) = \sum_i t_i\Big(\mathsf{n}_i(w)\mathbb{m}^{\mathbb{R}}_{d_N}((b,s),w,\mathsf{t})(u)\\ &\frac{1}{N}\sum_{\substack{e_{ij}\in\mathsf{W}_{k,n},\\ w_i=w_j=x_i}}\prod_d d^{\mathcal{K}_d(e_{ij},b)}D_N(b,s)\mathsf{Tr}\left(\mathbb{E}\left[w^{\otimes}([\mathsf{U}^{\mathbb{R}}_N]^1(ut_q)\otimes\cdots\otimes[\mathsf{U}^{\mathbb{R}}_N(ut_q))]^q\right]\circ\rho^{\mathbb{R}}_{d_N}(e_{ij}\circ b)\right)\\ &-\frac{1}{N}\sum_{\substack{\tau_{ij}\in\mathsf{T}_{k,n},\\ w_i=w_j=x_i}}\prod_d d^{\mathcal{K}_d(\tau_{ij},b)}D_N((b,s))\mathsf{Tr}\left(\mathbb{E}\left[w^{\otimes}([\mathsf{U}^{\mathbb{R}}_N]^1(ut_q)\otimes\cdots\otimes[\mathsf{U}^{\mathbb{R}}_N(ut_q))]^q\right]\circ\rho^{\mathbb{R}}_{d_N}(\tau_{ij}\circ b)\right)\Big)\end{aligned}$$

We orient the coloured Brauer diagrams $e_{ij}\circ b$ and $\tau_{ij}\circ b$ using the $\diamond$ operator. We multiply each terms in the first sum, respectively the second sum of the last equation by the normalization factor $D_N(\tau_{ij}\diamond(b,s))$, respectively $D_N(e_{ij}\diamond(b,s))$ to get, with $r\in\{e,\tau\}$:

$$\frac{\prod_{d\in\{d_N\}}d^{\mathcal{K}_d(r_{ij},b)}D_N(b,s)}{ND_N(r_{ij}\diamond b)}\mathbb{m}^{\mathbb{R}}_{d_N}(r_{ij}\diamond b,w,\mathsf{t}) = N^{\mathsf{nc}(b^{\bullet}\vee r^{\bullet}_{ij})-\mathsf{nc}(b^{\bullet}\vee 1)-1}\mathsf{r}_N\left(\mathring{r}_{ij}\diamond\mathring{b},\mathring{b}\right)f_{d_N}(r_{ij}\diamond(b,s),w,\mathsf{t}).$$

The functional δ_{Δ_k} is the indicator function of the set of coloured oriented Brauer diagrams in $\mathcal{OB}^{d_1}_k$ that are diagonally coloured,

$$\delta_{\Delta_k}((b,s)\otimes w) = \prod_{i=1}^q\delta_{c_b(i)=c_b(i')}.$$

Since the family $\{L^{\mathbb{K}}_{i,N}, i\leq N\}$ is a commuting family of operators, we get the following formula for $\mathbb{m}^{\mathbb{K}}_{d(N)}(\mathsf{t})$,

$$\mathbb{m}^{\mathbb{K}}_{d(N)}(\mathsf{t})(1) = \delta_{\Delta_k}\circ\prod_{i=1}^q e^{t_iL^{\mathbb{K}}_{i,N}} = \delta_{\Delta_k}\circ\exp\left(\sum_{i=1}^q t_iL^{\mathbb{K}}_{i,N}\right). \tag{2.58}$$

We draw the reader's attention on the fact that the domain of definition of the statistic $\mathbb{m}^{\mathbb{K}}_{d(N)}(\mathsf{t})$ and the generators $L^{\mathbb{K}}_{i,N}$ with $i\leq q$ rests on the kernel $\mathsf{Ker}(d_N)$ of the dimension function d_N. This prevents us to simply let N tends to infinity in the formulae (2.55) and (2.56) without further assumption on the sequence d_N. Nevertheless, under the assumption made in Proposition 2.36 on the sequence $(d_N)_{N\geq 1}$ the aforementioned issue does not show up; the generators $L^{\mathbb{K}}_{i,N}$ are defined on the real vector space $\mathbb{R}\left[\mathcal{B}^{\mathsf{d}_1}_k\right]$ included in all the spaces $\mathbb{R}\left[\mathcal{B}^{\mathsf{ker}(d_N)}_k\right]$, $N\geq 1$.

Let $1\leq i,j\leq k$ two integers. The quantities $\mathring{\tau}_{ij}\diamond\mathring{b}$ and $\mathring{e}_{ij}\diamond\mathring{b}$ are computed in the algebra $\mathcal{B}_k(\mathsf{r}_N)$. To make explicit this dependence we write instead for the next few lines $\mathring{\tau}\diamond_{\mathsf{r}_N}\mathring{b}$ and

$\overset{\circ}{e}_{ij} \diamond_{\mathsf{r}_N} \overset{\circ}{b}$. Since the limiting ratios r_i are all positive,

$$\mathsf{r}_N\left(\overset{\circ}{e}_{ij} \diamond_{\mathsf{r}_N} \overset{\circ}{b}, \overset{\circ}{b}\right) \underset{N\to+\infty}{\longrightarrow} \mathsf{r}\left(\overset{\circ}{e}_{ij} \diamond_{\mathsf{r}} \overset{\circ}{b}, \overset{\circ}{b}\right),\ \mathsf{r}_N\left(\overset{\circ}{\tau}_{ij} \diamond_{\mathsf{r}_N} \overset{\circ}{b}, \overset{\circ}{b}\right) \underset{N\to+\infty}{\longrightarrow} \mathsf{r}\left(\overset{\circ}{\tau}_{ij} \diamond_{\mathsf{r}} \overset{\circ}{b}, \overset{\circ}{b}\right).$$

We repeat the discussion we made in Section 2.5 in which the convergence of the statistics $\mathbb{m}_d^{\mathbb{K}}(t), t \geq 0$ was investigated as the dimension d tends to infinity: the sums over the elementary non-mixing diagrams r in equation (2.55) and (2.56) localize over the set of diagrams that create a cycle or a loop if multiplied with b: $r \in \mathsf{T}_k^+(b) \cup \mathsf{W}_k^+(b)$.

We are non convinced that we can let N tends to infinity in equation (2.55) and (2.56). For each Brauer diagram in $\mathcal{B}_k^{d_1}$, and word w in M_2,

$$L_{i,N}^{\mathbb{R}}(b \otimes w) \underset{N\to+\infty}{\longrightarrow} L_{i,\mathsf{r}}(b \otimes w),\ L_{i,N}^{\mathbb{H}}(b \otimes w) \underset{N\to+\infty}{\longrightarrow} L_{i,\mathsf{r}}(b \otimes w),$$

with the generator $L_{i,\mathsf{r}}$ defined for $b \in \mathcal{OB}_k^{\mathsf{r}}$:

$$L_{i,\mathsf{r}}(b \otimes w) = -\frac{1}{2}\mathsf{n}_i(w)(b,w) + \sum_{\substack{e_{ij} \in \mathsf{W}_{k,n}^+(b) \\ w_i = w_j = x_i}} \mathsf{r}(\overset{\circ}{e}_{ij} \diamond \overset{\circ}{b}, \overset{\circ}{b})(e_{ij} \diamond b \otimes w) + \sum_{\substack{\tau_{ij} \in \mathsf{T}_{k,n}^+(b), \\ w_i = w_j = x_i}} \mathsf{r}(\overset{\circ}{\tau}_{ij} \diamond \overset{\circ}{b}, \overset{\circ}{b})(\tau_{ij} \diamond b \otimes w). \tag{2.59}$$

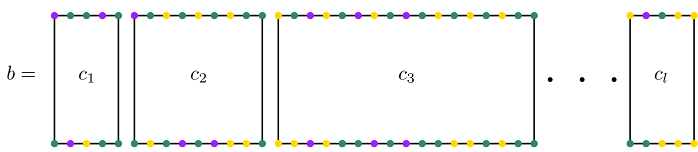
For the complex case, if w is a word in $\overline{\mathsf{M}}_2$ and $b \in \mathcal{OB}_k^{d_1}$, if we let N tends to infinity, we obtain the convergence for each integer $1 \leq i \leq n$ of $\overline{L}_{i,N}^{\mathbb{C}}(b,w)$ to $\overline{L}_{i,\mathsf{r}}(b \otimes w)$ with $L_{i,\mathsf{r}}$ defined by the equation

$$\overline{L}_{i,\mathsf{r}}(b \otimes w) = -\frac{1}{2}\mathsf{n}_i(w)(b \otimes w) + \sum_{\substack{e_{ij} \in \mathsf{W}_{k,n}^+(b) \\ w_i \neq w_j \\ w_i, w_j \in \{x_i, \bar{x}_i\}}} \mathsf{r}(\overset{\circ}{e}_{ij} \diamond \overset{\circ}{b}, \overset{\circ}{b})(e_{ij} \diamond b \otimes w) + \sum_{\substack{\tau_{ij} \in \mathsf{T}_{k,n}^+(b), \\ w_i = w_j \\ w_i, w_j \in \{x_i, \bar{x}_i\}}} \mathsf{r}(\overset{\circ}{\tau}_{ij} \diamond \overset{\circ}{b}, \overset{\circ}{b})(\tau_{ij} \diamond b \otimes w). \tag{2.60}$$

Set $L_{\mathsf{r}} = \sum_{i=1}^q t_i L_{i,\mathsf{r}}$ and $\overline{L}_{\mathsf{r}} = \sum_{i=1}^q t_i \overline{L}_{i,\mathsf{r}}$. Since the generators $L_{i,N}^{\mathbb{K}}, i \leq q, \mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and $L_{i,\mathsf{r}}, \bar{L}_{i,\mathsf{r}}$ act on finite dimensional spaces, we can let N tends to infinity in equation (2.58) and get, for $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$

$$\mathbb{m}_{d_N}^{\mathbb{K}}(b,w,\mathrm{t})(1) \underset{N\to+\infty}{\longrightarrow} \delta_{\Delta_k} \circ e^{\sum_{i=1}^q t_i L_{i,\mathsf{r}}(b \otimes w)},\ \mathbb{m}_{d_N}^{\mathbb{C}}(b \otimes w,\mathrm{t})(1) \underset{N\to+\infty}{\longrightarrow} \delta_{\Delta_k} \circ e^{\sum_{i=1}^q t_i \overline{L}_{i,\mathsf{r}}(b \otimes w)}.$$

We want now to empasize an important feature of the generators L_{r} and $\overline{L}_{\mathsf{r}}$ that is most easily discussed in the case $q = 1$. If $c_1, \ldots, c_l$ are irreductible Brauer diagrams of sizes $k_1, \ldots, k_l$ we denote by $c_1 \otimes \cdots \otimes c_l$ the coloured Brauer diagram of size $k = k_1 + \cdots + k_l$ which cycles are the diagrams $c_1, \ldots, c_l$ (in this order):



Owing to the definition of the operator L_{r}, for words $w_i \in \mathsf{M}_q(k_i)$, $1 \leq i \leq l$,

$$L_{\mathsf{r}}(b \otimes w) = L_{\mathsf{r}}(c_1 \otimes w_1) \otimes \cdots \otimes c_k \otimes w_k + c_1 \otimes L_{\mathsf{r}}(c_2) \otimes \cdots \otimes c_k + \cdots c_1 \otimes c_2 \otimes \cdots \otimes L_{\mathsf{r}}(c_k).$$

This last equation implies $\mathbb{m}_{\mathsf{r}}(c_1, w_1) \cdots \mathbb{m}_{\mathsf{r}}(c_l, w_l) = \mathbb{m}(b \otimes w_1 \cdots w_l)$. This factorization property extends to general Brauer diagrams:

$$\mathbb{m}_{\mathsf{r}}(b,w) = \mathbb{m}_{\mathsf{r}}(b_{V_1}, w_{V_1}) \cdots \mathbb{m}_{\mathsf{r}}(b_{V_p}, w_{V_p}) \tag{2.61}$$

where $b^\bullet \vee 1 = \{V_1, \ldots, V_p\}$, w_V is the word w restricted to V, $w_V = w_{i_1} \cdots w_{i_k}$ if $V = \{i_1, \ldots, i_k\}$ and b_V is the part of b that is contained in V. Of course, the same property holds for $\overline{\mathbb{m}}_{\mathsf{r}}$.

2.6.4. Convergence in high dimensions of the rectangular extraction processes. In this section, we prove that Proposition 2.36 implies the convergence in non-commutative distribution of the quantum processes $U_{d_N}^{\mathbb{K}}$ and $U_{d_N}^{\mathbb{K}}$ toward free Lévy processes. Let $N,n \geq 1$ be integers and for each $N \geq 1$, d_N a partition of N into n parts. In this section, we interpret the convergence proved in Section 2.6.2 of the statistics $f_{d_N}^{\mathbb{K}}$ as N tends to infinity in term of convergence of the rectangular extractions process $U_{d_N}^{\mathbb{K}}$ and $U_{\{d_N\}}^{\mathbb{K}}$ in non-commutative distribution.

Theorem 2.37. *Let $n \geq 1$ an integer. For each integer $N \geq 1$, pick a partition d_N of N into n parts. Let $\mathbb{K}$ be one of the three divisions algebras $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Assume that as N tends to infinity, there exists positive real numbers $r_i \in]0,1]$, for $1 \leq i \leq n$, such that*

$$\frac{d_N(i)}{N} \underset{N\to+\infty}{\longrightarrow} r_i,\ 1 \leq i \leq n.$$

As the dimension N tends to infinity, the non-commutative distribution of $U_{d_N}^{\mathbb{K}}$ converges to a $\mathcal{D}_n$ amalgamated free semi-group.

Prior to proving Theorem 2.37, we first settle some notations. We use the symbol E_{r} to denote the amalgamated free semi-group which existence stated in Theorem 2.37 with $\mathsf{r} = (r_1,\ldots,r_n)$. Recall that E_{r} is a free amalgamated semi-group means:

$$\mathsf{E}_{\mathsf{r}}(t+s) = \mathsf{E}_{\mathsf{r}}(t) \,\dot{\sqcup}_{\mathcal{D}_n}\, \mathsf{E}_{\mathsf{r}}(s).$$

Also a formula for the generator $\mathcal{L}_{\mathsf{r}}$ of this semi-group can be read of the formula for the operator $\overline{L}_{1,\mathsf{r}}$. Let $x = u_{\alpha_1,\alpha_2}^{\varepsilon(1)} \cdots u_{\alpha_{k-1},\alpha_k}^{\varepsilon(k)}$, the Brauer diagram b in the following equation is defined as in equation (2.67) and (2.69) below,

$$\mathcal{L}_{\mathsf{r}}(x) = -\frac{1}{2}k\delta_{\Delta_k}(b) + \sum_{e_{ij}\in \mathsf{W}_{k,n}^{+}(b)} \mathsf{r}(\mathring{e}_{ij} \diamond \mathring{b}, \mathring{b})\delta_{\Delta_k}(e_{ij}\diamond b) + \sum_{\tau_{ij}\in \mathsf{T}_{k,n}^{+}(b)} \mathsf{r}(\mathring{\tau}_{ij} \diamond \mathring{b}, \mathring{b})\delta_{\Delta_k}(\tau_{ij}\diamond b). \tag{2.62}$$

Let $i,j \leq k$. The way we chose to orient $e_{ij} \circ b$ and $\tau_{ij} \circ b$, using the operator $\diamond$ was quite arbitrary. However, the generator $\mathcal{L}_{\mathsf{r}}$ does no depend on such a choice, since δ_{Δ_k} is the support function of diagonally coloured Brauer diagrams. We fix, once for all, a division algebra $\mathbb{K}$ and to lighten the notation, we drop the symbol $\mathbb{K}$ in the notations introduced so far.

We focus on the cases $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$. In fact, the function $\overline{\mathbb{m}}_{\mathsf{r}}$ is equal to the function $\mathbb{m}_{\mathsf{r}}$ on the linear span of tensors $(b,s)\otimes \overline{w}$ with $((b,s),\overline{w})$ a pair of compatible word and diagrams meaning that:

$$\mathbb{m}_{\mathsf{r}}((b,s),w) = \mathsf{m}((b,s)\otimes \overline{w})$$

for any word $w \in \mathsf{M}_q(k)$ such that for all integer $1 \leq i \leq k$, $\overline{w}_i \in \{w_i, \overline{w}_i\}$. The joint distribution of the random variables $\left[\mathsf{U}_N^{\mathbb{C}}\right]^{(1)},\ldots,\left[\mathsf{U}_N^{\mathbb{C}}\right]^{(q)}$ is contained in the range of the statistic $\mathbb{m}_{d_N}^{\mathbb{C}}$ restricted to the linear span tensor of compatible words and diagrams, hence the limiting distribution of the process $U_{d_N}^{\mathbb{C}}$ is equal to the limiting distribution of $U_{d_N}^{\mathbb{R}}$ (and equal to the limiting distribution of $U_{d_N}^{\mathbb{H}}$). We continue with a small reminder. In Section 2.6.2, we introduced $\mathcal{M}_{d_N}$ as the rectangular probability space $\left(\mathcal{M}_N(L^\infty(\Omega,\mathcal{F},\mathbb{P},\mathbb{K})),\ \mathbb{E}_{d_N}, \frac{d_N}{N}\right)$. For all couple of integers α,β in the interval $[1,\ldots,n]$ we denote by $\mathcal{M}_{d_N}(\alpha,\beta)$ the compressed space $p_\alpha \mathcal{M}_{d_N} p_\beta$.

The conditional expectation $\mathbb{E}_{d_N}$ is a $\mathcal{D}_n$-bimodule map and for each integer $k \geq 1$, it defines an other $\mathcal{D}_n$-bimodule map $\mathbb{E}_{d_N}^k : \mathcal{M}_{d_N} \otimes_{\mathcal{D}_n} \cdots \otimes_{\mathcal{D}_n} \mathcal{M}_{d_N} \to \mathcal{D}_n$ by the formula:

$$\mathbb{E}_{d_N}^k(M_1 \otimes \cdots \otimes M_k) = \mathbb{E}^{d_N}(M_1 \cdots M_k).$$

To the family of maps $\left(\mathbb{E}_{d_N}^k\right)_{k\geq 1}$ is associated a multiplicative functional, also denoted $\mathbb{E}_{d_N}$, on the set of non crossing partitions (of any size). To study asymptotic amalgamated freeness, it is more convenient to work with the cumulant functional $\{c_\pi^{d_N} : \mathcal{M}_{d_N} \otimes_{\mathcal{D}_n} \cdots \otimes_{\mathcal{D}_n} \mathcal{M}_{d_N} \to \mathcal{D}_n,\ \pi \in \mathsf{NC}_k\}$. These cumulant functions are obtained by mean of a Möebius transformation, for

$r_1,\dots,r_k \in \mathcal{M}_{d_N}$,

$$\mathbb{E}^{\pi}_{d_N}(r_1 \otimes_{\mathcal{D}_n} \cdots \otimes_{\mathcal{D}_n} r_n) = \sum_{\alpha\leq\pi} c^{\alpha}_{d_N}(r_1 \otimes_{\mathcal{D}_n} \cdots \otimes_{\mathcal{D}_n} r_k),\ \pi \in \mathsf{NC}_k,$$

or equivalently:

$$c^{\pi}_{d_N}(r_1 \otimes \cdots \otimes r_k) = \sum_{\alpha\leq\pi} \mu(\alpha,\pi)\mathbb{E}^{\alpha}_{d_N}(r_1 \otimes \cdots \otimes r_k).$$

Asymptotic freeness of the semi-group E_{d_N} is equivalent to asymptotic amalgamated freeness of the increments of the process $U^{\mathbb{K}}_{\mathsf{d}}$, which can be checked on the cumulants. Proof of Theorem 2.37 is thus divided into two (big) steps :

1. For each time $t \geq 0$ and integer $k \geq 1$, we prove the convergence of $\mathsf{E}^{d_N}_k(t)$, as N tends to infinity.
2. For all times $s_1 < t_1 \leq s_2 < t_2 \ldots \leq s_q < t_q$, we prove that the cumulants
$$c^k_{d_N}\left(\left[\mathsf{U}^{\mathbb{K}}_N(s_{j_1},t_{j_1})\right]^{\varepsilon(1)} \otimes_{\mathcal{D}_n} \cdots \otimes_{\mathcal{D}_n} \left[\mathsf{U}^{\mathbb{K}}_N(s_{j_k},t_{j_k})\right]^{\varepsilon(k)}\right)$$
of increments of the process $U^{\mathbb{K}}_{\mathsf{d}}$ converges to 0 for any k-tuples $\varepsilon(1),\dots,\varepsilon(k) \in \{1,\star\}$ if there exists two non equal integers $j_a \neq j_b$ in the sequence $j_1,\dots,j_k \in \{1,\dots,q\}$.

Point 1. shows no difficulties, it is a simple corollary of Proposition 2.36 we stated and proved in Section 2.6.

The second point needs more precisions. Prior to expound the proof of point 2., we sketch it. With the notations introduced in point 2., set $\mathsf{t} = (t_{j_1} - s_{t_{j_1}},\dots,t_{j_q} - s_{j_q})$.

Let $\alpha_0,\alpha_1,\dots,\alpha_k$ a k-tuple of integers in $[1,\dots,n]$ and $\varepsilon(1),\dots,\varepsilon(k) \in \{1,\star\}$. The first step is about finding a Brauer diagram $b \in \mathcal{B}_k$ and a word $w \in \mathsf{M}_q(k)$ for which the asymptotic

$$\mathbb{m}^{\mathbb{K}}_{d_N}(b,w,\mathsf{t})p_{\alpha_0} - \mathbb{E}^{\pi}_{d_N}\left(p_{\alpha_0}\left[\mathsf{U}^{\mathbb{K}}_N(s_{j_1},t_{j_1})\right]^{\varepsilon(1)} p_{\alpha_1} \otimes \cdots \otimes p_{\alpha_{k-1}}\left[\mathsf{U}^{\mathbb{K}}_N(s_{j_k},t_{j_k})\right]^{\varepsilon(k)} p_{\alpha_k}\right) \underset{N\to+\infty}{\longrightarrow} 0 \tag{2.63}$$

holds.

Since $\mathsf{U}^{\mathbb{K}}_N$ is a Lévy process, we can substitute to the set of increments $\{\mathsf{U}^{\mathbb{K}}_N(s_1,t_1),\dots,\mathsf{U}^{\mathbb{K}}_N(s_q,t_q)\}$, a set of independent copies of the process $U^{\mathbb{K}}_{\mathsf{d}}$ evaluated at times $t_1 - s_1,\dots,t_q - s_q$ to compute the cumulants. This step leans on *condensation* property of $U^{\mathbb{K}}_{d_N}$ we exposed in at the end of the last section, see equation (2.63). We then write $\mathbb{m}_{\mathsf{r}}(b,w,\mathsf{t})$ (respectively $\overline{\mathbb{m}}_{\mathsf{r}}(b,w,\mathsf{t})$) as a sum:

$$\mathbb{m}_{\mathsf{r}}(b,w,\mathsf{t}) = \sum_{\gamma\leq\pi} c_{\gamma}(\alpha,w,\mathsf{t}). \tag{2.64}$$

To that end we use the differential systems that are satisfied by the limiting statistic $\mathbb{m}_{\mathsf{r}}$, and formulae (2.55), (2.56) found in Section 2.6.2 for the generators to give an explicit formula for the coefficients $c_{\gamma}(\alpha,\beta,\mathsf{t})$, $\gamma \in \mathsf{NC}_k$. From equation (2.63) and (2.64), we infer that:

$$\mathbb{E}^{\pi}_{d_N}\left(p_{\alpha_0}\left[\mathsf{U}^{\mathbb{K}}_N(s_{j_1},t_{j_1})\right]^{\varepsilon(1)} p_{\alpha_1} \otimes_{\mathcal{D}_n} \cdots \otimes_{\mathcal{D}_n} p_{\alpha_{k-1}}\left[\mathsf{U}^{\mathbb{K}}_N(s_{j_k},t_{j_k})\right]^{\varepsilon(k)} p_{\alpha_k}\right) \underset{N\to+\infty}{\longrightarrow} \sum_{\gamma\leq\pi} c_{\gamma}(\alpha,\mathsf{t}). \tag{2.65}$$

Since this last equality is valid for all non-crossing partitions π in NC_k, we can apply Möebius transformation to both side of (2.65) and deduce that:

$$c^k_{d_N}\left(\left[\mathsf{U}^{\mathbb{K}}_N(s_{j_1},t_{j_1})\right]^{\varepsilon(1)} \otimes_{\mathcal{D}_n} \cdots \otimes_{\mathcal{D}_n} \left[\mathsf{U}^{\mathbb{K}}_N(s_{j_k},t_{j_k})\right]^{\varepsilon(k)}\right) \underset{N\to+\infty}{\longrightarrow} c_k(\alpha,\beta,\mathsf{t}). \tag{2.66}$$

We begin, of course, with the first step. Let $\pi \in \mathsf{NC}_k$ a non-crossing partition and write $\pi = \{c_1,\dots,c_p\}$. The linear order on $[1,\dots,k]$ along with the partition π define a permutation σ_π of $[1,\dots,k]$: the cycles of π are the blocks c_i's, $i \leq p$, endowed with the natural cyclic order. We define a non-coloured Brauer diagram by:

$$b^{\bullet} = \prod_{\substack{1\leq i\leq k:\\ \varepsilon(i)=\star}} \mathsf{Tw}^{\bullet}(\sigma_\pi). \tag{2.67}$$

The permutation σ_b associated with b and defined in Section 2.4 is, equal to σ_π. We now define the colourization of the non-coloured Brauer diagram $b^\bullet$. Set $\mathsf{i}' = (\alpha_0, \ldots, \alpha_{k-1})$, $\mathsf{j}' = (\alpha_1, \ldots, \alpha_k)$. By using i, j and ε, we define a colourization i of the bottom line of $b^\bullet$ and an other one, which we call j of the bottom line of $b^\bullet$ by setting:

(2.68) $$\mathsf{i}_i = \mathsf{i}'_i, \text{ if } \varepsilon_i = 1,\ \mathsf{i}_i = \mathsf{j}'_i \text{ if } \varepsilon_i = \star, \text{ and } \mathsf{j}_i = \mathsf{j}'_i, \text{ if } \varepsilon_i = 1,\ \mathsf{j}_i = \mathsf{i}'_i \text{ if } \varepsilon_i = \star.$$

Finally, define the colourization c in $\{1, \ldots, k, 1', \ldots, k'\}$ by $c(i) = \mathsf{j}_i$ and $c(i') = \mathsf{j}_i$ for $1 \leq i \leq k$. We can not affirm that for all π, the colourization c is an admissible colourization of the non-coloured Brauer diagram $b^\bullet$ defined by equation (2.67). Hence, define the element b in $\mathbb{R}[\mathcal{B}_k]$ as:

(2.69) $$b = (b^\bullet, c) \text{ if } c \in C(b^\bullet) \text{ and } b = 0 \text{ if } c \notin C(b^\bullet).$$

Let w be the word $w = x_{j_1} \cdots x_{j_k}$. We now prove that asymptotic (2.63) holds. We recall first basic properties of the maps $\mathbb{E}^k_{d_N}$, $k \geq 1$. Let $k \geq 1$ an integer and two finite sequences $\gamma_0, \ldots, \gamma_{k-1}$ and $\beta_1, \ldots, \beta_k$ of integers in the interval $[1, \ldots, n]$. Since $\mathbb{E}^k_{d_N}$ is a $\mathcal{D}_n$-bi-module map, $\mathbb{E}^k_{d_N}(a_1 \otimes_{\mathcal{D}_n} \cdots \otimes_{\mathcal{D}_n} a_k) = 0$ with $a_i \in \mathcal{M}_{d_N}(\gamma_i, \beta_i)$ if there exists at least one integer $0 \leq i \leq k-1$ such that $\gamma_{i+1} \neq \beta_i$ (with the convention $\gamma_{k=1} = \gamma_0$) and $\mathbb{E}^k_{d_n}(a_1 \otimes_{\mathcal{D}_n} \cdots \otimes_{\mathcal{D}_n}) = \mathbb{E}^k_{d_n}(a_1 \otimes_{\mathcal{D}_n} \cdots \otimes_{\mathcal{D}_n}) p_{\beta_0}$. Let $\beta_0, \ldots, \beta_k \in [1, \ldots, n]$ a finite sequence of integers. A direct induction proves that for any non-crossing partition $\gamma \in \mathsf{NC}_k$, the map $\mathbb{E}^{d_n}_\gamma$ evaluates to zero on all k-tuples of elements $(a_1, \ldots, a_k)$ with a_i in the compressed algebra $\mathcal{M}_{d_N}(\beta_i, \beta_{i+1})$, $0 \leq i \leq k$, if there exists a block $V = \{v_1 < \ldots < v_s\}$ of γ such that $\beta_{v_1 - 1} \neq \beta_{v_k}$. Focusing on the tuple α and non-crossing partition π we chose, requiring that $\alpha_{v_1-1} = \alpha_{v_k}$ for all $\{v_1 < \cdots < v_k\}$ of π is the same as demanding that the colourization c is in $C(b^\bullet)$, which means $(b^\bullet, c) \in \mathcal{B}_k$. $\mathbb{E}^\pi_{d_N}\left(p_{\alpha_0}\left[\mathsf{U}^{\mathbb{K}}_N(s_{j_1}, t_{j_1})\right]^{\varepsilon(1)} p_{\alpha_1} \otimes_{\mathcal{D}_n} \cdots \otimes_{\mathcal{D}_n} p_{\alpha_{k-1}}\left[\mathsf{U}^{\mathbb{K}}_N(s_{j_k}, t_{j_k})\right]^{\varepsilon(k)} p_{\alpha_k}\right) = 0$ for all integers $N \geq 1$ if, with the above notation, $\alpha_{v_1-1} \neq \alpha_{v_k}$ for at least one block of π. We write $\mathbb{m}(b, w, \mathsf{t})$ as a sum over non-crossing partitions. It cumbersome to introduce some new notations to give explicit formulas for the coefficients $c_\gamma(\alpha, \mathsf{t}, w)$. First, we introduce

$$\mathsf{R}^+_s(b) = \{(r_{i_1,j_1}, \ldots, r_{i_s,j_s}) \in (\mathsf{R}_k)^s : r_{i_l,j_l} \in \mathsf{T}^+ \cup \mathsf{W}^+(r_{i_{l+1},j_{l+1}} \circ \cdots \circ r_{i_s,j_s} \circ b)\}.$$

Let us give a geometric interpretation of the set $\mathsf{R}^+_s(b)$. First, define a graph $\mathcal{G}$ that have as vertices the set $\mathcal{B}_k$ and by considering two Brauer diagrams as adjacent in this graph if one is obtained from this other by concatenation with a transposition or a projector. This graph have loops, since $e \circ e = e$. These loops can be broken if instead of considering as vertices the set $\mathcal{B}_k$, we replace it with the central extension $\overline{\mathcal{B}_k}$ and requiring for two Brauer elements in $\overline{\mathcal{B}_k}$ to be neighbours if one is obtained from the other by multiplication with a transposition / projection. For example, we would have $ee = (e, \mathsf{o})$ for some loop o that belongs to $\overline{\mathcal{G}}$. A tuple r in $\mathsf{R}^+_s(b)$ is a path in $\overline{\mathcal{G}}$) that starts at $(b, \emptyset)$ a visit successively Brauer elements that have one more loop or cycle comparing to the last one visited. We insist on the fact that the set $\mathsf{R}^+_s(b)$ is not solely determined by the partition π but depends also on the sequence ε that was use to twist the diagram (this twist are responsible for loops that may be created along a path). To be more precise, we are interested in a subset of paths in $\mathsf{R}^+_s(b)$ that have increments constrained by the word w,

$$\mathsf{R}^+_s(b, w) = \{(r_{i_1,j_1}, \ldots, r_{i_s,j_s}) \in \mathsf{R}^+_s(b) : w_{i_1} = w_{j_1}, \cdots, w_{i_s} = w_{j_s}\}.$$

We are now splitting the set $\mathsf{R}^+_s(b, w)$ according to the cycle partition of the end-point of a path, and define for this the function:

$$\gamma_s\left((r_{i_1,j_1}, \ldots, r_{i_s,j_s})\right) = \left((r^\bullet_{i_1,j_1} \circ \ldots \circ r^\bullet_{i_s,j_s}) \vee \mathbf{1}_k\right) \cap [1, \ldots, k].$$

For β a partition of $[1, \ldots, k]$, set $\mathsf{R}^+_s(b, w, \beta) = \{\gamma_s = \beta\} \cap \mathsf{R}^+_s(b, w)$.

Lemma 2.38. *Let b a Brauer diagram and denote by π the trace of the partition $b^\bullet \vee 1$ on $[1, \ldots, k]$. Assume that π is non crossing, then, for all tuple $\mathsf{r} \in \mathsf{R}^+_s(b)$, the partition $\gamma_s(\mathsf{r})$ is non-crossing and $\gamma_s(\mathsf{r}) \leq \pi$. In addition for all words $w \in \mathsf{M}_q$, $\mathsf{R}^+_s(b, w, \alpha) = \emptyset$ if there at least one couple of integers $1 \leq i, j \leq k$ with $i \sim_\pi j$ and $w_i \neq w_j$.*

In the following, we set $\mathsf{t}^{\mathsf{r}} = t_{i_1} \cdots t_{i_s}$. By using formulae proved in the previous section for the generators $L_{i,\mathsf{r}}$, we write as a sum over $\mathsf{R}_s^+(b,w)$ the exponential $\exp\left(\sum_{i=1}^q t_i L_{i,\mathsf{r}}\right)$ to obtain the following expression:

$$\exp\left(\sum_{i=1}^q t_i L_{i,\mathsf{r}}\right)(b,w) = e^{\frac{-1}{2}\sum_{i=1}^q t_i \mathsf{n}_i(w)}$$
$$\times \sum_{s=0}^{\infty} \sum_{\mathsf{r}\in \mathsf{R}_s^+(b,w)} \frac{\mathsf{t}^{\mathsf{r}}}{s!} \prod_{k=1}^{s} \mathsf{r}(r_{i_k,j_k} \diamond \ldots \diamond r_{i_s,j_s}, r_{i_{k+1},j_{k+1}} \diamond \ldots \diamond r_{i_s,j_s} \diamond b)\cdot(r_{i_1,j_1} \diamond \ldots \diamond r_{i_s,j_s} \diamond b, w)$$

Next, we split the sum over $\mathsf{R}_s^+(b)$ in the last equation into sums over the level sets of γ_s, each of these sum defines an operator $L_{\mathsf{r}}(\beta,\mathsf{t})$, $\beta \in \mathsf{NC}_k$

$$L_{\mathsf{r}}(\beta,\mathsf{t})(b,w) = \sum_{s=0}^{\infty} \frac{\mathsf{t}^{\mathsf{r}}}{s!} \sum_{\mathsf{r}\in \mathsf{R}_s^+(b,w,\alpha)} \prod_{k=1}^{s} \mathsf{r}(r_{i_k,j_k} \diamond \ldots \diamond r_{i_s,j_s}, r_{i_{k+1},j_{k+1}} \diamond \ldots \diamond r_{i_s,j_s} \diamond b)(r_{i_1,j_1} \diamond \ldots \diamond r_{i_s,j_s} \diamond b, w).$$

Finally, we obtain the following expression for $\mathbb{m}(b,w,\mathsf{t})$:

$$\mathbb{m}(b,w,\mathsf{t}) = \delta_{\Delta_k}\left(\exp\left(\sum_{i=1}^q t_i L_{i,\mathsf{r}}\right)(b,w)\right) = e^{-\frac{1}{2}\sum_{i=1}^q t_i \mathsf{n}_i(w)} \sum_{\beta\leq\pi} \delta_{\Delta_k}(L_{\mathsf{r}}(\beta,\mathsf{t})(b,w)).$$

It remains to show that for each non-crossing partition β less than π, $\delta_{\Delta_k}(\mathsf{L}_{\mathsf{r}}(\beta,\mathsf{t})(b,w))$ does not depends on π. As a matter of fact, the set $\mathsf{R}_s^+(b,w,\beta)$ does not depends on π and is determined by ε and β. Furthermore, owing to its definition, $\delta_{\Delta_k}(b)$ is a function of the colourization c (which lies on α).

Set $c_\beta(\alpha,w,\varepsilon) = \delta_{\Delta_k}(L_{i,\mathsf{r}}(\beta,\mathsf{t})((\mathbf{0}_k,c),w))$. Lemma 2.38 implies $c_{\mathbf{1}}(\alpha,w,\varepsilon) = 0$ if the word w contains to different letters. The method that was used to prove theorem 2.37 can be applied verbatim to prove the following theorem, thus the proof is left to the reader.

Theorem 2.39. *Let $n \geq 1$ an integer and for each integer N greater than one, let d_N be a partition of N into n parts.*

We assume that for all integer $i \leq n$, the ratio $r_N(i) = \frac{d_N(i)}{N}$ converges as N tends to infinity to a positive value r_i less than one. We assume further that the kernel of the partition d_1 is finer than the kernels $\mathsf{Ker}(d_N)$, $N \geq 1$. Let $\mathbb{K}$ be one the three divisions algebras $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

As the dimension N tends to infinity, the non-commutative distribution of $U^{\mathbb{K}}_{\{d_N\},\mathsf{ker}(d_1)}$ converges to a $\mathcal{D}_{d_1}$-amalgamated free semi-group.

CHAPTER 3

Pseudo-unitary Brownian diffusions in high dimensions

In this chapter, we give a definitin of Brownian motions on spaces of pseudo-Hermitian matrices and pseudo-unitary matrices are, and study the convergence in non-commutative distribution of these processes under the assumption the dimension normalized signature of the metric converges. Our study includes the real case, the complex case and the quaternionic cases.

Contents

3.1. Introduction

This work deal with diffusions on non necessarily compact Lie groups. We begin by reviewing results concerning Brownian diffusions on space of matrices in high dimensions. In the 1950's, Wigner postulated that the spectral distribution of a large random matrix constitutes a useful model to describe the energy levels of an atom within a compound containing a lot of non-localized electrons. Wigner proved that the spectral distribution of a Hermitian random matrix with independent and appropriately scaled identically distributed entries converges, as the dimension of the matrix tends to infinity, to the celebrated semi-circle distribution. To prove the convergence, he used what is nowadays known as the moments method. Stated in the terminology of non-commutative probability, Wigner proved the convergence in non-commutative distribution of a properly renormalized Hermitian matrix. The theory of free probability originally developed by Voiculescu has proven to be relevant for studying large random matrices, see [41], [3]. In fact, instead of just considering a large random matrix, one can investigate the convergence in high dimensions of a properly renormalized Brownian motion on the space of Hermitian matrices which law is invariant by conjugation under the unitary group. Speicher proved that asymptotically, the increments of such a Hermitian Brownian motion are free. Combining the work of Wigner and Speicher, we deduce that as the dimension tends to infinity, the Hermitian Brownian motion converges to a free Lévy process known as a free Brownian motion.

Later on, Biane proved a similar result for the complex unitary Brownian motion, and T. Lévy in [41] generalised this result to Brownian diffusions on the orthogonal and compact symplectic groups. T. Kemp and G. Cébron investigated in [?] the convergence in non-commutative distribution of a properly scaled Brownian motion named by a (r,s)-Brownian motion and denoted $B_{r,s}$ on the space of complex invertible matrices. Giving meaning to the notion of Brownian diffusions on such non-compact group is challenging. In [?], the authors show that asymptotically, $B_{r,s}$ is a free process and named the limiting process the free (r,s)-Brownian motion. In addition, in [?], the author studied fluctuations for the convergence of $B_{r,s}$ and proved that magnitude of the latter scaled as the inverse of the dimension, with the profile of remaining noise being Gaussian.

Recently, authors raised far more complex questions related to convergence of the spectral distribution of a random Gaussian invertible matrix. In fact, since matrices in a Brownian motion's path on $\mathrm{GL}_n(\mathbb{C})$ are almost surely non normal, convergence in non-commutative distribution does not imply convergence of the spectral measure. Interested readers will find material related to Brown measures in [34].

The present Chapter deal with Brownian diffusions on a class of non-compact real Lie algebras: algebras of anti-symmetric (or anti-Hermitian) matrices with respect to split metrics. Motivation for such a study is found in the physical theory of PT-quantum mechanics in which Hamiltonians of dynamical systems are pseudo-symmetric, see for example [7].

To introduce our work, let us review how to define a Brownian motion on general locally compact semi-simple Lie groups. In the compact case, one can define a Brownian motion in a natural way. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. We first pick a scalar product on $\mathfrak{g}$. We then solve the differential equation that rolls without slipping a standard linear Brownian motion on the Lie algebra, with covariance given by the chosen scalar product. The first to apply this method to integrate any local martingale on the Lie algebra $\mathfrak{g}$ was Lépingle in his seminal article [33]. When invariance for the law of the Brownian motion is required under the left and the right translation of the group on itself, choices for the scalar product are dramatically reduced. In fact, up to multiplication by a positive real number, on simple Lie group, only one scalar product has the required invariance: the negative of the Killing form of the Lie algebra.

For non-compact semi-simple Lie groups, requiring invariance for the law of the diffusion by left and right translations is far too restrictive as there may not be any conjugation invariant scalar products: the Killing form is not definite. A solution to this problem is to pick a splitting of the Lie algebra into a positive and negative subspace for the Killing form. The splittings we consider are in one-to-one correspondence with Cartan involutions. This leads to the introduction of a family of Brownian diffusions indexed by the Cartan involutions of the Lie algebra and relative speeds of diffusion.

In [35], the author investigated how to define a Brownian diffusion on the complexification of the group of unitary matrices, resulting in the group of complex invertible matrices $\mathrm{GL}_n(\mathbb{C})$. The first part of this work offers a generalisation of this construction. In a second part, we use this generalisation to construct diffusions on pseudo-unitary groups of real, complex and quaternionic matrices $\mathbb{U}(p,q,\mathbb{K})$. Most of these diffusions form a family indexed by two real numbers which are the diffusion's speed in respectively non-compact and compact directions. We name each of these diffusions a (v,c) pseudo-unitary Brownian motion.

In the same way, the author in [35] considered a two parameters family of diffusions on $\mathrm{GL}_n(\mathbb{C})$. It has since been shown in [35] that the classification of unitarily invariant scalar products on the Lie algebra of matrices $\mathfrak{gl}_n(\mathbb{C})$ implies instead that Brownian diffusions on $\mathfrak{gl}_n(\mathbb{C})$ are naturally indexed by three parameters. This point will be further discussed in these notes. Finally, our work is divided into two parts, we should focus first on the asymptotic in large dimension of a Brownian motion on the group of split pseudo-unitary matrices $\mathbb{U}(p,p,\mathbb{K}), p \geq 1, \mathbb{K} = \mathbb{R}, \mathbb{H}$ or $\mathbb{C}$ viewed as a non-commutative process with values in a Zhang algebra. This structure algebra is new and is a pseudo-unitary version of Voiculescu's dual unitary group. In the second part, we treat the general case of convergence of the pseudo-unitary Brownians motions in high dimensions under the hypothesis that the normalized signature of the underlying metric converges. Our present work is summed up in the content of the next theorem.

Theorem. *As the dimension of tends to infinity, the (v,c) real, complex and quaternionic pseudo-unitary Brownian motions converge in non-commutative distribution to the same limit which is a free Lévy process.*

In the split case, we prove that the limiting process is solution of a free stochastic differential equation and we name it a (v,c) free Brownian motion. We also provide a Schürmann triple for this process (a non-commutative version of the famous Lévy triple). To prove the aforementioned convergence, we prove a Schur–Weyl duality that involves the algebra of Brauer diagrams with coloured links.

The split pseudo-orthogonal groups $\mathbb{U}(p,p,\mathbb{R})$ are, in a way, maximally non-compact. We consider as the main outcome of this article the fact that free probability appears as the natural framework to describe convergence of the mixed moments of Brownian diffusions on $\mathbb{U}(p,p,\mathbb{K})$, as it does when we study convergence of a Brownian diffusion on the compact group $\mathbb{U}(0,2p,\mathbb{R})$.

Outline. Stochastic Calculus on group of matrices and free stochastic calculus are introduced in 3.2.

In Section 3.3, we make a brief remainder on Cartan Decomposition and define diffusions associated with a Cartan decomposition of semi-simple Lie algebra. In section 3.4, we define the Lie algebra of pseudo-Hermitian matrices with real, complex and quaternionic entries and the Lie groups of pseudo-unitary matrices. Bi-coloured Brauer diagrams, the combinatorial tool that is used to find a differential equation satisfied by the moments of split pseudo-unitary diffusions are defined in Section 3.5. We prove in Section 3.6 the convergence in high dimensions of a set of statistics for the split pseudo-unitary diffusions. Finally, in Section 3.7, we define the pseudo-unitary dual group, the free pseudo-unitary diffusion and prove our first main result Theorem 3.43. The Schürmann triple of the free pseudo-unitary Brownian motion is computed in Section 3.7.3. Finally, the general case is treated in Section 3.9 in which our second main result is Stated in Theorem 3.49.

3.2. Stochastic calculus on groups of matrices and free stoch. calc.

We expose in this section the basic material needed to define pseudo-unitary Brownians diffusions. We begin with reviewing stochastic calculus for group valued processes and end with free stochastic calculus.

3.2.1. Stochastic calculus on groups of matrices. Let $n \geq 1$ an integer. Let $\mathbb{K}$ be one of the three division algebras $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Let $G \subset GL_n(\mathbb{K})$ be a finite dimensional real connected Lie subgroup of the group of invertible matrices of dimension $n \times n$ and denote by $\mathfrak{g}$ be its Lie algebra. For each $E \in \mathfrak{g}$, the associated left invariant vector field on G is denoted ∂_E. For any real smooth function f on G and all $g \in G$,

$$\partial_\xi(f)(g) = \frac{d}{dt} f(g\exp(t\xi)),\ (f \in C^\infty(G),\ g \in G).$$

Set $p = \dim(G)$. Let $E_1, \ldots, E_p$ be p elements in $\mathfrak{g}$ and $W(t) = (W_k(t))_{1 \leq k \leq p}$ be a $\mathbb{R}^p$ valued Brownian motion. Pick an element $g \in G$. For all $t \geq 0$, we denote by $\sigma(W_u, u \leq t)$ the $\sigma-$ field generated by the Brownian motion W up to time t. The following result goes back, at least, to McKean, see [].

Proposition 3.1 ([28]). *There exists an unique $\sigma(W(u), u \leq t)$ measurable continuous G valued process $\Lambda = (\Lambda_t)_{t \geq 0}$ solution of the following Itô stochastic differential equation*

$$\begin{cases} d\Lambda_t = \sum_{k=1}^p \Lambda_t (E_k dW_k(t)) + \frac{1}{2}\Lambda_t \left(\sum_{k=1}^p E_k^2\right) dt, \\ \Lambda_0 = g. \end{cases} \tag{3.1}$$

Proof. See [28]. □

Proposition 3.2 ([28]). *With the notation of the last proposition, the right increments of the process Λ solution of the stochastic differential equation (3.1) are independent and homogeneous: for any $s \geq 0$, the process $\left(\Lambda_s^{-1}\Lambda_{t+s}\right)_{t \geq 0}$ has the same law as the process Λ and is independent of the σ-field generated by the $\mathbb{R}^d$-valued Brownian motion up to time s.*

Proof. See [28]. □

Proposition 3.3. *We keep the notations of the last two propositions. Let ϕ be a function of class C^2 on G. For all $t \geq 0$,*

$$\phi(\Lambda_t) = \phi(g) + \int_0^t \sum_{k=1}^p \partial_{E_k} f(\Lambda_s) dW_k(s) + \frac{1}{2}\int_0^t \sum_{k=1}^p \partial_{E_k}^2 f(\Lambda_s) ds. \tag{3.2}$$

Proof. See [28]. □

The second order differential operator $\mathcal{L} = \frac{1}{2}\sum_{k=1}^p \partial_{E_k}^2$ is called the generator of the diffusion Λ. Let $k \geq 1$ an integer. The k^{th} tensor product of Λ is the process $\Lambda^{\otimes k} = \left(\Lambda_t^{\otimes k}\right)_{t \geq 0}$ taking values in the k fold tensor product $\mathcal{M}_n(\mathbb{K})$. For a matrix $A \in \mathcal{M}_n(\mathbb{K})$, we let $A_i = 1 \otimes \cdots 1 \otimes \underset{i}{A} \otimes 1 \otimes \cdots \otimes 1 \in \mathcal{M}_n(\mathbb{K})^{\otimes k}$ and for $A \otimes B \in \mathcal{M}_n(\mathbb{K}) \otimes \mathcal{M}_n(\mathbb{K})$, we let $(A \otimes B)_{i,j} = 1 \otimes \cdots \otimes 1 \otimes \underset{i}{A} \otimes \cdots \otimes \underset{j}{B} \otimes \cdots \otimes 1 \in \mathcal{M}_n(\mathbb{K})^{\otimes k}$.

Proposition 3.4. *With the notation introduced so far, there exists a martingale M valued in $\mathcal{M}_n(\mathbb{K})$ such that for all time $t \geq 0$ and $k \geq 1$,*

$$d\Lambda^{\otimes k}(t) = \Lambda^{\otimes k}(t) \frac{1}{2} \sum_{l=1}^p \sum_{i=1}^k \left(E_l^2\right)_i \mathrm{d}t + \Lambda^{\otimes k}(t) \sum_{1 \leq i < j \leq k} \sum_{l=1}^p (E_l \otimes E_l)_{i,j} \mathrm{d}t + \mathrm{d}M_t. \tag{3.3}$$

For each time $t \geq p$, the bivector $\Lambda^{\otimes k}(t)\left(\sum_{l=1}^p \sum_{i=1}^k \left(E_l^2\right)_i + \frac{1}{2}\sum_{1 \leq i < j \leq k} \sum_{l=1}^p (E_l \otimes E_l)_{i,j}\right)$ is referred to as *the quadratic variation at time t* of the process $\Lambda^{\otimes k}$. In Section 3.6 we use the stochastic calculus developed in this section to build the pseudo-unitary Brownian motion. Assume now that $\mathfrak{g}$ is endowed with a real scalar product, $\langle \cdot, \cdot \rangle$. If the the vectors $E_i's$ form an orthonormal basis of the Lie algebra $\mathfrak{g}$, the bivector $\sum_{i=1}^p E_i \otimes E_i \in \mathfrak{g} \otimes \mathfrak{g}$ is named the *Casimir* and is denoted $\mathrm{C}^{\mathfrak{g}}$. For each time $t \geq 0$,

$$\mathrm{d}\Lambda^{\otimes k}(t) = \Lambda^{\otimes k}(t) \frac{1}{2} \sum_{i=1}^k m(\mathrm{C}^{\mathfrak{g}})_i \mathrm{d}t + \Lambda^{\otimes k}(t) \sum_{1 \leq i < j \leq k} \mathrm{C}^{\mathfrak{g}}_{i,j} \mathrm{d}t + \mathrm{d}M_t.$$

where $m(C^{\mathfrak{g}})$ is the image of the Casimir by the multplication map $m : \mathcal{M}_n(\mathbb{K}) \otimes \mathcal{M}_n(\mathbb{K}) \to \mathcal{M}_n(\mathbb{K})$. In the next section we review free stochastic calculus. To put it in few words free stochastic calculus is a infinite dimensional stochastic calculus and became increasingly popular since the 1980s with the work of Voiculescu, Speicher and Biane.

3.2.2. Free stochastic calculus. For an introduction to non-commutative probability and free probability theory, we draw the reader's attention to the monograph [44] in which he will will find a comprehensiv exposition of the material that is exposed in this sectio. For the constructin of the stochastic calculus with respect to a free additive Brownian motion, consult the seminal article [10] and for an introduction to free additive (semi-circular) Brownian motion, the reader is directed to the sections 1.1,1.2 and 1.3 of [36].

Let $(\mathcal{A}, \tau)$ be a faithful, tracial $W^\star$-probability space. A *free semi-circular Brownian motion* is a self adjoint stochastic process $x = (x_t)_{t\geq 0}$ in $\mathcal{A}$ such that $x(0) = 0$, $\tau\left(x(1)x(1)^\star\right) = 1$, and the additive increments of x are stationary and freely independent. For all times $0 \leq t_1 \leq t_2 < +\infty$, the random variable $x(t_2) - x(t_1)$ has the same distribution as the random variable $x(t_2 - t_1)$ and $x(t_2) - x(t_1)$ is free from the $W^\star$-algebra $W^\star\{x(t) : 0 \leq t \leq t_1\}$. For each $t \geq 0$, the distribution of $x(t)$ is a compactly supported measure. Freeness and stationarity of the increments imply that

$$\text{for all } u, v \geq 0, n \geq 1,\ \tau((x(u+v) - x(u))^n) = \frac{1}{2\pi v} \int_{-2\sqrt{v}}^{2\sqrt{t}} u^n \sqrt{4v - u^2}\mathrm{d}u.$$

Let $x = (x(t))_{t\geq 0}$ and $y = (y(t))_{t\geq 0}$ be two freely independent semi-circular Brownian motions, then the process $z = (z_t)_{t\geq 0}$ defined by $z_t = \frac{1}{\sqrt{2}}(x_t + \mathrm{i}y_t), (t \geq 0)$ is a *circular* Brownian motion. If $\left(H^N(t)\right)_{t\geq 0}$ is a Hermitian Brownian motion, then the process $\left(H^N(t)\right)_{t\geq 0}$ converges as $N \to +\infty$ in non commutative distribution to a free semi-circular Brownian motion. This means that for all polynomials f in p non-commuting variables and times $0 \leq t_1 < \ldots < t_p$.

$$\mathbb{E}\left(f\left(\mathrm{tr}\left(H^N(t_1)\cdots H^N(t_p)\right)\right)\right) \underset{N\to+\infty}{\longrightarrow} \tau(f(x(t_1), \ldots, x(t_p))), \tag{3.4}$$

see for example [3].

We continue our concise overview of free probability theory with the definition of the free stochastic integral. Let $x = (x_t)_{t\geq 0}$ be a free Brownian motion, For all time $t \geq 0$, set $\mathcal{F}_t = W^\star\{x_s,\ s \leq t\}$ and the filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$. A bi-process $A = (A_t)_{t\geq 0}$ is an $\mathcal{F}$-adapted process that takes values in the algebraic tensor product $\mathcal{A} \otimes \mathcal{A}$. Let $A = (A_t)_{t\geq 0}$ be a continuous bi-process. For an element $W = \sum_i w_i^{(1)} \otimes w_i^{(2)}$ and an element $x \in \mathcal{A}$, we use the notation $W\sharp x = \sum_i w_i^{(1)} x w_i^{(2)}$ and the adjoint $W^\star$ of W is $W^\star = \sum_i \left(w_i^{(1)}\right)^\star \otimes \left(w_i^{(2)}\right)^\star$. The norm of the process A is defined by

$$\|U\|_{L^2(\tau)} = \int_0^1 \sqrt{\tau(A_s A_s^\star)}\mathrm{d}s.$$

The *free Itô* integral of A with respect to x, denoted

$$\int_0^t A_s \sharp \mathrm{d}x_s,\ t \geq 0$$

is the limit, in $L^2(\tau)$ norm, of the Riemann sums $\sum_{i,j} a_i(t_{j-1})(x(t_j) - x(t_{j-1}))b_i(t_{j-1})$ over a partition $0 = t_0 < t_1 < \ldots < t_n = t$ as the mesh of the partition tends to 0. We have the following estimate,

$$\left\| \int_0^1 A_s \sharp \mathrm{d}x_s \right\|_{L^2(\tau)} = \|A\|_{L^2(\tau)}. \tag{3.5}$$

For all Lipschitz continuous functions f and h on $\mathbb{C}$, there exists an unique continuous $\mathcal{F}$-adapted process $(b_t)_{t\geq 0}$ with values in $\mathcal{A}$ such that

$$db(t) = (f(b(t)) \otimes 1)\sharp \mathrm{d}x(t) + h(b(t))dt \tag{3.6}$$

and $b(0) = 1$. Equation 3.6 is referred to as a *free stochastic differential equation* and the solution of such an equation is a *free itô process*.

Proposition 3.5. *Let $(\mathcal{A}, \tau)$ be a $W^\star$ probability space. Let $(x_t)_{t\geq 0}$ and $(y_t)_{t\geq 0}$ be two freely independent free semi-circular Brownian motions, adapted with respect to a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$. Let $\theta = (\theta(t))_{t\geq 0}$, $\theta_+ = (\theta_+(t))_{t\geq 0}$ and $\theta_2 = (\theta_2(t))_{t\geq 0}$ be $\mathcal{F}$-adapted processes. Then*

$$\tau\left(\int_0^t \theta_+(s)dx(s)\theta_2(s)\right) = \tau\left(\int_0^t \theta_+(s)dy_s\theta_s\right) = 0,$$
$$\int_0^t dx(s)\theta(s)dx(s) = \int_0^t dy(s)\theta(s)dy(s) = \int_0^t \tau(\theta(s))ds,$$
$$\int_0^t dx(s)\theta(s)dy(s) = \int_0^t dy(s)\theta(s)dx(s) = 0.$$

Moreover if θ_+ and θ_2 are free Itô processes, then the following Itô products rules hold:

$$d(\theta_+(t)\theta_2(t)) = d\theta_+(t)\theta_2(t) + \theta_+(t)d\theta_2(t) + d\theta_+(t)d\theta_2(t),$$
$$\theta_+(t)dx(t)\theta_2(t)dt = \theta_+dy(t)\theta_2(t)dt = 0.$$

Having explained a little about stochastic calculus and free stochastic, we introduce next the state space of the diffusions we are interested in, that is the group of pseudo-unitary matrices. The stochastic differential equations the pseudo-unitary Brownian motions are solution of are also presented in Section 3.4.

3.3. Cartan Involutions and associated diffusions

3.3.1. Cartan decomposition and invariant scalar products. For a detailed introduction on Cartan involutions, Cartan decompositions at Lie algebra and Lie group levels, classification of real semi-simple Lie algebras, the reader is invited to consult the monograph [37]. In this section we make a brief reminder on some of these notions, the main outcomes are first, the definition 3.7 of a diffusion associated with a Cartan decomposition and Proposition 3.6 on enumeration of scalar products on $\mathfrak{u}(p,q,\mathbb{K})$, $p,q \geq 1$ that are invariant with respect to the Adjoint action of a maximally compact subgroup of $\mathbb{U}(p,q,\mathbb{K})$, $p,q \geq 1$. In what follows, the complexification of a real Lie algebra $\mathfrak{g}$ is denoted $\mathfrak{g}^{\mathbb{C}}$ and the real Lie algebra obtained by restriction of scalars to the field of real numbers is denoted $\mathfrak{g}^{\mathbb{R}}$.

Let $\mathfrak{g}$ be a real semi-simple Lie algebra, G a Lie group such that $\mathrm{Lie}(G) = \mathfrak{g}$ and element $g \in G$. The differential at the identity of G of the conjugation by g map, $x \to gxg^{-1}$, is denoted $Ad_g : \mathfrak{g} \to \mathfrak{g}$ and is called the Adjoint action. Since we assumed $\mathfrak{g}$ to be semi-simple, its Killing form of $\mathfrak{g}$, denoted K,

$$\mathsf{K}(X,Y) = \mathsf{Tr}_{\mathfrak{k}}([X,[Y,\cdot]]), X, Y \in \mathfrak{g}$$

is non degenerate. A *Cartan involution* of $\mathfrak{g}$ is an involutive automorphism $\theta : \mathfrak{g} \to \mathfrak{g}$ such that the symmetric bilinear form K_θ defined by

$$\mathsf{K}_\theta(X,Y) = \mathsf{K}(\theta(X),Y),\ X,Y \in \mathfrak{g} \tag{3.7}$$

is *definite positive*. The bilinear form K_θ is symmetric in its arguments since θ is an algebra automorphism: it implies that $B(\theta(X),\theta(Y)) = B(X,Y)$, $X,Y \in \mathfrak{g}$. An *inner automorphism* of $\mathfrak{g}$ is an automorphism equal to Ad_g with $g \in G$. According to [37], Corollary 6.18 any real semi-simple Lie algebra $\mathfrak{g}_0$ has a Cartan involution and the corollary 6.19 of the same monograph tells us about the structure of the set of Cartan Involutions of $\mathfrak{g}_0$; two Cartan involutions θ_0 and θ_1 of $\mathfrak{g}$ are conjugated via an inner automorphism.

From now on, we assume that $\mathfrak{g}$ is a complex Lie algebra. A *compact real form* of $\mathfrak{g}$ is a real Lie sub-algebra $\mathfrak{u} \subset \mathfrak{g}$ such that:

- the complexification $\mathfrak{u}^{\mathbb{C}}$ is equal to $\mathfrak{g}$,
- there exists a compact Lie group which Lie algebra is $\mathfrak{u}$.

There exists a close connection between compact real forms of $\mathfrak{g}$ and its Cartan involutions. In fact, let $\mathfrak{u}$ be a compact real form and τ the conjugation with respect to $\mathfrak{u}$:

$$\tau(U_1 + iU_2) = U_1 - iU_2,\ U_1, U_2 \in \mathfrak{u}.$$

It turns out that τ is a Cartan involution and any Cartan involution of $\mathfrak{g}^{\mathbb{R}}$ is the conjugation with respect to a compact real form of $\mathfrak{g}$.

Amongst remarkable real forms of a complex Lie algebra $\mathfrak{g}$, in addition to the compact ones, those that are maximally non-compact and named *split* real forms are studied in the literature. To define what a split real form is we need to introduce the notions of roots system, restricted roots system, Cartan sub-algebra. All these notions are standard in real Lie theory and are encountered for the classification of complex semi-simple Lie algebras, real semi-simple algebras, it is out of scope for the present work to explain with details all these notions and the reader is directed, once again, to the monograph [37]. Let us, nevertheless, give some examples. A split real form of the complex Lie algebra of traceless matrices with complex entries, $\mathfrak{sl}(\mathbb{C})$, is given by the real Lie algebra of traceless matrices with real entries. This procedure of restriction of scalars to find a split real form applies for other Lie algebras of matrices. Recall that $\mathfrak{u}(p,q,\mathbb{H})^{\mathbb{C}} = \mathfrak{u}(p+q,0)^{\mathbb{C}} = \mathfrak{sp}_{p+q}(\mathbb{C})$. The Lie algebra $\mathfrak{sp}_{p+q}(\mathbb{C})$ is the Lie algebra of complex matrices with dimension $2(p+q)\times 2(p+q)$ that preserve the skew complex bilinear linear form w defined by $w(X,Y) = \sum_{i=1}^{2(p+q)} X_i Y_{p+q+i} - \sum_{i=1}^{2(p+q)} X_{p+q+i} Y_i,\ X, Y \in \mathbb{C}^{2n}$. The split real form of $\mathfrak{sp}(\mathbb{C})$ is the sub-Lie algebra of real points of $\mathfrak{sp}_n(\mathbb{C})$; the Lie algebra of real matrices with dimension $2n \times 2n$ that preserve the form w restricted to $\mathbb{R}^{2n} \subset \mathbb{C}^{2n}$. This procedure does not work with the algebras $o(p,q)^{\mathbb{C}} = \mathfrak{o}(p+q,\mathbb{C})$ since the real points of this algebra is the compact Lie algebra $\mathfrak{o}(p+q,\mathbb{R})$. Instead, the Lie algebra $\mathfrak{o}(p,p)$ and $\mathfrak{o}(p,p+1)$ are the split real forms, of respectively, the complex Lie algebra $\mathfrak{o}(2p)(\mathbb{C})$ and $\mathfrak{o}(2p+1)(\mathbb{C})$ of anti-symmetric matrices complex entries.

A Cartan involution θ of $\mathfrak{g}$ yields an eigenspace decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},\ \text{with } \ker(\theta + 1) = \mathfrak{p},\ \ker(\theta - 1) = \mathfrak{k} \tag{3.8}$$

into its $+1$ and -1 eigenspaces. In addition,

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}, [\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k},\ [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}. \tag{3.9}$$

The two spaces $\mathfrak{k}_0$ and $\mathfrak{p}_0$ are orthogonal with respect to the Killing form K and the scalar product K_θ. Since K_θ is definite positive,

$$\mathsf{K} \text{ is definite positive on } \mathfrak{p},\ \mathsf{K} \text{ is definite negative on } \mathfrak{k}. \tag{3.10}$$

A pair $(\mathfrak{k},\mathfrak{p})$ of linear sub-spaces of $\mathfrak{g}$ that satisfies (3.8), (3.9) and (3.10) is called a *Cartan pair*. Conversely, given a Cartan pair we construct a Cartan involution by using the formula 3.8. We give now two important examples. The Lie algebra $\mathfrak{u}_n$ of complex anti-Hermitian matrices is a compact real form of $\mathfrak{sl}_n(\mathbb{C})$ and the associated Cartan involution is $\theta(X) = -X^\star$. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}$ be a division algebra. The second example is of particular interest for the present work. We define an injection of the Lie algebra $\mathfrak{u}(p,\mathbb{K}) \oplus \mathfrak{u}(q,\mathbb{K})$ into the Lie algebra $\mathfrak{u}(p,q,\mathbb{K})$ by setting

$$\mathfrak{k} = \mathfrak{u}(p,\mathbb{K}) \oplus \mathfrak{u}(q,\mathbb{K}),\ \mathfrak{p} = \left\{ \begin{bmatrix} 0 & S \\ S^\star & 0, \end{bmatrix},\ S \in \mathcal{M}_{p+q}(\mathbb{K}) \right\}$$

The notion of Cartan involution was introduced, in particular, by mean of equation 3.7. In equation 3.7, K denoted the Killing form $\mathfrak{g}$. We can actually substitute to the Killing form of $\mathfrak{g}$ any Ad-invariant non degenerate bilinear form on K such that K_θ is symmetric bilinear:

$$\mathsf{K}([X,Y],Z) = \mathsf{K}(X,[Y,Z]),\ \mathsf{K}(\theta(X),Y) = \mathsf{K}(\theta(Y),X)\ X,Y,Z \in \mathfrak{g}.$$

Let $(\mathfrak{k},\mathfrak{p})$ a Cartan pair of $\mathfrak{g}$ and denote by K the compact sub-group of G such that $\mathrm{Lie}(K) = \mathfrak{k}$. At the Lie group level, this Cartan decomposition yields the following decomposition of G. Let K be the Lie subgroup having $\mathfrak{k}$ as Lie algebra. There exists an involutive automorphism Θ of G, which differential is θ, which set of fixed points is K. In addition,

1. the mapping:

$$\begin{cases} \phi: & K \times \mathfrak{p} \to G \\ & (k,p) \mapsto k\exp(p) \end{cases} \text{ is a diffeomorphism,}$$

2. The group is K a maximal subgroup of G containing the center Z_G such that K/Z_G is compact.

For the groups $\mathfrak{u}(p,q,\mathbb{K})$, the Lie group Cartan decomposition associated with the cartan pairs defined above involves the compact group $\mathbb{U}(p,\mathbb{K})\times\mathbb{U}(q,\mathbb{K})$ and

$$\exp(\mathfrak{p}) = \begin{bmatrix} 1 & S^\star \\ S & 1 \end{bmatrix}\begin{bmatrix} \mathrm{ch}(S^\star S) & 0 \\ 0 & \mathrm{ch}(SS^\star) \end{bmatrix}. \tag{3.11}$$

This Cartan decomposition yields a two parameters family of non-degenerate positive scalar products: $\{\mathsf{K}_{\theta,c,v},\ c>0,\ v>0\}$ on $\mathfrak{g}$ by setting for all $k_1,\ k_2\in\mathfrak{k}$ and $p_1,\ p_2\in\mathfrak{p}$,

$$K_{\theta,c,v}(k_1+p_1,k_2+p_2) = \frac{1}{v}K(p_1,p_2) - \frac{1}{c}K(k_1,k_2),\ c>0,\ v>0. \tag{3.12}$$

The scalar products (3.12) are invariant by the Adjoint action of G restricted to K, such scalar products are called Ad_K invariant. We may insist on the fact that the scalar products $\{\mathsf{K}_{\theta,c,pa}, c, pa>0\}$ are not invariant by the full Adjoint action of G.

They are at least two natural questions to ask. First, how does $B_{\theta,c,pa}$ depends on the Cartan involution θ ? Are all Ad_K invariant scalar products of the form (3.12) ? Let θ' be an other Cartan involution, $g\in G$ be such that $\theta' = Ad_g\circ\theta\circ Ad_{g^{-1}}$ and $\left(\mathfrak{k}_0',\mathfrak{p}_0'\right)$ be the associated Cartan pair. Then,

$$\mathsf{K}_{Ad_g\theta Ad_{g^{-1}},c,v} = \mathsf{K}_{\theta,c,pa}\left(Ad_{g^{-1}},\ Ad_{g^{-1}}\right),\ c,\ v>0. \tag{3.13}$$

The first question is answered. Now we turn our attention to the second one. Let $\tilde{\mathsf{K}}$ be an Ad_K invariant scalar product on $\mathfrak{g}$. There exists an endomorphism $m:\mathfrak{g}\to\mathfrak{g}$ such that

$$\mathsf{K}_{\theta,1,1}(m(X),m(Y)) = \tilde{\mathsf{K}}(X,Y),\ X,Y\in\mathfrak{g}.$$

Since both $\tilde{\mathsf{K}}$ and $\mathsf{K}_{\theta,1,1}$ are non degenerate and $Ad(K)$ invariant, m is an intertwiner for the Adjoint representation of K on $\mathfrak{g}_0$ in addition the two subspaces $\mathfrak{k}_0$ and $\mathfrak{p}_0$ are stable by the Adjoint representation/ The map m can thus be written as

$$m = m_{\mathfrak{g}}^{\mathfrak{g}} + m_{\mathfrak{p}}^{\mathfrak{p}} + m_{\mathfrak{g}}^{\mathfrak{p}} + (m_{\mathfrak{g}}^{\mathfrak{p}})^\star \tag{3.14}$$

with $(m_{\mathfrak{g}}^{\mathfrak{p}})^\star\in\mathsf{End}_K(\mathfrak{p},\mathfrak{k})$ the transpose with respect to $\mathsf{K}_{\theta,1,1}$ of $m_{\mathfrak{g}}^{\mathfrak{p}}\in\mathsf{End}_K(\mathfrak{k},\mathfrak{p})$ and $m_{\mathfrak{k}}^{\mathfrak{k}}\in\mathsf{End}_K(\mathfrak{k},\mathfrak{k})$, $m_{\mathfrak{p}}^{\mathfrak{p}}\in\mathsf{End}_K(\mathfrak{p},\mathfrak{p})$. We focus now on the families of real Lie algebra we are interested in.

Proposition 3.6. *Let $p,q\geq 1$ two integers. The scalar products on $\mathfrak{u}(p,q,\mathbb{K})$ invariant by the Adjoint action of the maximal compact subgroup $\mathbb{U}(p,\mathbb{K})\times\mathbb{U}(q,\mathbb{K})\subset\mathbb{U}(p,q,\mathbb{K})$ are all of the form:*

$$B_{\mathfrak{u}(p\oplus q,\mathbb{K})} + \frac{1}{v}K_{|\mathfrak{p}\times\mathfrak{p}}, v>0,$$

where $B_{\mathfrak{u}(p\oplus q,\mathbb{K})}$ is a $\mathbb{U}(p,\mathbb{K})\times\mathbb{U}(q,\mathbb{K})$ invariant scalar product on $\mathfrak{u}(p,\mathbb{K})\oplus\mathfrak{u}(q,\mathbb{K})$:

$$B_{\mathfrak{u}(p\oplus q,\mathbb{K})} = -\mathsf{Tr}\Big((A^+(d_1^+)+c^+h_1^+)(A^+(d_2^+)+c^+h_2^+)\Big) - \mathsf{Tr}\Big((A^-(d_1^-)+c^-h_1^-)(A^-(d_2^-)+c^-h_2^-)\Big),\ \textit{if } p\neq q,$$

with $h_i^+\in\mathfrak{su}(p,\mathbb{K})$, d_i^+ a diagonal matrix with purely imaginary entries in $\mathfrak{u}(p,\mathbb{K})\subset\mathfrak{u}(p,\mathbb{K})\otimes\mathfrak{u}(q,\mathbb{K})$, $i\in\{1,2\}$ and A^+ is a real linear endomorphism on the real span of the imaginary elements of the standard real basis of $\mathbb{K}$ ($\mathbb{R}\mathsf{i}$ if $\mathbb{K}=\mathbb{C}$ and $\mathbb{R}[\mathsf{i},\mathsf{j},\mathsf{k}]$ if $\mathbb{K}=\mathbb{H}$) (and accordingly for h_i^-,d_i^- and A^-).

Proof. Let $p,q\geq 1$ two integers satisfying $p+q\geq 3$. In the following, to light the notation we drop references to $\mathbb{K}$. It is a known fact that $\mathfrak{su}(n)$, $n\geq 1$ is simple, thus the Adjoint action of $\mathbb{SU}(p)\times\mathbb{SU}(q)$ on its Lie algebra is irreductible. The Lie algebra $\mathfrak{u}(n)$ is however not simple: the Adjoint representation of $\mathbb{U}(n)$ splits as :

$$\mathfrak{u}(n) = \mathfrak{su}(n)\oplus\bigoplus_{\gamma\in\mathbb{I}(\mathbb{K})}\gamma I_n$$

where $\mathbb{I}(\mathbb{K})$ denotes the imaginary elements of a real basis of $\mathbb{K}$. Set $D_{\mathbb{K}} = \bigoplus_{\gamma\in\mathbb{I}(\mathbb{K})}\gamma I_n$. An intertwinner m of the Adjoint representation of $\mathbb{U}(n)$ looks like:

$$m = cp_{\mathfrak{su}(n)} + A\circ p_{D_{\mathbb{K}}}$$

where $A \in \mathsf{End}(D_{\mathbb{K}})$ and $p_{\mathfrak{su}(n)}, p_{D_{\mathbb{K}}}$ are the projectors on the isotypic summands. To show that $\mathfrak{p}$ is an irreducible $Ad\,(\mathbb{U}(p)\times\mathbb{U}(q))$ module, we define first

$$S: \begin{array}{ccc} V_p\otimes V_q & \to & \mathfrak{p} \\ e_i\otimes f_j & \mapsto & e^j\otimes f_i + f^i\otimes e_j \end{array}.$$

with $V_p = \mathbb{R}^p$, $V_q = \mathbb{R}^q$ and $e = (e_i)_{i\le p}$ and $f = (f_j)_{j\le q}$ the standard basis of V_p and V_q. Let $(o_1, o_2) \in \mathbb{U}(p)\times U(q)$, then

$$S((o_1\otimes o_2)(e_i\otimes e_j)) = e^j\circ o_1^{-1}\otimes o_2(e_i) + e^i\circ o_2^{-1}\otimes o_1(e_j) = Ad_{o_1\times o_2}(S(e_i\otimes e_j)). \tag{3.15}$$

Hence $\mathfrak{p}$ and $V_p\otimes V_q$ are isomorphic as $\mathrm{Ad}(\mathbb{U}(p)\times\mathbb{U}(q))$ modules and thus $\mathfrak{p}$ is irreducible. To prove that $\mathfrak{p}$ and $\mathfrak{u}(p)\oplus\mathfrak{u}(q)$ are not isomorphic, we compute dimensions. Let $\beta_{\mathbb{K}}$ be the real dimension of $\mathbb{K}$ on $\mathbb{R}$. Then,

$$\begin{aligned}\mathsf{dim}\,(\mathfrak{u}(p,\mathbb{K})\oplus\mathfrak{u}(q,\mathbb{K})) &= \frac{\beta_{\mathbb{K}}}{2}p^2 + \left(\frac{\beta_{\mathbb{K}}}{2}-1\right)p + \frac{\beta_{\mathbb{K}}}{2}q^2 + \left(\frac{\beta_{\mathbb{K}}}{2}-1\right)q \\ &= pq\beta_{\mathbb{K}} + \frac{\beta_{\mathbb{K}}}{2}(p-q)^2 + \left(\frac{\beta_{\mathbb{K}}}{2}-1\right)p + \left(\frac{\beta_{\mathbb{K}}}{2}-1\right)q \\ &= \mathsf{dim}(\mathfrak{p}) + \frac{\beta_{\mathbb{K}}}{2}(p-q)^2 + \left(\frac{\beta_{\mathbb{K}}}{2}-1\right)p + \left(\frac{\beta_{\mathbb{K}}}{2}-1\right)q.\end{aligned}$$

This last equation proves that $\mathsf{dim}\,(\mathfrak{u}(p,\mathbb{K})\oplus\mathfrak{u}(q,\mathbb{K})) > \mathsf{dim}(p)$. In the case where $p\neq q$, owing to Schur Lemma, the intertwiner m introduced before the statement of the proposition, more specifically its decomposition, reduces to

$$m = c_+\mathrm{id}_{\mathfrak{su}(p)}\oplus A^+\oplus c_+\mathrm{id}_{\mathfrak{su}(p)}\oplus A^-\oplus c_-\mathrm{id}_{\mathfrak{su}(q)}\oplus v\mathrm{id}_{\mathfrak{p}}.$$

for some constants c, v and $A^\pm \in \mathsf{End}(D^\pm_{\mathbb{K}})$.

This achieves the proof. ☐

Thus, in this work, the invariant scalar products that are used to construct a family of Brownian diffusions is a small subset of the set all scalar products invariant by the conjugation action of the maximal compact subgroup being genuinely larger in the symplectic case. $\mathbb{U}(p,\mathbb{K})\times\mathbb{U}(q,\mathbb{K})$, which

3.3.2. Diffusions associated with a Cartan decomposition. Cartan decompositions of a real Lie algebra yield diffusions on the corresponding Lie groups.

Definition 3.7. Let $\mathfrak{g}$ be a real Lie algebra and a connected real Lie group G such that $\mathrm{Lie}(G)=\mathfrak{g}$. Let θ be a Cartan involution on $\mathfrak{g}$. Let v, c be two strictly positive real numbers and let $\left(E^\theta_{c,v}(k)\right)_{k\le\mathsf{dim}(\mathfrak{g})}$ be an orthonormal basis of $\mathfrak{g}$ with respect to the bilinear form $B_{\theta,c,v}$ defined by equation (3.12). Let W be a $\mathsf{dim}(\mathfrak{g})$ be a Brownian motion. The diffusion $\Lambda_{\theta,c,v} = (\Lambda_{\theta,c,v}(t))_{t\ge0}$ is the solution of the stochastic differential equation

$$\begin{aligned} \mathrm{d}\Lambda_{\theta,c,v}(t) &= \Lambda_{\theta,c,v}(t)\sum_{k=1}^{\mathsf{dim}(\mathfrak{g})}\mathrm{dW}^k(t)E^\theta_{c,v}(k) + \Lambda_{\theta,c,v}(t)\left(\frac{1}{2}\sum_{k=1}^{\mathsf{dim}(\mathfrak{g})}E^\theta_{c,p}(k)^2\right)\mathrm{d}t, \\ \Lambda_{\theta,c,v}(0) &= 1\in G. \end{aligned}\tag{3.16}$$

In the article [35], the author studies asymptotic in large dimension of the diffusion $\Lambda_{\theta,c,v}$ on $GL_N(\mathbb{C})$ with

$$\mathfrak{g} = \mathcal{ML}_N(\mathbb{C}),\ \theta(X) = -X^\star,\ B(X,Y) = N\mathsf{Tr}(XY),\ X,Y\in\mathcal{M}_N(\mathbb{C}) \tag{3.17}$$

and proved the convergence of the diffusion $\Lambda_{\theta,c,v}(s)$ for each values of the parameter c and p_a to a free stochastic process.

In this notes, we are interested in the asymptotic as the dimension tends to infinity of the diffusion $\Lambda_{\theta,v,c}$ on the group $\mathbb{U}(p,q,\mathbb{K})$ with $\mathbb{K}=\mathbb{R},\mathbb{H},\mathbb{C}$ and the involution θ given by $\theta(X) = -I_{p,q}X^\star I_{p,q}$, where $X^\star$ denotes the hermitian conjugates of X and $I_{p,q}$ is the diagonal matrix with p one in the upper left corner and q minus one in the lower right corner. The bilinear form K we choose is invariant by the maximal compact subgroup $\mathbb{U}(p)\times\mathbb{U}(q)\subset\mathbb{U}(p,q,\mathbb{K})$ of all block diagonal matrices in $\mathbb{U}(p,q,\mathbb{K})$ and proportional to the Killing form on the lie algebras $u(p) \in \mathfrak{u}(p,q,\mathbb{K})$, $\mathfrak{u}(q) \subset \mathfrak{u}(p,q,\mathbb{K})$.

3.4. Pseudo-antihermitian and pseudo-unitary Brownian diffusions

In this section, we introduce the pseudo-unitary groups as groups of matrices of endomorphisms preserving a non-degenerate bilinear form. The lie algebras of these groups are algebras of pseudo-antihermitian matrices. With the help of the previous section, in which Brownian diffusions associated with a Cartan decompositions are defined, we built Brownian diffusions on the pseudo-unitary groups.

3.4.1. Pseudo-antihermitien and pseudo-unitary matrices. For an introduction to the theory of quadratic forms and much more, we invite the reader to consult the monograph [20]. In the sequel, $\mathbb{K}$ denotes one of the three division algebras $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$. In this section we define the Lie group of pseudo-unitary matrices $\mathbb{U}(p,q,\mathbb{K})$ with $p,q \geq 1$ two integers. A matrix is pseudo-unitary if it is the matrix in a pseudo-orthonormal basis of an isometry with respect to a non necessarily definite metric. These groups, unlike the group of unitary matrices, are not compact. We begin with the real case.

Let (V,Q) be a real quadratic space, V is a real vector space of dimension $n \geq 1$ and Q is a symmetric bilinear form on V:

$$Q(x,y) = Q(y,x),\ Q(\lambda x + y, z) = \lambda Q(x,z) + Q(y,z), x,y,z \in V, \lambda \in \mathbb{R}.$$

The group $O(Q)$ is the subgroup of the group of linear isomorphisms of V that preserve the bilinear form Q. In symbols,

$$\forall u \in \mathsf{End}(V),\ u \in O(Q) \Leftrightarrow \forall x,y \in V,\ Q(u(x),u(y)) = Q(x,y).$$

We impose the *non-degeneracy* of the quadratic form Q which means that no non-trivial vectors are orthogonal to the full space,

$$\forall x \in V,\ (\forall y \in V,\ Q(x,y) = 0) \Leftrightarrow x = 0.$$

A basic result of the theory of quadratic form is the existence of a basis $(e_i, 1 \leq i \leq n)$ of V made of pseudo-orthogonal vectors, that is, a basis $\mathsf{e} = (e_1,\ldots,e_n)$ of V such that there exists integers $a_1,\ldots,a_n$ in $\{-1,1\}$ such that for all $x_1,\ldots,x_n \in \mathbb{R}$ the equality

$$Q\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_i a_i x_i^2$$

holds. The matrix $[Q]_{e_1,\ldots,e_n} = \left(Q(e_i,e_j)\right)_{1\leq i,j\leq n}$ of the quadratic form in the basis e is diagonal with the diagonal entries equal to either 1 or -1.

We let p be the number of 1 and q the number of -1 on the diagonal. The pair (p,q), which does not depends on the pseudo-orthogonal basis, is named the *signature* of the quadratic form Q. The numbers p and q have a geometric meaning. A *positive* vector for Q is a vector v satisfying $Q(v,v) > 0$, whereas a negative vector w is a vector w of V satisfying $Q(w,w) < 0$. An *isotropic* vector is a vector w with null Q-norm: $Q(v,v) = 0$. The quadratic form is said to be *definite* if all vectors of V are either postive are negative for Q. The integer p is the dimension of the maximal subspace on which Q is definite positive, whereas q is the dimension of the subspace of V on which Q is definite negative. The matrix U of an isometry $u \in O(Q)$ in a pseudo-orthogonal basis satisfies the relation

$$U^t I_{p,q} U = I_{p,q}, \tag{3.18}$$

with $I_{p,q}$ the diagonal matrix with the first p upper left entries equal to 1 and the other entries on the diagonal equal to -1. We denote by $\mathbb{U}(p,q,\mathbb{R})$ the group of matrices satifying relation (3.18). Let $A \in \mathcal{M}_n(\mathbb{R})$ and write A in block form as

$$A = \begin{bmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{bmatrix} \text{ with } A_{++}, A_{--} \in \mathcal{M}_{p,p}(\mathbb{R}),\ A_{+-} \in \mathcal{M}_{p,q}(\mathbb{R}), A_{-+} \in \mathcal{M}_{q,p}(\mathbb{R}),$$

We define the involution $\star' : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ by setting,

$$A^{\star'} = \begin{bmatrix} A^\star_{++} & A^\star_{+-} \\ A^\star_{-+} & A^\star_{--} \end{bmatrix} = \begin{bmatrix} A^t_{++} & -A^t_{-+} \\ -A^t_{+-} & A^t_{--} \end{bmatrix}, \ A \in \mathcal{M}_n(\mathbb{R}) \tag{3.19}$$

Using the involution $\star'$, the equation (3.19) is restated as $A^\star = I_{p,q}A^tI_{p,q}$. The Lie algebra $\mathfrak{u}(p,q,\mathbb{R})$ of $\mathbb{U}(p,q,\mathbb{R})$ is

$$\mathfrak{u}(p,q,\mathbb{R}) = \{A \in \mathcal{M}_n(\mathbb{R}) : A^tI_{p,q} + I_{p,q}A = 0\}. \tag{3.20}$$

Wo now turn our attention to definition of the complex and quaternionic counterpart of $\mathbb{U}(p,q,\mathbb{R})$. The major difference, compared with the real case, is the existence of non trivial positive unital involutions on $\mathbb{C}$ and $\mathbb{H}$. On the two divisions algebras $\mathbb{C}$ and $\mathbb{H}$ we denote by the same symbol $^-$ the usual conjugations:

$$\begin{array}{lll} \text{for all } x,y \in \mathbb{R}, & \text{if } z = x+iy, & \bar{z} = x - iy, \\ \text{for all } x,y,z,t \in \mathbb{R}, & \text{if } q = t + x\mathrm{i} + y\mathrm{j} + z\mathrm{k}, & \bar{q} = t - (x\mathrm{i} + y\mathrm{j} + z\mathrm{k}). \end{array}$$

For $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$ and p,q two integers, we define the pseudo-unitary group $\mathbb{U}(p,q,\mathbb{K})$ with coefficients in $\mathbb{K}$ as the subgroup comprising all matrices of $GL_n(\mathbb{K}) = GL_n(\mathbb{R}) \otimes \mathbb{K}$ satisfying

$$I_{p,q}\left(^- \otimes {}^t\right)(U)I_{p,q}U = I_{p,q}.$$

For $U \in \mathcal{M}_n(\mathbb{K})$, if $U^{\star'} = I_{p,q}\left({}^t \otimes \bar{z}\right)I_{p,q}$, then U belongs to $U(p,q,\mathbb{K})$ if and only if $U^{\star'}U = UU^{\star'} = I_n$. The Lie algebra $\mathfrak{u}(p,q,\mathbb{K})$ is the algebra of anti-symmetric matrices for the involution $\star$, in symbols

$$\mathfrak{u}(p,q,\mathbb{K}) = \{A \in GL_n(\mathbb{K}) : A^{\star'} + A = 0\}.$$

It is true that, as for the real case, the matrices in $\mathbb{U}(p,q,\mathbb{C})$ and in $\mathbb{U}(p,q,\mathbb{H})$ represent endomorphisms that preserves a $\mathbb{R}$-bilinear form. For the complex case, this bilinear form is

$$Q(x,y) = \bar{x}_1y_1 + \ldots + \bar{x}_py_p - \left(\bar{x}_{p+1}y_{p+1} + \cdots + \bar{x}_ny_n\right),$$

which is an *Hermitian* form. For the quaternionic case, the standard form reads

$$Q(x,y) = \bar{x}_1y_1 + \cdots + \bar{x}_py_p - \left(\bar{x}_{p+1}y_{p+1} + \cdots + \bar{x}_ny_n\right), x,y \in \mathbb{H}^n.$$

The dimensions of these Lie groups are summarised in the table 3.21.

(3.21)

$\mathbb{U}(p,q,\mathbb{R})$	$\mathbb{U}(p,q,\mathbb{C})$	$\mathbb{U}(p,q,\mathbb{H})$
$\frac{1}{2}n(n-1)$	n^2	$n(2(n+1)+1)$

We make a point on the terminlogy we use to desginate the groups $\mathbb{U}(p,q,\mathbb{K})$. We insist on the fact that these groups are matrix groups, in the real case, an isometry of a quadratic space (V,Q) with Q a metric having signature (p,q) has a matrix in a pseudo-orthogonal basis in the group $\mathbb{U}(p,q,\mathbb{K})$, however if the basis is not pseudo-orthogonal it is not true anymore. However, an other set of basis for V as remarkable properties. In virtue of Witt's decomposition Theorem, the quadratic space (V,Q) splits as a sum of two quadratic spaces V_a, Q_a, on which the form is definite and V_h, Q_h on which the form is *hyperbolic*, or *split* meaning that its signature is (p',p'). On such hyperbolic space a pseudo-orthonormal basis has the same number of negative and positive vectors. If v and w' are, respectively, a negative and a positive vector, the matrix of the quadratic form Q in the basis $(x = \frac{1}{2}(v+w), y = \frac{1}{2}(v-w))$ is:

$$[Q]_{x,y} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

If we pair negative and positive vectors of a pseudo-orthonormal basis, we obtain an other normal form for Q:

$$[Q] \sim \begin{bmatrix} I_{p-q} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \text{ if } p \geq q.$$

If $q \leq p$, the upper left $1^{'}s$ are replaced by the same number of -1. Such a basis will be called a Witt basis. The change of coordinate matrix from a pseudo-orthogonal basis to a Witt basis is not a pseudo-orthogonal matrix. Hence, the group of matrices comprising all matrices of isometries is isomorphic to $\mathbb{U}(p,q,\mathbb{R})$ by an inner automorphism, which do not preserves the two parameter family of scalar products which is built from the Killing form as explained in the last section. Also already mentioned, this two parameters family of scalars products are not even invariant by the full group $\mathbb{U}(p,q,\mathbb{K})$. When its comes to defining objects that depends to on this quadratic structure on $\mathfrak{u}(p,q)$, such as the diffusions defined in the last section, we have to be aware that there an huge amngst of choices that need to be done.

The following table show Lie algebra's types of complexifications of the real Lie algebras $\mathfrak{u}(p,q)$ with $p,q \geq 1$ depending on the values of the integers p and q (it does only depends on the sum n).

$\mathfrak{u}(p,q,\mathbb{C})^{\mathbb{C}}$	$\mathfrak{u}(p,q,\mathbb{R})^{\mathbb{C}}$, $n=2l$	$\mathfrak{u}(p,q,\mathbb{R})^{\mathbb{C}}$, $n=2l+1$	$\mathfrak{u}(p,q,\mathbb{H})^{\mathbb{C}}$
A_n	D_l	C_l	B_n

To draw the last table, assuming that the classification of the complex Lie algebras $\mathfrak{u}(n,0,\mathbb{K})$, $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ *or* $\mathbb{H}$ is known, we only have to prove that type of $\mathfrak{u}(p,q,\mathbb{K})^{\mathbb{C}}$ does only depends on n. Let $\mathbb{K}$ equal to one of the three divisions algebras $\mathbb{R},\mathbb{C}$ or $\mathbb{H}$ and set $V = \mathbb{K}^n$, V is a real vector space. Denote by B the standard real bilinear form on V (one of the three defined before) and denote by $B^{\mathbb{C}}$ the complex bi-linear form induced by B on $V^{\mathbb{C}} = V \oplus iV$. The complex of the Lie algebra $\mathfrak{u}(p,q,\mathbb{K})^{\mathbb{C}}$ is isomorphic to the complex Lie algebra of complex and $\mathbb{K}$ linear endomorphisms of $V^{\mathbb{C}}$ such that

$$B^{\mathbb{C}}(u(x),y) = -B^{\mathbb{C}}(x,u(y)),\ x,y \in V \text{ and } u(V) \subset V. \tag{3.22}$$

Note that the class of algebras isomorphic to the latter does not depends on B since there is only one equivalence class of complex bilinear forms on $V^{\mathbb{C}}$. This proves that the type of $\mathfrak{u}(p,q,\mathbb{K})^{\mathbb{C}}$ does only depends on the dimension n.

We introduce notations that will be needed in the next sections. Let $p,q \geq 1$ be integers and set $n = p+q$. We denote by $(E_{kl})_{1\leq k,l\leq n}$ the standard basis of $\mathcal{M}_n(\mathbb{R})$. The standard basis $(E_{kl})_{1\leq k,l\leq n}$ is divided into four subsets: $\left\{E^{+,+}_{i,j},\ i \leq p, j \leq p\right\}$, $\left\{E^{-,-}_{i,j},\ i \leq q, j \leq q\right\}$, $\left\{E^{+,-}_{i,j},\ i \leq p, j \leq q\right\}$, $\left\{E^{-,+}_{i,j},\ i \leq q, j \leq p\right\}$ of matrices in $\mathcal{M}_n(\mathbb{R})$ with

$$\begin{aligned} &E^{+,+}_{ij} = E_{i,j}, && 1 \leq i,j \leq p, && E^{+,-}_{ij} = E_{i,p+j}, && 1 \leq i \leq p,\ 1 \leq j \leq q \\ &E^{-,-}_{ij} = E_{p+i,p+j}, && 1 \leq i,j \leq q, && E^{-,+}_{ij} = E_{p+i,j}, && 1 \leq i \leq q,\ 1 \leq j \leq p \end{aligned}$$

Similarly, the standard basis of symmetric (resp anti-symmetric) matrices of dimension $n \times n$ is cut into three subsets $S^{+,+}, S^{-,-}$ and $S^{+,-}$ (resp. $A^{+,+}, A^{-,-}$ and $A^{+,-}$) of matrices,

$$\begin{aligned} &S^{+,-}_{i,j} = E^{+,-}_{ij} + E^{-,+}_{ji}, && A^{+,-}_{i,j} = E^{+,-}_{ij} - E^{-,+}_{ji}, && i \leq p,\ j \leq q, \\ &A^{+,+}_{i,j} = E^{+,+}_{ij} - E^{+,+}_{ji}, && S^{+,+}_{i,j} = E^{+,+}_{ij} + E^{+,+}_{ji}, && i \leq p,\ j \leq p, \\ &A^{-,-}_{i,j} = E^{-,-}_{ij} - E^{+,+}_{ji}, && S^{-,-}_{i,j} = E^{-,-}_{ij} + E^{+,+}_{ji}, && i \leq q,\ j \leq q. \end{aligned}$$

We define the following linear subspaces of $\mathcal{M}_n(\mathbb{R})$,

$$\begin{aligned} &s_{+,-} = \mathbb{R}[S^{+,-}], && a_{+} = \mathbb{R}[A^{+,+}], && a_{-} = \mathbb{R}[A^{-,-}] \\ &a_{+,-} = \mathbb{R}[A^{+,-}], && s_{+} = \mathbb{R}[S^{+,+}], && s_{-} = \mathbb{R}[S^{-,-}]. \end{aligned} \tag{3.23}$$

Among the spaces in equation (3.23), only $\mathfrak{a}_{+,-}$, $\mathfrak{a}_+$ and $\mathfrak{a}_-$ are Lie sub-algebras of $(\mathcal{M}_n(\mathbb{R}), [\,\cdot\,,\,\cdot\,])$ with $[\,\cdot\,,\,\cdot\,]$ the standard commutator on $\mathcal{M}_n(\mathbb{R})$. Let us draw examples with $p = 1, q = 3$,

$$A(+,-)_{1,3} = \left[\begin{array}{c|ccc} 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right],\ S(+,-)_{1,2} = \left[\begin{array}{c|ccc} 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

3.4.1.1. *Gaussian ensemble of pseudo-Hermitian matrices.* Having defined the group of pseudo-unitary matrices(with entries in $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$) we proceed with defining a Brownian diffusion on such groups. First, we expose the fundamental object for constructing such diffusions. We introduce here real, complex and quaternionic Gaussian ensembles of pseudo-Hermitian matrices.

Let p and q two integers and set $n = p+q$. Let $\mathbb{K}$ be one of the three divisions algebras $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. The random matrix's laws we are interested in have random entries distributed as gaussian laws. The law of such matrix is characterized by the data of its *variance profile and a covariance profile*. Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and $(X_n(i,j))_{1\leq i,j\leq n} \in L^{\infty}(\Omega, \mathcal{A}, \mathbb{P}) \otimes \mathcal{M}_{n,n}(\mathbb{K})$ a random matrix such that each entrie $X_n(i,j)$, $i \leq n$, $j \leq n$ is independent from $X_n(k,l)$ with $k \neq j, l \neq i$. Define the variance profile $(g_n(i,j))_{i,j\leq n}$ and the covariance profile $(\tau_n(i,j))_{i,j\leq n}$ of X_n by

$$\mathbb{E}\left[X_n(i,j)\overline{X_n(i,j)}\right] = \frac{1}{n}g_{ij}^2,\ \mathbb{E}\left[X_n(i,j)X_n(j,i)\right] = \frac{1}{n}g_{ji}g_{ji}\tau_{ij}.$$

We assume that almost surely X_n is pseudo-Hermitian with respect to the standard split pseudo-Hermitian metric of signature (p,q) on $\mathbb{K}^{2p}$. As a consequence, the covariance profile τ of X_n is given by:

$$\tau_{i,j} = \tau_{p+i,p+j} = -n\mathbb{E}[X_N(i,j)] \text{ and } \tau_{p+i,j} = \tau_{i,p+j} = n\mathbb{E}[X_n(i,j)], \text{ for all } 1 \leq i \leq q, 1 \leq j \leq p.$$

We assume further that the variance profile g has the block form:

$$g = \begin{bmatrix} cJ_{p,p} & vJ_{p,q} \\ vJ_{p,q} & cJ_{q,q} \end{bmatrix} \in \mathcal{M}_{n,n}(\mathbb{R}),\ J_{p,q} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \in \mathcal{M}_{p,q}(\mathbb{R}).$$

We name X_n a *Gaussian pseudo-Hermitian matrix with compact standard deviation c and non-compact standard deviation v*. In Section 3.7, we investigate the convergence of the moments of X_{2p} as $p \to +\infty$.

3.4.2. Pseudo-antihermitian and pseudo-unitary Brownian diffusions. In the sequel, we treat separately the construction of the real, complex and quaternionic pseudo-unitary and pseudo-unitary diffusions. We make the convention that tensor products, otherwise specified are take over the field of real numbers. In the complex case, however, we will see that it is more natural to take tensor product over the complex field.

3.4.2.1. *Brownian motions on the pseudo-orthogonal groups and on their lie algebras.* Let $p, q \geq 1$ be two integers. The Lie algebra $\mathfrak{o}(p,q) = \mathbb{U}(p,q,\mathbb{R})$ splits as, with $[s_{+,-}, s_{+,-}] \subset a_+ \otimes a_-$:

$$\mathfrak{o}(p,q) = s_{+,-} \oplus a_+ \oplus a_-. \tag{3.24}$$

Let $v > 0$ and $c > 0$ be two strictly positive real numbers, we endow $\mathfrak{o}(p,q)$ with the scalar product:

$$B^{\mathbb{R}}_{v,c}(k_1^+ + k_1^- + p_1, k_2^+ + k_2^- + p_2) = -\frac{p}{2c}\mathrm{Tr}\left(k_1^+, k_+^2\right) - \frac{q}{2c}\mathrm{Tr}\left(k_1^-, k_-^2\right) + (\frac{n}{4v})\mathrm{Tr}(p_1, p_2). \tag{3.25}$$

where $k_i^+ \in a_+$, $k_i^- \in a_-$, $p_i \in s_{+,-}$, $i \in \{1,2\}$. The fact that $B^{\mathbb{R}}_{v,c}$ is a scalar product is implied by Lemma 3.9, which is stated and proved at the end of the section, in addition it is trivial to prove invariance of $B^{\mathbb{R}}_{v,c}$ with respect to the conjugation action of $O(p) \times O(q) \subset O(p,q)$. An orthonormal basis $\beta^{p,q}_{v,c}$ for $B^{\mathbb{R}}_{c,v}$ is given by the formula (3.26). Recall that the three sets of matrices in the following equation have been defined in Section 3.4.

$$\beta_{v,c}^{p,q} = \left(\frac{\sqrt{v}}{\sqrt{\frac{n}{2}}}\right) S^{-,+} \bigcup \left(\frac{\sqrt{c}}{\sqrt{p}}\right) A^{+,+} \bigcup \left(\frac{\sqrt{c}}{\sqrt{q}}\right) A^{-,-}. \tag{3.26}$$

The pseudo-antisymmetric Brownian motion $\mathsf{W}_{\mathbb{R}}^{p,q}$ is defined from a $\dim_{\mathbb{R}}(\mathfrak{o}(p,q))$ dimensional Brownian motion $(W_t^{\gamma})_{\gamma\in\beta_{v,c}^{p,q}}$ on a probability space $(\Omega,\mathcal{F},\mathbb{P})$,

$$\mathsf{W}_{\mathbb{R}}^{p,q} = \sum_{\gamma\in\beta_{v,c}^{p,q}} W^{\gamma}\gamma. \tag{3.27}$$

The law of $\mathsf{W}_{\mathbb{R}}^{p,q}$ is only invariant by conjugation by any matrix in $O(p)\times O(q) \subset O(p,q)$. This prevent us to define in this way a pseudo-antisymmetric Brownian motion on the Lie algebra of pseudo-antihermitian endomorphisms, in fact we must pick first a pseudo-orthogonal basis but the set of all such frames is not an orbit under the action of $O(p)\times O(q)$, there are in fact (at least intuitively) $O(p,q)$ of them. By using the Cartan decomposition of $O(p,q)$ described in Section 3.3, we see that $O(p,q)/O(p)\times O(q)$ is diffeomorphic to the set in equation 3.11, Section 3.3. For most of the points $S \in O(p,q)/O(p)\times O(q)$, the scalar product $B_{v,c}^{\mathbb{R}}$ is not invariant by Ad_S. Hence, defining a Brownian pseudo-antihermitien diffusion on saying it has the law of (3.27) does not make any sense, we must first pick a preferred basis. That why we talk about a diffusion on a Lie algebra of pseudo-antihermitien matrices. As we mentionned in Section 3.4, there is an other set of preferred basis associated with a pseudo-metric, which is the set of Witt's basis. Remark that a Witt basis is not related to a pseudo-orthogonal basis by mean of a pseudo-orthogonal endomorphism. The set of matrices in a Witt basis of pseudo-antihermitien endomorphism is a Lie algebra on which a Brownian diffusion can also be defined using the procedure we used to define $\mathsf{W}_{\mathbb{R}}^{p,q}$.

It would be desirable to construct a Brownian diffusion having a law that has the biggest invariance group as possible. This can be done by averaging the scalar product $B_{v,c}^{\mathbb{R}}$. We can demand invariance by the permutation group $\mathfrak{S}_n$ acting on $\mathbb{R}^n$ by permuting the vectors of the canonical basis. In fact,

$$B_{v,c}^{1,\mathbb{R}}(\cdot,\cdot) = \sum_{\sigma\in\mathfrak{S}_n} B_{v,c}^{\mathbb{R}}(Ad_{\sigma}(\cdot), Ad_{\sigma}(\cdot)) \tag{3.28}$$

has the required invariance. However, averaging over the group $O(p,q)$ the scalar product $B_{v,c}$ is much more difficult. First, there is no preferred Haar measure on $O(p,q)$. Secondly the group is not compact so that we need to take of possible divergence. We want to construct a pseudo-orthogonal Brownian motion by solving the stochastic differential equation (3.16). Remark that the Cartan involution θ appearing in (3.16) is given by, $\theta : X \mapsto -X^{\star'}$.

Lemma 3.8. *With the notation introduced so far,*

$$\sum_{\gamma\in\beta_{v,c}^{p,q}} \gamma^2 = -c\left(\frac{p-1}{p}\right) I_p - c\left(\frac{q-1}{q}\right) I_{p,n} + 2v\left(\frac{q}{n} I_p + \frac{p}{n} I_{p,n}\right)$$

Proof. Let us prove that $\sum_{\substack{1\le a\le q\\1\le b\le p}} \left(S_{ab}^{-,+}\right)^2 = qI_p + pI_{p,n}$ by direct computations. First

$$\left(S_{ab}^{-,+}\right)^2 = \left(E_{ab}^{-,+} + E_{ab}^{+,-}\right)^2 = E_{aa}^{--} + E_{bb}^{++}.$$

Summing over the indices $1 \le a \le q$ and $1 \le b \le p$ gives the result. The reader is invited to check by similar computations that $\sum_{1\le a\le p, 1\le b\le p} \left(A_{ab}^{+,+}\right)^2 = (p-1)I_p$. □

The (left) *pseudo-orthogonal Brownian motion* (with speed parameters c, v) is the solution $\Lambda_{\mathbb{R}}^{p,q}$ of the following stochastic differential equation (derived from equation (3.16)):

$$\text{(3.29)} \qquad \begin{aligned} &\mathrm{d}\Lambda_{\mathbb{R}}^{p,q}(t) = \Lambda_{\mathbb{R}}^{p,q}(t)\mathrm{d}\mathrm{W}_{\mathbb{R}}^{p,q} + \frac{1}{2}\Lambda_{\mathbb{R}}^{p,q}(t)\left(-c\left(\frac{p-1}{p}\right)I_p - c\left(\frac{q-1}{q}\right)I_{p,n} + 2v\left(\frac{q}{n}I_p + \frac{p}{n}I_{p,n}\right)\right)\mathrm{d}t,\\ &\mathrm{W}_{\mathbb{R}}^{p,q}(0) = I_n \end{aligned}$$

Owing to the results exposed in Setion 3.2, for each time $t \geq 0$, the matrix $\Lambda_{\mathbb{R}}^{p,q}$ is in $O(p,q)$, and a right invariant version of the pseudo-antisymmetric Brownian motion is $\left[\Lambda_{\mathbb{R}}^{p,q}\right]^{\star}$. For any time $t \geq 0$, the matrix is not normal, in particul there is no simple relations between its spectrum and its moments. We end this section with a formula for the quadratic variation of the process $\Lambda_{\mathbb{R}}^{p,q}$, (The Casimir of the scalar product $B_{v,c}^{\mathbb{R}}$). With the notations introduced so far,

$$\text{(3.30)} \qquad \mathrm{C}_{p,q}^{\mathbb{R}} = \frac{v}{p}(\mathrm{T}^{-,+} + \mathrm{T}^{+,-} + \mathrm{W}^{+,-} + \mathrm{W}^{-,+}) + \frac{c}{p}(\mathrm{W}^{+,+} - \mathrm{T}^{+,+}) + \frac{c}{q}(\mathrm{W}^{-,-} - \mathrm{T}^{-,-}).$$

We defined the bilinear form $B_{v,c}^{\mathbb{R}}$ and claimed that it is a scalar product. We should now relate him to the Killing form of $O(p,q)$ and its Cartan decomposition that is defined in Section 3.3.

Lemma 3.9. *The Killing form* K *of the Lie algebra* $\mathbb{U}(p,q,\mathbb{R})$ *is non-degenerate. Furthermore,* K *is definite positive on the real vector space* $s_{+,-}$ *and definite negative on the two real vector spaces* a_+ *and* a_-*. The decomposition* (3.24) *is orthogonal and the following formula holds*

$$\mathsf{K}(S_{gh}^{-,+}, S_{ij}^{-,+}) = 2(n-2),\ \mathsf{K}(A_{hj}^{+,+}, A_{ab}^{+,+}) = -2(p-2),\ \mathsf{K}(A_{gi}^{-,-}, A_{cd}^{-,-}) = -2(q-2).$$

for all integers $1 \leq g,i,c,d \leq q$ *and* $1 \leq h,j,a,b \leq p$.

Proof. We leave to the reader the verification, by a direct calculation, that K is definite negative on $a_+ \oplus a_-$. One can also notice that $a_+ \oplus a_-$ is the Lie algebra of the compact group $\mathbb{U}(p,0,\mathbb{R}) \times \mathbb{U}(q,0,\mathbb{R}) \subset \mathbb{U}(p,q,\mathbb{R})$ and is thus compact. We prove that K is positive definite on the space $s_{+,-}$.

$$\begin{aligned}
ad(E_{ij}^{-,+})\left(E_{kl}^{-,+}\right) &= [E_{ij}^{-,+}, E_{kl}^{-,+}] + [E_{ij}^{-,+}, E_{lk}^{+,-}] + [E_{ji}^{+,-}, E_{kl}^{-,+}] + [E_{ji}^{+,-}, E_{lk}^{+,-}]\\
&= \delta_{jl}E_{i,k}^{-,-} - \delta_{ki}E_{l,j}^{+,+} + \delta_{ik}E_{j,l}^{+,+} - \delta_{lj}E_{ki}^{-,-}\\
&= \delta_{jl}E_{ik}^{-,-} + \delta_{ik}E_{jl}^{+,+}\\
ad(E_{gh}^{-,+})\left(E_{lj}^{+,+}\right) &= [E_{gh}^{-,+}, E_{lj}^{+,+}] - [E_{gh}^{-,+}, E_{jl}^{+,+}] + [E_{hg}^{+,-}, E_{lj}^{+,+}] - [E_{hg}^{+,-}, E_{jl}^{+,+}]\\
&= \delta_{hl}E_{gj}^{-,+} - \delta_{hj}E_{gl}^{-,+} - \delta_{hj}E_{lg}^{+,-} + \delta_{hl}E_{jg}^{+,-}\\
&= \delta_{hl}\left(E_{gj}^{-,+}\right) - \delta_{hj}\left(E_{gl}^{-,+}\right)\\
ad(E_{gh}^{-,+})\left(E_{ik}^{-,-}\right) &= -\delta_{kg}E_{ih}^{-,+} + \delta_{ig}E_{kh}^{-,+} + \delta_{gi}E_{hk}^{+,-} - \delta_{gk}E_{hi}^{-,+}\\
&= \delta_{ig}\left(E_{kh}^{-,+}\right) - \delta_{kg}(E_{ih}^{-,+})
\end{aligned}$$

From the last equations, it follows that

$$\begin{aligned}
\sum_{kl}\langle ad(E_{gh}^{-,+}) \circ ad(E_{ij}^{-,+})(E_{kl}^{-,+}), E_{kl}^{-,+}\rangle &= \sum_{kl}\langle \delta_{ik}\delta_{hl}E_{gj}^{-,+} - \delta_{ik}\delta_{hj}E_{gl}^{-,+}\\
&\qquad + \delta_{jl}\delta_{ig}E_{kh}^{-,+} - \delta_{jl}\delta_{kg}E_{ih}^{-,+}, E_{kl}^{-,+}\rangle\\
&= p + q - 2.
\end{aligned}$$

By similar computations, we convince ourselves that $\mathsf{K}(E_{gh}^{-,+}, E_{ij}^{-,+}) = q + p - 2 + q - 1 + p - 1 = 2(n-2)$. ∎

3.4.2.2. *Brownian motion on the pseudo-unitary groups.* Let $p,q \geq 1$ be two integers. We will be more brief in that section, all the remarks we made regarding the law of the pseudo-orthogonal Brownian motions applies for the pseudo-unitary Brownian motions that is defined in this section. The Lie algebra $\mathfrak{u}(p,q)=\mathfrak{u}(p,q,\mathbb{C})$ splits as follows:

$$\mathfrak{u}(p,q)=\mathrm{i}a_{+,-}+s_{+,-}+\mathrm{i}s_{+}+a_{+}+\mathrm{i}s_{-}+a_{-}.$$

Set $\mathfrak{k}=\mathrm{i}s_{+}+a_{+}+\mathrm{i}s_{-}+a_{-}$ and $\mathfrak{p}=\mathrm{i}a_{+,-}+s_{+,-}$. The Lie algebra $\mathfrak{k}$ is the compact Lie algebra of block diagonal anti-Hermitian matrices, $\mathfrak{k}=\mathfrak{u}(p)\oplus\mathfrak{u}(q)$. The two parameters family of scalar products $\left(B^{\mathbb{C}}_{c,v}\right)_{c>0,v>0}$ (that is associated with the Cartan involution $X\mapsto -X^{\star'}$ and the Kiling form on $\mathbb{U}(p,q)$) is defined by the equation:

$$B^{\mathbb{C}}_{v,c}(p_1+k_1^{+}+k_1^{-},p_2+k_2^{+}+k_2^{-})=-\frac{p}{c}\mathrm{Tr}(k_1^{+}k_2^{+})-\frac{q}{c}\mathrm{Tr}(k_1^{-}k_2^{-})+\frac{n}{2v}\mathrm{Tr}(p_1p_2)$$

with $p_1,p_2\in\mathfrak{p}$, $k_1^{1},\ k_2^{1}\in\mathfrak{u}(p)$, $k_1^{-1},k_2^{-1}\in\mathfrak{u}(q)$ and $v>0,c>0$. We fix for the remaining of this section two parameters $v>0$ and $c>0$. An orthogonal basis for the scalar product $B^{\mathbb{C}}_{v,c}$ is given by:

$$\beta^{p,q}_{v,c}=\left\{\frac{\sqrt{c}}{\sqrt{2p}}\mathrm{i}S^{++},\ \frac{\sqrt{c}}{\sqrt{2q}}\mathrm{i}S^{-,-},\ \frac{\sqrt{c}}{\sqrt{2p}}A^{+,+},\ \frac{\sqrt{c}}{\sqrt{2q}}A^{-,-},\ \frac{\sqrt{v}}{\sqrt{n}}S^{+,-},\mathrm{i}\frac{\sqrt{v}}{\sqrt{n}}A^{+,-}\right\}.$$

The *pseudo-antihermitien Brownian motion* $\mathsf{W}^{p,q}_{\mathbb{C}}$ is defined by picking a $\dim_{\mathbb{C}}(\mathfrak{u}(p,q))$ dimensional Brownian motion $(W^{\gamma}_t)_{\gamma\in\beta^{p,q}_{v,c}}$ on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and setting

$$\mathsf{W}^{p,q}_{\mathbb{C}}=\sum_{\gamma\in\beta^{p,q}_{v,c}}W^{\gamma}\gamma.$$

In order to write the stochastic differential equation the diffusion $\Lambda^{p,q}_{\mathbb{R}}$ is solution of, we compute first the sum of the squares of the matrices in the basis $\beta^{p,q}_{v,c}$.

Lemma 3.10. $\sum_{\gamma\in\beta^{p,q}_{v,c}}\gamma^2=\frac{2}{n}\left(pI_{p,n}+qI_p\right)-cI_n$

Proof. We prove that $\sum_{S\in S^{+,-}}S^2-\sum_{A\in A^{+,-}}A^2=\left(pI_{p,n}+qI_p\right)$. It is a simple computation:

$$\begin{aligned}\sum_{S\in S^{+,-}}S^2-\sum_{A\in A^{+,-}}A^2&=\sum_{a,b}\left(E^{+,-}_{ab}+E^{-+}_{ba}\right)^2-\sum_{a,b}\left(E^{+-}_{ab}-E^{-+}_{ba}\right)^2\\&=2\left(qI_p+pI_{p,n}\right).\end{aligned}$$

□

The complex pseudo-unitary diffusion $\left(\Lambda^{p,q}_{\mathbb{C}}(t)\right)_{t\geq 0}$ on $\mathbb{U}(p,q,\mathbb{C})\subset\mathcal{M}_n(\mathbb{C})$ is the solution of the stochastic differential equation

$$\mathrm{d}\Lambda^{p,q}_{\mathbb{C}}(t)=\Lambda^{p,q}_{\mathbb{C}}(t)\mathrm{d}\mathsf{W}^{p,q}_{\mathbb{C}}(t)+\Lambda^{p,q}_{\mathbb{C}}(t)\left(\frac{v}{n}\left(pI_{p,n}+qI_p\right)-\frac{c}{2}I_n\right)\mathrm{dt},\ t>0. \tag{3.31}$$

with the initial condition $\Lambda^{p,q}_{\mathbb{C}}(0)=I_n$.

Set $\mathsf{C}^{00}_{p,q}=\mathsf{C}^{\mathbb{C}}_{p,q}$. Since derivatives of polynomial functions of the process $\Lambda^{p,q}_{\mathbb{C}}$ and its complex conjugate are needed in the sequel, we introduce in addition to the quadratic variation $\mathsf{C}^{00}_{p,q}$ the bi-vector $\mathsf{C}^{01}_{p,q}$ in $\mathfrak{u}(p,q)\otimes\mathfrak{u}(p,q)$ satisfying

$$\mathrm{d}\left(\Lambda^{p,q}_{\mathbb{C}}(t)\otimes\overline{\Lambda^{p,q}_{\mathbb{C}}(t)}\right)=\mathrm{d}\Lambda^{p,q}_{\mathbb{C}}(t)\otimes\overline{\Lambda^{p,q}_{\mathbb{C}}(t)}+\mathrm{d}\overline{\Lambda^{p,q}_{\mathbb{C}}(t)}\otimes\Lambda^{p,q}_{\mathbb{C}}(t)+\left(\Lambda^{p,q}_{\mathbb{C}}(t)\otimes\overline{\Lambda^{p,q}_{\mathbb{C}}(t)}\right)\mathsf{C}^{01}_{p,q}. \tag{3.32}$$

Closed formulae for the Casimir $\mathsf{C}^{00}_{p,q}$ and $\mathsf{C}^{01}_{p,q}$ are easily obtained:

$$\begin{aligned}\mathsf{C}^{00}_{p,q}=\frac{v}{n}(1\otimes 1+\mathrm{i}\otimes\mathrm{i})(\mathsf{E}^{+-}+\mathsf{E}^{-+})+\frac{v}{n}(1\otimes 1-\mathrm{i}\otimes\mathrm{i})(\mathsf{T}^{+,-}+\mathsf{T}^{+,-})\\+\frac{c}{2p}(1\otimes 1+\mathrm{i}\otimes\mathrm{i})\mathsf{E}^{++}+\frac{c}{2q}(1\otimes 1+\mathrm{i}\otimes\mathrm{i})\mathsf{E}^{--}-\frac{c}{2q}(1\otimes 1-\mathrm{i}\otimes\mathrm{i})\mathsf{T}^{-,-}\\-\frac{c}{2p}(1\otimes 1-\mathrm{i}\otimes\mathrm{i})\mathsf{T}^{++},\end{aligned}$$

$$\begin{aligned}\mathsf{C}^{01}_{p,q} = \frac{v}{n}(1\otimes 1 - \mathrm{i}\otimes \mathrm{i})(\mathsf{E}^{+-}+\mathsf{E}^{-+}) + \frac{v}{n}(1\otimes 1 + \mathrm{i}\otimes \mathrm{i})(\mathsf{T}^{+-}+\mathsf{T}^{-+})\\ + \frac{c}{2p}(1\otimes 1 - \mathrm{i}\otimes \mathrm{i})\mathsf{E}^{+,+} + \frac{c}{2q}(1\otimes 1 - \mathrm{i}\otimes \mathrm{i})\mathsf{E}^{-,-} - \frac{c}{2q}(1\otimes 1 + \mathrm{i}\otimes \mathrm{i})\mathsf{T}^{--}\\ - \frac{c}{2p}(1\otimes 1 + \mathrm{i}\otimes \mathrm{i})\mathsf{T}^{++}.\end{aligned}$$

The same formula holds for $C^{10}_{p,q}$ defined by the relation $\mathrm{d}\overline{\Lambda}^{\mathbb{C}}_{p,q}(t)\otimes \mathrm{d}\Lambda^{\mathbb{C}}_{p,q} = \left(\overline{\Lambda}^{\mathbb{C}}_{p,q}\otimes \Lambda^{\mathbb{C}}_{p,q}(t)\right)C^{10}_{p,q}$. All tensor products we wrote so far are taken over the field of real numbers. If instead the Casimir are seen as bi-vectors in tensor products over the field of complex numbers, that is in $\mathcal{M}_n(\mathbb{C})\otimes_{\mathbb{C}}\mathcal{M}_n(\mathbb{C})$, the formulae for $\mathsf{C}^{00}_{p,q}$ and $\mathsf{C}^{01}_{p,q}$ simplify dramatically. With the notations introduced so far,

$$\mathsf{C}^{00}_{p,q} = \frac{2v}{n}(\mathsf{T}^{+-}+\mathsf{T}^{-+}) - \frac{c}{p}\mathsf{T}^{++} - \frac{c}{q}\mathsf{T}^{++},\ \mathsf{C}^{01}_{p,q} = \frac{2v}{n}(\mathsf{E}^{+-}+\mathsf{E}^{-+}) + \frac{c}{p}\mathsf{E}^{++} + \frac{c}{q}\mathsf{E}^{--}. \tag{3.33}$$

We end this section with the computation of the Killing form on $U(p,q)$.

Lemma 3.11. *The Killing form* K *of* $\mathfrak{u}(p,q)$ *is definite positive on* $\mathfrak{s}_{+,-}\oplus \mathrm{i}\mathfrak{a}_{+,-}$. *In addition, the spaces* $\mathfrak{p}$ *and* $\mathfrak{k}$ *are mutually orthogonal and*

$$\mathsf{K}(S^{+,-}_{ab},S^{+,-}_{ij}) = (3(n)-2)\delta_{ai}\delta_{bj},\ \mathsf{K}(\mathrm{i}A^{+,-}_{ij},\mathrm{i}A^{+,-}_{ab}) = 2\delta_{ia}\delta_{jb}(n+2).$$

for all integers $1\le a\le p, 1\le i\le p, 1\le b\le q, 1\le j\le q$.

Proof. We show first that $s_{+,-}$ and $a_{+,-}$ are orthogonal subspaces.

$$\begin{aligned}&[s_{+,-},[ia_{+,-},is_{+}]]\subset a_1+a_-,\ [s_{+,-},[ia_{+,-},is_{-}]]\subset a_++a_-,\ [s_{+,-},[ia_{+,-},s_{+,-}]]\subset a_{+,-},\\ &[s_{+,-},[ia_{+,-},ia_{+,-}]]\subset s_{+,-},\ [s_{+,-},[ia_{+,-},a_{+}]]\subset s_++s_-,\ [s_{+,-},[ia_{+,-},a_{-}]]\subset s_++s_-\end{aligned}$$

Thus $\mathsf{K}(ia_{+,-},s_{+,-})=0$.

We show first that K is definite positive on $s_{+,-}$. The trace over $s_{+,-}$ of $\mathrm{ad}(S^{+,-}_{ij})\circ \mathrm{ad}(S^{+,-}_{kl})$ is equal to $(n-2)\delta_{ik}\delta_{jl}$ and the trace over a_+ and a_- is equal to $p-1+q-1$. We compute the trace of $\mathrm{ad}(S^{+,-}_{ij})\circ\mathrm{ad}(S^{+,-}_{kl})$ over $a_{+,-}$ and over s_+,s_-. First, over $a_{+,-}$, one has:

$$[E^{-,+}_{ij}+E^{-,-}_{ji},E^{+,-}_{kl}-E^{-,+}_{lk}] = \delta_{ik}\left(E^{-,-}_{jl}+E^{-,-}_{lj}\right)-\delta_{lj}\left(E^{++}_{ik}+E^{++}_{ki}\right), \tag{3.34}$$

from which it follows that:

$$\begin{aligned}[E^{+,-}_{ab}+E^{-,+}_{ba},[E^{-,+}_{ij}+E^{-,-}_{ji},E^{+,-}_{kl}-E^{-,+}_{lk}]] = \delta_{ik}\delta_{bj}(E^{+,-}_{al}-E^{-,+}_{la})+\delta_{ik}\delta_{bl}(E^{-,+}_{aj}-E^{-,+}_{ja})\\ -\delta_{lj}\delta_{ka}(E^{-,+}_{bi}-E^{-,+}_{ib})-\delta_{lj}\delta_{ia}(E_{bk}-E_{kb}).\end{aligned}$$

Summing over the indices $k\le q, l\le p$ gives:

(3.35)

$$\begin{aligned}&\frac{1}{2}\sum_{kl}\langle[E^{+,-}_{ab}+E^{-,+}_{ba},[E^{-,+}_{ij}+E^{-,-}_{ji},E^{+,-}_{kl}-E^{-,+}_{lk}]],E^{-,+}_{kl}-E^{-,+}_{lk}\rangle\\ &\qquad=\sum_{kl}\delta_{ik}\delta_{bj}\delta_{ak}\delta_{ll}+\delta_{ik}\delta_{bl}\delta_{ak}\delta_{lj}+\delta_{lj}\delta_{ka}\delta_{lb}\delta_{ki}+\delta_{lj}\delta_{ia}\delta_{bl}\delta_{kk} = (p+q-2)\delta_{ia}\delta_{bj}.\end{aligned}$$

Let $1\le a,j\le p$ and $1\le b,i\le q$ integers. We compute the trace of $[E^{+,-}_{ab}+E^{-,+}_{ba},[E^{-,+}_{ij}+E^{-,+}_{ji},\cdot]]$ over s_+ and s_-. First, for $1\le k,l\le p$, we have:

$$[E^{-,+}_{ij}+E^{-,+}_{ji},E^{+,+}_{kl}+E^{+,+}_{lk}] = \delta_{li}(E^{-,+}_{jk}-E^{+,-}_{kj})+\delta_{ki}(E^{-,+}_{jl}-E^{+,-}_{lj}).$$

Further,

$$[E^{+,-}_{ab}+E^{-,+}_{ba},[E^{-,+}_{ij}+E^{-,+}_{ji},E^{+,+}_{kl}+E^{+,+}_{lk}]] = \delta_{li}\delta_{bj}(E^{++}_{ak}+E^{++}_{ka})+\delta_{ki}\delta_{bj}(E^{++}_{al}+E^{++}_{la})+\cdots$$

where $\cdots$ stand for an element in $s_{-,-}$. Thus, summing over $1\le k,l\le q$ gives:

$$\frac{1}{2}\sum_{kl}\langle[E^{+,-}_{ab}+E^{-,+}_{ba},[E^{-,+}_{ij}+E^{-,+}_{ji},E^{-,-}_{kl}+E^{-,-}_{lk}]],E^{-,-}_{kl}+E^{-,-}_{lk}\rangle = \delta_{a,i}\delta_{b,j}(1+p).$$

Similar computations lead to the following formula for the trace over s_-:

$$\frac{1}{2}\sum_{kl}\langle[E^{+,-}_{ab}+E^{-,+}_{ba},[E^{-,+}_{ij}+E^{-,+}_{ji},E^{-,-}_{kl}+E^{-,-}_{lk}]],E^{-,-}_{kl}+E^{-,-}_{lk}\rangle=\delta_{a,i}\delta_{b,j}(1+q).$$

Finally, summing over the partial traces we have computed so far gives: $\mathsf{K}(S^{+,-}_{ab},S^{+,-}_{ij})=(3(n)-2)\,\delta_{ai}\delta_{bj}$. We show now that $\left(\mathrm{i}A^{+,-}_{ij}\right)_{ij}$ is an orthogonal basis of $\mathrm{i}a_{-,+}$ for the Killing form. As previously, we divide the computations of the trace of $\mathrm{ad}(A^{+,-}_{ij})\circ\mathrm{ad}(A^{+,-}_{ab})$ into contributions of partial traces over $s_{+,-},a_{+,-},a_+,a_-$ and $s_{+,-}$. We begin with the trace over $s_{+,-}$. Let $1\le i,k\le p$, $1\le j,l\le q$ integers. We have

$$[E^{+,-}_{ij}-E^{-,+}_{ji},E^{+,-}_{kl}+E^{-,+}_{lk}]=\delta_{lj}(E^{++}_{ki}+E^{++}_{ik})-\delta_{ik}(E^{++}_{jl}+E^{++}_{lj}).$$

Hence, for $1\le a\le p, 1\le b\le q$, one has:

$$\begin{aligned}&[E^{+,-}_{ab}-E^{-,+}_{ba},[E^{+,-}_{ij}-E^{-,+}_{ji},E^{+,-}_{kl}+E^{-,+}_{lk}]=\\&=-\delta_{lj}\delta_{ak}(E^{+,-}_{ib}+E^{-,+}_{b,i})-\delta_{ia}\delta_{lj}(E^{+,-}_{kb}+E^{-,+}_{bk})-\delta_{ik}\delta_{bj}(E^{+,-}_{al}+E^{-,+}_{l,a})-\delta_{ik}\delta_{lb}(E^{+,-}_{aj}+E^{-,+}_{ja}).\end{aligned}$$

It follows that $\sum_{1\le k\le p,1\le l\le q}\langle[E^{+,-}_{ab}-E^{-,+}_{ba},[E^{+,-}_{ij}-E^{-,+}_{ji},E^{+,-}_{kl}+E^{-,+}_{lk}],E^{+,-}_{kl}+E^{-,+}_{lk}\rangle=-(n+2)$. For the contribution of the trace over $a_{+,-}$, we have first

$$[E^{+,-}_{ij}-E^{+,-}_{ji},E^{+,-}_{kl}-E^{+,-}_{lk}]=\delta_{lj}(E^{++}_{ki}-E^{++}_{ik})+\delta_{ik}(E^{-,-}_{lj}-E^{-,-}_{jl}).$$

Hence,

$$\begin{aligned}&[E^{-,+}_{ab}-E^{-,+}_{ba},[E^{+,-}_{ij}-E^{+,-}_{ji},E^{+,-}_{kl}-E^{+,-}_{lk}]]=\\&\delta_{lj}\delta_{ak}(E^{-,+}_{ib}-E^{-,+}_{bi})+\delta_{lj}\delta_{ai}(E^{-,+}_{bk}-E^{+,-}_{kb})+\delta_{ik}\delta_{bl}(E^{+,-}_{aj}-E^{-,+}_{ja})-\delta_{ik}\delta_{bj}(E^{+,-}_{al}-E^{-,+}_{la}).\end{aligned}$$

It implies that

$$\sum_{kl}\langle[E^{-,+}_{ab}-E^{-,+}_{ba},[E^{+,-}_{ij}-E^{+,-}_{ji},E^{+,-}_{kl}-E^{+,-}_{lk}]],E^{+,-}_{kl}-E^{+,-}_{lk}\rangle=-(2-n)\delta_{ai}\delta_{bj}.$$

For the contribution of the trace over a_+, we have first

$$[E^{+,-}_{ij}-E^{+,-}_{ji},E^{+,+}_{kl}-E^{+,+}_{lk}]=\delta_{ik}(E^{+,-}_{lj}-E^{-,+}_{jl})-\delta_{il}(E^{+,-}_{kj}-E^{-,+}_{jk})$$

Hence,

$$[E^{+,-}_{ab}-E^{-,+}_{ba},[E^{+,-}_{ij}-E^{+,-}_{ji},E^{++}_{kl}-E^{++}_{lk}]]=\delta_{ik}\delta_{jb}(E^{++}_{la}-E^{++}_{al})-\delta_{il}\delta_{jb}(E^{++}_{ka}-E^{++}_{ak})+\cdots$$

We sum over integers $1\le k\le p$, $1\le l\le q$ the last equation to find:

$$\frac{1}{2}\sum_{kl}\langle[E^{++}_{ab}-E^{++}_{ba},[E^{+-}_{ij}-E^{+-}_{ji},E^{++}_{kl}-E^{++}_{lk}]],E^{++}_{kl}-E^{++}_{lk}]\rangle=(1-p)\delta_{ia}\delta_{jb}.$$

The contribution of the trace over a_- is given by the same formula. It remains to compute the partial trace over $\mathrm{i}s_-$ and $\mathrm{i}s_+$. For the contribution of the trace over $\mathrm{i}s_+$, it holds that:

$$[E^{++}_{ij}-E^{-+}_{ji},E^{++}_{kl}+E^{++}_{lk}]=-\delta_{ik}(E^{+-}_{lj}+E^{-+}_{jl})-\delta_{li}(E^{+-}_{kj}+E^{-+}_{jk}).$$

It follows that,

$$[E^{+-}_{ab}-E^{-+}_{ba},[E^{++}_{ij}-E^{-+}_{ji},E^{++}_{kl}+E^{++}_{lk}]]=-\delta_{ik}\delta_{jb}(E^{++}_{la}+E^{++}_{al})-\delta_{li}\delta_{jb}(E^{++}_{ka}+E^{++}_{ak})+\cdots$$

Thus the contribution is equal to $-(1+p)$. Summing up all the contributions we get the following formula

$$\mathsf{K}(\mathrm{i}A^{+,-}_{ij},\mathrm{i}A^{+,-}_{ab})=2\delta_{ia}\delta_{jb}(n+2).$$

$\square$

3.4.2.3. *Brownian motion on the pseudo-symplectic groups.* In this section we define the quaternionic counterpart of the processes introduced in the last two sections. We proceed as we did for the real and complex cases. Let $p,q \geq 1$ be integers. The Lie algebra of pseudo-symplectic matrices $\mathfrak{sp}(p,q) = \mathfrak{u}(p,q,\mathbb{H})$ splits as:

$$\mathfrak{sp}(p,q) = s_{+,-} + \mathsf{i}a_{+,-} + \mathsf{j}a_{+,-} + \mathsf{k}a_{+,-} + \mathsf{i}s_{+} + \mathsf{j}s_{+} + \mathsf{k}s_{+} + a_{+} + \mathsf{i}s_{-} + \mathsf{j}s_{-} + \mathsf{k}s_{-} + a_{-}. \tag{3.36}$$

Put $\mathfrak{k} = a_+ \oplus \mathsf{i}s_+ \oplus \mathsf{j}s_+ \oplus \mathsf{k}s_+ \oplus a_- \oplus \mathsf{i}s_- \oplus \mathsf{j}s_- \oplus \mathsf{k}s_-$ and $\mathfrak{p} = s_{+,-} + \mathsf{i}a_{+,-} + \mathsf{j}a_{+,-} + \mathsf{k}a_{+,-}$. The Lie algebra $\mathfrak{k}$ is a compact Lie algebra of block diagonal antiHermitian matrices. The two parameters family of scalar products $B^{\mathbb{H}}_{c,v}$, $c > 0$, $v > 0$ is defined by:

$$\begin{aligned} B^{\mathbb{H}}_{c,v}&(k_1^+ + k_+^- + p_1,\, k_2^+ + k_2^- + p_2) \\ &= -\frac{2p}{c}\mathcal{R}e(\mathsf{Tr}(k_1^+k_2^+)) - \frac{2q}{c}\mathcal{R}e(\mathsf{Tr}(k_1^-k_2^-)) + \frac{n}{v}\mathcal{R}e((\mathsf{Tr}(p_1p_2))) \end{aligned} \tag{3.37}$$

with $k_1^+, k_2^+ \in \mathfrak{sp}(p), k_1^-, k_2^- \in \mathfrak{sp}(q)$, $p_1, p_2 \in \mathfrak{p}$. An orthonormal frame $\beta^{p,q}_{v,c}$ for the quadratic form $B^{\mathbb{H}}_{c,v}$ is

$$\beta^{p,q}_{v,c} = \left\{ \frac{\sqrt{c}}{2\sqrt{p}}\gamma S^{+,+},\ \frac{\sqrt{c}}{2\sqrt{p}}A^{+,+},\ \frac{\sqrt{c}}{2\sqrt{q}}\gamma S^{-,-},\ \frac{\sqrt{c}}{2\sqrt{q}}A^{-,-},\ \frac{\sqrt{v}}{\sqrt{2n}}\gamma A^{+,-},\ \frac{\sqrt{v}}{\sqrt{2n}}S^{+,-},\ \gamma \in \{\mathsf{i},\mathsf{j},\mathsf{k}\} \right\}.$$

The pseudo-symplectic Brownian motion $\mathsf{W}^{p,q}_{\mathbb{H}}$ is defined from a $\mathsf{dim}_{\mathbb{H}}(\mathfrak{sp}(p,q))$ dimensional Brownian motion $(W^{\gamma}_t)_{\gamma \in \beta^{p,q}_{v,c}}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and by setting

$$\mathsf{W}^{p,q}_{\mathbb{H}} = \sum_{\gamma \in \beta^{p,q}_{v,c}} W^{\gamma}\gamma.$$

The quaternionic pseudo-unitary diffusion denoted $\Lambda^{p,q}_{\mathbb{H}}$ is solution of the stochastic differential equation(derived from equation (3.16)):

$$\begin{aligned} \mathsf{d}\Lambda^{p,q}_{\mathbb{H}}(t) &= \Lambda^{p,q}_{\mathbb{H}}(t) \sum_{\beta \in \beta^{p,q}_{v,c}} \beta \mathsf{d}W^{\beta}(t) \\ &+ \frac{1}{2}\Lambda^{p,q}_{\mathbb{H}}(t)\left(-\frac{c(p+3)}{p}I_p - \frac{c(q+3)}{q}I_{p,n} + 2v\left(\frac{p}{n}I_{p,n} + \frac{q}{n}I_p\right)\right)\mathsf{d}t. \end{aligned} \tag{3.38}$$

with initial condition $\Lambda^{p,q}_{\mathbb{H}}(0) = I_n$. With the notations introduced so far,

$$\begin{aligned} \mathsf{C}^{\mathbb{H}}_{p,q} = \frac{c}{4p}\Big(&(1\otimes 1 + \mathsf{i}\otimes\mathsf{i} + \mathsf{j}\otimes\mathsf{j} + \mathsf{k}\otimes\mathsf{k})\,\mathsf{E}^{++} - (1\otimes 1 - \mathsf{i}\otimes\mathsf{i} - \mathsf{j}\otimes\mathsf{j} - \mathsf{k}\otimes\mathsf{k})\,\mathsf{T}^{++}\Big) \\ + \frac{c}{4q}\Big(&(1\otimes 1 + \mathsf{i}\otimes\mathsf{i} + \mathsf{j}\otimes\mathsf{j} + \mathsf{k}\otimes\mathsf{k})\,\mathsf{E}^{--} - (1\otimes 1 - \mathsf{i}\otimes\mathsf{i} - \mathsf{j}\otimes\mathsf{j} - \mathsf{k}\otimes\mathsf{k})\,\mathsf{T}^{--}\Big) \\ + \frac{v}{2n}\Big(&(1\otimes 1 + \mathsf{i}\otimes\mathsf{i} + \mathsf{j}\otimes\mathsf{j} + \mathsf{k}\otimes\mathsf{k})(\mathsf{E}^{+-} + \mathsf{E}^{-+}) \\ &+ (1\otimes 1 - \mathsf{i}\otimes\mathsf{i} - \mathsf{j}\otimes\mathsf{j} - \mathsf{k}\otimes\mathsf{k})(\mathsf{T}^{-+} + \mathsf{T}^{+-})\Big) \end{aligned} \tag{3.39}$$

By introducing $\mathsf{Re}^{\mathbb{H}} = \sum_{\gamma \in I(\mathbb{H})} \gamma \otimes \gamma$ and $\mathsf{Im}^{\mathbb{H}} = \sum_{\gamma \in I(\mathbb{H})} \gamma \otimes \gamma^{-1}$, we get the more readable formula:

$$\mathsf{C}^{\mathbb{H}}_{p,q} = \frac{c}{4p}\mathsf{Re}^{\mathbb{H}}\mathsf{E}^{++} - \mathsf{Im}^{\mathbb{H}}\mathsf{T}^{++} + \frac{c}{4p}\mathsf{Re}^{\mathbb{H}}\mathsf{E}^{--} - \mathsf{Im}^{\mathbb{H}}\mathsf{T}^{--} + \frac{v}{2n}\left(\mathsf{Re}^{\mathbb{H}}(\mathsf{E}^{+-} + \mathsf{E}^{-+}) + \mathsf{Im}^{\mathbb{H}}(\mathsf{T}^{-+} + \mathsf{T}^{+-})\right). \tag{3.40}$$

Lemma 3.12. *The Killing form* K *of* $\mathfrak{sp}(p,q)$ *is definite positive on* $\mathfrak{p} = s_{+,-} + \mathsf{i}a_{+,-} + \mathsf{j}a_{+,-} + \mathsf{k}a_{+,-}$ *and definite negative on* $\mathfrak{u}(p) \oplus \mathfrak{u}(q) = \mathsf{i}s_+ + \mathsf{j}s_+ + \mathsf{k}s_+ + a_+ + \mathsf{i}s_- + \mathsf{j}s_- + \mathsf{k}s_- + a_-$*. The direct sum decomposition 3.36 is orthogonal for the Killing form* K*, in addition, an orthogonal frame for* K *is given by*

$$\{\gamma A^{+,-}, \gamma S^+, \gamma S^-, S^{+,-}, A^+, A^-\}.$$

Proof. To prove that the direct sum decomposition 3.36 is orthogonal, we use the computations made in the last section for the complex case, in particular in 3.4.2.2. We obtain that

$s_{+,-}$ is orthogonal to each of the three subspaces $\mathrm{i}a_{+,-}$, $\mathrm{j}a_{+,-}$ and $\mathrm{k}a_{+,-}$. Because the bracket on $\mathcal{M}_n(\mathbb{R})\otimes\mathbb{H}$ is $\mathbb{H}$-linear, one has

$$\mathrm{Tr}\Big(\mathrm{ad}(\gamma_+ A_{ij}^{+,-})\circ \mathrm{ad}(\gamma_2 A_{ab}^{+,-})\Big) = \mathrm{Tr}\Big(\gamma_1\gamma_2 \mathrm{ad}(A_{ij}^{+,-})\circ \mathrm{ad}(A_{ab}^{+,-})\Big).$$

Let $\gamma_1,\gamma_2 \in \{\mathrm{i},\mathrm{j},\mathrm{k}\}$ and $1\leq a,i\leq p$, $1\leq b,j\leq q$ four integers. Since $[a_{+,-},a_{+,-}]\subset a_{+,+}+a_{-,+}$ and $[a_{+,-},a_{+}+a_{-}]\subset a_{+,-}$, we have $\gamma_1\gamma_2\mathrm{ad}(A_{ab}^{+,-})\circ\mathrm{ad}(A_{ij}^{+,-})(\gamma_3 a_{+,-})\subset \gamma_3 a_{+,-}$, if and only if $\gamma_1\gamma_2=1$ which implies $\gamma_1=\gamma_2$. Moreover, $\gamma_1\gamma_2\mathrm{ad}(A_{ij}^{1,-1})\circ\mathrm{ad}(A_{ab}^{1,-1})(s_{+,-})\subset s_{+,-}$ if and only if $\gamma_1\gamma_2\in\mathbb{R}$ which implies $\gamma_1=\gamma_2$. Eventually,

$$\mathrm{Tr}(\gamma_1\gamma_2\mathrm{ad}(A_{ab}^{+,-}\circ A_{ij}^{+,-})\neq 0 \Leftrightarrow \gamma_1=\gamma_2.$$

We leave to the reader the verification that the family of matrices in Lemma 3.12 is orthogonal for the Killing form K of $\mathfrak{sp}(p,q)$ by making use of the computations we made for the complex case. We have in particular

$$\mathsf{K}(\gamma A_{ij}^{-,+},\ \gamma A_{ab}^{-,+}) = \delta_{ai}\delta_{bj}2(n-2),\ \mathsf{K}(S_{ij}^{+,-},\ S_{ab}^{+,-}) = 7n-6.$$

□

To end this section, we rewrite the equations that define the pseudo-unitary Brownian motions in a more concise way. First, introduce:

$$\beta^{\mathbb{K}} = \dim_{\mathbb{R}}(\mathbb{K})\beta^{\mathbb{K}}_{p+q} = -1 + \frac{1-\dim_{\mathbb{R}}(\mathbb{K})}{\dim_{\mathbb{R}}(\mathbb{K})(p+q)}I_{p+q},\ \alpha_{p,q} = 2\frac{p}{p+q}I_p + 2\frac{q}{p+q}I_{p,p+q}$$

Given two speed parameters $v,c>$, the scalar product on the lie algebra $\mathfrak{u}(p,q,\mathbb{K})$ we consider takes the form:

$$B^{\mathbb{K}}_{v,c}(k_1^-+k_1^++p_1,k_2^-+k_2^++p_2) = -\frac{\beta p}{2c}\mathcal{R}e\mathrm{Tr}(k_1^+k_2^+) - \frac{\beta q}{2c}\mathcal{R}e\mathrm{Tr}(k_1^-k_2^-) + \frac{\beta\frac{p+q}{2}}{2v}\mathcal{R}e\mathrm{Tr}(p_1p_2)$$

and the defining stochastic differential equation of $\Lambda^{\mathbb{K}}_{p,q}$ is written as:

$$\mathrm{d}\Lambda^{\mathbb{K}}_{p,q}(t) = \Lambda^{\mathbb{K}}_{p,q}(t)\mathrm{d}W^{\mathbb{K}}_{p,q}(t) + \frac{1}{2}\Lambda^{\mathbb{K}}_{p,q}(t)\Big(v\alpha_{p,q}+c\beta^{\mathbb{K}}_{p+q}\Big)\mathrm{d}t. \tag{3.41}$$

3.5. Bicoloured Brauer diagrams and invariant polynomials

3.5.1. Bi-coloured Brauer diagrams. In that section, we introduce the notion of bi-coloured Brauer diagrams and take the opportunity to make a remainder on Brauer diagrams. We strive to motivate all the definitions that are introduced. However, we are aware that the combinatorics developed here may seem to be quite raw but is absolutely fundamental for our work. A bicoloured Brauer diagrams is, of course, a coloured Brauer diagram (as defined in Chapter 2) which is coloured only with two colours. Some of the material that is introduced in that section has thus already been expounded in Chapter 2. Brauer. It is easier to do computations with bi-coloured Brauer diagrams as with general coloured Brauer diagram and we take the opportunity to discuss representations of the uncoloured Brauer algebra that arise from injection of the latter in the algebra of bi-coloured Brauer diagrams. We introduce for that a new function on the set of oriented coloured diagram that is related to the involution $\star'$.

Definition 3.13 (Brauer diagram). A Brauer diagram of size k of is a fixed point free involution of the set $\{1,\ldots,k,1',\ldots,k'\}$.

The set of all Brauer diagrams of size k is denoted by $\mathcal{B}^{\bullet}_k$ (The superscript $\bullet$ is used to make a clear difference between Brauer diagrams and the notion of bi-coloured Brauer diagram introduced below). A Brauer diagram σ of size k is alternatively seen as a map of as a partition of the set $\{1,\ldots,k,\,1',\ldots,k'\}$: two integers $1\leq i,j\leq k$ are related if and only if $i=\sigma(j)$. Such a partition associated with a Brauer diagram is depicted as follows. We draw first k vertices on a line labelled by the integers in $1,\ldots,k$ from left to right and k other vertices on an other line down to the first one and labelled by the integers in $\{1,\ldots,k'\}$, we add strands that connect two integers if they are in the same block of σ (or if one is the image of the other by the map σ.) In the sequel we freely make the identification between a Brauer diagrams (a map), a partition with blocks of cardinal 2 and its diagram. We denote by the symbol $\mathbf{1}_k$ the Brauer diagram that is pictured as in Fig. 1.

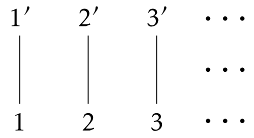


Figure 1. The identity element of the algebra of coloured Brauer diagram

We denote by $\mathsf{nc}(p)$ the number of blocks of a partition p of the set $\{1,\ldots,k,1',\ldots,k'\}$. A *cycle* of a Brauer diagram $b^\bullet$ is a block of the partition $b^\bullet \vee 1$, the number of cycles of σ is thus $\mathsf{nc}(\sigma \vee \mathbb{1}_k)$

Let $k \geq 1$ and $p \neq q \geq 1$ two integers. We start with a few basic definitions and notations. We use $i' = 2k+i$ for $1 \leq i \leq k$ and denote by $\{1,\ldots,k,1',\ldots,k'\}$ the interval of integers $[\![1,2k]\!]$. A *colouring* of a set S is a function $c : S \longrightarrow \{+,-\}$. Denote by $\mathcal{C}_{2k}$ the set of all colourings of $\{1,\ldots,k,1',\ldots,k'\}$. The *dimension function* associated with a colouration is the function

$$\begin{array}{rccl} d_c: & \{1,\ldots,k,1',\ldots,k'\} & \rightarrow & \{q,p\} \\ & i & \mapsto & \left\{\begin{array}{lll} p & \text{if} & c(i)=+ \\ q & \text{if} & c(i)=- \end{array}\right. . \end{array}$$

We now come to the main definition on this section.

Definition 3.14. A *bi-coloured Brauer diagram* is a pair $(b^\bullet, c)$ with $b^\bullet$ a Brauer diagram and c a colouring satisfying $d_c \circ b^\bullet = d_c$.

The set of all bi-coloured Brauer diagrams is denoted $\mathcal{B}_k$ and we use $\mathbb{K}[\mathcal{B}_k]$ for the $\mathbb{K}$ vector space with basis $\mathcal{B}_k$. The set $\mathcal{B}_k$ does depends on the two integers p,q, althought they do not appear in the notation. If $p=q$, integers connected by link of a bicoloured Brauer diagram may be coloured with two different colours. We may add this dependency to the notation we chose for the set of bicolourd Brauer diagram if needed. If b is a bi-coloured Brauer diagram, $b^\bullet$ stands for its underlying Brauer diagram and c_b is the colouring. A bi-coloured Brauer diagram is conveniently depicted as follows (see Figure 2 and Figure 3). We define now on the vector

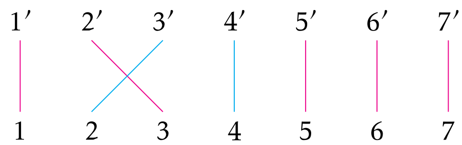


Figure 2. A bi-coloured Brauer diagram. We draw in cyan the links coloured with $+$ and in magenta the links coloured with $-$.

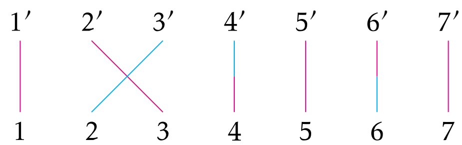


Figure 3. If $p=q$, the same link can be coloured with two different colors. We draw in cyan the links coloured with $+$ and in magenta the links coloured with $-$.

space $\mathbb{K}[\mathcal{B}_k]$ an algebra structure. Let $b_1, b_2 \in \mathcal{B}_k$ two Brauer diagrams. We stack b_1 over b_2, if we look at the non-coloured components of b_1 and b_2, we obtain a diagram $c^\bullet$ which contains, eventually, closed connected components. Now, if the colourization of the bottom line of b_1 matches the colourization of the upper line of b_2, we set $b_1 \circ b_2$ equal to the Brauer diagram obtained by colouring the bottom line (resp. upper line) of $c^\bullet$ with the colourization of the bottom line of b_1 (resp. the upper line of b_2) and removing all the closed components of $b^\bullet$. Otherwise, we set $b_1 \circ b_2 = 0$

Suppose $b_1 \circ b_2 \neq 0$. Since the colourizations of b_1 and b_2 match, the closed connected components that have been removed were eventually bi-coloured, but this happens only if $p = q$ so that we can associate to each loop either the dimension p either the dimension q.

Assume that $p \neq q$. We let $\mathcal{K}_p(b_1, b_2)$(resp. $\mathcal{K}_q(b_1, b_2)$) be the number of closed connected components which (resp −) of $b_1 \circ b_2$. The product $b_1 b_2$ is subsequently defined by

$$b_1 b_2 = p^{\mathcal{K}_p(b_1,b_2)} q^{\mathcal{K}_q(b_1,b_2)} (b_1 \circ b_2).$$

We prefer to denote the two functions $\mathcal{K}_p$ and $\mathcal{K}_q$, respectively, by $\mathcal{K}_+$ and $\mathcal{K}_-$ since these two functions do not really depends on the two integers p and q. If $p = q$, we need only the function $\mathcal{K}(b_1, b_2)$ that counts the number of loops that were removed for $b^\bullet$, the product between b_1 and b_2 is subsequently defined by

$$b_1 b_2 = p^{\mathcal{K}(b_1,b_2)} b_1 \circ b_2.$$

We use the notation $\mathcal{B}_k(p,q)$ for the algebra we just defined. In Fig. 4 and 5, we drew two examples of products. The unit of $\mathcal{B}_k(p,q)$ by summing over all the colourizations c of the uncoloured Brauer diagram $\mathbf{1}_k$ that are diagonal, $c_b(i) = c_b(i')$ for all integer $1 \leq i \leq k$. We denote by $C(\mathbf{1}_k)$ the comprising all such colourizations. If $c \in C(\mathbf{1}_k)$, we denote by p_c the coloured Brauer diagram $(\mathbf{1}_k, c)$. Then $p_c, c\mathbf{1}_c$ is a complete and mutually orthogonal set of projectors of $\mathcal{B}_k(p,q)$.

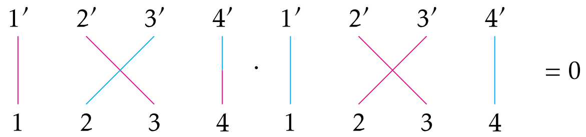


FIGURE 4.

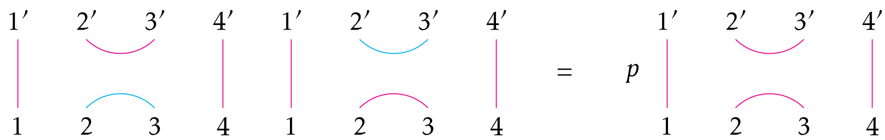


FIGURE 5.

Let $N \geq 1$ an integer. Recall that the real span of $\mathcal{B}_k^\bullet$ is endowed with an algebra structure, which have $\mathbf{1}_k$ as unit and defined as follows. If $b_1^\bullet$ are two uncoloured Brauer diagrams, we stack $b_1^\bullet$ over $b_2^\bullet$ to form a diagram $c^\bullet$ that may have closed connected components, we denote by $\mathcal{K}(b_1, b_2)$ the number of these components. We remoce these components to form the concatenation $b_1 \circ b_2$. The product $b_1 b_2$ is defined by the formula:

$$b_1^\bullet b_2^\bullet = N^{\mathcal{K}(b_1,b_2)} b_1 \circ b_2.$$

We denote by $\mathcal{B}_k^\bullet(N)$ the algebra we just defined. We should make an intensive use of the following fundamental relation, that is easily proved by a drawing (see [30]):

$$\mathsf{nc}((b_1^\bullet \circ b_2^\bullet) \vee 1) = \mathsf{nc}(b_1^\bullet \vee b_2^\bullet) + \mathcal{K}(b_1^\bullet, b_2^\bullet). \tag{3.42}$$

3.5.2. Representation. Let $k \geq 1$ an integer. If $\mathsf{i} = \left(i_j\right)_{1 \leq j \leq k}$ is a k-tuple of integers, we denote by $\mathsf{Ker}(\mathsf{i})$ the partition of the set $\{1, \ldots, k\}$ equal to the set of all level sets of i. Also, if i, j are two sequences of integers of length k, $\mathsf{ker}(\mathsf{i}, \mathsf{j})$ is the partition equal to the set of all level sets of the function defined on $\{1, \ldots, k, 1', \ldots, k'\}$ equal to i on $\{1, \ldots, k\}$ and j on $\{1', \ldots, k'\}$.

We define a representation of the algebra $\mathcal{B}_k(p,q)$ on the k-fold tensor product $(\mathbb{R}^{p+q})^{\otimes k}$ by setting

$$\begin{array}{rccc} \rho_{p,q}: & \mathcal{B}_k(p,q) & \to & \mathsf{End}(\mathbb{R}^{p+q}) \\ & (b^\bullet, c_b) & \to & \displaystyle\sum_{\substack{\mathsf{i},\mathsf{j} \in \{1,\ldots,p+q\}^k, \\ b \prec \mathsf{ker}(\mathsf{i},\mathsf{j})}} E_{i_1,j_1}^{c(1'),c(1)} \otimes \cdots \otimes E_{i_k,j_k}^{c(k'),c(k)} \end{array}$$

We turn our attention to the definition of the three real representations, $\rho^{\mathbb{R}}_{p,q}, \rho^{\mathbb{C}}_{p,q}$ and $\rho^{\mathbb{H}}_{p,q}$ that will be used later to define statistics of the pseudo-unitary Brownian motions. For the real and complex case, we set $\rho^{\mathbb{R}}_{p,q} = \rho^{\mathbb{C}}_{p,q} = \rho_{p,q}$. The representation $\rho^{\mathbb{C}}_{p,q}$ of the real algebra $\mathcal{B}_k(p,q)$ defines a representation, denoted by the same symbol, of the complex algebra $\mathcal{B}_k(p,q)\otimes\mathbb{C}$.

For the quaternionic case, it is more complicated, a real linear representation $\rho^{\mathbb{H}}$ of the algebra of Brauer diagrams $\mathcal{B}^{\bullet}_k(-2(p+q))$ on $(\mathbb{H}^n)^{\otimes k}$ is defined in [41], equation (36) as a convolution of two representations: the representation $\rho_{p,q}$ of $\mathcal{B}^{\bullet}_k(p,q)$ and a representation γ of $\mathcal{B}^{\bullet}_k(-2)$ which commutes with $\rho_{p,q}$. We explain how a representation of the bi-coloured Brauer algebra $\mathcal{B}_k(-2(p+q))$ is defined similarly as a convolution product of two representations. Let m be the multiplication map on the space of all $\mathbb{H}$ linear endomorphisms $\mathsf{End}_{\mathbb{H}}(\mathbb{H}^n)^{\otimes_{\mathbb{R}}k}$ and let $s,t>0$ be two integers. The key observation is the existence of a morphism $\Delta^{s,t}_{st} : \mathcal{B}^{\bullet}_k(st) \to \mathcal{B}^{\bullet}_k(s)\times\mathcal{B}_k(t)$ which is the real linear extension of the set function defined on $\mathcal{B}^{\bullet}_k$ by $\Delta^{s,t}_{st}(b) = b\otimes b$, $b\in\mathcal{B}^{\bullet}_k$. The representation $\rho^{\mathbb{H}}_n$ of the algebra $\mathcal{B}^{\bullet}_k(-2n)$ defined by Lévy in [41] is the convolution product:

$$\rho^{\mathbb{H}}_n = m\circ\left(\rho^{\mathbb{R}}\otimes\gamma\right)\circ\Delta^{(p+q),-2}_{-2(p+q)}.$$

The definition of a coloured version of the representation $\rho^{\mathbb{H}}_n$ is ensured by the existence of coloured version $\Delta^{(p,q),-2}_{-2p,-2q} : \mathcal{B}_k(-2p,-2q) \to \mathcal{B}_k(p,q)\times\mathcal{B}^{\bullet}_k(-2)$ of $\Delta^{s,t}_{st}$, namely, for $b\in\mathcal{B}_k$:

$$\Delta^{(p,q),-2}_{-2p,-2q}(b) = b\otimes b^{\bullet}. \tag{3.43}$$

The map $\Delta^{(p,q),-2}_{-2p,-2q}$ is a morphism from $\mathcal{B}_k(-2p,-2q)$ to $\mathcal{B}_k(p,q)\otimes\mathcal{B}^{\bullet}_k(-2)$. The representation $\rho^{\mathbb{H}}_{p,q}$ is defined by the equation:

$$\rho^{\mathbb{H}}_{p,q} = m\circ\left(\rho^{\mathbb{R}}_{p,q}\otimes\gamma\right)\circ\Delta^{-2p,-2q}_{-2(p+q)}. \tag{3.44}$$

An explicit expression for γ is given at the end of the next section.

3.5.3. Orienting and cutting a Brauer diagram. In this section, we define the notion of oriented Brauer diagram. We define here a new function associated with an oriented Brauer diagram that is closely related to the involution $\star'$.

Let $b = (b^{\bullet}, c_b)$ a bi-coloured Brauer diagram. To the colouring c we associate the function $\varepsilon^c : \{1,\ldots,k\} \to \{-1,1\}$ defined by $\varepsilon^c(i) = 1$ if $c(i) = c(i+k)$ and $\varepsilon^c(i) = -1$ otherwise.

To the partition $b^{\bullet}$ we associate a graph Γ_b: the vertices are the points $\{1,\ldots,k,1',\ldots,k'\}$ and the edges are the links of the partition $b^{\bullet}$ together with the vertical edges $\{x,x'\}$, $x\leq k$. Each of the connected components of this graph is a loop and we pick an orientation of these loops. To that orientation of Γ_b, we associate a function $s : \{1,\ldots,k\} \to \{-1,1\}$ defined as follows. Let $i\in\{1,\ldots,k\}$ an integer, we set $s(i) = 1$ if the edge that belongs to b which contains i is incoming at i in the chosen orientation of Γ_b and -1 otherwise. Of course an orientation of Γ_b is completely known through its associated sign function s, thus we will in the sequel freely identify these last two objects. See Fig. 6 for examples of oriented Brauer diagram. Note that the sign function s is solely determined by its values on a subset of integers $[\![1,n]\!]$ that intersect non trivially each of its the cycle of $b^{\bullet}$. The set of oriented, uncoloured Brauer diagrams is denoted $\mathcal{OB}^{\bullet}_k$ while the set of oriented coloured Brauer diagram is denoted $\mathcal{O}(\mathcal{B}_k)$. We use the notation $b^s = (b,s)$ for an oriented Brauer diagram with sign function s and the set of oriented Brauer diagrams is denoted $\mathcal{OB}_k$. To each oriented Brauer diagram (b,s) there are two associated permutations $\Sigma_{(b,s)}$ and $\sigma_{(b,s)}$ defined as follows. An oriented Brauer diagram (b,s) is naturally a permutation $\Sigma_{(b,s)}$ of the set $\{1,\ldots,k,1',\ldots,k'\}$. The cycles of the permutation $\sigma_{(b,s)}$ are the traces on $\{1,\ldots,k\}$ of the cycles of $\Sigma_{(b,s)}$.

Finally, we define the function $\varepsilon^{(b,s)} : [\![1,n]\!] \to \{-1,1\}$ that is specific to the bicoloured case, let $1\leq i\leq n$ an integer,

$$\begin{aligned} \varepsilon^{(b,s)}(i) &= -1 && \text{if } \varepsilon^c(i) = s(i) = -1,\\ \varepsilon^{(b,s)}(i) &= 1 && \textit{otherwise}. \end{aligned}$$

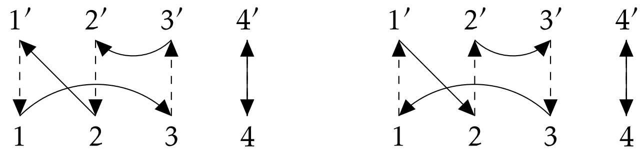


FIGURE 6. Two orientations of the same Brauer diagram

As it will be clear later on, when stating the Schur–Weyl duality we are interested in, the function $\varepsilon^{(b,s)}$ takes into account the fact than if computing the (pseudo-) conjugate of a matrix $A \in \mathcal{M}_n(\mathbb{R})$, $A^{\star'}$ is computed by transposing the block of A and multplying the off diagonal block by the coefficient -1.

The subset of permutations $\mathcal{S}_k \in \mathcal{B}_k$ is defined as the set of diagrams that represents a permutation, that are the diagram σ that send the set $\{1,\ldots,k\}$ to $\{1',\ldots,k'\}$. It is easily proved that a Brauer diagram b is a permutation diagram if and only if any orientation of b is constant on the cycles on b.

There is no canonical injection of the set of unortient Brauer diagrams into the set of oriented Brauer diagrams. Also, real span of the set $\mathcal{OB}_k$ does not seems to be endowed with an algebra structure that would turn the canonical projection from the set of oriented Brauer diagrams to the set of Brauer diagrams into an algebra morphism. However, it is important to choose a way to orient the concatenation $b \circ b_1$ (if $b_1 \circ b \neq 0$) of two Brauer diagrams b and b_1 knowing an orientation s of b_1. We choose quite arbitrarily to orient the cycles of $b \circ b_1$ as follows. We define the sign function $s_{b \circ b_1}$ as the unique sign function that defines an orientation of $b \circ b_1$ that is equal to s_{b_1} on the minimum of each cycle of $b \circ b_1$. The oriented brauer diagram that is defined in this way is denoted $b \diamond b_1$.

PROPOSITION 3.15. *Let b be a non-coloured Brauer diagram. Let $i \neq j \in \{1,\ldots,k\}$ two integers. If i,j do not lie in the same cycle of $b^\bullet$ ($i \nsim_{b^\bullet \vee 1} j$) then* $\mathsf{nc}(e^\bullet_{ij} \vee b^\bullet) = \mathsf{nc}(e_{ij} \vee 1) - 1$. *If i and j are in the same cycle of $b^\bullet$, we have for any orientation s of $b^\bullet$:*

$$\begin{aligned} &\textit{if } s(i)s(j) = -1, && \mathsf{nc}(b^\bullet \vee e^\bullet_{ij}) = \mathsf{nc}(b^\bullet \vee 1) + 1,\\ &\textit{if } s(i)s(j) = 1, && \mathsf{nc}(b^\bullet \vee \tau^\bullet_{ij}) = \mathsf{nc}(b^\bullet \vee 1) + 1. \end{aligned}$$

Let $b^\bullet$ be a Brauer diagram. We end this section by quoting the explicit formula for γ, from [41]:

$$\gamma(b^\bullet) = \frac{1}{(-2)^n} \sum_{\gamma_1,\ldots,\gamma_n \in \mathsf{I}(\mathbb{H}} \prod_{i_1,\ldots,i_n} (-2\mathcal{R}e)(\gamma_{i_1} \cdots \gamma_{i_s}) \gamma_1^{-s^\bullet_b} \otimes \gamma_n^{-s^\bullet_n}$$

where $I(\mathbb{H}) = \{\mathsf{i},\mathsf{j},\mathsf{k}\}$, $s^\bullet_b$ is the orientation of $b^\bullet$ that is positive on the minimum of each cycle of $b^\bullet$.

3.5.4. Central extension of the algebra of coloured Brauer diagrams. Let $p \neq q$ two integers. It will appear in Section 3.9, in which we investigate the asymptotic behaviour of Brownian pseudo-unitary matrices on $\mathbb{U}(p,q,\mathbb{K})$, it will be necessary to keep track of the dimensions the are associated to loops eventually created if two bi-coloured Brauer diagrams are multiplied together. In fact, from now, it is not possible to do so: from the definition of the algebra structure on $\mathcal{B}_k(p,q)$, from a loop produced by multiplication of two diagrams result a factor multiplying the concatenation of the two diagrams. It there are least two loops that are created, it not possible to find back the colourizations of each of these loops from this multiplication factor. This is the main reason for introducing a central extension of the algebra $\mathcal{B}_k(p,q)$.

Let p and q be two non equal integers. The central extension $\mathring{\mathcal{B}}_k(p,q)$ is, as a vector space, equal to the direct sum $\mathbb{R}[\mathcal{B}_k] \oplus \mathbb{R}[\mathsf{o}_+,\mathsf{o}_-]$. The set $\mathsf{o} = \{\mathsf{o}_d, d \in \{\mathsf{d}\}\}$ of commuting variables is referred to as the set of *loops variables*. Two elements $b \oplus P$ and $b' \oplus Q$ in $\mathcal{B}_k \oplus \mathbb{R}[\mathsf{o}_+,\mathsf{o}_-]$ are multiplied as follows:

$$(b \oplus P) \cdot (b' \oplus Q) = \left(b \circ b', PQ \times \mathsf{o}_+^{\mathcal{K}_+(b,b')} \mathsf{o}_-^{\mathcal{K}_-(b,b')}\right). \tag{3.45}$$

In the sequel, we will mainly deal with operators that are defined on a subalgebra of $\mathring{\mathcal{B}}_k(p,q)$. If (α_-,α_+) is a multi-index, we denote by o_{+-}^{α} the monomial $\mathrm{o}_+^{\alpha_+}\mathrm{o}_-^{\alpha_-}$. We define the set of *diagrams with loops* as

$$\mathring{\mathcal{B}}_k = \{b \oplus \mathrm{o}_{-+}^{\alpha},\ b \in \mathcal{B}_k, \alpha \in \mathbb{N}^{(-,+)}.$$

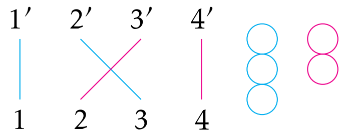


Figure 7. A diagram with loops, we use the same colours code as in Fig. 2.

The space $\mathring{\mathcal{B}}_k(p,q)$ projects onto the algebra of bi-coloured Brauer diagrams. The projection $\mathring{\pi} : \mathring{\mathcal{B}}_k(-+) \to \mathcal{B}_k(-+)$ specializes a loop variable $\mathrm{o}_+, \mathrm{o}_-$ to the corresponding dimension:

$$\mathring{\pi}((b, P(\mathrm{o}_+,\mathrm{o}_-)) = P(p,q)b.$$

We draw in Fig. 8 the short exact sequence of algebra morphism.

$$0 \longrightarrow \mathring{\mathcal{B}}_k(p,q) \xrightarrow{\mathring{\pi}} \mathcal{B}_k(p,q) \longrightarrow 0.$$

Figure 8. A central extension of the algebra of bi-coloured Brauer diagrams.

If $b \in \mathcal{B}_k$ is a bi-coloured Brauer diagram, we denote by $\mathring{b}$ the element $(b,1)$ in the $\mathring{\mathcal{B}}_k$. In this way we define a section from $\mathcal{B}_k$ to $\mathring{\mathcal{B}}_k$ which is however not an algebra morphism (the central extension $\mathring{\mathcal{B}}_k(p,q)$ is not split).

We finish with the definition of the functions that justify alone the introduction of this central extension. In the last section, we gave orientation to Brauer diagrams. We will do the same for Brauer diagrams with loops in a consistent way. We recall that we denote by $\mathcal{OB}_k(p,q)$ the vector space with basis the set $\mathcal{OB}_k$ of all oriented coloured Brauer diagrams. We denote by $\mathcal{O}\mathring{\mathcal{B}}_k(p,q)$ the vector space $\mathcal{OB}_k(p,q) \oplus \mathbb{R}[\mathrm{o}_-,\mathrm{o}_+]$.

Let $s \in \{-,+\}$ and $\tilde{b} = ((b,s), \mathrm{o}_+^{\alpha_+}\mathrm{o}_-^{\alpha_-}) \in \mathcal{O}\mathring{\mathcal{B}}_k(p,q)$, the function fnc_s counts the number of loops variables coloured with s and the number of cycles of b whose minimum m or minimum m' coloured with s, depending on the orientation of the cycle. More formally:

$$\begin{aligned}\mathsf{fnc}_s\left(\tilde{b}\right) = n_d + \sharp\{\{i_1 < \ldots < i_k\} \in b^{\bullet} \vee 1 : c_b(i_1) = +) \text{ and } s(m) = 1\} \\ + \sharp\{\{i_1 < \ldots < i_k\} \in b^{\bullet} \vee 1 : c_b(i_1') = -) \text{ and } s(m) = -1\}.\end{aligned}$$

In the last section, given an oriented Brauer diagram (b_1,s) and a Brauer diagram b, we defined the oriented Brauer diagram $b \diamond (b_1,s)$. This operation can be lifted to $\mathcal{O}\mathring{\mathcal{B}}_k(\mathrm{d})$:

$$((b,P) \diamond ((b_1,s),Q) = \left(b \diamond (b_1,s), PQ \times \mathrm{o}_+^{\mathcal{K}_+(b,b_1)}\mathrm{o}_-^{\mathcal{K}_-(b,b_1)}\right), \tag{3.46}$$

where $(b,P) \in \mathring{\mathcal{B}}_k(p,q)$, $((b_1,s),Q) \in \mathcal{O}\mathring{\mathcal{B}}_k(p,q)$. In the split case, if $p = q$, althought we will not need a central extension of $\mathcal{B}_k(p,q)$ o perform the computations, we only indicate however that a central extension of $\mathcal{B}_k(p,p)$ is obtaind by adding only one loop variable:

$$\mathring{\mathcal{B}}_k(p,p) = \mathcal{B}_k(p,p) \oplus \mathbb{R}[\mathrm{o}].$$

The product on $\mathring{\mathcal{B}}_k(p,p)$ and the operation $\diamond$ are defined by equations 3.45 and 3.46.

3.5.5. Subsets of bicoloured Brauer diagrams. Let $k \geq 1$ an integer. We introduce now subsets of Brauer diagrams that will be used in Section 3.6 to express generators of differential systems satisfied by statistics of the pseudo-unitary Brownian motions. The set of bi-coloured Brauer diagrams that have links coloured with only one sign is denoted $\mathcal{B}_{k,2}$.

The *bi-coloured projectors* or simply the *projectors* $e_{ij} = (e^\bullet_{ij}, c_{e_{ij}}) \in \mathcal{B}_{k,2}$ with $1 \leq i < j \leq k$ are defined by

$$e^\bullet_{ij} = \{\{i,j\},\{i',j'\}\} \cup \{\{x,x'\},\ x \neq i,j,\ 1 \leq x \leq k\},$$

with $c_{e_{ij}}$ a colouration such that $d_c \circ e^\bullet_{ij} = d_c$. The *bi-coloured transpositions* or simply the *transpositions* $\tau_{ij} = (\tau^\bullet_{ij}, c_{\tau_{ij}})$ are defined by

$$\tau^\bullet_{ij} = \{\{i,j'\},\ \{j,i'\}\} \cup \{\{x,x'\},\ x \neq i,j,\ 1 \leq x \leq k\}.$$

with $c_{e_{ij}}$ a colouration such that $d_c \circ e^\bullet_{ij} = e_{ij}$. The set of projectors is denoted W_k and the set of transpositions is denoted T_k.

We define also the sets of *bi-coloured exclusive transpositions* $\mathsf{T}^{\neq}_k$ and the set of *bi-coloured exclusive projectors* $\mathsf{W}^{\neq}_k$ by setting

$$\mathsf{T}^{\neq}_k = \{(\tau^\bullet_{ij}, c_{\tau_{ij}}) \in \mathsf{T}_k : c(i) \neq c(j)\},\ \mathsf{W}^{\neq}_k = \{(e^\bullet_{ij}, c_{e_{ij}}) \in \mathsf{W}_k : c(i) \neq c(i')\}.$$

The sets of bi-coloured diagonal transpositions and bi-coloured diagonal projectors are defined by

$$\mathsf{T}^{=}_k = \{(\tau^\bullet_{ij}, c) \in \mathsf{T}_k : c(i) = c(j)\},\ \mathsf{W}^{=}_k = \{(e^\bullet_{ij}, c) \in \mathsf{W}_k : c(i) = c(i')\}.$$

Because of the normalization factors we chose for the scalar products $B^{\mathbb{K}}$, $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, we have to introduce two more subsets of T_k and W_k:

$$\mathsf{T}^{=}_k(+) = \{(\tau^\bullet_{ij}, c) \in \mathsf{T}^{=}_k : c(i) = +\},\ \mathsf{W}^{=}_k(+) = \{(e^\bullet_{ij}, c) \in \mathsf{W}^{=}_k : c(i) = +\},$$
$$\mathsf{T}^{=}_k(-) = \{(\tau^\bullet_{ij}, c) \in \mathsf{T}^{=}_k : c(i) = -\},\ \mathsf{W}^{=}_k(-) = \{(e^\bullet_{ij}, c) \in \mathsf{W}^{=}_k : c(i) = -\}.$$

Put $\mathsf{E}^{=}(+) = \mathsf{T}^{=}_k(+) \cup \mathsf{W}^{=}_k(+)$ and $\mathsf{E}^{=}(-) = \mathsf{T}^{=}_k(+) \cup \mathsf{W}^{=}_k(-)$. In Figure 9 are pictured examples of elements of the subsets defined above.

Figure 9. On the left, a coloured exclusive transposition and on the right a diagonal transposition

To compute asymptotic of the statistics of the pseudo-unitary diffusions that are defined in Section 3.6, and give readable formulae, we introduce the following sets of Brauer diagrams. Let $b \in \mathcal{B}_k$, we define the subsets

$$\mathsf{T}^{\neq,+}_k(b) = \{\tau \in \mathsf{T}^{\neq}_k : \tau b \neq 0,\ \mathsf{nc}(b^\bullet \vee \tau^\bullet) = \mathsf{nc}(b^\bullet \vee 1) + 1\},$$
$$\mathsf{W}^{\neq,+}_k(b) = \{e \in \mathsf{W}^{\neq}_k : eb \neq 0,\ \mathsf{nc}(b^\bullet \vee e^\bullet) = \mathsf{nc}(b^\bullet \vee 1) + 1\}.$$

and

$$\mathsf{T}^{=,+}_k(b) = \{\tau \in \mathsf{T}^{=}_k : \tau b \neq 0,\ \mathsf{nc}(b^\bullet \vee \tau^\bullet) = \mathsf{nc}(b^\bullet \vee 1) + 1\},$$
$$\mathsf{W}^{=,+}_k b = \{e \in \mathsf{W}^{=}_k : eb \neq 0,\ \mathsf{nc}(b^\bullet \vee e^\bullet) = \mathsf{nc}(b^\bullet \vee 1) + 1\}.$$

Transpositions and projectors are called *elementary diagrams*, a generic elementary diagram is written r. To denote unions of subsets of transpositions or projections that share the same properties, we use the symbols that were introduced for these sets in which T or W is replaced by R, eg $\mathsf{R}^{=}_k = \mathsf{T}^{=}_k \cup \mathsf{W}^{=}_k$. *Non-mixing* diagrams are Brauer diagrams that have their links coloured with a single color: two integers that are in the same block of a Brauer diagram share the same colour, either + either +. To the symbols used to denote the subsets of Brauer diagrams we introduced, we add the subscript 2 to denote its intersection with the set of non-mixing Brauer diagrams. Finall, to express mean of tensor polynomials of the pseudo-unitary Brownian motions, we introduce the notation $\tau_{ij}^{\varepsilon,\varepsilon'}$ for the sum in $\mathbb{R}[\mathcal{B}_k]$ of the coloured transpositions in $\mathcal{B}_k$ whose blocks $\{i,j+k\}$ and $\{j,i+k\}$ are coloured, respectively, by ε and ε'. Similarly, the notation $e^{\varepsilon,\varepsilon'}$ is used for the sum of the coloured projectors e_{ij} in $\mathcal{B}_k$ whose blocks $\{i,j\}$ and $\{i+k,j+k\}$ are coloured respectively by ε and ε'. In symbols,

$$\tau_{ij}^{\varepsilon,\varepsilon'} = \sum_{\substack{t\in\mathsf{T}_{k,n},\ t^\bullet=\tau^\bullet_{ij},\\ c_t(i)=\varepsilon, c_t(i')=\varepsilon'}} t, \quad e_{ij}^{\varepsilon,\varepsilon'} = \sum_{\substack{e\in\mathsf{W}_{k,n},\ e^\bullet=e^\bullet_{ij},\\ c_e(i)=\varepsilon, c_e(i')=\varepsilon'}} e.$$

With these definitions, we have for $\tau^{\pm,\pm}, e^{\pm,\pm}\in\mathcal{B}_2$,

$$\begin{aligned}\rho^{\mathbb{R}}_{p,q}(\tau^{\pm,\pm}) = \rho^{\mathbb{C}}_{p,q}(\tau^{\pm,\pm}) = \mathsf{T}^{\pm,\pm}, \rho^{\mathbb{H}}_{p,q}(\tau^{\pm,\pm}) = -\frac{1}{2}\mathsf{Re}^{\mathbb{H}}\mathsf{T}^{\pm,\pm}\\ \rho^{\mathbb{R}}_{p,q}(e^{\pm,\pm}) = \rho^{\mathbb{C}}_{p,q}(e^{\pm,\pm}) = \mathsf{E}^{\pm,\pm}, \rho^{\mathbb{H}}_{p,q}(e^{\pm,\pm}) = -\frac{1}{2}\mathsf{Im}^{\mathbb{H}}E^{\pm,\pm}.\end{aligned} \tag{3.47}$$

3.5.6. Algebra of bi-coloured Brauer diagrams and polynomial invariants. We will, quite briefly, without providing all details, define isomorphisms between spaces of invariants and subalgebras of $\mathcal{B}(p,q)$. We fix two integers $p,q\geq 1$. Let $\mathbb{K}$ be one of the three division algebras $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. We discuss so-called Schur–Weyl dualities for the groups $\mathbb{U}(p,q,\mathbb{K})$ and the maximal compact subgroup $\mathbb{U}(p,\mathbb{K})\times\mathbb{U}(q,\mathbb{K})$ of $\mathbb{U}(p,q,\mathbb{K})$. We recall that if written in the standard basis of $\mathbb{R}^{p+q}$, a matrix of an element $\mathbb{U}(p,\mathbb{K})\times\mathbb{U}(q,\mathbb{K})\subset\mathbb{U}(p,q,\mathbb{K})$ is a block diagonal matrix; the upper left matrix has dimension $p\times p$ and is unitary, the lower right matrix has dimension $q\times q$ and is also unitary. We denote by $\mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^{p+q})$ be the algebra of $\mathbb{K}$-linear endomorphisms acting on $\mathbb{K}^{p+q}$. In the case $\mathbb{K}=\mathbb{H}$, $\mathrm{Hom}_{\mathbb{H}}(\mathbb{H}^{p+q})$ is the space of endomorphisms commuting with left action by multiplication of $\mathbb{H}$ on $\mathbb{H}^{p+q}$.

The group $\mathbb{U}(p,q,\mathbb{K})$ has a natural action nat on $\mathbb{K}^{p+q}$ which can be folded to obtain an action on $(\mathbb{K}^{p+q})^{\otimes_{\mathbb{R}}k}$, $k\geq 1$:

$$U\cdot(v_1\otimes\cdots\otimes v_k) = \mathsf{nat}(U)(v_1)\otimes_{\mathbb{R}}\cdots\otimes_{\mathbb{R}}\mathsf{nat}(U)(v_k) = U(v_1)\otimes_{\mathbb{R}}\cdots\otimes_{\mathbb{R}}U(v_k),\ U\in\mathbb{U}(p,q,\mathbb{K}). \tag{3.48}$$

In the last equation, we wrote tensor product over $\mathbb{R}$. In the complex case, it is more natural to take tensor product over $\mathbb{C}$. In the following, to lighten to notation we use the following convention: for the real and quaternionic case $\otimes$ denotes the tensor product over $\mathbb{R}$ while for the complex case, $\otimes$ denotes the tensor product over $\mathbb{C}$. In this work, what we call a Schur–Weyl duality for the group $\mathbb{U}(p,q,\mathbb{K})$ is an isomorphism between the algebra of endomorphisms $\mathrm{Hom}_{\mathbb{H}}$ that commutes with the action (3.48) and an algebra that have a combinatorial interpretation. More on this is explained below. To state these Schur–Weyl dualities, we must first introduce in addition to the real algebra $\mathcal{B}_k(p,q)_{\mathbb{R}}=\mathcal{B}_k(p,q)$, the complex algebra $\mathcal{B}_k(p,q)_{\mathbb{C}}=\mathcal{B}_k(p,q)\otimes\mathbb{C}$ and the real algebra $\mathcal{B}_k(p,q)_{\mathbb{H}}=\mathcal{B}_k(p,q)\otimes\mathbb{H}$. We state first a Schur–Weyl duality for the pseudo-orthogonal groups $\mathbb{U}(p,q,\mathbb{R})$. Lemma 3.16 relates the algebra of bi-coloured Brauer diagrams $\mathcal{B}_k(p,q)_{\mathbb{R}}$ and real linear functions on $\mathcal{M}_n(\mathbb{R})^{\otimes k}$ that are invariant by the k folded conjugacy action of $U(p,q,\mathbb{R})$.

For a permutation $\sigma\in\mathcal{S}_k$, we write $(i_1,\ldots,i_k)\prec\sigma$ if $(i_1,\ldots,i_k)$ is a cycle of σ. For a matrix $A\in\mathcal{M}_n(\mathbb{R})$, we denote by $A(+,+)$ (resp. $A(-,-)$) the upper left block in A (resp. the lower right block of size $q\times q$ in A). We also denote by $A(+,-)$ (resp.$A(-,+)$) is the upper right block in A of size $p\times q$ (resp. the lower left block of size $q\times p$). In the following lemma, we use the notation $[A]^1=A$, $[A]^{-1}=A^t$, $A\in\mathcal{M}_{p+q}(\mathbb{R})$.

Lemma 3.16 ([41]). *Let $A_1,\ldots,A_k$ be real matrices of size $(p+q)\times(p+q)$. Let (b,s) an oriented Brauer diagram and denote by $\sigma_{(b,s)}$ the associated permutation. Then*

$$\mathsf{Tr}\Big((A_1\otimes\ldots\otimes A_k)\circ\rho_{p,q}(b)\Big) = \prod_{\substack{(i_1,\ldots,i_p)\\ \prec\sigma_{(b,s)}}}\mathsf{Tr}\Big(\Big[A_{i_1}(c(i_1'),c(i_1))\Big]^{s_b(i_1)}\cdots\Big[A_{i_p}\Big(c(i_p'),c(i_p)\Big)\Big]^{s_b(i_p)}\Big).$$

The Lemma 3.16 implies the following equality. Let $A_1,\ldots,A_k\in\mathcal{M}_n(\mathbb{R})$. Let (b,s) an oriented bi-coloured Brauer diagram. We introduce the notation $s_b^{\star}(i)=\star'$ if $s_b(i)=-1$ and

$s_b^\star(i) = 1$ if $s_b(i) = 1$. The function $\varepsilon^{(b,s)}$ is defined in the previous sub-section.

$$\begin{aligned}\left(\prod_{i\leq k}\varepsilon^{(b,s)}(i)\right)&\mathsf{Tr}\Big((A_1\otimes\cdots\otimes A_k)\circ\rho_{p,q}(b)\Big)\\ &=\prod_{(i_1,\ldots,i_p)\prec\sigma_{(b,s)}}\mathsf{Tr}\Big(\big(A_{i_1}(c(i_1'),c(i_1))\big)^{s_b^\star(i_1)}\cdots\big(A_{i_p}(c(i_p'),c(i_p))\big)^{s_b^\star(i_s)}\Big)\end{aligned}\tag{3.49}$$

The real algebra $\mathrm{Hom}_{\mathbb{U}(p,q,\mathbb{R})}\left((\mathbb{R}^{p+q})^{\otimes k}\right)$ of $\mathbb{U}(p,q,\mathbb{R})$ invariant is the real algebra of all endomorphisms acting on $(\mathbb{R}^{p+q})^{\otimes k}$ and commuting with the k-fold tensor product of the natural action of $\mathbb{U}(p,q,\mathbb{R})$ on $\mathbb{R}^{p+q}$, in symbols:

$$A\in\mathsf{End}_{\mathbb{U}(p,q,\mathbb{R})}\left((\mathbb{R}^{p,q})^{\otimes k}\right)\Leftrightarrow O^{\otimes k}AO^{-1^{\otimes k}}=A,\ \forall O\in\mathbb{U}(p,q,\mathbb{R}).$$

We now prove the existence of an isomorphism between a sub-algebra of the algebra of coloured Brauer diagrams $\mathcal{B}_k(p,q)_{\mathbb{R}}$ and the algebra of $\mathbb{U}(p,q,\mathbb{R})$-invariants we just defined.

We define a projection π of the vector space of oriented Brauer diagrams, $\mathcal{OB}_k(p,q)$ valued in $\mathcal{B}_k(p,q)$ by $\pi:\mathcal{OB}_k\to\mathcal{B}_k,(b,s)\mapsto\left(\prod_{i\leq k}\varepsilon^{(b,s)}(i)\right)b$. We choose to inject the set of non-oriented Brauer diagram in the set of oriented coloured Brauer diagrams by picking for each cycle of a non-oriented diagram the unique orientation such that the vertical edge at the minimum of the cycle is outgoing.

For all $b^\bullet\in\mathcal{B}_k$, we define $\mathsf{C}(b^\bullet)$ as the set of colourings c constant on the blocks of $b^\bullet$. We let $\Delta\mathcal{B}_k(p,q)_{\mathbb{R}}$ be the sub-algebra of $\mathcal{B}_k(p,q)_{\mathbb{R}}$ generated by the subset $\Delta\mathcal{B}_k$ of $\mathcal{B}_k$ and defined by

$$\Delta\mathcal{B}_k=\left\{\sum_{c\in\mathsf{C}(b^\bullet)}\pi\left((b^\bullet,c,s)\right),(b,s)\in\mathcal{OB}_k^\bullet\right\}.$$

Proposition 3.17. *Let $p,q\geq 1$ two integers,*

$$\rho_{p,q}^{\mathbb{R}}\left(\Delta\mathcal{B}_k^{\mathbb{R}}(p,q)\right)=\mathsf{End}_{U(p,q,\mathbb{R}))}\left((\mathbb{R}^n)^{\otimes k}\right).\tag{3.50}$$

Proof. The stated proposition is a simple consequence of the Theorem 7 in [5] and equation 3.49. ☐

We refer to the proposition 3.17 as a Schur–Weyl duality between the algebra $\Delta\mathcal{B}_k(p,q)_{\mathbb{R}}$ and the group $U(p,q,\mathbb{R})$. We state without proving that a Schur–Weyl duality holds between the full algebra of bi-coloured Brauer diagrams $\mathcal{B}_k$ and the compact group $\mathbb{U}(p,\mathbb{R})\times\mathbb{U}(q,\mathbb{R})$. The proof relies on basic results of invariant theory we consider however out of scope to develop for the present work. This duality is nevertheless important, since the real pseudo-unitary Brownian motion with parameter v,c defined in Section 3.4 have its quadratic variation equal to the Casimir of the metric $B_{v,c}^{\mathbb{R}}$, which is a bivector in $\mathrm{Hom}_{\mathbb{U}(p,q,\mathbb{R})}((\mathbb{R}^{p+q})^{\otimes 2})$, owing to$\mathbb{U}(p,\mathbb{R})\times\mathbb{U}(q,\mathbb{R})$ invariance of the metric. Computations of Section 3.6 to study asymptotic of the pseudo-unitary Brownian motions are made efficient because of combinatorial formulae for the Casimirs, that are implied by the forthcoming Proposition 3.18, with no proof.

Proposition 3.18. *Let $p,q\geq 1$ two integers, for all integer $k\geq 1$,*

$$\mathrm{Hom}_{\mathbb{U}(p,\mathbb{R})\times\mathbb{U}(q,\mathbb{R})}\left((\mathbb{R}^{p+q})^{\otimes k}\right)=\rho_{p,q}^{\mathbb{R}}(\mathcal{B}_k(p,q)).$$

We investigate now a Schur–Weyl duality for the quaternionic and complex pseudo-unitary group.

In [5], authors proved that the set of $U(p,q,\mathbb{C})$ invariant polynomial functions (polynomial in the matrix entries and their conjugate) is equal to the algebra generated by the functions:

$$(A_1,\ldots,A_s)\mapsto\mathsf{Tr}\left(P(A_1,\ldots,A_s,A_1^{\star'},\ldots A_s^{\star'})\right),\ P\in\mathbb{R}\left[X_1,\ldots,X_s,\bar{X}_1,\ldots,\bar{X}_s\right],\ s\geq 1.\tag{3.51}$$

See Section 3.4 for the definition of $\star'$. Let $s\geq 1$ an integer and let w a word in $\overline{\mathsf{M}}_k(s)$. The word w is said to be compatible with a Brauer diagram $b^\bullet$ if for any orientation s of $b^\bullet$, one has

$$\bar{w}_i=w_j\Leftrightarrow s(i)s(j)=-1,\ \text{ for all } 1\leq i,j\leq k.\tag{3.52}$$

Using Lemma 3.49, the algebra generated by the set of functions (3.51) is equal to the algebra generated by the functions

$$(A_1,\ldots,A_s) \mapsto \mathsf{Tr}\left(w^{\otimes}\left(A_{i_1} \otimes \ldots \otimes A_{i_s}\right) \circ \rho_{p,q}\left(\sum_{(c,s)\in \mathsf{C}(b^\bullet)} \pi(b^\bullet, c, s) \right)\right)$$

where $(w, b^\bullet)$ is a pair of compatible words and diagrams of arbitrary size k and $i_1,\ldots,i_k \in \{1,\ldots,s\}$. We remind the reader that $w^{\otimes}(A_1 \otimes \cdots \otimes A_k)$ is the evaluation on the word w of the unique morphism of $\overline{\mathsf{M}}_k$ such that $x_i^{\otimes}(A_1 \otimes \ldots \otimes A_k) = A_i$ and $\bar{x}_i^{\otimes}(A_1 \otimes \ldots \otimes A_k) = \bar{A}_i$. Hence, the complex algebra of polynomials functions on $\mathcal{M}_n(\mathbb{C})$ in the matrix entries (excluding the conjugates) is generated by

$$(A_1,\ldots,A_k) \mapsto \mathsf{Tr}\left(\left(A_{i_1} \otimes \ldots \otimes A_{i_s}\right) \circ \rho^{\mathbb{C}}_{p,q}\left(\sum_{(c,s)\in \mathsf{C}(\sigma^\bullet)} \pi(\sigma^\bullet, c, s)\right)\right),\ \sigma \in \mathcal{S}_s,\ s \geq 1.$$

Proposition 3.19. *Let $p, q \geq 1$ two integers,*

$$\rho^{\mathbb{C}}_{p,q}(\Delta\mathcal{B}_k(p,q)_{\mathbb{C}}) = \mathsf{End}_{\mathbb{U}(p,q,\mathbb{C})}\left((\mathbb{C}^{p+q})^{\otimes k}\right). \tag{3.53}$$

We now deal with the quaternionic case. In equation (3.51), if $\star'$ stands now for the hermitian conjugation on $\mathcal{M}_{p+q}(\mathbb{H})$,the real algebra generated by the functions (3.51) is a proper sub-algebra of the algebra of all $U(p,q,\mathbb{H})$ invariants function, see [5] (where the notation $Sp(p,q) = U(p,q,\mathbb{H})$ is used. The following lemma is a direct corollary of the lemma 2.6 in [41].

Lemma 3.20. *Let $A_1,\ldots,A_n \in \mathcal{M}_n(\mathbb{H})$, (b,s) an oriented bi-coloured Brauer diagram,*

$$(\mathcal{R}e \circ \mathsf{Tr})^{\otimes k}\left((A_1 \otimes \cdots \otimes A_k) \circ \rho^{\mathbb{H}}\left(\left(\prod_{i\leq k} \varepsilon^{(b,s)}(i)\right) b\right)\right) =$$
$$\prod_{(i_1,\ldots,i_s)\in \sigma_{(b,s)}} (\mathcal{R}e \circ \mathsf{Tr})\left(\left(A_{i_1}(c_{i_1}, c_{i_1'})\right)^{\varepsilon_b^\star(i_1)} \cdots \left(A_{i_s}(c_{i_s}, c_{i_s'})\right)^{\varepsilon_b^\star(i_s)}\right).$$

with $\varepsilon_b^\star(i) = \star'$ if $s(i) = -1$ and $\varepsilon_b^{\star'}(i) = 1$ if $s(i) = 1$.

The following proposition is a simple corollary of Lemma 3.20 and final result in [5].

Proposition 3.21. *Let p, q integers and $k \geq 1$ a third one,*

$$\rho^{\mathbb{H}}(\Delta\mathcal{B}_k{}^{\mathbb{H}}) \subset \mathsf{End}_{\mathbb{U}(p,q,\mathbb{H})}\left((\mathbb{H}^{p+q})^{\otimes k}\right).$$

We state also, without proving it, that the ranges of the representations $\rho^{\mathbb{C}}_{p,q}$ and $\rho^{\mathbb{H}}_{p,q}$ are included, respectively, into the algebras of endomorphisms of $\mathbb{C}^{p+q}$ that commute with the k-folded natural action of $\mathbb{U}(p,\mathbb{C})\times\mathbb{U}(q,\mathbb{C})$ on $(\mathbb{C}^{p+q})^{\otimes k}$ respectively, the algebra of endomorphisms acting on $\mathbb{H}^{p+q}$ that commutes with the k-folded natural action of $\mathbb{U}(p,\mathbb{H}) \times \mathbb{U}(p,\mathbb{H})$ on $(\mathbb{H}^{p+q})^{\otimes k}$. The Casimir associated with the scalar products $B^{\mathbb{K}}$ are invariant bi-vectors and belong to the ranges of, respectively, $\rho^{\mathbb{K}}_{p,q}$. This last remark is important, since it allows a combinatorial interpretation of the differential systems satisfied by statistics of the pseudo-unitary Brownian motions we are interested in.

3.5.7. Other representations of the uncoloured Brauer algebra. Let $k \geq 1$ an integer and $p, q \geq 1$ two integers, that stand for the signature of a metric. For $i \leq k-1$, set $\tau_i = \tau^\bullet_{i,i+1}$ and $e_i = e^\bullet_{i,i+1}$. The algebra $\mathcal{B}_k{}^\bullet(p+q)$ is generated by the element τ_i, e_i, $1 \leq i \leq k-1$ to the relations:

$$\begin{gathered}
\tau_i^2 = 1,\ e_i^2 = (p+q)e_i,\ \tau_i e_i = e_i \tau_i = e_i,\\
\tau_i\tau_j = \tau_j\tau_i,\ e_ie_j = e_je_i,\ \tau_ie_j = e_j\tau_i,\ |i-j| > 1,\\
\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1},\ e_ie_{i+1}e_i = e_i,\ e_{i+1}e_ie_{i+1} = e_{i+1},\\
\tau_ie_{i+1}e_i = \tau_{i+1}e_i,\ e_{i+1}e_i\tau_{i+1} = e_{i+1}\tau_i.
\end{gathered} \tag{3.54}$$

There exists a canonical injection Δ' of the algebra of non-coloured Brauer diagrams $\mathcal{B}_k(p+q)$ into the algebra of coloured Brauer diagams $\mathcal{B}_k(p,q)$ that is obtained by tracing over colourizations. We recall the notation $\mathcal{B}_{k,2}$ for diagrams having links that are coloured with a single colour in the set $\{-,+\}$. If $b^\bullet$ is an uncoloured Brauer diagram, we denote by $C(b^\bullet)$ the subset of colourizations of $\{-,+\}^{2k}$ that makes $(b^\bullet,c)$ a bicoloured Brauer diagram of $\mathcal{B}_{k,2}$. We define an injection $\Delta' : \mathcal{B}_k(p+q)\to\mathcal{B}_k(p,q)$ by setting:

$$\Delta'(b^\bullet) = \sum_{c\in C(b^\bullet)} (b^\bullet,c),\ b^\bullet\in\mathcal{B}_k^\bullet.$$

Simple computations show that Δ' does define a morphism. Its values on the generator s of the Brauer algebra of equation (3.54) are given by:

$$\Delta'(\tau_i) = \tau_i^{++}+\tau_i^{--}+\tau_i^{-+}+\tau_i^{+-},\ \Delta'(e_i) = e_i^{++}+e_i^{--}+e_i^{-+}+e_i^{+-}.$$

We claim that there are other, less trivial ways, to inject the algebra of non-coloured Brauer diagrams into the algebra of coloured Brauer diagrams. For a non-coloured brauer diagram $b^\bullet$, we denote by $s^\circ(b^\bullet)$ the orientation of $b^\bullet$ equal to 1 on the minimum of the cycles of $b^\bullet$. We define next $\Delta(b^\bullet)=\sum_{c\in C(b^\bullet)}\pi\left(\left(b^\bullet,c,s_b^\bullet\right)\right)$.

Proposition 3.22. *There exists an unique morphisms from the algebra of non coloured Brauer diagrams $\mathcal{B}_k^\bullet(p+q)$ into the algebra $\Delta\mathcal{B}_k(p,q)$ equal to Δ on the generators $\{\tau_i,e_i,i\leq k-1\}$ of $\mathcal{B}_k{}^\bullet(p+q)$.*

Attention, Proposition 3.22 does not states that Δ is a morphism. The morphisms of Proposition 3.22 is the unique one that extends the values

$$\Delta(\tau_i) = \tau_i^{++}+\tau_i^{--}+\tau_i^{+-}+\tau_i^{-+},\ \Delta(e_i) = e_i^{++}+e_i^{--}-e_i^{+-}-e_i^{-+}. \tag{3.55}$$

Proof. Let $i\leq k$. To prove the existence of a morphism that takes the same values as Δ on the generators $\tau_j,e_j,j\leq 1$, we should prove that these values satisfy the defining relations (3.54) of the algebra $\mathcal{B}_k^\bullet(p+q)$. We will not perform the needed computations to prove that the relations on the second line of equation (3.54) are satisfied by the images $\Delta(\tau_i)$, e_i, $i\leq k$. Let $s,s'\in\{-,+\}$. We denote by $p_i^{ss'}$ the sum of projectors $p_i^{ss'}=\sum_{\substack{c\in C(1)\\ c(i)=s,c(i')=s'}}(1,c)$. We begin with the first line of (3.54). Let $1\leq i\leq k-1$ an integer.

$$\begin{aligned}
\Delta(\tau_i)^2 &= (\tau_i^{++}+\tau_i^{--}+\tau_i^{+-}+\tau_i^{-+})^2 = \left((\tau_i^{++})^2+(\tau_i^{--})^2+\tau_i{+-}\tau_i^{-+}+\tau_i^{-+}\tau_i^{+-}\right)\\
&= p_i^{++}+p_i^{--}+p_i^{+-}+p_i^{-+} = 1_{\mathcal{B}_k},\\
\Delta(e_i)^2 &= (e_i^{++}+e_i^{--}-e_i^{+-}-e_i^{-+})^2\\
&= (e_i^{++})^2+(e_i^{--})^2-e_i^{-+}e_i^{++}-e_i^{++}e_i^{+-}-e_i^{+-}e_i^{--}-e_i^{--}e_i^{-+}+e_i^{+-}e_i^{-+}+e_i^{-+}e_i^{+-}\\
&= pe_i^{++}+qe_i^{--}-pe_i^{-+}-pe_i^{+-}-qe_i^{+-}-qe_i^{-+}+qe_i^{++}+pe_i^{--}\\
&= (p+q)\Delta(e_i)
\end{aligned}$$

Checking the remaining relations on the first line of equation (3.54) show no difficulties:

$$\begin{aligned}
\Delta(\tau_i)\Delta(e_i) &= (\tau_i^{++}+\tau_i^{--}+\tau_i^{+-}+\tau_i^{-+})(e_i^{++}+e_i^{--}-e_i^{+-}-e_i^{-+})\\
&= (\tau_i^{++}+\tau_i^{--})(e_i^{++}+e_i^{--}-e_i^{+-}-e_i^{-+})\\
&= \tau_i^{++}e_i^{++}-\tau_i^{++}e_i^{+-}-\tau_i^{--}e_i^{-+}+\tau_i^{--}e_i^{--}\\
&= e_i^{++}-e_i^{+-}+e_i^{-+}+e_i^{--}\\
&= \Delta(e_i),\\
\Delta(\tau_i)\Delta(e_i) &= (e_i^{++}+e_i^{--}-e_i^{+-}-e_i^{-+})(\tau_i^{++}+\tau_i^{--}+\tau_i^{+-}+\tau_i^{-+})\\
&= (e_i^{++}+e_i^{--}-e_i^{+-}-e_i^{-+})(\tau_i^{++}+\tau_i^{--})\\
&= e_i^{++}\tau_i^{++}+e_i^{--}\tau_i^{--}-e_i^{+-}\tau_i^{--}-e_i^{-+}\tau_i^{++}\\
&= e_i^{++}+e_i^{--}-e_i^{+-}-e_i^{-+})\\
&= \Delta(e_i).
\end{aligned}$$

We proceed with the verification of the relations in the fourth line of (3.54). We introduce a few more notations to perform the computations. Let $b^{\bullet}$ a Brauer diagram and assume that $b^{\bullet}$ has a cycle of length three and has no other cycles of length strictly greater than one, finally let $l = \{l_1 < l_2 < l_3\}$ be the support of the unique cycle of length three of $b^{\bullet}$. Let $s \in \{-,+\}^3$ and $s' \in \{-,+\}^3$ two sets of signs, we define the Brauer diagram $b_s^{s'} = \sum_{\substack{c \in C(b^{\bullet}) \\ c(l)=s, c(l')=s'}} (b,c)$. With these notations, the computations we perform to check the relations of the fourth line simplify:

$$\begin{aligned}
\Delta(\tau_i)\Delta(e_{i+1})\Delta(e_i) = (\tau_i^{++} + \tau_i^{--} + \tau_i^{+-} + \tau_i^{-+})& \\
(e_{i+1}^{++} + e_{i+1}^{--} - e_{i+1}^{+-} - e_{i+1}^{-+})&(e_i^{++} + e_i^{--} - e_i^{+-} - e_i^{-+}) \\
= (\tau_i^{++} + \tau_i^{--} + \tau_i^{+-} + \tau_i^{-+})&((e_{i+1}e_i)_{+++}^{+++} - (e_{i+1}e_i)_{--+}^{+++} + (e_{i+1}e_i)_{---}^{---} \\
&- (e_{i+1}e_i)_{++-}^{---} - (e_{i+1}e_i)_{---}^{-++} + (e_{i+1}e_i)_{++-}^{-++} - (e_{i+1}e_i)_{+++}^{+--} + (e_{i+1}e_i)_{--+}^{+--} \\
= ((\tau_i e_{i+1}e_i)_{+++}^{+++} &- (\tau_i e_{i+1}e_i)_{--+}^{+++} + (\tau_i e_{i+1}e_i)_{---}^{---} - (\tau_i e_{i+1}e_i)_{++-}^{---} \\
&- (\tau_i e_{i+1}e_i)_{+-+}^{---} - (\tau_i e_{i+1}e_i)_{++-}^{+-+} - (\tau_i e_{i+1}e_i)_{+++}^{-+-} + (\tau_i e_{i+1}e_i)_{--+}^{-+-} \\
= ((\tau_i e_{i+1}e_i)_{+++}^{+++} &- (\tau_{i+1}e_i)_{--+}^{+++} + (\tau_{i+1}e_i)_{---}^{---} - (\tau_{i+1}e_i)_{++-}^{---} \\
&- (\tau_{i+1}e_i)_{---}^{+-+} - (\tau_{i+1}e_i)_{++-}^{+-+} - (\tau_{i+1}e_i)_{+++}^{-+-} + (\tau_{i+1}e_i)_{--+}^{-+-}
\end{aligned}$$

Let $c \in \{+,-\}^{2k}$ a colourization. The function $\varepsilon^{(\tau_{i+1}e_i, c, s^{\bullet}_{(\tau_{i+1}e_i, s)})}$ associated with the oriented colour diagram $(\tau_{i+1}e_i, s)$ satisfies $\varepsilon^{(\tau_{i+1}e_i, c, s^{\bullet}_{(\tau_{i+1}e_i)})}(j) = 1,\ j \neq i+1$ and $\varepsilon^{(\tau_{i+1}e_i, c, s^{\bullet}_{(\tau_{i+1}e_i)})}(i+1) = 1$ if $c(i+1) = c((i+1)')$ and $\varepsilon^{(\tau_{i+1}e_i, c, s^{\bullet}_{(\tau_{i+1}e_i)})} = -1$ if $c(i+1) \neq c((i+1)')$. A simple expansion of the product $\Delta(\tau_{i+1})\Delta(e_{i+1})$ using formula 3.56 leads to the desired relation $\Delta(\tau_i)\Delta(e_{i+1})\Delta(e_i) = \Delta(\tau_{i+1})\Delta(e_i)$. Similar computations are performed to check the second relation:

$$\begin{aligned}
\Delta(e_{i+1})\Delta(e_i)\Delta(\tau_{i+1}) = (e_{i+1}^{++} + e_{i+1}^{--} - e_{i+1}^{+-} - e_{i+1}^{-+})&(e_i^{++} + e_i^{--} - e_i^{+-} - e_i^{-+}) \\
&(\tau_i^{++} + \tau_i^{--} + \tau_i^{+-} + \tau_i^{-+}) \\
= ((e_{i+1}e_i)_{+++}^{+++} - (e_{i+1}e_i)_{--+}^{+++} &+ (e_{i+1}e_i)_{---}^{---} - (e_{i+1}e_i)_{++-}^{---} - (e_{i+1}e_i)_{---}^{-++} \\
+ (e_{i+1}e_i)_{++-}^{-++} - (e_{i+1}e_i)_{+++}^{+--} &+ (e_{i+1}e_i)_{--+}^{+--})(\tau_i^{++} + \tau_i^{--} + \tau_i^{+-} + \tau_i^{-+}) \\
= ((e_{i+1}e_i\tau_{i+1})_{+++}^{+++} - (e_{i+1}e_i\tau_{i+1})_{-+-}^{+++} &+ (e_{i+1}e_i\tau_{i+1})_{+-+}^{-++} - (e_{i+1}e_i\tau_{i+1})_{---}^{-++} \\
+ (e_{i+1}e_i\tau_{i+1})_{---}^{---} - (e_{i+1}e_i\tau_{i+1})_{+-+}^{---} &- (e_{i+1}e_i\tau_{i+1})_{+++}^{+--} + (e_{i+1}e_i\tau_{i+1})_{-+-}^{+--} \\
= ((e_{i+1}\tau_i)_{+++}^{+++} - (e_{i+1}\tau_i)_{--+}^{+++} &+ (e_{i+1}\tau_i)_{+-+}^{-++} - (e_{i+1}\tau_i)_{---}^{-++} \\
+ (e_{i+1}\tau_i)_{---}^{---} - (e_{i+1}\tau_i)_{+-+}^{---} &- (e_{i+1}\tau_i)_{+++}^{+--} + (e_{i+1}\tau_i)_{--+}^{+--}.
\end{aligned}$$

Let c a colourization in $C(\tau_{i+1}e_i)$. The function $\varepsilon^{(e_{i+1}\tau_i, c, s_{(\tau_{i+1}e_i, s)})}$ associated with the oriented colour diagram $(e_{i+1}\tau_i, s)$ satisfies $\varepsilon^{(e_{i+1}\tau_i, c, s^{\bullet}_{(e_{i+1}\tau_i)})}(i) = 1,\ i \neq 2$ and $\varepsilon^{(e_{i+1}\tau_i, c, s^{\bullet}_{(e_{i+1}\tau_i)})}(2) = 1$ if $c(i+2) = c((i+2)')$ and $\varepsilon^{(e_{i+1}\tau_i, c, s^{\bullet}_{(e_{i+1}\tau_i)})} = -1$ if $c(i+2) \neq c((i+2)')$. Once again, it is now simple to check that $\Delta(\tau_i)\Delta(e_{i+1})\Delta(e_i) = \Delta(\tau_{i+1})\Delta(e_i)$ by expanding the right hand side of the last equation. We end the proof the proposition with the verification of the braid relations in line two middle lines of equation (3.54). The first relation show no difficulties, we focus on the two remaining ones. One one hand,

$$\begin{aligned}
&\Delta(e_i)\Delta(e_{i+1})\Delta(e_i) = \\
&\qquad (e_i^{++} + e_i^{--} - e_i^{+-} - e_i^{-+})(e_{i+1}^{++} + e_{i+1}^{--} - e_{i+1}^{+-} - e_{i+1}^{-+})(e_i^{++} + e_i^{--} - e_i^{+-} - e_i^{-+}) \\
&= ((e_ie_{i+1})_{+++}^{+++} - (e_ie_{i+1})_{+--}^{+++} + (e_ie_{i+1})_{---}^{---} - (e_ie_{i+1})_{-++}^{---} - (e_ie_{i+1})_{---}^{++-} + (e_ie_{i+1})_{-++}^{++-} \\
&\qquad - (e_ie_{i+1})_{+++}^{--+} + (e_ie_{i+1})_{+--}^{--+})(e_i^{++} + e_i^{--} - e_i^{+-} - e_i^{-+}) \\
&= (e_ie_{i+1}e_i)_{+++}^{+++} - (e_ie_{i+1}e_i)_{--+}^{+++}(e_ie_{i+1}e_i)_{---}^{---} - (e_ie_{i+1}e_i)_{++-}^{---} \\
&\qquad - (e_ie_{i+1}e_i)_{---}^{++-} + (e_ie_{i+1}e_i)_{++-}^{++-} - (e_ie_{i+1}e_i)_{+++}^{--+} + (e_ie_{i+1}e_i)_{--+}^{--+},
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
&\Delta(e_{i+1})\Delta(e_i)\Delta(e_{i+1}) = \\
&\qquad (e_{i+1}^{++}+e_{i+1}^{--}-e_{i+1}^{+-}-e_{i+1}^{-+})(e_i^{++}+e_i^{--}-e_i^{+-}-e_i^{-+})(e_{i+1}^{++}+e_{i+1}^{--}-e_{i+1}^{+-}-e_{i+1}^{-+}) \\
&= ((e_{i+1}e_i)_{+++}^{+++}-(e_{i+1}e_i)_{--+}^{+++}+(e_{i-1}e_i)_{---}^{---}-(e_{i+1}e_i)_{++-}^{---}-(e_{i+1}e_i)_{---}^{-++}+(e_{i+1}e_i)_{-++}^{++-} \\
&\qquad -(e_{i+1}e_i)_{+++}^{+--}+(e_{i+1}e_i)_{--+}^{+--})(e_{i+1}^{++}+e_{i+1}^{--}-e_{i+1}^{+-}-e_{i+1}^{-+}) \\
&= (e_{i+1}e_ie_{i+1})_{+++}^{+++}-(e_{i+1}e_ie_{i+1})_{--+}^{+++}(e_{i+1}e_ie_{i+1})_{---}^{---}-(e_{i+1}e_ie_{i+1})_{++-}^{---} \\
&\qquad -(e_{i+1}e_ie_{i+1})_{---}^{++-}+(e_{i+1}e_ie_{i+1})_{++-}^{++-}-(e_{i+1}e_ie_{i+1})_{+++}^{--+}+(e_{i+1}e_ie_{i+1})_{--+}^{--+}.
\end{aligned}$$

Thus $\Delta(e_i)\Delta(e_{i+1})\Delta(e_i)=\Delta(e_{i+1})\Delta(e_i)\Delta(e_{i+1})$. The proof is complete. □

We now want to draw the attention of the reader to a point that should be interesting to develop. First, we remark that the choice of an orientation s_b for each brauer diagram $b\in\mathcal{B}_k{}^\bullet$ defines, a map $s:\mathcal{B}_k\mapsto\{-1,1\}^k$ and, secondly, an injection ι^s of $\mathcal{B}_k^\bullet$ into $\mathcal{OB}_k^\bullet$ by setting $\iota^s(b)=(b,s_b)$, $b\in\mathcal{B}_k{}^\bullet$. We call s an *orientation section* of $\mathcal{B}_k$. The map s and its $\mathbb{R}$-linear extension to $\mathbb{R}\left[\mathcal{B}_k^\bullet\right]$ are denoted by the same symbol. Let s an orientation section of $\mathcal{B}_k^\bullet$, we define the map $\Delta^s:\mathbb{R}\left[\mathcal{B}_k^\bullet\right]\to\mathbb{R}\left[\mathcal{B}_k^\bullet\right]$,

$$\Delta^s(b^\bullet)=\sum_{c\in\mathsf{C}(b^\bullet)}\pi((b^\bullet,c,s_b)).$$

Note that if $s=s^\bullet$, $\Delta^s=\Delta$. Now, an interesting question would be to know for which section s the map Δ^s is a morphism. We make a slight digression. So far, we have defined two injection of the algebra of non-coloured brauer diagram, the map Δ' and the map Δ. The map Δ is valued into $\Delta\mathcal{B}_k(p,q)$. There an unique map Δ^s, if at all, that is a morphism since for any orientation section,

$$\Delta^s(\tau_i)=\tau_i^{++}+\tau_i^{--}+\tau_i^{+-}+\tau_i^{-+},\ \Delta^s(e_i)=e_i^{++}+e_i^{--}-e_i^{+-}-e_i^{-+}. \tag{3.56}$$

There is a third injection of the un-coloured Brauer algebra, this is the content of the next proposition.

Proposition 3.23. *With the notation introduced so far, there exists a morphism $\tilde\Delta$ from $\mathcal{B}_k^\bullet(p+q)$ to $\mathcal{B}_k(p,q)$ such that*

$$\tilde\Delta(\tau_i)=(\tau_i^{++}+\tau_i^{--}-\tau_i^{+-}-\tau_i^{-+}),\ \tilde\Delta(e_i)=e_i^{++}+e_i^{--}-e_i^{+-}-e_i^{-+}.$$

Proof. Let $i\leq k-1$. We have to check seven relations. We check the relations on the first line of (3.54). First, one has

$$\tilde\Delta(\tau_i)^2=(\tau_i^{++}+\tau_i^{--}-\tau_i^{+-}-\tau_i^{-+})^2=p_i^{++}+p_i^{--}+p_i^{+-}+p_i^{-+}=1$$

and

$$\begin{aligned}
\tilde\Delta(e_i)\tilde\Delta(\tau_i)&=(\tau_i^{++}+\tau_i^{--}-\tau_i^{+-}-\tau_i^{-+})(e_i^{++}+e_i^{--}-e_i^{+-}-e_i^{-+})\\
&=\tau_i^{++}e_i^{++}+\tau_i^{--}e_i^{--}-\tau_i^{++}e_i^{+-}-\tau_i^{--}e_i^{-+}=\tilde\Delta(e_i)
\end{aligned}$$

We focus on the braid relations on the third line of equation (3.54). First,

$$\begin{aligned}
\tilde\Delta(\tau_i)\tilde\Delta(\tau_{i+1})\tilde\Delta(\tau_i)&=(\tau_i^{++}+\tau_i^{--}-\tau_i^{+-}-\tau_i^{-+})(\tau_{i+1}^{++}+\tau_{i+1}^{--}+-\tau_{i+1}^{+-}-\tau_{i+1}^{-+})\\
&\qquad\times(\tau_i^{++}+\tau_i^{--}-\tau_i^{+-}-\tau_i^{-+})\\
&=(\tau_i\tau_{i+1}\tau_i)_{+++}^{+++}-(\tau_i\tau_{i+1}\tau_i)_{++-}^{-++}+(\tau_i\tau_{i+1}\tau_i)_{--+}^{+--}\\
&\qquad+(\tau_i\tau_{i+1}\tau_i)_{++-}^{-++}+(\tau_i\tau_{i+1}\tau_i)_{-+-}^{-+-}+(\tau_i\tau_{i+1}\tau_i)_{-++}^{++-}\\
&\qquad+(\tau_i\tau_{i+1}\tau_i)_{+-+}^{+-+}+(\tau_i\tau_{i+1}\tau_i)_{+--}^{--+}
\end{aligned}$$

The right hand side of the last equation is manifestly invariant by the substitution $i\to i+1$ and $i+1\to i$ owing to the braid relations. We turn our attention to the relations at the last line of equation (3.54).

$$\begin{aligned}\tilde{\Delta}(\tau_i)\tilde{\Delta}(e_{i+1})\tilde{\Delta}(e_i) &= \tilde{\Delta}(\tau_i)((e_{i+1}e_i)^{+++}_{+++} - (e_{i+1}e_i)^{+++}_{--+} + (e_{i+1}e_i)^{---}_{---} - (e_{i+1}e_i)^{---}_{++-} \\ &\quad - (e_{i+1}e_i)^{-++}_{---} + (e_{i+1}e_i)^{-++}_{++-} - (e_{i+1}e_i)^{+--}_{+++} + (e_{i+1}e_i)^{+--}_{--+}) \\ &= (\tau_{i+1}e_i)^{+++}_{+++} - (\tau_{i+1}e_i)^{+++}_{--+} + (\tau_{i+1}e_i)^{---}_{---} - (\tau_{i+1}e_i)^{---}_{++-} \\ &\quad (\tau_{i+1}e_i)^{+-+}_{---} + (\tau_{i+1}e_i)^{+-+}_{++-} + (\tau_{i+1}e_i)^{-+-}_{+++} - (\tau_{i+1}e_i)^{-+-}_{--+}\end{aligned} \tag{3.57}$$

$$\begin{aligned}\tilde{\Delta}(\tau_{i+1})\tilde{\Delta}(e_i) &= (\tau_{i+1}e_i)^{+++}_{+++} - (\tau_{i+1}e_i)^{+++}_{--+} + (\tau_{i+1}e_i)^{---}_{---} - (\tau_{i+1}e_i)^{---}_{++-} \\ &\quad + (\tau_{i+1}e_i)^{+-+}_{---} - (\tau_{i+1}e_i)^{+-+}_{++-} + (\tau_{i+1}e_i)^{-+-}_{+++} + (\tau_{i+1}e_i)^{-+-}_{--+}\end{aligned} \tag{3.58}$$

We have $\tilde{\Delta}(\tau_{i+1})\tilde{\Delta}(e_i) = \tilde{\Delta}(\tau_i)\tilde{\Delta}(e_{i+1})\tilde{\Delta}(e_i)$. Similar computations show also that $\tilde{\Delta}(e_{i+1})\tilde{\Delta}(\tau_i) = \tilde{\Delta}(e_{i+1})\tilde{\Delta}(e_i)\tilde{\Delta}(\tau_{i+1})$. □

We define two representations of the Brauer algebra $\mathcal{B}^{\bullet}_k(p,q)$ by setting,

$$\rho^{(1)}_{p,q}(b^{\bullet}) = \rho_{p,q} \circ \Delta(b^{\bullet}), \rho^{(2)}_{p,q}(b^{\bullet}) = \rho^{(2)}_{p,q} \circ \tilde{\Delta}(b^{\bullet}).$$

The author doe not know if these two representations are isomorphic and isomorphic to representation $\rho^{\bullet}_{p+q}$.

3.6. Statistics of the split unitary Brownian diffusions

In this section we begin to investigate one the main question of the present chapter, which is the asymptotic behaviour of the non-commutative distribution of the split pseudo-unitary Brownian motions. We define a statistic, that is a function on the algebra Brauer diagrams which range comprise the non-commutative distribution of the split pseudo-unitary Brownian motions. We show that this statistic satisfies a differential equation which generator converges as the dimension tends to infinity. We fix the two speed parameters $v > 0$ and $c > 0$.

3.6.1. Definitions. Let $p \geq 1$ an integer and $k \geq 1$ an other and $\mathbb{K}$ be one of the three division algebras $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$. We set $\beta_{\mathbb{R}} = \beta_{\mathbb{C}} = p$ and $\beta_{\mathbb{H}} = -2p$. To ease the exposition, we denote by $\mathrm{Tr}_{\mathbb{K}}$ the usual matrix trace if $\mathbb{K}$ equal $\mathbb{R}$ or $\mathbb{C}$ and we set $\mathrm{Tr}_{\mathbb{H}} = \mathcal{R}e \circ \mathrm{Tr}$. Define next

$$\mathrm{m}^{\mathbb{K}}_p(b) : \begin{cases} \mathcal{M}_{2p}(\mathbb{K})^{\otimes k} & \to & \mathbb{K} \\ A_1 \otimes \cdots \otimes A_k & \mapsto & \beta_{\mathbb{K}}^{-\mathrm{nc}(b^{\bullet} \vee 1)} \mathrm{Tr}_{\mathbb{K}}\left((A_1 \otimes \cdots \otimes A_k) \circ \rho^{\mathbb{K}}_{p,p}(b)\right). \end{cases}$$

In the sequel, if $A \in \mathcal{M}_{2p}(\mathbb{K})^{\otimes k}$, we use the shorter notation $\mathrm{m}^{\mathbb{K}}_p[A]$ for the function, $b \mapsto \mathrm{m}^{\mathbb{K}}_p(b)(A)$. We denote by $\overline{\mathrm{M}_1}$ the free monoid generated by the set $\{x_1, \bar{x}_1\}$. If $w \in \overline{\mathrm{M}_1}$ is a word and $A \in \mathcal{M}_n(\mathbb{C})$ a complex matrix, the symbol $w^{\otimes}(A^{\otimes k})$ stands for the monomial in $\mathcal{M}_N(\mathbb{C})^{\otimes C}$ obtained by the substitution rule: $x_1 \mapsto A$ and $\overline{x}_1 \mapsto \overline{A}$. If $t \geq 0$ is a time, we use the notation:

$$\mathbb{m}^{\mathbb{K}}_p(b,t) = \mathrm{m}^{\mathbb{K}}_p(b)\left(\mathbb{E}\left[\left(\Lambda^{p,p}_{\mathbb{K}}(t)\right)^{\otimes k}\right]\right), \mathbb{K} = \mathbb{R} \text{ or } \mathbb{H} \text{ and } \mathbb{m}^{\mathbb{C}}_p(b,t) = \mathrm{m}^{\mathbb{C}}_p(b)\left(\mathbb{E}\left[\left(\Lambda^{p,p}_{\mathbb{C}}(t)\right)^{\otimes k}\right]\right).$$

Remember that the Casimir $\mathrm{C}^{\mathbb{K}}_{p,p}$ is a bi-vector in $\mathfrak{u}(p,p) \otimes \mathfrak{u}(p,p)$ related to the quadratic variation of the diffusions $\Lambda^{p,p}_{\mathbb{K}}$ by:

$$\mathrm{d}\left(\Lambda^{p,p}_{\mathbb{K}}\right)^{\otimes k}(t) = \sum_{i=1}^{k} \Lambda^{p,p}_{\mathbb{K}}(t)^{\otimes(i-1)}\left(\mathrm{d}\Lambda^{p,p}_{\mathbb{K}}(t)\right)\Lambda^{p,p}_{\mathbb{K}}(t)^{\otimes(k-i)} + \sum_{1 \leq i < j \leq k} \left(\mathrm{C}^{\mathbb{K}}_{p,p}\right)_{ij} \Lambda^{p,p}_{\mathbb{K}}(t)^{\otimes k},$$

for all time $t \geq 0$. A combinatorial formula for the Casimir $\mathrm{C}^{\mathbb{K}}_{p,p}$ is given in the following proposition. Such a formula exists owing to invariance of the scalar product $B^{\mathbb{K}}_{v,c}$ by the conjugation action of the maximal compact subgroup $\mathbb{U}(p,\mathbb{K}) \times \mathbb{U}(q,\mathbb{K})$ in $\mathbb{U}(p,q,\mathbb{K})$. The following proposition is a downward consequence of formulas (3.30), (3.33), (3.40) and (3.47).

Proposition 3.24. *Let $p \geq 1$ an integer. With the notations introduced so far,*

$$\mathsf{C}^{\mathbb{R}}_{p,q} = \frac{v}{p}\left(\rho^{\mathbb{R}}_{p,p}(\tau^{+-}+\tau^{-+}+e^{+-}+e^{-+})\right)+\frac{c}{p}\left(\rho^{\mathbb{R}}_{p,p}(e^{++}+e^{--}-(\tau^{++}+\tau^{--}))\right)$$
$$\mathsf{C}^{00}_{p,q} = \frac{v}{p}\rho^{\mathbb{C}}_{p,p}(\tau^{-+}+\tau^{+-})-\frac{c}{p}\rho^{\mathbb{C}}_{p,p}(\tau^{++}+\tau^{--}),$$
$$\mathsf{C}^{01}_{p,q} = \frac{v}{p}\rho^{\mathbb{C}}_{p,p}(e^{+,-}+e^{-+})+\frac{c}{p}\rho^{\mathbb{C}}_{p,p}(e^{++}+e^{-,-}),$$
$$\mathsf{C}^{\mathbb{H}}_{p,q} = -\frac{c}{2p}\rho^{\mathbb{H}}_{p,p}(e^{+,+}-\tau^{++}+e^{-,-}-\tau^{--})-\frac{v}{2p}\rho^{\mathbb{H}}_{p,p}(e^{+-}+\tau^{+-}).$$

Proposition 3.25 is a direct corollary of Proposition 3.24 and is obtained by a simple application of the Itô formula. The combinatorial formulae of Proposition 3.25 for mean of tensor polynomials of pseudo-unitary Brownian diffusions are proof's cornerstones of Theorem 3.43.

Proposition 3.25. *Let $p,k \geq 1$ be integers. Let $t > 0$ a time and a word $w \in \overline{\mathsf{M}}_1(k)$,*

$$\mathbb{E}\left[w^{\otimes}\left(\Lambda^{p,p}_{\mathbb{C}}(t)\right)\right] = \exp\left[\frac{v-c}{2}I^{\otimes k}_{2p}\right]$$
$$\times\exp\left[\frac{v}{p}\Big(\sum_{\substack{\tau_{ij}\in\mathsf{T}^{\neq}_{k,2}\\ w_i=w_j}}\tau_{ij}+\sum_{\substack{e_{ij}\in\mathsf{W}^{\neq}_{k,2}\\ w_i\neq w_j}}e_{ij}\Big)+\frac{c}{p}\Big(-\sum_{\substack{\tau_{ij}\in\mathsf{T}^{=}_{k,2}\\ w_i=w_j}}\tau_{ij}+\sum_{\substack{e_{ij}\in\mathsf{W}^{=}_{k,2}\\ w_i\neq w_j}}e_{ij}\Big)\right],$$

$$\mathbb{E}\left[\Lambda^{p,p}_{\mathbb{R}}(t)^{\otimes k}\right] = \exp\left[\left(-\frac{c}{2}\left(\frac{p-1}{p}\right)+\frac{v}{2}\right)I^{\otimes k}_{2p}\right],$$
$$\times\exp\left[\frac{v}{p}\Big(\sum_{\tau_{ij}\in\mathsf{T}^{\neq}_{k,2}}\tau_{ij}+\sum_{e_{ij}\in\mathsf{W}^{\neq}_{k,2}}e_{ij}\Big)+\frac{c}{p}\Big(-\sum_{\tau_{ij}\in\mathsf{T}^{=}_{k,2}}\tau_{ij}+\sum_{e_{ij}\in\mathsf{W}^{=}_{k,2}}e_{ij}\Big)\right]$$

$$\mathbb{E}\left[\Lambda^{p,p}_{\mathbb{H}}(t)^{\otimes k}\right] = \exp\left[\left(-\frac{c(p+3)}{2p}+\frac{v}{2}\right)I_{2p}\right]$$
$$\times\exp\left[-\frac{v}{2p}\Big(\sum_{\tau_{ij}\in\mathsf{T}^{=}_{k,2}}\tau_{ij}+\sum_{e_{ij}\in\mathsf{W}^{=}_{k,2}}e_{ij}\Big)-\frac{c}{2p}\Big(-\sum_{\tau_{ij}\in\mathsf{T}^{\neq}_{k,2}}\tau_{ij}+\sum_{e_{ij}\in\mathsf{W}^{\neq}_{k,2}}e_{ij}\Big)\right].$$

Now, we focus on finding asymptotic of the statistics $\mathfrak{m}^{\mathbb{K}}_p(t)$ Proposition 3.26 implies our main result, Theorem 3.43, as it will be shown in Section 3.7. We first define two linear operators L and $\overline{L}$ acting on, respectively, $\mathcal{B}_k(p,p)$ and $\mathcal{B}_k(p,p)\otimes\mathbb{R}\left[\overline{\mathsf{M}}_1(k)\right]$. Let b be a bicoloured Brauer diagram and w a word in $\overline{\mathsf{M}}_1(k)$, we set

$$L(b) = \frac{v-c}{2}b + c\Big(-\sum_{\tau\in\mathsf{T}^{=,+}_{k,2}(b)}\tau\circ b+\sum_{e\in\mathsf{W}^{=,+}_{k,2}(b)}(e\circ b)\Big)+v\Big(\sum_{\tau\in\mathsf{T}^{\neq,+}_{k}(b)}\tau\circ b+\sum_{e\in\mathsf{W}^{\neq,+}_{k}(b)}e\circ b\Big) \tag{3.59}$$

and

$$\overline{L}(b\otimes w) = \frac{v-c}{2}b + c\Big(-\sum_{\substack{\tau_{ij}\in\mathsf{T}^{=,+}_{k,2}(b)\\ w_i=w_j}}\tau_{ij}\circ b\otimes w+\sum_{\substack{e_{ij}\in\mathsf{W}^{=,+}_{k,2}(b)\\ w_i\neq w_j}}e_{ij}\circ b\otimes w\Big)$$
$$+v\Big(\sum_{\substack{\tau_{ij}\in\mathsf{T}^{\neq,+}_{k}(b)\\ w_i=w_j}}\tau_{ij}\circ b\otimes w+\sum_{\substack{e_{ij}\in\mathsf{W}^{\neq,+}_{k}(b)\\ w_i\neq w_j}}e_{ij}\circ b\otimes w\Big). \tag{3.60}$$

Before stating our main proposition (Proposition 3.26), we relate the two operators L and $\bar{L}$. Let $b \in \mathcal{B}_k$ be a coloured Brauer diagram. Recall that the orientation of b defined by the sign function $s^{\bullet}_b$ has been defined in Section 3.5. A word $w \in \overline{\mathsf{M}}_1(k)$ is said to be compatible with the diagram b if

$$s_b(i)s_b(j) = -1 \Leftrightarrow w_i \neq w_j \text{ for all } 1\leq i,j\leq k. \tag{CR}$$

Denote by $C \subset \mathbb{R}\left[\mathcal{B}_k \times \overline{\mathsf{M}}_1(k)\right]$ the real linear span of the set of compatible pairs of word and diagrams in $\overline{\mathsf{M}}_k \times \mathcal{B}_k$. As proved in Section 3.5,

$$\forall b \in \mathcal{B}_k,\ \forall e \in \mathsf{W}_k^+(b),\ \forall 1 \leq i,j \leq k,\ i \sim_{b\circ e} j,\ s_{b\circ e}(i)s_{b\circ e}(j) = -1 \Leftrightarrow s_b(i)s_b(j) = -1$$
$$\forall b \in \mathcal{B}_k,\ \forall \tau \in \mathsf{T}_k^+(b),\ \forall 1 \leq i,j \leq k,\ i \sim_{b\circ \tau} j,\ s_{b\circ \tau}(i)s_{b\circ \tau}(j) = -1 \Leftrightarrow s_b(i)s_b(j) = -1.$$

These last two equations implies that the operator $\overline{L}$ stabilizes the vector space C. In addition,

$$\overline{L}(b\otimes w) = L(b)\otimes w, \text{ with } (b,w) \in C. \tag{3.61}$$

3.6.2. Convergence in high dimensions. We now state our main proposition. Let δ_Δ be the support function of the set of diagonal bi-coloured Brauer diagrams (the bi-coloured diagrams (b,C) such that $c(i) = c(i')$, $1 \leq i \leq k$).

Proposition 3.26. *Let $t \geq 0$ be a time. As the dimension $p \to +\infty$, the statistics $\mathfrak{m}_p^{\mathbb{R}}(t)$ and $\mathfrak{m}_p^{\mathbb{H}}(t)$ converge pointwise on $\mathcal{B}_k(p,p)$ to the same limit which we denote $\mathfrak{m}(t)$. In addition, $t \mapsto \mathfrak{m}(t)$ is the solution of the differential equation:*

$$\frac{d}{dt}\mathfrak{m}(t,b) = \mathfrak{m}(t)(L(b)),\ \mathfrak{m}(0) = \delta_\Delta.$$

As the dimension $p \longrightarrow +\infty$, the statistic $\mathfrak{m}_p^{\mathbb{C}}$ converges pointwise on $\mathbb{R}\left[\overline{\mathsf{M}}_1(k)\right]\otimes \mathcal{B}_k(p,p)$. The limit $\overline{\mathfrak{m}}$ is the solution of the differential equation

$$\frac{d}{dt}\overline{\mathfrak{m}}(t,b\otimes w) = \mathfrak{m}(t)(\overline{L}(b\otimes w)),\ \mathfrak{m}(0) = \delta_\Delta.$$

Proof. Let $p \geq 1$ an integer. We begin with the real case. First, we find a differential system satisfied by the statistic $\mathfrak{m}_p^{\mathbb{R}}$. Let $t \geq 0$ be a time and $b \in \mathcal{B}_k$ a Brauer diagram. By using formulas of Proposition 3.25 for the mean of tensor polynomials in the pseudo-orthogonal diffusions and the formula 3.98, we get

$$\begin{aligned}\frac{d}{dt}\mathfrak{m}_p^{\mathbb{R}}(b,t) = \frac{k}{2}&\left(-c\left(\frac{p-1}{p}\right)+v\right)\mathfrak{m}_p^{\mathbb{R}}(b,t)\\ &+\frac{v}{p}\left(\sum_{\tau\in\mathsf{T}_{k,2}^{\neq}} p^{\mathsf{nc}(\tau^\bullet\vee b^\bullet)-\mathsf{nc}(b^\bullet)}\mathfrak{m}_p^{\mathbb{R}}(\tau\circ b,t)+\sum_{e\in\mathsf{W}_{k,2}^{\neq}} p^{\mathsf{nc}(e^\bullet\vee b^\bullet)-\mathsf{nc}(b^\bullet)}\mathfrak{m}_p^{\mathbb{R}}(e\circ b,t)\right)\\ &+\frac{c}{p}\left(-\sum_{\tau\in\mathsf{T}_{k,2}^{=}} p^{\mathsf{nc}(\tau^\bullet\vee b^\bullet)-\mathsf{nc}(b^\bullet)}\mathfrak{m}_p^{\mathbb{R}}(\tau\circ b,t)+\sum_{e\in\mathsf{W}_{k,2}^{=}} p^{\mathsf{nc}(e^\bullet\vee b^\bullet)-\mathsf{nc}(b^\bullet)}\mathfrak{m}_p^{\mathbb{R}}(e\circ b,t)\right).\end{aligned}$$

We denote by $L_p^{\mathbb{R}}$ the operator on acting on $\mathcal{B}_k(p,p)$ such that $\frac{d}{dt}\mathfrak{m}^{\mathbb{R}}(b,t) = \mathfrak{m}_p^{\mathbb{R}}(L_p^{\mathbb{R}}(b),t)$. In Section 3.5, we proved that $\mathsf{nc}(\tau^\bullet\vee b^\bullet)-\mathsf{nc}(b^\bullet) \in \{-1,0,1\}$, $b \in \mathcal{B}_k$, $\tau \in \mathsf{T}_k$. The same property holds if in place of a transposition τ we have a projector e. Hence, if we let p tends to $+\infty$, the generator $L_p^{\mathbb{R}}$ converges pointwise to L. Since $\mathcal{B}_k(p,q)$ is finite dimensional, it implies the convergence of the solution:

$$\mathfrak{m}_p^{\mathbb{R}}(t,b) = \exp(tL_p^{\mathbb{R}})(\delta_\Delta)(b) \underset{p\to+\infty}{\longrightarrow} \exp(tL)(\delta_\Delta)(b) = \mathfrak{m}(b,t),\ b \in \mathcal{B}_k.$$

The complex case does not show extra difficulties. Let $w \in \overline{\mathsf{M}}_1(k)$ a word. If We apply, once again, the formulas in Proposition 3.25, to get:

$$\begin{aligned}\frac{d}{dt}\mathfrak{m}_p^{\mathbb{C}}(b\otimes w,t) = \frac{k}{2}&(v-c)\,\mathfrak{m}_p^{\mathbb{C}}(b\otimes w,t)\\ +\frac{v}{p}&\left(\sum_{\substack{\tau_{ij}\in\mathsf{T}_{k,2}^{\neq}\\ w_i=w_j}} p^{\mathsf{nc}(\tau_{ij}^\bullet\vee b^\bullet)-\mathsf{nc}(b^\bullet)}\mathfrak{m}_p^{\mathbb{C}}(\tau_{ij}\circ b\otimes w,t)+\sum_{\substack{e_{ij}\in\mathsf{W}_{k,2}^{\neq}\\ w_i\neq w_j}} p^{\mathsf{nc}(e_{ij}^\bullet\vee b^\bullet)-\mathsf{nc}(b^\bullet)}\mathfrak{m}_p^{\mathbb{C}}(e_{ij}\circ b\otimes w,t)\right)\\ +\frac{c}{p}&\left(-\sum_{\substack{\tau_{ij}\in\mathsf{T}_{k,2}^{=}\\ w_i=w_j}} p^{\mathsf{nc}(\tau_{ij}^\bullet\vee b^\bullet)-\mathsf{nc}(b^\bullet)}\mathfrak{m}_p^{\mathbb{C}}(\tau_{ij}\circ b\otimes w,t)+\sum_{\substack{e_{ij}\in\mathsf{W}_{k,2}^{=}\\ w_i\neq w_j}} p^{\mathsf{nc}(e_{ij}^\bullet\vee b^\bullet)-\mathsf{nc}(b^\bullet)}\mathfrak{m}_p^{\mathbb{C}}(e_{ij}\circ b\otimes w,t)\right).\end{aligned}$$

If we define the operator $L_p^{\mathbb{C}}$ acting on $\mathcal{B}_k(p)\oplus\mathbb{R}\left[\overline{\mathsf{M}}_1(k)\right]$ by the equation $\frac{d}{dt}\mathbb{m}^{\mathbb{C}}(b\otimes w,t)=\mathbb{m}_p^{\mathbb{C}}(L^{\mathbb{C}}(b,w),t)$, $b\in\mathcal{B}_k$, $w\in\mathsf{M}_k, t\geq 0$, the same argument used for the real case shows that as $p\to+\infty$, the operator $L_p^{\mathbb{C}}$ converges pointwise to the operator $\overline{L}$ defined in equation (3.60). □

3.7. Pseudo-unitary dual groups and free split pseudo Brownian diffusions

In this section we define pseudo-unitary versions of the Voiculescu dual unitary groups. This dual groups are Zhang algebras in the category of involutive probability spaces, the are isomorphic to the Voiculescu dual groups as algebra, but not as involtuve algebras. The pseudo-unitary dual groups are the structure groups from the finite dimensional processes associated with the pseudo-unitary Brownian motions we defined earlier. In this section we define also the process that we name pseudo-untiary free Brownian motion which is, given speed paramaters $v,c>0$, a free Levy process. Infinitiesimal versions of these dual groups are also considered, that we name pseudo-antihermitien dual groups.

3.7.1. Pseudo-unitary and pseudo-antihermitien dual groups. In this section, we introduce pseudo-antihermitien and pseudo-Unitary dual groups.

3.7.1.1. *Free products and free Zhang algebras.* The reader is directed to Chapter 1 for a detailed overview on free product of algebras and the notion of Zhang algebras. Let us recall some basic definitions. If A and B are two algebras, a word in A and B is an expression of the form $w_1w_2\cdots w_n$ with $w_i\in A\cup B$ where each a_i is either an element of A or an element of B. Such a word may be reduced using the following operations:

- Remove an instance of the identity element (of either A or B).
- Replace a pair of the form a_1a_2, $a_i\in A$ by its product in A, or a pair b_1b_2, $b_i\in B$ by its product in B.

Every reduced word is an alternating product of elements of A and elements of B,

$$a_1b_1a_2b_2\cdots a_kb_k,\text{ with } a_i\in A \text{ and } b_i\in B,\ 1\leq i\leq k.$$

The free product $A\sqcup B$ is the algebra whose elements are linear combinations of the reduced words in A and B under the operation of concatenation followed by reduction. There exists two canonical injections ι_A and ι_B of the algebras A and B into their free products. The free product of A and B enjoys a universal property. In fact, if C is a third algebra and a (resp. b) is a morphism from A (resp. B) to C, then there exits a unique morphism $a\sqcup b$ from the free product $A\sqcup B$ with values in C such that $(a\sqcup b)\circ\iota_A=a$ and $(a\sqcup b)\circ\iota_B=b$. We will sometimes consider the free product of an algebra A with itself. In this case, we use the notation $a_{|1}$ (respectively $a_{|2}$) for the elements of the first copy of A in $A\sqcup A$ (respectively, for the second copy)

In the sequel, $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. An involutive Zhang algebra over $\mathbb{K}$ is an unital involutive associative algebra with unit $(\mathcal{A},\eta)$, $(\eta:\mathbb{K}\to A)$ along with three structural morphisms

$$S:\mathcal{A}\to\mathcal{A}\text{ (Antipode)},\ \Delta:\mathcal{A}\to\mathcal{A}\sqcup\mathcal{A}\text{ (Coproduct)},\ \varepsilon:\mathcal{A}\longrightarrow\mathbb{K}\text{ (Counit)}.$$

satisfying the the four relations:

$$(\varepsilon\sqcup 1)\circ\Delta(x)=x,\ (1\sqcup\varepsilon)\circ\Delta(x)=x,$$
$$(S\sqcup 1)\circ\Delta(x)=\varepsilon(x)\mathbf{1},\ (1\sqcup S)\circ\Delta(x)=\varepsilon(x)\mathbf{1},\ x\in\mathcal{A}.$$

where we have written $\mathbf{1}=\eta(1)$. Below we define the pseudo-unitary group, which is a Zhang algebra, let us however present a more elementary example. The algebra $\mathbb{C}\langle X_1,\ldots,X_n\rangle$ of all polynomials with complex coefficients and non-commuting indeterminates $X_1,\ldots,X_n$ is a Zhang algebra if equipped with the coproduct, co-unit and antipode defined by

$$\varepsilon(X_i)=0,\ \varepsilon(1)=1,\ \Delta(X_i)=X_i|_1+X_i|_2,\ S(X_i)=-X_i.$$

The Zhang algebra $(\mathbb{C}\langle X_1,\ldots,X_n\rangle,\ \varepsilon,\ \Delta,\ S)$ is a free version of the unshuffle bi-algebra, see [27].

3.7.1.2. *pseudo-antihermitien free Zhang algebra.* We denote by $\mathfrak{h}(-,+)$ the free complex unital algebra generated by four elements $h_+, h_-, h_{+-}, h^\star_{+-}$. The algebra $\mathfrak{h}(+,-)$ is endowed with the following involutive structure:

$$\star(h_+) = -h_+,\ \star(h_-) = -h_-,\ \star(h_{+-}) = h^\star_{+-}.$$

We refer to the product $\star$ as the *transposition*. In addition to the anti-morphism $\star$, a second involutive anti-morphism on the algebra $\mathfrak{h}(-,+)$ is defined by setting

$$\star'(h_+) = -h_+,\ \star'(h_-) = -h_-,\ \star'(h_{+-}) = -h^\star_{+-}.$$

A concise way to define the involution $\star'$ would have been:

$$\begin{bmatrix} h_+ & h^\star_{+-} \\ h_{+,-} & h_- \end{bmatrix} + \left(\star' \otimes {}^t\right)\left(\begin{bmatrix} h_+ & h^\star_{+-} \\ h_{+,-} & h_- \end{bmatrix}\right) = 0.$$

We define now a coproduct and an antipode that turn $\mathfrak{h}(-,+)$ into a free involutive Zhang algebra. Let $S : \mathfrak{h}(-,+) \to \mathfrak{h}(-,+)$ and $\Delta : \mathfrak{h}(-,+) \to \mathfrak{h}(-,+) \sqcup \mathfrak{h}(-,+)$ be the unique $\star$-morphisms that take the following values on the generators:

$$S(h_+) = -h_+,\ S(h_-) = -h_-,\ S(h_{+,-}) = -h_{+,-},$$

and

$$\Delta(h_+) = h_+|_1 + h_+|_2,\ \Delta(h_-) = h_-|_1 + h_+|_2,\ \Delta(h_{+,-}) = h_{+-}|_1 + h_{+-}|_2.$$

The counit $\varepsilon : \mathfrak{h} \to \mathbb{C}$ is the unique $\star$-morphism such that

$$\varepsilon(h_+) = \epsilon(h_-) = \varepsilon(h_{+-}) = \varepsilon(h^\star_{+-}) = 0.$$

It is easy to check the defining relations of a Zhang algebra for the three morphisms S, Δ, ε.

3.7.1.3. *Pseudo-unitary free Zhang algebra.* The pseudo-unitary dual group defined in this section integrates, in a vague meaning, the Zhang algebra $\mathfrak{h}(-,+)$.

Definition 3.27 (The pseudo-orthogonal dual group). We denote by $(\mathcal{O}(-,+),\eta)$ the real associative algebra with unit η generated by eight elements $o_{++}, o_{--}, o_{+-}, o_{-+}, o^t_{++}, o^t_{--}, o^t_{+-}, o^t_{-+}$ subject to the relations

$$\begin{bmatrix} o_{++} & o_{+-} \\ o_{-+} & o_{--} \end{bmatrix} \cdot \begin{bmatrix} o^t_{++} & -o^t_{-+} \\ -o^t_{+-} & o^t_{--} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \begin{bmatrix} o^t_{++} & -o^t_{-+} \\ -o^t_{+-} & o^t_{--} \end{bmatrix} \cdot \begin{bmatrix} o_{++} & o_{+-} \\ o_{-+} & o_{--} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.62}$$

To define a morphism ϕ from the pseudo-unitary dual group $\mathcal{O}(-,+)$ into an algebra C by specifying its values on the generators of $\mathcal{O}\langle n\rangle$, we use the compact notation:

$$\phi\left(\begin{bmatrix} o_{++} & o_{+-} \\ o_{-+} & o_{--} \end{bmatrix}\right) = \begin{bmatrix} \phi(o_{++}) & \phi(o_{+-}) \\ \phi(o_{-+}) & \phi(o_{--}) \end{bmatrix},\ \phi\left(\begin{bmatrix} o^t_{++} & o^t_{+-} \\ o^t_{-+} & o^t_{--} \end{bmatrix}\right) = \begin{bmatrix} \phi(o^t_{++}) & \phi(o^t_{+-}) \\ \phi(o^t_{-+}) & \phi(o^t_{--}) \end{bmatrix}$$

and check that

$$\begin{bmatrix} \phi(o_{++}) & \phi(o_{+-}) \\ \phi(o_{-+}) & \phi(o_{--}) \end{bmatrix} (\star' \otimes {}^t) \begin{bmatrix} \phi(o_{++}) & \phi(o_{+-}) \\ \phi(o_{-+}) & \phi(o_{--}) \end{bmatrix} = (\star' \otimes {}^t) \begin{bmatrix} \phi(o_{++}) & \phi(o_{+-}) \\ \phi(o_{-+}) & \phi(o_{--}) \end{bmatrix} \begin{bmatrix} \phi(o_{++}) & \phi(o_{+-}) \\ \phi(o_{-+}) & \phi(o_{--}) \end{bmatrix} = I_2.$$

In the sequel, we put

$$O = \begin{bmatrix} o_{++} & o_{+-} \\ o_{-+} & o_{--} \end{bmatrix},\ O^t = \begin{bmatrix} o^t_{++} & o^t_{+-} \\ o^t_{-+} & o^t_{--} \end{bmatrix},\ I_{+,-} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathcal{M}_2(\mathcal{O}(-,+)).$$

For example, the equation $\Delta(O) = O_{|1} O_{|2}$ in the forthcoming proposition should be read

$$\begin{bmatrix} \Delta(o_{++}) & \Delta(o_{+-}) \\ \Delta(o_{-+}) & \Delta(o_{--}) \end{bmatrix} = \begin{bmatrix} o_{++|1} o_{++|2} + o_{+-|1} o_{-+|2} & o_{++|1} o_{+-|2} + o_{+-|1} o_{--|2} \\ o_{-+|1} o_{++|2} + o_{--|1} o_{-+|2} & o_{-+|1} o_{+-|2} + o_{--|1} o_{--|2} \end{bmatrix}.$$

We define the three morphisms $\Delta : \mathcal{O}(-,+) \to \mathcal{O}(-,+) \sqcup \mathcal{O}(-,+)$, $S : \mathcal{O}(-,+) \to \mathcal{O}(-,+)$ and $\varepsilon : \mathbb{R} \to \mathcal{O}(-,+)$ as the unique morphisms that takes the following values on the generators

$$\begin{aligned} &\Delta(O) = O_{|1} O_{|2},\ \Delta(O^t) = O^t_{|2} O^t_{|1}, \\ &S(O) = I_{-,+} O^t I_{-,+},\ S(O^t) = I_{-,+} O I_{-,+}, \\ &\varepsilon(O) = I_2,\ \varepsilon(O^t) = I_2. \end{aligned} \tag{3.63}$$

Proposition 3.28. *$(\mathcal{O}(-,+),\eta,\Delta,\varepsilon,S)$ is a Zhang algebra. In addition there exists an unique involutive anti-linear anti-morphism $\star:\mathcal{O}(-,+)\to\mathcal{O}(-,+)$ such that*

$$(\ {}^{t}\otimes\star)O=\begin{bmatrix} o^{\star}_{++} & o^{\star}_{-+}\\ o^{\star}_{+-} & o^{\star}_{--}\end{bmatrix}=\begin{bmatrix} o^{t}_{++} & o^{t}_{-+}\\ o^{t}_{+-} & o^{t}_{--}\end{bmatrix}.$$

Proof. Let us check that the prescribed values (3.63) for the coproduct Δ, the antipode S and the counit ε on the generators of $\mathcal{U}(-,+)$ do define morphisms. First, we write the relation 3.62 in the more compact form:

$$OS(O)=S(O)O=1, S(O)=O^{-1}.$$

The equation (3.63) defines morphisms S,Δ,ε on the free algebra $\mathcal{F}(8)$ generated by $\{o_{\varepsilon,\varepsilon'},o^{t}_{\varepsilon,\varepsilon'},\varepsilon,\varepsilon'\in\{-,+\}\}$. To show that this morphisms factorize over the quotient $\mathcal{F}(8)/I$ with I the ideal generated by the relations (3.62), we prove that this ideal is stable by the three morphisms S,Δ,ε.

$$\Delta(S(O))=\Delta\left(\begin{bmatrix} o_{++} & -o_{-+}\\ -o_{+-} & o_{--}\end{bmatrix}\right)=\begin{bmatrix} o_{++}|_1 o_{++}|_2+o_{+-}|_1 o_{-+}|_2 & -o_{-+}|_1 o_{++}|_2-o_{--}|_1 o_{-+}|_2\\ -o_{++}|_1 o_{+-}|_2-o_{+-}|_1 o_{--}|_2 & o_{-+}|_1 o_{+-}|_2+o_{--}|_1 o_{--}|_2\end{bmatrix}$$

Thus $\Delta(S(O))=m\circ(S\sqcup S)(\Delta O)$ with m the unique morphism $A\sqcup A\to A\sqcup A$ equal to the identity on A, thus

$$\Delta(O)\Delta(S(O))=O_{|1}O_{|2}S(O)_{|2}S(O)_{|1}=1,\ \Delta(S(O))\Delta(O)=S(O)_{|1}S(O)_{|2}O_{|2}O_{|1}=1$$

Thus the first equation in 3.63 defines a morphism Δ. For the antipode S and the counit ε, it is straightforward to prove that the defining relations in (3.63) are compatibles with. □

On the pseudo unitary dual group $\mathcal{O}(-,+)$, in addition to the involution $\star$ defined in Definition 3.28, there exists a second involutive anti-morphism $\star':\mathcal{O}(-,+)\to\mathcal{O}(-,+)$ that takes the following values on the generators O:

$$o^{\star'}_{+-}=-o^{t}_{+-},\ o^{\star'}_{-+}=-o^{t}_{-+},\ o^{\star'}_{++}=o^{t}_{++},\ o^{\star'}_{--}=o^{t}_{--}$$

Hence the defining relations of the algebra $\mathcal{O}(-,+)$ are rephrased using the operator $\star'$ by

$$\left(\star'\otimes{}^{t}\right)(O)\cdot O=I_2,\ O\cdot\left(\star'\otimes{}^{t}\right)(O)=I_2,\ S(O)=O^{\star}$$

We defined the algebra $\mathcal{O}(-,+)$ as a real algebra, in commutative probability the structure algebras are usually complex algebras. In the sequel, random variables that are defined take value in either real or complex vector space. Of course, for the random variables that take values in a real vector space we can consider them as begin valued in the complexification of their target space. However, we prefer not to extend scalars if it is not needed. For complex valued random variables, we may consider them as being defined over the complexification $\mathcal{U}(-,+)=\mathcal{O}(-,+)\otimes\mathbb{C}$ of the Zhang algebra $\mathcal{U}(-,+)$. We call $\mathcal{U}(-,+)$ the pseudo-unitary dual algebra. The antipode, coproduct and counit are extended as complex linear maps and the two involution $\star$ and $\star'$ are extended as anti-linear maps.

Let us draw a comparison between the algebra $\mathcal{U}(-,+)$ and the celebrated unitary Voiculescu dual group $\mathcal{U}(2)$. The Voiculescu unitary dual group $\mathcal{U}(2)$ is the real or complex unitary algebra generated by eight elements $o_{i,j}$, $1\le i\le 2$, $1\le j\le 2$ and $u^{\star}_{ij}$, $1\le i\le 2$, $1\le j\le 2$. The algebra $\mathcal{U}(2)$ is turned into an involutive algebra if we define an anti-linear involutive anti-morphism $\star$ on $\mathcal{U}(2)$ by setting $(o_{ij})^{\star}=u^{\star}_{ji}$. The Zhang algebra $\mathcal{U}(-,+)$ is isomorphic to the Voiculescu dual group $\mathcal{U}(2)$ but these two last algebras are not isomorphic as involutive bi-algebras if $\mathcal{U}(-,+)$ is equipped with the involutive anti-morphism $\star$. All the random variables on $\mathcal{U}(-,+)$ that are considered in the sequel are random variables with respect to the anti-morphism $\star$ and their distribution define $\star$-positive linear forms on $\mathcal{U}(-,+)$.

We relate now the pseudo-unitary dual groups and the pseudo-antihermitien dual groups. To that end, we introduce a formal variable h and consider a formal version of the pseudo-antihermitien dual groups. We set

$$H=\begin{bmatrix} h_{+} & h^{\star}_{+-}\\ h_{+,-} & h_{-}\end{bmatrix},\ O^{h}=\exp(hH)\in(\mathfrak{h}(-,+)\otimes\mathcal{M}_2(\mathbb{C}))[[h]]\simeq\mathcal{M}_2(\mathfrak{h}(-,+)[[h]]).$$

We extend the involutive anti-morphisms $\star$ and $\star'$ as anti-morphisms on $\mathfrak{h}(-,+)[[h]]$ by setting $h^\star = h^{\star'} = h$.

Proposition 3.29. *The real $\star$-algebra generated by the components of $O^{\mathfrak{h}}$ is isomorphic to $\mathcal{U}(-,+)$.*

Proof. The exponential map $\exp : \mathcal{M}_2(\mathfrak{h}(-,+)[[h]]) \to \mathcal{M}_2(\mathfrak{h}(-,+)[[h]])$ is $\star' \otimes {}^t$ morphism. Also, $(\star' \otimes {}^t)(O^h) = I_{-+}(\star \otimes {}^t)O^h I_{-+}$. In addition,

$$O^h \cdot \left(\star' \otimes {}^t\right)\left(O^h\right) = \exp(hH)\exp\left(h\left(\star' \otimes {}^t\right)(H)\right) = \exp(hH)\exp(-hH) = I_2.$$

□

3.7.2. The limiting free Lévy processes. In this subsection, we define a two parameters family of free Lévy processes, which we name *free split pseudo-unitary Brownian motions*, and denoted $(V_{v,c})_{v>0,c>0}$ on the pseudo-unitary dual group $\mathcal{U}(-,+)$ as solutions free stochastic differential equations driven by *free pseudo-antihermitien Brownian motions.* Let $v > 0, c > 0$ be two positive real numbers. In the sequel, we drop the superscripts v, c of all notations.

Let $x = (x_t)_{t\geq 0}$ be a circular Brownian motion in a von Neumann algebra with state $(\mathcal{A}, \tau)$. Let $a = (a_t)_{t\geq 0}$ and $b = (b_t)_{t\geq 0}$ be two semi-circular Brownian motions in $(\mathcal{A}, \tau)$ such that $\{a, b, x\}$ is a family of free processes. Defin

$$\mathsf{W}(t) = \sqrt{c}\begin{bmatrix} \mathrm{i}a_t & 0 \\ 0 & \mathrm{i}b_t \end{bmatrix} + \sqrt{v}\begin{bmatrix} 0 & x_t \\ x_t^\star & 0 \end{bmatrix} \in \mathcal{A} \otimes \mathsf{M}_2(\mathbb{C}). \tag{3.64}$$

For each $t \geq 0$, to $\mathsf{W}(t)$ is associated a random variable $W(t) : \mathfrak{h}(-,+) \to \mathcal{A}$,

$$\begin{array}{rccc} W(t) : & \mathfrak{h}(-,+) & \to & \mathcal{A} \\ & \begin{bmatrix} h_+ & h_{+-}^\star \\ h_{+,-} & h_- \end{bmatrix} & \mapsto & \mathsf{W}(t) \end{array} \tag{3.65}$$

The non-commutative process W is named the *free pseudo-antihermitien Brownian motion*. We prove in the sequel that this process is the limit in high dimensions of the complex pseudo-unitary Brownian motion we defined earlier with same speed parameters v, c on the Lie algebra $\mathfrak{u}(p,p)$ of pseudo-antihermitien matrices. Consider now the free stochastic differential system

$$\begin{aligned} &\mathrm{d}\mathsf{V}(t) = \mathsf{V}(t)\mathrm{d}\mathsf{W}(t) + \frac{1}{2}(v-c)\mathsf{V}(t)\mathrm{d}t \\ &\mathsf{V}(0) = I_2, \end{aligned} \tag{3.66}$$

with $\mathsf{V}(t) \in \mathcal{A} \otimes \mathcal{M}_2(\mathbb{C})$, and set

$$\mathsf{V}(t) = \begin{bmatrix} \mathsf{V}(t)_{++} & \mathsf{V}(t)_{-,+} \\ \mathsf{V}(t)_{+-} & \mathsf{V}(t)_{--} \end{bmatrix}.$$

Albeit the entries of the matricial process W are processes with free increments, it does not implies that W is a free process in $(\mathcal{M}_2(\mathbb{C}) \otimes \mathcal{A}, \mathbb{E} \otimes \tau)$. It is true however that V is a process with free invcrements in the amalgamated space $\mathcal{M}_2(\mathbb{A}$ with a $\mathcal{M}_2(\mathbb{C})$ bimodule structure given by left and right multiplication. The conditional expectation is:

$$E((m_{i,j})_{1\leq i,j\leq 2}) = (\tau(m_{i,j}))_{1\leq i,j\leq 2}$$

Let us very briefly recall how such a free stochastic equation can be solved. Let us define the sequence $\left(\mathsf{V}^{(n)}\right)_{n\geq 0}$ of processes via Picard's iterative procedure, for each $t \geq 0$,

$$\begin{cases} V^{(0)}(t) = I_2, \\ \mathsf{V}^{(n+1)}(t) = I_2 + \displaystyle\int_0^t \mathsf{V}^{(n)}(u)\mathrm{d}\mathsf{W}(u) + \frac{1}{2}(v-c)\int_0^t \mathsf{V}^{(n)}(u)\mathrm{d}u. \end{cases}$$

For each $t \geq 0$, we let I_t be the operator acting on $\mathcal{A} \otimes \mathcal{M}_2(\mathbb{C})$ such that $\mathsf{V}^{(n+1)}(t) = I_t\left(\mathsf{V}^{(n)}\right)$. Let $1 \leq i, j \leq 2$. With the definitions introduced in 3.2.2 we have the following estimate

$$\begin{aligned} \left\|I_t(V)_{ij} - I_t(W)_{ij}\right\|_{L^2(\tau)} &\leq t \max_{1\leq k,l\leq 2}\left\|V_{k,l} - W_{k,l}\right\|_{L^2(\tau)} + \frac{t}{2}|(v-c)| \max_{1\leq k,l\leq 2}\left\|V_{k,l} - W_{k,l}\right\|_{L^2(\tau)} \\ &\leq A_t \max_{1\leq k,l\leq 2}\left\|V_{k,l} - W_{k,l}\right\|_{L^2(\tau)} \end{aligned}$$

with $A_t = t + \frac{t}{2}\max_{1\le k,l\le 2}|v-c|$. Now pick $\delta \in [0,1]$ such that $A_t < 1$ for all $t \le \delta$. For each $t \in [0,\delta]$ the operator I_t is a contraction, by using the Picard's fixed point theorem this ensures the existence of $V(t)$ for every time $t \in [0,\delta]$. Besides, the following uniform estimates holds

$$\sup_{t\in[0,\delta]} \|I_t(V) - I_t(W)\| \le A_\delta \max_{1\le i,j\le 2} \left\|V_{ij} - W_{ij}\right\|_{L^2(\tau)}.$$

The fixed point theorem of Picard can thus be applied for the operator $I : \mathcal{B}_2^a \mapsto C^0([0,\delta],\mathcal{A}), I = (I_t)_{t\le\delta}$ to obtain a continuous process solution of the equation 3.66. We remark that δ does not depend on the initial condition V(0). Hence the Picard's iteration procedure can be applied to extend the solution initially defined on $[0,\delta]$ to the interval $[0,2\delta]$ by solving the system

$$\begin{cases} \mathsf{V}^{(0)}(t) = \mathsf{V}(\delta), \\ \mathsf{V}^{(n+1)}(t) = I_2 + \displaystyle\int_0^t \mathsf{V}^{(n)}(u)\mathrm{d}\mathsf{W}(u) + \frac{1}{2}(v-c)\int_0^t \mathsf{V}^{(n)}(u)\mathrm{d}u. \end{cases}$$

for all $0 \le t \le \delta$. If T is defined as the larger time for which the solution of the system (3.66) is defined on $[0,T[$, the latter reasoning prove that $T = 1$. We now prove that for each $t \ge 0$, the four components of the matrix $\mathsf{V}(t)$ satisfy for each time $t \ge 0$ the defining relations of the pseudo-unitary dual group and hence V is well defined as a morphism from $\mathcal{U}(-,+)$ to $\mathcal{A}$.

To show that there exists a process $V = (V(t))_{t\ge 0}$ on $\mathcal{U}(-,+)$ such that $V(t)(o_{\varepsilon,\varepsilon'}) = \mathsf{V}(t)_{\varepsilon,\varepsilon'}$, $\varepsilon,\varepsilon' \in \{-,+\}$, we prove that the four components of the matrix $\mathsf{V}(t)$ satisfy the defining relations of the pseudo-unitary group for all positive time $t \ge 0$.

Lemma 3.30. *With the notation introduced so far*

$$\begin{bmatrix} \mathsf{V}_{++}(t)^{\star'} & \mathsf{V}_{-+}(t)^{\star'} \\ \mathsf{V}_{-+}(t)^{\star'} & \mathsf{V}_{--}(t)^{\star'} \end{bmatrix}\begin{bmatrix} \mathsf{V}(t)_{++} & \mathsf{V}(t)_{-,+} \\ \mathsf{V}(t)_{+-} & \mathsf{V}(t)_{--} \end{bmatrix} = \begin{bmatrix} \mathsf{V}(t)_{++} & \mathsf{V}(t)_{-,+} \\ \mathsf{V}(t)_{+-} & \mathsf{V}(t)_{--} \end{bmatrix}\begin{bmatrix} \mathsf{V}_{++}(t)^{\star'} & \mathsf{V}_{-+}(t)^{\star'} \\ \mathsf{V}_{-+}(t)^{\star'} & \mathsf{V}_{--}(t)^{\star'} \end{bmatrix} = I_2.$$

Proof. We check that the relation $\left(\star' \otimes {}^t\right)(\mathsf{V}(t))\mathsf{V}(t)$ holds for all $t \ge 0$. Write $\mathsf{W} = \mathsf{W}^\kappa + \mathsf{W}^\pi$ with

$$\mathsf{W}^\kappa(t) = \sqrt{c}\begin{bmatrix} \mathrm{i}a_t & 0 \\ 0 & \mathrm{i}b_t \end{bmatrix},\ \mathsf{W}^\pi(t) = \sqrt{v}\begin{bmatrix} 0 & x_t \\ x_t^\star & 0 \end{bmatrix},\ t \ge 0.$$

We apply the free Itô formula. There exists a matricial valued free local martingale $M = (M(t))_{t\ge 0}$ such that for all $t \ge 0$

$$(\star)\quad \begin{aligned} \mathrm{d}\left(\left(\star'\otimes{}^t\right)\mathsf{V}(t)\mathsf{V}(t)\right) &= M_t + (v-c)\left(\star'\otimes{}^t\right)\mathsf{V}(t)\mathsf{V}(t) \\ &\quad - \mathrm{d}\mathsf{W}^\kappa(t)\left((\star'\otimes t)\mathsf{V}(t)\mathsf{V}(t)\right)\mathrm{d}\mathsf{W}^\kappa(t) - \mathrm{d}\mathsf{W}^\pi(t)(\star'\otimes t)\mathsf{V}(t)\mathsf{V}(t)\mathrm{d}\mathsf{W}^\pi(t) \\ &= M_t + (v-c)\left(\star'\otimes{}^t\right)\mathsf{V}(t)\mathsf{V}(t) + c\begin{bmatrix} \mathrm{d}a_t & 0 \\ 0 & \mathrm{d}b_t \end{bmatrix}\left(\star'\otimes{}^t\right)\mathsf{V}(t)\mathsf{V}(t)\begin{bmatrix} \mathrm{d}a_t & 0 \\ 0 & \mathrm{d}b_t \end{bmatrix} \\ &\quad - v\begin{bmatrix} 0 & \mathrm{d}x_t \\ \mathrm{d}x_t^\star & 0 \end{bmatrix}\left(\star'\otimes{}^t\right)\mathsf{V}(t)\mathsf{V}(t)\begin{bmatrix} 0 & \mathrm{d}x_t \\ \mathrm{d}x_t^\star & 0 \end{bmatrix} \\ &= M_t + (v-c)\left(\star'\otimes{}^t\right)\mathsf{V}(t)\mathsf{V}(t) + \begin{bmatrix} f(t) & 0 \\ 0 & g(t) \end{bmatrix}\mathrm{d}t, \end{aligned}$$

with the two functions $f, g : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$f(t) = c\tau\left(\left(\star'\otimes{}^t\right)\mathsf{V}(t)\mathsf{V}(t)\right)_{++} - v\tau\left(\left(\star'\otimes{}^t\right)\mathsf{V}(t)\mathsf{V}(t)\right)_{--},$$
$$g(t) = c\tau\left(\left(\star'\otimes{}^t\right)\mathsf{V}(t)\mathsf{V}(t)\right)_{--} - v\tau\left(\left(\star'\otimes{}^t\right)\mathsf{V}(t)\mathsf{V}(t)\right)_{++}.$$

Put $k_+(t) = \tau\left(\left(\star'\otimes{}^t\right)\mathsf{V}(t)\mathsf{V}(t)\right)_{++}$ and $k_-(t) = \tau\left(\left(\star'\otimes{}^t\right)\mathsf{V}(t)\mathsf{V}(t)\right)_{--}$. If we apply $\tau\otimes 1$ to both sides of the equation ($\star$), we obtain the differential system:

$$\begin{cases} \frac{d}{dt}(k_+(t) + k_-(t)) &= 0 \\ \frac{d}{dt}(k_+(t) - k_-(t)) &= -2(p+c)(k_+(t) - k_-(t)), \end{cases},\quad k_+(0) + k_-(0) = 2,\ k_+(0) = k_-(0).$$

The solution of the last differential system is $k_+(t) = k_-(t) = 1$, hence $f(t) = g(t) = c - v$ for all $t \geq 0$. By injecting in the equation ($\star$) the formulas for the two functions f and g we prove that

$$\begin{aligned} &\mathrm{d}\left((\star' \otimes t)(\mathsf{V}(t))\mathsf{V}(t)\right) = (v-c)(\star' \otimes t)(\mathsf{V}(t)\mathsf{V}(t) - I_2)\mathrm{d}t \\ &(\star' \otimes t)\mathsf{V}(0)\mathsf{V}(0) = I_2. \end{aligned}$$

Since the above system admits an unique solution and since the function $t \mapsto I_2$ is a solution, we obtain that $\left(\star' \otimes^t\right)\mathsf{V}(t)\ \mathsf{V}(t) = I_2$ for all $t \geq 0$. One can do the same computations to prove that $\mathsf{V}(t)(\star' \otimes t)\mathsf{V}(t) = I_2$. □

It exists a morphism $V(t)$ from the pseudo-unitary dual Voiculescu group to $\mathcal{A}$ such that:

$$V(t)(u_{\varepsilon,\varepsilon'}) = \mathsf{V}(t)_{\varepsilon,\varepsilon'},\ \varepsilon,\varepsilon' \in \{-,+\}. \tag{3.67}$$

The convolution product on the space of $\star$ morphisms from the pseudo-unitary dual group $\mathcal{U}(-,+)$ to the algebra $\mathcal{A}$ is denoted by $\times$ and defined by the equations $A \times B = A \sqcup B \circ \Delta$, $A, B \in \mathrm{End}(\mathcal{O}(-,+), \mathcal{A})$.

Lemma 3.31. *The process V is a free Lévy process.*

Proof. We prove, first, that the non-commutative distributions of the increments are stationary: for every pair $0 \leq s < t$ of times, $V(s,t) = V(s) \circ S \times V(t) = V(s)^{-1} \times V(t)$ has the same law as $V(t-s)$. We have the relation

$$\begin{bmatrix} V(s)^{-1} \times V(t)(u_{++}) & V(s)^{-1} \times V(t)(u_{+-}) \\ V(s)^{-1} \times V(t)(u_{-+}) & V(s)^{-1} \times V(t)(u_{--}) \end{bmatrix} = \mathsf{V}(s)^{-1}\mathsf{V}(t).$$

By differentiating in the time variable t we get

$$\mathsf{V}(s)^{-1}\mathrm{d}\mathsf{V}(t) = \mathsf{V}(s)^{-1}\mathsf{V}(t)\mathrm{d}\mathsf{W}(t) + \frac{1}{2}(v-c)\mathsf{V}(s)^{-1}\mathsf{V}(t)\mathrm{d}t. \tag{3.68}$$

Thus the two processes $[s,+\infty] \ni t \to \mathsf{V}(s)^{-1}\mathsf{V}(t)$ and $[s,+\infty] \ni t \to \mathsf{V}(t-s)$ satisfy the same free stochastic differential equation with the same initial condition, hence $\mathsf{V}(t-s) = \mathsf{V}(s)^{-1}\mathsf{V}(t)$ for all times $s,t \geq 0$ with $t \geq s$. Now we prove that the four components of $\mathsf{V}(s)^{-1}\mathsf{V}(t)$ are free from $\mathcal{A}_s = vN\left(\mathsf{W}(u)_{\varepsilon,\varepsilon'}, u \leq s, \varepsilon,\varepsilon' \in \{-,+\}\right)$. The process $[s,+\infty[\ni t \mapsto \mathsf{V}(s)^{-1}\mathsf{V}(t)$ is a solution of the free stochastic differential equation (3.68), hence it can be approximated by the sequence of matricial processes $\gamma^{(n)} = \left(\gamma_t^{(n)}\right)_{t\geq s}$ defined by

$$\gamma^{(n+1)}(t) = I_2 + \int_s^t \gamma^{(n)}(u)\mathrm{d}\mathsf{W}(u) + \frac{1}{2}(v-c)\int_s^t \gamma^{(n)}(u)\mathrm{d}u \tag{3.69}$$

for the norm $\|A\| = \max_{\varepsilon\varepsilon' \in \{-,+\}}\left(\|A_{\varepsilon,\varepsilon'}\|_{L^2(\tau)}\right)$ as shown previously. A simple recurrence shows that the components of $\gamma^{(n)}$ for $n \geq 1$ are free from the algebra $\mathcal{A}_s$. It is true for $n = 0$. Let us now assume that the components of $\gamma^{(n)}$ are free from $\mathcal{A}_s$. Since the components of the increments $\mathsf{W}(v) - \mathsf{W}(u)$ with $v > u \geq s$ are free from $\mathcal{A}_s$, we have that the components $\int_s^t \gamma^{(n)}(u)\mathrm{d}\mathsf{W}_{v,c}(u)$ are free from $\mathcal{A}_s$ for all $t \geq s$, hence the components of $\gamma^{(n+1)}$ are free from $\mathcal{A}_s$. □

3.7.3. Schürmann triples. For detailed introduction on the notion of Schürmann triple, the reader is directed to [29]. Schürmann triples classify laws of quantum Lévy processes: it is the non commutative counterpart of Lévy triples for classical Lévy processes (a diffusion matrix, a drift and a Lévy measure). A Schürmann triple $(\pi, \eta, \mathcal{L})$ is the data of

- $\pi : \mathcal{U}(-,+) \to D$ $\star$-representation of the algebra $\mathcal{U}(-,+)$ on a Hilbert space $(D, \langle\ ,\ \rangle)$,
- $\eta : \mathcal{U}(-,+) \to D$ a π-ϵ cocycle:
$$\eta(ab) = \pi(a)\eta(b) + \eta(a)\epsilon(b),\ a, b \in \mathcal{U}(-,+)$$
- $\mathcal{L} : \mathcal{U}(-,+) \to \mathbb{C}$ is a generator which is an ϵ-ϵ cocycle
$$\mathcal{L}(a^\star b) = \epsilon(a)\mathcal{L}(a) + \mathcal{L}(a)\epsilon(b) + \langle \eta(a), \eta(b)\rangle, a, b \in \mathcal{U}(-,+)$$

In this section we compute a Schürmann triple for the free split Hermitian Brownian motions and the free split unitary Brownian motions.

3.7.3.1. *Schürmann triple for the free Hermitian Brownian motion.* In that section we compute the Schürmann triple associated with the generator $\mathcal{G}$ of the pseudo-Hermitian Brownian motion W. We recall that W depends on two parameters v and c, our notations do not show this dependence. Let us fix the two parameters $v \geq 0$ and $c \geq 0$ for the remaining of this section.

Lemma 3.32. *The words on the generators of $\mathfrak{h}(-,+)$ for which the evaluation of the generator $\mathcal{G} : \mathfrak{h}(-,+) \to \mathbb{C}$ of the free Lévy process W is not zero are*

$$\mathcal{G}(h_+^2) = \mathcal{G}(h_-^2) = -1,\ \mathcal{G}(h_{+-}h_{+-}^\star) = 1.$$

Proof. For a time $t > 0$, we name μ_t the distribution of $W(t)$. Let $t \geq 0$ and w a word on the generators of $\mathfrak{h}(-,+)$. We denote by $NC_2(w)$ the set of all non-crossing matchings of the interval $[\![1, \mathsf{length}(w)]\!]$ such that, for all $\pi \in NC_2(w)$:

$$\forall 1 \leq i \neq j \leq \mathsf{length}(w),\ i \sim_\pi j \implies w_i = w_j^\star \tag{3.70}$$

By using the moment-cumulant formula and the fact that $\{a^+, a^-, x\}$ is a free family, we prove that

$$\mu_t(w) = \sum_{\pi \in NC_2(w)} k_\pi(W(t)(w)).$$

Since the order two cumulants $k_2(a_t^+, a_t^+)$, $k_2(a_t^-, a_t^-)$ and $k_2(x_t, x_t^\star)$ are linear in the time variable t, $\frac{d}{dt}\mu_t = 0$ if the length of w is greater or equal to three. The three equalities of the proposition are proved easily. □

In the next proposition, we use the following notations:

$$\varepsilon_{++} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ \varepsilon_{+-} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ \varepsilon_{-+} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ \varepsilon_{--} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Proposition 3.33. *There exists a $\star$-representation $\pi : \mathfrak{h}(-,+) \to \mathcal{M}_2(\mathbb{C})$ and a $\varepsilon - \pi$ cocycle $\eta : \mathfrak{h}(-,+) \to \mathbb{C}$ such that*

$$\begin{aligned} &\pi = 0, \\ &\eta(h_{-+}) = \sqrt{v}\varepsilon_{-+},\ \eta(h_{-+}^\star) = \sqrt{v}\varepsilon_{-+},\ \eta(h_{++}) = \sqrt{c}\varepsilon_{++}. \end{aligned} \tag{3.71}$$

With respect to the scalar product $\langle\ ,\ \rangle = \mathsf{Tr}(\cdot^t\cdot)$ and the involution $\star$ on $\mathfrak{h}(-,+)$, the triple $(\pi, \eta, \mathcal{G})$ is a Schürmann triple.

Proof. For the cocycle η defined equation 3.71, we have the formula $\eta(w_0 \cdots w_n) = \eta(w_0)$, $w_i \in \mathfrak{h}(-,+)$, $i \leq n$. Let us prove that the generator $\mathcal{G}$ satisfies the equation

$$\mathcal{G}(a^\star b) = \varepsilon(a)\mathcal{G}(b) + \mathcal{G}(a)\varepsilon(b) + \langle \eta(a),\ \eta(b)\rangle. \tag{3.72}$$

Since the algebra $\mathfrak{h}(-,+)$ is freely generated by $\{h_+, h_-, h_{-+}, h_{-+}^\star\}$, there exists an Hermitian functional $\tilde{\mathcal{G}}$ on $\mathfrak{h}(-,+)$ such that equation 3.72 holds for $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{G}}(h_+) = \tilde{\mathcal{G}}(h_-) = \tilde{\mathcal{G}}(h_{-+}) = \tilde{\mathcal{G}}(h_{-+}^\star) = 0$. We now prove that $\tilde{\mathcal{G}}$ is equal to $\mathcal{G}$.

$$\begin{aligned} &\tilde{\mathcal{G}}(h_+^2) = -\tilde{\mathcal{G}}((h_+)^\star h_+) = -\tilde{\mathcal{G}}(h_+)\varepsilon(h_+) - \varepsilon(h_+)\tilde{\mathcal{G}}(h_+) - \langle \eta(h_+),\ \eta(h_+)\rangle = -1 \\ &\tilde{\mathcal{G}}(h_-^2) = -\tilde{\mathcal{G}}((h_+)^\star h_+) = -\tilde{\mathcal{G}}(h_-)\varepsilon(h_-) - \varepsilon(h_-)\tilde{\mathcal{G}}(h_-) - \langle \eta(h_-),\ \eta(h_-)\rangle = -1 \\ &\tilde{\mathcal{G}}(h_{-+}h_{-+}^\star) = \tilde{\mathcal{G}}(h_{-+})\varepsilon(h_{-+}) + \varepsilon(h_{-+})\tilde{\mathcal{G}}(h_{-+}) + \langle \eta(h_{-+}),\ \eta(h_{-+})\rangle = 1 \end{aligned} \tag{3.73}$$

Let $w = abc$ be a word of length three on the generators of $\mathfrak{h}(-,+)$ then $\tilde{\mathcal{G}}(abc) = \varepsilon(a^\star)\tilde{\mathcal{G}}(bc) + \tilde{\mathcal{G}}(a)\varepsilon(bc) + \langle \eta(a^\star),\ \eta(bc)\rangle = 0$. We prove by induction that for any word of length greater or equal to three, we have $\tilde{\mathcal{G}}(w) = 0$. Let $n \geq 1$ be an integer. Assume that $\tilde{\mathcal{G}}(w) = 0$ for all words w of length less or equal to $n-1$ and let $u = aw$ be a word of length n with w a word of length $(n-1)$ and a of length 1. From the cocycle property of the generator $\tilde{\mathcal{G}}$, we have:

$$\tilde{\mathcal{G}}(u) = \tilde{\mathcal{G}}(aw) = \varepsilon(a^\star)\tilde{\mathcal{G}}(w) + \tilde{\mathcal{G}}(a^\star)\varepsilon(w) + \langle \eta(a^\star),\ \eta(w)\rangle = 0 \tag{3.74}$$

From the equations (3.73) and (3.74), we deduce that $\tilde{\mathcal{G}} = \mathcal{G}$ □

3.7.3.2. *Schürmann triple for the free split unitary Brownian motion.* Let $v,c > 0$ and $k \geq 1$ an integer. A bi-coloured Brauer diagram $b = (b_{nc}, C_b)$ acts on the vector space generated by k-tuples with entries in $\{-,+\}$ by

$$b \cdot (i_1, \ldots, i_k) = \delta_{c_b' = \mathsf{i}} c_b,\ \mathsf{i} = (i_1, \ldots, i_k),\ i_j \in \{-,+\},\ j \leq k. \tag{3.75}$$

Let $\mathcal{L}$ be the hermitian linear form on $\mathcal{U}(-,+)$ that takes on an element $u_{i_1,j_1}^{\varepsilon(1)} \cdots u_{i_k,j_k}^{\varepsilon(k)}$, $i_j \in \{-,+\}$, $j \leq k$, $\varepsilon \in \{1,\star\}^k$ the real value

$$\begin{aligned}\mathcal{L}\left(u_{i_1,j_1}^{\varepsilon_1} \cdots u_{i_k,j_k}^{\varepsilon_k}\right) = \frac{k(v-c)}{2}\delta_{\mathsf{i},\mathsf{j}} + v \sum_{\substack{\tau_{a,b} \in \mathsf{T}_{k,2}^{\neq},\\ \varepsilon(a)=\varepsilon(b)}} \delta_{\tau \cdot \mathsf{i},\mathsf{j}} + v \sum_{\substack{e_{ab} \in \mathsf{W}_{k,2}^{\neq}\\ \varepsilon(a)\neq\varepsilon(b)}} \delta_{e_{ab}\cdot \mathsf{i},\mathsf{j}} \\ + -c \sum_{\substack{\tau_{ab} \in \mathsf{T}_{k,2}^{=}\\ \varepsilon_a = \varepsilon_b}} \delta_{\tau_{ab}\cdot \mathsf{i},\mathsf{j}} + c \sum_{\substack{e_{ab} \in \mathsf{W}_{k,2}^{=}\\ \varepsilon_a \neq \varepsilon_b}} \delta_{e_{ab}\cdot \mathsf{i},\mathsf{j}}.\end{aligned} \tag{3.76}$$

We check that the hermitian form $\mathcal{L}$ does exist. Let $(i,j) \in \{+,-\}^2$ two colours and define the functional $\varepsilon : \{-,+\}^2 \to \{-1,1\}$ by $\varepsilon(a,a) = 1$, $\varepsilon(a,b) = -1$ for $a \neq b \in \{-,+\}$, then

$$\sum_{k \in \{+,-\}} \mathcal{L}(u_{ik}(u_{jk})^{\star'}) = (v-c)\delta_{ij} + v \sum_{e_{12} \in \mathsf{W}_{k,2}^{\neq}} \sum_{k \in \{+,-\}} \varepsilon(k,j)\delta_{e_{12}\mathsf{l},\mathsf{c}} + c \sum_{e_{12} \in \mathsf{W}_{k,2}^{=}} \sum_{k \in \{+,-\}} \varepsilon(k,j)\delta_{e_{12}\mathsf{l},\mathsf{c}}$$

with $\mathsf{l} = (i,j)$ and $\mathsf{c} = (k,k)$. Since $\sum_k \varepsilon(k,j)\delta_{e_{12}^{exc}\mathsf{l},\mathsf{c}} = -\delta_{ij}$ and $\sum_k \varepsilon(k,j)\delta_{e_{12}^{diag}\mathsf{l},\mathsf{c}} = \delta_{ij}$, we get that

$$\sum_{\varepsilon \in \{+,-\}} \mathcal{L}\left((u_{ik})(u_{jk})^{\star'}\right) = \mathcal{L}(1) = 0.$$

Proposition 3.34. *There exists a $\star$-representation $\pi : \mathcal{U}(-,+) \to \mathcal{M}_2(\mathbb{C})$ and a cocycle $\eta : \mathcal{U}(-,+) \to \mathbb{C}$ such that*

$$\begin{aligned} i \neq j,\ \eta_{v,c}(u_{ij}) = \sqrt{v}\varepsilon_{ij},\ \eta_{v,c}(u_{ij}^t) = \sqrt{v}\varepsilon_{ji},\ \pi_{v,c}(u_{ij}) = \delta_{ij}1, \\ i = j,\ \eta_{v,c}(u_{ii}) = \sqrt{c}\varepsilon_{ii},\ \eta_{v,c}(u_{ii}^t) = -\sqrt{c}\varepsilon_{ii},\ \pi_{v,c}(u_{ii}) = \delta_{ii}1. \end{aligned} \tag{3.77}$$

With respect to the scalar product $\langle\ ,\ \rangle = \mathrm{Tr}(\cdot^t\cdot)$ and the involution $\star$ on $\mathcal{U}(-,+)$, the triple $(\pi, \eta, \mathcal{L})$ is a Schürmann triple.

Proof. We need to check first that such η and π exists, it is trivial for π. The prescription of the values of η in (3.77) and the requirement for η to be ε-ε cocyle does define a morphism from the free group $\mathcal{F}_8$ generated by $\{u_{\varepsilon,\varepsilon'}, o_{\varepsilon,\varepsilon'}^t \varepsilon, \varepsilon' \in \{-,+\}\}$. To be able to factorize the morphism η by the canonical projection $\pi : \mathcal{F}_8 \to \mathcal{U}(-,+)$ we have to check that the 8 relations in $\mathcal{O}(-,1)$ are satisfied by the values of η. We will only show that two of them are satisfied. The reader is invited to check by itself the remaining ones. We need to show that

$$\eta(u_{++}u_{++}^t - u_{+-}u_{+-}^t) = 0,\ \eta(u_{-+}u_{++}^t - u_{--}u_{+-}^t) = 0. \tag{$\star$}$$

Owing to η is an π-ε cocyle, we have

$$\eta(u_{++}u_{++}^t - u_{+-}u_{+-}^t) = \pi(u_{++})\eta(u_{++}^t) + \eta(u_{++})\varepsilon(u_{++}^t) - \pi(u_{+-})\eta(u_{+-}^t) - \eta(u_{+-}^t)\varepsilon(u_{+-}^t) = 0.$$

The first relation in ($\star$) is satisfied, we check now that the second one is also satisfied. One has

$$\eta(u_{-+}u_{++}^t - u_{--}u_{+-}^t) = \eta(u_{-+})\varepsilon(u_{++}) - \pi(u_{--})\eta(u_{+-}^t) = \sqrt{p}(\varepsilon_{-+} - \varepsilon_{+-}) = 0.$$

The morphism η from the free group $\mathcal{F}_8$ is factorizable by the canonical projection π, otherwise stated η descends to a morphism, also named η on the pseudo-unitary group $\mathcal{U}(-,+)$ that takes the values (3.77) on the generators of $\mathcal{O}(-,+)$. It is not difficult to find an explicit expression for the cocycle η evaluated on $u^{\varepsilon(1)}_{i_1j_1}\cdots u^{\varepsilon(k)}_{i_kj_k}$,

$$\eta(u^{\varepsilon_1}_{i_1j_1}\cdots u^{\varepsilon_k}_{i_kj_k}) = \sum_{l\leq k}\delta_{i_1j_1}\cdots\delta_{i_l\neq j_l}\begin{cases}\sqrt{v}\varepsilon_{i_lj_l} & \text{if } \varepsilon(l)=1\\ \sqrt{v}\varepsilon_{j_li_l} & \text{if } \varepsilon(l)=t\end{cases}\delta_{i_{l+1}j_{l+1}}\cdots\delta_{i_kj_k}$$
$$+\sum_{l\leq k}\delta_{i_1j_1}\cdots\delta_{i_lj_l}\begin{cases}\sqrt{c}\varepsilon_{i_lj_l} & \text{if } \varepsilon(l)=1\\ -\sqrt{c}\varepsilon_{i_lj_l} & \text{if } \varepsilon(l)=t\end{cases}\delta_{i_{l+1}j_{l+1}}\cdots\delta_{i_kj_k}.$$

We only have to check that

$$(\star') \qquad -\langle\eta(a^\star),\eta(b)\rangle = \varepsilon(a)\mathcal{L}(b)-\mathcal{L}(ab)+\mathcal{L}(a)\varepsilon(b),\ a,b\in\mathcal{U}(-,+).$$

Assume that there exists an operators $\tilde{\mathcal{L}}$ that satisfy ($\star'$) and $\tilde{\mathcal{L}}(u_{ik}) = \frac{(v-c)}{2}\delta_{ik},\ i,k\in\{+,-\}$. The values of $\tilde{\mathcal{L}}$ on $\{u_{ij},\ i,j\in\{-,+\}\}$ are sufficient to determine the value of the operator $\mathcal{L}$ on any element of the pseudo unitary dual group. We show that $\mathcal{L}=\tilde{\mathcal{L}}$. This is done by induction on the length of a word on $\{u_{ij},u^t_{ij},i,j\in\{+,-\}\}$. Assume that the relation $\mathcal{L}^1(w)=\tilde{\mathcal{L}}^1(w)$ holds for all the words on the generators of length less or equal to k, then

$$\tilde{\mathcal{L}}(u^{\varepsilon_1}_{i_1,j_1}\cdots u^{\varepsilon_k}_{i_kj_k}) = \varepsilon(u_{i_2,j_2}\cdots u_{i_2j_2})\mathcal{L}(u_{i_1,j_1})+\varepsilon(u_{i_1j_1})\mathcal{L}(u_{i_2j_2}\cdots u_{i_{k+1}j_{k+1}})$$
$$+\langle\eta(u^{1-\varepsilon_1}_{i_1j_1}),\eta(u_{i_2j_2}\cdots u_{i_{k+1}j_{k+1}})\rangle.$$

We apply the induction's hypothesis to get

$$\tilde{\mathcal{L}}(u^{\varepsilon_1}_{i_1,j_1}\cdots u^{\varepsilon_k}_{i_kj_k}) = \frac{(k+1)(p-c)}{2}\delta_{i_1j_1}\delta_{i_2j_2}\cdots\delta_{i_{k+1}j_{k+1}}+v\delta_{i_1j_1}\sum_{\substack{\tau_{ij}\in\mathsf{T}^{\mathrm{exc}}_k\\ i\geq 2\\ \varepsilon(i)=\varepsilon(j)}}\delta_{\tau_{ij}\cdot \mathrm{i},\mathrm{j}}+v\delta_{i_1,j_1}\sum_{\substack{e_{ij}\in\mathsf{W}^{\neq}_{k,2}\\ i\geq 2\\ \varepsilon(i)\neq\varepsilon(j)}}\delta_{e_{ij}\cdot\mathrm{i},\mathrm{j}}$$
$$-c\delta_{i_1j_1}\sum_{\substack{\tau_{ij}\in\mathsf{T}^{diag}_k,\\ i\geq 2,\\ \varepsilon(i)=\varepsilon(j)}}\delta_{\tau_{ij}\cdot\mathrm{i},\mathrm{j}}+-c\delta_{i_1j_1}\sum_{\substack{e_{ij}\in\mathsf{W}^{=}_{k,2}\\ i\geq 2,\\ \varepsilon(k)\neq\varepsilon(l)}}\delta_{e_{ij}\cdot\mathrm{i},\mathrm{j}}+\langle\eta(u^{1-\varepsilon_1}_{i_1j_1}),\eta(u_{i_2j_2}\cdots u_{i_{k+1}j_{k+1}})\rangle.$$

From the explicit formula for the cocyle η, one has

$$\langle\eta(u^{\neg\varepsilon(1)}_{i_1j_1}),\eta(u_{i_2j_2}\cdots u_{i_{k+1}j_{k+1}})\rangle =$$
$$\sqrt{v}\sum_{2\leq l\leq k+1}\delta_{i_2j_2}\cdots\delta_{i_l\neq j_l}\begin{cases}\langle\eta(u^{\neg\varepsilon(1)}_{i_1j_1}),\varepsilon_{i_lj_l})\rangle & \text{if } \varepsilon_l=1\\ \langle\eta(u^{\neg\varepsilon(1)}_{i_1j_1}),\varepsilon_{j_li_l})\rangle & \text{if } \varepsilon(l)=t\end{cases}\delta_{i_{l+1}j_{l+1}}\cdots\delta_{i_{k+1}j_{k+1}}$$
$$+\sqrt{c}\sum_{2\leq l\leq k+1}\delta_{i_2j_2}\cdots\delta_{i_l=j_l}\begin{cases}\langle\eta(u^{\neg\varepsilon(1)}_{i_1j_1}),\varepsilon_{i_lj_l})\rangle & \text{if } \varepsilon(l)=1\\ \langle\eta(u^{\neg\varepsilon(1)}_{i_1j_1}),\varepsilon_{j_li_l})\rangle & \text{if } \varepsilon(l)=t\end{cases}\delta_{i_{l+1}j_{l+1}}\cdots\delta_{i_{k+1}j_{k+1}}$$

We distinguish four cases:

(1) $\varepsilon_l=1,\varepsilon_1=1,\ \delta_{i_l\neq j_l}\langle\eta(u^t_{i_1j_1})\varepsilon_{i_lj_l})\rangle = \delta_{i_1\neq j_1}\delta_{i_l\neq j_l}\delta_{j_li_1}\delta_{i_lj_1} = \delta_{i_l\neq i_1}\delta_{j_li_1}\delta_{i_lj_1}$
(2) $\varepsilon_l=1,\varepsilon_1=t,\ \delta_{i_l\neq j_l}\langle\eta(u_{i_1j_1})\varepsilon_{i_lj_l})\rangle = \delta_{i_1\neq j_1}\delta_{i_l\neq j_l}\delta_{i_li_1}\delta_{j_lj_1} = \delta_{i_1\neq j_1}\delta_{i_l\neq j_1}\delta_{i_li_1}\delta_{j_lj_1}$
(3) $\varepsilon_l=t,\varepsilon_1=1,\ \delta_{i_l\neq j_l}\langle\eta(u_{i_1j_1})\varepsilon_{i_lj_l})\rangle = \delta_{i_1\neq j_1}\delta_{i_l\neq j_l}\delta_{j_li_1}\delta_{i_lj_1} = \delta_{i_l\neq i_1}\delta_{j_li_1}\delta_{i_lj_1}$
(4) $\varepsilon_l=t,\varepsilon_1=t,\ \delta_{i_l\neq j_l}\langle\eta(u_{j_1i_1})\varepsilon_{j_li_l})\rangle = \delta_{i_l\neq j_l}\delta_{j_li_1}\delta_{i_lj_1} = \delta_{i_l\neq j_l}\delta_{j_1\neq i_1}\delta_{i_li_1}\delta_{j_lj_1}$

Similar formulas hold for $i_l=j_l$. Hence,

$$\langle\eta(u^{1-\varepsilon_1}_{i_1j_1}),\eta(u_{i_2j_2}\cdots u_{i_{k+1}j_{k+1}})\rangle =$$
$$v\sum_{\substack{\tau_{1l}\in\mathsf{T}^{\neq,+}_k(,)\\ j\geq 2,\\ \varepsilon_1=\varepsilon_l}}\delta_{\tau_{1l}\cdot\mathrm{i},\mathrm{j}}+p\sum_{\substack{e_{1l}\in\mathsf{W}^{+,(2)}_{k,exc}\\ l\geq 2,\\ \varepsilon_1\neq\varepsilon_l}}\delta_{e_{1l}\cdot\mathrm{i},\mathrm{j}}-c\sum_{\substack{\tau_{1l}\in\mathsf{T}^{\neq,+}_k(,)\\ j\geq 2,\\ \varepsilon_1=\varepsilon_l}}\delta_{\tau_{1l}\cdot\mathrm{i},\mathrm{j}}-c\sum_{\substack{e_{1l}\in\mathsf{W}^{+,(2)}_{k,exc}\\ l\geq 2,\\ \varepsilon_1\neq\varepsilon_l}}\delta_{e_{1l}\cdot\mathrm{i},\mathrm{j}}$$

Thus, we proved that $\mathcal{L} = \tilde{\mathcal{L}}$. To complete the proof we only need to show that such an $\tilde{\mathcal{L}}$ exists. An application of ($\star'$) leads to

$$\begin{aligned}\sum_k \mathcal{L}(u_{ik}u_{jk}^{\star'}) &= \sum_k \langle \eta(o_{ik}^t), \eta(o_{jk}^t)\varepsilon(j,k)\rangle + (v-c)\delta_{ij} \\ &= v\sum_{k\neq i,j} \delta_{ij}\delta_{kk}\varepsilon(j,k) + c\delta_{ij}\varepsilon(i,j) + \delta_{ij}(p-c) = 0\end{aligned}$$

□

Lemma 3.35. *The generator of the process V is $\mathcal{L}$.*

Proof. The proof is proceeds by induction on the length of words on the generators of $\mathcal{U}(-,+)$ and their transpose. A simple application of the free it formula gives

$$\frac{d}{dt}_{|t=0}\tau(V(t))(u_{i,j}^{\varepsilon}) = \frac{1}{2}(v-c)\tau(V(0)(u_{ij}^{\varepsilon})) = \frac{1}{2}(v-c)\delta_{ij},\ i,j\in\{-,+\},\ \varepsilon\in\{1,t\}.$$

We assume now that the relation

$$\frac{d}{dt}_{|t=0}\tau(V(t)(w)) = \mathcal{L}(w)$$

holds for all the words w on the generators and their transpose of $\mathcal{U}(-,+)$ whose length is less than an integer $n\geq 0$. We pick a word w of length $n+1$. We apply the free itô formula to get

$$\begin{aligned}&\mathrm{d}\tau\left(V(t)(w_1\cdots w_n)V(t)(w_{n+1})\right)\\ =&\tau\left(\mathrm{d}\left(V(t)(w_1\cdots w_n)\right)V(0)(w_{n+1})\right) + \tau\left(V(0)(w_1\cdots w_n)\mathrm{d}V(t)(w_{n+1})\right)\\ &+\tau\left(\mathrm{d}\left(V(t)(w_1\cdots w_n)\right)\mathrm{d}V(t)(w_{n+1})\right)\\ =&\frac{n+1}{2}(p-c)\delta_{\mathrm{i,j}}\mathrm{dt} + v\sum_{\substack{\tau\in\mathsf{T}_{n,exc}^{+,(2)}\\ \varepsilon_k=\varepsilon_l}}\delta_{\tau\cdot\mathrm{i,j}} + v\sum_{\substack{e\in\mathsf{W}_{n,exc}^{+,(2)}\\ \varepsilon_k\neq\varepsilon_l}}\delta_{e\cdot\mathrm{i,j}} - c\sum_{\substack{\tau\in\mathsf{T}_{n,diag}^{+,(2)}\\ \varepsilon_k=\varepsilon_l}}\delta_{\tau\cdot\mathrm{i,j}} + c\sum_{\substack{e\in\mathsf{W}_{n,diag}^{+,(2)}\\ \varepsilon_k\neq\varepsilon_l}}\delta_{e\cdot\mathrm{i,j}}\\ &+\sum_{k=1\ldots n}\tau\left(\delta_{i_1,j_1}\cdots\delta_{i_{k-1},j_{k-1}}\left(\mathrm{d}V(t)(w_k)\mathrm{d}W(t)(w_{n+1})\right)_{t=0}\delta_{i_{k+1},j_{k+1}}\cdots\delta_{i_n,j_n}\right)\end{aligned}$$

To compute $(\mathrm{d}V(t)(w_k)\mathrm{d}W(t)(w_{n+1}))_{t=0}$, there are four cases to distinguish : we give prove a formula for $\mathrm{d}V(t)(u_{ij}^{\varepsilon(1)})\mathrm{d}W(t)(u_{kl}^{\varepsilon(2)})_{|t=0}$ that depends on the values of $\varepsilon(1)\in\{-,+\}$ and $\varepsilon(2)\in\{-,+\}$.

- If $\varepsilon(1)=\varepsilon_2=1$,

$$\begin{aligned}\mathrm{d}V(t)(u_{ij}^{\varepsilon(1)})\mathrm{d}W(t)(u_{kl}^{\varepsilon(2)})_{|t=0} &= \sum_{a=1\ldots 2}\mathrm{d}W(t)(u_{aj})\mathrm{d}W(t)(u_{kl})_{|t=0}\\ &= c\delta_{a=k}\delta_{a,j}\delta_{kl}\mathrm{dt} + v\delta_{a\neq j}\delta_{k\neq l}\delta_{a\neq k}\mathrm{dt}\\ &= c\delta_{a=k}\delta_{a,j}\delta_{k,l}\mathrm{dt} + v\delta_{a\neq k}\delta_{l,a}\delta_{j,k}\mathrm{dt}.\end{aligned}$$

- If $\varepsilon_2=1$ and $\varepsilon_1=\star$, then

$$\begin{aligned}\mathrm{d}V(t)(u_{ij}^{\varepsilon_1})\mathrm{d}W(t)(u_{kl}^{\varepsilon(2)})_{|t=0} &= \sum_{a=1\ldots 2}\mathrm{d}W(t)(u_{aj})\mathrm{d}W(t)(u_{kl}^{t})_{|t=0}\\ &= -c\delta_{a=k}\delta_{a,j}\delta_{k,l}\mathrm{dt} + v\delta_{a\neq j}\delta_{a\neq k}\delta_{a=k}\mathrm{dt}\\ &= c\delta_{a=k}\delta_{a,j}\delta_{k,l}\mathrm{dt} + v\delta_{a\neq k}\delta_{l,a}\delta_{j,k}\mathrm{dt}.\end{aligned}$$

- The two remaining cases are treated likewise and are left to the reader.

Hence,

$$\sum_{k=1\dots n} \tau(\delta_{i_1,j_1}\cdots\delta_{i_{k-1},j_{k-1}}(\mathrm{d}V(t)(w_k)\mathrm{d}W(t)(w_{n+1}))_{t=0}\,\delta_{i_{k+1},j_{k+1}}\cdots\delta_{i_n,j_n})$$
$$= v\sum_{\substack{\tau_{i,n+1}\in \mathsf{T}^{+,(2)}_{n+1,exc}\\ \varepsilon_i=\varepsilon_{n+1}}} \delta_{\tau_{i,n+1}\cdot\mathrm{i},\mathrm{j}} + p\sum_{\substack{e_{i,n+1}\in \mathsf{W}^{+,(2)}_{n=1,exc}\\ \varepsilon_1\neq\varepsilon_{n+1}}} \delta_{e_{i,n+1}\cdot\mathrm{i},\mathrm{j}}$$
$$- c\sum_{\substack{\tau_{i,n+1}\in \mathsf{T}^{+,(2)}_{n+1,diag}\\ \varepsilon_i=\varepsilon_{n+1}}} \delta_{\tau_{i,n+1}\cdot\mathrm{i},\mathrm{j}} + c\sum_{\substack{e_{i,n+1}\in \mathsf{W}^{+,(2)}_{n+1,diag}\\ \varepsilon_i\neq\varepsilon_{n+1}}} \delta_{e_{i,n+1}\cdot\mathrm{i},\mathrm{j}}.$$

The proof is complete. □

At that point, we computed two Schürmann triples, one for the free pseudo-hermitian Brownian motion W and one for the pseudo-unitary Brownian motion V. The complex involutive algebra generated by the coefficients of the matrix $O^h = \exp(hH) \in \mathfrak{h}(-,+)[[h]]\otimes\mathcal{M}_2(\mathbb{C})$ is isomorphic to the formal dual pseudo-unitary group $\mathcal{U}^h(-,+)$ (but not as a bi-algebras).

The following proposition draw comparisons between the two computed Schürmann triples. To make a clear distinctions between those two triples, we use the notations $\left(\pi^\star,\eta^\star,\mathcal{G}\right)$ for the Schürmann triple associated with W and $(\pi,\eta,\mathcal{L})$ for the one of the pseudo-unitary Brownian motion. Each of the map in the triple $\left(\pi^\star,\eta^\star,\mathcal{G}\right)$ extends to a $\mathbb{C}[[h]]$ linear map over $\mathfrak{h}(-,+)[[h]]$. We add a subscript h to denote these extensions. On the algebra generated by the coefficient of O^h, we define a Schürmann triple $(\pi_h,\eta_h,\mathcal{L}_h)$ by requiring for these map $\mathbb{C}[[h]]$-linearity and setting

$$\eta_h(O^h_{++}) = h\sqrt{v}\varepsilon_{++},\ \eta_h(O^h_{--}) = h\sqrt{v}\varepsilon_{--},\ \eta_h(O^h_{+-}) = h\sqrt{c}\varepsilon_{+-},\ \eta_h(O^h_{-+}) = h\sqrt{c}\varepsilon_{-+},$$
$$\eta_h((O^h_{++})^t) = -h\sqrt{v}\varepsilon_{++},\ \eta_h((O^h_{--})^t) = -h\sqrt{v}\varepsilon_{--},\ \eta_h((O^h_{+-})^t) = hc\varepsilon_{+-},\ \eta_h((O^h_{-+})^t) = hc\varepsilon_{-+},$$

and $\pi_h(O^h_{\varepsilon,\varepsilon'}) = \delta_{\varepsilon=\varepsilon'}$, $\mathcal{L}_h(u_{\varepsilon,\varepsilon'}) = \delta_{\varepsilon,\varepsilon'}\frac{h(v-c)}{2}$. If,formally, h is set equal to one, we find back for $(\pi_h,\eta_h,\mathcal{L}_h)$ the Schürmann triple defined earlier for W. This *formal* triple $(\pi_h,\eta_h,\mathcal{L}_h)$ corresponds to a formal dilatation of time $t\mapsto ht$.

Proposition 3.36. *On $\mathcal{O}^h(-,+)$, $\left(\pi^\star_h,\eta^\star_h,\mathcal{G}_h\right) = (\pi_h,\eta_h,\mathcal{L}_h)$.*

Proof. It is sufficient to prove that $\pi^\star_h = \pi$ and $\eta^\star_h = \eta$ from the co-cycles property of the generators. Recall that $H = \begin{bmatrix} h_+ & h^\star_{+-}\\ h_{+,-} & h_-\end{bmatrix}$. For the two representation $\pi^\star_h$ and π_h, this is trivial:

$$\pi^\star(O^h) = \exp\left(h\pi^\star_h(H)\right) = I = \pi_h(O^h).$$

Concerning the two co-cycles $\eta^\star_h$ and η_h, one has

$$\eta^\star_h(O^h) = \eta^\star_h(\exp(hH)) = \sum_k \frac{h^k}{k!}\eta^\star(H^k) = \eta^\star_h(I) + h\eta^\star_h(H) + \sum_{k\geq 3}^{\infty}\frac{h^k}{k!}\eta^\star_h(H)$$

Since $\varepsilon(H) = 0$, the cocyle property of $\eta^\star_h$ implies the equation $\forall k\geq 2$, $\eta^\star_h(H^k) = 0$. Hence, $\eta^\star_h(O^h) = h\begin{bmatrix}\sqrt{c} & \sqrt{v}\\ \sqrt{v} & \sqrt{c}\end{bmatrix} = \eta_h(O^h)$. This concludes the proof. □

Definition 3.37. A Lévy process on a bi-algebra $\mathcal{B}$ is called Gaussian process if the relation generating functional L vanishes on all triple products of elements of $\ker(\varepsilon)$.

The last definition can be rephrased in terms of other elements of the Schürmann triple (π,η,L). With the notation

$$K_m = \langle a_1\cdots a_m,\ a_i\in\ker(\varepsilon),\ i\leq m\rangle,$$

one can easily show that the following conditions are equivalent.

(1) $L_{|K_3} = 0$,

(2) $\eta_{|K_2} = 0$,
(3) $\pi(a) = \varepsilon(a)$ for $a \in \mathcal{B}$.

A direct corollary of Propositions 3.34 and 3.33 is the following one.

PROPOSITION 3.38. *W and V are two Gaussian free Lévy processes.*

3.8. Convergence of the split pseudo-antihermitian Brownian motion and of the split pseudo-unitary Brownian motion

Pick $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and a time $t \geq 0$, we define the random variable $V_p^{\mathbb{K}}$ on the pseudo-orthogonal dual group $\mathcal{O}(-,+)$ by setting

$$\begin{array}{cccc} V_p^{\mathbb{K}}(t): & \mathcal{O}(-,+) & \to & \mathcal{M}_2(\mathbb{R}) \otimes \left(L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \otimes \mathcal{M}_p(\mathbb{K})\right) \\ & \begin{bmatrix} u_{++} & u_{-+} \\ u_{-+} & u_{--} \end{bmatrix} & \mapsto & \begin{bmatrix} \Lambda_{\mathbb{K}}^{p,p}(t)_+^+ & \Lambda_{\mathbb{K}}^{p,p}(t)_-^+ \\ \Lambda_{\mathbb{K}}^{p,p}(t)_+^- & \Lambda_{\mathbb{K}}^{p,p}(t)_-^- \end{bmatrix} \end{array}. \tag{3.78}$$

The main result on which Theorems 3.43 and 3.42 relies on is the following one.

THEOREM 3.39. *Pick a division algebra $\mathbb{K}$ equal to $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Let $(A_{N,1}, \ldots, A_{N,n})_{N \geq 1}$ and $(B_{N,1}, \ldots, B_{N,n})_{N \geq 1}$ be two sequences of random matrices with coefficients in $\mathbb{K}$. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be two families of elements of a non commutative probability space $(\mathcal{A}, \tau)$. Assume that the convergence in non-commutative distribution*

$$(A_{N,1}, \ldots, A_{N,n}) \to (a_1, \ldots, a_n) \text{ and } (B_{1,N}, \ldots, B_{N,n}) \to (b_1, \ldots, b_n)$$

holds. Assume also that for all N, given a random matrix U distributed according to the Haar measure on $U(N, \mathbb{K})$ and independent of $(A_{N,1}, \ldots, A_{N,n}, B_{N,1}, \ldots, B_{N,n})$, the two families $\left(A_{N,11}, \ldots, A_{N,n}, B_{N,1}, \ldots, B_{N,n}\right)$ and $\left(UA_{N,1}U^{-1}, \ldots, UA_{N,n}U^{-1}, b_{N,1}, \ldots, B_{N,n}\right)$ have the same distribution. Then, the families $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are free.

3.8.1. Convergence of the split antihermitian Brownian motion in high dimensions. We focus on the complex case. Before stating a result related to convergence of Gaussian process, we adress the case of a single Gaussian random variable. We defined X_{2p} as a pseudo-Hermitian random matrix with complex independent Gaussian distributed random entries up to the symmetries. The entries of the two diagonal blocks of size dimension $p \times p$ of X_{2p} have variance equal to $\frac{c}{p}$. The entries of the upper right block of X_{2p} have variance $\frac{v}{p}$.

Let $(\mathcal{A}, \phi)$ be a von Neumann algebra with cyclic trace. We assume that in the algebra $\mathcal{A}$ there exists two projectors p_+ and p_- with $\phi(p_+) = \phi(p_-) = \frac{1}{2}$. Let $a^+ \in p_+\mathcal{A}p_+$, $a^- \in p_-\mathcal{A}p_-$ be two(mutually free) semi-circular elements and $x \in p_+\mathcal{A}p_-$ a circular element (free from $\{a, b\}$). On $\mathcal{A}$, we define the condition expectation $E : \mathcal{A} \to \mathcal{M}_2(\mathbb{C})$ by

$$E(a) = \frac{\phi(p_+ a p_-)}{\phi(p_+)} p_+ + \frac{\phi(p_- a p_-)}{\phi(p_-)} p_-$$

Set

$$P_p^+ = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}, \; P_p^- = \begin{bmatrix} 0 & 0 \\ 0 & I_p \end{bmatrix} \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})) \otimes \mathcal{M}_{2p}(\mathbb{C}).$$

In the next proposition, the random matrix X_{2p} is viewed as an element of the operator-valued non-commutative probability space $\left(L^\infty(\Omega, \mathcal{A}, \mathbb{P}) \otimes \mathcal{M}_{2p}(\mathbb{C}), \, E_{2p}\right)$ with the conditional expectation $E_{2p} : L^\infty(\Omega, \mathcal{A}, \mathbb{P}) \otimes \mathcal{M}_{2p}(\mathbb{C}) \mapsto \mathcal{M}_2(\mathbb{C})$ satisfying

$$E_{2p}(A) = \left(\mathbb{E} \otimes \left(\frac{1}{p}\mathrm{Tr}\right)\right)\left[P_p^+ A P_p^+\right] P_p^+ + \left(\mathbb{E} \otimes \left(\frac{1}{p}\mathrm{Tr}\right)\right)\left[P_p^- A P_p^-\right] P_p^-.$$

PROPOSITION 3.40. *As $p \to +\infty$, X_{2p} converges in non-commutative distribution to the element $\begin{bmatrix} ca & vx^\star \\ vx & cb \end{bmatrix}$ of the operator valued probability space $(\mathcal{A}, E_2)$.*

Proof. The matrix X_{2p} is written as $X_{2p} = \begin{bmatrix} \sqrt{\frac{c}{p}} X_p^+ & \sqrt{\frac{v}{p}} \left(X_p^{-,+}\right)^\star \\ \sqrt{\frac{v}{p}} X_p^{-,+} & \sqrt{\frac{c}{p}} X_p^- \end{bmatrix}$ with X_p^+, X_p^-, $X_p^{+,-}$ matrices of size $p \times p$ filled with standard independent Gaussian random variables. The matrix X_{2p} is alternatively written using random Gaussian Hermitian matrices as

$$\sqrt{2c} P_p^+ H^+(p) P_p^+ + \sqrt{2c} P_p^- H^-(p) P_p(-) + \sqrt{2v}\mathrm{i}\left(P_p(+) H^{+,-}(p) P_p^- + P_p^+ H^{+,-}(p) P_p^-\right)$$

with H_p^+ a random matrix drawn from the Gaussian Unitary Ensemble with variance $\frac{1}{2p}$ that have the matrix $\frac{1}{\sqrt{2p}} X_p^+$ at the upper left corner, X_p^- is defined similarly and has the matrix $\frac{1}{\sqrt{2p}} X_p^-$ at the lower right corner. The matrix $H_p^{+,-}$ is drawn from the Gaussian Unitary Ensemble, has variance $\frac{1}{2p}$ and has the matrix $X_p^{-,+}$ at its lower left corner and $\left(X_+^{-,+}\right)^\star$ at its upper right corner.

In the sequel, all random matrices are considered in the probability space $\left(\mathcal{M}_{2p}(\mathbb{C}), \frac{1}{p}\mathbb{E} \otimes \mathsf{Tr}\right)$. The random variables P_p^+ and P_p^- converge to p_+ and p_- are asymptotically free. Since the joint distribution of $\{H_p^+, H_p^-, H_p^{+,-}\}$ is invariant by conjugation by any unitary matrix in the unitary group $U(2p)$ and since the matrices $H_p^+, H_p^-, H_p^{+,-}$ are independent from $\{P_+^p, P_-^p\}$ for each $p \geq 1$, by using Theorem 3.39 we prove that

$$\{H_p^+, H_p^-, H_p^{+,-}\} \text{ is asymptoticaly free from } \{P_p^+, P_p^-\}.$$

By using, once again, Theorem 3.39, one proves that the random matrices H^+, H^- and $H^{+,-}$ are asymptotically mutually free random variables. Hence, the random variables $\{H_p^+, H_p^-$ and $H_p^{+,-}\}$ if seen in $\left(L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \otimes \mathcal{M}_{2p,2p}(\mathbb{C}), \mathbb{E} \otimes \frac{1}{2p}\mathsf{Tr}\right)$ converges in non-commutative distribution to $\{a^+, a^-, x\}$. This concludes the proof. □

In the last proposition, the random matrix X_{2p} is seen as an element of the operator valued probability space $L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \otimes \mathcal{M}_{2p}(\mathbb{C}), E_{2p}$. The last proposition does not state the convergence of the moments $\frac{1}{p}(\mathbb{E} \otimes \mathsf{Tr})\left[X_p^+ X_p^-\right]$. In fact, the matrices X_p^+, X_p^- are injected in the algebra $\mathcal{M}_{2p,2p} \otimes L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ as

$$\tilde{X}_{2p} = \begin{bmatrix} X_p^+ & 0 \\ 0 & 0 \end{bmatrix}, \; \tilde{X}_p = \begin{bmatrix} 0 & 0 \\ 0 & X_p^- \end{bmatrix}.$$

The product of the last two matrices is equal to zero.

To the matrix X_p is associated a random variable from the algebra $\natural(-,+)$ and valued in the probability space $\left(\mathcal{M}_{p,p}(\mathbb{C}) \otimes L^\infty(\Omega, \mathcal{F}, \mathbb{P}), \frac{1}{p}(\mathbb{E} \otimes \mathsf{Tr})\right)$. These random variable j_{X_n} is defined as the unique $\star$-morphism such that:

$$j_{X_{2p}}(h_+) = \mathrm{i}X_p^{++}, \; j_{X_{2p}}(h_-) = \mathrm{i}X_p^{--}, \; j_{X_{2p}}(h_{+-}) = \mathrm{i}X_p^{+-}.$$

The following proposition straighten the convergence proved in proposition 3.40.

Proposition 3.41. *As $p \to +\infty$, the random variable $j_{X_{2p}}$ converges to $W(1)$*

Proof. The matrices X_p^+ and X_p^- converge to semi-circular elements. Since for all $p \geq 1$ X_p^+ is independent from X_p^-, X_p^+ is asymptotically free from X_p^-. Hence, $\{X_p^+, X_p^-\}$ converges in distribution to $\{a^+, a^-\}$. The matrix X_p^{+-} converges in non-commutative distribution to the circular element x. In addition, X_p^{+-} is independent from $\{X_p^+,\, X_p^-\}$, by using Theorem 3.39, we prove that X_p^{+-} is asymptotically free from $\{X_p^+,\, X_p^-\}$. To sum up, the mixed moments of $\{X_p^{+-}, X_p^{++}, X_p^{--}\}$ converges to the mixed moments of $\{a^+, a^-, x\}$. This concludes the proof. □

Since $W(t)$ has the same non commutative distribution as $\sqrt{t}W(1)$ and since $W_p^{\mathbb{C}}(t)$ has the same non-commutative distribution as $\sqrt{t}W_p^{\mathbb{C}}(1)$, the last proposition implies that

$$W_p^{\mathbb{C}}(t) \xrightarrow[p \to +\infty]{\text{n.c}} W(t), \text{ for all } t \geq 0.$$

THEOREM 3.42. *Let $\mathbb{K}$ a finite dimensional division algebra. As $p \longrightarrow +\infty$, $W_p^{\mathbb{C}}(t)$ converges to W.*

PROOF. The increment between times s and t of the processes $W^{p,q}$ and W are denoted by $W^{p,q}(s,t)$ and $W(s,t)$:

$$W_p(s,t) = \Big(W_p(t) \otimes \Big(W_p(s) \circ S\Big)\Big) \circ \Delta,\ W(s,t) = (W(t) \otimes (W(s) \circ S)) \circ \Delta.$$

We prove that increments of $W^{p,q}$ converge to the increments of W. Let $p \geq 1$ be an integer and $s_1 < t_1 \leq s_2 < t_2 \ldots s_p < t_p$ be a tuple of times and $w_1, \ldots, w_p \in \mathfrak{h}(-,+)$. The proof is done by induction on p. Assume that $\{W_p^{\mathbb{C}}(s_1,t_1)(w_1), \ldots, W_p^{\mathbb{C}}(s_{p-1},t_{p-1})(w_p)\}$ converge to $\{W_{\mathbb{C}}(s_1,t_1)(w_1), \ldots, W_{\mathbb{C}}(s_p,t_p)(w_p)$. We proved that $W_p^{\mathbb{C}}(s_p,t_p)$ converge in non-commutative distribution to $W(s_p,t_p)$. Since $W_p^{\mathbb{C}}(s_p,t_p)(w_p)$ is independent from $\{W_p^{\mathbb{C}}(s_1,t_1)(w_1)\ldots$ $W_p^{\mathbb{C}}(s_{p-1},t_{p-1})(w_{p-1})\}$ and since for any unitary matrix $U \in U(2p)$ we have

$$\begin{aligned}\{W_p^{\mathbb{C}}(s_1,t_1)(w_1), \ldots, W_p^{\mathbb{C}}(s_{p-1},t_{p-1})(w_{p-1}), UW^{p,q}(s_p,t_p)(w_p)U^\star\}\\ \sim \{W_p^{\mathbb{C}}(s_1,t_1)(w_1), \ldots, W_p^{\mathbb{C}}(s_{p-1},t_{p-1})(w_{p-1}), W^{p,q}(s_p,t_p)(w_p)\},\end{aligned}$$

we use Theorem 3.39 and the induction hypothesis to prove that the family $\{W_p^{\mathbb{C}}(s_1,t_1)(w_1),\ldots$ $W_p^{\mathbb{C}}(s_{p-1},t_{p-1})(w_{p-1}), W^{p,q}(s_p,t_p)(w_p)\}$ is asymptotically free. This concludes the proof for the complex case. □

3.8.2. Convergence of the split pseudo-unitary Brownian motion.

THEOREM 3.43. *For each $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, the split process $V_p^{\mathbb{K}}$ converges in non-commutative distribution, as $p \to +\infty$, to the process V.*

PROOF. Let $t \geq 0$. We prove the convergence in non-commutative distribution of $V_p^{\mathbb{R}}(t)$ to $V(t)$, the quaternionic and complex cases are left to the reader. Let $o = o_1^{\varepsilon_1} \cdots o_k^{\varepsilon_k}$ be a word on the generators of $\mathcal{O}(-,+)$. Let $t \geq 0$, the distribution of $V(t)$ is denoted $\nu(t)$ and the distribution of $V_p(t)$ is denoted $\nu_p(t)$. Since $t \to \nu(t)$ is a (free) semi-group, we have

$$\frac{d}{dt}\nu(t) = \mathcal{L}(\nu(t)), \nu(0) = \delta_{\Delta_0}.$$

Owing to formula 3.49, there exists a Brauer diagram $b_o \in \mathcal{B}_k$ such that $\nu_p(t)(o) = \mathbb{m}_p^{\mathbb{R}}(b_o, t)$. The diagram b_o can be chosen so that

$$b_o = \prod_{i \leq k : \varepsilon_i = \star} \mathsf{Tw}_i(b_{(1,\ldots,k)})$$

with $b_{(1,\ldots,k)}$ the diagram that pictures the permutation $(1,\ldots,k)$. Define $\tilde{L} \in \mathcal{U}(-,+)^\star$ by $\tilde{L}(o) = L(b_o)$ with o as before. We use Proposition 3.26. As $p \to +\infty$, $\mathbb{m}_p^{\mathbb{R}}(b_o,t)$ converges to $\mathbb{m}(b_o,t)$. The function $\mathbb{m}$ satisfies $\frac{d}{dt}\mathbb{m}(t) = L(\mathbb{m}(t))$, $t \geq 0$. In addition, the function $\tilde{\mathbb{m}}(t) \in \mathcal{U}(-,+)^\star$ defined by $\tilde{\mathbb{m}}(t)(o) = \mathbb{m}(b_o,t)$ with o as before satisfies the differential equation $\frac{d}{dt}\tilde{\mathbb{m}}(t) = \tilde{L}(\tilde{\mathbb{m}}(t))$, $t \geq 0$. Thus, to prove the result, we have to show that $\tilde{L} = \mathcal{L}$. This last equality is a simple consequence of the caracterisation 2.22 of the set $\mathsf{T}^+(b)$ and $\mathsf{W}^+(b)$ is $b \in \mathcal{B}_k$. To prove the convergence of the multi-dimensional marginals of $V_p^{\mathbb{R}}$ to the multi-dimensional marginals of V, we use the reasoning developed in Proposition 3.42. There is no difficulties, this part of the proof is thus left to the reader. The quaternionic case does not show either extra difficulties and is not done for brevity. For the complex case, remark that given a word $o = o_1^{\varepsilon_1} \cdots o_k^{\varepsilon_k}$ on the generators of $\mathcal{O}(-,+)$ we can define a Brauer diagram b_o as before and a word $w_o \in \bar{\mathsf{M}}_k$ by setting $w_o = x_1^{\varepsilon_1} \cdots x_k^{\varepsilon_k}$. The pair (b_o, w_o) is then a compatible pair and the result relies on equation (3.61). □

3.9. Pseudo-unitary Brownian motions in high dimensions: the general case

In the first part of this Chapter, we studied split pseudo-unitary Brownian motions in high dimensions, proved the convergence in non-commutative distributions and exhibited a free stochastic differential equation the limiting process is solution of. We adress now the general case for which the underlying metric is not assumed to be split. Let $p,q>0$ two integers. Let $\mathbb{K}$ be one of the three division algebras $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. We are concerned with the convergence in non-commutative distribution of the process $\Lambda_{\mathbb{K}}^{p,q}$ in the limit

$$\text{(H)}\qquad p,q\to+\infty \quad \text{with} \quad \frac{p}{q}\to\lambda>0.$$

We would like to put the emphasize on the fact that the ratio $\frac{p}{q}$ is assumed to converge to a strictly positive number. We suspect the case $\lambda=0$ to cover multiple asymptotic regimes. In the previous work, related to the convergence of the split process $\Lambda_{\mathbb{K}}^{p,p}$, we settled a framework of non-commutative probability by defining, in particular, the structure Zhang algebra $\mathcal{O}(-,+)$ we called the pseudo-unitary dual group lying in the category of involutive algebra we denoted in the first Chapter 1, $\mathrm{Alg}^{\star}$. For the general case ($\lambda\neq 1$), we define a structure Zhang algebra for the process $\Lambda_{\mathbb{K}}^{p,q}$ lying in the category of involutive algebras bi-module. We introduced in Section 3.9.1 the rectangular pseudo-unitary algebra $\mathcal{RO}(-+)$ and make a brief reminder on rectangular probability spaces. In Section 3.9.2, we define the statistics of the process $\Lambda_{\mathbb{K}}^{p,q}$ which asymptotic under Hypothesis (H) is the main concern of the present Section.

3.9.1. The rectangular pseudo-unitary algebra. In this section, we define the structure algebra $\mathcal{RO}(-+)$ for the process $\Lambda_{\mathbb{K}}^{p,q}$ and we introduce the reader with the notion of rectangular probability spaces, of non-commutative conditional expectation and the related notion of convergence.

Definition 3.44. The algebra $\mathcal{RO}(-,+)$ is the real unital algebra generated by four elements o,o^t,p_+,p_- subject to the relations:

$$p_+^2=p_+,\ p_-^2=p_-,\ p_+p_-=p_-p_+=0.$$

$$\begin{bmatrix} o_{++} & o_{+-}\\ o_{-+} & o_{--}\end{bmatrix}\begin{bmatrix} o_{++}^t & -o_{+-}^t\\ -o_{-+}^t & o_{-+}^t\end{bmatrix}=\begin{bmatrix} o_{++}^t & -o_{+-}^t\\ -o_{-+}^t & o_{-+}^t\end{bmatrix}\begin{bmatrix} o_{++} & o_{+-}\\ o_{-+} & o_{--}\end{bmatrix}=\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}\in\mathcal{M}_2(\mathbb{C})\otimes\mathcal{RO}(-+),$$

with the notations:

$$o_{++}=p_+op_+,\ o_{--}=p_-op_-,\ o_{+-}=p_+op_-,\ o_{-+}=p_-op_+,$$
$$o_{++}^t=p_+o^tp_+,\ o_{--}^t=p_-o^tp_-,\ o_{+-}^t=p_+o^tp_-,\ o_{-+}^t=p_-o^tp_+.$$

Denote by $\mathcal{R}$ the algebra generated by p_+ and p_-. The two elements p_+ and p_i are projectors which, together, form a resolution of the identity of $\mathcal{RO}(-+)$. The algebra $\mathcal{R}$ is a commutative subaglebra of $\mathcal{RO}(-+)$ and acts by left and right multiplication on $\mathcal{RO}(-+)$ turning this latter algebra into a $\mathcal{R}$-bimodule algebra. Moreover, if endowed with the involutive anti-morphism $\star:\mathcal{RO}(-+)\to\mathcal{RO}(-+)$ that takes the following values on the set of generators $\mathcal{RO}(-+)$:

$$\star(p_+)=p_+,\ \star(p_-)=p_-,\ \star(o)=o^t,\tag{3.79}$$

the algebra $\mathcal{RO}(-+)$ is turned into an involutive $\mathcal{R}$-bi-module algebra. The complexification $\mathcal{RU}(-,+)=\mathcal{RO}(-+)\otimes\mathbb{C}$ is an involutive algebra if one requires that equation (3.79) defines an anti-linear anti-morphism.

For each time $t\geq 0$, the non-commutative random variable $V_{p,q}^{\mathbb{K}}:\mathcal{RO}(-+)\to\mathcal{M}_{p+q}(\mathbb{K})$ is the unique $\mathbb{R}$-linear $\star$-morphism such that:

$$V_{p,q}^{\mathbb{K}}(o)=\Lambda_{\mathbb{K}}^{p,q}(t),\ V_{p,q}^{\mathbb{K}}(p_+)=I_p,\ V_{p,q}^{\mathbb{K}}(p_-)=I_{p,q}.\tag{3.80}$$

The rectangular unitary algebra is a Zhang algebra in the Category $\mathrm{Alg}^{\star}(\mathcal{R})$ (see Chapter 1 for a detailed introduction on Zhang alebras). The structural morphisms Δ,ε and S are defined as the unique involutive morphisms of $\mathrm{Alg}^{\star}(\mathcal{R})$ that extend:

$$\begin{aligned}
&\Delta : \mathcal{RO}(-,+) \to \mathcal{RRO}(-,+) \sqcup_{\mathcal{D}} \mathcal{RO}(-,+), && \Delta(o) = o_1 o_2,\ \Delta(p_\pm) = p_\pm,\\
&S : \mathcal{RO}(-,+) \to \mathcal{RO}(-,+), && S(o) = o^t_{++} + o^t_{--} - o^t_{+-} - o^t_{-+}, S(p_\pm) = p_\pm\\
&\varepsilon : \mathcal{RO}(-,+) \to \mathcal{RO}(-,+), && \varepsilon(o) = 1, \varepsilon(p_\pm) = p_\pm.
\end{aligned}$$

The process $V^{\mathbb{K}}_{p,q}$ is a quantum stochastic process with structure Zhang algebra the rectangular pseudo-unitary dual group, it is however not a Lévy process. To define the non-commutative distribution of $V^{\mathbb{K}}_{p,q}$, we introduce a conditional expectation on $\mathcal{M}_n(L^\infty(\Omega,\mathcal{F},\mathbb{P}))$. Before that, we discuss generalities on rectangular probability spaces.

Let $n \geq 1$ be an integer. Consider a $\star$-algebra $\mathcal{A}$ endowed with a family $(p_1,\dots,p_n)$ of non-zero self-adjoint projectors, $\forall i \leq n, p_i^2 = p_i$, that are pairwise orthogonal, $\forall 1 \leq i \neq j \leq n$, $p_i p_j = p_j p_i = 0$, and such that $p_1 + \ldots + p_n = 1$. Any element $x \in \mathcal{A}$ can then be represented as

$$x = \begin{bmatrix} x_{11} & \dots & x_{1n} \\ \vdots & & \vdots \\ x_{d1} & \dots & x_{nn} \end{bmatrix},\ \forall 1 \leq i,j \leq n,\ x_{ij} = p_i x p_j.$$

Let $1 \leq k \leq n$ be an integer and assume that the sub-algebra $p_k \mathcal{A} p_k$ is endowed with a tracial state ϕ_k satisfying

$$\phi_k(p_k) = 1,\ x,y \in p_k\mathcal{A}p_k,\ \phi_k(xy - yx) = 0.$$

We assume further the existence of a sequence $(\rho_k,\ k \leq n)$ of non-negative real numbers with at most one equal to zero such that for all $k,l \in [1,\dots,n]$, $x \in p_k\mathcal{A}p_l$, $y \in p_l\mathcal{A}p_k$,

$$\rho_k \phi_k(xy) = \rho_l \phi_l(yx).$$

Elements of the union of $p_k\mathcal{A}p_l$ are called *simple elements*. The triple $(\mathcal{A}, p_1,\dots,p_n,\phi_1,\dots,\phi_n)$ is named a $(\rho_1,\dots,\rho_n)$ *rectangular probability space*. Denote by $\mathcal{R}$ the linear span of the $p_k^{'}s$, then $\mathcal{R}$ is an algebra and the function $E : \mathcal{A} \longrightarrow \mathbb{C}^n$, which maps $x \in \mathcal{A}$ to

$$E(x) = \sum_{k=1}^{n} \phi_k(x_{kk}) p_k$$

is a *conditional expectation* from $\mathcal{A}$ to $\mathcal{D}$,

$$E(1) = 1 \text{ and } \forall (d,a,d') \in \mathcal{D}\times\mathcal{A}\times\mathcal{D}, E(da'd') = dE(a)d'.$$

Definition 3.45. In a $(\rho_1,\dots,\rho_n)$-rectangular probability space, the $\mathcal{R}$-distribution of a family $(a_j)_{j\in J}$ of simple elements is the function which maps any polynomial P in the non-commutative variables $(X_j, X_j^\star)_{j\in J}$ with coefficients in the algebra $\mathcal{R}$, to $E(P(a_j,a_j^\star)_{j\in J})$.

Note that all the $\mathcal{R}'^s$ of different rectangular spaces can all be identified with $\mathbb{C}^n$.

Definition 3.46. If for all p, $(\mathcal{A}^p, p_n^1,\dots,p_n^p,\phi_1^p,\dots,\phi_n^p)$ is a $(\rho_1^p,\dots,\rho_n^p)$ probability space such that

$$(\rho_{1,p},\dots,\rho_{n,p}) \underset{p\to+\infty}{\longrightarrow} (\rho_1,\dots,\rho_n)$$

a family $(a_j(n))_{j\in J}$ of simple elements of $\mathcal{A}^p$ is said to converge in $\mathcal{D}$-distribution, when p goes to infinity, to a family $(a_j)_{j\in J}$ of elements of $\mathcal{A}$ if the $\mathcal{R}$-distributions converge point-wise.

Put $\rho_+ = p, \rho_- = q$, $p_+ = I_p$, $p_- = I_{p,q}$ and set $\mathrm{Tr}_{\mathbb{R}} = \mathrm{Tr}_{\mathbb{C}} = \mathrm{Tr}, \mathrm{Tr}_{\mathbb{H}} = \mathcal{R}e\mathrm{Tr}$. Define the two states $\phi_+^{\mathbb{K}} = \mathbb{E}\otimes \frac{1}{p}\mathrm{Tr}_{\mathbb{K}}$ (resp. $\phi_-^{\mathbb{K}} = \mathbb{E}\otimes\frac{1}{q}\mathrm{Tr}_{\mathbb{K}}$) on the space of matrices $p_+\mathcal{M}_{p+q}(L^\infty(\Omega,\mathcal{F},\mathbb{K}))p_+$ (resp. $p_-\mathcal{M}_{p+q}(L^\infty(\Omega,\mathcal{F},\mathbb{K}))p_-$) which coefficients are all zero except for those in the upper left block of dimensions $p\times p$ (resp. which coefficients are all zero except for those in the lower right block of dimensions $q\times q$). Matrices in $p_\pm\mathcal{M}_{p+q}(L^\infty(\Omega,\mathcal{F},\mathbb{K}))p_\pm$ are pictured as

$$p_+\mathcal{M}_{p+q}(L^\infty(\Omega,\mathcal{F},\mathbb{K}))p_+ = \left\{\begin{bmatrix} A & 0 \\ 0 & 0\end{bmatrix},\ A \in \mathcal{M}_p(L^\infty(\Omega,\mathcal{F},\mathbb{K}))\right\},$$

$$p_-\mathcal{M}_{p+q}(L^\infty(\Omega,\mathcal{F},\mathbb{K}))p_- = \left\{\begin{bmatrix} 0 & 0 \\ 0 & A\end{bmatrix},\ A \in \mathcal{M}_q(L^\infty(\Omega,\mathcal{F},\mathbb{K}))\right\}.$$

A conditional expectation on $(\mathcal{A},\rho_+,\rho_-,\phi_+,\phi_-)$ is defined by

$$E^{\mathbb{K}}_{p,q}(X)=\begin{bmatrix}\phi^{\mathbb{K}}_+(X^+_+) & 0\\ 0 & \phi^{\mathbb{K}}_-(X^-_-)\end{bmatrix},\ X=\begin{bmatrix}X^+_+ & X^+_-\\ X^-_+ & X^-_-\end{bmatrix}. \tag{3.81}$$

For each time $t\geq 0$, the conditional expectation $E^{\mathbb{K}}_{p,q}(t)$ on $\mathcal{RO}(-,+)$ is the pull-back of the expectation $E^{\mathbb{K}}_{p,q}$ by the random variable $V^{\mathbb{K}}_{p,q}$: $\mathbb{E}^{\mathbb{K}}_{p,q}(t)=E^{\mathbb{K}}_{p,q}\circ V^{\mathbb{K}}_{p,q}$. We define the two states $\phi_\pm(t)$ by the equation:

$$\mathbb{E}^{\mathbb{K}}_{p,q}(t)(w)=\phi^{p,q}_\pm(t)(w)p_\pm \text{ for } w\in p_\pm\mathcal{RO}(-+)_\pm.$$

With these definitions,

$$\left(\mathcal{RO}(-+),\ \phi^{p,q}_+(t),\ \phi^{p,q}_-(t),\ \frac{p}{p+q},\ \frac{q}{p+q}\right) \tag{3.82}$$

is a rectangular probability space of for all $t\geq 0$. Let $l\geq 1$ an integer, and $(s_1<t_1\leq\ldots s_l<t_l)$ a tuple of times. The amalgamated free product $\mathcal{RO}(-+)^{\sqcup s}=\mathcal{RO}(-+)\sqcup_{\mathcal{R}}\cdots\sqcup_{\mathcal{R}}\mathcal{RO}(-+)$ is a rectangular probability space. In fact, $\mathcal{R}$ is canonically injected in $\mathcal{RO}(-+)^{\sqcup s}$ and $\{p_+,p_-\}$ is a complete set of projectors. We define a conditional expectation $\mathbb{E}_{p,q}(s_1,t_1,\ldots,s_l,t_l)$ on $\mathcal{RO}(-+)^{\sqcup l}$ by setting:

$$\mathbb{E}^{p,q}(s_1,t_1,\ldots,s_l,t_l)(w)=\mathbb{E}_{p,q}\Big[\Big(V^{\mathbb{K}}_{p,q}(s_1,t_1)\sqcup\ldots\sqcup V^{\mathbb{K}}_{p,q}(s_l,t_l)\Big)(w)\Big],\ w\in\mathcal{RO}(-+)^{\sqcup l}. \tag{3.83}$$

We use the symbol $\phi^{p,q}_\pm(s_1,t_1,\ldots,s_l,t_l)$ to denote the restriction of the conditional expectation $E^{p,q}_s(s_1,t_1,\ldots,s_l,t_l)$ to $p_\pm\mathcal{RO}(-+)^{\sqcup_{\mathcal{D}}s}p_\pm$.

In the next sections we are concerned with the convergence of the conditional expectation $\mathbb{E}_{p,q}(t)^{\mathbb{K}}$ under the hypothesis (H). We show in particular that under (H), $\mathbb{E}_{p,q}$ is an asymptotic amalgamated free convolution semi-group.

3.9.2. The normalized statistics of the Brownian pseudo-unitary diffusions. Let K be one of the three divisions algebras $\mathbb{R},\mathbb{C}$ and $\mathbb{H}$. Let $k,p,q\geq 1$ be integers, set $n=p+q$ and let $\mathrm{t}=(t_1,\ldots,t_s)\in\mathbb{R}^q_+$ be a tuple of times. We pick in addition s independent copies

$$\mathbf{\Lambda}^{\mathbb{K}}_{p,q}=\left(\left[\Lambda^{p,q}_{\mathbb{K}}\right]^1,\ldots,\left[\Lambda^{p,q}_{\mathbb{K}}\right]^s\right)$$

of the pseudo-unitary diffusion $\Lambda^{p,q}_{\mathbb{K}}$. In that section, we define normalized statistics $\mathbbm{m}^{\mathbb{K}}_{p,q}(\mathrm{t}):\mathcal{OB}_k(p,q)\to\mathbb{R}$ of the family $\left(\left[\Lambda^{p,q}_{\mathbb{K}}\right]^1,\ldots,\left[\Lambda^{p,q}_{\mathbb{K}}\right]^s\right)$ that are defined on the set of oriented Brauer diagrams which asymptotic under hypothesis (H) will be investigated in the next section. To that end, we use the functions fnc_+ and fnc_- defined in Section 3.5.

Set $\mathbf{\Lambda}^{\mathbb{K}}_{p,q}(t_1,\ldots,t_q)=\left(\left[\Lambda^{p,q}_{\mathbb{K}}\right]^1(t_1),\ldots,\left[\Lambda^{p,q}_{\mathbb{K}}\right]^s(t_s)\right)$. We explain how to construct the statistic $\mathbbm{m}^{\mathbb{K}}_{p,q}(\mathrm{t})$, they will be defined as mean of a function $\mathrm{m}^{\mathbb{R}}_{p,q}(\mathrm{t})$ that is introduced below.

First, remember that if $A_1,\ldots,A_k$ are matrices of size $n\times n$ with entries in $\mathbb{K}$ and (b,s) is an oriented Brauer diagram,

(3.84)

$$\mathsf{Tr}\Big(\rho^{\mathbb{K}}(b)\circ(A_1\otimes\cdots\otimes A_k)\Big)=\prod_{(i_1,\ldots,i_s)\in\sigma_b}\mathsf{Tr}\Big([A(i_1)(c_b(i_1),c_b(i_1'))]^{s(i_1)}\cdots[A(i_s)(c_b(i_s),c_b(i_s'))]^{s(i_s)}\Big).$$

To define the quantity $\mathrm{m}^{\mathbb{R}}_{p,q}((b,s))[A_1,\ldots,A_k]$, we pick a suitable normalization of (3.84) that will break the cylicity of the trace. Since the dimensions of the matrices in the product in the right hand side of (3.84) are not all equal, a linear order on the cycles of σ_b needs to be picked prior to normalize (3.84), it amongst to decide which block in the product sits at the first position. Let $(i_1,\ldots,i_s)\in\sigma_b$ be a cycle of σ_b. We endow $\{i_1,\ldots,i_s\}$ with the linear order that is left out by cutting the minimum of $\{i_1,\ldots,i_q\}$ out of the cyclic order $\left(i_1,\ldots,i_q\right)$. This linear order we defined allows us to associate a word on the blocks of the matrices $A's$ to each cycle of σ_b. Now to decide which dimension of the first block we use as a normalization, we use the orientation s, if s is negative on the minimum of the cycle, we normalize by the number of columns of the blocks, if instead s is positive, we normalize by the number of lines. Let us make a small digression concerning the combinatorial object associated with an oriented

Brauer diagram and a linear order on each of its cycle that allows for the computation of the normalization of (3.84) we sketched. A linear order on a cycle of b is conveniently pictured as a dot, either on the top line or in the bottom line on the diagram that picture b. Of course, two dots lying at the vertical to each other represent the same linear order.

The normalization of (3.84) is pictured by placing this puncture on the minimum of each cycle c of b according to Fig 10.

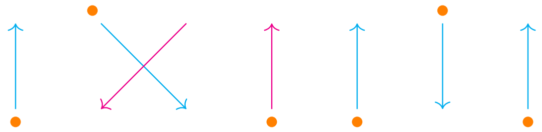


FIGURE 10. An oriented coloured Brauer diagrams. The punctures • indicate the dimension used in each cycle to normalize

The three representations $\rho^{\mathbb{R}}_{p,q}$, $\rho^{\mathbb{C}}_{p,q}$ and $\rho^{\mathbb{H}}_{p,q}$ are defined in Section 3.5.2. We define for an oriented Brauer diagram $(b,s) \in \mathcal{OB}_k$ and matrices $A_1,\ldots,A_k \in \mathcal{M}_{p+q}(\mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$:

$$\mathsf{m}^{\mathbb{K}}_{p,q}((b,s))(A_1,\ldots,A_k) = \frac{1}{q^{\mathsf{fnc}_-(\mathring{b},s)}}\frac{1}{p^{\mathsf{fnc}_+(\mathring{b},s)}}\mathsf{Tr}^{\otimes k}(\rho^{\mathbb{K}}_{p,q}(b)\circ(A_1\otimes\ldots\otimes A_k)). \tag{3.85}$$

3.9.3. The pseudo-unitary Brownian motions in high dimensions. This section is devoted to the proof of our main result, contained in Theorem 3.48, stated below. First, we give formulae for the mean of tensor monomials of the pseudounitary diffusions that will be used to prove Theorem 3.49.

3.9.3.1. *Mean of polynomials of the pseudo-unitary Brownian motions.* Let $\mathbb{K}$ be one of the three division algebras $\mathbb{R}$,$\mathbb{C}$ or $\mathbb{H}$. We recall that the Casimirs $\mathsf{C}^{\mathbb{K}}_{p,q}$ of the scalar product $B^{\mathbb{K}}_{v,c}$ defined in Section 3.4 is related to the quadratic variation of the Brownian motion on $\mathbb{U}(p,q,\mathbb{K})$ through

$$\left(\Lambda^{p,q}_{\mathbb{K}}(t)^{-1}\otimes\Lambda^{p,q}_{\mathbb{K}}(t)^{-1}\right)\left(\mathrm{d}\Lambda^{p,q}_{\mathbb{K}}\otimes\mathrm{d}\Lambda^{p,q}_{\mathbb{K}}\right)(t) = \mathsf{C}^{\mathbb{K}}_{p,q},\ t\geq 0.$$

Using the Sweedler notation, if $A = A_{(1)}\otimes A_{(2)}$ is a bivector in $\mathcal{M}_N(\mathbb{K})\otimes\mathcal{M}_N(\mathbb{K})$ and ι_i is the canonical injection of the i^{th} factor of $\mathcal{M}_N(\mathbb{K})$ in $\mathcal{M}_N(\mathbb{K})^{\otimes k}$, we denote by A_{ij} the tensor $\iota_i(A_{(1)})\iota_j(A_{(2)})$. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$. By using the classical matricial Itô formula, we compute the derivative of the process $\left(\Lambda^{p,q}_{\mathbb{K}}\right)^{\otimes k}$:

$$\mathrm{d}\left(\Lambda^{p,q}_{\mathbb{K}}\right)^{\otimes k}(t) = \sum_{i=1}^{k}\Lambda^{p,q}_{\mathbb{K}}(t)^{\otimes(i-1)}\otimes\mathrm{d}\Lambda^{p,q}_{\mathbb{K}}(t)\otimes\Lambda^{p,q}_{\mathbb{K}}(t)^{\otimes(k-i)} + \left(\sum_{1\leq i<j\leq k}\left(\mathsf{C}^{\mathbb{K}}_{p,q}\right)_{ij}\right)\Lambda^{p,q}_{\mathbb{K}}(t)^{\otimes k}.$$

For the complex case, we have in addition, for $w\in\bar{\mathsf{M}}_1(k)$:

$$\mathrm{d}w^{\otimes}\left(\left(\Lambda^{p,q}_{\mathbb{C}}\right)^{\otimes k}\right)(t) = \sum_{i=1}^{k}w^{\otimes}\left(\Lambda^{p,q}_{\mathbb{C}}(t)^{\otimes(i-1)}\otimes\left(\mathrm{d}\Lambda^{p,q}_{\mathbb{C}}(t)\right)\otimes\Lambda^{p,q}_{\mathbb{C}}(t)^{\otimes(k-i)}\right)$$
$$+\left(\sum_{1\leq i<j\leq k}(w_iw_j)^{\otimes}\left(\mathsf{C}^{\mathbb{K}}_{p,q}\right)_{ij}\right)w^{\otimes}\left(\Lambda^{p,q}_{\mathbb{C}}(t)^{\otimes k}\right).$$

We use the algebra of coloured Brauer diagram introduced in Section 3.5 and the representations $\rho^{\mathbb{K}}_{p,q}$ we defined in the same section to give combinatorial formulae for the Casimirs. This is the content of the next proposition. The following proposition is a downward consequence of equations (3.30), (3.33), (3.40) and (3.47).

PROPOSITION 3.47. *Let $p,q \geq 1$ two integers. Recall that $\mathsf{C}^{00}_{p,q} = \mathsf{C}^{p,q}_{\mathbb{C}}$ and $\mathsf{C}^{01}_{p,q} = (x_1\bar{x}_1)^{\otimes k}\left(\mathsf{C}^{p,q}_{\mathbb{C}}\right)$.*

$$\mathsf{C}^{\mathbb{R}}_{p,q} = \frac{2v}{p+q}\left(\rho^{\mathbb{R}}_{p,q}(\tau^{+-}+\tau^{-+}+e^{+-}+e^{-+})\right)+\frac{c}{p}\rho^{\mathbb{R}}_{p,q}(e^{++}-\tau^{++})+\frac{c}{q}\rho^{\mathbb{R}}_{p,q}(e^{--}-\tau^{--})$$
$$\mathsf{C}^{00}_{p,q} = \frac{2v}{p+q}\rho^{\mathbb{C}}_{p,q}(\tau^{-,+}+\tau^{+,-})-\frac{c}{p}\rho^{\mathbb{C}}_{p,q}(\tau^{++})-\frac{c}{q}\rho^{\mathbb{C}}_{p,q}(\tau^{-,-}),$$
$$\mathsf{C}^{01}_{p,q} = \frac{2v}{p+q}\rho^{\mathbb{C}}_{p,q}(e^{+,-}+e^{-,+})+\frac{c}{p}\rho^{\mathbb{C}}_{p,q}(e^{+,+})+\frac{c}{q}\rho^{\mathbb{C}}_{p,q}(e^{-,-}),$$
$$\mathsf{C}^{\mathbb{H}}_{p,q} = \frac{c}{-2p}\left(\rho^{\mathbb{H}}(e^{+,+})-\rho^{\mathbb{H}}(\tau^{+,+})\right)+\frac{c}{-2q}\left(\rho^{\mathbb{H}}(e^{-,-})-\rho^{\mathbb{H}}(\tau^{-,-})\right)+\frac{v}{-(p+q)}\rho^{\mathbb{H}}_{p,q}(e^{+,-}+\tau^{+,-}).$$

3.9.3.2. *Convergence in high dimensions of the pseudo-unitary diffusions.*

THEOREM 3.48. *For each $t \geq 0$, under the hypothesis (H), the sequence of rectangular probability spaces (3.82) converges.*

For each time $t \geq 0$, we denote by

$$\left(\mathcal{RO}(-+), \phi^{\lambda}_{+}(t), \phi^{\lambda}_{-}(t), \frac{1}{1+\lambda}, \frac{\lambda}{1+\lambda}\right)$$

the limiting rectangular probability space of Theorem 3.48 and by $\mathbb{E}_{\lambda}(t)$ the limit of the conditional expectation. Theorem 3.48 is a corollary of Theorem 3.49 stated below.

THEOREM 3.49. *Let $l \geq 1$ an integer and let $(s_1,t_1,\ldots,s_l,t_l) \in \mathbb{R}^l_+$ be a tuple of times. The rectangular probability space $\left(\mathcal{RO}(-,+)^{\sqcup_{\mathcal{D}} l}, \phi^{p,q}_{+}(s_1,t_1,\ldots,s_l,t_l), \phi^{p,q}_{-}(s_1t_1,\ldots,s_l,t_l), \frac{p}{p+q}, \frac{q}{p+q}\right)$ converges under hypothesis (H). In addition, if we denote by $\mathbb{E}_{\lambda}(s_1,t_1,\ldots,s_l,t_l)$ the limit conditional expectation,*

$$\mathbb{E}_{\lambda}(s_1,t_1,\ldots,s_l,t_l) = \mathbb{E}_{\lambda}(t_1-s_1)\,\dot{\sqcup}\cdots\dot{\sqcup}\,\mathbb{E}_{\lambda}(t_1-s_1).$$

Otherwise stated, the quantum process $V^{\mathbb{K}}_{p,q}$ converges to a free with amalgamation over $\mathcal{R}$ Lévy process.

The rest of this section is devoted to the proof of theorem 3.49. If w is a word in $\mathsf{M}_l(k)$, we denote by $w^{\otimes}_{\mathbb{R}}(ut)$(resp. $w^{\otimes}_{\mathbb{H}}(ut)$) the monomial in $\mathcal{M}_{p+q}(\mathbb{R})^{\otimes k}$ (resp. in $\mathcal{M}_{p+q}(\mathbb{H})$) that is obtained by the substitution $x_i \mapsto \left[\Lambda^{p,q}_{\mathbb{R}}\right]^i(ut_i)$ (resp. $x_i \mapsto \left[\Lambda^{p,q}_{\mathbb{H}}\right]^i(ut_i)$). For a word $w \in \overline{\mathsf{M}}_s(k)$ of length k, define $w^{\otimes}_{\mathbb{C}}(\mathrm{t})$ as the monomial in $\mathcal{M}_{p+q}(\mathbb{C})^{\otimes k}$ obtained by making the substitutions $x_i \mapsto \left[\Lambda^{p,q}_{\mathbb{C}}\right]^i(t_i)$ and $\bar{x}_i \mapsto \overline{\left[\Lambda^{p,q}_{\mathbb{C}}\right]^i}(t_i)$. Now, we write down a differential equation satisfied by the statistics :

$$\mathbb{m}^{\mathbb{K}}_{p,q}(\mathrm{t})(u,w,b) = \mathbb{E}\left[\mathsf{m}^{\mathbb{K}}_{p,q}(w^{\otimes}_{\mathbb{K}}(u\mathrm{t}))\right],\ u\in[0,1], w\in \mathsf{M}_s(k)(w\in\overline{\mathsf{M}}_s(k) \text{ if } \mathbb{K}=\mathbb{C}), b\in\mathcal{OB}_k \tag{3.86}$$

For each time $u\in[0,1]$, the statistic $\mathbb{m}^{\mathbb{C}}_{p,q}(\mathrm{t})(u)$ define a function, denoted by the same symbols, on the tensor product $\mathbb{R}[\mathcal{OB}_k]\otimes\mathbb{R}[\mathsf{M}_l(k)]$ (as always, on $\mathbb{R}[\mathcal{OB}_k]\otimes\mathbb{R}\left[\overline{\mathsf{M}}_l(k)\right]$ for the complex case). Let $\tilde{b}_1$ and $\tilde{b}_2$ be two oriented Brauer diagrams with loops, and set

$$\rho^{p,q}(b_1,b_2) = \left(\frac{p}{p+q}\right)^{\mathsf{fnc}_+(b_1)-\mathsf{fnc}_+(b_2)}\frac{q}{p+q}^{\mathsf{fnc}_-(b_1)-\mathsf{fnc}_-(b_2)}.$$

Let $\mathbb{K}=\mathbb{R},\ \mathbb{H}$. We express, as in Chapter 2, the derivative of $\mathbb{m}^{\mathbb{K}}_{p,q}$ (with respect to the time variable $u\in[0,1]$) in the following form:

$$\frac{d}{du}\mathbb{m}^{\mathbb{K}}_{p,q}(u,w,b) = \mathbb{m}^{\mathbb{K}}_{p,q}(L^{\mathbb{K}}_{p,q}((b\otimes w),u),\ L^{\mathbb{K}}_{p,q} = \sum_{i=1}^{l} t_i L^{\mathbb{K},i}_{p,q}((b\otimes w),u)$$

where each $L^{\mathbb{K},i}_{p,q}$ denotes an operator acting on $\mathbb{R}[\mathcal{OB}_k]\otimes\mathbb{R}[\mathsf{M}_l(k)]$(on $\mathbb{R}[\mathcal{OB}_k]\otimes\mathbb{R}\left[\overline{\mathsf{M}}_l(k)\right]$ for the complex case).

Let $1 \leq i \leq s$ an integer. We define first two operators $C^{\mathbb{K},i}_{p,q}$ and $P^{\mathbb{K},i}_{p,q}$ acting $\mathcal{B}_k(p,q)$. The operator $C^{\mathbb{K},i}_{p,q}$ accounts for the derivative of the mean $\mathbb{E}\left[w^{\otimes}_{\mathbb{K}}(ut)\right]$ in the compact direction $\mathbb{U}(p,\mathbb{K}) \times \mathbb{U}(q,\mathbb{K})$ for the independent copy $\left[\Lambda^{p,q}_{\mathbb{K}}\right]^i$. The operator $P^{\mathbb{K},i}_{p,q}$ has a similar interpretation, it accounts for the derivative of $w^{\otimes}_{\mathbb{K}}(ut)$ in non-compact directions.

The set of Brauer diagrams that have as non coloured component the identity diagram is a complete set of projectors of the algebra $\mathcal{B}_k$ and we write p_k for this set. The set P_k is indexed by the set of colouration of the bottom line of its elements, that we denote C_k. Let $r \in \mathsf{W}_k \cup \mathsf{T}_k$ we define the quantity $(-1)^r$ by setting it equal to 1 if $r \in \mathsf{W}_k$ and to 1 if $r \in \mathsf{T}_k$. The computations are very similar to the one we made in Chapter 2, Section 2.6. Hence, our presentation will be more succint.

Let $\mathbb{K}$ be equal to $\mathbb{R}$ or $\mathbb{H}$, we treat separately the complex case. Let (b,s) an oriented Brauer diagram, a word w in $\mathsf{M}_s(k)$, we define the operator $C^{\mathbb{K},i}_{p,q}$ as:

$$
\begin{aligned}
(3.87)\qquad C^{\mathbb{K},i}_{p,q}((b,s)\otimes w) = &-\left[\frac{c}{2}\sum_{p\in P_k}\alpha^{\mathbb{K}}_i(p,w)\right](p\circ b,s)\otimes w\\
&+\frac{c(p+q)}{p}\sum_{\substack{r_{ab}\in(\mathsf{T}^{=}_k\cup\mathsf{W}^{=}_k)(+)\\ w_{a,b}=x_i}}(-1)^{r_{ab}}n^{\mathsf{nc}(b^{\bullet}\vee\mathbf{1}_k)-\mathsf{nc}(b^{\bullet}\vee\mathbf{1}_k)-1}\rho^{p,q}(\mathring{r}_{ab}\diamond(\mathring{b},s),(\mathring{b},s))(r_{ab}\diamond(b,s)\otimes w)\\
&+\frac{c(p+q)}{q}\sum_{\substack{r_{ab}\in(\mathsf{T}^{=}_k\cup\mathsf{W}^{=}_k)(-)\\ w_{a,b}=x_i}}(-1)^{r_{ab}}n^{\mathsf{nc}(b^{\bullet}\vee\mathbf{1}_k)-\mathsf{nc}(b^{\bullet}\vee\mathbf{1}_k)-1}\rho^{p,q}(\mathring{r}_{ab}\diamond(\mathring{b},s),(\mathring{b},s))(r_{ab}\diamond(b,s)\otimes w).
\end{aligned}
$$

with, for a projector $p \in P_k$ and a word $w \in \mathsf{M}_l(k)$,

$$
\alpha^{\mathbb{K}}_i(p,w) = \sum_{\substack{j\in c_p\\ w_j=x_i}}\alpha^{\mathbb{K}}_j \text{ and } \alpha^{\mathbb{R}}_+ = \frac{p-1}{p},\ \alpha^{\mathbb{R}}_- = \frac{q-1}{q},\ \alpha^{\mathbb{H}}_- = \frac{p+3}{p},\ \alpha^{\mathbb{H}}_- = \frac{q+3}{q}.
$$

The operator $P^{\mathbb{K},i}_{p,q}$ acting on $\mathbb{R}[\mathsf{M}_s(k)]\otimes\mathcal{B}_k(p,q)$ is defined by

$$
\begin{aligned}
P^{\mathbb{K},i}_{p,q}((b,s)\otimes w) &= \frac{v}{2}\sum_{p\in P_k}\beta_i(p)(p\circ b,s)\otimes w\\
(3.88)\qquad &+2v\sum_{\substack{r_{ab}\in\mathsf{T}^{\neq}_k\cup\mathsf{W}^{\neq}_k\\ w_{a,b}=x_i}}n^{\mathsf{nc}(b^{\bullet}\vee\mathbf{1}_k)-\mathsf{nc}(b^{\bullet}\vee\mathbf{1}_k)-1}\rho^{p,q}(\mathring{r}_{ab}\diamond(\mathring{b},s),(\mathring{b},s))(r_{ab}\circ b,s)\otimes w,
\end{aligned}
$$

where $\beta_i(p) = \sum_{\substack{j\in c_p\\ w_j=x_i}}\beta_j$, $\beta_+ = \frac{p}{\frac{p+q}{2}}$, $\beta_- = \frac{q}{\frac{p+q}{2}}$.

We treat now the complex case and define for a letter $x_i \in \overline{\mathsf{M}}_2$, $1 \leq i \leq q$, a pair of integers $(a,b) \in [1,\ldots,k]\times[1,\ldots,k]$ $e_{ab}(x_i) = \overline{x}_i$ and $\tau_{ab}(x_i) = \overline{x_i}$. Let w a word in $\overline{\mathsf{M}}_s(k)$ and define

$$
\begin{aligned}
(3.89)\qquad C^{\mathbb{C},i}_{p,q}(b,s)\otimes w) = &-\left[\frac{c}{2}\sum_{p\in P_k}\alpha^{\mathbb{C}}_i(p,w)\right]((p\circ b,s)\otimes w)\\
&+\frac{c(p+q)}{p}\sum_{\substack{r_{ab}\in(\mathsf{T}^{=}_k\cup\mathsf{W}^{=}_k)(+)\\ r_{ab}(w_a)=w_b\\ w_{a,b}\in\{x_i,\bar{x}_i\}}}(-1)^{r_{ab}}n^{\mathsf{nc}(b^{\bullet}\vee\mathbf{1}_k)-\mathsf{nc}(b^{\bullet}\vee\mathbf{1}_k)-1}\rho^{p,q}(\mathring{r}_{ab},\mathring{r}_{ab}\diamond b)(r_{ab}\diamond(b,s)\otimes w)\\
&+\frac{c(p+q)}{q}\sum_{\substack{r_{ab}\in(\mathsf{T}^{=}_k\cup\mathsf{W}^{=}_k)(-)\\ r_{ab}(w_a)=w_b\\ w_{a,b}\in\{x_i,\bar{x}_i\}}}(-1)^{r_{ab}}n^{\mathsf{nc}(b^{\bullet}\vee\mathbf{1}_k)-\mathsf{nc}(b^{\bullet}\vee\mathbf{1}_k)-1}\rho^{p,q}(\mathring{r}_{ab},\mathring{r}_{ab}\diamond\mathring{b})(r_{ab}\diamond(b,s)\otimes w),
\end{aligned}
$$

where, for a projector $p \in P_k$,

$$\alpha_i^{\mathbb{C}}(p,w) = \sum_{\substack{j \in c_p \\ w_j \in \{x_i, \bar{x}_i\}}} \alpha_i^{\mathbb{C}} \text{ and } \alpha_+^{\mathbb{C}} = \alpha_-^{\mathbb{C}} = 1.$$

In addition, we define the operator $P_{p,q}^{\mathbb{C},i}$ by:

$$\begin{aligned} P_{p,q}^{\mathbb{C},i}((b,s)\otimes w) &= \frac{v}{2}\sum_{p\in P_k} \beta^{\mathbb{C}}(p,w)((p\circ b,s)\otimes w) \\ &+ \frac{2v}{p+q} \sum_{\substack{r_{ab}\in (\mathsf{T}_k^{\neq}\cup \mathsf{W}_k^{\neq})(+) \\ r_{ab}(w_a)=w_b}} \rho^{p,q}(\mathring{r}_{ab}\diamond \mathring{b}, \mathring{b})((r_{ab}\diamond b,s)\otimes w)), \end{aligned} \tag{3.90}$$

where $\beta^{\mathbb{C}}(p,w) = \sum_{\substack{j\in c_p \\ w_j \in \{x_i,\bar{x}_i\}}} \beta_i^{\mathbb{C}}$, $\beta_+^{\mathbb{C}} = \frac{p}{\frac{p+q}{2}}$, $\beta_-^{\mathbb{C}} = \frac{q}{\frac{p+q}{2}}$.

We denote by δ_{Δ_k} the function that extends linearly to $\mathcal{B}_k(p,q)$ the support function of the set of diagonally coloured Brauer diagram. By using formulae of Proposition 3.47 in Section 3.9.3.1 for the mean of $V_{p,q}^{\mathbb{K}}(t)$, we obtain for the derivative of the statistic $\mathfrak{m}_{p,q}^{\mathbb{K}}$, with (b,s) an oriented Brauer diagram and w a word in M_q (resp. in $\overline{\mathsf{M}}_q(k)$) if $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$ (resp. $\mathbb{K} = \mathbb{C}$):

$$\begin{aligned} &\frac{d}{du}\mathfrak{m}_{p,q}^{\mathbb{K}}((b,s)\otimes w,u) = \sum_{i=1}^{q} t_i\left(\mathfrak{m}_{p,q}^{\mathbb{K}}(C_{p,q}^{\mathbb{K},i}((b,s),w),u)\right) + t_i\left(\mathfrak{m}_{p,q}^{\mathbb{K}}(P_{p,q}^{\mathbb{K},i}((b,s)\otimes w),u)\right), \\ &\mathfrak{m}_{p,q}^{\mathbb{K}}(b,\mathrm{t})(0) = \delta_{\Delta_k}. \end{aligned} \tag{3.91}$$

Set $P_{p,q}^{\mathbb{K},i} = \sum_{i=1}^{l} t_i P_{p,q}^{\mathbb{K},i}$ and $C_{p,q}^{\mathbb{K},i} = \sum_{i=1}^{l} t_i C_{p,q}^{\mathbb{K},i}$. This last equation is obtained by computations that are very similar to the one we performed and expounded in Chapter 2 to study convergence of rectangular extraction of an Unitary Brownian motion are thus not repeated here.

We let $p,q \to +\infty$ under hypothesis (H) and prove the convergence of the operators $C_{p,q}^{\mathbb{K},i}$ and $P_{p,q}^{\mathbb{K},i}$ for $1 \leq i \leq q$. We focus on proving the convergence of the operator $C_{p,q}^{\mathbb{K},i}$. For each projector $p \in P_k$, the coefficient $\alpha^{\mathbb{K}}(p,w)$ converges to $\mathsf{n}_i(w)$, the number of letters equal to x_i in the word w. Each of the two last sums in the equation (3.87) and in equation (3.88) is divided into three sums according to the values of the quantity $\mathsf{nc}(b^\bullet \vee r_{ab}^\bullet) - \mathsf{nc}(b^\bullet \vee 1) - 1 \in \{-1,0,1\}$, $\left(r_{ab} \in \mathsf{T}_k^{=} \cup \mathsf{W}_k^{=}\right)$.

Let $r_{ab} \in (\mathsf{T}_k^{=} \cup \mathsf{W}_k^{=})(+)$ an elementary bi-coloured Brauer diagram.

1. If $\mathsf{nc}(b^\bullet \vee r_{ab}{}^\bullet) = \mathsf{nc}(b^\bullet \vee 1) - 1$, two cycles of b are merged into one cycle of the diagram $r_{ab}^\bullet \circ b^\bullet$ and $n^{\mathsf{nc}(b^\bullet \vee \mathbf{1}_k) - \mathsf{nc}(b^\bullet \vee \mathbf{1}_k)-1}\rho^{p,q}(\mathring{r}\diamond \mathring{b},\mathring{b}) = O\left(\frac{1}{(p+q)}\right)$.
2. If $\mathsf{nc}(b^\bullet \vee r_{ab}{}^\bullet) = \mathsf{nc}(b^\bullet \vee 1)$ no cycles or loops are created if the diagram $b^\bullet$ is multiplied by $r_{ab}^\bullet$, thus $n^{\mathsf{nc}(b^\bullet \vee \mathbf{1}_k) - \mathsf{nc}(b^\bullet \vee \mathbf{1}_k)-1}\rho^{p,q}(\mathring{r}_{ab}\diamond \mathring{b},\mathring{b}) = O(1)$.
3. If $\mathsf{nc}(b^\bullet \vee r_{ab}^\bullet) = \mathsf{nc}(b^\bullet) + 1$ then a cycle of b is cut into two cycles of $r_{ab}^\bullet \circ b^\bullet$ or a loop is created. In that case, the quantity $n^{\mathsf{nc}(b^\bullet \vee \mathbf{1}_k) - \mathsf{nc}(b^\bullet \vee \mathbf{1}_k)-1}\rho^{p,q}(\mathring{r}_{ab}\diamond \mathring{b},\mathring{b})$ converges to a number different from zero.

The same discussion can be held to analyse the pointwise convergence of the sums over elementary diagrams in $(\mathsf{T}_k \cup \mathsf{W}_k)(-)$ in equation (3.87) and the pointwise convergence of the operator $P_{p,q}^{\mathbb{K},i}$. We let $C_\lambda^{\mathbb{K},i}$ and $P_\lambda^{\mathbb{K},i}$ be the limits of, respectively, $C_{p,q}^{\mathbb{K},i}$ and $P_{p,q}^{\mathbb{K},i}$. To give a formula for $C_\lambda^{\mathbb{K}}$ and $P_\lambda^{\mathbb{K}}$ we introduce the function, with the notation of Section 3.5,

$$\begin{array}{rccc} \rho_\lambda : & \mathcal{O}\mathring{\mathcal{B}}_k \times \mathcal{O}\mathring{\mathcal{B}}_k & \to & \mathbb{R} \\ & (b,b_1) & \mapsto & \left(\frac{\lambda}{1+\lambda}\right)^{\mathsf{fnc}_+(b)-\mathsf{fnc}_+(b_1)}\left(\frac{1}{1+\lambda}\right)^{\mathsf{fnc}_-(b)-\mathsf{fnc}_-(b_1)} \end{array}$$

Since we assumed λ to be positive, the function ρ_λ is well defined. With the help of ρ_λ, we can give simple formulae for the operators $C_\lambda^{\mathbb{K},i}$ and $P_\lambda^{\mathbb{K},i}$, $1 \leq i \leq l$. Let $w \in \mathsf{M}_l(k)$ and $c \in \{-,+\}^k$.

We denote by $\sharp(\pm, c, x_i, w)$ the cardinal of the set $\{j \in [1,\ldots,k] : c(j) = \pm,\ w_j = x_i\}$. The following formulae for the operators $C_\lambda^{\mathbb{K},i}$ and $P_\lambda^{\mathbb{K},i}$ hold, with $(b,s) \in \mathcal{OB}_k$ and $w \in \mathsf{M}_l(k)$:

$$\begin{aligned} C_\lambda^{\mathbb{K},i}((b,s)\otimes w) &= -\frac{c\sharp(x_i,w)}{2}((b,s)\otimes w) \\ &+ c(1+\frac{1}{\lambda}) \sum_{\substack{r_{ab}\in \mathsf{T}_{k,2}^{=,+}(b,+)\cup \mathsf{W}_{k,2}^{=,+}(b,+) \\ w_{a,b}=x_i}} (-1)^{r_{ab}} \rho_\lambda(\mathring{r}_{ab} \diamond (\mathring{b},s), b)(r_{ab}\diamond (b,s)\otimes w) \\ &+ c(1+\lambda) \sum_{\substack{r_{ab}\in \mathsf{T}_{k,2}^{=,+}(b,-)\cup \mathsf{W}_{k,2}^{=,+}(b,-) \\ w_{a,b}=x_i}} (-1)^{r_{ab}} \rho_\lambda(\mathring{r}_{ab} \diamond \mathring{b}, \mathring{b})(r_{ab}\diamond ((b,s)\otimes w) \end{aligned} \tag{3.92}$$

$$\begin{aligned} \mathcal{P}_\lambda^{\mathbb{K},i}((b,s)\otimes w) &= v\left(\frac{\sharp(+,c_b,x_i,w)}{1+\lambda} + \frac{\lambda\sharp(-,c_b,x_i,w)}{1+\lambda}\right)((b,s)\otimes w) \\ &+ 2v \sum_{\substack{r_{ab}\in \mathsf{T}_{k}^{\neq,+}(b)\cup \mathsf{W}_{k}^{\neq,+}(b) \\ w_{a,b}=x_i}} \rho^\lambda(\mathring{r}_{ab}\diamond \mathring{b}, \mathring{b})(r_{ab}\diamond (b,s)_\otimes w). \end{aligned} \tag{3.93}$$

Concerning the complex case, the operators $C_{p,q}^{\mathbb{C},i}$ and $P_{p,q}^{\mathbb{C},i}$, $1 \le i \le l$ converge pointwise to operators $C_\lambda^{\mathbb{C}}$, $P_\lambda^{\mathbb{C}}$ acting on the linear span of $\mathcal{OB}_k \times \overline{\mathsf{M}}_l(k)$ and are defined by:

$$\begin{aligned} C_\lambda^{\mathbb{C},(i)}((b,s)\otimes w) &= -\frac{c\sharp(x_i,w))}{2}((b,s)\otimes w) \\ &+ c(1+\frac{1}{\lambda}) \sum_{\substack{r_{ab}\in \mathsf{T}_{k,2}^{=,+}(b,+)\cup \mathsf{W}_{k,2}^{=,+}(b,+) \\ r_{ab}(w_a)=w_b \\ w_{a,b}\in\{x_i \bar{x}_i\}}} (-1)^{r_{ab}} \rho_\lambda(\mathring{r}_{ab}\diamond \mathring{b}, \mathring{b})(r_{ab}\diamond (b,s)\otimes w) \\ &+ c(1+\lambda) \sum_{\substack{r_{ab}\in \mathsf{T}_{k,2}^{=,+}(b,-)\cup \mathsf{W}_{k,2}^{=,+}(b,-) \\ r_{ab}(w_a)=w_b \\ w_{a,b}\in\{x_i,\bar{x}_i\}}} (-1)^{r_{ab}} \rho_\lambda(\mathring{r}_{ab}\diamond \mathring{b}, \mathring{b})(r_{ab}\diamond (b,s)\otimes w), \end{aligned} \tag{3.94}$$

$$\begin{aligned} P_\lambda^{\mathbb{C},(i)}((b,s)\otimes w) &= v\left(\frac{\sharp(+,c_b,x_i,w)}{1+\lambda} + \frac{\lambda\sharp(-,c_b,x_i,w)}{1+\lambda}\right)((b,s)\otimes w) \\ &+ 2v \sum_{\substack{r_{ab}\in \mathsf{T}_{k}^{\neq,+}(b,+)\cup \mathsf{W}_{k}^{\neq,+}(b,+) \\ r_{ab}(w_a)=w_b \\ w_{a,b}\in\{x_i,\bar{x}_i\}}} \rho_\lambda(\mathring{r}_{ab}\diamond \mathring{b}, \mathring{b})(r_{ab}\diamond (b,s)\otimes w) \end{aligned} \tag{3.95}$$

We summarize in the next proposition our discussion.

Proposition 3.50. *Let $\mathbb{K}$ be one of the three division algebras $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.*

Put $C_\lambda^{\mathbb{K}} = \sum_{i=1}^s t_i C_\lambda^{\mathbb{K},(i)}$ and $P_\lambda^{\mathbb{K}} = \sum_{i=1}^s t_i P_\lambda^{\mathbb{K},(i)}$. The statistic $\mathfrak{m}_{p,q}^{\mathbb{K}}$ converges point-wise to a function $\mathfrak{m}_\lambda^{\mathbb{K}}(\mathrm{t}) \in \mathcal{F}(\mathcal{B}_k)$ with the one parameter family of functions $u \mapsto \mathfrak{m}_\lambda^{\mathbb{K}}(\mathrm{t})(u)$ solution of

$$\begin{cases} \dfrac{d}{du}\mathfrak{m}_\lambda^{\mathbb{K}}(u\mathrm{t}) = \left(C_\lambda^{\mathbb{K}} + P_\lambda^{\mathbb{K}}\right)\left(\mathfrak{m}_\lambda^{\mathbb{K}}(u\mathrm{t})\right), \\ \mathfrak{m}_\lambda^{\mathbb{K}}(0, b\otimes w) = \delta_\Delta(b\otimes w). \end{cases}$$

Remark. *Since $C_\lambda^{\mathbb{R}} = C_\lambda^{\mathbb{H}}$ and $P_\lambda^{\mathbb{R}} = P_\lambda^{\mathbb{H}}$, we have $\mathfrak{m}_\lambda^{\mathbb{R}} = \mathfrak{m}_\lambda^{\mathbb{H}}$.*

Now, Theorem 3.49 is a corollary of Proposition 3.50. In fact, the method we used in Chapter 2, Section 2.6 for proving Theorem 2.37 applies verbatim for the case at stake.

In Chapter 4 we investigate the speed of convergence of the non-commutative distribution of the process $V_{p,q}^{\mathbb{K}}$. To that end, we make use of properties of the operators $P_{p,q}^{\mathbb{K}}$ and $C_{p,q}^{\mathbb{K}}$ that are exposed in Chapter 4. We take the opportunity now to introduce notations that are used in Chapter 4.

To a colourization $c \in \{-,+\}^k$, we associate the projector p_c which underlying diagram is the identity of $\mathcal{B}_k^\bullet(p+q)$, the colourization of the bottom line being equal to the colourization of the upper line and equal to c. Set first

$$\alpha_{p,q}^{\mathbb{K},i}(c,w) = \alpha_i^{\mathbb{K}}(p_c,w),\ \beta_{p,q}^{\mathbb{K},i}(c,w) = \beta_i^{\mathbb{K}}(p_c,w).$$

where $c \in \{-,+\}^k$ and w is a word in $\mathsf{M}_l(k)$ (resp. in $\overline{\mathsf{M}}_l(k)$ if $\mathbb{K}=\mathbb{R}$ or $\mathbb{H}$ (resp if $\mathbb{K}=\mathbb{C}$). Let b a bicoloured Brauer diagram. In addition to the set $\mathsf{W}_{k,2}^{=,+}(b,\pm)$, $\mathsf{T}_{k,2}^{=,+}(b,\pm)$ and $\mathsf{T}_k^{\neq,+}(b)$, we introduce other subsets of $\mathsf{T}_k \cup \mathsf{W}_k$, that depend also on the diagram b, obtained by counting cycles or loops that are created or deleted if we multiply b by an element of these sets. Set $\varepsilon(+)=1, \varepsilon(-)=-1$ and $\varepsilon(0)=0$. Let $s \in \{-,0,+\}$, we define

$$\mathsf{T}_{k,2}^{\neq,s}(b) = \{\tau \in \mathsf{T}_{k,2}^{\neq} : \tau b \neq 0,\ \mathsf{nc}(b^\bullet \vee \tau^\bullet) = \mathsf{nc}(b^\bullet \vee 1) + \epsilon(s)\},$$
$$\mathsf{W}_{k,2}^{\neq,s}(b) = \{e \in \mathsf{W}_{k,2}^{\neq} : eb \neq 0,\ \mathsf{nc}(b^\bullet \vee e^\bullet) = \mathsf{nc}(b^\bullet \vee 1) + \epsilon(s)\},$$

and the diagonal counterparts of the latter sets:

$$\mathsf{T}_{k,2}^{=,s}(b) = \{\tau \in \mathsf{T}_{k,2}^{=} : \tau b \neq 0,\ \mathsf{nc}(b^\bullet \vee \tau^\bullet) = \mathsf{nc}(b^\bullet \vee 1) + \epsilon(s)\},$$
$$\mathsf{W}_{k,2}^{=,s}(b) = \{e \in \mathsf{W}_{k,2}^{=} : eb \neq 0,\ \mathsf{nc}(b^\bullet \vee e^\bullet) = \mathsf{nc}(b^\bullet \vee 1) + \epsilon(s)\}.$$

Finally, denote by $\mathsf{T}_{k,2}^{=,s}(b,\pm)$ the subset coloured transpositions in $\mathsf{T}_{k,2}^{=,s}(b)$ having its two non-vertical links coloured with $\pm$, $\mathsf{T}_{k,2}^{=,s}(b,\pm) = \mathsf{T}_{k,2}^{=,s}(b) \cap \mathsf{T}_{k,2}^{=}(\pm)$ and similarly set $\mathsf{W}_{k,2}^{=,s}(b,\pm) = \mathsf{W}_{k,2}^{=,s}(b) \cap \mathsf{W}_{k,2}^{=}(\pm)$, $\mathsf{R}_{k,2}^{=,s}(b,\pm) = \mathsf{T}_{k,2}^{=,s}(b,\pm) \cup \mathsf{W}_{k,2}^{=,s}(b,\pm)$, $\mathsf{R}_{k,2}^{\neq,s}(b) = \mathsf{T}_{k,2}^{\neq,s}(b) \cup \mathsf{W}_{k,2}^{\neq,s}(b)$. By using these sets, we write each operator $L_{p,q}^{\mathbb{K},i}$ as a sum of four operators, two of them converging to 0 at different rates.

Suppose first that $\mathbb{K}=\mathbb{R}$ or $\mathbb{H}$ and pick an integer $1 \leq i \leq l$. We define an operator $N_{p,q}^{\mathbb{K},s,i}$ acting on $\mathcal{OB}_k(p,q) \otimes \mathbb{R}[\mathsf{M}_l(k)]$ and indexed by a sign $s \in \{-,0,+\}$:

$$\begin{aligned} N_{p,q}^{\mathbb{K},s,i}((b,s_b)\otimes w) = &-\frac{c(p+q)^{-\varepsilon(s)+2}}{p} \sum_{\substack{r_{ab} \in \mathsf{R}_{k,2}^{=,s}(b,+) \\ w_{a,b}=x_i}} (-1)^{r_{ab}} \rho^{p,q}(\mathring{r}_{ab}, \mathring{r}_{ab} \diamond \mathring{b})(r_{ab} \diamond (b,s_b) \otimes w) \\ &-\frac{c(p+q)^{-\varepsilon(s)+2}}{q} \sum_{\substack{r_{ab} \in \mathsf{R}_{k,2}^{=,s}(b,-) \\ w_{a,b}=x_i}} (-1)^{r_{ab}} \rho^{p,q}(\mathring{r}_{ab}, \mathring{r}_{ab} \diamond \mathring{b})(r_{ab} \diamond (b,s_b) \otimes w) \\ &+\frac{2(p+q)^{-\varepsilon(s)+2} v}{p+q} \sum_{\substack{r_{ab} \in \mathsf{R}_{k,2}^{\neq,s}(b) \\ w_{a,b}=x_i}} \rho^{p,q}(\mathring{r}_{ab} \diamond \mathring{b}, \mathring{b})(r_{ab} \diamond (b,s_b) \otimes w). \end{aligned} \tag{3.96}$$

Note that as p,q tend to infinity, under the hypothesis (H), the operator $N_{p,q}^{\mathbb{K},s,i}$ converges to a manifestly non trivial operator, that the role of the factor $(p+q)^{-\varepsilon(s)+1}$ we put in front of the sums. In fact, for the case $s=-$, with obvious notations, $\rho^{p,q}(\mathring{r}_{ab}, \mathring{r}_{ab} \diamond \mathring{b})$ converges to zero at speed $\frac{1}{(p+q)^2}$, so that $(p+q)^2 \rho^{p,q}(\mathring{r}_{ab}, \mathring{r}_{ab} \diamond \mathring{b})$ converges to a non-null number. We use these limiting operators, which we denote $N_\lambda^{\mathbb{K},s,i}$, in the next chapter.

Having introduced all the necessary definitions, we can write in a convenient and tractable form (for the investigation of the fluctuations) the generator $L_{p,q}^{\mathbb{K}} = C_{p,q}^{\mathbb{K}} + P_{p,q}^{\mathbb{K}}$ of the differential system the statistics $\mathbb{m}_{p,q}^{\mathbb{K}}(\mathrm{t})$ are solution of. For each integer $1 \leq i \leq l$, the operator $L_{p,q}^{\mathbb{K},i} = C_{p,q}^{\mathbb{K},i} + P_{p,q}^{\mathbb{K},i}$ is written as, with $(b,s_b) \in \mathcal{OB}_k$ and $w \in \mathsf{M}_l(k)$ (resp. $w \in \overline{\mathsf{M}}_l(k)$ if $\mathbb{K}=\mathbb{C}$),

$$\begin{aligned} L_{p,q}^{\mathbb{K},i}((b,s_b),w) = &\ c \sum_{c \in C_k} \alpha_{p,q}^{\mathbb{K},i}(c,w)(((\mathbf{1}_k,c)\circ b,s_b),w) + v \sum_{c \in C_k} \beta_{p,q}^{\mathbb{K},i}(c,w)(((\mathbf{1}_k,c)\circ b,s_b)\otimes w) \\ &+ N_{p,q}^{\mathbb{K},+,i}((b,s_b),w) + \frac{1}{p+q} N_{p,q}^{\mathbb{K},0,i}((b,s_b)\otimes w) + \frac{1}{(p+q)^2} N_{p,q}^{\mathbb{K},-,i}((b,s_b)\otimes w). \end{aligned} \tag{3.97}$$

Informally, the operators $N^{\mathbb{K},0,i}_{p,q}$, $N^{\mathbb{K},0,i}_{\lambda}$ mix colours without creating cycles or loops and the operator $N^{\mathbb{K},-,i}_{p,q}$, $N^{\mathbb{K},-,i}_{\lambda}$ merge two cycles. In our study of the fluctuations, we will see that microscopic fluctuations are governed by $N^{\mathbb{K},-,i}_{\lambda}$ while macroscopic fluctuations are governed by $N^{\mathbb{K},0,i}_{\lambda}$.

3.10. Two questions related to the spectrum of a Gaussian pseudo-Hermitian matrix

3.10.1. Spectrum of a pseudo-Hermitian matrix. Let H be a be a pseudo-hermitian operator on V. If B is a definite positive metric, the operator H is diagonalisable in a orthonormed basis and has a real spectrum. If the metric B has simultaneously negative and positive directions, it exists *isotropic* vectors for the metric B: vectors $v \in V$ such that $B(v,v) = 0$. These istropic vectors makes the diagonalisation of B-hermitian operators more difficult as we should see now. Pick an eigenvalue $\lambda \in \mathsf{Sp}(H)$ of H and an eigenvector v_λ associated with λ. In the sequel, we denote by $H^\star$ the adjoint of H with respect to the metric B: $B(H(x),y) = B(x,H^\star(y)), x,y \in V$. As we suppose B non-degenerate, the bilinear for B induces an anti-linear isomorphism $\eta_B : V \to V^\star$. By definition, $H^\star = \eta_B^{-1} \circ H^t \circ \eta_B$.

Lemma 3.51. *The spectrum of H is stable by complex conjugation.*

Proof. First, it is well known that the spectrum of H^t is equal to the spectrum of H and since η_B is anti-linear, the spectrum of $H^\star$ is the complex conjugate of H^t. $\square$

Lemma 3.52. *Let $\lambda \in \mathsf{Sp}(H)$ and $\nu \in \mathsf{Sp}(H)$ such that $\overline{\lambda} \neq \nu$, then the eigenspaces $E_\lambda(H)$ and $E_\nu(H)$ are B-orthogonal.*

Proof. Let $v_\lambda \in E_\lambda(H)$ and $v_\nu \in E_\nu(H)$ be two eigenvectors of H, $B(H(v_\lambda),v_\nu) = \overline{\lambda}B(v_\lambda,v_\nu) = B(v_\lambda,H(v_\nu)) = \nu B(v_\lambda,v_\nu)$ which implies

$$(\lambda - \overline{\nu})B(v_\lambda,v_\nu) = 0 \tag{3.98}$$

$\square$

We split the spectrum of the operator H into two subsets:

$$\mathsf{Sp}(H) = (\mathsf{Sp}(H) \cap \mathbb{R}) \cup (\mathsf{Sp}(H) \backslash \mathsf{Sp}(H) \cap \mathbb{R}) \tag{3.99}$$

and write $\mathsf{Sp}_{\mathbb{R}}(H) = \mathsf{Sp}(H) \cap \mathbb{R}$, $\overline{\mathsf{Sp}}(H)$ for the complementary of this latter set in the set $\mathsf{Sp}(H)$. We call an eigenspace associated with an eigenvalue in $\overline{\mathsf{Sp}}(H)$ *non-real*. As a consequence of equation (3.98), the non-real eigenspaces of H are isotropic subspaces:

$$\forall \lambda \in \overline{\mathsf{Sp}}(H),\ \forall v,w \in E_\lambda(H),\ B(v,w) = 0. \tag{3.100}$$

For each eigenvalue $\lambda \in \mathsf{Sp}(H)$, the pairing $B : E_\lambda(H) \times E_{\overline{\lambda}}(H) \to \mathbb{C}$ is non-degenerate, thus for each eigenvector $v_\lambda \in E_\lambda(H)$, there exists an unique eigenvector $v^\sharp_\lambda$ of H for the eigenvalue $\overline{\lambda}$ such that $B(v_\lambda,v^\sharp_\lambda) = 1$. It is not true that all $B-$ Hermitian operators are diagonalizable in a B-orthonormal basis (are even diagonalizable), in fact the B-orthogonal of an eigenspace contains this eigenspace if it is associated with a non-real eigenvalue where as the dimension of the orthogonal of any space is the codimension of that space. Let us give an example, in dimension 2. Let $\mathbb{H} = (\mathbb{R}^2, B)$ be the standard hyperbolic plane, the matric of metric B is written in the canonical basis of $\mathbb{R}^2$ as

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

A matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the matrix in the standard basis of $\mathbb{R}^2$ of a B-hermitian operator if and only if:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \overline{a} & \overline{c} \\ \overline{d} & \overline{d} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{3.101}$$

which implies $A = \begin{bmatrix} a & b \\ c & a \end{bmatrix}$ with $b, c \in \mathbb{R}$. Hence the nilpotent matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is B-Hermitian, but not diagonalizable.

Nevertheless, assume further that H is a diagonalizable operator. We should see that its non real spectrum is constrained by B and find a normal form H.

Let us recall that the *Witt index*, $\mathsf{Witt}(B)$, of the bilinear form B is equal to $2\min(r,s)$ where (r,s) is the signature of B. The Witt index has a geometric interpretation : it is the maximal dimension of totally isotropic subspaces of B. In the next lemma, we denote by $m_H(\lambda)$ the multiplicity of the eigenvalue λ of H.

Lemma 3.53. *Let H be a diagnalizable B-hermitian operator, then*

$$\sum_{\lambda \in \overline{\mathsf{Sp}}(H)} m_H(\lambda) \leq \mathsf{Witt}(B).$$

The lemmma 3.53 will be a consequence of the following lemma. Denote by $\mathsf{CoWitt}(B) = r + s - \mathsf{Witt}(B)$.

$$\overline{m}(H) = \sum_{\lambda \in \overline{\mathsf{Sp}}(H)} m_\lambda(H),\ m(H) = \sum_{r \in \mathsf{Sp}_{\mathbb{R}}} m_r(H). \tag{3.102}$$

Lemma 3.54. *Let H be a diagonalizable B-Hermitian operator. There exists a basis (e,f) of V with $e = (e_1, \dots, e_{m(H)})$ and $f = (f_1, \dots, f_{\overline{m}(H)})$ such that B is diagonal in this basis with the properties:*

$$H(e_i) = re_i \text{ for some } r \in \mathsf{Sp}_{\mathbb{R}}(H) \text{ and } H(f_j) = \lambda f_j \text{ for some } \lambda \in \overline{\mathsf{Sp}}(H),$$

$$B(e_{2i}, e_{2i-1}) = 1 \text{ and } B(e_{2i}, e_j) = 0, \text{ for all } j \neq 2i-1, i \leq \frac{1}{2}\overline{m}(H). \tag{3.103}$$

Proof. Let λ be a non-real eigenvalue of H. Let v_λ be a eigenvector of H and $v_\lambda^\sharp \in E_{\overline{\lambda}}$ be such that $B(v_\lambda, v_\lambda^\sharp) = 1$. Set $E_{\lambda,\overline{\lambda}} = E_\lambda \oplus E_{\overline{\lambda}}$ and $H = \mathbb{C}v_\lambda + \mathbb{C}v_\lambda^\sharp$. The orthogonal of $v_\lambda^\sharp$ in E_λ is of dimension $m_H(\lambda) - 1$ and contained in $H^\perp \cap E_{\lambda,\overline{\lambda}}$, the same holds for the orthognal v_λ in $E_{\overline{\lambda}}$. By counting dimensions, we see that $H^\perp \cap E_{\lambda,\overline{\lambda}} = v_\lambda^\perp \cap E_{\overline{\lambda}} \oplus v_\lambda^{\sharp\perp} \cap E_\lambda$. Since $v_\lambda \notin v_\lambda^{\sharp\perp} \cap E_\lambda$ and $v_\lambda^\sharp \notin v_\lambda^\perp \cap E_{\overline{\lambda}}$, the pairing $B : v_\lambda^{\sharp\perp} \cap E_\lambda \times v_\lambda^\perp \cap E_{\overline{\lambda}}$. A direct induction shows the existence of a basis $e_1^\lambda, \dots, e_{m_\lambda(H)}$ of E_λ (resp. $e_1^{\overline{\lambda}}, \dots, e_{m_{\overline{\lambda}}(H)}$ of $E_{\overline{\lambda}}$) such that

$$B(e_i^\lambda, e_i^{\overline{\lambda}}) = 1 \text{ and } B(e_i^\lambda, e_j^{\overline{\lambda}}) = 0,\ B(e_i^\lambda, e_i^\lambda) = 0.$$

Set $(f_1, \dots, f_{\overline{m}(H)}) = \left(e_1^{\lambda_1}, e_1^{\overline{\lambda}_1} e^{\overline{\lambda_1}}_{m_{\lambda_1}(H)}, \dots, e_1^{\lambda_q}, e_1^{\overline{\lambda}_q} \dots, e^{\lambda_q}_{m_{\lambda_q}(H)}, e^{\overline{\lambda}_q}_{m_{\overline{\lambda}_q}(H)}\right)$ where $q = \frac{1}{2}\overline{m}(O)$ and $\lambda_1, \dots, \lambda_q$ is an enumeration of the non-real spectrum of H in the upper half plane. We construct now the e basis, it is much more simple. Let $r \in \mathsf{Sp}_{\mathbb{R}}(H)$ be a real eigenvalue of H. The bilinear form B restricted to E_r is non degenerate, hence we can find a pseudo-orthonormal basis of E_r, let e^r such a basis. We set $e = (e^{r_1}, \dots, e^{r_s})$ where $r_1, \dots, r_s$ is a enumeration of the real eigenvalues then (e,f) is a basis that have the properties we were looking for. $\square$

$$\mathsf{def}(H) = \mathsf{Witt}(B) - \overline{m}(H)$$

3.10.2. Spectrum of pseudo-orthogonal operators. Let O be an isometry of (V,B). In the last section we proved that the spectrum of a B-Hermitian operator is stable by conjugation. We prove in this section invariance properties for the spectrum of O and diagonalisation result.

Lemma 3.55. *The spectrum of O is stable by conjugation-inversion: If $\lambda \in \mathsf{Sp}(O)$ then $\frac{1}{\overline{\lambda}}$ is in the spectrum of O. In addition, if $\lambda, \nu \in \mathsf{Sp}(O)$ with $\lambda \neq \frac{1}{\overline{\lambda}}$ then the eigenspaces $E_\lambda(O)$ and $E_\nu(O)$ are B-orthogonal.*

PROOF. By definition $B(O(x),O(y)) = B(x,y)$ for all $x,y \in V$, hence $O^\star = B^{-1} \circ O^t \circ B = O^{-1}$. Since $Sp(O^{-1}) = Sp(O)^{-1}$ and $Sp(O^\star) = Sp(O)^\star$ we get $\mathsf{Sp}(O) = \overline{Sp(O)}^{-1}$. Let $\lambda, \nu \in \mathsf{Sp}(O)$ such that $\overline{\lambda}\nu \neq 1$ then $\overline{\lambda}\nu B(v_\lambda, v_\nu) = B(v_\lambda, v_\nu)$ implies $B(v_\lambda, v_\nu) = 0$. □

If $\lambda \in Sp(O)$ with $|\lambda| \neq 1$ then the E_λ is an isotropic subspace of (B,V), in addition the pairing $B : E_\lambda(O) \times E_{\frac{1}{\overline{\lambda}}}(O)$ is non-degenerate. Denote by $\mathsf{Sp}_{\mathbb{U}}(O)$ the intersection of the spectrum of O with the unit circle, $\overline{\mathsf{Sp}}O$ its complementary in $\mathsf{Sp}(O)$ and

$$\overline{m}(O) = \sum_{\lambda \in \overline{\mathsf{Sp}}(O)} m_\lambda(O),\ m(O) = \sum_{u \in \mathsf{Sp}_{\mathbb{U}}(O)} m_u(O).$$

LEMMA 3.56. *Let O be a diagonalizable isometry of (V,B). There exists a basis (e,f) of V with $e = (e_1,\ldots,e_{m(O)})$ and $f = (f_1,\ldots,f_{\overline{m}(O)})$ such that B is diagonal in this basis with the properties:*

$$O(e_i) = re_i \text{ for some } r \in \mathsf{Sp}_{\mathbb{U}}(O) \text{ and } O(f_j) = \lambda f_j \text{ for some } \lambda \in \overline{\mathsf{Sp}}(O),$$

$$B(e_{2i}, e_{2i-1}) = 1 \text{ and } B(e_{2i}, e_j) = 0, \text{ for all } j \neq 2i-1, i \leq \frac{1}{2}\overline{m}(O). \tag{3.104}$$

PROOF. The proof goes as the proof of the similar statement for pseudo-hermitian operators, see Lemma 3.54. □

3.10.3. Conjecture: Real eigenvalues of a gaussian random pseudo-Hermitian matrix. We denote by the symbol dH the lebesgues measure on the space of pseudo-hermitian matrices which gives to the convex hull of the basis $\{S^{+,-}, \mathrm{i}A^{+,-}, A^{++}, A^{--}, \mathrm{i}S^{++}, \mathrm{i}S^{--}\}$ a volume equal to *one*. We denote by $X_{p,q}$ the random matrix having the block form:

$$X_{p,q} = \begin{bmatrix} \frac{1}{\sqrt{p}} X^{++} & \frac{\sqrt{2}}{\sqrt{p+q}} X^{+-}_{p,q} \\ \frac{\sqrt{2}}{\sqrt{p+q}} X^{-+}_{p,q} & \frac{1}{\sqrt{q}} X^{--}_{p,q} \end{bmatrix}$$

where $X^{++}_{p,q}$ and $X^{--}_{p,q}$ are standard Gaussian Hermitian matrices of dimensions, respectively, $p \times p$ and $q \times q$ and $X^{-+}_{p,q}$ is a matrix with dimensions $p \times q$ filled with independent complex standard Gaussian random variables.

CONJECTURE 3.57. As p,q tends to infinity, with $\frac{p}{q} \to \lambda$, $\mathsf{def}(X_{p,q})$ tends to 0 almost surely. Or equivalently, the ratio of real eigenvalues of $X_{p,q}$ tends to $\frac{1-\lambda}{1+\lambda}$.

We will present numerical simulations that suggest 3.57 holds. But first, let us show that 3.57 can not be strenghten to hold for finite p,q: It is not true that the ratio of real eigenvalues of the random matrix $X_{p,q}$ is almost surely equal to $\frac{|p-q|}{p+q}$. We foud a simple counter example in dimension two. We consider the space $\mathbb{C}^2$ equipped with the (hyperbolic) Hermitian quadratic form which matrix in the standard basis of $\mathbb{R}^2$ is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. We show that for the Lebesgues measure dH, the measure of the set of pseudo-Hermitian matrices having real eigenvalues is strictly positive (in fact, infinite). A pseudo-Hermitian matrix is of the form $\begin{bmatrix} \mathrm{i}x & \overline{a} \\ a & \mathrm{i}y \end{bmatrix}$, $x,y \in \mathbb{R}$ and $a \in \mathbb{C}$ and the Lebesgues measure dH is simply $dxdyda$. The characteristic polynomial of such matrix reads $X^2 - \mathrm{i}(x+y)X - xy - |a|^2$ which discriminant is $-(x^2+y^2) + 4|a|^2$. Hence, the volume of pseudo-Hermitian matrices having real eigenvalues is the volume of the set $\{|a|^2 \geq (x+y)^2\} \subset \mathbb{C} \times \mathbb{R}^2$ for the Lebesgues measure $dxdyda$.

3.10.4. Conjecture: Convergence of the spectrum of a large random pseudo-hermitian matrix. We investigated in this chapter the convergence in non-commutative distribution of Gaussian pseudo-Hermitian matrices and pseudo-orthogonal matrices. We now would like to introduce some numerical results regarding the convergence of the spectrum of such matrices. We are not stating in this section any theoretical results. First, for a complex matrix A of

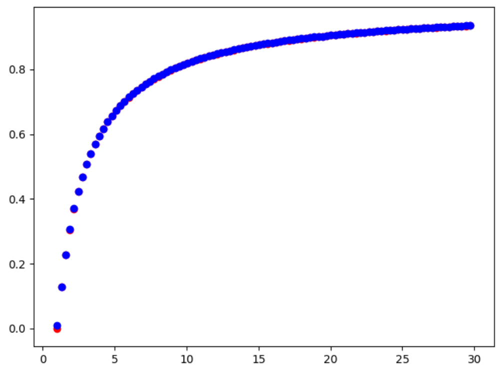


FIGURE 11. The horizontal axis is labelled with the values of $\frac{1}{\lambda}$ $\lambda \in [1, \frac{1}{3}]$, the vertical axis is labelled with the values of $\frac{1-\lambda}{1+\lambda}$. For this simulations we took q larger than 5000.

dimensions $N \times N$, we define its empirical spectral distribution μ_A as the measure putting a charge equal to $\frac{1}{N}$ at each eigenvalue of A (counted with multiplicities),

$$\mu_A = \frac{1}{N} \sum_{\lambda \in \mathsf{Sp}(A)} \delta_\lambda. \tag{3.105}$$

If A is a random matrix, μ_A is a random measure. A result that goes back to Wigner asserts, in our settings, that if B is definite positive quadratic form, the moments of the random measure $\mu_{X_{p,0}}$ converges, in mean, as p tends to infinity:

$$\mathbb{E}\left[\int_{\mathbb{C}} z^s \overline{z}^t \mathrm{d}\mu_{X_{p,0}}(\mathrm{d}z\mathrm{d}\overline{z})\right] \to \int_{\mathbb{C}} z^s z^t \nu(\mathrm{d}z\mathrm{d}\overline{z}). \tag{3.106}$$

where ν is compactly supported measure on the real liner known as the semi-circular Brownian motion. The original proof relies on the moment method: If A is a normal matrix (A commutes with its Hermitian conjugate $A^\star = \overline{A^t}$)), the moments of the measure μ_A are equal to the moments of the matrix A:

$$\int_{\mathbb{C}} z^s \overline{z}^t \mathrm{d}\mu_{X_{p,0}}(\mathrm{d}z\mathrm{d}\overline{z}) = \frac{1}{N} \mathsf{Tr}(A^s {A^\star}^t). \tag{3.107}$$

With the terminology of non-commutative probability, the non-commutative distribution of A coincides with its spectral distribution. It is not true anymore if the matrix A is not normal. The equality 3.107 relies on the Spectral Theorem, if A and $A^\star$ commutes then, first, A and $A^\star$ are diagonalizable and secondly, A and $A^\star$ are simultaneously diagonalizable in an orthonormal basis. Study the convergence of the spectral distribution of non-normal random matrix is a hard problem, see for example [] where the authors, after decades of investigation, succeeded in proving the convergence of a random matrix filled with independent entries

The random matrix $X_{p,q}$ is not an Hermitian matrix with respect to a positive definite scalar product but still, what remains of the moment method ? Or the spectral Theorem ? Almost surely, the matrix $X_{p,q}$ is diagonalizable and have simple eigenvalues (it is a standard fact: matrices with at least a non-simple eigenvalue are roots of a polynomial function). Let $\varepsilon_1^+, \ldots, \varepsilon_p^+, \varepsilon_1^-, \ldots, \varepsilon_q^-$ be the standard basis of $\mathbb{C}^{p+q}$:

$$B(\varepsilon_i^+, \varepsilon_j^+) = \delta_{ij},\ B(\varepsilon_k^-, \varepsilon_l^-) = \delta_{kl},\ B(\varepsilon_i^+, \varepsilon_k^+) = 0,$$

and denote by $J_{p,q}$ the change of coordinate matrix from the standard basis of $\mathbb{C}^{p,q}$ to the basis $\varepsilon_1^+, \ldots, \varepsilon_{p-q}^+, \frac{1}{2}(\varepsilon_{p-q+1}^+ + \varepsilon_1^-), \frac{1}{2}(\varepsilon_{p-q+1}^+ - \varepsilon_1^+), \frac{1}{2}(\varepsilon_p^+ + \varepsilon_q^-), \ldots, \frac{1}{2}(\varepsilon_p^+ - \varepsilon_q^+)$. Set $\tilde{X}_{p,q} = J_{p,q} X_{p,q} J_{p,q}^{-1}$. Owing to lemma 3.54,

$$\mathsf{Tr}(\tilde{X}_{p,q}^s (\tilde{X}_{p,q}^\star)^t) = \mathsf{Tr}(U D_{\mathsf{r},\lambda}^{s+t} U^\star) = \sum_{\lambda \in \mathsf{Sp}(O)} \lambda^{s+t}.$$

Hence, the moments $\mathbb{E}\left[\int_{\mathbb{C}} z^s \mathrm{d}\mu_{X_{p,q}}(\mathrm{d}z\mathrm{d}\bar{z})\right]$ can be computed from the non-commutative distribution of $\tilde{X}_{p,q}$ but since the spectrum of $X_{p,q}$ contains non-real eigenvalues (due to the existence of isotropic vectors for B), these moments do not fully characterize the measure $\mu_{p,q}$.

In Fig 12 we displayed numerical simulations of the spectrum of a Gaussian pseudo-Hermitian matrix. Each cell corresponds to a different value of the ratio $\frac{p}{q}$. These simulations sugggest that the distribution of the non-real part of the spectrum converges to an uniform law wich support seems to be an ellipsoid with a band parallel to real axis removed. The width of this band is a function of the parameter λ, which is zero if $\lambda = 1$. Thus, in the split case, it seems that the spectrum of $X_{p,p}$ converges to the free circular distribution (on 12), the support of the measure looks like an ellipse: it is just a matter of normalization of the variances of the blocks of $X_{p,p}$).

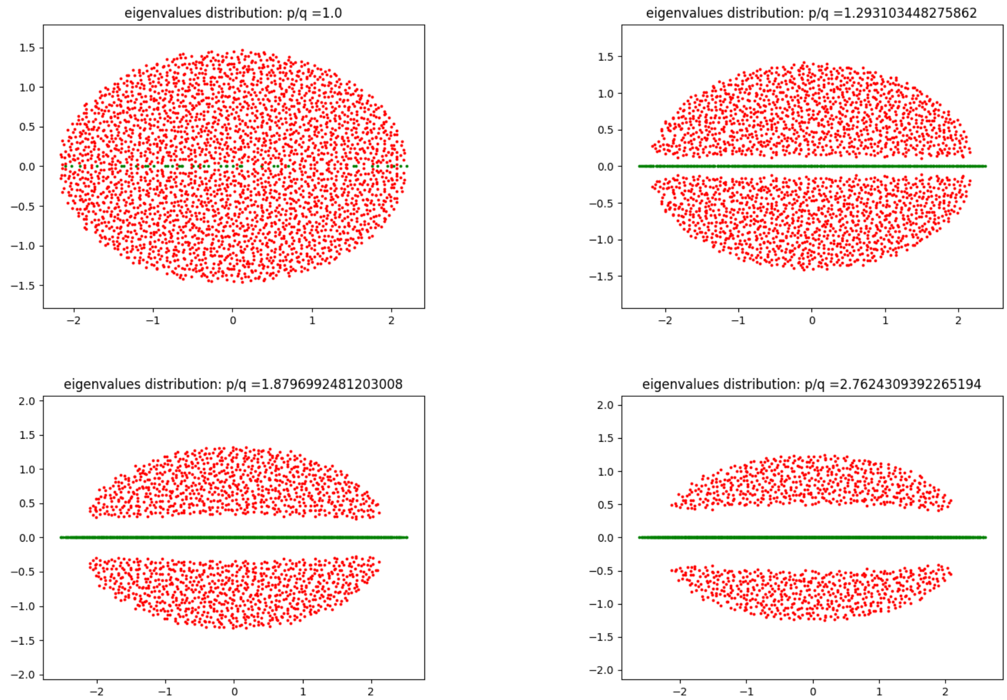


Figure 12. Spectrum of a Gaussian pseudo-Hermitian matrix. The real part of the spectrum is displayed in green.

CHAPTER 4

Fluctuations for the convergence of Brownian pseudo-unitary matrices in high dimensions

Contents

4.1. Introduction

Let $\mathbb{K}$ be one of the three division algebras $\mathbb{R},\mathbb{C}$ and $\mathbb{H}$. We introduced in the two previous chapters the notion of pseudo-unitary Brownian diffusions, which were denoted $\Lambda_{\mathbb{K}}^{p,q}$, $p,q \leq 1$. Pseudo-unitary diffusions depend on two speeds parameters v,c that correspond to diffusion rates in compact and non compact directions in the pseudo-unitary group $\mathbb{U}(p,q,\mathbb{K})$. We fix once for all, these speeds. A pseudo-unitary or a pseudo-hermitian A matrix is cut into four blocks according to the scheme in Figure 1.

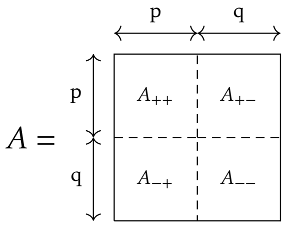


Figure 1. Block structures of a pseudo-unitary – pseudo-hermitian matrix.

Let q an integer and $t_1,\ldots,t_q$ a tuple of times. The present chapter is devoted to the investigation of the magnitudes and profile random variables' fluctuations (4.1) as $p,q \to +\infty$ in the regime $\frac{p}{q} \to \lambda > 0$.

$$\mathrm{Tr}\Big(\Lambda_{\mathbb{K}}^{p,q}(t_1)_{s_1,s_1'}\cdots\Lambda_{\mathbb{K}}^{p,q}(t_2)_{s_k,s_k'}\Big) - \mathbb{E}\Big[\mathrm{Tr}\Big(\Lambda_{\mathbb{K}}^{p,q}(t_{q-1})_{s_1,s_1'}\cdots\Lambda_{\mathbb{K}}^{p,q}(t_q)_{s_k,s_k'}\Big)\Big] \tag{4.1}$$

with $s_i,s_i' \in \{-,+\}$, $1 \leq i \leq k$. In fact, random variables as in equation (4.1) form a converging sequence; moments of any order converge to 0.

In the two last chapters, we proved the convergence of the non-commutative distribution of the pseudo-unitary Brownian motions. In addition, we proved asymptotic freeness in the split case ($\lambda = 1$) and constructed the free limiting process as solution of a free stochastic differential equation. In the general case, amalgamated freeness was stated, with no proof, we argued that the technique we used to prove asymptotic amalgamated freeness for rectangular extraction of a unitary Brownian difussions applies for the case at hand. In particular, these last results imply existence of scalars $\nu_\lambda((s,s'),t)$ with s and s' two finite sequences of signs with same length and $t \geq 0$ a time, such that

$$\mathbb{E}\Big[\Lambda_{\mathbb{K}}^{p,q}(t)_{s_1,s_i'}\cdots\Lambda_{\mathbb{K}}^{p,q}(t)_{s_q,s_q'}\Big] \to \nu_\lambda((s,s'),t). \tag{4.2}$$

The semi-group $\{\nu_\lambda(\cdot,\cdot')(t), t \geq 0\}$ is an (amalgamated free semi-group) on a suitable free bi-algebra (consult the last chapter for more details.) Related to the question of a central limit theorem for (4.1) is the determination of the convergence rate (and remaining asymptotic noise) of the distribution of the pseudo-unitary Brownian motions to their free counterpart. In our case, it consist in finding the right exponent $s > 0$ and law Z so that, for each time $t > 0$,

$$(p+q)^s\Big(\mathbb{E}\Big[\Lambda_{\mathbb{K}}^{p,q}(t)_{s_1,s_i'}\cdots\Lambda_{\mathbb{K}}^{p,q}(t)_{s_q,s_q'}\Big] - \nu_\lambda((s,s'),t)\Big) \xrightarrow{law} Z. \tag{4.3}$$

[The two integers p and q, the signature of the metric, tends to infinity and we assume that their ratio $\frac{p}{q}$ tends to a strictly positive scalar $\lambda \in]0,1]$. As we explained in the last chapter, we exclude the case $\lambda = 0$ since we suspect this case to branch to multiples regime. The main result of the present chapter is Theorem 4.7 which states that random variables (4.1) converge to 0 at speed $(p+q)$ and the remaining noise is asymptotically Gaussian. We provide the reader with a formula for the covariance of the noise.

We make a brief overview of results that are available related to speed of convergence of non-commutative distributions of random matrices on Lie groups (compact or not). Brownian motions on compact unitary groups $\mathbb{U}(N,\mathbb{K})$ have been studied extensively since the work of Biane, see [10], [8]. As the dimension $N \to +\infty$, the unitary Brownian motion on $\mathbb{U}(N,\mathbb{K})$ converges to a free stochastic process, the so-called free unitary Brownian motion. Fluctuations at a single time for this convergence was first studied by Lévy and Maïda in [?]. In [18], the author studied the speed of convergence using a combinatorial method related to the theory of second order freeness developped by Speicher, Collins, Sniady in [43], [42] and [16]. The main outcome of [18] is that the non commutative distribution of a unitary Brownian motion on $\mathbb{U}(N,\mathbb{K})$ converges at speed N and the remaining noise is Gaussian. Later, Kemp and Cébron defined a two parameters family of (s,t) Brownian motions, denoted $B_{s,t}$ on the group $GL_N(\mathbb{K})$ of invertible matrices. The authors proved in [14] that as the dimension N tends to infinity, the non-commutative distribution of $B_{s,t}$ converges to a Free stochastic process. In [14], the autor proved that the convergence's speed of $B_{s,t}$ is linear in the dimension and the remaining noise is also Gaussian.

To prove our main result Theorem 4.7, we use the same method that was used by the authors in [14]. This method is name *Carré du Champ* and consists in writing the generators of the pseudo-unitary as a sum of a derivation and a second order operator. This method provides also a formula for the covariance of the noise, in contrary to the combinatorial method used in [18] which provides a differential equation for the covariance of the normalized fluctuations.

Outline. In Section 4.2, we define the notion of first and second order operator on an unital commutative algebra. In Section 4.2, we introduce the notion of Fock space of bi-coloured Brauer diagrams. In Section 4.2.2.3 we show that the statistics $\mathrm{m}^{\mathbb{K}}_{p,q}$ and the generators $L^{\mathbb{K}}_{p,q}$ of the pseudo-unitary diffusions extends to the full Fock space of bi-coloured Brauer diagrams. Our main result is proved in Section 4.3.

4.2. Fock space of coloured oriented Brauer diagrams

Let $\mathbb{K}$ be one the three division algebras $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. This section is devoted to the settlement of the necessary algebraic framework for proving Theorem 4.7. The main outcome being the two operators, $\mathcal{D}^{\mathbb{K}}_{p,q}$ and $\mathcal{S}^{\mathbb{K}}_{p,q}$ that are respectively first and second order operator on a commutative algebra that generate a system of differential equations the statistics of the pseudo-unitary diffusions are solutions of.

In the sequel, we use the notations

$$\begin{aligned} D^{\mathbb{K}}_{p,q}((b,s)\otimes w) &= c\sum_{c\in C_k}\alpha^{\mathbb{K}}_{p,q}(c,w)(((\mathbf{1}_k,c)\circ b),s)\otimes w + v\sum_{c\in C_k}\beta^{\mathbb{K}}_{p,q}(c,w)((\mathbf{1}_k,c)\circ b,s)\otimes w \\ &\quad + N^{\mathbb{K},+}_{p,q}((b,s)\otimes w) + \frac{1}{p+q}N^{\mathbb{K},0}_{p,q}((b,s)\otimes w), \\ S^{\mathbb{K}}_{p,q} &= \frac{1}{(p+q)^2}N^{\mathbb{K},-}_{p,q}((b,s)\otimes w). \end{aligned} \tag{4.4}$$

4.2.1. First and second order operators on an unital commutative algebra. We give a brief overview on what a second order operator is and important relations that are used for proving Theorem 4.7. Let (A,η) be a finite dimensional unital commutative algebra.

An operator $D : A \to A$ on a commutative algebra A is a *first order operator* also named a *derivation* if for all $a,b\in A$, $D(ab) = aD(b)+D(a)b$. We denote by $\mathcal{D}er(A)$ the space of derivations on A. The following lemma states that the space of derivations is the tangent space of algebra morphisms of A.

Lemma 4.1. *If $D : A \to A$ be a first order operator then e^D is an algebra morphism.*

Proof. Let $a,b\in A$. A simple recurrence shows that $D^n(ab) = \sum_{k\geq 1}^{n}\binom{n}{k}D^k(a)D^{n-k}(b)$. Hence

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{1}{n!}D^n(ab) &= \sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{1}{n!}\binom{n}{k}D^k(a)D^{n-k}(b) = \sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\frac{1}{n!}\binom{n}{k}D^k(a)D^{n-k}(b) \\ &= \sum_{k=1}^{\infty}\sum_{n\geq k}\frac{1}{k!(n-k)!}D^k(a)D^{n-k}(b) = e^D(a)e^D(b). \end{aligned}$$

Hence, $e^D(ab) = e^D(a)e^D(b)$. $\square$

A *second order operator* L is an operator $L: A \to A$ such that for all $a,b,c \in A$,

$$L(abc) = L(ab)c + aL(bc) + L(ac)b - L(a)bc - aL(b)c - abL(c).$$

We associate to a second order operator L an operator $\Gamma_L : A \times A \to A$ defined by $\Gamma_L(a,b) = \frac{1}{2}(L(ab) - aL(b) - L(a)b)$, with $a,b \in A$. Note that since A is a commutative algebra, the operator Γ_L is symmetric in its arguments.

In the next lemma, we use the notation $a_1 \cdots a_{i-1}\hat{a}_i a_{i+1} \cdots a_k$ for the word obtained from the word $a_1 a_2 \cdots a_k$ in which the letter a_i has been erased erased.

Lemma 4.2. *For all $a,b,c \in A$ we have $\Gamma_L(ab,c) = \Gamma_L(a,c)b + a\Gamma_L(b,c)$. In addition, for all $a_1,\dots,a_k$ in A,*

$$L(a_1 \cdots a_k) = \sum_{i=1}^{k} a_1 \cdots a_{i-1}\hat{a}_i a_{i+1} \cdots a_k + \sum_{1 \le i < j \le k} a_1 \cdots \hat{a}_i \cdots \hat{a}_j \cdots a_k \Gamma_L(a_i, a_j).$$

The first assertion of the last lemma is re-stated on saying that for any $c \in A$, $\Gamma_L(\cdot, c)$ is a derivation of A.

Proof. Let $a,b,c \in A$. For the first assertion, one has

$$\begin{aligned}\Gamma_L(ab,c) &= \frac{1}{2}(L(abc) - abL(c) - L(ab)c) = \frac{1}{2}(aL(bc) + L(ac)b - L(a)bc - aL(b)c - 2abL(c)) \\ &= \frac{1}{2}((L(ac)b - L(a)cb - abL(c)) + (aL(bc) - aL(b)c - abL(c))) \\ &= \frac{1}{2}(\Gamma_L(a,c)b + a\Gamma_L(b,c))\end{aligned}$$

The proof of the second assertion is done by recurrence. □

As an application of the last lemma we have

$$L(a^n b) = na^{n-1}nL(a) + a^n L(b) + n(n-1)a^{n-2}b\Gamma_L(a,a) + na^n\Gamma_L(a,b) \tag{4.5}$$

for all $n \ge 1$ and $a,b \in A$. This formula is used in the proof of our main theorem, Theorem 4.7.

4.2.2. Fock space of Brauer diagrams and statistics of pseudo-unitary diffusions. In this section, $\mathbb{K}$ denotes either the field of complex numbers, $\mathbb{C}$, either the field of real numbers $\mathbb{R}$.

4.2.2.1. *Fock space of Brauer diagrams.* We first settle our notations, that are designed to give formulae that covers the real, complex and quaternionic cases at the same time.

Let $k \ge 1$ an integer. We set $\mathcal{B}_k^{\mathbb{R}} = \mathbb{R}[\mathcal{OB}_k] \otimes \mathbb{R}[\mathsf{M}_l(k)]$, $\mathcal{B}_\infty^{\mathbb{R}} = \bigoplus_{k=1}^{\infty} \mathbb{R}[\mathcal{OB}_k] \otimes \mathbb{R}\left[\overline{\mathsf{M}_l}(k)\right]$ and $\mathcal{B}_k^{\mathbb{C}} = \mathbb{R}[\mathcal{OB}_k] \otimes \mathbb{R}\left[\overline{\mathsf{M}}_l(k)\right]$, $\mathcal{B}_\infty^{\mathbb{C}} = \bigoplus_{k=1}^{\infty} \mathbb{R}[\mathcal{OB}_k] \otimes \mathbb{R}\left[\overline{\mathsf{M}}_l(k)\right]$ and denote by $T\left[\mathcal{B}_\infty^{\mathbb{K}}\right]$ the tensor algebra over $\mathcal{B}_\infty^{\mathbb{K}}$. The subspace of $T\left[\mathcal{B}^{\mathbb{K}}\right]$ of homogeneous tensors are denoted

$$T_p\left[\mathcal{B}_\infty^{\mathbb{K}}\right] = (\mathcal{B}_\infty^{\mathbb{K}})^{\otimes p} \subset T(\mathbb{K}[\mathcal{B}_\infty]).$$

In addition, we set $\mathcal{B}_\infty = \sum_{k=1}^{\infty} \mathbb{R}[\mathcal{OB}_k]$ and $T[\mathcal{B}_\infty]$ stands for the tensor algebra over $\mathcal{B}_\infty$. We recall that an irreducible oriented coloured Brauer diagram is an oriented diagram which underlying non-coloured component has only one cycle.

We denote by $\mathcal{OB}_{k,\mathrm{irr}}$ the subset of $\mathcal{OB}_k$ comprising all irreducible oriented coloured Brauer diagrams. We set $\mathcal{B}_{\infty,\mathrm{irr}}^{\mathbb{R}} = \bigoplus_{k \ge 1} \mathbb{R}[\mathcal{B}_{k,\mathrm{irr}}] \otimes \mathbb{R}[\mathsf{M}_l(k)]$, $\mathcal{B}_{k,\mathrm{irr}}^{\mathbb{C}} = \bigoplus_{k \ge 1} \mathbb{R}[\mathcal{B}_{k,\mathrm{irr}}] \otimes \mathbb{R}\left[\overline{\mathsf{M}}_l(k)\right]$.

To define the Fock space of Brauer diagrams we introduce the symmetrisation operator Sym acting on $T\left[\mathcal{B}_\infty^{\mathbb{K}}\right]$,

$$\mathrm{Sym}\left(b_1 \otimes \cdots \otimes b_q\right) = \sum_{\sigma \in \mathcal{S}_q} b_{\sigma^{-1}(1)} \otimes \cdots \otimes b_{\sigma^{-1}(q)},$$

on an homogeneous tensor $b_1 \otimes \cdots \otimes b_q \in T\left[\mathcal{B}^{\mathbb{K}}\right]$ of degree $q \ge 1$. We denote by $\mathcal{F}\left[\mathcal{B}_\infty^{\mathbb{K}}\right] \subset T\left[\mathcal{B}_\infty^{\mathbb{K}}\right]$ the Fock space of symmetric tensors, that is the image of $T\left[\mathcal{B}_\infty^{\mathbb{K}}\right]$ by Sym. The space $\mathcal{F}\left[\mathcal{B}_\infty^{\mathbb{K}}\right]$

is endowed with the algebra structure given by tensor product followed by symmetrisation, explicitly:

$$c_1 c_2 \cdots c_s = \mathsf{Sym}(c_1 \otimes \cdots \otimes c_s),\ c_i \in \mathcal{F}\left[\mathcal{B}_\infty^{\mathbb{K}}\right],\ i \leq s.$$

If E is an endomorphism of $\mathcal{B}_\infty^{\mathbb{K}}$ (linear or anti-linear) we denote by $T(E)$ (resp. $\bar{\mathcal{F}}(E)$), the morphism (resp. anti-morphism) defined by

$$T(E)(b_1 \otimes \cdots \otimes b_s) = E(b_1) \cdots E(b_s),\ \bar{T}(E)(b_1 \otimes \cdots \otimes b_s) = E(b_s) \cdots E(b_1).$$

On the Fock space $\mathcal{F}\left[\mathcal{B}_\infty^{\mathbb{K}}\right]$, $T(E)$ is equal to $\bar{T}(E)$, we denote the restriction to $\mathcal{F}\left[\mathcal{B}_\infty^{\mathbb{K}}\right]$ of these two endomorphisms by the same symbol $\mathcal{F}(E)$.

The Fock spaces $\mathcal{F}\left[\mathcal{B}_\infty^{\mathbb{K}}\right]$ and $\mathcal{F}\left[\mathcal{B}_{\mathrm{irr},\infty}^{\mathbb{K}}\right]$ are involutive algebras, the involution is trivial in the cases $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$. In the complex case, the free monoid $\overline{\mathsf{M}}_l$ is equipped with the involution that switch a letter x_i (resp. $\overline{x}_i$) to $\overline{x}_i$ (resp. x_i), $1 \leq i \leq l$. The Fock space $\mathcal{F}(\mathcal{B}_\infty^{\mathbb{C}})$ with the involution that sends, with obvious notations, $b \otimes w$ to $b \otimes w^\star$.

4.2.2.2. *Observables of the Fock space of Brauer diagrams.* Let $k \geq 1$ an integer. The statistics $\mathsf{m}_{p,q}^{\mathbb{K}}$ are defined on the space $\mathcal{B}_k^{\mathbb{K}}$. We should emphasize this dependence by adding a subscript k from now on. We explain how to construct an observable, for which we use the symbol $\mathsf{m}_{p,q}^{\mathbb{K}}$, on the Fock space $\mathcal{F}\left[\mathcal{B}_\infty^{\mathbb{K}}\right]$ starting from the family $\left(\mathsf{m}_{p,q,k}^{\mathbb{K}}\right)_{k\geq 1}$ defined in the previous section.

In the following, $\mathbb{K}$ denotes the field of complex numbers, the field of real numbers or the quaternionic division algebra. We set $V = \mathbb{K}^{p+q}$, V is a left $\mathbb{K}$-module of rank $p+q$ (it is a vector space if $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, but as $\mathbb{H}$ is not a commutative algebra, we use a terminology that suits for all the three cases) and $A \in \mathrm{Hom}_{\mathbb{K}}(V)$ a $\mathbb{K}$-linear endomorphism of V. The following convention is used in the sequel. In the cases $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$, $T(V)$ is the real tensor algebra over V, meaning that tensor products are taken over the field of real numbers. If $\mathbb{K} = \mathbb{C}$, $T(V)$ denotes the tensor product over the field of complex numbers. The endomorphism $T(A)$ acting on $T(V)$ belongs to the space $\mathsf{End}_0(T(V))$ of endomorphisms that preserve the homogeneous degree of a tensor. If $A \in \mathsf{End}_0(T(V))$, we denote by $[A]_k$ its restriction to the space $T_k(V)$ of homogeneous tensors of degree k.

We should first make the space on which the linear form $\mathsf{m}_{p,q,k}^{\mathbb{K}}$ acts independent from the integer $k \geq 1$. Let (b,s) an oriented Brauer diagram, the $\mathbb{R}$-linear form $\mathsf{m}_{p,q,k}^{\mathbb{K}}((b,s))$ acting on $\mathcal{M}_{p+q}(\mathbb{K})^{\otimes k}$ is extended to $\mathsf{End}_0(T(V))$ by setting, with a slight abuse of notations,

$$\mathsf{m}_{p,q,k}^{\mathbb{K}}((b,s))[A] = \mathsf{m}_{p,q,k}^{\mathbb{K}}(b)[[A]_k],\ A \in \mathsf{End}_0(V).$$

Now let $k_1, \ldots, k_s \geq 1$ be a partition of k. The tensor product $\mathbb{R}\left[\mathcal{B}_{k_1}\right] \otimes \mathbb{R}\left[\mathcal{B}_{k_s}\right]$ is injected into the space $\mathbb{R}[\mathcal{B}_k]$ by mean of the map $\theta_{k_1,\ldots,k_s}$,

$$\theta_{k_1,\ldots,k_s}((b_1,s_1) \otimes \cdots \otimes ((b_p,s_p)) = \left(\bigcup_{i=0}^{s-1}\left(\sum_{j=1}^{i-1} k_j + b_i\right), c\right)$$

with $i + S = \{i+s, s \in S\}$, $S \in \mathcal{P}(\mathbb{N})$ and the colourisation c satisfies $c(j) = c_i(j)$, $c(j') = c_i(j')$ if $j \in]\!]k_{i-1}, k_i]\!]$ with the convention $k_0 = 0$, $1 \leq i \leq p$.

The map $\mathsf{m}_{p,q,k}^{\mathbb{K}} : \mathbb{R}[\mathcal{B}_k] \to \left(\mathsf{End}_0(T(V))\right)^\star$ is pulled back by the mean of $\tilde{\theta}_{k_1,\ldots,k_s}$ to define a map on $\mathbb{R}\left[\mathcal{B}_{k_1}\right] \otimes \cdots \otimes \mathbb{R}\left[\mathcal{B}_{k_s}\right]$ valued in $\left(\mathsf{End}_0(T(V))\right)^\star$; $\mathsf{m}_{p,q}^{\mathbb{K}} : T(\mathcal{B}_\infty) \to \mathbb{K}$ is subsequently defined as:

$$\mathsf{m}_{p,q}^{\mathbb{K}}(b_1 \otimes \cdots \otimes b_s) = \mathsf{m}_{p,q,k}^{\mathbb{K}}\left(\theta_{k_1,\ldots,k_s}(b_1 \otimes \cdots \otimes b_s)\right), b_1 \otimes \cdots \otimes b_s \in \mathcal{B}_{k_1}^{\mathbb{K}} \otimes \cdots \otimes \mathcal{B}_{k_s}^{\mathbb{K}}.$$

From now on we fix a finite sequence of matrices $\mathsf{A} = (A_1, \ldots, A_l)$ with entries in $\mathbb{K}$ of length l and define for a tensor $((b_1,s_1) \otimes w_1) \otimes \cdots \otimes ((b_p,s_p) \otimes w_p) \in T\left[\mathcal{B}_\infty^{\mathbb{K}}\right]$

$$\mathsf{m}_{p,q}^{\mathbb{K}}(\mathsf{A})(((b_1,s_1) \otimes w_1) \otimes \cdots \otimes ((b_p,s_p) \otimes w_p)) = \mathsf{m}_{p,q}^{\mathbb{K}}((b_1,s_1) \otimes \cdots \otimes (b_p,s_p))\left[w^{\otimes}(A_1, \ldots, A_l)\right].$$

From the very definition of the maps in the sequence $\left(\mathsf{m}^{\mathbb{K}}_{p,q,k}\right)_{k\geq 1}$,

$$\mathsf{m}^{\mathbb{K}}_{p,q,k_1+k_2}((b_1,s_1)\otimes(b_2,s_2))(A\otimes B)=\mathsf{m}^{\mathbb{K}}_{p,q,k_1}(b_1)(A)\mathsf{m}^{\mathbb{K}}_{p,q,k_2}(b_2)(B), \tag{4.6}$$

with $A\in\mathcal{M}_{p+q}(\mathbb{K})^{\otimes k_1}$ and $B\in\mathcal{M}_{p+q}(\mathbb{K})^{\otimes k_2}$. The following proposition is direct corollary of equation (4.6).

Proposition 4.3. *Let $p,q\geq 1$ two integers, and a matrix $A\in\mathsf{End}_{\mathbb{K}}(\mathbb{K}^{p+q})$, the map $\mathsf{m}^{\mathbb{K}}_{p,q}[T(A)]$ is a character of $T[\mathcal{B}_\infty]$.*

As a consequence of the last proposition, since for all $A\in\mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^{p+q})$, $\mathsf{m}^{\mathbb{K}}_{p,q}[T(A)]\circ\star=\overline{\mathsf{m}^{\mathbb{K}}_{p,q}[T(A)]}$, $\mathsf{m}^{\mathbb{K}}_{p,q}$ is a random variable on the algebraic probability space $\left(\mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_\infty\right],\star\right)$.

In the following, we use the notation $\mathsf{End}^{\mathsf{sym}}_0(V)\subset\mathsf{End}_0(T(V))$ for the space of endomorphisms acting on $T(V)$ that commute with the action by permutation on $T(V)$:

$$A\in\mathsf{End}^{\mathsf{sym}}_0(T(V))\Leftrightarrow A(\sigma\cdot(v_1\otimes\cdots\otimes v_q))=\sigma\cdot A(v_1\cdots v_q),\ v_i\in V,\ i\leq s,\ \sigma\in\mathcal{S}_q.$$

Equation (4.6) implies that for all endomorphism $A\in\mathsf{End}^{\mathsf{sym}}_0(V)$,

$$\mathsf{m}^{\mathbb{K}}_{p,q}(b_1\otimes\cdots\otimes b_p)[A]=\mathsf{m}^{\mathbb{K}}_{p,q}((b_{\sigma(1)}\otimes\cdots\otimes b_{\sigma(p)})[A]. \tag{4.7}$$

with $b_1\otimes\cdots\otimes b_p\in T[\mathcal{B}_\infty]$. Thus, if S is a symmetric operator belonging to $\mathsf{End}^{\mathsf{sym}}_0(T(V))$, it holds that $\mathsf{m}^{\mathbb{K}}_{p,q}[S]\circ\mathsf{Sym}=\mathsf{m}^{\mathbb{K}}_{p,q}[S]$. This property is also satisfied by the map, $\mathsf{m}^{\mathbb{K}}_{p,q}(\mathsf{A})$,

$$\mathsf{m}^{\mathbb{K}}_{p,q}(\mathsf{A})\circ\mathsf{Sym}=\mathsf{m}^{\mathbb{K}}_{p,q}(\mathsf{A}).$$

In the next section, we are concerned with the random variables:

$$\mathsf{m}^{\mathbb{K}}_{p,q}(\cdot)\left[\Lambda^{p,q}_{\mathbb{K}}(ut_1),\ldots,\Lambda^{p,q}_{\mathbb{K}}(ut_q)\right] \tag{4.8}$$

with $(t_1,\ldots,t_q)$ a tuple of times and a real $u\in[0,1]$.

4.2.2.3. *Generators of the pseudo-unitary diffusions as Fock space's operators.* Let $\mathbb{K}$ be one of the three division algebras $\mathbb{R},\mathbb{C}$ and $\mathbb{H}$. We are now in position to give a slightly different point of view, nevertheless needed, on the statistics of the pseudo-unitary diffusion that were introduced in the last two chapters. The distribution of the random variables (4.8) is

$$\mathbb{m}^{\mathbb{K}}_{p,q}(x,\mathsf{t})(u)=\mathbb{E}\left[\mathsf{m}^{\mathbb{K}}_{p,q}\left(\Lambda^{p,q}_{\mathbb{K}}(ut_1),\ldots,\Lambda^{p,q}_{\mathbb{K}}(ut_l)\right)(x)\right],\ u\in[0,1]. \tag{4.9}$$

The purpose of this section is to compute the derivative of 4.9 and to define a first order operator $D^{\mathbb{K}}_{p,q}$ and a second order operator $L^{\mathbb{K}}_{p,q}$, both acting on $\mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_{\mathrm{irr},\infty}\right]$, such that

$$\frac{d}{du}\mathbb{m}^{\mathbb{K}}_{p,q}(x,\mathsf{t})(u)=\mathbb{m}^{\mathbb{K}}_{p,q}((\mathcal{D}^{\mathbb{K}}_{p,q}+\mathcal{L}^{\mathbb{K}}_{p,q})(x),\mathsf{t})(u).$$

To define $\mathcal{D}^{\mathbb{K}}_{p,q}$ and $\mathcal{L}^{\mathbb{K}}_{p,q}$, we first define the map type that, roughly speaking, cuts a diagram according to its cycles. It is a bit technical to define this map, even if its action on a Brauer diagram is simple to understand. Let b be a bicoloured oriented Brauer diagram of size k and w a word in $\mathsf{M}_l(k)$ (or in $\overline{\mathsf{M}}_l(k)$ if $\mathbb{K}=\mathbb{C}$). Let $\sigma_{(b,s)}$ be the associated permutation of $[\![1,\ldots,k]\!]$ and let σ be a cycle of $b^\bullet$. The cycle σ is a block of $b^\bullet\vee\mathbf{1}_k$, its intersection with $[\![1,\ldots,k]\!]$ is the support of a cycle $\bar\sigma$ of $\sigma_{(b,s)}$. In the previous chapters, we endowed the support of $\bar\sigma$ with a linear order $\prec$ that is left by the cyclic order induced by σ if we cut out the minimum m of the support:

$$i\prec j\Leftrightarrow i=\bar\sigma^k(m),j=\bar\sigma^q(m),\ k<q,\ i,j\in\sigma\wedge[\![1,\ldots,k]\!].$$

The exists an unique non decreasing bijection $\theta_\sigma:([\![1,\ldots,\ell(\sigma)]\!],<)\to(\sigma\wedge[\![1,\ldots,k]\!],\prec)$. We denote by θ'_σ the bijection defined by $\theta'_\sigma(a)=\theta_\sigma(a)$, $\theta_\sigma(a')=\theta_\sigma(a)'$, $a\leq\ell(\sigma)$. The type $\mathsf{type}(b,w)$ of the pair (b,w) is a tensor in $\mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_{\mathrm{irr},\infty}\right]$ defined by

$$\mathsf{type}(b)=\prod_{\sigma\in b^\bullet\vee 1}(b^\bullet\circ\theta'_\sigma,\ c_b\circ\theta'_\sigma,w_{\theta_\sigma(1)}\cdots w_{\theta_\sigma(\ell(\sigma))}).$$

The map type is extended linearly to $\mathcal{B}_{\infty}^{\mathbb{K}}$. As a final ingredient for the definition of the maps $\mathcal{D}_{p,q}^{\mathbb{K}}$ and $\mathcal{L}_{p,q}^{\mathbb{K}}$ we are ultimately interested in, we introduce a (second) graduation on $T\left[\mathcal{B}^{\mathbb{K}}\right]$ which is defined through the degree map deg:

$$\mathrm{deg}((b_1\otimes w_1)\otimes\cdots\otimes(b_s\otimes w_s)) = k_1+\ldots+k_s,$$

where $(b_1\otimes w_1)\otimes\cdots\otimes(b_s\otimes w_s)\in T\left[\mathcal{B}_{\infty}^{\mathbb{K}}\right]$. Put $\mathcal{F}\left[\mathcal{B}_{\infty}^{\mathbb{K}}\right]_k^{\mathrm{deg}} = \{\mathrm{deg}=k\}$, $k\geq 1$ and define the operator $\theta^k : \mathcal{F}\left[\mathcal{B}_{\infty}^{\mathbb{K}}\right]_k \to \mathcal{B}_k^{\mathbb{K}}$ by

$$\theta^k = \bigoplus_{k_1+\cdots+k_s=k} \theta_{k_1,\ldots,k_s}.$$

We are now ready to define the two operators $\mathcal{S}_{p,q}^{\mathbb{K}}$, $\mathcal{D}_{p,q}^{\mathbb{K}}$ on $T\left[\mathcal{B}^{\mathbb{K}}\right]$ by setting, with the notation introduced in equation (4.4),

$$\mathcal{S}_{p,q}^{\mathbb{K}} = \bigoplus_{k\geq 1} \mathrm{type}\circ S_{p,q,k}^{\mathbb{K}}\circ\theta^k,\ \mathcal{D}_{p,q}^{\mathbb{K}} = \bigoplus_{k\geq 1}\mathrm{type}\circ D_{p,q,k}^{\mathbb{K}}\circ\theta^k. \tag{4.10}$$

We seek for operators that are defined over the Fock space $\mathcal{F}\left[\mathcal{B}_{\infty}^{\mathbb{K}}\right]$, of course we can restrict each of the two operators in 4.10 but the following lemma asserts in addition that it is possible to factorize $D_{p,q}^{\mathbb{K}}$ by Sym.

Lemma 4.4. *Let $k_1,\ldots,k_s$ integers and put $k_1+\cdots+k_s=k$. In addition, let $b_1\otimes\cdots\otimes b_s\in\mathcal{B}_{k_1}\otimes\cdots\otimes\mathcal{B}_{k_s}$. For any permutation of σ of $[\![1,s]\!]$, one has $D_{p,q}^{\mathbb{K}}\left(b_{\sigma^{-1}(1)}\otimes\cdots\otimes b_{\sigma^{-1}(s)}\right) = b_1\otimes\cdots\otimes b_s$.*

The following lemma is central.

Lemma 4.5. *The operator $D_{p,q}^{\mathbb{K}} : \mathcal{F}\left[\mathcal{B}_{irr,\infty}^{\mathbb{K}}\right]\to\mathcal{F}\left[\mathcal{B}_{irr,\infty}^{\mathbb{K}}\right]$ is a first order operator while $S_{p,q}^{\mathbb{K}} : \mathcal{F}\left[\mathcal{B}_{irr,\infty}^{\mathbb{K}}\right]\to\mathcal{F}\left[\mathcal{B}_{irr,\infty}^{\mathbb{K}}\right]$ is a second order operator, besides, for all $x\in\mathcal{F}\left[\mathcal{B}_{irr,\infty}^{\mathbb{K}}\right]$,*

$$\frac{d}{du}\mathbb{m}_{p,q}^{\mathbb{K}}(x,\mathfrak{t})(u) = \mathbb{m}_{p,q}^{\mathbb{K}}((\mathcal{D}_{p,q}^{\mathbb{K}}+\mathcal{S}_{p,q}^{\mathbb{K}})(x),\mathfrak{t})(u),\ u\in[0,1].$$

Proof. We assume $l=1$. The computations that are expounded below in that case are performed for each operator $\mathcal{S}_{p,q,k}^{\mathbb{K},i}$ obtained by replacing $S_{p,q}^{\mathbb{K}}$ in equation (4.10) by $\mathcal{S}_{p,q}^{\mathbb{K},i}$ to treat the general case $\ell>1$. In addition, we focus on the cases $\mathbb{K}=\mathbb{R}$ or $\mathbb{H}$ and since

$$\mathbb{m}_{p,q}^{\mathbb{K}}(\Lambda_{\mathbb{K}}^{p,q}(ut),\ldots,\Lambda_{\mathbb{K}}^{p,q}(ut))(b\otimes w) = \mathbb{m}_{p,q}^{\mathbb{K}}(b)\left[T(\Lambda_{\mathbb{K}}^{p,q}(ut))\right],$$

a generator of a differential system satisfied by the statistics would be naturally defined on the Fock space $\mathcal{F}[\mathcal{B}_{\infty}]$. Fortunately, the operators $\mathcal{D}_{p,q}^{\mathbb{K}}$ and $\mathcal{S}_{p,q}^{\mathbb{K}}$ are in fact acting on $\mathcal{F}[\mathcal{B}_{\infty}]$, since $\mathcal{B}^{\mathbb{K}}$ is isomorphic to $\mathcal{B}_{\infty}$, owing to the fact that for each integer $k\geq 1$ there is only one word of length k in M_1. In the forthcoming computations we omit the maps θ^k and $\theta_{k_1,\ldots,k_s}$ with $k_1,\ldots,k_s\geq 1$ to ease the exposition.

We show first that $D_{p,q}^{\mathbb{K}}$ restricted to the Fock space $\mathcal{F}\left[\mathcal{B}_{\mathrm{irr},\infty}^{\mathbb{K}}\right]$ is a derivation. Let $k_1,k_2\geq 1$ integers. Let $c_1\in\mathcal{B}_{k_1}$ and $c_2\in\mathcal{B}_{k_2}$ be two oriented irreducible Brauer diagrams. The element $\mathcal{D}_{p,q,k_1+k_2}^{\mathbb{K}}(c_1\otimes c_2)$ is computed using equation (4.4):

$$\begin{aligned}D_{p,q}^{\mathbb{K}}(c_1\otimes c_2) = c\sum_{c\in C_k}\alpha_{p,q}^{\mathbb{K}}(c)(\mathbf{1}_k,c)\diamond(c_1\otimes c_2) + v\sum_{c\in C_k}\beta_{p,q}^{\mathbb{K}}(c)(\mathbf{1}_k,c)\diamond(c_1\otimes c_2)\\ + N_{p,q}^{\mathbb{K},+}(c_1\otimes c_2) + \frac{1}{p+q}N_{p,q}^{\mathbb{K},0}(c_1\otimes c_2).\end{aligned}$$

The operators $N_{p,q}^{\mathbb{K},+}$ and $N_{p,q}^{\mathbb{K},0}$ are defined as sums over non-oriented Brauer diagrams acting on $\mathcal{OB}_{k_1+k_2}$ by mean of the operator $\diamond$. The ranges of these sums are contained in the subset of elementary diagrams which associated permutations have their only non trivial cycle contained either in the interval $[\![1,k_1]\!]$ or $[\![k_1,k_1+k_2]\!]$. Hence, $N_{p,q}^{\mathbb{K},0}(c_1\otimes c_2) = N_{p,q}^{\mathbb{K}}(c_1)\otimes c_2 + N_{p,q}^{\mathbb{K}}(c_1)\otimes c_2$ and the same holds for $N_{p,q}^{\mathbb{K},+}(c_1\otimes c_2)$.

We apply the map type, owing to the property $\mathrm{type}(b\otimes c_1)=\mathrm{type}(b)\mathrm{type}(c_1)$, we have $\mathrm{type}\circ N^{\mathbb{K},0}_{p,q}(c_1\otimes c_2)=(\mathrm{type}\circ N^{\mathbb{K}}_{p,q}(c_1))\mathrm{type}(c_2)+(\mathrm{type}\circ N^{\mathbb{K}}_{p,q})(c_1)\mathrm{type}(c_2)$.

The first two sums in equation (4.2.2.3) are evaluated as:

$$\sum_{c\in\mathcal{C}_k}\alpha^{\mathbb{K}}_{p,q}(c)(\mathbf{1}_k,c)\diamond(c_1\otimes c_2)=\alpha\left(\left(c_{c_1\otimes c_2}(1),\ldots,c_{c_1\otimes c_2}(k)\right)\right)c_1\otimes c_2,$$

and the same for the second sum. From the very definition of the coefficient $\alpha(c_b(1),\ldots,c_b(k_1+k_2))$, one has

$$\alpha\left(c_{c_1\otimes c_2}(1),\ldots,c_{c_1\otimes c_2}(k_1+k_2)\right)=\alpha\left(c_{c_1}(1),\ldots,c_b(c_1)\right)+\alpha\left(c_{c_2}(k_1+1),\ldots,c_{c_2}(k_1+k_2)\right).$$

It is now pretty downward to infer that $\mathcal{D}^{\mathbb{K}}_{p,q}$ is a derivation. We prove now that $S^{\mathbb{K}}_{p,q}$ is a second order operator. To that end, we shall give a formula for $S^{\mathbb{K}}_{p,q}(b)$, $b\in\mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_{\infty}\right]$. Let $k_1\geq 1$ and $k_2\geq 1$ two integers. We define the permutations $\alpha^{(k_1,k_2)}:[\![k_1,\,k_2]\!]\to[\![k_1,\,k_2]\!]$ by

$$\alpha^{(a,b)}(k)=k+k_2 \text{ if } k\leq k_1 \text{ and } \alpha^{(a,b)}(k)=k-k_1 \text{ if } k>k_1,\ 1\leq k\leq k_1+k_2.$$

The conjugation by the permutation $\theta^{(k_1,k_2)}$, seen as a Brauer diagram in $\mathcal{B}_{k_1+k_2}(p,q)$, is denoted $c_{\alpha^{(k_1,k_2)}}$. We assume first that it exists two sets T and E,

$$T=\left\{\tau^{a,b}_{p,q}:\mathcal{B}_{k_1}\times\mathcal{B}_{k_2}\to\mathcal{B}_{k_1+k_2},\ \tau\in\mathsf{T}_{k_1+k_2},\ k_1,k_2\in\mathbb{N}^\star\right\},$$

$$E=\left\{e^{a,b}_{p,q}:\mathcal{B}_{k_1}\times\mathcal{B}_{k_2}\to\mathcal{B}_{k_1+k_2},\ e\in\mathsf{W}_{k_1+k_2},\ k_1,k_2\in\mathbb{N}^\star\right\},$$

and a third set $S(c_1,c_2)\subset\mathcal{B}_{k_1+k_2}$ depending on two Brauer diagrams $c_1\in\mathcal{B}_{k_1}$ and $c_2\in\mathcal{B}_{k_2}$ so as to, with

$$\bar{\tau}^{k_1,k_2}_{p,q}(c_1,c_2)=\frac{1}{2}\left(\tau^{k_1,k_2}_{p,q}(c_1,c_2)+e^{k_2,k_1}_{p,q}(c_2,c_1)\right),\ \bar{e}^{k_1,k_2}_{p,q}(c_1,c_2)=\frac{1}{2}\left(e^{k_1,k_2}_{p,q}(c_1,c_2)+e^{k_2,k_1}_{p,q}(c_2,c_1)\right),$$

$$c_{\alpha^{(k_1,k_2)}}\left(S\left(c_1\otimes c_2\right)\right)=S\left(c_{\alpha^{(k_1,k_2)}}\left(c_1\otimes c_2\right)\right)=S\left(c_2\otimes c_1\right),$$

and $(c_{\alpha^{(k_1,k_2)}}(\bar{\tau}))^{k_1,k_2}_{p,q}(c_1,c_2)=\bar{\tau}^{k_2,k_1}_{p,q}(c_2,c_1)$, $(c_{\alpha^{(k_1,k_2)}}(e))^{k_1,k_2}_{p,q}(c_1,c_2)=\bar{e}^{k_2,k_1}_{p,q}(c_2,c_1)$. In addition, we assume that the three aforementioned sets are chosen so that the operator $S^{\mathbb{K}}_{p,q}$ can be written

$$S^{\mathbb{K}}_{p,q}(c_1\cdots c_k)=\sum_{1\leq i<j\leq k}\ \sum_{r\in S(c_i,c_j)}\bar{r}^{k_i,k_j}_{p,q}(c_i,c_j)c_1\cdots\hat{c}_i\cdots\hat{c}_j\cdots c_k,\ c_1\cdots c_k\in\mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_{\mathrm{irr},\infty}\right]. \tag{4.11}$$

Let $k_1,\ldots,k_s>0$ be integers and $c_i\in\mathcal{B}^{\mathrm{irr}}_{k_i}$, $i\leq s$. Put $S(i,j)=S\left(c_{\alpha^{(k_j,k_i)}}\left(\left(c_j\otimes c_i\right)\right)\right)$, $i,j\leq s$. The right hand side of equation (4.11) defines an operator on the Fock space $\mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_{\mathrm{irr},\infty}\right]$ since

$$\begin{aligned}\sum_{1\leq i<j\leq s}\ \sum_{r\in S(i,j)}\bar{r}^{k_i,k_j}_{p,q}(c_i,c_j)c_1\cdots\hat{c}_i\cdots\hat{c}_j\cdots c_s&=\sum_{1\leq i<j\leq s}\ \sum_{r\in S(i,j)}\bar{r}^{k_i,k_j}_{p,q}(c_i,c_j)c_1\cdots\hat{c}_i\cdots\hat{c}_j\cdots c_p\\&=\sum_{1\leq i<j\leq s}\ \sum_{r\in S(i,j)}\bar{r}^{k_j,k_i}_{p,q}(c_j,c_i)c_1\cdots\hat{c}_i\cdots\hat{c}_j\cdots c_s.\end{aligned}$$

It simple computations to prove that right hand side of (4.11) is a second order operator. To prove that $S^{\mathbb{K}}_{p,q}$ it remains to show that T,E and S exists so that equality (4.11) holds. Let $c_1\in\mathcal{B}_{k_1}$, $c_2\in\mathcal{B}_{k_2}$ and define

$$\tau^{k_1k_2}_{p,q}(c_1,c_2)=\rho^{p,q}(\mathring{\tau}\diamond(c_1\mathring{\otimes}c_2),c_1\mathring{\otimes}c_2)\tau\diamond(c_1\otimes c_2)$$
$$e^{k_1k_2}_{p,q}(c_1,c_2)=\rho^{p,q}(\mathring{\tau}\diamond(c_1\mathring{\otimes}c_2),c_1\mathring{\otimes}c_2)e\diamond(c_1\otimes c_2).$$

and the set $S(c_1,c_2)=\{r_{ij}\in \mathsf{W}_{k_1+k_2}\cup \mathsf{T}_{k_1+k_2} : 1\leq i\leq k_1,\ k_1<j<k_1+k_2\}$. Let $c_1\cdots c_s\in\mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_{\infty}\right]$, then

$$\begin{aligned}\frac{1}{s!}\sum_{\sigma\in\mathcal{S}_s}\left(S^{\mathbb{K}}_{p,q}\cdot\sigma\right)(c_1\otimes\cdots\otimes c_s) &= \sum_{i<j}\sum_{\sigma\in\mathcal{S}_s}\tau(c_{\sigma(i)},c_{\sigma(j)})c_{\sigma(1)}\cdots\hat{c}_{\sigma(i)}\cdots\hat{c}_{\sigma(j)}\cdots c_{\sigma(s)}\\ &= \frac{1}{s!}\sum_{i<j}\sum_{k<l}\sum_{\sigma:\sigma(i)=k,\sigma(j)=l}\tau(c_k,c_l)c_{\sigma(1)}\cdots\hat{c}_k\cdots\hat{c}_l\cdots c_{\sigma(s)}\\ &\quad+\frac{1}{s!}\sum_{i<j}\sum_{l<k}\sum_{\sigma:\sigma(i)=k,\sigma(j)=l}\tau(c_k,c_l)c_{\sigma(1)}\cdots\hat{c}_k\cdots\hat{c}_l\cdots c_{\sigma(s)}\\ &= \frac{1}{s!}\sum_{i\neq j}\sum_{k<l}\sum_{\sigma:\sigma(i)=k,\sigma(j)=l}\bar{\tau}(c_k,c_l)c_{\sigma(1)}\cdots\hat{c}_k\cdots\hat{c}_l\cdots c_{\sigma(s)}\\ &= \sum_{k<l}\bar{\tau}(c_k,c_l)c_1\cdots\hat{c}_k\cdots\hat{c}_l\cdots c_s\end{aligned}$$

Let $c_1\cdots c_s\mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_{\mathrm{irr},\infty}\right]$ and set $k=\deg$. We end the proof by computing the derivative of $\mathfrak{m}^{\mathbb{K}}_{p,q}(c_1\cdots c_s,t)$:

$$\begin{aligned}\frac{d}{du}\mathfrak{m}^{\mathbb{K}}_{p,q}(c_1\cdots c_s,\mathrm{t})(u) &= \frac{1}{s!}\sum_{\sigma\in\mathfrak{S}_s}\frac{d}{du}\mathfrak{m}^{\mathbb{K}}_{p,q,s}\left(c_{\sigma^{-1}(1)}\otimes\cdots\otimes c_{\sigma^{-1}(s)},\mathrm{t}\right)(u)\\ &= \frac{1}{s!}\sum_{\sigma\in\mathcal{S}_s}\mathfrak{m}^{\mathbb{K}}_{p,q,k}\left((S^{\mathbb{K}}_{p,q,s}+D^{\mathbb{K}}_{p,q,k})\left(c_{\sigma^{-1}(1)}\otimes\cdots\otimes c_{\sigma^{-1}(s)}\right),\mathrm{t}\right)(u)\\ &= \frac{1}{s!}\sum_{\sigma\in\mathfrak{S}_s}\mathfrak{m}^{\mathbb{K}}_{p,q,k}\left(\mathsf{type}\left((S^{\mathbb{K}}_{p,q,k}+D^{\mathbb{K}}_{p,q,s})\left(c_{\sigma^{-1}(1)}\otimes\cdots\otimes c_{\sigma^{-1}(s)}\right)\right),\mathrm{t}\right)(u)\\ &= \mathfrak{m}^{\mathbb{K}}_{p,q}\left(\mathcal{D}^{\mathbb{K}}_{p,q}(c_1\cdots c_s)+\mathcal{S}^{\mathbb{K}}_{p,q}(c_1\cdots c_s),t\right)(u).\end{aligned}$$

□

4.3. Microscopic fluctuations

This section deals with the main concern of the present chapter, which is to give the fluctuations' magnitude and the remaining asymptotic noise for the convergence of the pseudo unitary diffusions to their free counterparts. Let $k\geq 1$ an integer, b an irreducible coloured Brauer diagram of size k and w a word of size k (either in $\overline{\mathsf{M}}_q(k)$, either in $\mathsf{M}_q(k)$. We define the shifting morphism $X^{\mathbb{K}}_{p,q}:\mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_{\mathrm{irr},\infty}\right]\to\mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_{\mathrm{irr},\infty}\right]$ as

$$X^{\mathbb{K}}_{p,q}(b\otimes w)=b\otimes w-\mathfrak{m}^{\mathbb{K}}_{p,q}(b\otimes w,\mathrm{t})\in\mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_{\mathrm{irr},\infty}\right].$$

If $q\geq 1$ is an integer, we denote by $\mathsf{P}_2([\![1,q]\!])$ the set of matchings of $[\![1,q]\!]$, that is the partitions of $[\![1,q]\!]$ which blocks having cardinals all equal and equal to two.

Definition 4.6 (Centered Gaussian state on $\mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_{\mathrm{irr},\infty}\right]$). A Centered Gaussian state on $\mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_{\mathrm{irr},\infty}\right]$ is a state Φ (positive and hermitian linear functional) for which it exists a symmetric function $\sigma:\mathcal{B}^{\mathbb{K}}_{\infty}\times\mathcal{B}^{\mathbb{K}}_{\infty}\to\mathbb{C}$ such that

$$\Phi(b_1\cdots b_q)=\sum_{\pi\in\mathsf{P}_2([\![1,q]\!])}\prod_{\{i,j\}\in\pi}\sigma(b_i,b_j),\ b_1\cdots b_s\in\mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_{\mathrm{irr},\infty}\right].$$

Our main theorem states that random variable $\mathrm{m}^{\mathbb{K}}_{p,q}(X^{\mathbb{K}}_{p,q}\circ(p+q)^N,\mathrm{t})$ converges to a Gaussian field on $\mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_{\mathrm{irr},\infty}\right]$. To write down a formula for the covariance of that field, we need to introduce two more operators. We could define these two operators that are denoted $D^{\mathbb{K}}_{\lambda}$ and $\mathcal{S}^{\mathbb{K}}_{\lambda}$ as the point-wise limit of, respectively, $D^{\mathbb{K}}_{p,q}$ and $(p+q)^2\mathcal{S}^{\mathbb{K}}_{p,q}$. We have not yet discussed of a norm on $\mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_{\infty}\right]$; (it is in infinite dimensional space so that normed topology are not all equals) so that prior to define these operators a lengthy discussion involving complicated materials on tensors product of Banach space must be conducted. To circumvent this issue, we prefer to give explicit formulae for $\mathcal{D}^{\mathbb{K}}_{\lambda}$ and $\mathcal{S}^{\mathbb{K}}_{\lambda}$. Convergence of, respectively $D^{\mathbb{K}}_{p,q}$ and $(p+q)^2\mathcal{S}^{\mathbb{K}}_{p,q}$ to $D^{\mathbb{K}}_{\lambda}$ and $\mathcal{S}^{\mathbb{K}}_{\lambda}$ is then used only if these operators on restricted on finite dimensional subspaces of $\mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_{\mathrm{irr},\infty}\right]$, in which normed topologies are all equivalent.

We define $\mathcal{D}^{\mathbb{K}}_{\lambda}$ and $\mathcal{S}^{\mathbb{K}}_{\lambda}$ as respectively, a derivation and second order operator. We should exploit the fact that a basis of $\mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_{\infty}\right]$ is a set of (commutative) words on the alphabet $\mathcal{B}^{\mathbb{K}}$. In fact, to define such operators, it is enough to set the values $i\mathcal{D}^{\mathbb{K}}_{\lambda}(b)$, $b\in\mathcal{B}^{\mathbb{K}}$ and $,\mathcal{S}^{\mathbb{K}}_{\lambda}(b_1b_2)$, $b_1\in\mathcal{B}_k$, $b_2\in\mathcal{B}_q$. From Chapter 3, the functional $\mathcal{O}\mathring{\mathcal{B}}_k\times\mathcal{O}\mathring{\mathcal{B}}_k$ satisfies:

$$\rho_{\lambda}(((b_1,s_1),\mathrm{o}^{\alpha}),((b_2,s_2),\mathrm{o}^{\beta}))=\left(\tfrac{1}{1+\lambda}\right)^{\mathrm{fnc}_+(((b_1,s_1),\mathrm{o}^{\alpha}),((b_2,s_2),\mathrm{o}^{\beta}))}\left(\tfrac{\lambda}{1+\lambda}\right)^{\mathrm{fnc}_-(((b_1,s_1),\mathrm{o}^{\alpha}),((b_2,s_2),\mathrm{o}^{\beta}))}$$

For two oriented Brauer diagrams with words $b_1\otimes w_1\in\mathcal{O}\mathcal{B}_k$ and $b_2\otimes w_2\in\mathcal{O}\mathcal{B}_q$, set for $1\leq i\leq q$ an integer,

$$D^{\mathbb{K},i}_{\lambda}(b_1\otimes w_1)=c\sum_{c\in C_k}\alpha^{i}_{\lambda}(c,w_1)\mathrm{type}((\mathbf{1}_k,c)\diamond b_1\otimes w_1)+v\sum_{c\in C_k}\beta^{i}_{\lambda}(c,w_1)\mathrm{type}(((\mathbf{1}_k,c)\diamond b_1)\otimes w_1)$$

$$\mathcal{S}^{\mathbb{K}}_{\lambda}(b_1)=0,$$

$$\begin{aligned}S^{\mathbb{K},i}_{\lambda}((b_1\otimes w_1)(b_2\otimes w_2))&=(1+\frac{1}{\lambda})c\sum_{r\in\mathrm{R}^{=,-}_{k+q}(b_1b_2)(-)}(-1)^r\rho_{\lambda}(\mathring{r},\mathring{r}\diamond\mathring{b_1b_2})\ \mathrm{type}((r\diamond(b_1\otimes b_2))\otimes w_1w_2)\\&+(1+\lambda)c\sum_{r\in\mathrm{R}^{=,-}_{k+q}(b_1b_2)(+)}(-1)^r\rho_{\lambda}(\mathring{e},\mathring{e}\diamond\mathring{b_1b_2})\ \mathrm{type}((r\diamond(b_1\otimes b_2))\otimes w_1w_2)\\&+2v\sum_{r\in\mathrm{R}^{\neq,-}_{k+q}(b_1b_2)}\rho_{\lambda}(\mathring{r},\mathring{e}\diamond\mathring{b_1b_2})\mathrm{type}((e\diamond(b_1\otimes b_2))\otimes w_1w_2).\end{aligned}$$

and $\mathcal{S}^{\mathbb{K}}_{\lambda}=\sum_{i=1}^{q}t_iS^{\mathbb{K},i}_{\lambda}$, $\mathcal{D}^{\mathbb{K}}_{\lambda}=\sum_{i=1}^{q}t_iD^{\mathbb{K},i}_{\lambda}$. Remember, we proved in Chapter 3 the point-wise convergence of the finite dimensional generator $\mathcal{L}^{\mathbb{K}}_{p,q}$ to $\mathcal{L}^{\mathbb{K}}_{\lambda}=\mathcal{D}^{\mathbb{K}}_{\lambda}+\mathcal{S}^{\mathbb{K}}_{\lambda}$.

In the following theorem, we use the notation $[S]=\mathrm{m}^{\mathbb{K}}_{p,q}(S)(\mathcal{F}(I_{\mathbb{K}^{p+q}}))$ since this last quantity does not depend on the integers p,q. Also, N denotes the number operator, $N(b_1\cdots b_s)=s(b_1\cdots b_s)$ with $b_1,\ldots,b_s\in\mathcal{B}^{\mathbb{K}}$.

Theorem 4.7. *The state $\mathbb{m}^{\mathbb{K}}_{p,q}(\mathrm{t})\circ X^{\mathbb{K}}_{p,q}\circ(p+q)^N$ on $\mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_{irr,\infty}\right]$ converges to a Gaussian state as $p,q\to+\infty$ with $\frac{p}{q}\to\lambda>0$. In addition, with respect to the limiting state, the covariance between two Brauer diagrams b,b' is given by*

$$\sigma_{\lambda,\mathrm{t}}(b,b')=\left[\int_0^1 2e^{u\mathcal{D}^{\mathbb{K}}_{\lambda}}\Gamma_{\mathcal{S}^{\mathbb{K}}_{\lambda}}\left(e^{(1-u)\mathcal{D}^{\mathbb{K}}_{\lambda}}b,e^{(1-u)\mathcal{D}^{\mathbb{K}}_{\lambda}}b'\right)\mathrm{d}u\right],\ b,b'\in\mathcal{B}^{\mathbb{K}}.$$

The next section is devoted to proving theorem 4.7.

4.3.1. Proof of Theorem 4.7. Let p,q two integers and $t\geq 0$ a time. For $n\in\mathbb{N}^{\star}$ and $t\geq 0$, let us denote by $\Delta_n(t)\subset\mathbb{R}^n$ the simplex

$$\Delta_n(t)=\{0\leq t_1\leq\cdots t_n\leq t\}.$$

We begin with the following fundamental Lemma.

Lemma 4.8. *For all $b \in \mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_\infty\right]$, one has for all time $t \geq 0$ and integers $p,q \geq 0$,*

$$e^{t\left(\mathcal{D}^{\mathbb{K}}_{p,q}+\frac{1}{(p+q)^2}\mathcal{S}^{\mathbb{K}}_{p,q}\right)}(b) = e^{t\mathcal{D}^{\mathbb{K}}_{p,q}}(b) + \sum_{n=1}^{k}\frac{1}{(p+q)^{2n}}\int_{u\in\Delta_n(t)} e^{u_n\mathcal{D}^{\mathbb{K}}_{p,q}}\mathcal{S}^{\mathbb{K}}_{p,q}e^{(u_{n-1}-u_n)\mathcal{D}^{\mathbb{K}}_{p,q}}\mathcal{S}^{\mathbb{K}}_{p,q}\cdots\mathcal{S}^{\mathbb{K}}_{p,q}e^{(t-u_1)\mathcal{D}^{\mathbb{K}}_{p,q}}(b)\mathrm{d}u + \frac{1}{(p+q)^{2(k+1)}}\int_{u\in\Delta(t)} e^{t_{k+1}(\mathcal{D}^{\mathbb{K}}_{p,q}+\mathcal{S}^{\mathbb{K}}_{p,q})}\mathcal{S}^{\mathbb{K}}_{p,q}e^{(u_{k-1}-u_k)\mathcal{D}^{\mathbb{K}}_{p,q}}\mathcal{S}^{\mathbb{K}}_{p,q}\cdots\mathcal{S}^{\mathbb{K}}_{p,q}e^{(t-u_1)\mathcal{D}^{\mathbb{K}}_{p,q}}(b)\mathrm{d}u.$$

Proof. The proof can be found in [14]. □

Define $\mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_\infty\right]_{\leq k} = \bigoplus_{q\leq k}\mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_\infty\right]_q$ and $\Phi_{p,q}(t) = \mathbb{E}\left[\mathrm{m}^{\mathbb{K}}_{p,q}(t)\circ X^{\mathbb{K}}_{p,q}\circ(p+q)^N\right]$ for $t \geq 0$. Let $t \geq 0$. To prove the convergence of the state $\Phi_{p,q}(t)$, it is sufficient to prove

$$\Phi_{p,q}(t)(b^m) \underset{p,q\to+\infty}{\longrightarrow} C_m\sigma_{t,\lambda}(b,b), b\in\mathcal{B}^{\mathbb{K}},\ m\geq 1,$$

with $C_m = \sharp\mathrm{P}_2(\llbracket 1,\ldots,m\rrbracket) = \frac{m!}{2^{\frac{m}{2}}\frac{m}{2}!}$ is m is even and 0 otherwise.

Let $b \in \mathcal{B}^{\mathbb{K}}$ and $k \geq 1$ the integer such that $b \in \mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_\infty\right]_k$.

We restrict the action of the operators $\mathcal{D}^{\mathbb{K}}_{p,q}$ and $\mathcal{S}^{\mathbb{K}}_{p,q}$ to the finite dimensional subspace $\mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_\infty\right]_{\leq km}$ introduced previously. The operators $\mathcal{S}^{\mathbb{K}}_{p,q}$ and $\mathcal{D}^{\mathbb{K}}_{p,q}$ stabilize $\mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_\infty\right]_{\leq km}$. We endow this space with a norm $\|\cdot\|$, denote by the same symbol the associated operator norm on the space of operators acting on $\mathcal{F}\left[\mathcal{B}^{\mathbb{K}}_\infty\right]_{km}$ and let

$$\|\mathcal{D}^{\mathbb{K}}\| = \sup_{p,q}\|\mathcal{D}^{\mathbb{K}}_{p,q}\| < +\infty,\ \|\mathcal{S}^{\mathbb{K}}\| = \sup_{p,q}\|S^{\mathbb{K}}_{p,q}\|.$$

From the definition of the operators $\mathcal{D}^{\mathbb{K}}_\lambda$ and $\mathcal{S}^{\mathbb{K}}_{p,q}$, we have

$$\mathcal{D}^{\mathbb{K}}_{p,q} \underset{p,q\to+\infty}{\overset{\|\cdot\|}{\longrightarrow}} \mathcal{D}^{\mathbb{K}}_\lambda,\ \mathcal{S}^{\mathbb{K}}_{p,q} \underset{p,q\to+\infty}{\overset{\|\cdot\|}{\longrightarrow}} \bar{\mathcal{S}}^{\mathbb{K}}_\lambda. \tag{4.12}$$

From Lemma 4.8, to prove Theorem 4.7, it is sufficient to study separately the limit as $p,q\to+\infty$ of the three quantities

1. $(p+q)^m\left[e^{t\mathcal{D}^{\mathbb{K}}_{p,q}}(X^{\mathbb{K}}_{p,q}(b)^m))\right]$,
2. $(p+q)^{m-2n}\left[\int_{u\in\Delta_n(t)} e^{t_n\mathcal{D}^{\mathbb{K}}_{p,q}}\mathcal{S}e^{(u_{n-1}-u_n)\mathcal{D}^{\mathbb{K}}_{p,q}}\mathcal{S}^{\mathbb{K}}_{p,q}\cdots\mathcal{S}^{\mathbb{K}}_{p,q}e^{(t-u_1)\mathcal{D}^{\mathbb{K}}_{p,q}}\left(X^{\mathbb{K}}_{p,q}(b)^m\right)\mathrm{d}u\right]$, $1\leq n\leq[\frac{m}{2}]$,
3. $(p+q)^{m-2[\frac{m}{2}]-1}\left[\int_{u\in\Delta_{k+1}(t)} e^{u_n\mathcal{D}^{\mathbb{K}}_{p,q}}\mathcal{S}^{\mathbb{K}}_{p,q}e^{(u_{n-1}-u_n)\mathcal{D}^{\mathbb{K}}_{p,q}}S^{\mathbb{K},(1)}_{p,q}\ldots S^{\mathbb{K},(1)}_{p,q}e^{(t-u_1)\mathcal{D}^{\mathbb{K}}_{p,q}}\left(X^{\mathbb{K}\ m}_{p,q}(b)\right)\mathrm{d}u\right]$.

For the first term (1), one has:

$$\begin{aligned}\left[e^{t\mathcal{D}^{\mathbb{K}}_{p,q}}\left(X^{\mathbb{K}}_{p,q}(b)^m\right)\right] &= \left[e^{t\mathcal{D}^{\mathbb{K}}_{p,q}}\left(\left(b-\left[e^{t\left(\mathcal{D}^{\mathbb{K}}_{p,q}+\frac{1}{(p+q)^2}\mathcal{S}^{\mathbb{K}}_{p,q}\right)}(b)\right]\right)^m\right)\right]\\ &= \left[\left(e^{t\mathcal{D}^{\mathbb{K}}_{p,q}}\left(b-\left[e^{t\left(\mathcal{D}^{\mathbb{K}}_{p,q}+\frac{1}{(p+q)^2}\mathcal{S}^{\mathbb{K}}_{p,q}\right)}(b)\right]\right)\right)^m\right]\\ &= \left[\left(e^{t\mathcal{D}^{\mathbb{K}}_{p,q}}(b)-\left[e^{t\mathcal{D}^{\mathbb{K}}_{p,q}}(b)+\frac{1}{(p+q)^2}\mathcal{S}^{\mathbb{K}}_{p,q}(b)+O(\frac{1}{(p+q)^2})\right]\right)^m\right]\\ &= O\left(\frac{1}{(p+q)^{2m}}\right).\end{aligned} \tag{4.13}$$

Hence, $(p+q)^m\left[e^{t\mathcal{D}^{\mathbb{K}}_{p,q}}(X^{\mathbb{K}}_{p,q}(b)^m)\right] = O\left(\frac{1}{(p+q)^m}\right)$.

We focus now on the second term (2.). The computations are very close to the ones done in [14], we should only mention how they can be adapted for our needs. The lemma 3.7 in [14] is applied for the operators at hands an furnish for each $n \leq m$ a remainder $R_n^{p,q} \in \mathcal{F}\left[\mathcal{B}_\infty^{\mathbb{K}}\right]_{km}$, bounded independently of p,q such that, with $(t_1,\ldots,t_n) \in \Delta(t)$:

$$
\begin{aligned}
&\mathcal{S}_{p,q}^{\mathbb{K}} e^{(t_{n-1}-t_n)\mathcal{D}_{p,q}^{\mathbb{K}}} \mathcal{S}_{p,q}^{\mathbb{K}} \cdots \mathcal{S}_{p,q}^{\mathbb{K}} e^{(1-t_1)\mathcal{D}_{p,q}^{\mathbb{K}}} (X_{p,q}^{\mathbb{K}}(b)^m) \\
&\qquad = \frac{m!}{(m-2n)!} \left(e^{(1-t_n)\mathcal{D}_{p,q}^{\mathbb{K}}} X_{p,q}^{\mathbb{K}}(b)\right)^{m-2n} e^{-t_n \mathcal{D}_{p,q}^{\mathbb{K}}} \left(\Gamma_{p,q}(t_1)\cdots\Gamma_{p,q}(t_n)\right) \\
&\qquad\quad + \left(e^{(1-t_n)\mathcal{D}_{p,q}^{\mathbb{K}}} X_{p,q}^{\mathbb{K}}(b)\right)^{k-2n+1} R_n^{p,q}.
\end{aligned} \tag{4.14}
$$

where $\Gamma(u) = e^{uD_{p,q}^{\mathbb{K}}} \Gamma(e^{(1-u)\mathcal{D}_{p,q}^{\mathbb{K}}} X_{p,q}^{\mathbb{K}}(b), e^{(1-u)\mathcal{D}_{p,q}^{\mathbb{K}}} X_{p,q}^{\mathbb{K}}(b))$. By using (4.13), we have

$$
\left[\left(e^{\mathcal{D}_{p,q}^{\mathbb{K}}} X_{p,q}^{\mathbb{K}}(b)\right)^{k-2n}\right] = O\left(\left(\frac{1}{(p+q)^2}\right)^{k-2n}\right), \left[\left(e^{\mathcal{D}_{p,q}^{\mathbb{K}}} X_{p,q}^{\mathbb{K}}(b)\right)^{k+1-2n}\right] = O\left(\left(\frac{1}{(p+q)^2}\right)^{k+1-2n}\right).
$$

By combining the last two estimates and equality (4.14), we obtain

$$
(p+q)^{m-2n}\left[\int_{\Delta_n(t)} e^{t_n \mathcal{D}_{p,q}^{\mathbb{K}}} \mathcal{S}_{p,q}^{\mathbb{K}} e^{(t_{n-1}-t_n)\mathcal{D}_{p,q}^{\mathbb{K}}} \mathcal{S}_{p,q}^{\mathbb{K}} \cdots \mathcal{S}_{p,q}^{\mathbb{K}} e^{(t-t_1)\mathcal{D}_{p,q}^{\mathbb{K}}} \left(X_{p,q}^{\mathbb{K}}(b)^m\right) \mathrm{d}t_1 \cdots \mathrm{d}t_n\right]
$$

is $O(\frac{1}{p+q})$ is $n < [\frac{m}{2}]$ and is equal to $m!\left[\int_{\Delta_n(t)} \Gamma_{p,q}(u_1)\cdots\Gamma_{p,q}(u_n)\mathrm{d}u\right]$ if $2n = m$. In the case $n = 2m$, since $\Gamma_{\mathcal{S}_\lambda^{\mathbb{K}}}(t_1)\cdots\Gamma_{\mathcal{S}_\lambda^{\mathbb{K}}}(t_n)$ is symmetric in the variables $t_1,\ldots,t_n$ and $S(1) = 0$ we get that

$$
(p+q)^{m-2n}\left[\int_{\Delta_n(t)} e^{t_n \mathcal{D}_{p,q}^{\mathbb{K}}} \mathcal{S}_{p,q}^{\mathbb{K}} e^{(u_{n+1}-u_n)\mathcal{D}_{p,q}^{\mathbb{K}}} \mathcal{S}_{p,q}^{\mathbb{K}} \cdots \mathcal{S}_{p,q}^{\mathbb{K}} e^{(t-u_1)\mathcal{D}_{p,q}^{\mathbb{K}}} (X_{p,q}^{\mathbb{K}}(b)^m)\mathrm{d}u\right]
$$

is equal to $\frac{2n!}{2^n n!}\left(2\int_0^t \Gamma_{p,q}(u)\mathrm{d}u\right)^n$ if $m = 2n$ and $O(\frac{1}{(p+q)})$ if not.

For the third term (3.), we remark that $m - 2\left[\frac{m}{2}\right] - 1 \leq -1$ and that

$$
\left[\int_{\Delta_n(t)} e^{u_n \mathcal{D}_{p,q}^{\mathbb{K}}} \mathcal{S}_{p,q}^{\mathbb{K}} e^{(u_{n-1}-u_n)\mathcal{D}_{p,q}^{\mathbb{K}}} \mathcal{S}_{p,q}^{\mathbb{K}} \cdots \mathcal{S}_{p,q}^{\mathbb{K}} e^{(t-u_1)\mathcal{D}} \left(X_{p,q}^{\mathbb{K}}(b)^m\right)\mathrm{d}u\right] \leq \|\mathcal{S}^{\mathbb{K}}\|^n e^{\left(\|\mathcal{D}_{p,q}^{\mathbb{K}}\| + \|\mathcal{S}^{\mathbb{K}}\|\right)}.
$$

Therefore, the third term is $O(\frac{1}{p+q})$. Eventually, we get

$$
\mathbb{E}\left[\mathrm{m}_{p,q}^{\mathbb{K}}(X_{p,q}^{\mathbb{K}} \circ (p+q)^N)(b^m), \mathrm{t})\right] = \frac{m!}{2^{\frac{m}{2}}(\frac{m}{2})!}\left(\int_0^t 2\Gamma_{p,q}(u)\mathrm{d}u\right)^{\frac{m}{2}} + O\left(\frac{1}{p+q}\right) \tag{4.15}
$$

if m is even and $O(\frac{1}{(p+q)})$ otherwise. From (4.12) we get

$$
\mathbb{E}\left[\mathrm{m}_{p,q}^{\mathbb{K}}(X_{p,q}^{\mathbb{K}} \circ (p+q)^N)(b^m), \mathrm{t})\right] \underset{p,q\to+\infty}{\longrightarrow} \frac{m!}{2^{\frac{m}{2}} \frac{m}{2}!}\left(\int_0^1 e^{u\mathcal{D}_\lambda^{\mathbb{K}}} \Gamma_{\mathcal{S}_\lambda^{\mathbb{K}}}\left(e^{(1-u)\mathcal{D}_\lambda^{\mathbb{K}}}(b), e^{(1-u)\mathcal{D}_\lambda^{\mathbb{K}}}(b)\right)\mathrm{d}u\right)^{\frac{m}{2}},
$$

if m is even and 0 otherwise.

4.4. Speed of convergence of the non-commutative distribution

In the last section we proved a central limit theorem; the random variables (4.1) converge to 0 at speed $(p+q)$ and we gave a formula for the covariance of the remaining Gaussian noise. In this section, we are concerned with the speed at which the split pseudo-unitary Brownian motion's distribution converges to the free Brownian pseudo-unitary Brownian motion's distribution. The split pseudo-unitary Brownian motion, as a non commutative process on the pseudo-unitary dual group is denoted $V_p^{\mathbb{K}}$ and its free counterpart is denoted V. The distributions at time t of these two random variables are denoted respectively $\nu_{p,q}^{\mathbb{K}}(t)$ and $\nu^{\mathbb{K}}(t)$. In a second part, we turn our attention to the more general question of finding the rate of convergence of the distribution the pseudounitary Brownian diffusion as the signature of the metric tends to infinity under the assumptions of the last section. Once again, the split case is treated first to expose the method, since computations are much more simple. In the last chapter, we prove the convergence

$$\frac{1}{p}\mathrm{Tr}\Big(V_p^{\mathbb{K}}(t)(u)\Big)-\nu^{\mathbb{K}}(t)(u)\xrightarrow{p\to+\infty},\ \text{for all } u\in\mathcal{U}(-,+).$$

in moments. Let $k\geq 1$ an integer. We leave the algebraic frameword we developed to study microscopic fluctuations. In the sequel, all Brauer diagrams and words are of fixed length $k\geq 1$ and operators acts on $\mathbb{R}[\mathcal{B}_k]\otimes\mathbb{R}[\mathsf{M}_1(k)]$ (or $\mathbb{R}[\mathcal{B}_k]\otimes\mathbb{R}\Big[\overline{\mathsf{M}_1}(k)\Big]$ if considering the complex case). In Chapter 3 we introduced two functions $\mathbb{m}$ and $\bar{\mathbb{m}}$ that are the solutions of differential systems with generators L and $\bar{L}$ (equal respectively, modulo the map type, to the generators $\mathcal{L}_1^{\mathbb{K}}$ and $\mathcal{L}_1^{\mathbb{C}}$ if restricted to $\mathbb{R}[\mathcal{B}_k]\subset\mathcal{F}\Big[\mathcal{B}_\infty^{\mathbb{K}}\Big]$) (or to $\mathbb{R}[\mathcal{B}_k]\otimes\mathbb{R}\Big[\overline{\mathsf{M}}_1(k)\Big]$ in the complex case), for all $b\in\mathcal{B}_k$ and $w\in\overline{\mathsf{M}}_2(k)$:

$$\frac{d}{dt}\mathbb{m}(b,t)=\mathbb{m}(L(b),t),\ \mathbb{m}(0)=\delta_\Delta \text{ and } \frac{d}{dt}\overline{\mathbb{m}}(b\otimes w,t)=\overline{\mathbb{m}}(\bar{L}(b\otimes w),t),\ \overline{\mathbb{m}}(0)=\delta_\Delta. \tag{4.16}$$

The two finite dimensional generators $L_{p,p}^{\mathbb{R}}$ and $L_{p,p}^{\mathbb{H}}$ converge to L while $L_{p,p}^{\mathbb{C}}$ converges to $\bar{L}$. It implies the pointwise convergence of the functions $\mathbb{m}_{p,p}^{\mathbb{R}},\mathbb{m}_{p,p}^{\mathbb{H}}$ to $\mathbb{m}$ and $\mathbb{m}_{p,p}^{\mathbb{C}}$ to $\bar{\mathbb{m}}$ as the dimension p tends to $+\infty$. The following lemma is key to find out the rate of convergence of $\mathbb{m}_{p,p}^{\mathbb{K}}$.

Lemma 4.9. *Let $d\geq 1$ be an integer. Let $\|\cdot\|$ be a norm of algebra on $M_d(\mathbb{C})$. Let A,B be two elements of $M_d(\mathbb{C})$. Then*

$$\|e^{A+B}-e^A\|\leq\|B\|e^{\max(\|A+B\|,\|A\|)}.$$

We shall apply the last lemma to the generators $L_{p,q}^{\mathbb{K}}$, L and $\bar{L}$. If $p=q$, the three operators $N_{p,p}^{\mathbb{K},+}$, $N_{p,p}^{\mathbb{K},-}$ and $N_{p,p}^{\mathbb{K},0}$ do not depend on p: in the sequel we drop the indices p in the symbols that stand for these operators. Consider first the real and quaternionic cases and write, with the notations introduced in the first section,

$$\begin{aligned}P_p^{\mathbb{K}}&=c\sum_{c\in C_k}\alpha_{p,p}^{\mathbb{K}}(c)(c,\mathbf{1}_k)\diamond+v\sum_{c\in C_k}\beta_{p,p}^{\mathbb{K}}(c)(c,\mathbf{1}_k)\diamond+N^{\mathbb{K},+}\\ M_p^{\mathbb{K}}&=\frac{1}{2p}N^{\mathbb{K},0}+\frac{1}{(2p)^2}N^{\mathbb{K},-}.\end{aligned}\tag{4.17}$$

for $\mathbb{K}=\mathbb{R}$ or $\mathbb{H}$. Let $t\geq 0$. In the split case, the two operators $P_p^{\mathbb{K}}$ and $M_p^{\mathbb{K}}$ have fairly simple expression, let $c\in C_k$,

$$\begin{aligned}&\alpha_{p,p}^{\mathbb{R}}(c)=-\frac{k}{2}\left(\frac{p-1}{p}\right), &&\alpha_{p,p}^{\mathbb{C}}(c)=-\frac{k}{2}, &&\alpha_{p,p}^{\mathbb{H}}(c)=-\frac{k}{2}\left(\frac{p+3}{p}\right),\\ &\beta_{p,p}^{\mathbb{R}}(c)=\frac{k}{2}, &&\beta_{p,p}^{\mathbb{C}}(c)=\frac{k}{2}, &&\beta_{p,p}^{\mathbb{H}}(c)=\frac{k}{2}.\end{aligned}$$

Since the coefficients $\alpha_{p,p}^{\mathbb{K}}(c)(\beta_{p,p}^{\mathbb{K}}(c))$ do not depend on the colourization c (resp. on the colourization c and the integer p), we should write $\alpha_p^{\mathbb{K}}(resp.\beta^{\mathbb{K}})$ in the sequel. The same remark applies for the coefficients $\rho_{p,p}^{\mathbb{K}}$ in the expansion of the operators $N_{p,p}^{\mathbb{K},s}$, $s\in\{-,0,+\}$ as sums over element diagrams.

Proposition 4.10. *As the dimension $p\to+\infty$, one has for each time $t\geq 0$,*

$$\mathbb{m}_{p,p}^{\mathbb{K}}(b,t)=\mathbb{m}(b,t)+O\left(\frac{1}{p}\right),\ b\in\mathcal{B}_k,\ \mathbb{K}=\mathbb{R}\text{ or }\mathbb{H},$$

and

$$\mathbb{m}_{p,p}^{\mathbb{C}}((b,w),t)=\overline{\mathbb{m}}((b,w),t)+O\left(\frac{1}{p^2}\right),\ (b\otimes w)\in\mathbb{R}[\mathcal{B}_k]\otimes\mathbb{R}\Big[\overline{\mathsf{M}}_2(k)\Big].$$

Proof. Asumme first $\mathbb{K}=\mathbb{R}$ or $\mathbb{H}$. Let $p \geq 1$ an integer. From $\mathbb{m}^{\mathbb{K}}_{p,p}(b,t) = \delta_{\Delta} \circ \exp\left(tL^{\mathbb{K}}_{p,p}\right)(b)$ and Lemma 4.9,

$$\|\mathbb{m}^{\mathbb{K}}_{p,p}(b,t) - \mathbb{m}(b,t)\| \leq \|\delta_{\Delta}\|\|e^{tL^{\mathbb{K}}_{p,q}} - e^{tL}\| \leq t\|L^{\mathbb{K}}_{p,p} - L\|e^{t\max(\|L\|,\|L^{\mathbb{K}}_{p,p}\|)},\ b \in \mathcal{OB}_k,\ t \geq 0. \tag{4.18}$$

From the formula for the operator L,

$$L(b) = \frac{k}{2}(v-c)b + c \sum_{r^{\bullet} \in \mathsf{T}^{=,+}_{k,2}(+,b) \cup \mathsf{W}^{=,+}_{k,2}(+,b)} (-1)^{r} r^{\bullet} \diamond b + v \sum_{r^{\bullet} \in \mathsf{T}^{\neq,+}_{k}(b) \cup \mathsf{W}^{\neq,+}_{k}(b)} r^{\bullet} \diamond b, \text{ with } b \in \mathcal{OB}_k,$$

we obtain the estimate

$$\|L^{\mathbb{K}}_{p,p} - L\| \leq \left\|\frac{k}{2}\left(v\left(1-\frac{p}{p-1}\right) - c\left(1-\frac{p-1}{p}\right)\right)\right\| + \|M^{\mathbb{K}}_{p}\|.$$

This estimate combined with $\left\|M^{\mathbb{K}}_{p}\right\| = \mathcal{O}(\frac{1}{p})$ and equation (4.18) implies the result.

For the complex case, a slight modification of previous computations leads to the estimate announced estimate. In fact, as we already noticed, $N^{\mathbb{C},0}_{p,p} = 0$ and thus $\|M^{\mathbb{C}}_{p,q} = O\left(\frac{1}{(2p)^2}\right)\|$, in addition, $\alpha^{\mathbb{C}}_{p,p}(c) = 1$. $\square$

One can states a sharper estimate than Proposition 4.10 gives. In fact, we prove in the next theorem that the sequence $p\left(\mathbb{m}^{\mathbb{K}}_{p,p}(t) - \mathbb{m}(t)\right)$, as a sequence of functions over $\mathbb{R}[\mathcal{OB}_k]$ (on $\mathbb{R}[\mathcal{OB}_k] \otimes \mathbb{R}\left[\overline{\mathsf{M}}_1(k)\right]$ in the complex case) in addition to being bounded, converges as $p \to +\infty$.

Theorem 4.11. *As $p \to +\infty$, for all time $t \geq 0$, $p\left(\mathbb{m}^{\mathbb{C}}(t) - \bar{\mathbb{m}}(t)\right)$ converges to 0.*

Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{H}$. As $p \to +\infty$, $p\left(\mathbb{m}^{\mathbb{K}}_{p,p}(t) - \mathbb{m}(t)\right)$ to the solution $\mathbb{m}^{\mathbb{K}}_{(1)}(t)$ of the following differential system:

$$\begin{aligned} &\frac{d}{dt}\mathbb{m}^{\mathbb{K}}_{(1)}(t) = \frac{k}{2}(v-c)\mathbb{m}^{\mathbb{K}}_{(1)}(t) + N^{\mathbb{K},+}\left(\mathbb{m}^{\mathbb{K}}_{(1)}(t)\right) + \frac{k}{2}\alpha^{\mathbb{K}}_{(1)}\mathbb{m}(t) + \frac{1}{2}N^{\mathbb{K},0}(\mathbb{m}(t)),\\ &\mathbb{m}^{\mathbb{K}}_{(1)}(0) = 0, \end{aligned} \tag{4.19}$$

with $\alpha^{\mathbb{R}}_{(1)} = 1$, $\alpha^{\mathbb{H}}_{(1)} = -3$.

Proof. Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{H}$. Let $b \in \mathcal{B}_k$ a Brauer diagram and $t \geq 0$ a time. Put $\mathbb{m}^{\mathbb{K},(1)}_{p}(b,t) = \mathbb{m}^{\mathbb{K}}_{p,p}(b,t) - \mathbb{m}(b,t)$. We compute the derivative of $\mathbb{m}^{\mathbb{K},(1)}_{p}$. In the following, we make a slight abuse of notation; the function $\mathbb{m}$ stands for $\mathbb{m}$ defined above if $\mathbb{K}=\mathbb{R}$ or $\mathbb{H}$ and for $\bar{\mathbb{m}}$ if $\mathbb{K}=\mathbb{C}$.

$$\begin{aligned} \frac{d}{dt}\mathbb{m}^{\mathbb{K},(1)}_{p} &= p\frac{d}{dt}\mathbb{m}^{\mathbb{K}}_{p,p}(t) - p\frac{d}{dt}\mathbb{m}(t) = pL^{\mathbb{K}}_{p,p}\left(\mathbb{m}^{\mathbb{K}}_{p,p}(t)\right) - pL(\mathbb{m}(t))\\ &= \left(p\alpha^{\mathbb{K}}_{p,p} + p\beta^{\mathbb{K}}_{p,p}\right)\mathbb{m}^{\mathbb{K}}_{p,p}(t) - \frac{k}{2}(v-c)p\mathbb{m}(t)\\ &\qquad + p\sum_{r \in (\mathsf{W}\cup\mathsf{T})^{+}} \rho^{\mathbb{K}}_{p,p}(r,\cdot)\mathbb{m}^{\mathbb{K}}_{p,p}(r\circ\cdot,t) - \mathbb{m}(r\circ\cdot,t)\\ &\qquad + \frac{p}{2p}N^{\mathbb{K},0}(\mathbb{m}^{\mathbb{K}}_{p,p}(t)) + \frac{p}{(2p)^2}N^{\mathbb{K},-}(\mathbb{m}^{\mathbb{K}}_{p,p}(t))\\ &= \left(\frac{k}{2}(v-c)\right)p\left(\mathbb{m}^{\mathbb{K}}_{p,p}(t) - \mathbb{m}(t)\right) + \left(\alpha^{\mathbb{K}}_{p,p} + \beta^{\mathbb{K}}_{p,p} - \frac{k}{2}(v-c)\right)\mathbb{m}^{\mathbb{K}}_{p,p}(t)\\ &\qquad + p\sum_{r \in (\mathsf{W}\cup\mathsf{T})^{+}} \rho^{\mathbb{K}}_{p,p}(r,\cdot)\mathbb{m}^{\mathbb{K}}_{p,p}(r\circ\cdot,t) - \mathbb{m}(r\circ\cdot,t)\\ &\qquad + \frac{1}{2}N^{\mathbb{K},0}_{p,q}(\mathbb{m}^{\mathbb{K}}_{p,p}(t)) + o(p) \end{aligned}$$

Furthermore, $\left(\alpha^{\mathbb{R}}_{p,p}+\beta^{\mathbb{R}}_{p,p}\right)-\frac{k}{2}(v-c)=c\frac{k}{2}\alpha^{\mathbb{K}}_{(1)}+o(p)$ with $\alpha^{\mathbb{H}}_{(1)}=-3,\ \alpha^{\mathbb{R}}_{(1)}=1,\ \alpha^{\mathbb{C}}_{(1)}=0$. The coefficients $\rho^{\mathbb{K}}_p$ are all equal to one. From the last equation, we get

$$\frac{d}{dt}\mathfrak{m}^{\mathbb{K},(1)}_{p,p}(t)=\frac{k}{2}(v-c)\mathfrak{m}^{\mathbb{K},(1)}_{p,p}(t)+\sum_{r\in(\mathsf{W}\cup\mathsf{T})^+}\mathfrak{m}^{\mathbb{K},(1)}_{p,p}(t)+\frac{k}{2}\alpha^{\mathbb{K}}_{(1)}\mathfrak{m}(t)+\frac{1}{2}N^{\mathbb{K},0}(\mathfrak{m}(t))+o(p)$$

We infer that $t\to\mathfrak{m}^{\mathbb{K},(1)}_{p,p}(t)$ converges, as $p\to+\infty$ to the the solution $\mathfrak{m}^{\mathbb{K}}_{(1)}$ of the non-autonomous system:

$$\begin{aligned}&\frac{d}{dt}\mathfrak{m}^{\mathbb{K}}_{(1)}(t)=\frac{k}{2}(v-c)\mathfrak{m}^{\mathbb{K}}_{(1)}(t)+N^{+}\left(\mathfrak{m}^{\mathbb{K}}_{(1)}(t)\right)+\frac{k}{2}\alpha^{\mathbb{K}}_{(1)}\mathfrak{m}(t)+\frac{1}{2}N^{\mathbb{K},0}(\mathfrak{m}(t)),\\&\mathfrak{m}^{\mathbb{K}}_{(1)}(0)=0.\end{aligned}\tag{4.20}$$

□

4.4.0.1. *Macroscopic fluctuations : the general case.* We now treat the general case. Let $k\geq 1$ an integer, that will be the size of all words and diagrams that are considered in this section. Let $\mathbb{K}$ be one of the three division algebras $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. As for the split case, we first find a bound on the rate of convergence of $\mathfrak{m}^{\mathbb{K}}_{p,q}$. Then, we sharpen this estimate. We prove in the last chapter, Chapter 3, that $\mathfrak{m}^{\mathbb{K}}_{p,q}$ is the solution of a differential system, with generator $L^{\mathbb{K}}_{p,q}=D^{\mathbb{K}}_{p,q}+S^{\mathbb{K}}_{p,q}$, see equation (4.4). We proved in the same chapter that if $p,q\to+\infty$ (under the hypothesis (H)), the operator $N^{\mathbb{K},\varepsilon}_{p,q}$, converges point-wise to an operator $N^{\mathbb{K},\varepsilon}_{\lambda}$ with $\varepsilon\in\{-,0,+\}$ (we have already used that point in the last section). To apply the same method we used for the split case, we have to find a bound of the norm $\|L^{\mathbb{K}}_{p,q}-L^{\mathbb{K}}_{\lambda}\|$. This will be done in two steps. First, we find a bound on

$$\|\alpha^{\mathbb{K}}_{p,q}(c)-\alpha^{\mathbb{K}}_{p,q}(c)\|\text{ and }\|\beta^{\mathbb{K}}_{p,q}-\beta^{\mathbb{K}}_{\lambda}\|,$$

where $\alpha^{\mathbb{K}}_{\lambda}$ (resp. $\beta^{\mathbb{K}}_{\lambda}$) denotes the limit of $\alpha^{\mathbb{K}}_{p,q}$ (resp. $\beta^{\mathbb{K}}_{p,q}$). Formulae for these last two quantities can be read of the formulae we gave for the operators $\mathcal{C}^{\mathbb{K},1}_{\lambda}$ and $\mathcal{P}^{\mathbb{K},1}_{\lambda}$ in Chapter 3:

$$\beta^{\mathbb{K}}_{\lambda}=-\frac{1}{2}\left(\sharp(+,c)\frac{1}{1+\lambda}+\sharp(-,c)\frac{\lambda}{1+\lambda}\right),\ \alpha^{\mathbb{K}}_{\lambda}=-\frac{k}{2}.$$

Hence,

$$\|\alpha^{\mathbb{K}}_{p,q}(c)-\alpha^{\mathbb{K}}_{\lambda}(c)\|=\mathcal{O}\left(\frac{1}{p+q}\right)\text{ and }\|\beta^{\mathbb{K}}_{p,q}(c)-\beta^{\mathbb{K}}_{\lambda}(c)\|=\mathcal{O}\left(\frac{p}{q}-\lambda\right).$$

This last two estimates give a first hint on the form of the bound for $\|L^{\mathbb{K}}_{p,q}-L^{\mathbb{K}}_{\lambda}\|$ we should expect to obtain, the rate of convergence of the ratio $\frac{p}{q}$ to λ has to be compared with an inverse linear speed. It remains to find a bound for the difference $\|N^{\mathbb{K},+}_{p,q}-N^{\mathbb{K},+}_{\lambda}\|$, since, from the the point-wise convergence of the operators $N^{\mathbb{K},-}_{p,q}$ and $N^{\mathbb{K},0}_{p,q}$, one has $\frac{1}{p+q}N^{\mathbb{K},-}_{p,q}=\mathcal{O}\left(\frac{1}{p+q}\right)$ and $\frac{1}{(p+q)^2}N^{\mathbb{K},-}=\mathcal{O}\left(\frac{1}{(p+q)^2}\right)$.

If (b,s) is an oriented coloured Brauer diagram and r is an elementary coloured diagram in $\mathsf{T}^{+}_{k,n}(b)$ or $\mathsf{W}^{+}_{k,n}(b)$, the coefficient $\rho_{\lambda}((b,s),r\diamond(b,s))$ (resp. $\rho_{p,q}(b,r\diamond b)$) is either equal to $\frac{\lambda}{1+\lambda}$ either to $\frac{1}{1+\lambda}$ (resp to $\frac{p}{p+q}$ or to $\frac{q}{p+q}$). Hence,

$$\|\rho_{\lambda}((b,s),r\diamond(b,s))-\rho_{p,q}((b,s),r\diamond(b,s))\|=\mathcal{O}\left(\frac{p}{q}-\lambda\right)\text{ and }N^{\mathbb{K},+}_{p,q}-N^{\mathbb{K},+}_{\lambda}=\mathcal{O}\left(\frac{p}{q}-\lambda\right)$$

The following proposition summarize the computations we made so far.

Proposition 4.12.

$$\mathfrak{m}^{\mathbb{K}}_{p,q}-\mathfrak{m}^{\mathbb{K}}_{\lambda}=\mathcal{O}\left(\left\|\frac{p}{q}-\lambda\right\|\vee\frac{1}{p+q}\right).$$

Theorem 4.13. *Let $\lambda \in]0,1]$ a real number. Assume that $(\frac{p}{q}-\lambda)(p+q) \to \gamma$ as $p,q \to +\infty$ with $\gamma \in \mathbb{R}_+$. If b and b' are two oriented Brauer diagrams, define:*

$$\rho^{(1)}_{\lambda}(b,b') = (\mathsf{fnc}_+(\mathring{b}) - \mathsf{fnc}_+(\mathring{b}'))\left(\frac{\lambda}{(1+\lambda)^2}\right)^{\mathsf{fnc}_+(\mathring{b})-\mathsf{fnc}_+(\mathring{b}')-1}\left(\frac{1}{1+\lambda}\right)^{\mathsf{fnc}_-(\mathring{b})-\mathsf{fnc}_-(\mathring{b}')}$$
$$+ (\mathsf{fnc}_-(\mathring{b}) - \mathsf{fnc}_-(\mathring{b}'))\left(\frac{-1}{(1+\lambda)^2}\right)^{\mathsf{fnc}_-(\mathring{b})-\mathsf{fnc}_-(\mathring{b}')-1}\left(\frac{\lambda}{1+\lambda}\right)^{\mathsf{fnc}_+(\mathring{b})-\mathsf{fnc}_+(\mathring{b}')}. \tag{4.21}$$

If $\mathbb{K}=\mathbb{R}$ or $\mathbb{H}$, the function $\mathbb{m}^{(1),\mathbb{K}}_{p,q}$ converges to the solution $\mathbb{m}^{(1),\mathbb{K}}_{\lambda}$ as $p,q\to\infty$ of the following differential system:

$$\frac{d}{dt}\mathbb{m}^{(1),\mathbb{K}}_{\lambda}(t)(b) = \mathbb{m}^{(1),\mathbb{K}}_{\lambda}(t)\Big(c\sum_{c\in C(\mathbf{1}_k)}\alpha^{\mathbb{K}}_{\lambda}(c)p_c\diamond b + v\sum_{c\in C(\mathbf{1}_k)}\beta^{\mathbb{K}}_{\lambda}(c)p_c\diamond b + N^{\mathbb{K},+}_{\lambda}(b)\Big)$$
$$+\mathbb{m}_{\lambda}(t)(b)\Big(N^{(1),\mathbb{K},+}_{\lambda}(b) + c\sum_{c\in C(\mathbf{1}_k)}\alpha^{(1),\mathbb{K}}_{\lambda}(c)p_c\diamond b + v\sum_{c\in C(\mathbf{1}_k)}\beta^{(1),\mathbb{K}}_{\lambda,\gamma}p_c\diamond b + N^{0,\mathbb{K}}_{\lambda}(b)\Big). \tag{4.22}$$

with initial condition $\mathbb{m}^{(1),\mathbb{K}}_{\lambda}(0)=0$ and where $N^{(1),\mathbb{K},+}_{\lambda}$ is an operator acting on the $\mathcal{OB}_k(p,q)$ and $\alpha^{(1),\mathbb{K}}_{\lambda}, \beta^{(1),\mathbb{K}}_{\lambda,\gamma}$ are two sets of real coefficients.

Proof. We begin the proof by giving an asymptotic expansion of the coefficients $\alpha^{\mathbb{K}}_{p,q}$ and $\beta^{\mathbb{K}}_{p,q}$,

$$\begin{aligned}\beta^{\mathbb{K}}_{p,q}(c)-\beta^{\mathbb{K}}_{\lambda}(c) &= \frac{1}{2}\left(\sharp(+,c)\left(\frac{q}{p+q}-\frac{1}{1+\lambda}\right)+\sharp(-,c)\left(\frac{p}{p+q}-\frac{\lambda}{1+\lambda}\right)\right)\\ &= -\frac{1}{2}\left(\sharp(+,c)\frac{1}{(1+\lambda)^2}+\sharp(-,c)\frac{\lambda}{(1+\lambda)^2}\right)\left(\frac{p}{q}-\lambda\right)+o\left(\frac{p}{q}-\lambda\right)\\ &= \beta^{(1),\mathbb{K}}_{\lambda,\gamma}(c)\frac{1}{p+q}+o\left(\frac{1}{p+q}\right),\end{aligned} \tag{4.23}$$

$$\begin{aligned}\alpha^{\mathbb{K}}_{p,q}+\frac{k}{2} &= \left(\frac{\sharp(+,c)}{2}\left(1+\frac{1}{\lambda}\right)+\frac{\sharp(-,c)}{2}(1+\lambda)(c)\right)\frac{\alpha^{\mathbb{K}}_{(1)}}{p+q}+o\left(\frac{1}{p+q}\right)\\ &= \alpha^{(1),\mathbb{K}}_{\lambda}(c)\frac{1}{p+q}+o\left(\frac{1}{p+q}\right)\end{aligned} \tag{4.24}$$

We give next an asymptotic expansion of the difference $N^{\mathbb{K},+}_{p,q}-N^{\mathbb{K},+}_{\lambda}$ in powers of $(\frac{p}{q}-\lambda)$. To that end, it would not be surprising for the attentive reader, especially in regard of the last two equations, that the function $\rho^{(1),\mathbb{K}}_{\lambda}$ is, at least, helpful to write a formula for the limit of the operator $(\frac{p}{q}-\lambda)^{-1}N^{\mathbb{K},+}_{p,q}-N^{\mathbb{K},+}_{\lambda}$. The function $\rho^{(1)}_{\lambda}$ is nothing more but the derivative of the function $x\mapsto\rho_x(b,b')$ at λ. The computations are not difficult and are left to the reader.

$$\begin{aligned}(p+q)\Big(N^{\mathbb{K},+}_{p,q}-N^{\mathbb{K},+}_{\lambda}\Big)(b) &= \frac{c\gamma}{\lambda^2}\sum_{r\in\mathsf{R}^{=,+}_{k,2}(b,+)}(-1)^r\rho_{\lambda}(\mathring{r}\circ\mathring{b},\mathring{b})r\circ b + c\gamma\sum_{r\in\mathsf{R}^{=,+}_{k,2}(b,-)}(-1)^r\rho_{\lambda}(\mathring{r}\circ\mathring{b},\mathring{b})r\circ b\\ &+c\gamma(1+\frac{1}{\lambda})\sum_{r\in\mathsf{R}^{=,+}_{k,2}(b,+)}(-1)^r\rho^{(1)}_{\lambda}(\mathring{r}\circ\mathring{b},\mathring{b})r\circ b\\ &+c\gamma(1+\lambda)\sum_{r\in\mathsf{R}^{=,+}_{k,2}(b,+)}(-1)^r\rho^{(1)}_{\lambda}(\mathring{r}\circ\mathring{b},\mathring{b})r\circ b\\ &+2v\gamma\sum_{r\in\mathsf{R}^{\neq,+}_{k,2}(b)}(-1)^r\rho^{(1)}_{\lambda}(\mathring{r}\circ\mathring{b},\mathring{b})r\circ b+o(1) = N^{\mathbb{K},+,(1)}_{\lambda}(b,s)+o(1).\end{aligned} \tag{4.25}$$

We compute the derivative of $\mathfrak{m}^{(1),\mathbb{K}}_{p,q} = (p+q)\left(\mathfrak{m}^{\mathbb{K}}_{p,q} - \mathfrak{m}_\lambda\right)$.

$$\begin{aligned}\frac{d}{dt}\mathfrak{m}^{(1),\mathbb{K}}_{p,q}(t)(b) &= \mathfrak{m}^{(1),\mathbb{K}}_{p,q}(t)\left(c\sum_{c\in C(\mathbf{1}_k)}\alpha^{\mathbb{K}}_\lambda(c)p_c\diamond b + v\sum_{c\in C(\mathbf{1}_k)}\beta^{\mathbb{K}}_\lambda(c)p_c\diamond b\right)\\ &+\mathfrak{m}^{\mathbb{K}}_{p,q}(t)\left(c\sum_c(p+q)(\alpha^{\mathbb{K}}_{p,q}(c)-\alpha^{\mathbb{K}}_\lambda(c))\,p_c\diamond b + v\sum_c(p+q)(\beta^{\mathbb{K}}_{p,q}(c)-\beta^{\mathbb{K}}_\lambda(c))\,p_c\diamond b\right)\\ &+\mathfrak{m}^{(1),\mathbb{K}}_{p,q}(t)(N^{\mathbb{K},+}_\lambda(b)) + \mathfrak{m}^{(1),\mathbb{K}}_{p,q}(t)((p+q)(N^{\mathbb{K},+}_{p,q}-N^{\mathbb{K},+}_\lambda)(b)) + \mathfrak{m}^{\mathbb{K}}_{p,q}(t)(N^{\mathbb{K},0}_{p,q}(b)).\end{aligned}$$

We let p,q tend to infinity in the last equation. Owing to equation (4.23) and (4.25), we get that $\mathfrak{m}^{(1),\mathbb{K}}_{p,q}$ converges to the solution $\mathfrak{m}^{(1),\mathbb{K}}_\lambda$ of the differential system (4.26). □

The last theorem deals with the macroscopic fluctuations for the real and quaternionic case. For the complex case, we have $N^{\mathbb{K},0}_{p,q} = 0$ for all $p,q\geq 1$. Also, $\alpha^{(1),\mathbb{K}}_\lambda = 0$. The computations we made for the real and quaternionic cases are repeated verbatim to prove the next theorem.

Theorem 4.14. *Under the same hypothesis of Theorem 4.13 and with the definitions and notations of the same Theorem, as p,q tend to infinity, $\mathfrak{m}^{(1),\mathbb{C}}_{p,q} = \mathfrak{m}^{\mathbb{C}}_{p,q}(t) - \overline{\mathfrak{m}}_\lambda(t)$ converges to the solution $\overline{\mathfrak{m}}^{(1)}_\lambda$ of the following differential system:*

$$\begin{aligned}\frac{d}{dt}\mathfrak{m}^{(1),\mathbb{C}}_\lambda(t)(b\otimes w) &= \mathfrak{m}^{(1),\mathbb{C}}_\lambda(t)\left(c\sum_{c\in C(\mathbf{1}_k)}\alpha^{\mathbb{C}}_\lambda(c)p_c\diamond b\otimes w + v\sum_{c\in C(\mathbf{1}_k)}\beta^{\mathbb{C}}_\lambda(c)p_c\diamond b\otimes w + N^{\mathbb{C},+}_\lambda(b\otimes w)\right)\\ &+\overline{\mathfrak{m}}_\lambda(t)\left(N^{(1),\mathbb{K},+}_\lambda(b) + v\sum_{c\in C(\mathbf{1}_k)}\beta^{(1),\mathbb{C}}_{\lambda,\gamma}p_c\diamond b\right), \qquad (4.26)\end{aligned}$$

$$\mathfrak{m}^{(1),\mathbb{C}}_\lambda(0)(b\otimes w) = 0.$$

where $b,w\in\mathcal{OB}_k\times\overline{\mathsf{M}}_1(k)$, $N^{(1),\mathbb{K},+}_\lambda$ is an operator acting on $\mathbb{R}\left[\overline{\mathsf{M}}_1(k)\right]\otimes\mathcal{OB}_k(p,q)$ equal to zero if $\gamma = 0$ and $\beta^{(1),\mathbb{C}}_{\lambda,\gamma}$ is defined by equation (4.23).

Bibliography

[1] Jiří Adámek, Horst Herrlich, and George E Strecker. Abstract and concrete categories. the joy of cats. 2004.

[2] Anton Alekseev, Harald Grosse, and Volker Schomerus. Combinatorial quantization of the hamiltonian chern-simons theory I. *Communications in Mathematical Physics*, 172(2):317–358, 1995.

[3] Greg W Anderson, Alice Guionnet, and Ofer Zeitouni. *An introduction to random matrices*, volume 118. Cambridge university press, 2010.

[4] Emil Artin. Theory of braids. *Ann. of Math*, 48(2):101–126, 1947.

[5] Helmer Aslaksen, Eng-Chye Tan, and Chen-bo Zhu. Invariant theory of special orthogonal groups. *Pacific Journal of Mathematics*, 168(2):207–215, 1995.

[6] Hélene Barcelo and Arun Ram. Combinatorial representation theory. *New perspectives in algebraic combinatorics (Berkeley, CA, 1996–97)*, 38:23–90, 1999.

[7] Carl M. Bender. Introduction to PT-symmetric quantum theory. 2005.

[8] Philippe Biane. Free brownian motion, free stochastic calculus and random matrices. free probability theory (waterloo, on, 1995), 1–19. *Fields Inst. Commun*, 12, 1997.

[9] Philippe Biane. Free probability for probabilists. In *Quantum probability communications, Vol. XI (Grenoble, 1998)*, QP-PQ, XI, pages 55–71. World Sci. Publ., River Edge, NJ, 2003.

[10] Philippe Biane and Roland Speicher. Stochastic calculus with respect to free brownian motion and analysis on wigner space. *Probability theory and related fields*, 112(3):373–409, 1998.

[11] Charles Bordenave, Djalil Chafaï, et al. Around the circular law. *Probability surveys*, 9, 2012.

[12] Eric Buffenoir and Philippe Roche. Two dimensional lattice gauge theory based on a quantum group. *Communications in mathematical physics*, 170(3):669–698, 1995.

[13] Guillaume Cébron, Antoine Dahlqvist, and Franck Gabriel. The generalized master fields. *Journal of Geometry and Physics*, 119:34–53, 2017.

[14] Guillaume Cébron and Todd Kemp. Fluctuations of brownian motions on GL_N. *arXiv preprint arXiv:1409.5624*, 2014.

[15] Guillaume Cébron and Michaël Ulrich. Haar states and lévy processes on the unitary dual group. *Journal of Functional Analysis*, 270(7):2769–2811, 2016.

[16] Benoit Collins, James A. Mingo, Piotr Sniady, and Roland Speicher. Second order freeness and fluctuations of random matrices, III. Higher order freeness and free cumulants. 2006.

[17] Charles Curry, Kurusch Ebrahimi-Fard, Dominique Manchon, and Hans Z. Munthe-Kaas. Planarly branched rough paths and rough differential equations on homogeneous spaces, 2018.

[18] Antoine Dahlqvist. Free energies and fluctuations for the unitary brownian motion. *Communications in Mathematical Physics*, 348(2):395–444, 2016.

[19] Antoine Dahlqvist and James Norris. Yang-mills measure and the master field on the sphere. *arXiv preprint arXiv:1703.10578*, 2017.

[20] Clément de Seguins Pazzis. *Invitation aux formes quadratiques*. Calvage & Mounet, 2011.

[21] Nizar Demni, Mathieu Guay-Paquet, and Alexandru Nica. Star-cumulants of free unitary brownian motion. *Advances in Applied Mathematics*, 69:1–45, 2015.

[22] József Dénes. The representation of a permutation as the product of a minimal number of transpositions and its connection with the theory of graphs. *Publ. Math. Inst. Hungar. Acad. Sci*, 4:63–70, 1959.

[23] Aurélien Deya and René Schott. On the rough-paths approach to non-commutative stochastic calculus. *Journal of Functional Analysis*, 265(4):594–628, 2013.

[24] Aurélien Deya and René Schott. On stochastic calculus with respect to q-brownian motion. *Journal of Functional Analysis*, 274(4):1047–1075, 2018.

[25] Bruce K Driver, Brian C Hall, and Todd Kemp. The brown measure of the free multiplicative brownian motion. *arXiv preprint arXiv:1903.11015*, 2019.

[26] Kurusch Ebrahimi-Fard and Frédéric Patras. The combinatorics of green's functions in planar field theories. 2015.

[27] Kurusch Ebrahimi-Fard and Frédéric Patras. Cumulants, free cumulants and half-shuffles. *Proc. A.*, 471(2176):20140843, 18, 2015.

[28] Jacques Franchi and Yves Le Jan. *Hyperbolic dynamics and Brownian motion: an introduction*. Oxford University Press, 2012.

[29] Uwe Franz. The theory of quantum lévy processes. *arXiv preprint math/0407488*, 2004.

[30] Franck Gabriel. Combinatorial theory of permutation-invariant random matrices i: Partitions, geometry and renormalization. *arXiv preprint arXiv:1503.02792*, 2015.

[31] Malte Gerhold, Stephanie Lachs, and Michael Schürmann. Categorial lévy processes. *arXiv preprint arXiv:1612.05139*, 2016.

[32] Nicolas Gilliers. Non-commutative gauge theories and zhang algebras. *arXiv preprint*, 2019.

[33] M Hakim-Dowek and Dominique Lépingle. L'exponentielle stochastique des groupes de lie. In *Séminaire de Probabilités XX 1984/85*, pages 352–374. Springer, 1986.

[34] Brian Hall and Todd Kemp. Brown measure support and the free multiplicative brownian motion, 2018.

[35] Todd Kemp. The large-n limits of brownian motions on the group of complex invertible matrices. *International Mathematics Research Notices*, 2016(13):4012–4057, 2015.

[36] Todd Kemp, Ivan Nourdin, Giovanni Peccati, Roland Speicher, et al. Wigner chaos and the fourth moment. *The Annals of Probability*, 40(4):1577–1635, 2012.

[37] Anthony W Knapp. *Lie groups beyond an introduction*, volume 140. Springer Science & Business Media, 2013.

[38] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry*, volume 1. New York, 1963.

[39] Thierry Lévy. *Yang-Mills measure on compact surfaces*. American Mathematical Soc., 2003.

[40] Thierry Lévy. Two-dimensional markovian holonomy fields. *arXiv preprint arXiv:0804.2230*, 2008.

[41] Thierry Lévy. The master field on the plane. *arXiv preprint arXiv:1112.2452*, 2011.

[42] James A. Mingo, Piotr Śniady, and Roland Speicher. Second order freeness and fluctuations of random matrices: II. unitary random matrices. 2004.

[43] James A. Mingo and Roland Speicher. Second order freeness and fluctuations of random matrices: I. Gaussian and wishart matrices and cyclic fock spaces. 2004.

[44] Alexandru Nica and Roland Speicher. *Lectures on the combinatorics of free probability*, volume 13. Cambridge University Press, 2006.

[45] Onorato Timothy O'Meara. *Introduction to quadratic forms*, volume 117. Springer, 2013.

[46] Michael Schürmann. *White noise on bialgebras*. Springer, 2006.

[47] Roland Speicher. *Combinatorial theory of the free product with amalgamation and operator-valued free probability theory*, volume 627. American Mathematical Soc., 1998.

[48] Zirô Takeda. Inductive limit and infinite direct product of operator algebras. *Tohoku Mathematical Journal, Second Series*, 7(1-2):67–86, 1955.

[49] Michaël Ulrich. Construction of a free lévy process as high-dimensional limit of a brownian motion on the unitary group. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, 18(03):1550018, 2015.

[50] Dan Voiculescu. Symmetries of some reduced free product C*-algebras. In *Operator algebras and their connections with topology and ergodic theory*, pages 556–588. Springer, 1985.

[51] Dan Voiculescu. Limit laws for random matrices and free products. *Inventiones mathematicae*, 104(1):201–220, 1991.

[52] James J Zhang. H-algebras. *Advances in Mathematics*, 89(2):144–191, 1991.