Shuffle algebra perspective on operator valued probability theory

30 mars 2020

Operator valued probability theory

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Definition (Distribution of random variables)

Let $a_1, \ldots, a_n \in \mathcal{A}$. The *distribution of* $a_1, \ldots, a_n$ is the collection of elements in B defined by :

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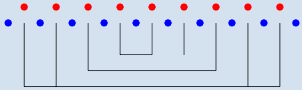
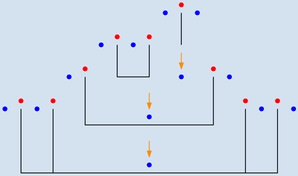
Definition (Free Multiplicative extension on NC.)





$$E_\pi(b_1, \ldots, b_{10}) = E(b_1 a b_2 a E(b_3 a E(b_4 a b_5 a E(b_6 a b_7)) a b_8) a b_9 a b_{10})$$

Operator valued probability theory

Definition (Boolean multiplicative extension)

Let IP be the poset of interval partitions, and write $I = I_1 \cdots I_p$ for $I = \{I_1, \ldots, I_p\} \in \mathrm{IP}$.

$$E_I(b_1, \ldots, b_{|I|}) = \prod_{i \in 1, \ldots, p} E(b_{\cdots + I_{j-1}+1} \cdots b_{\cdots + I_j})$$

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Definition (Boolean and Free cumulants)

$$E(b_1, \ldots, b_{n+1}) = \sum_{\pi \in \mathsf{NC}(n)} \kappa_\pi(b_1, \ldots, b_{n+1}) = \sum_{\beta \in \mathrm{IP}(n)} \beta_\pi(b_1, \ldots, b_{n+1}).$$

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Free and Boolean cumulants linearize Free and Boolean operator valued independance.

Shuffle approach to scalar probability theory

Double bar construction

$$H = \bar{T}(T(\mathcal{A})).$$

$$\emptyset, \quad a_1 \cdots a_n, \quad a^1_1 \cdots a^1_{n_1} \,\Big|\, a^2_1 \cdots a^2_{m_1}$$

$$\Delta^{\sqcup\!\sqcup}(\cdot) = \emptyset \otimes \cdot + \cdot \otimes \emptyset + \bar{\Delta}(\cdot) = \emptyset \otimes \cdot + \cdot \otimes \emptyset + \Delta^{\prec}(\cdot) + \Delta^{\succ}(\cdot).$$

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$\mathrm{Hom}_{Vect_C}(H, \sqcup\!\sqcup)$ is a monoid and $G = \mathrm{Hom}_{Alg}(H, \sqcup\!\sqcup)$ is a group.

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$$\exp_{\prec}(k) = 1_\star + \sum_{n \geq 1} k^{\prec n}, \quad \exp_{\succ}(k) = 1_\star + \sum_{n \geq 1} k^{\succ n}$$

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$$\exp_{\prec}(k)^{-1} = \exp_{\succ}(-k).$$

Shuffle and non-commutative probability theory

A $\star$-algebra $\mathcal{A}$ and an expectation $E : \mathcal{A} \to \mathbb{C}$.

$$
\begin{aligned}
&\mathsf{M} \in G, && \mathsf{M}(a_1 \otimes \cdots \otimes a_n) = E(a_1 \cdot_{\mathcal{A}} \cdots \cdot_{\mathcal{A}} a_n) \\
&\mathsf{k} \in \mathrm{Lie}(G), && \mathsf{k}(a_1 \otimes \cdots \otimes a_n) = \kappa(a_1, \ldots, a_n) \\
&\mathsf{b} \in \mathrm{Lie}(G), && \mathsf{b}(a_1, \ldots, a_n) = \beta(a_1 \otimes \cdots \otimes a_n)
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$$\mathsf{M} = \varepsilon + \mathsf{k} \prec \mathsf{M}, \quad \mathsf{M} = \varepsilon + \mathsf{M} \succ \mathsf{b}$$

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Ebrahimi-Fard, K., Patras, F. Cumulants, free cumulants and half-shuffles.

Ebrahimi-Fard, K., Patras, F. Monotone, free, and boolean cumulants : a shuffle algebra approach.

Relation between Möbius inversion and Shuffles

Ebrahimi-Fard, K., Foissy, L., Kock, J., Patras, F. Operads of (noncrossing) partitions, interacting bialgebras, and moment-cumulant relations.

Shuffle Approach $\Longrightarrow$ Gap insertion operad of non-crossing partitions

Operad $\mathcal{NC} \longrightarrow$ incidence bi-algebra (N, Δ) on words on non-crossing partitions :

$$\Delta(\pi) = \sum_{\pi = q \circ (p_1, \ldots, p_n)} q \otimes (p_1 \otimes \ldots \otimes p_n) = \Delta_{\prec}^{+}(\pi) + \Delta_{\succ}^{+}(\pi).$$

$$\begin{array}{ccl} f = (E(a^n))_{n \geq 1} & \longrightarrow & F : \mathrm{NC} \to \mathbb{C},\ \text{multiplicative} \\ & \searrow & \\ & & \mathrm{F} : N \to \mathbb{C},\ \mathrm{F} = \varepsilon_{\mathrm{N}} + f \prec \mathrm{F}. \end{array}$$

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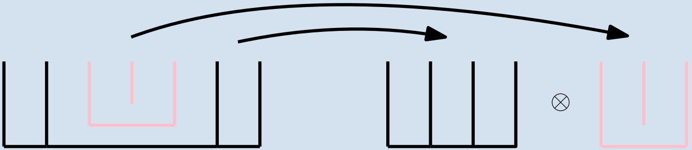
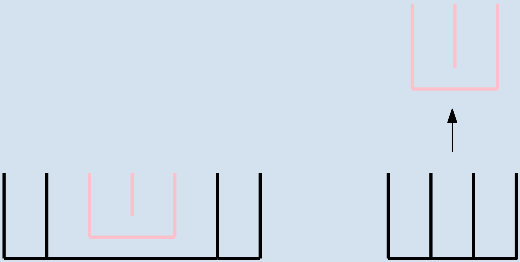
Möbius inversion $\Longrightarrow$ Block substitution operad

Shuffle operadic approach to operator valued cumulants and moments

☒ Express multiplicativity of $\{E_\pi,\ \pi \in \mathrm{NC}\}$.
Define a decomposition map Δ that presevers linear order of the "legs" of a non-crossing partition.

⊗ Give a Lie theoretic perspective, with a group of morphisms and a Lie algebra of infinitesimal morphisms and a fixed point equation for $\{E_\pi,\ \pi \in \mathrm{NC}\}$.





Duoidal Category of bigraded collections

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$$(C \otimes D)_{n,m} = \bigoplus_{\substack{n_c + n_d = n \\ m_c + m_d = m}} C_{n_c, m_c} \otimes D_{n_d, m_d},\ \ (C \boxtimes D)_{n,m} = \bigoplus_k C_{n,k} \otimes D_{k,m}$$

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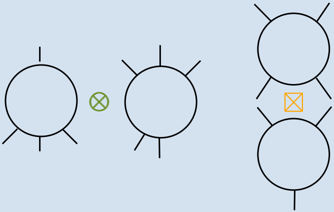

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$$(\mathbb{C}_{\boxtimes})_{n,m} = \delta_{n=m}\mathbb{C}, \quad (\mathbb{C}_{\otimes})_{n,m} = \delta_{n=m=0}\mathbb{C}$$

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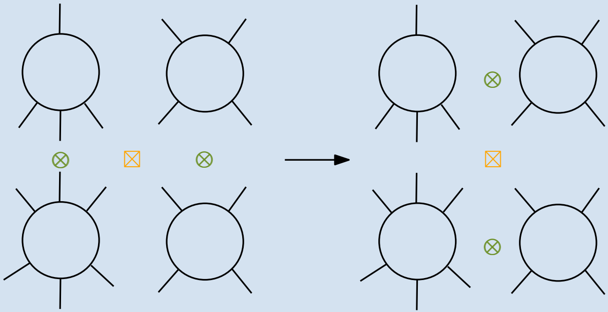
Vallette, B. A Koszul duality for props. Transactions of the American Mathematical Society.

Lax property



Lax property

Consequences :

The category $\mathrm{Alg}_{\otimes}$ of horizontal algebras endowed with $\boxtimes$ is monoidal with unit $\mathbb{C}_{\boxtimes}$.
The category $\mathrm{CoAlg}_{\boxtimes}$ of vertical co-algebras endowed with $\otimes$ is monoidal with unit $\mathbb{C}_{\otimes}$.

$$(A, m_{\otimes}^{A}),\ (B, m_{\otimes}^{B}),\quad m_{\otimes}^{A\boxtimes B} = (m_{\otimes}^{A} \boxtimes m_{\otimes}^{B}) \circ R$$

$$(A, \Delta^{\boxtimes}),\ (B, \Delta^{\boxtimes}),\quad \Delta_{A\otimes B}^{\boxtimes} = R \circ (\Delta_{A}^{\boxtimes} \otimes \Delta_{B}^{\boxtimes})$$

$\boxtimes\otimes$ - bialgebras

Proposition

A bi-graded collection C is a ($\boxtimes$-co|$\otimes$-al)gebra if and only if it is a ($\otimes$-al|$\boxtimes$-co)gebra.

$$\Delta^{\boxtimes} : C \to C \boxtimes C, \ m^{\otimes} : C \otimes C \to C, \ \varepsilon : C \to \mathbb{C}_{\boxtimes}.$$

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$$\begin{tikzcd}
C \otimes C \arrow[r, "\Delta\otimes\Delta"] \arrow[dd, "m^{\otimes}"] & (C\boxtimes C)\otimes(C\boxtimes C) \arrow[d, "R"] \\
& (C\otimes C)\boxtimes(C\otimes C) \arrow[d, "m^{\otimes}\boxtimes m^{\otimes}"] \\
C \arrow[r, "\Delta"] & C\boxtimes C
\end{tikzcd}
\qquad
\begin{tikzcd}
C\boxtimes C \arrow[r, "\varepsilon\boxtimes\mathrm{id}"] & \mathbb{C}_{\boxtimes}\boxtimes C \arrow[d] \\
C \arrow[u, "\Delta^{\boxtimes}"] \arrow[r, "id"] & C
\end{tikzcd}$$

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$$\nabla^{\boxtimes} : C \boxtimes C \to C, \; S : C \to C, \; \eta : \mathbb{C}_{\boxtimes} \to C$$

$$\nabla^{\boxtimes} \circ (S \boxtimes \mathrm{id}_C) \circ \Delta^{\boxtimes} = \varepsilon \circ \eta, \quad \nabla^{\boxtimes} \circ (\mathrm{id}_C \boxtimes S) \circ \Delta^{\boxtimes} = \varepsilon \circ \eta$$

An unshuffle ($\boxtimes$-co)($\otimes$-al)gebra

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A ($\boxtimes$-co)($\otimes$-al)gebra $(\bar{\mathcal{C}} = \mathcal{C} \oplus \mathbb{C}_{\boxtimes}, \Delta^{\boxtimes}, m_{\otimes}, \nabla^{\boxtimes})$

$$\Delta(c) = \bar{\Delta}(c) + c \boxtimes 1_m + 1_n \boxtimes c, \quad \bar{\Delta} = \Delta^{\boxtimes}_{\prec} + \Delta^{\boxtimes}_{\succ},$$

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$$(\Delta^{\boxtimes}_{\prec,\succ} \circ \rho)(p \otimes q) = m_{\otimes}^{C \boxtimes C} \circ (\Delta^{\boxtimes}_{\prec,\succ} \otimes \Delta)(p \otimes q), \quad p \notin \mathbb{C}_{\boxtimes}, \ q \in C.$$

Monoid of Horizontal morphisms

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$$\alpha, \beta \in \mathrm{Hom}_{\mathrm{Alg}_{\otimes}}(C, T_{\otimes}(\mathrm{Hom}(B))).$$

$$\alpha \star \beta = T_{\otimes}(\nabla^{\boxtimes}_{\mathrm{Hom(B)}}) \circ (\alpha \boxtimes \beta) \circ \Delta^{\boxtimes} \in \mathrm{Hom}_{\mathrm{Alg}_{\otimes}}$$

⚠

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⚠ If C is Hopf, $\alpha \in \mathrm{Hom}_{\mathrm{Alg}_{\boxtimes}} \subset \mathrm{Hom}_{\mathrm{Alg}_{\otimes}}$ then $\alpha^{-1} = \alpha \circ S$.
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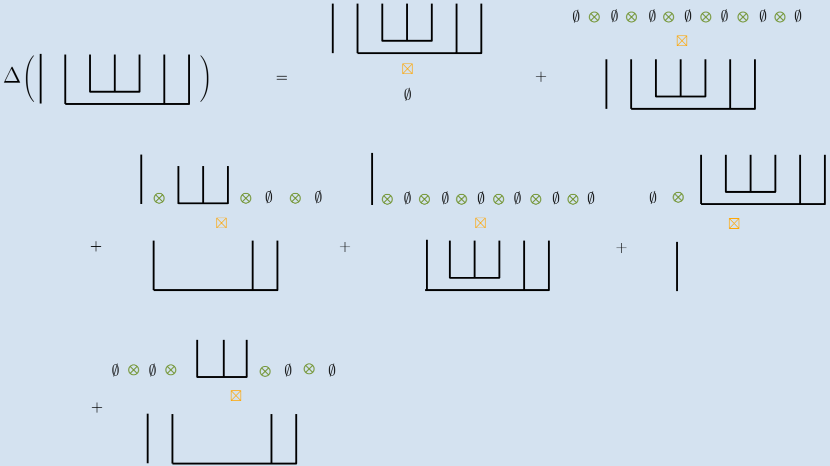
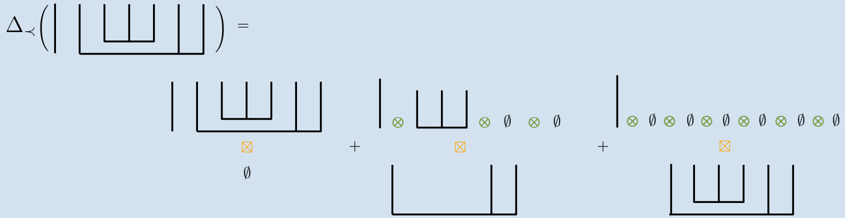
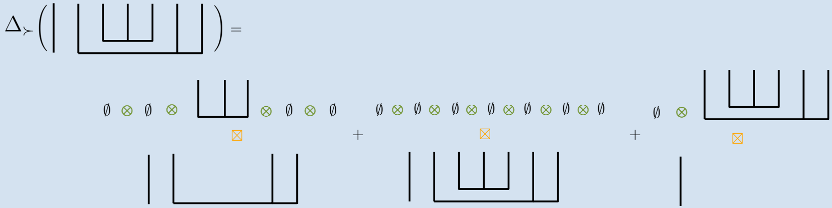
⚠ $S^2 \neq \mathrm{id}_C$, $(\alpha^{-1})^{-1} \neq \alpha^{-1} \circ S$.

Example



Unshuffling the gap insertion operad





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An antipode $S : T_{\otimes}(NC) \to T_{\otimes}(NC)$.

$$S(\pi) = (-1)^{nb(\pi)} \delta_{\pi \in PI}\ \pi.$$

$$\eta(1_m) = \emptyset^m.$$

Half shuffle exponentials

Infinitesimal character

$$\mathsf{k} : T_{\otimes}(NC) \to T_{\otimes}(\mathrm{Hom}(B)),$$
$$\mathsf{k}(\emptyset^p \pi \emptyset^q) = \emptyset^p \mathsf{k}(\pi) \emptyset^q.$$

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Proposition (Free morphisms)

$$\mathsf{K} = \eta \circ \varepsilon + k \prec \mathsf{K}.$$

$$\mathsf{K}(\alpha \circ (\beta_1, \ldots, \beta_{|\alpha|})) = K(\alpha) \circ (\mathsf{K}(\beta_1), \ldots, \mathsf{K}(\beta_{|\alpha|}))$$
$$\Leftrightarrow$$
$$k(\pi) = \delta_{\sharp\pi=1} k(\pi),\ \mathsf{k}(\mathbf{1}_n) \circ_{n+1} \mathsf{k}(\mathbf{1}_m) = \mathsf{k}(\mathbf{1}_m) \circ_1 \mathsf{k}(\mathbf{1}_n)$$

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Proposition (Boolean morphisms)

$$\mathsf{B} = \eta \circ \varepsilon + \mathsf{B} \succ k$$
$$B(\pi) = 0,\ \pi \notin IP.$$
$$B(I \circ (I_1, \ldots, I_p)) = B(I) \circ B(I_1), \ldots, B(I_p).$$
$$\Leftrightarrow$$
$$k(\pi) = \delta_{\sharp\pi=1} k(\pi),\ \mathsf{k}(\mathbf{1}_n) \circ_{n+1} \mathsf{k}(\mathbf{1}_m) = \mathsf{k}(\mathbf{1}_m) \circ_1 \mathsf{k}(\mathbf{1}_n)$$

Takk skal du ha !