Rough differential equations in $C^\star$ algebras

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Bergen meeting, 30.10.2020

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The problem

The class $\mathcal{C}$ of differential equations

Let $(\mathcal{A}, \cdot, \|\cdot\|)$ be a $C^\star$ algebra. Consider the following class of differential equations :

$$\mathrm{d}Y_t = a(Y_t) \cdot \mathrm{d}X_t \cdot b(Y_t), \;\; t \in [0,1].$$

- $X : [0,1] \to \mathcal{A}$ is a continuous path with Hölder regularity $0 < \alpha < 1$,

$$\|X_t - X_s\| \prec |t-s|^\alpha, \;\; t,s \in [0,1]$$

- two smooth functions a and b.

▷ The problem is well-posed if X is a smooth path... or $\alpha = 1$.
▷ "Strong theory" for solving this kind of equations may have important applications in free probability theory.

Non-commutative probability theory

Examples of equations in the class $\mathcal{C}$ emerge in non-commutative free probability.

Non commutative probability space (Voiculescu '85)

A probability space is a pair $(\mathcal{A}, \phi)$ with $\mathcal{A}$ a $C^\star$ algebra (in fact a von Neumann algebra...) and ϕ a positive linear map.

For example, classical probability theory corresponds to choosing for $\mathcal{A}$ the algebra of essentially bounded random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\phi = \mathbb{E}$.

Whereas in classical probability, there is an unique way to define independence between two random variables X and Y, there are multiple ways to do in non-commutative probability.

Various notions of classical probability have counterparts in n-c. probability, such as (n-c.) stochastic processes, conditional expectations, martingales, Markov processes...

Free stochastic differential equations

Freeness is a notion of independence, that is a set of rules for computing a joint ditribution knowing the distributions of the marginales. Freeness has been introduced by Voiculescu in 85 and later Speicher in 93 sorted out the underlying combinatorics, defining free cumulants $(k_n(a))$ of a random variable a.

Free Brownian motion

Let $(\mathcal{F}_t)_{t\geq 0}$ be a growing family of vN algebras. The free Browian motion is a process $w_t : [0,1] \rightarrow \mathcal{A}$, that is a collection of self-adjoint elements and

$$k_2^{free}(w_t) = t,\ k_{n\geq 3}^{free}(w_t) = 0,\ w_t - w_s \text{ is free from } \mathcal{F}_s$$

For each $t > 0$, the law of w_t is the semi-circular law with parameter t,

$$\phi(f(w_t)) = \int_{[-2,2]} f(x) \frac{1}{4\pi t} \sqrt{4t - x^2} \mathrm{d}x$$

The Free Brownian motion is $\frac{1}{2}$ Hölder

Free stochastic calculus of Biane and Speicher

▷ Biane and Speicher, 1998, Kümmerer and Speicher 1992

$$\mathrm{d}Y_t = Y_0 + \int_0^t a(Y_t) \cdot \mathrm{d}w_t \cdot b(Y_t)$$

Free stochastic integral

If A_t and B_t are two adapted processes, the Riemann sums

$$\lim_{\pi \downarrow 0} \sum_{i=1}^{n} A_{t_i}(w_{t_{i+1}} - w_{t_i}) B_t = \int_s^t A_u \mathrm{d}w_u B_u$$

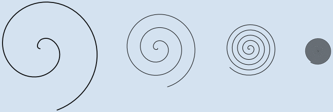
exists in L^2 norm... and in the operator norm ! ($p = \infty$ Burkholder-David-Gundy inequality)

These Riemann sums converges also for the q-deformed free Brownian motion... Donati-Martin 2003.

▷ The equation $\mathrm{d}Y_t = a(Y_t) \cdot \mathrm{d}w_t \cdot b(Y_t)$ is meaningful.

Rough-path principles

▷ Chen '77, Lyons '88, Gubinelli '2010 ...



Controlled differential equations

V a finite dimensional vector space.

$$\mathrm{d}Y_t = \sum_{i=1}^{d} V_i(Y_t)\mathrm{d}X_t^i \Leftrightarrow \int_s^t V_i(Y_u)\mathrm{d}X_u,\ Y_t \in V,\ \ t \in [0,1].$$

$$Y_t^j - Y_s^j = \sum_{k=1}^{N} \sum_{j_1,\dots,j_k} V_{j_1} \cdots V_{j_k}^j \int_{s<t_1<\dots<t_k<t} \mathrm{d}X_{t_1}^{i_1} \cdots \mathrm{d}X_{t_k}^{i_k} + R_N(s,t)$$

▷ How to define $\int_s^t Y_u \mathrm{d}X_u$ if X is irregular ? and for which integrands Y ?
▷ The space of integrands should be stable by composition with a vector field V_i

Beyond Young integration : Sewing lemma

How to define $I_t - I_s = \int_s^t f(t) \cdot \dot{g}(t) dt$?

If g is C^1 or with bounded variations, one can use *Riemann-Stieljes integration*.
If both f and g are Hölder continuous, with exponent α, respectively β with $\alpha + \beta > 1$, one can use *Young integration*.

... What if $\alpha + \beta < 1$?

$$
\begin{aligned}
I_t - I_s &= f(s)(g(t) - g(s)) + \int_s^t (f(t) - f(s)) \dot{g}(s) \mathrm{d}s \qquad (1) \\
&= f(s)(g(t) - g(s)) - R_{st}, \;\; R_{st} = O(|t - s|^{1+})
\end{aligned}
$$

Therefore, I is the unique function such that (1) holds, $A_{st} = f(s)(g_t - g_s)$ is the germ. Besides

$$
\delta_{sut} R = R_{st} - R_{su} - R_{ut} = (f(t) - f(s))(g(t) - g(s)) = A_{st} - A_{su} - A_{ut} = \delta_{sut} A
$$

and $\delta_{sut} R \prec |t - s|^2$.

Beyond Young Integration : Sewing Lemma

Sewing Lemma

If $A_{st} \prec |t-s|^{\alpha}$ is *quasi-additive* $(\delta_{sut} A_{st} \prec |t-s|^{1+\epsilon})$ then

$$\lim_{\pi \downarrow 0} \sum_{[s,t] \in \pi} A_{st} = I_t - I_s \text{ for some } I, \;\; I_t - I_s = A_{st} + R_{st}$$

Notice also that if

$$f(t) = F(g(t)) = F(g(s)) + F'(g(s))(g_t - g_s) + F''(g(s))\frac{1}{2}(g(t) - g(s))^2 + \cdots$$

We say that increments of f are *controlled* by g, then

$$\int_s^t f(g(u)\mathrm{d}g(u) = F(g(s)) \int_s^t \mathrm{d}X_u + F'(g(s)) \int_{\Delta^2(s,t)} \mathrm{d}X_{t_1} \mathrm{d}X_{t_2} + \cdots$$

$$= \sum_{k=1}^{n} F^{(k)}(g(s)) \mathrm{Sign}_{st}^{k+1}(g) + \cdots$$

(weakly) (geometric) Rough Paths

Let X be a α-Hölder path in V and set $N = \lfloor \frac{1}{\alpha} \rfloor$.

Geometric rough path (Lyons 1998)

A geometric rough path above X is

$$\mathbb{X}_{st} = (1, X_t - X_s, \mathbb{X}^2_{st}, \dots, \mathbb{X}^k_{st}, \dots) \in \hat{T}(V), \;\; \mathbb{X}^k_{st} \in V^{\otimes k}$$

reproducing algebraic / analytical properties of the iterated integrals of a bounded variations path X,

▷ (Chen relation) $\quad \mathbb{X}^N_{st} = \sum_{k=0}^N \mathbb{X}^k_{su} \otimes \mathbb{X}^{N-k}_{ut}$

$$\int_{s<t_1<t_2<t_3<t} \mathrm{d}X_{t_1} \mathrm{d}X_{t_2} \mathrm{d}X_{t_3} =$$

$$\cdots + \int_{s<t_1<t_2<u} \mathrm{d}X_{t_1} \mathrm{d}X_{t_2} \int_s^t \mathrm{d}X_{t_3} + \int_{s<t_1<u} \int_{u<t_2<t_3<t} \mathrm{d}X_{t_1} \mathrm{d}X_{t_2} \mathrm{d}X_{t_3}$$

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▷ (Shuffle relation)(geometricity)

$$\mathbb{X}^i_{st}\mathbb{X}^j_{st} = \int_{s<t_1<t_2<t} \mathrm{d}X^i_s \mathrm{d}X^j_s + \int_{s<t_1<t_2<t} \mathrm{d}X^j_s \mathrm{d}X^i_s$$

▷ $|\mathbb{X}^k_{st}| \prec |t-s|^{k\alpha}$

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reproducing algebraic / analytical properties of the iterated integrals of a bounded variations path X,

- ▷ $\mathbb{X}_{\cdot,\cdot}$ is a two parameters trajectory on a group $G \subset \hat{T}(V)$ and $\mathbb{X}_{st} = \mathbb{X}_s^{-1} \otimes \mathbb{X}_t$
- ▷ A rough path is determined by its N-jet $(1, X_t - X_s, \ldots, \mathbb{X}^{(N)}_{st})$.

Rough integral

Set $N = [\frac{1}{\alpha}]$.

Rough Integral

Let $\mathbb{X}$ be a α-rough path

$$A_{st} = \sum_{k=1}^{N-1} F^{(k)}(X_s)\mathbb{X}_{st}^{k+1} = F(X_s)(X_t - X_s) + F^{(1)}(X_s)\mathbb{X}_{st}^2 + \cdots + F^{(N-1)}(X_s)\mathbb{X}_{st}^N$$

satisfies the condition of the Sewing lemma,

$$\int_s^t F(X_\cdot)\mathrm{d}_\cdot \mathbb{X}_{\cdot,\cdot} = F^{(1)}(X_s)(X_t - X_s) + F^{(1)}(X_s)\mathbb{X}_{st}^2 + \cdots + R_{st}$$

▷ The good notion is the one of controlled rough path (Gubinelli 2010)

$$Y^{(k)}(t) = Y_s^{(k)} + Y_s^{(k+1)}(X_t - X_s) + \cdots + Y_s^N \mathbb{X}_{st}^{N-k} + R_{st}$$

Deya & Schott approach to n-c. stochastic calculus

Rough path theory works well with finite dimensional state spaces...What about replacing V with a Banach space ? In our case, with a $C^\star$ algebra ? The algebraic tensor product is not a complete normed space anymore...

Projective and spatial tensor products

The projective tensor product is the completion of $\mathcal{A} \otimes \mathcal{A}$ with respect to the norm

$$\|x\| = \min_{x=\sum_i a_i \otimes b_i} \sum_{i=1}^{n} \|a_i\|\|b_i\|$$

The spatial tensor product $\mathcal{A} \otimes_\sigma \mathcal{A}$ is the completion $\mathcal{A} \otimes \mathcal{A}$ seen as an subalgebra of $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ for the operator norm.

Let $(\mathcal{B}, \|\|)$ be a Banach algebra containing $\mathcal{A} \otimes \mathcal{A}$, and $\|a \otimes b\| = \|a\|\|b\|$ then $\mathcal{A}\hat{\otimes}\mathcal{A} \subset B$.

Free Lévy area

Free Lévy's area (C. Donati-Martin 2001)

There exists a free Lévy area in the in the spatial tensor product $\mathcal{A} \otimes_\sigma \mathcal{A}$ above the Free Brownian motion.

$$\int_S^t \mathrm{d}w_t \otimes \mathrm{d}w_t \in \mathcal{A} \otimes_\sigma \mathcal{A}$$

Deya & Schott 2016

The controlled differential equation $\mathrm{d}Y_t = a(Y_t) \cdot \mathrm{d}w_t$ always has a solution.

Deya & Schott 2016

Let $(\mathcal{A}, \mu, \|\cdot\|, \star, 1)$ be a von Neumann algebra accomodating a free Brownian process, then the multiplication map μ is not continuous for the spatial topology.

Free Lévy area

Victoir 2001

There is no Lévy area in the projective tensor product above the free Brownian motion.

We are facing a problem : The field $x \mapsto a(x) \otimes b(x)$ is not continuous for the topology containing the Free Lévy area.

Free Lévy area

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Is it the end of the story ?

What do we look for ?

We have to define a notion of rough paths, controlled paths *tailored* to the class $\mathcal{C}$...suitable to define rough integration.

$$\mathrm{d}_x a\ (Y) = \partial_x a|_{(1)} \cdot Y \cdot \partial_x a|_{(2)} = \partial_x a \sharp Y,\ \partial_x a \in \mathcal{A} \otimes \mathcal{A}^{op},\ \ Y \in \mathcal{A}.$$

$$\begin{aligned} Y_t = {} & Y_s + a(Y_s)(X_t - X_s)b(Y_s) \\ & + \int_{s<t_1<t_2<t} \Big([\partial a(Y_s) \cdot (a(Y_s) \otimes b(Y_s))] \sharp \mathrm{d}X_{t_1} \Big) \cdot \mathrm{d}X_{t_2} \cdot b(Y_s) \\ & + \int_{s<t_1<t_2<t} a(Y_s) \cdot \mathrm{d}X_{t_2} \cdot \Big(\big[\big(a(Y_s) \otimes b(Y_s) \big) \cdot \partial b(Y_s) \big] \sharp \mathrm{d}X_{t_1} \Big) \\ & + \text{reeeeaaallyyyyyy messy terms} \end{aligned}$$

with $A \otimes B \sharp X = A \cdot X \cdot B$. In infinite dimensions, the data of the map $\mathbb{X}^2 : A \mapsto \mathbb{X}^2_{st}|_{(1)} \cdot A \cdot \mathbb{X}^2_{st}|_{(2)}$ is weaker than the data of $\mathbb{X}^2_{st}$!

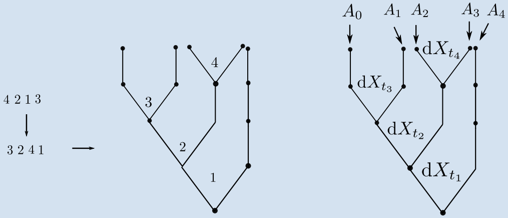
Non-commutative rough paths

The signature as operators in $\mathrm{End}_{\mathcal{A}}$

Let $A_0, \ldots, A_n \in \mathcal{A}$ and define

$$\mathbb{X}^{\sigma}_{s,t}(A_0, \ldots, A_n) = \int_s^t \int_s^{t_2} \cdots \int_s^{t_{n-1}} A_0 \cdot \mathrm{d}X_{\sigma \cdot \mathbf{t_1}} \cdot A_1 \cdots A_{n-1} \cdot \mathrm{d}X_{\sigma \cdot \mathbf{t}_n} \cdot A_n, \;\; \sigma \in \mathcal{S}_n$$

We choose to encode a permutation by a tree, see the following figure.

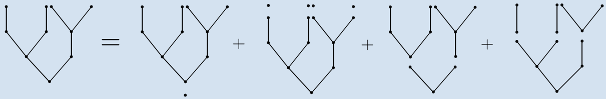


Non-commutative rough paths

How is the Chen relation written for these operators ?

$$\int_{s<t_3<t_2<t_1<t} A_0 \cdot \mathrm{d}X_{t_2} \cdot A_1 \cdot \mathrm{d}X_{t_1} \cdot A_2 \cdot \mathrm{d}X_{t_3} \cdot A_3$$
$$= \int_{\Delta^{(3)}(s,u)} A_0 \cdot \mathrm{d}X_{t_2} \cdot A_1 \cdot \mathrm{d}X_{t_1} \cdot A_2 \cdot \mathrm{d}X_{t_3} \cdot A_3 + \int_{\Delta^{(3)}(u,t)} A_0 \cdot \mathrm{d}X_{t_2} \cdot A_1 \cdot \mathrm{d}X_{t_1} \cdot A_2 \cdot \mathrm{d}X_{t_3} \cdot A_3$$
$$+ \int_{t_1\in\Delta^{(1)}(u,t)} \int_{(t_2,t_3)\in\Delta^{(2)}(u,t)} A_0 \cdot \mathrm{d}X_{t_2} \cdot A_1 \cdot \mathrm{d}X_{t_1} \cdot A_2 \cdot \mathrm{d}X_{t_3} \cdot A_3$$
$$+ \int_{(t_1,t_2)\in\Delta^{(2)}(s,u)} \int_{t_3\in\Delta^{(1)}(u,t)} A_0 \cdot \mathrm{d}X_{t_2} \cdot A_1 \cdot \mathrm{d}X_{t_1} \cdot A_2 \cdot \mathrm{d}X_{t_3} \cdot A_3$$

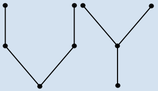
We can pictorially represents the above sum as



It appears that we need partial contraction operators !

Non-commutative rough paths

To each leveled forest we associate a partial contraction operator



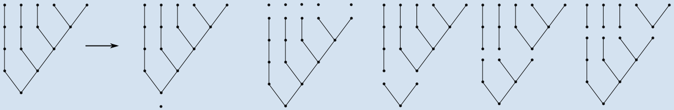
$$\longrightarrow \quad (A_0, \dots, A_3) \mapsto \int_s^t A_0 \mathrm{d}X_{t_1} A_1 \otimes A_2 \mathrm{d}X_{t_2} A_3$$

Proposition

Let $s < u < t < T$ three times. Let τ be an almost binary tree. Then,

$$\mathbb{X}^{\tau}_{s,t} = \sum_{\tau' \subset \tau} \mathbb{X}^{\tau'}_{ut} \circ \left[\mathbb{X}^{\tau \setminus \tau'}_{su} \right]$$

We find back the usual Chen relation by looking at comb trees :



Model for non-commutative rough differential equations

$$\mathsf{LPBT}(\mathcal{A}) = \bigoplus_{\tau \in \mathsf{LPBT}} \mathcal{A}^{\otimes|\tau|}$$

A model (in the sense of Hairer's regularity structure)

$$\begin{array}{rccc} \bar{\mathbb{X}}_{st} : & \bigoplus_{\tau \in \mathrm{Perm.}} \mathcal{A}^{\otimes|\tau|} & \to & \bigoplus_{\tau \in \mathrm{Perm}} \mathcal{A}^{\otimes|\tau|}, \\ & (A^{\otimes|\tau|} \cdot \tau) & \mapsto & \sum_{\tau' \subset \tau} \mathbb{X}_{st}^{\tau \setminus \tau'}(A^{\otimes|\tau|}) \cdot \tau' \end{array}$$

- $\bar{\mathbb{X}}_{st} = \bar{\mathbb{X}}_{ut} \circ \bar{\mathbb{X}}_{su}$
- $\bar{\mathbb{X}}_{st}$ is invertible and $\bar{\mathbb{X}}_{st} = \bar{\mathbb{X}}_{0t} \circ \bar{\mathbb{X}}_{0s}^{-1}$
- $\bar{\mathbb{X}}_{st} = \sum_{k=0}^{\infty} \bar{\mathbb{X}}_{st}^{(k)}$, $\|\bar{\mathbb{X}}_{st}^{k}\| \prec |t-s|^{k\alpha}$, $\mathbb{X}^{(k)}$ kills the k last generations of a tree.

Model for non-commutative rough differential equations

$$
\begin{array}{rccc}
\mathsf{L}: & \mathsf{LPBT}(\mathcal{A}) \,\hat{\circ}\, \mathsf{LPBT}(\mathcal{A}) & \to & \mathsf{LPBT}(\mathcal{A}) \\
& U \cdot \alpha \otimes X_1 \tau_1 \otimes \cdots \otimes X_{\sharp\alpha} \tau_{\sharp\alpha} & \mapsto & \displaystyle\sum_{\alpha, \tau_1, \ldots, \tau_{\sharp\alpha}} L(U^{\tau} \otimes X_1 \otimes X_{\sharp\tau}) \cdot \tau_1 ⧢ \cdots ⧢ \tau_{\sharp\alpha}
\end{array}
$$

(weak) Geometricity

For any pair of times $s < t$,

$$\mathsf{L} \circ (\mathrm{id} \,\hat{\circ}\, \mathbb{X}_{st}) = \mathbb{X}_{st} \circ \mathsf{L}.$$

$$
\begin{tikzcd}
\int_s^t A_0 \mathrm{d} X_u A_1 \arrow[d] \\
\int_{s<t_1<t_2<t} B_0 \mathrm{d} X_{t_2} B_1 \mathrm{d} X_{t_1} B_2
\end{tikzcd}
$$

... can be written as a sum of iterated integrals

Model for non-commutative rough differential equations

What is strong geometricity ?

$$\int_s^t A_0 \mathrm{d}X_u A_1$$

$$\int_{s<t_1<t_2<t} B_0 \mathrm{d}X_{t_2} B_1 \mathrm{d}X_{t_1} B_2$$

... can be written as a sum of iterated integrals, but this is meaningless in a probabilistic settings !

▷ If X is smooth, then strong geometricity holds and all the iterated integrals of X can be packed into a trajectory of a group (a kind of convolution group). However, if X is trully irregular, then it is too much to require existence of such a trajectory packing the data we need on X to build a strong solution theory !

On going works

- We can define controlled rough paths in this settings, aka generalized Taylor expansion,
- In the smooth setting, we can use higher category theory to create a bi-algebra (B, Δ) (a duoid in a 2-monoidal category) such that

$$\bar{\mathbb{X}}_{st} = (\mathrm{id} \boxtimes \mathbb{X}_{st}) \circ \Delta$$

- We build a rough integral,
- We prove existence of solutions to

$$\mathrm{d}Y_t = a(Y_t) \cdot \mathrm{d}X_t \cdot b(Y_t)$$

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