# ES_APPM 346 Project 3

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# Contents

# 1 Written Assignment

## 1.1 Problem One

Below are the variables in this system labeled as either algebraic or differential.

$$\begin{aligned} e_1 &\leftarrow algebraic \\ e_2 &\leftarrow differential \\ I_v &\leftarrow algebraic \end{aligned}$$

Note that $t$ is not included above because it is an independent variable.

## 1.2 Problem Two

In order to find the index of the differential algebraic system, one must count how many times the algebraic equations must be differentiated to get ordinary differential equations of the algebraic variables. It turns out one only needs to differentiate once meaning that the index of the differential algebraic system is one.

Recall the differential algebraic system:

$$\frac{de_2}{dt} = \frac{1}{RC}(e_1 - e_2) \tag{1}$$

$$0 = -I_v + \frac{1}{R}(e_1 - e_2) \tag{2}$$

$$0 = V(t) + e_1 \tag{3}$$

Differentiating Equations 2 and 3 just once gives the differential equations of the algebraic variables.

$$\begin{aligned} \frac{de_2}{dt} &= \frac{1}{RC}(e_1 - e_2) \\ \frac{de_1}{dt} &= -\frac{dV}{dt} \\ \frac{dI_v}{dt} &= \frac{de_1}{dt}\frac{1}{R} - \frac{de_2}{dt}\frac{1}{R} \\ &= \frac{1}{R}(-\frac{dV}{dt} - \frac{1}{RC}(e_1 - e_2)) \end{aligned}$$

## 1.3 Problem Three

Below is the exact solution for the system for when $V(t) = Asin(wt)$.

**(a)** Using Equation 3 to determine $e_1$

$$\begin{aligned} 0 &= V(t) + e_1 \\ e_1 &= -V(t) \\ &= -Asin(wt) \end{aligned}$$

**(b)** Using Equation 1 to determine $e_2$

$$\begin{aligned}\frac{de_2}{dt} &= \frac{1}{RC}(e_1 - e_2)\\ &= \frac{1}{RC}(-Asin(wt) - e2)\\ &= \frac{-A}{RC}sin(wt) - \frac{e_2}{RC}\frac{de_2}{dt} + \frac{e_2}{RC} = \frac{-A}{RC}sin(wt)\end{aligned}$$

Integrating Factor Method

$$e_2 = \rho^{-1}[\int Q(t)\rho dt + K]$$

$$\begin{aligned}\frac{de_2}{dt} + P(t) * e_2 &= Q(t)\\ P(t) &= \frac{1}{RC}\\ Q(t) &= \frac{-A}{RC}sin(wt)\\ \rho(t) &= exp(\int P(t)dt)\\ &= exp(\int \frac{1}{RC}dt)\\ &= exp(\frac{t}{RC})\\ e_2 &= e^{\frac{-t}{RC}}[\underbrace{\int Q(t)e^{\frac{t}{RC}}dt}_{\text{B}} + K]\end{aligned}$$

Integration by Parts

$$\begin{aligned}B &= \int \frac{-A}{RC}sin(wt)e^{\frac{t}{RC}}dt\\ &= \frac{-A}{RC}\int sin(wt)e^{\frac{t}{RC}}dt\end{aligned}$$

$$u = sin(wt)\ dv = e^{\frac{t}{RC}}dt$$
$$du = w * cos(wt)dt\ v = RCe^{\frac{t}{RC}}$$

$$\begin{aligned}B &= \frac{-A}{RC}\int sin(wt)e^{\frac{t}{RC}}dt\\ &= \frac{-A}{RC}[sin(wt)RCe^{\frac{t}{RC}} - RCw\underbrace{\int cos(wt)e^{\frac{t}{RC}}dt}_{\text{D}}]\end{aligned}$$

Integration by Parts: Second Time

$$D = \int cos(wt)e^{\frac{1}{RC}}dt$$

$$u = cos(wt)\ dv = e^{\frac{t}{RC}}dt$$
$$du = -w * sin(wt)dt\ v = RCe^{\frac{t}{RC}}$$

$$D = cos(wt)RCe^{\frac{t}{RC}} + RCw\int sin(wt)e^{\frac{t}{RC}}dt$$
$$\int sin(wt)e^{\frac{t}{RC}}dt = \frac{sin(wt)RCe^{\frac{t}{RC}} - (RC)^2we^{\frac{t}{RC}}cos(wt)}{1+(wRC)^2}$$

Plugging into $e_2$, one gets the final result where $K$ is a constant.

$$e_2 = -A(\frac{sin(wt) - wRCcos(wt)}{1+(wRC)^2}) + Ke^{\frac{-t}{RC}}$$

**(c)** Using Equation 2 to determine $I_v$

$$\begin{aligned} 0 &= -I_v + \frac{1}{R}(e_1 - e_2) \\ I_v &= \frac{1}{R}(e_1 - e_2) \\ &= \frac{1}{R}[-Asin(wt) + A(\frac{sin(wt) - wRCcos(wt)}{1+(wRC)^2}) - Ke^{\frac{-t}{RC}}] \end{aligned}$$

Below are the possible initial conditions at $t = 0$ when $e_2(0) = E_2$.

$$e_1(0) = -Asin(w(0)) = 0$$
$$I_v(0) = \frac{1}{R}(e_1(0) - e_2(0)) = \frac{1}{R}(0 - E_2) = \frac{-E_2}{R}$$

If the initial conditions are $e_1(0) = e_2(0) = I_v(0) = 0$, then below is the unique solution for $e_1,\ e_2$, and $I_v$.

Note that $e_1$ is already a unique solution that satisfies the initial condition.

$$e_1 = -Asin(wt)$$

To find a unique solution for $e_2$, one must solve for $K$.

$$\begin{aligned}
e_2 &= -A(\frac{sin(wt) - wRCcos(wt)}{1+(wRC)^2}) + Ke^{\frac{-t}{RC}} \\
0 &= -A(\frac{sin(w(0)) - wRCcos(w(0))}{1+(wRC)^2}) + Ke^{\frac{-(0)}{RC}} \\
0 &= -A(\frac{-wRC}{1+(wRC)^2}) + K \\
K &= \frac{-ARCw}{1+(wRC)^2} \\
e_2 &= -A(\frac{sin(wt) - wRCcos(wt)}{1+(wRC)^2}) - \frac{ARCw}{1+(wRC)^2}e^{\frac{-t}{RC}}
\end{aligned}$$

Plugging what one finds for $e_1$ and $e_2$ into $I_v$ gives the final unique solution one desires.

$$\begin{aligned}
I_v &= \frac{1}{R}(e_1 - e_2) \\
&= \frac{1}{R}(-Asin(wt) + A(\frac{sin(wt) - wRCcos(wt)}{1+(wRC)^2}) + \frac{ARCw}{1+(wRC)^2}e^{\frac{-t}{RC}})
\end{aligned}$$

## 1.4 Problem Four

Here is the system in the standard form.

$$\begin{aligned}
\boldsymbol{y'} &= \boldsymbol{F}(t, \boldsymbol{y}, \boldsymbol{z}) \\
\boldsymbol{0} &= \boldsymbol{G}(t, \boldsymbol{y}, \boldsymbol{z})
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{y} &= [e_2] \\
\boldsymbol{z} &= [e_1; I_v] \\
\boldsymbol{F} &= [\frac{1}{RC}(e_1 - e_2)] \\
\boldsymbol{G} &= [-I_v + \frac{1}{R}(e_1 - e_2); V(t) + e_1]
\end{aligned}$$

Below are additional elements that will aid in the construction of the Jacobian and residual matrix.

$$\begin{aligned}
\frac{\delta \boldsymbol{F}}{\delta \boldsymbol{y}} &= [\frac{-1}{RC}] \\
\frac{\delta \boldsymbol{F}}{\delta \boldsymbol{z}} &= [\frac{1}{RC} \quad 0] \\
\frac{\delta \boldsymbol{G}}{\delta \boldsymbol{y}} &= [\frac{-1}{R}; 0] \\
\frac{\delta \boldsymbol{G}}{\delta \boldsymbol{z}} &= [\frac{1}{RC} \quad -1; 1 \quad 0]
\end{aligned}$$

Using the above, here is the Jacobian and residual matrix for the function.

$$J = \begin{bmatrix} 1 + \gamma\frac{\Delta t}{RC} & -\gamma\frac{\Delta t}{RC} & 0 \\ \frac{-1}{R} & \frac{1}{R} & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$r = \begin{bmatrix} y_{n+1} - \gamma\frac{\Delta t}{RC}(e_1 - e_2) \\ -I_v + \frac{1}{R}(e_1 - e_2) \\ V(t) + e_1 \end{bmatrix} - B$$

# 2 Programming Assignment

## 2.1 Problem One

After building a quasi-Newton method solver $[y, z] = daesolver(k, t, gam, dt, B, yn, zn, tol)$ where
**Inputs**

- k = [R,C,w,A] parameter values
- t = $t_{n+1}$ when solving for $y_{n+1}$
- gam = the gamma value
- dt = time step size
- B = constant vector
- yn = initial value for y
- zn = initial value for z
- tol = tolerance for the error in the residual

**Outputs**

- y = $y_{n+1}$
- z = $z_{n+1}$

the solver was tested with the line

$$[y, z] = daesolver([1, 1, pi, 0.01], 1e - 3, 1, 1e - 3, 0, 0, [0; 0], 1e - 12)$$

and yielded the expected results of $y = -3.1384e - 08$ and $z = 1.0e - 04 * [-.3142; -.3138]$.

## 2.2 Problem Two

After programming a Backward Difference Formula 1 method with $k = [1, 1, pi, 10^{-2}]$ and with initial conditions $e_1 = e_2 = I_v = 0$ for 1000 steps, $N$, one gets the following solution.

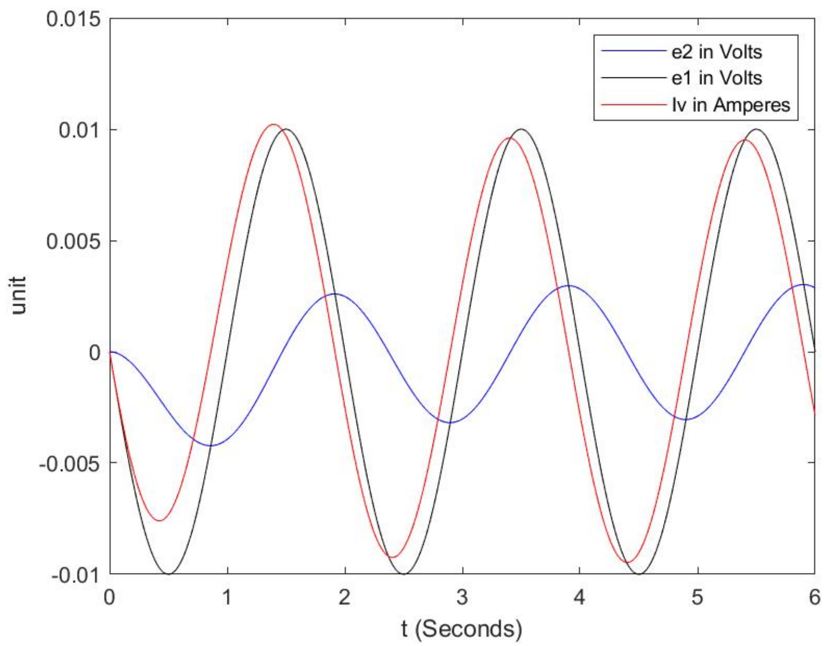


Figure 1: Numerical solution after using Backwards Difference Formula 1 method with N=1000

If one increases the number of steps to 2000 and 4000, one gets the following graphs, respectively.

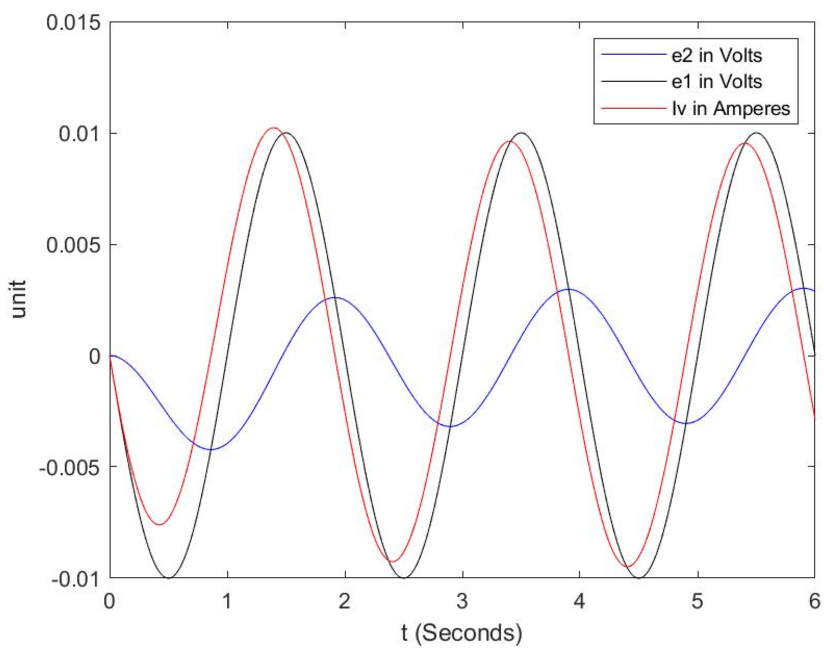


Figure 2: Numerical solution after using Backwards Difference Formula 1 method with N=2000

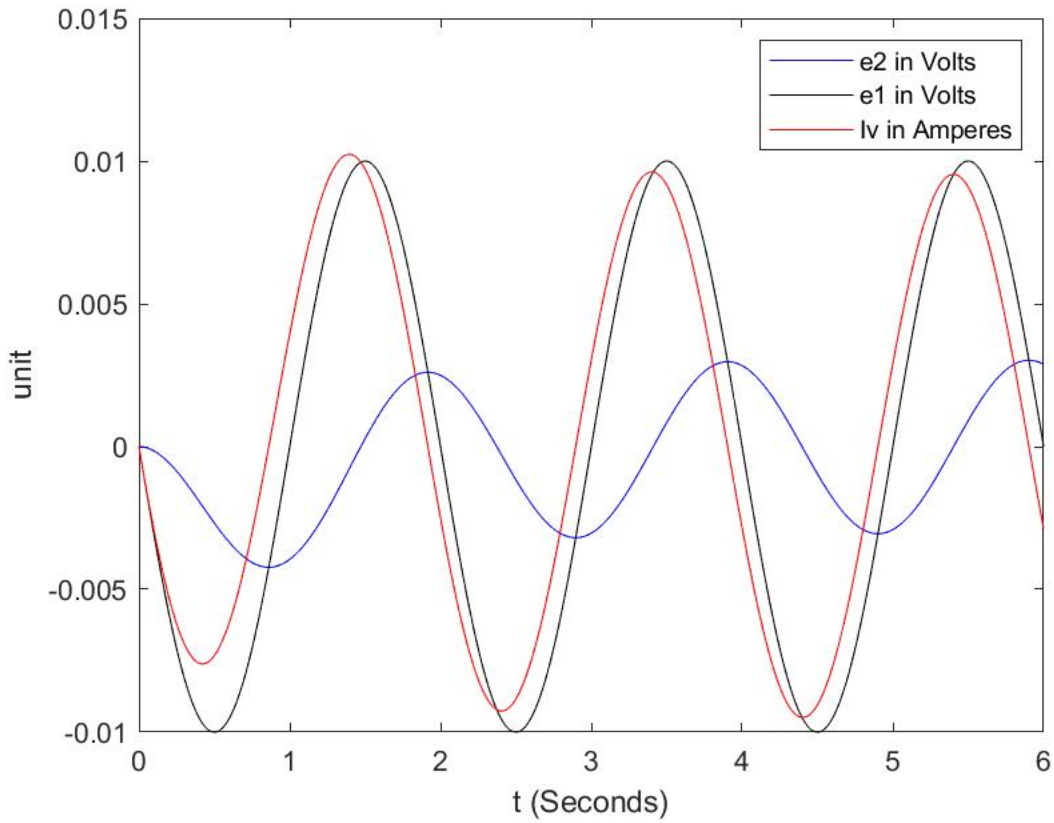


Figure 3: Numerical solution after using Backwards Difference Formula 1 method with N=4000

While they may look identical to the naked eye, they actually vary quite a bit. If one assumes that Figure 3 is the actual solution and finds the 2-norm error of Figure 1 and 2, the following table is obtained. Let Figure 1 be $u_1$, Figure 2 be $u_2$, and Figure 3 be $u_3$.

Table 1: Error of $u_1$ and $u_2$

| | $u_1 - u_3$ | $u_2 - u_3$ | $\frac{u_1-u_3}{u_2-u_3}$ |
|---|---|---|---|
| $e_1$ | 0 | 0 | NaN |
| $e_2$ | 1.1884e-05 | 3.9604e-06 | 3.0007 |
| $I_v$ | 1.1884e-05 | 3.9604e-06 | 3.0007 |

In a first order accurate method, one should expect the error to cut down by a factor of 2 as the number of steps are doubled. Looking at the ratio between the two errors, this method does better but is still less than a factor of 4 showing that the method is not second order accurate but rather first order accurate.

## 2.3 Problem Three

After programming a Backward Difference Formula 2 method with $k = [1, 1, pi, 10^{-2}]$ and with initial conditions $e_1 = e_2 = I_v = 0$ for 1000 steps, $N$, one gets the following solution.

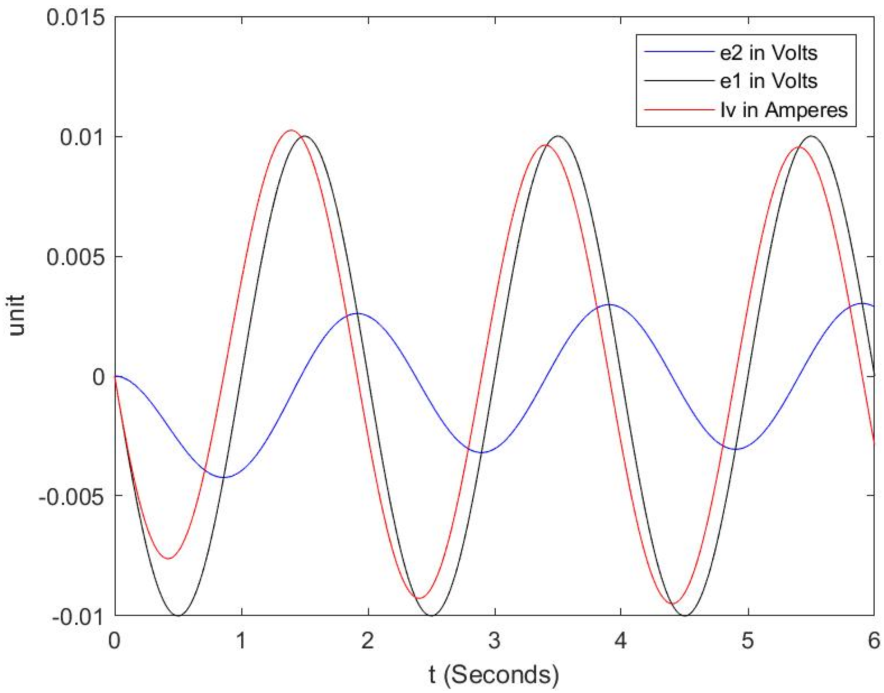


Figure 4: Numerical solution after using Backwards Difference Formula 2 method with N=1000

If one increases the number of steps to 2000 and 4000, one gets the following graphs, respectively.

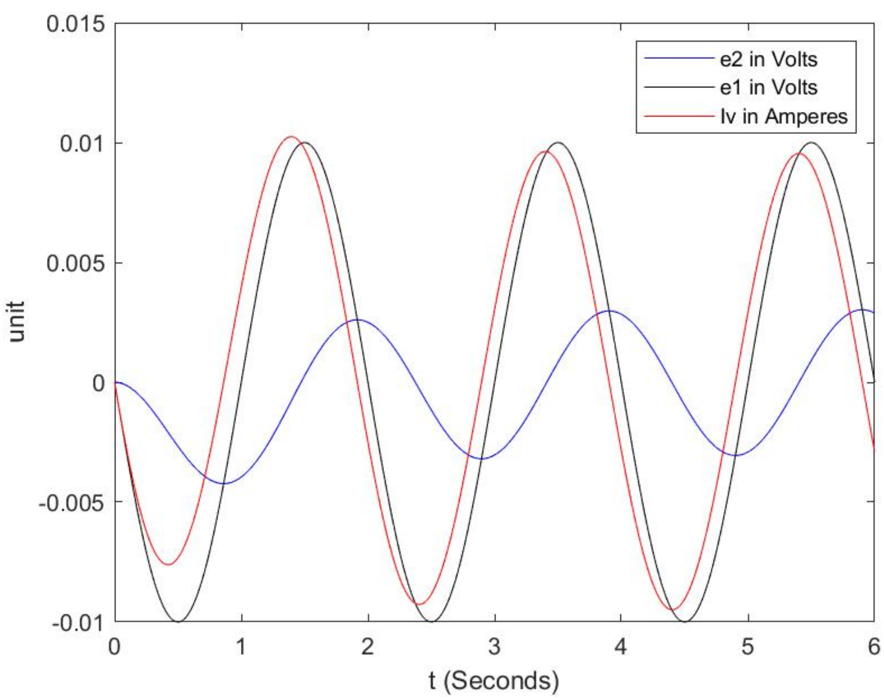


Figure 5: Numerical solution after using Backwards Difference Formula 2 method with N=2000

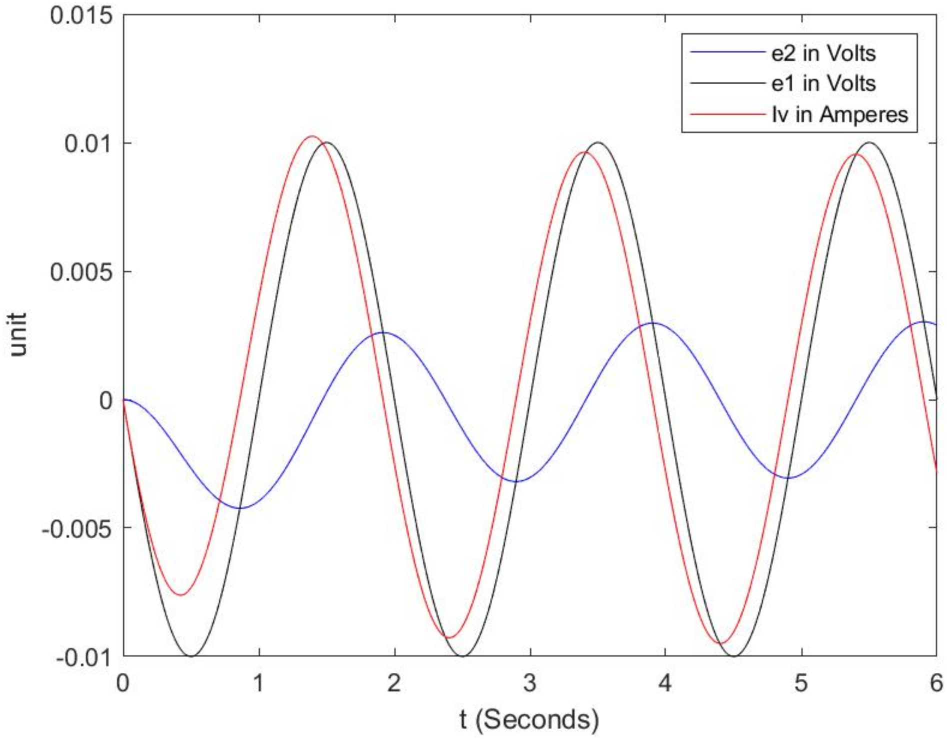


Figure 6: Numerical solution after using Backwards Difference Formula 2 method with N=4000

While they may look identical to the naked eye, they actually vary quite a bit. If one assumes that Figure 6 is the actual solution and finds the 2-norm error of Figure 4 and 5, the following table is obtained. Let Figure 4 be $u_4$, Figure 5 be $u_5$, and Figure 6 be $u_6$.

Table 2: Error of $u_4$ and $u_5$

| | $u_4 - u_6$ | $u_5 - u_6$ | $\frac{u_4 - u_6}{u_5 - u_6}$ |
|---|---|---|---|
| $e_1$ | 0 | 0 | NaN |
| $e_2$ | 2.6540e-07 | 5.2850e-08 | 5.0218 |
| $I_v$ | 2.6540e-07 | 5.2850e-08 | 5.0218 |

first, notice how the values of error between the supposed "actual" solution and the other two solutions are much smaller when compared to the errors in Table 1. Furthermore, in a second order accurate method, one should expect the error to cut down by a factor of four when the number of steps are doubled. In this case looking at the ratio of the errors, the method seems to have done better than that showing that this method is second order accurate.

## 2.4 Problem Four

To make a better program, an adaptive time stepper solver has been created to optimize the step size. Setting the tolerance of the program to $tol = 10^{-5}$, one gets the following graphs showing the numerical solution and the time step over time.

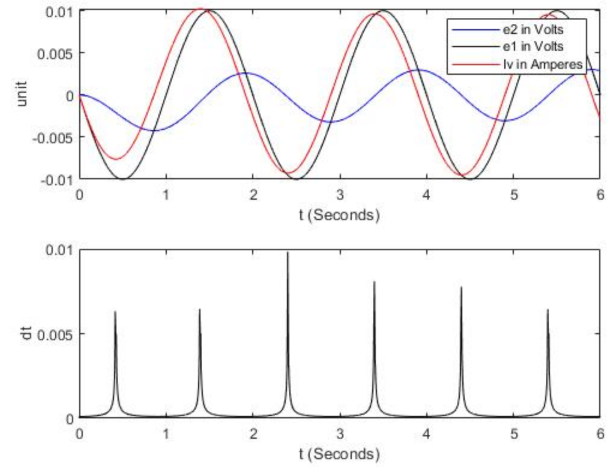


Figure 7: Numerical solution after using an adaptive time stepper solver

Looking at the lower plot of Figure 7, it seems that for the most part the time steps are very small but at around .5, 1.5, 2.5, 3.5, 4.5, and 5.5 seconds, there are spikes for larger time steps.

Let's compare this with the exact solution and determine if the tolerance, $tol = 10^{-5}$, was actually met.

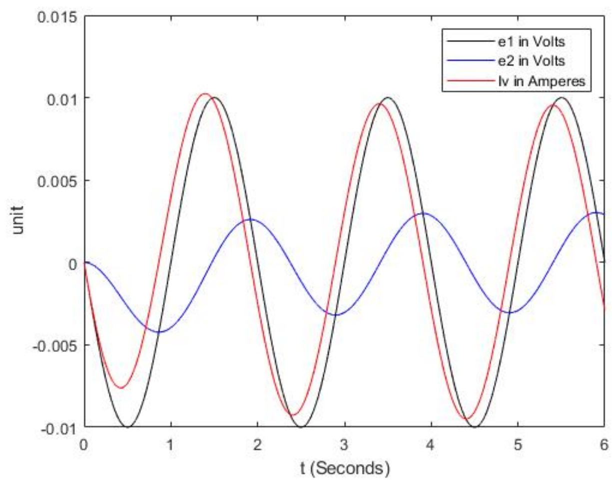


Figure 8: Exact Analytical Solution of the System

To determine if the adaptive solver solution is within the tolerance, the value at t=6 will be taken from both this numerical solution and the exact solution and will be used to find the 2-norm error. Let u* be the adaptive time stepper numerical solution and u be the exact solution.

Table 3: Error of Adaptive Solver

| u* - u | tol |
|---|---|
| $4.4430e-07$ | $1.0000e-05$ |

Seeing that the error of the final value at $t = 6$ is less than the tolerance, one can say the tolerance is indeed met.