ES_APPM 346 Project 1

Northwestern University

Nicolas Guerra
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Contents

1 Written Project

1.1 Problem One - Part (a)

An integral to represent total momentum of the fluid in a control volume V can be as written below.

$$total\ momentum = \iiint_V \rho \boldsymbol{u} dV$$

1.2 Problem One - Part (b)

$$\rho \boldsymbol{u}(\boldsymbol{u} \cdot \boldsymbol{n})$$

The only way momentum can change in V is if it flows in or out. Since $\boldsymbol{u}$ is the velocity of the fluid, the scalar projection of $\boldsymbol{u}$ onto the normal of the surface, $\boldsymbol{n}$, without any normalization is calculated. Multiplying this length by momentum which is a vector scales the momentum to demonstrate its outward flow of the control volume.

1.3 Problem One - Part (c)

$$-\oiint_{\delta V} p\boldsymbol{n} dS$$

Force is the change of momentum over time. If there were no change in momentum, pressure would just be zero. This inward force called pressure, p, applied to the surface of the control volume shows that there is a loss in momentum.

1.4 Problem One - Part (d)

To Derive the conservation of momentum equation, the previous known parts will be put into one equation.

$$\frac{\delta}{\delta t}[\iiint_V \rho \boldsymbol{u} dV] + \oiint_{\delta V} \rho \boldsymbol{u}(\boldsymbol{u} \cdot \boldsymbol{n}) dS = -\oiint_{\delta V} p\boldsymbol{n} dS$$

After applying the Divergence theorem to have common triple integrals, one gets the following.

$$\iiint_V \frac{\delta \boldsymbol{u}}{\delta t} \rho dV + \iiint_V \rho \boldsymbol{u}(\boldsymbol{u} \cdot \nabla) dV = -\iiint_V p\nabla dV$$

One must then group the equation to have just one integral.

$$\iiint_V \frac{\delta \boldsymbol{u}}{\delta t} \rho + \rho \boldsymbol{u}(\boldsymbol{u} \cdot \nabla) + p\nabla dV = 0$$

After setting the integrand to zero, one gets here.

$$\frac{\delta \boldsymbol{u}}{\delta t} \rho + \rho \boldsymbol{u}(\boldsymbol{u} \cdot \nabla) + p\nabla = 0$$

Bringing $p\nabla$ to the other side of the equal sign and dividing both sides by ρ yields the desired result.

$$\frac{\delta \boldsymbol{u}}{\delta t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \frac{-p\nabla}{\rho}$$

1.5 Problem Two

To get from $\frac{\delta \boldsymbol{u}}{\delta t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \frac{-p\nabla}{\rho}$ to $\frac{\delta \xi}{\delta t} + (\boldsymbol{u} \cdot \nabla)\xi = 0$, the curl of the equation must be taken. To do so, I will apply the curl property that when taking the curl of an equation one can also take the curl of individual parts and add them together. Below is the curl of $\frac{\delta \boldsymbol{u}}{\delta t}$ where $\boldsymbol{u} = (u, v, 0)$.

$$\nabla \times \frac{\delta \boldsymbol{u}}{\delta t} = \begin{pmatrix} i \; j \; k \\ \frac{\delta}{\delta x} \; \frac{\delta}{\delta y} \; \frac{\delta}{\delta z} \\ \frac{\delta u}{\delta t} \; \frac{\delta v}{\delta t} \; 0 \end{pmatrix}$$
$$= \hat{i}(-\frac{\delta^2 v}{\delta t \delta z}) + \hat{j}(\frac{\delta^2 u}{\delta t \delta z}) + \hat{k}(\frac{\delta^2 v}{\delta x \delta t} - \frac{\delta^2 u}{\delta y \delta t})$$

Since $\boldsymbol{u}$ is in 2D, the vorticity points out of the page, so therefore only the $\hat{k}$ component will be taken. By letting ξ be the following, one can perform the subsequent substitution for the $\hat{k}$ component.

$$\xi = \frac{\delta v}{\delta x} - \frac{\delta u}{\delta y}$$
$$\frac{\delta \xi}{\delta t} = (\nabla \cdot \boldsymbol{u}) \cdot \hat{k}$$

To take the curl of the next part $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$, it must first be expanded and then one can proceed with taking the curl.

$$\nabla \times ((\nabla \cdot \boldsymbol{u})\boldsymbol{u}) = \nabla \times (((\frac{\delta}{\delta x}, \frac{\delta}{\delta y}, \frac{\delta}{\delta z}) \cdot < u, v, 0 >)\boldsymbol{u})$$
$$= \nabla \times ((\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y}) < u, v, 0 >)$$
$$= \begin{pmatrix} i \; j \; k \\ \frac{\delta}{\delta x} \; \frac{\delta}{\delta y} \; \frac{\delta}{\delta z} \\ u(\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y}) \; v(\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y}) \; 0 \end{pmatrix}$$
$$= \hat{i}(-\frac{\delta v}{\delta z}(\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y})) + \hat{j}(\frac{\delta u}{\delta z}(\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y})) + \hat{k}(\frac{\delta v}{\delta x}(\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y}) - \frac{\delta u}{\delta y}(\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y}))$$

Again, since we are only taking the $\hat{k}$ component, one gets the following which can be partly substituted by ξ and $\boldsymbol{u} \cdot \nabla$.

$$\frac{\delta v}{\delta x}(\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y}) - \frac{\delta u}{\delta y}(\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y}) = (\frac{\delta v}{\delta x} - \frac{\delta u}{\delta y})(\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y}) = (\boldsymbol{u} \cdot \nabla)\xi$$

To take the curl of the third and final part $\frac{-p\nabla}{\rho}$, one can notice that, since ρ is a constant and $-p\nabla$ is a scalar, this is a conservative vector field. Thus, the curl of any conservative vector field is zero. Putting all three parts into one equation yields our desired result.

$$\frac{\delta \xi}{\delta t} + (\boldsymbol{u} \cdot \nabla)\xi = 0$$

Note: To show that $(\xi\hat{k} \cdot \nabla)\boldsymbol{u} = 0$, one must expand the equation. Since $\boldsymbol{u}$ is in 2D, only the $\hat{k}$ component will be taken.

$$(\xi \cdot \nabla)\boldsymbol{u} =< 0, 0, \frac{\delta v}{\delta x} - \frac{\delta u}{\delta y} > \cdot (\frac{\delta}{\delta x}, \frac{\delta}{\delta y}, \frac{\delta}{\delta z})\boldsymbol{u} = \frac{\delta}{\delta z}(\frac{\delta v}{\delta x} - \frac{\delta u}{\delta y}) < u, v, 0 > \xRightarrow{\hat{k} \text{ component}} 0$$

1.6 Problem Three

To find the radius and center of $|1 + \Delta t| \leq 1$, Δt will be substituted with $a + bi$. Then, the conjugate of the complex number will be multiplied to get a real valued inequality that a and b must satisfy.

$$|1 + \Delta t| \leq 1$$
$$|1 + a + bi| \leq 1$$
$$|((1 + a) + bi)| \leq 1$$
$$((1 + a) + bi)((1 + a) - bi) \leq 1$$
$$(1 + a)^2 + b^2 \leq 1$$

The result is the equation of a shaded circle centered at (-1,0) with a radius of 1.

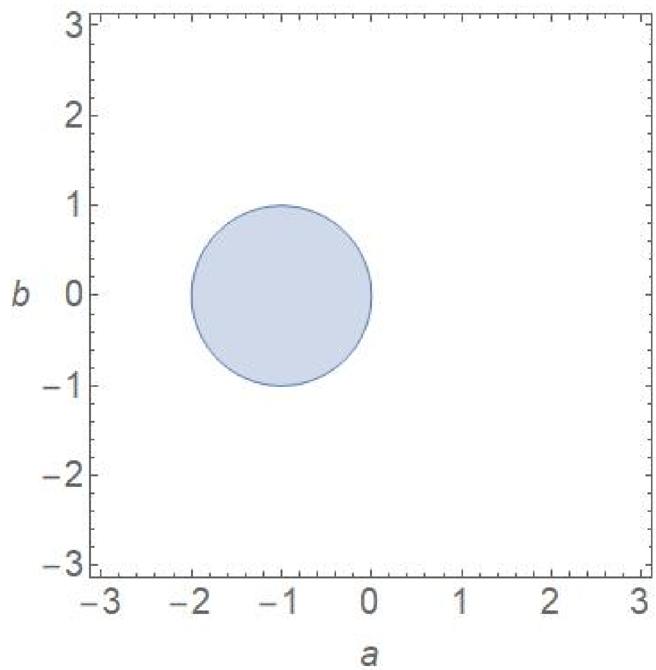

Figure 1: Illustration of the stability region for Euler's method

Furthermore, to find the stability region for Backward Euler, one must do the following calculations.

$$y_{n+1} = y_n + \Delta t * F(t_{n+1}, y_{n+1})$$

Plug in $y' = \lambda y$.

$$y_{n+1} = y_n + \Delta t(\lambda y_{n+1})$$
$$y_{n+1} = \frac{1}{1 - \lambda \Delta t} y_n$$

There is stability when $|y_{n+1}| \leq |y_n|$. Therefore, the solution is stable when:

$$|1 - \Delta t| \geq 1$$

To be able to graph such an inequality, one must let Δt be equal to $a + bi$. Then, like before, multiply by the conjugate to get a real valued inequality that a and b must satisfy.

$$|1 - \Delta t| \geq 1$$
$$|1 - a - bi| \geq 1$$
$$|(1 - a) + (-b)i| \geq 1$$
$$((1 - a) + (-b)i)((1 - a) - (-b)i) \geq 1$$
$$(1 - a)^2 + b^2 \geq 1$$

The final result is the equation of a circle centered at $(1, 0)$ with a radius of 1 and the outside area being shaded.

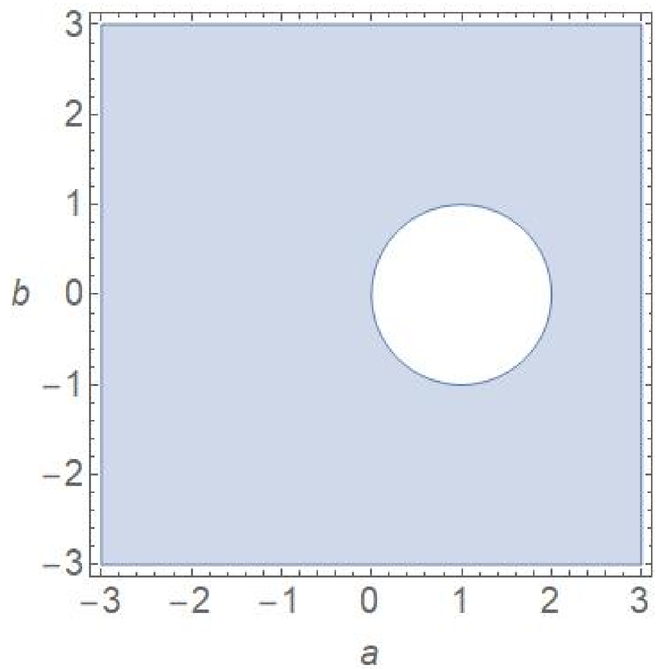

Figure 2: Illustration of the stability region for Backward Euler's method

To find the stability region for Improved Euler, one must perform the following calculations.

$$y_{n+1} = y_n + \frac{\Delta t}{2}(F(t_n, y_n) + F(t_{t+1}, y_n + \Delta t * F(t_n, y_n)))$$

Plug in $y' = \lambda y$.

$$y_{n+1} = y_n + \frac{\Delta t}{2}(\lambda y_n + \lambda(y_n + \Delta t \lambda y_n))$$
$$y_{n+1} = y_n + \frac{\Delta t}{2}\lambda y_n + \frac{\lambda \Delta t}{2} y_n + \frac{(\Delta t \lambda)^2}{2} y_n$$
$$y_{n+1} = y_n + \lambda \Delta t y_n + \frac{(\Delta t \lambda)^2}{2} y_n$$
$$= (1 + \lambda \Delta t + \frac{(\Delta t \lambda)^2}{2}) y_n$$

There is stability when $|y_{n+1}| \leq |y_n|$. Therefore, the solution is stable when:

$$|1 + \lambda\Delta t + \frac{(\Delta t\lambda)^2}{2}| \leq 1$$

To be able to graph such an inequality, one must let Δt be equal to $a + bi$. Then, like before, multiply by the conjugate to get a real valued inequality that a and b must satisfy.

$$|1 + \lambda\Delta t + \frac{(\Delta t\lambda)^2}{2}| \leq 1$$

$$|1 + (a + bi) + \frac{(a + bi)^2}{2}| \leq 1$$

$$|1 + a + bi + \frac{a^2}{2} + abi - \frac{b^2}{2}| \leq 1$$

$$|(\frac{a^2}{2} - \frac{b^2}{2} + 1 + a) + (b + ab)i| \leq 1$$

$$((\frac{a^2}{2} - \frac{b^2}{2} + 1 + a) + (b + ab)i)((\frac{a^2}{2} - \frac{b^2}{2} + 1 + a) - (b + ab)i) \leq 1$$

$$(\frac{a^2}{2} - \frac{b^2}{2} + 1 + a)^2 + (b + ab)^2 \leq 1$$

The final result returns the following graph displaying the stability region for Improved Euler.

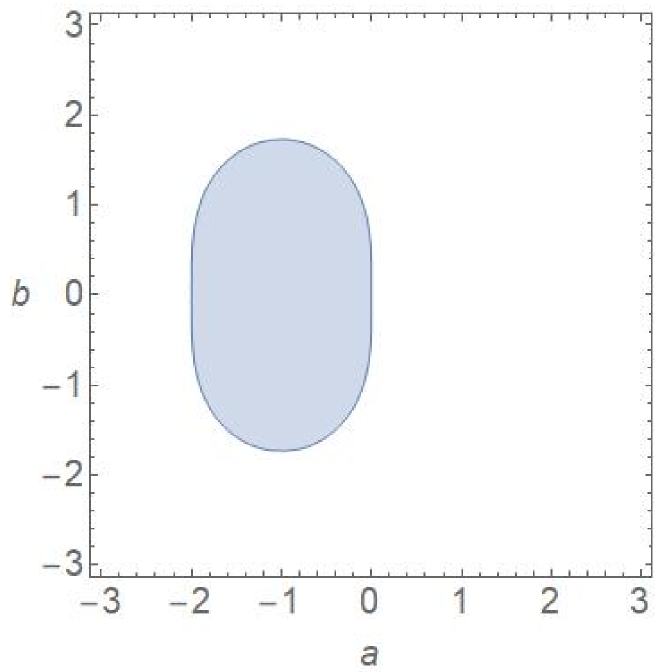

Figure 3: Illustration of the stability region for Improved Euler's method

1.7 Problem Four

The following is a derivation for Adams-Moulton 2 using a linear interpolant that passes through the points $F(t_n, y_n)$ and $F(t_{n+1}, y_{n+1})$.

$$\overrightarrow{y_{n+1}} = \int_{tn}^{t_{n+1}} \overrightarrow{p(t)}dt + \overrightarrow{y_n}$$

where $\overrightarrow{p(t)}$ is the Lagrange interpolating polynomial below.

$$\overrightarrow{p_2(t)} = \frac{t - t_{n+1}}{t_n - t_{n+1}} * \overrightarrow{F(t_n, \overrightarrow{y_n})} + \frac{t - t_n}{t_{n+1} - t_n} * \overrightarrow{F(t_{n+1}, \overrightarrow{y_{n+1}})}$$

One must then substitute in this polynomial.

$$\overrightarrow{y_{n+1}} - \overrightarrow{y_n} = \int_{t_n}^{t_{n+1}} \overrightarrow{p_2(t)}dt$$

$$\overrightarrow{y_{n+1}} - \overrightarrow{y_n} = \int_{t_n}^{t_{n+1}} \frac{t - t_{n+1}}{t_n - t_{n+1}} * \overrightarrow{F(t_n, \overrightarrow{y_n})} + \frac{t - t_n}{t_{n+1} - t_n} * \overrightarrow{F(t_{n+1}, \overrightarrow{y_{n+1}})}dt$$

$$\overrightarrow{y_{n+1}} - \overrightarrow{y_n} = \int_{t_n}^{t_{n+1}} \frac{t - t_{n+1}}{t_n - t_{n+1}} * \overrightarrow{F(t_n, \overrightarrow{y_n})}dt + \int_{t_n}^{t_{n+1}} \frac{t - t_n}{t_{n+1} - t_n} * \overrightarrow{F(t_{n+1}, \overrightarrow{y_{n+1}})}dt$$

Using u-substitution, one can solve the integrals.

$$\overrightarrow{y_{n+1}} - \overrightarrow{y_n} = \int_{t_n - t_{n+1}}^{0} \frac{u}{t_n - t_{n+1}} * \overrightarrow{F(t_n, \overrightarrow{y_n})}du + \int_{0}^{t_{n+1} - t_n} \frac{w}{t_{n+1} - t_n} * \overrightarrow{F(t_{n+1}, \overrightarrow{y_{n+1}})}dw$$

$$\overrightarrow{y_{n+1}} - \overrightarrow{y_n} = -\frac{(t_n - t_{n+1})^2}{2(t_n - t_{n+1})}\overrightarrow{F(t_n, \overrightarrow{y_n})} + \frac{(t_{n+1} - t_n)^2}{2(t_{n+1} - t_n)}\overrightarrow{F(t_{n+1}, \overrightarrow{y_{n+1}})}$$

Let $\Delta t = t_{n+1} - t_n$.

$$\overrightarrow{y_{n+1}} - \overrightarrow{y_n} = \frac{\Delta t^2 \overrightarrow{F(t_n, \overrightarrow{y_n})}}{2\Delta t} + \frac{\Delta t^2 \overrightarrow{F(t_{n+1}, \overrightarrow{y_{n+1}})}}{2\Delta t}$$

After rearranging the terms, one obtains the final equation for Adams-Moulton 2.

$$\overrightarrow{y_{n+1}} = \overrightarrow{y_n} + \frac{\Delta t}{2}(\overrightarrow{F(t_n, \overrightarrow{y_n})} + \overrightarrow{F(t_{n+1}, \overrightarrow{y_{n+1}})})$$

One should notice that this result is identical to the trapezoidal rule method.

2 Vortex Project

2.1 Problem One - Part (a)

In order to show that the fluid particle moves in a circle around the vortex, the equation for its position will be derived by eliminating the variable t. To do so, one must divide $\frac{dy}{dt}$ by $\frac{dx}{dt}$ and integrate both sides.

$$\frac{dy}{dt} * \frac{dt}{dx} = \frac{\Gamma}{2\pi} * \frac{x}{x^2+y^2} * \frac{2\pi}{\Gamma} * \frac{x^2+y^2}{-y}$$
$$\frac{dy}{dx} = \frac{-x}{y}$$
$$\int -y dy = \int x dx$$
$$\frac{-y^2}{2} + c_1 = \frac{x^2}{2} + c_2$$

Rearranging both sides and combining the constants as a single constant, one finds the equation of a circle centered at $(0,0)$ with radius $\sqrt{C}$ showing that the path of the particle is indeed circular around the vortex.

$$x^2 + y^2 = 2(c_1 - c_2)$$
$$x^2 + y^2 = C$$

To find the time it takes for the particle to make one full revolution around the origin, we must convert the differential equations from Cartesian to polar coordinates. This can be achieved by plugging in $x(t) = r_0 cos(\alpha t)$ and $y(t) = r_0 sin(\alpha t)$, where r_0 is the initial radius of the particle's path and α is the frequency of the particle around the origin, into either one of the differential equations. One will come to the same result no matter which equation they use.

$$\frac{dx}{dt} = \frac{\Gamma}{2\pi} * \frac{-y}{x^2+y^2}$$
$$-\alpha r_0 sin(\alpha t) = \frac{\Gamma}{2\pi} * \frac{-r_0 sin(\alpha t)}{r_0^2(cos^2(\alpha t) + sin^2(\alpha t))}$$
$$-\alpha r_0 sin(\alpha t) = \frac{\Gamma}{2\pi} * \frac{-sin(\alpha t)}{r_0}$$
$$\alpha = \frac{\Gamma}{2\pi r_0^2}$$

Since the radius of the particle's path is $\sqrt{C}$, and the given initial points of x and y are (x_0, y_0), this means that the initial radius is $r_0 = \sqrt{x_0^2 + y_0^2}$.

Moreover, to calculate the period, T, of the function, one must divide 2π by the frequency found.

$$T = 2\pi * \frac{2\pi r_0^2}{\Gamma}$$
$$T = \frac{4\pi^2 r_0^2}{\Gamma}$$

2.2 Problem One - Part (b)

Using the value of $\Gamma = 1$, the value of the period is approximately $T \approx 39.48$. Using Euler's method with varying numbers of steps, $N = 50,\ N = 100,\ N = 200,\ N = 400$, one finds the following graph.

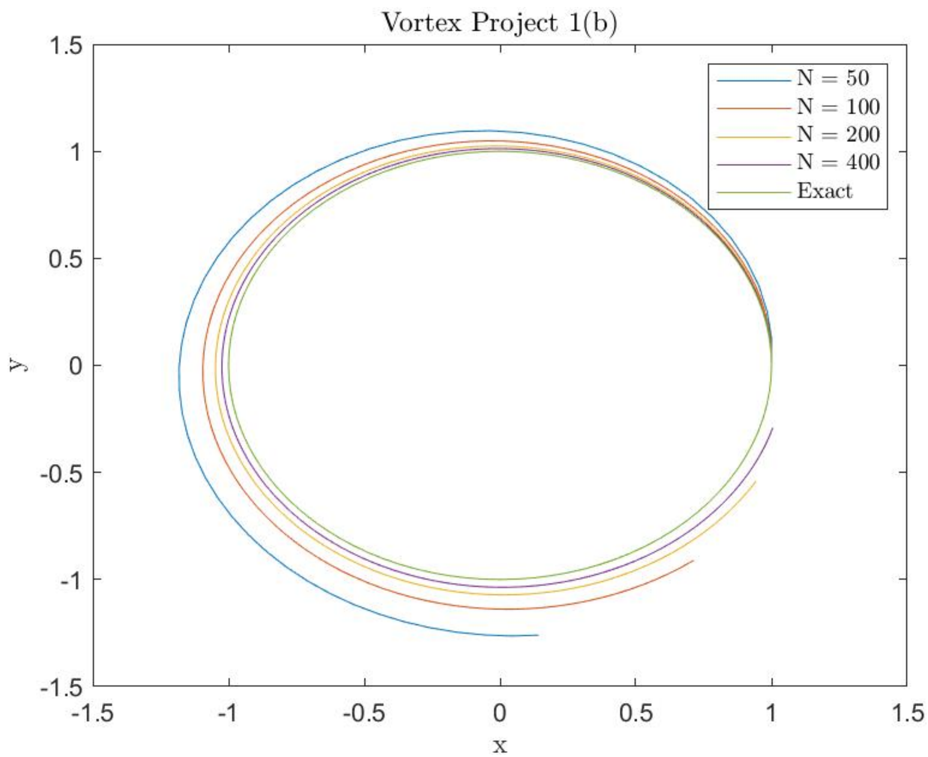

Figure 4: Numerical solution of problem 1(b) using Euler's method with varying step sizes

The exact solution is a perfect circle. When $N = 400$, the solution is almost correct, but then starts to drift off. As N decreases, the amount the solution drifts increases. When $N = 50$, one can see the solution drift the earliest here.

To find the error with each value of N, I will use the given formula:

$$error = ||exact - computed|| = \sqrt{(1 - x_N)^2 + (0 - y_N)^2}$$

Table 1: Error corresponding to each N for Euler's method

N = 50	N = 100	N = 200	N = 400
1.5243	0.9546	0.5438	0.2917

After noting the error of each N, one can see that as the N value increases, the error decreases.

2.3 Problem One - Part (c)

In order to apply 4^{th} order Runge-Kutta, the function $rk\_nicolasguerra(N, T, x, y, g, p, q)$ will be called. It takes the following inputs and returns the following outputs.

Inputs:

- N is the number of steps
- T is the final time

- x is a sized-m column vector of initial x-coordinates of vortices
- y is a sized-m column vector of initial y-coordinates of vortices
- g is a sized-m column vector of the strengths of the vortices
- p is a sized-n column vector of initial x-coordinates of particles
- q is a sized-n column vector of initial y-coordinates of particles

Outputs:

- x_out returns an m x N matrix of each vortex's x-component path
- y_out returns an m x N matrix of each vortex's y-component path
- p_out returns an n x N matrix of each particle's x-component path
- q_out returns an n x N matrix of each particle's y-component path

By plugging in $x = [0],\ y = [0],\ g = [1],\ p = [1]$, and $q = [0]$ into the function above and using a for loop to iterate the process with $N = 50,\ 100,\ 200$, and 400, one gets the graph below.

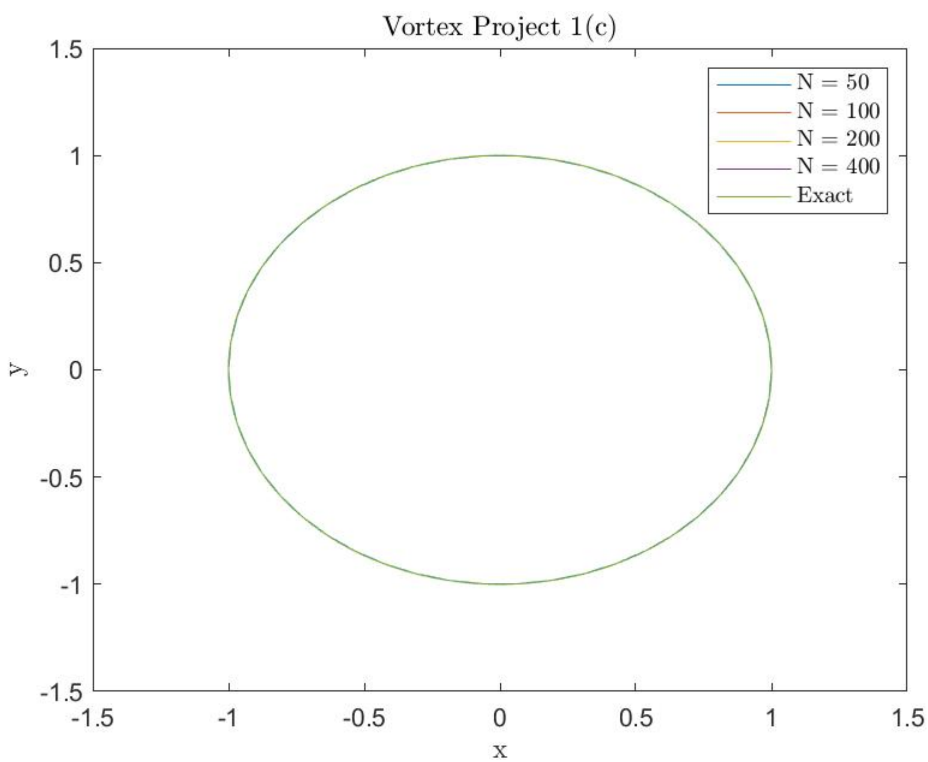

Figure 5: Numerical solution of problem 1(c) using 4^{th} order Runge-Kutta with varying step sizes

Looking at this graph first glance, it seems as if there is only one circle. This is a good sign showing that Runge-Kutta provides accurate results with little steps compared to Euler's method. To demonstrate that there is still error, here is a figure of the same graph zoomed in.

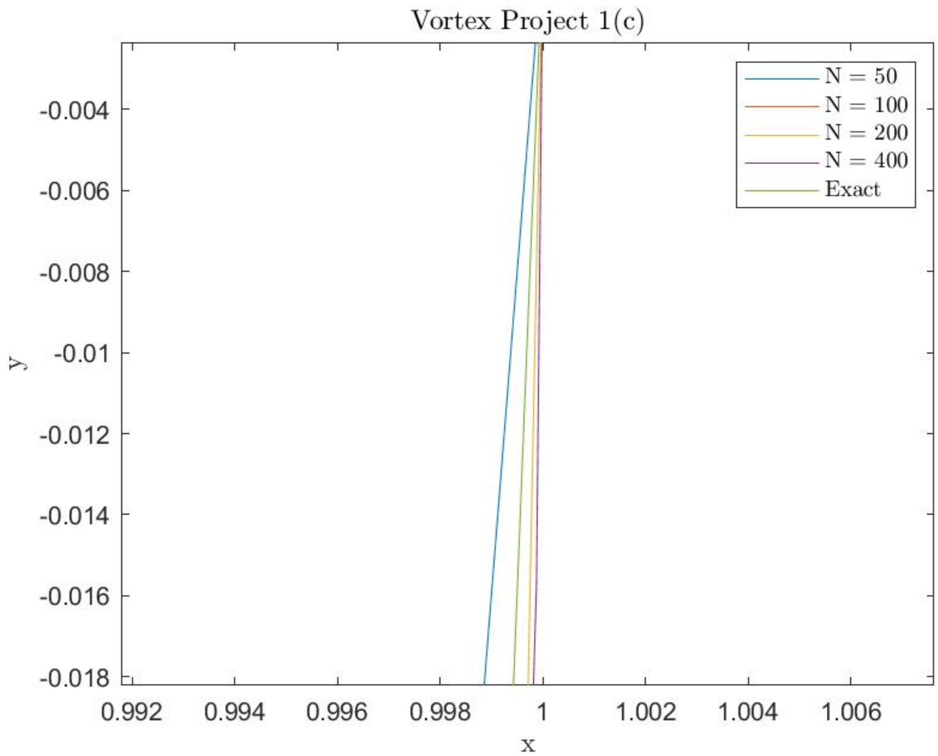

Figure 6: Zoomed-in numerical solution of problem 1(c) using 4^{th} order Runge-Kutta with varying step sizes

Now the error is clearly visible; however, still extremely small when compared to the error obtained from Euler's method. Below is the numerical error for 4^{th} order Runge-Kutta. One can see that such values are minuscule after looking at the error from Euler's method.

Table 2: Error corresponding to each N for 4^{th} order Runge-Kutta

N = 50	N = 100	N = 200	N = 400
5.0099e-05	2.9911e-06	1.8271e-07	1.1288e-08

2.4 Problem Two - Part (a)

By plugging the following values into $rk\_nicolasguerra()$, Figure 7 is obtained.

- x = [0;.4;-0.2;-.2]
- y = [0;0;.3464;-.3464]
- g = [-1;-.667;-.667;-.667]
- p = [1]
- q = [0]
- T = 20
- N = 400

*Note: In order to calculate N, T is divided by the given $\Delta t = 0.05$.

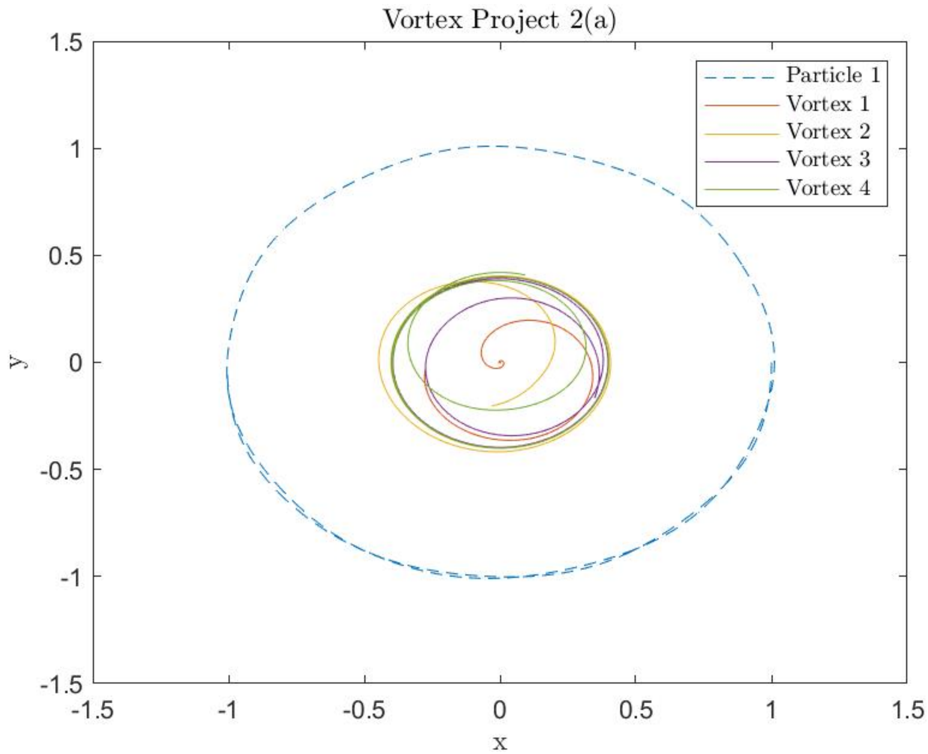

Figure 7: Numerical solution using 4^{th} order Runge-Kutta for when several vortices are close together

Looking at the graph, one can see that the particle behaves very similarly to problem one. Since the vortices are close together and far away from the particle, the sum of their strengths can be thought of as a point source centered at $(0, 0)$.

2.5 Problem Two - Part (b)

By plugging the following values into $rk\_nicolasguerra()$, Figure 8 is obtained.

- x = [0;1]
- y = [0;0]
- g = [1;-1]
- p = [-.5;0;.5;1;1.5]
- q = [.2;.2;.2;.2;.2]
- T = 20
- N = 400

*Note: In order to calculate N, T is divided by the given $\Delta t = 0.05$.

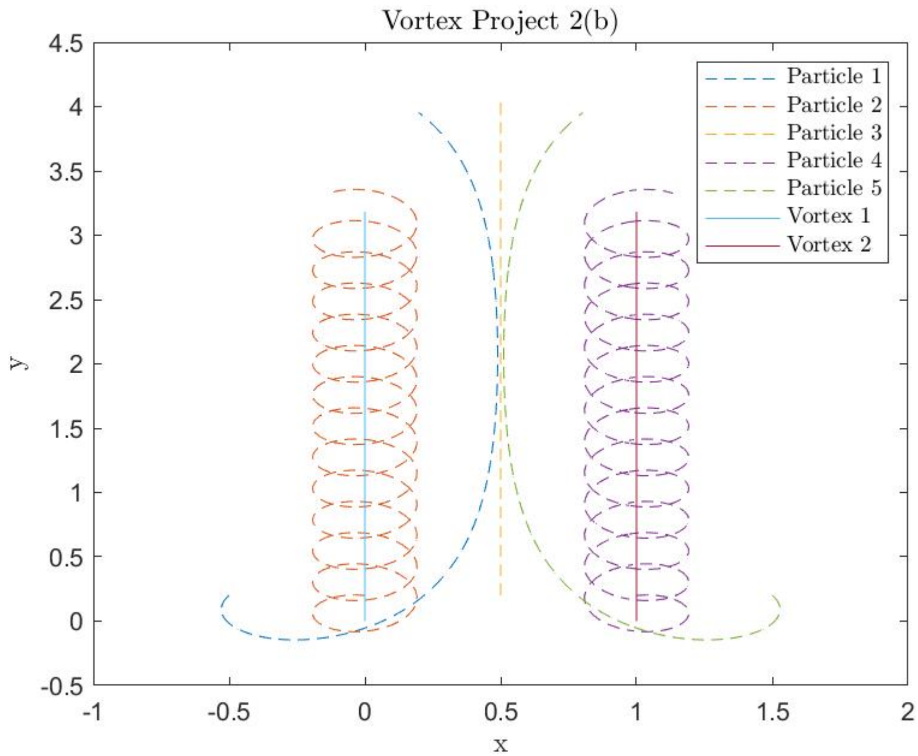

Figure 8: Numerical solution using 4^{th} order Runge-Kutta for when the vortices will propagate indefinitely

Looking at the graph, one can see that the particles closest to the vortices travel circularly with a relatively fast frequency. The particle that started directly in the middle between the two vortices seems to travel in a straight line. This would make sense since it would be pushed and pulled equally by both vortices. The particles that started further away from the vortices seem to travel circularly too but with a larger radius and slower frequency.

2.6 Problem Two - Part (c)

By plugging the following values into $rk\_nicolasguerra()$, Figure 9 is obtained.

- x = [-.5;.5;0;-.5;.5;0]
- y = [0;0;0;1;1;1]
- g = [5;-5;.2;-5;5;-.2]
- p = [0]
- q = [.5]
- T = 20
- N = 2000

*Note: In order to calculate N, T is divided by the given $\Delta t = 0.01$.

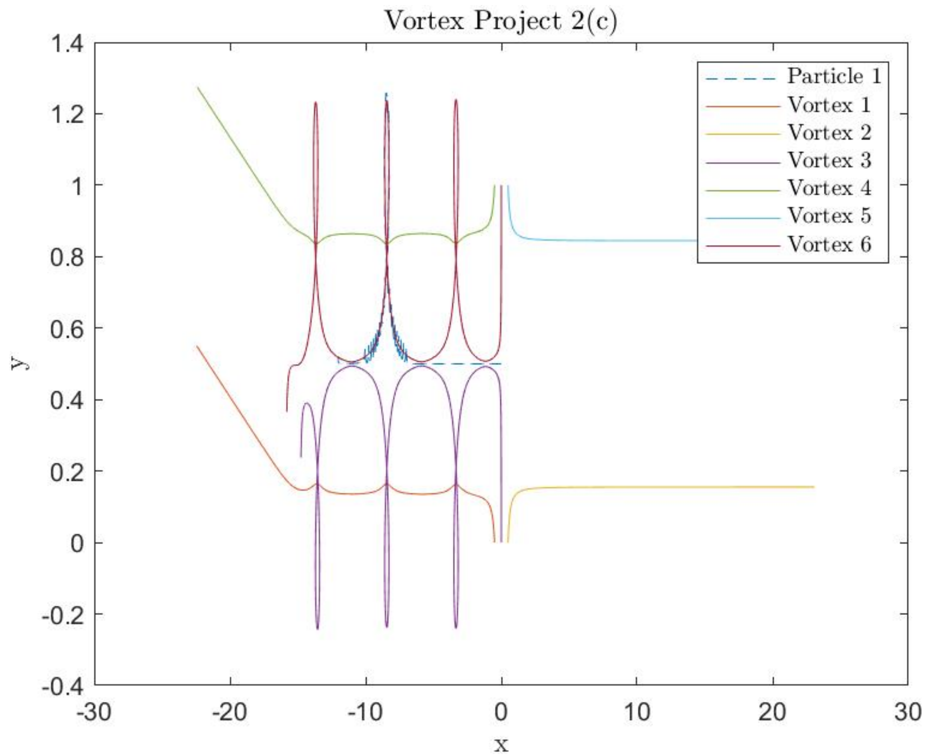

Figure 9: Numerical solution using 4^{th} order Runge-Kutta for when the vortices approach each other

In the beginning stages of the solution, the vortices seem horizontally symmetric and the particle moves horizontally to the left. However, later on, it seems as if the solution starts to shift and swerve and no longer have this symmetry. It seems at about $x = -8$, the numerical solution steers off track. After playing with the simulation and adjusting the size N, it seems that the value of $N = 2000$ seems to be too small to continue having that symmetry so it is set "off course" so to say. Another possibility is 4^{th} order Runge-Kutta is not apt enough for this specific situation.