Homework #1

ES/AM 346-0

Due Jan 25, 2021

Introduction to Navier-Stokes Equations for Fluid Flow

One could argue that the quintessential example of successful mathematical modeling is the Navier-Stokes Equations for Fluids. Starting from some basic assumptions, equations can be derived that predict the flow of fluids remarkably well, and have withstood the test of time (almost 200 years). The entire system will not be derived here, but we will derive a special case, for inviscid, incompressible flow. To begin, we assume that the fluid is a continuous medium, and not the sum of many individual atoms. Making this assumption immediately puts a framework about the type of length scales for which the theory will be relevant: large systems, like coffee in your coffee cup, and not very small systems, like water flowing in small spaces such as seepage in soils, or even smaller scales where the vibrations of individual atoms affect the local behavior of other molecules.

With that assumption in hand, suppose we have a small region in space where we wish to know the amount and the state of the fluid within that region, which we will call a *control volume*, V.

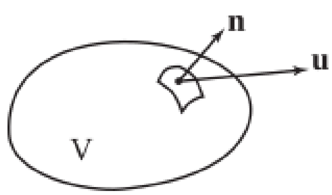

Fig: Sample control volume V with surface normal $\mathbf{n}$ and fluid velocity $\mathbf{u}$

Let us begin by developing a model for conservation of mass. The actual amount of mass in the control volume at a given point in time could be described as the sum of the density of the fluid at all the points inside V. If we define $\rho(t, \mathbf{x})$ to be the density function, then the mass at time t is given by

$$\text{mass} = \iiint_V \rho(t, \mathbf{x})\, dV.$$

For conservation of mass, we want to understand how the mass inside the control volume changes in time. Clearly, the change in time of the mass is then given by

$$\text{rate of change of mass} = \frac{d}{dt} \iiint_V \rho(t, \mathbf{x})\, dV = \iiint_V \frac{\partial \rho}{\partial t}(t, \mathbf{x})\, dV$$

We assume that mass isn't spontaneously created or destroyed, i.e. it is conserved, so the only way that the mass can change in V is by flowing in or out of the control volume. Let $\mathbf{u}(t, \mathbf{x})$ be the velocity of the fluid at time t, and position $\mathbf{x}$, then the mass flowing *out* of V is given by the surface integral of the normal component of the velocity over the surface of V:

$$\text{rate of mass loss} = \oiint_{\partial V} \rho \mathbf{u} \cdot \mathbf{n}\, dS.$$

Therefore, putting the two terms together, we have that

$$\iiint_V \frac{\partial \rho}{\partial t}\, dV = -\oiint_{\partial V} \rho \mathbf{u} \cdot \mathbf{n}\, dS.$$

If we apply the divergence theorem to the integral on the right, we get

$$\iiint_V \frac{\partial \rho}{\partial t}\, dV = -\iiint_V \nabla \cdot (\rho \mathbf{u})\, dV.$$

Combining the two integrals, we then have

$$\iiint_V \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u})\, dV = 0 \tag{1}$$

Note that the control volume V is an arbitrary region of space in the flow field, and can be taken to be any region. In particular, if there is a point in the flow where, say, the integrand is positive, then by continuity, there is a small region around that point must be positive. If we choose that small region to be our control volume, then the equation is violated (because the integrand is greater than zero everywhere in V, and hence equation (1) is violated. Therefore, it must be that the integrand is zero everywhere in the flow field. In other words, we have derived an equation for mass conservation of a fluid:

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \cdot \mathbf{u}) = 0.$$

If we further assume that the density of the fluid, ρ, is constant (a reasonable assumption for the fluid in your coffee mug, not so reasonable for the water in the ocean), then the mass conservation equation simplifies further:

$$\nabla \cdot \mathbf{u} = 0.$$

This is often referred to as the *incompressibility condition.*

A similar argument can be made to model conservation of momentum, i.e. the sum of $\rho \mathbf{u}$ inside V should be conserved. Problem 1 in the assigned problems concerns deriving the equation for conservation of momentum for an inviscid fluid:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p.$$

Point Vortex Method

Vortex methods are one example of a method used to model incompressible fluid flow. From problem 1, the momentum equation for inviscid incompressible fluid flow is given by

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p \tag{2}$$

where $\mathbf{u}$ is the fluid velocity and ρ is the fluid density. The incompressibility condition gives $\nabla \cdot \mathbf{u} = 0$, and we make the additional assumption that ρ is constant. Assume the flow is two-dimensional, i.e. $\mathbf{u} = (u, v, 0)^T$, and take the curl of Equation (2) to produce the vorticity equation

$$\frac{\partial \xi}{\partial t} + \mathbf{u} \cdot \nabla \xi + \xi \nabla \cdot \mathbf{u} = 0 \tag{3}$$

where $\xi = (\nabla \times \mathbf{u}) \cdot \hat{\mathbf{k}}$ is the fluid vorticity. The incompressibility condition, $\nabla \cdot \mathbf{u} = 0$ reduces Equation (3) to

$$\frac{\partial \xi}{\partial t} + (\mathbf{u} \cdot \nabla)\xi = 0. \tag{4}$$

Equation (4) tells us that the vorticity is transported passively by the fluid velocity $\mathbf{u}$, while at the same time, through the definition of ξ, the vorticity determines the velocity. Given ξ in the fluid domain, we can compute the fluid velocity $\mathbf{u} = (u, v, 0)^T$ by the following formulae:

$$u(x,y) = \iint \frac{-(y-y')}{2\pi((x-x')^2+(y-y')^2)}\xi(x',y',t)\,dx'\,dy' \tag{5}$$

$$v(x,y) = \iint \frac{(x-x')}{2\pi((x-x')^2+(y-y')^2)}\xi(x',y',t)\,dx'\,dy' \tag{6}$$

This formulation suggests the following simple approximation for the fluid flow. Suppose the vorticity is confined to single point sources $\xi = \xi_1 + \xi_2 + \cdots + \xi_n$ where

$$\xi_i(x,y,t) = \Gamma_i\delta(x - x_i(t), y - y_i(t)). \tag{7}$$

Here, $(x_i(t), y_i(t))$ is the location of the i^{th} vortex, and $\delta(x,y)$ is the two-dimensional δ function. Plugging this into equations (5), (6), we get

$$u(x,y,t) = \frac{1}{2\pi}\sum_{i=1}^{n}\frac{-\Gamma_i(y-y_i(t))}{(x-x_i(t))^2+(y-y_i(t))^2} \tag{8}$$

$$v(x,y,t) = \frac{1}{2\pi}\sum_{i=1}^{n}\frac{\Gamma_i(x-x_i(t))}{(x-x_i(t))^2+(y-y_i(t))^2}. \tag{9}$$

Equations (8), (9) are valid everywhere except for precisely at a vortex because of the denominator being zero. However, if we assume that a vortex does not cause its own motion (a reasonable assumption), then we can eliminate the zero denominator and describe the motion of the vortices in terms of their location at points (x_j, y_j) and strength Γ_j:

$$\frac{dx_j}{dt} = u(x_j,y_j,t) = \frac{1}{2\pi}\sum_{\substack{i=1\\ i\neq j}}^{n}\frac{-\Gamma_i(y_j-y_i)}{(x_j-x_i)^2+(y_j-y_i)^2} \tag{10}$$

$$\frac{dy_j}{dt} = v(x_j,y_j,t) = \frac{1}{2\pi}\sum_{\substack{i=1\\ i\neq j}}^{n}\frac{\Gamma_i(x_j-x_i)}{(x_j-x_i)^2+(y_j-y_i)^2}. \tag{11}$$

Similarly, any fluid particle located at $(x(t), y(t))$ and not at a vortex can also be tracked using equations (8), (9):

$$\frac{dx}{dt} = u(x,y,t) = \frac{1}{2\pi}\sum_{i=1}^{n}\frac{-\Gamma_i(y-y_i)}{(x-x_i)^2+(y-y_i)^2} \tag{12}$$

$$\frac{dy}{dt} = v(x,y,t) = \frac{1}{2\pi}\sum_{i=1}^{n}\frac{\Gamma_i(x-x_i)}{(x-x_i)^2+(y-y_i)^2}. \tag{13}$$

Given a set of n vortices ξ_i located at (x_i, y_i) with strength Γ_i for $i = 1,\ldots,n$, and a set of m fluid particles ϕ_i located at $(\hat{x}_i, \hat{y}_i)$ for $i = 1, \ldots, m$, Equations (10)–(13) describe a system of ordinary differential equations which describes incompressible, inviscid two-dimensional fluid flow in an infinite domain. In the problems below, you will solve this system numerically.

Written Exercises

1. Conservation of Momentum. For our simplicity, we will assume that there are no significant body forces, e.g. no gravity, and no viscosity (i.e. the fluid has no resistance to shearing, e.g. the opposite of syrup). The momentum of the fluid at a point is given by $\rho\mathbf{u}$. Fluid pressure, represented by p is an inward force on the surface of the control volume.

(a) Write down an integral that represents the total momentum of the fluid in a control volume V.

(b) Explain why the expression $\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})$ describes the outflow of momentum from the control volume at a point on the surface of the control volume.

(c) The pressure p is an inward force applied to the surface of a control volume. Explain why the expression $-\oiint_{\partial V} p\mathbf{n}\, dS$ will contribute to the change in momentum of the fluid inside the control volume. (Hint: what is the relationship between forces and momentum, Mr. Newton?)

(d) Assemble the pieces that are involved in the changes of momentum inside the control volume to derive the conservation of momentum equation:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p$$

where we assume the fluid is still incompressible.

2. Carry out the calculation to derive Equation (4) from Equation (2) including showing that $(\xi\hat{\mathbf{k}}\cdot\nabla)\mathbf{u} = 0$.

3. Region of Absolute Stability. The region of absolute stability for a method is important to understand when studying stability of numerical methods. In a nutshell, it is the set of values $\lambda\Delta t$ for which $|y_{n+1}| \leq |y_n|$, when solving the equation $y' = \lambda y$ and where λ may be complex valued. For example, Euler's method for this equation is

$$y_{n+1} = y_n + \lambda\Delta t y_n = (1+\lambda\Delta t)y_n$$

Therefore, the absolute stability condition is satisfied provided $|1+\lambda\Delta t| \leq 1$. Show that the region of absolute stability for Euler's method is a disk centered at $\lambda\Delta t = -1$ with radius 1 in the complex plane. Compute the region of absolute stability for Backward Euler: $y_{n+1} = y_n + \Delta t F(t_{n+1}, y_{n+1})$ and for Improved Euler: $y_{n+1} = y_n + \frac{\Delta t}{2}(F(t_n, y_n) + F(t_{n+1}, y_n + \Delta t F(t_n, y_n)))$. (Note: when computing the region of absolute stability, assume $\lambda\Delta t = \alpha + i\beta$ for some real values α, β, and then derive a real valued inequality that α and β must satisfy.)

4. Deriving Adams-Moulton 2. Derive the implicit multi-step method Adams-Moulton 2 by approximating the function $F(t, y(t))$ by a linear interpolant that passes through the points $F(t_n, y_n)$ and $F(t_{n+1}, y_{n+1})$. Show that the resulting method turns out to be the trapezoidal rule method we saw in class. (Extra credit: Derive Adams-Bashforth 3, where the approximating polynomial of F is a quadratic that interpolates F at the time points t_{n-2}, t_{n-1}, t_n.)

Vortex Project

1. Let's begin with a simple problem, a single vortex, and a single fluid particle. Suppose the vortex has strength Γ, and is located at $(0,0)$. Because it is the only vortex, the vortex will remain stationary through this example. The particle that moves is the fluid particle. Assume it begins at $(x(0), y(0)) = (x_0, y_0)$. In this example, the evolution equations for the particle simplify to

$$\frac{dx}{dt} = \frac{\Gamma}{2\pi}\frac{-y}{x^2+y^2} \tag{14}$$

$$\frac{dy}{dt} = \frac{\Gamma}{2\pi}\frac{x}{x^2+y^2} \tag{15}$$

(a) Show that the fluid particle moves in a circle around the vortex, and compute the time T required for the particle to make one full trip around the origin. (Hint: Show $x(t) = r_0\cos(\alpha t)$, $y(t) = r_0\sin(\alpha t)$ solves system (14) for some constants r_0, α, then determine the value of r_0, α.)

(b) For $\Gamma = 1$, $(x_0, y_0) = (1, 0)$, use Euler's method to solve system (14), (15) for time $0 \le t \le T$ where T is the value computed in problem 1a. Use $N = 50$, 100, 200, and 400 steps (i.e. $\delta t = T/50$, $T/100$, etc.) Plot all of the trajectories on one graph. Determine the error in the computed solution at time T by measuring

$$\text{error} = \|\text{exact} - \text{computed}\| = \sqrt{(1 - x_N)^2 + (0 - y_N)^2}. \tag{16}$$

(c) Repeat problem 1b using the fourth-order Runge-Kutta method.

Note: Your Runge-Kutta method should be written in the form of a function with the following syntax:

```
function rk_yourname(N, T, x, y, g, p, q)
```

where `N` is the number of time steps, `T` is the terminal time, and `x`, `y`, `g`, `p`, and `q` are column vectors as described below. A listing of your program should be included in your submitted assignment, and the corresponding Matlab .m file must be emailed to me.

2. In this problem, we will put the vortices in motion. Use Runge-Kutta 4 to solve the full system (10–13) for each of the following cases. In each case, plot the trajectories of all vortices and particles on the same graph.

   To assist you in this problem, I have written the derivative function as a Matlab function `F`. Suppose you have m vortices located at (x_i, y_i), with strength g_i, and also n particles located at the points (p_i, q_i). Define $\mathbf{x} = (x_1, x_2, \ldots, x_m)^T$, and similarly define $\mathbf{y}$, $\mathbf{g}$, $\mathbf{p}$, and $\mathbf{q}$. Then the full system can be written as

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}_x(\mathbf{x}, \mathbf{y}, \mathbf{g}, \mathbf{p}, \mathbf{q}) \tag{17}$$

$$\frac{d\mathbf{y}}{dt} = \mathbf{F}_y(\mathbf{x}, \mathbf{y}, \mathbf{g}, \mathbf{p}, \mathbf{q}) \tag{18}$$

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}_p(\mathbf{x}, \mathbf{y}, \mathbf{g}, \mathbf{p}, \mathbf{q}) \tag{19}$$

$$\frac{d\mathbf{q}}{dt} = \mathbf{F}_q(\mathbf{x}, \mathbf{y}, \mathbf{g}, \mathbf{p}, \mathbf{q}) \tag{20}$$

   The function I've provided does all four of these equations in one line invoked from Matlab this way:

```
[Fx,Fy,Fp,Fq] = F(x,y,g,p,q)
```

   Thus, Euler's method using this function for problem 2(a) below could be written as

```
x=[0;0.4;-0.2;-0.2];
y=[0;0;0.3464;-0.3464];
g=[-1;-0.667;-0.667;-0.667];
p=[1];
q=[0];
dt=0.05;
num_steps=20/dt;
for k=1:num_steps
  [Fx,Fy,Fp,Fq] = F(x(:,k),y(:,k),g,p(:,k),q(:,k));
  x(:,k+1) = x(:,k) + dt*Fx;
  y(:,k+1) = y(:,k) + dt*Fy;
  p(:,k+1) = p(:,k) + dt*Fp;
  q(:,k+1) = q(:,k) + dt*Fq;
end
figure(1);
hold off
for k=1:4
  plot(x(k,:),y(k,:),'b-');
  hold on
end
for k=1:1
  plot(p(k,:),q(k,:),'r-');
end
```

This program stores all of the point locations as columns in the matrices **x**, **y**, **p**, **q** and the resulting trajectories can be plotted as indicated at the end of the program where the vortices have blue lines and the particles have red lines.

(a) When several vortices are close together, from far away their net strength is approximately the sum of their strengths. To illustrate this, put a single vortex of strength 4 at the origin, and put three vortices of strength -1 near the origin. If we put a particle at $(1, 0)$ as in problem 1, then the fluid particle should roughly behave the same as in problem 1. Try this with $\Delta t = 0.05$, for $0 \le t \le 20$.

	$x_i(0)$	$y_i(0)$	Γ_i
vortex 1	0	0	-1
vortex 2	0.4	0	-0.667
vortex 3	-0.2	0.3464	-0.667
vortex 4	-0.2	-0.3464	-0.667
particle 1	1	0	

(b) Vortex pairs can often be observed that will propagate along indefinitely (since there's no viscosity to slow them down), for example, in the wake of a paddle when rowing a boat. In this example, a vortex pair is set up with several particles to show the flow field. For this, use $\Delta t = 0.05$, for $0 \le t \le 20$.

	$x_i(0)$	$y_i(0)$	Γ_i
vortex 1	0	0	1
vortex 2	1	0	-1
particle 1	-0.5	0.2	
particle 2	0	0.2	
particle 3	0.5	0.2	
particle 4	1	0.2	
particle 5	1.5	0.2	

(c) What happens when two vortices approach each other? In this case, two vortices are being forced together by other stronger vortices. Try this set of initial conditions and identify obvious problems with the computed results. Use $\Delta t = 0.01$, for $0 \leq t \leq 20$.

	$x_i(0)$	$y_i(0)$	Γ_i
vortex 1	-0.5	0	2
vortex 2	0.5	0	-2
vortex 3	0	0	0.2
vortex 4	-0.5	1	-2
vortex 5	0.5	1	2
vortex 6	0	1	-0.2
particle 1	0	0.5	