# ES_APPM 346 Project 4

NORTHWESTERN UNIVERSITY

NICOLAS GUERRA
MARCH 8, 2021

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# 1 Written Assignment

## 1.1 Problem One

**(a)** To modify the system for when pelicans have an additional constant food supply that is available beside the tilapia, let $f$ be the additional constant food. One must add $c * f$ to $\frac{dP}{dt}$. $c$ is the rate at which the number of pelicans are added to the population for every $f$ eaten. Multiplying these together gives the amount pelican population grows because of this additional food. $\frac{dS}{dt}$ and $\frac{dI}{dt}$ stay the same.

$$\begin{aligned}
\frac{dS}{dt} &= rS(1 - \frac{S+I}{k}) - \lambda SI \\
\frac{dI}{dt} &= \lambda SI - \mu I - \frac{mIP}{I+a} \\
\frac{dP}{dt} &= \frac{\theta IP}{I+a} - \delta P + \underline{c * f}
\end{aligned}$$

**(b)** To modify the system for when infected tilapia are able to reproduce at the same rate as healthy tilapia and all the offspring are also infected at birth, one must add a logistic model element to $\frac{dI}{dt}$. This way, the number of infected tilapia can increase, but there is still a carrying capacity limiting further growth in population. $\frac{dS}{dt}$ and $\frac{dP}{dt}$ stay the same.

$$\begin{aligned}
\frac{dS}{dt} &= rS(1 - \frac{S+I}{k}) - \lambda SI \\
\frac{dI}{dt} &= \lambda SI - \mu I - \frac{mIP}{I+a} + rI(1 - \frac{S+I}{k}) \\
\frac{dP}{dt} &= \frac{\theta IP}{I+a} - \delta P
\end{aligned}$$

**(c)** To modify the system for when infected tilapia are able to reproduce at the same rate as healthy tilapia and half the offspring are infected, and half are healthy, one must add half a logistic model element from part (b) to both $\frac{dS}{dt}$ and $\frac{dI}{dt}$. This way, the number of infected tilapia and healthy tilapia can increase, but there is still a carrying capacity limiting further growth in population. $\frac{dP}{dt}$ stays the same.

$$\begin{aligned}
\frac{dS}{dt} &= rS(1 - \frac{S+I}{k}) - \lambda SI + \frac{rI}{2}(1 - \frac{S+I}{k}) \\
\frac{dI}{dt} &= \lambda SI - \mu I - \frac{mIP}{I+a} + \frac{rI}{2}(1 - \frac{S+I}{k}) \\
\frac{dP}{dt} &= \frac{\theta IP}{I+a} - \delta P
\end{aligned}$$

## 1.2 Problem Two

To establish stability for the critical points, the Jacobian is calculated.

$$J = \begin{bmatrix} r - \frac{2rS}{k} - \frac{Ir}{k} - \lambda I & -\frac{Sr}{k} - \lambda S & 0 \\ \lambda I & \lambda S - \mu - \frac{mPa}{(I+a)^2} & \frac{-mI}{I+a} \\ 0 & \frac{\theta Pa}{(I+a)^2} & \frac{\theta I}{I+a} - \delta \end{bmatrix}$$

To see if the critical point $S = I = P = 0$ is stable, let's plug this point into the Jacobian. If there is a positive value in the eigenvalues, the critical point is unstable.

$$J = \begin{bmatrix} r & 0 & 0 \\ 0 & -\mu & 0 \\ 0 & 0 & -\delta \end{bmatrix}$$

Since this is has all zeros in the lower left triangle, the eigenvalues would be the elements of the diagonal. $\lambda^*$ will be used to notate the eigenvalues. Thus, the eigenvalues are:

$$\begin{aligned} \lambda* &= r \\ \lambda* &= -\mu \\ \lambda* &= -\delta \end{aligned}$$

Since $r > 0$, the critical point is unstable.

To see if the critical point $S = k \, and \, I = P = 0$ is stable, let's plug this point into the Jacobian. If there is a positive value in the eigenvalues, the critical point is unstable.

$$J = \begin{bmatrix} -r & -r - \lambda k & 0 \\ 0 & \lambda k - \mu & 0 \\ 0 & 0 & -\delta \end{bmatrix}$$

Since this is has all zeros in the lower left triangle, the eigenvalues would be the elements of the diagonal. $\lambda$' will be used to notate the eigenvalues. Thus, the eigenvalues are:

$$\begin{aligned} \lambda* &= -r \\ \lambda* &= \lambda k - \mu \\ \lambda* &= -\delta \end{aligned}$$

If $\lambda k < \mu$, the critical point is stable. If $\lambda k > \mu$, the critical is unstable.

### 1.3 Problem Three

To find the critical point where $S = 0$ and $I \neq 0$, set $\frac{dP}{dt} = 0$ and $\frac{dI}{dt} = 0$.

$$\begin{aligned} \frac{dP}{dt} &= 0 \\ 0 &= \frac{\theta IP}{I + a} - \delta P \\ &= (\frac{\theta I}{I + a} - \delta)P \\ &= P \\ P &= 0 \end{aligned}$$

$$
\begin{aligned}
\frac{dI}{dt} &= 0 \\
0 &= -\mu I - \frac{mIP}{I+a} \\
\frac{mIP}{I+a} &= -\mu I \\
mIP &= -\mu I^2 - \mu a I
\end{aligned}
$$

Plug in $P = 0$.

$$
\begin{aligned}
\mu I^2 + I(mp + \mu a) &= 0 \\
\mu I^2 + I\mu a &= 0 \\
I(\mu I + \mu a) &= 0 \\
\mu I + \mu a &= 0 \\
I &= -a
\end{aligned}
$$

In summary, the desired critical point is $S = 0$, $I = -a$, and $P = 0$. This critical point is not physically possible because this would mean that there would be a negative population of infected tilapia, when there is no such thing as negative population.

### 1.4 Problem Four

Below are the critical points found after using MATLAB.
**(a)**

$$
\begin{aligned}
S &= 75.6723 \\
I &= 73.2353 \\
P &= 6.4915
\end{aligned}
$$

To see if this point is stable, let's take a look at its eigenvalues.

$$
\begin{aligned}
\lambda* &= -0.1481 + 5.1068i \\
\lambda* &= -0.1481 - 5.1068i \\
\lambda* &= -0.0816 + 0.0000i
\end{aligned}
$$

Since all the real parts of the eigenvalues are less than zero, this means that this point is stable.

**(b)**

$$
\begin{aligned}
S &= 253.5294 \\
I &= 73.2353 \\
P &= 67.2396
\end{aligned}
$$

To see if this point is stable, let's take a look at its eigenvalues.

$$\lambda* = -8.6531 + 0.0000i$$
$$\lambda* = 1.6226 + 5.1639i$$
$$\lambda* = 1.6226 - 5.1639i$$

Since not all the real parts of the eigenvalues above are less than zero but rather some are positive, this point is unstable.

# 2 Programming Assignment

## 2.1 Problem One

**(a)**

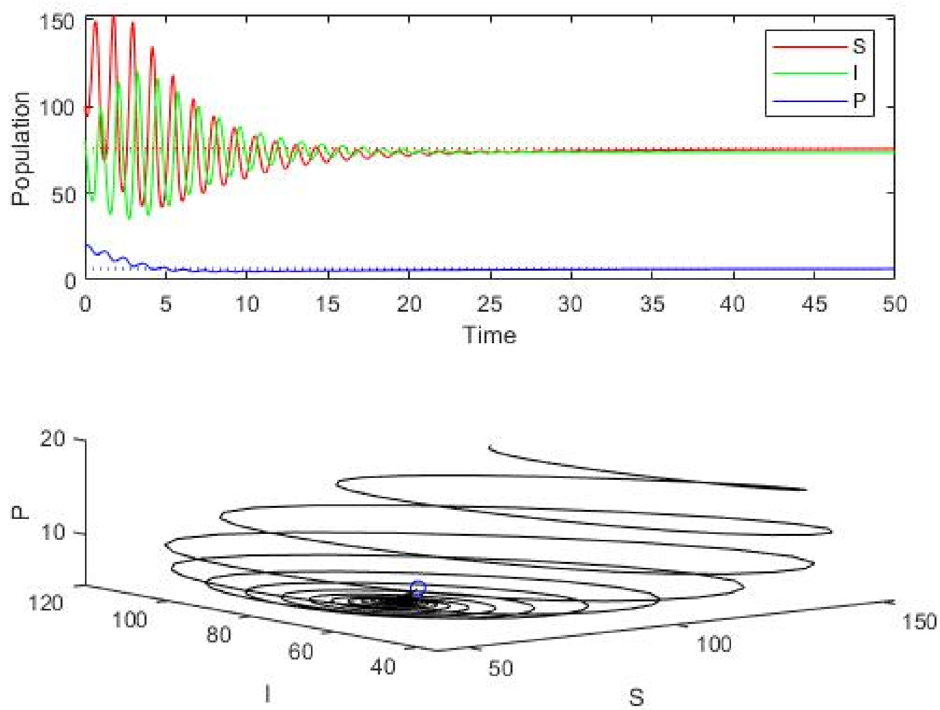


Figure 1: S, I, and P over time and phase plot in space when r=7 and N=1000

In the above plot, the pelican population seems to have gone towards zero while the susceptible and infected tilapia stabilize at about population = 75. The phase plot shows the a spiral downwards where the populations hit the critical point. Now let's see what happens when $N$ is increased.

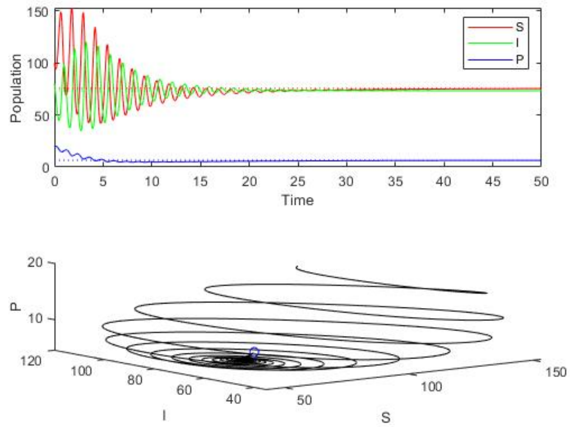


Figure 2: S, I, and P over time and phase plot in space when r=7 and N=2000

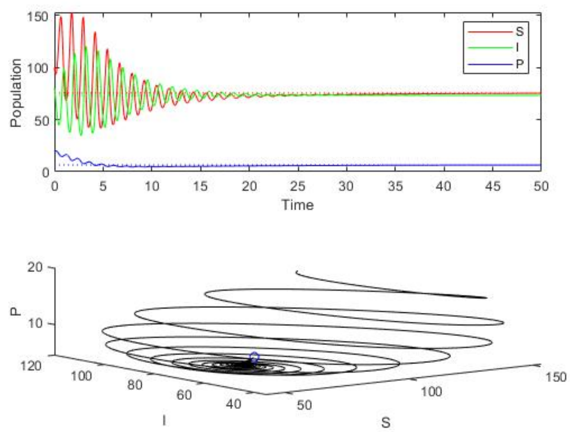


Figure 3: S, I, and P over time and phase plot in space when r=7 and N=4000

To the naked eye, Figure 1, 2, and 3 seem identical; however, one can take a closer look by assuming Figure 3 is the "actual" solution. Taking the final values, below are the errors. Let Figure 1 be $u_1$, Figure 2 be $u_2$, and Figure 3 be $u_3$.

Table 1: Error of $u_1$ and $u_2$

| | $u_1 - u_3$ | $u_2 - u_3$ | $\frac{u_1-u_3}{u_2-u_3}$ |
|---|---|---|---|
| $S$ | 2.9982e-05 | 2.5472e-06 | 11.7706 |
| $I$ | 2.4597e-05 | 8.3999e-07 | 29.2825 |
| $P$ | 3.2052e-05 | 5.8993e-07 | 54.3319 |

By increasing the number of steps by a factor of two, the error seems to decrease tremendously by a factor of 11, 29, and even 54 for S, I, and P respectively showing the error converges well.

**(b)**

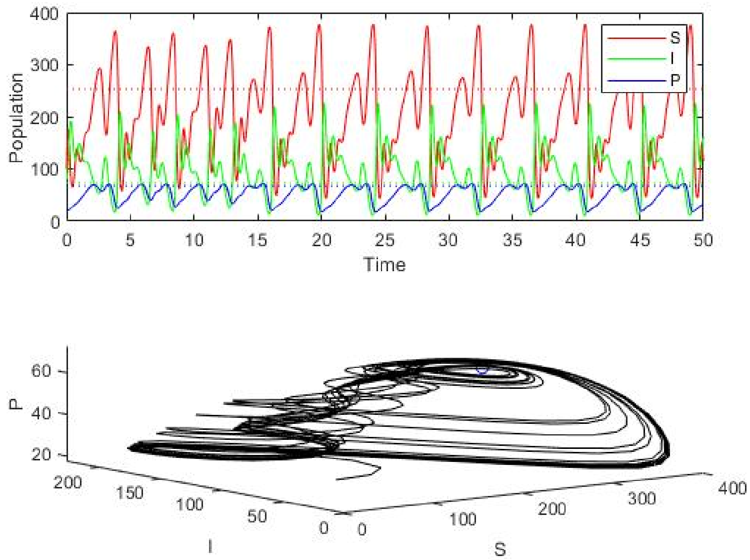


Figure 4: S, I, and P over time and phase plot in space when r=24 and N=1000

In the above plot, the pelican population's limit seems to be at about its critical point. The Susceptible tilapia populations appears to revolve around its critical point while the infected tilapia population seems to be most of the time above its critical point. The phase plot shows the populations spiraling upwards where it then hits its critical point. Let's see what happens when $N$ is increased.

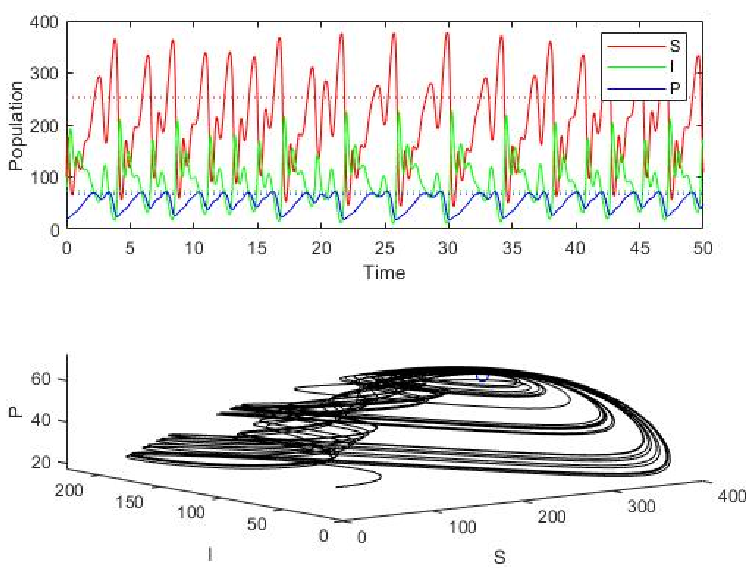


Figure 5: S, I, and P over time and phase plot in space when r=24 and N=4000

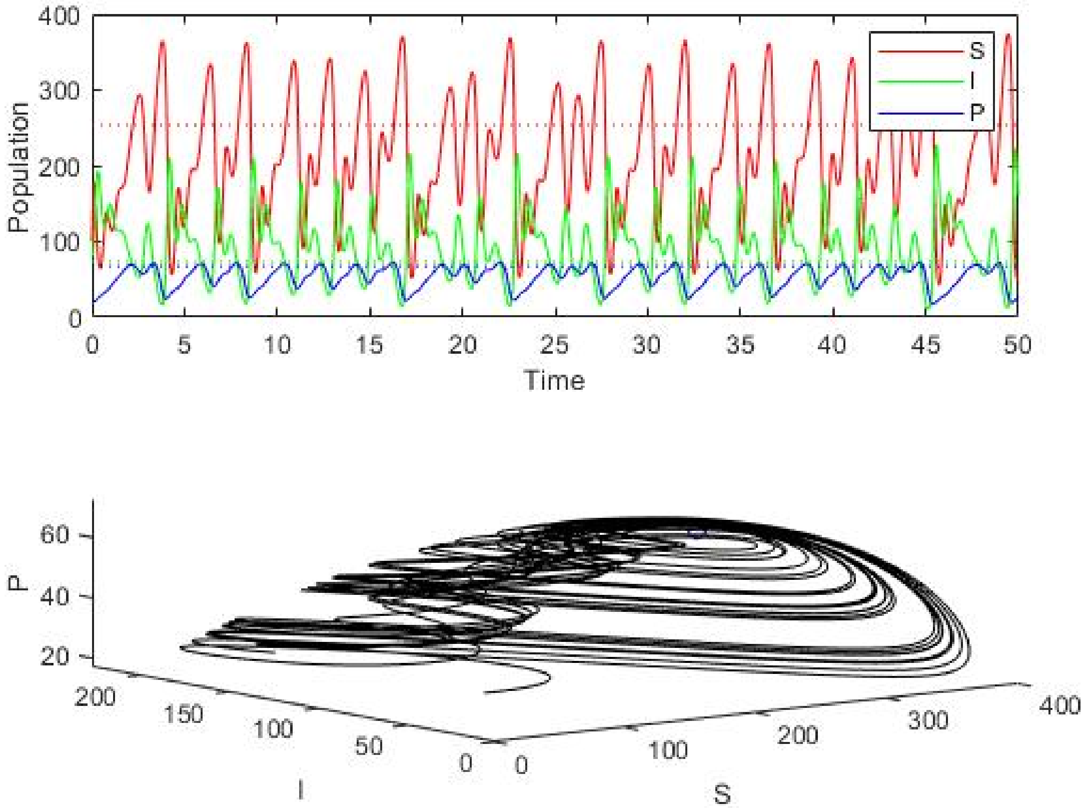


Figure 6: S, I, and P over time and phase plot in space when r=24 and N=4000

To the naked eye, Figure 4, 5, and 6 seem identical; however, one can take a closer look by assuming Figure 6 is the "actual" solution. Taking the final values, below are the errors. Let Figure 4 be $u_4$, Figure 5 be $u_5$, and Figure 6 be $u_6$.

Table 2: Error of $u_4$ and $u_5$

| | $u_4 - u_6$ | $u_5 - u_6$ | $\frac{u_4-u_6}{u_5-u_6}$ |
|---|---|---|---|
| $S$ | 67.1586 | 61.3599 | 1.0945 |
| $I$ | 2.2396 | 11.2638 | 0.1988 |
| $P$ | 7.3857 | 20.6697 | 0.3573 |

When r=24, it seems as though when $N$ is increased, the solution improves little to none and sometimes worsens. This may be because the parameter $r$ from the logistic element is much higher than from the previous three simulations making it more difficult for the simulation to improve as $N$ increases. It seems like the simulation is quite chaotic and, with $N = 4000$, it is hard to make the error converge.

### 2.2 Problem Two

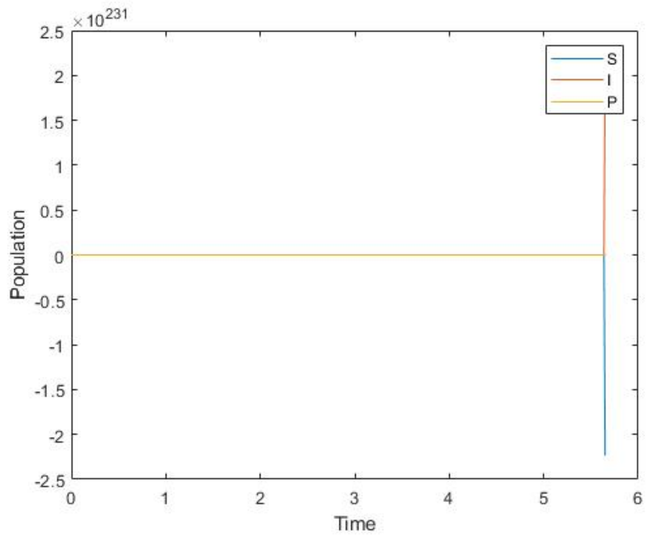


Figure 7: S, I, and P over time when r=7 and N=4000 using Euler-Maruyama

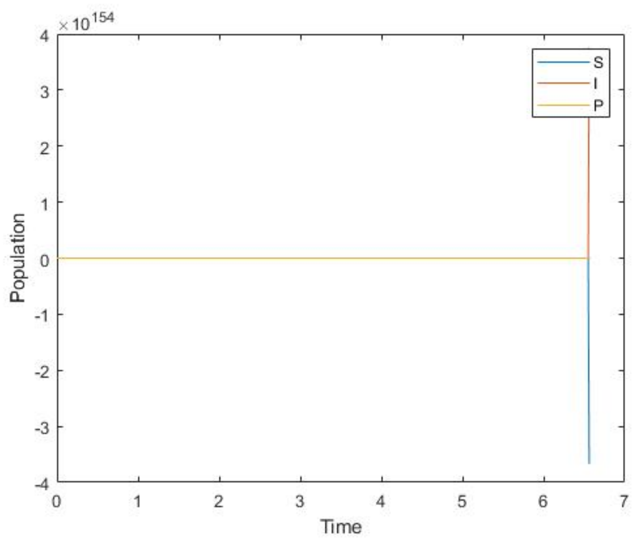


Figure 8: S, I, and P over time when r=24 and N=4000 using Euler-Maruyama

Looking at both figures, the graphs seem to have the same trends with the populations hovering around zero and then the infected tilapia explodes upwards while the susceptible tilapia shoots downward. In the stochastic case, there is a negative population that you do not see in the deterministic version. This is definitely non-physical because there is no such thing as a negative population in nature. This could be due to the fact that this simulation has a random number generator that can produce large negative values which throws off our simulation and makes it impractical. This didn't happen in the deterministic case because it does not have this added random element. In the next problem, this will be addressed by cutting down the time step in such a case.

## 2.3 Problem Three

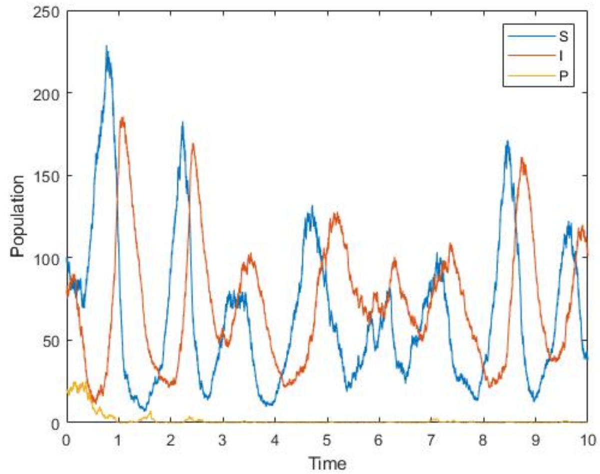


Figure 9: S, I, and P over time when r=7 and N=4000 using Euler-Maruyama with varying time steps

When comparing this stochastic simulation to its analogous deterministic simulation, they are extremely similar. Comparing between Figure 1 and Figure 9, it is easy to see that they both share the pelican decreasing initially and then hovering at zero. In both simulations, both tilapia populations start with a lot of volatility and then hover around a population of 75. The biggest difference are the jagged curves which were characterized by the randomness in the stochastic simulation.

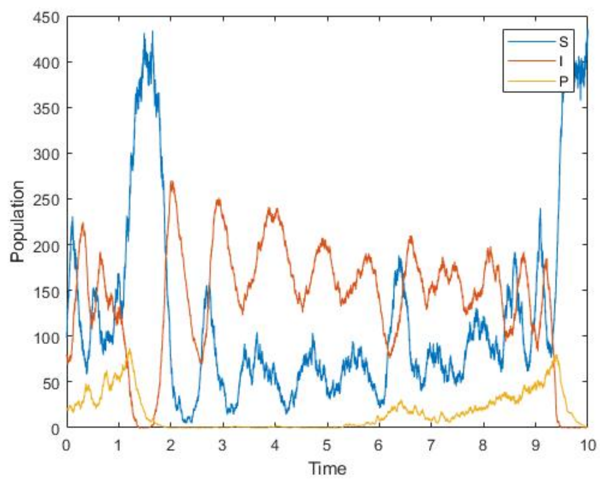


Figure 10: S, I, and P over time when r=24 and N=4000 using Euler-Maruyama with varying time steps

When comparing this stochastic simulation with its analogous deterministic version, they are again extremely similar. In both, the pelican population, increases steadily and once it hits its critical point, it crashes down to zero and starts over again. Also in both, the susceptible tilapia population has the largest peaks out of all the three populations with the infected tilapia population oscillating with a "swing" in between. Again, the biggest difference are the jagged curves which were characterized by the randomness in the stochastic simulation.

## 2.4 Problem Four

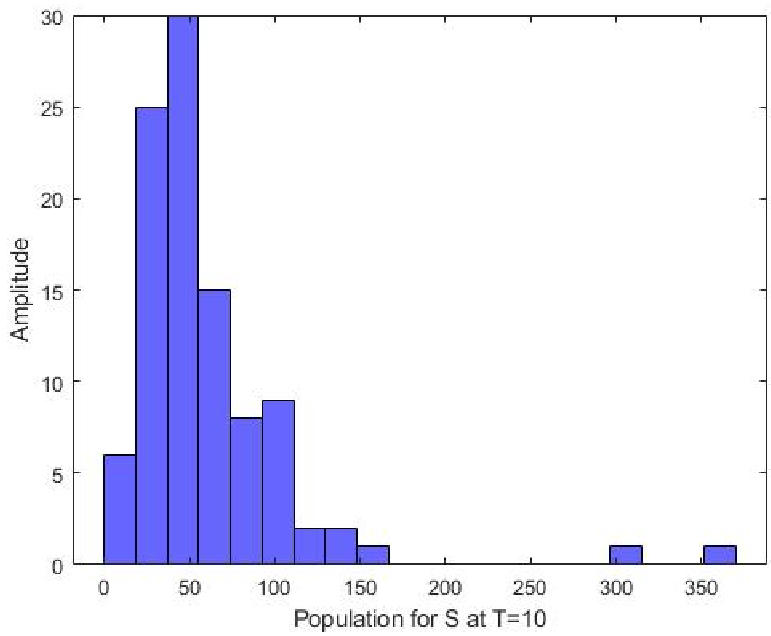


Figure 11: Histogram of S at T=10 when r=7, N=4000, and trials=100

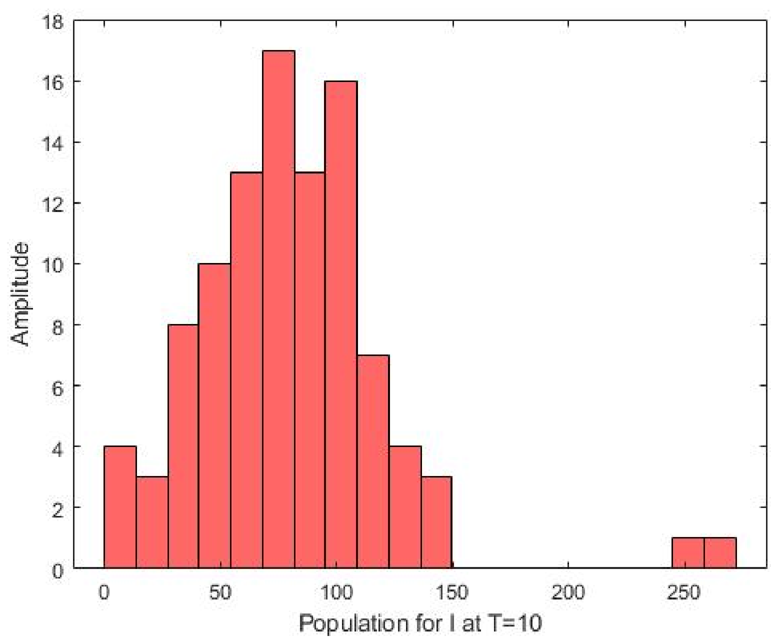


Figure 12: Histogram of I at T=10 when r=7, N=4000, and trials=100

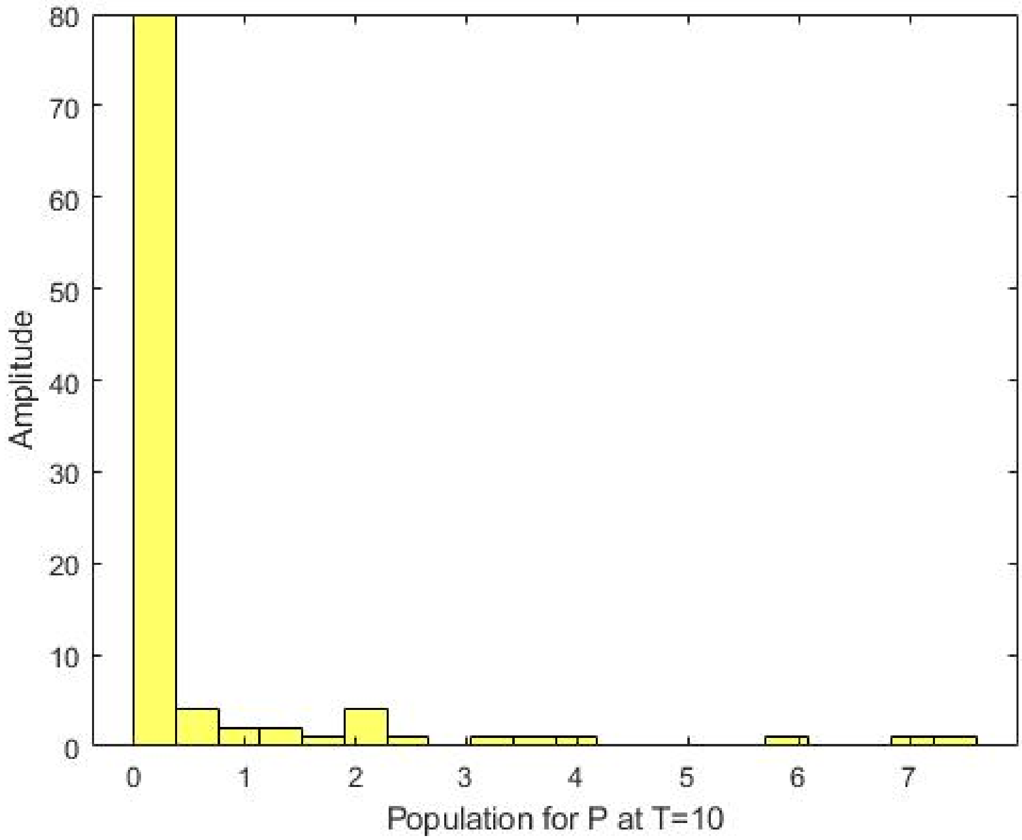


Figure 13: Histogram of P at T=10 when r=7, N=4000, and trials=100

Table 3: Mean and Variance of S, I, and P at T=10

| | Mean | Variance |
|---|---|---|
| $S$ | 60.5237 | 2.5960e+03 |
| $I$ | 79.4851 | 1.7431e+03 |
| $P$ | 0.5369 | 1.8666 |

After reviewing the above histograms and studying the mean and the variance of $S$, $I$, and $P$, one can expect that at T=10 the population of susceptible tilapia is about 61, the population of infected tilapia is expected to be roughly 79, and the population of pelicans is expected to be about 1. Looking at the histogram from S, the distribution seems to be skewed right but not as skewed right as the Figure 13. It seems that many times, the pelican population at T=10 has basically disappeared. Since the variance for Figure 13 is smaller, the spread of the distribution is a lot less. Looking at Figure 11 and Figure 12, the variance is a lot bigger meaning a wider spread across many values.

Using a stochastic simulation gives someone a good idea about where a certain element may end up. However, this does not give one explicit answer but rather gives a distribution of many paths with different probabilities of that path occurring.