Nicolas Kress

# All Barchi No Bitey

February 4, 2026

Springer Nature

# Contents

**1 Introduction** . . . . . . . . . . 1

**2 The $\mathbb{Z}_p$ favoritism tree** . . . . . . . . . . 5
2.1 Definition of infinite trees . . . . . . . . . . 5
2.2 Preliminary results on trees . . . . . . . . . . 7
2.3 The $\mathbb{Z}_p$ tree . . . . . . . . . . 11

**3 The Barchi set $\mathbb{T}_p$** . . . . . . . . . . 13
3.1 The Barchi set $\mathbb{T}_p$ . . . . . . . . . . 14
3.2 First properties of $\mathbb{T}_p$ . . . . . . . . . . 15
3.2.1 Partial order . . . . . . . . . . 15
3.2.2 Metric over $\mathbb{T}_p$ . . . . . . . . . . 16
3.3 Topology over $\mathbb{T}_p$ . . . . . . . . . . 18
3.3.1 Results on convergence . . . . . . . . . . 19
3.3.2 Completeness and compactness . . . . . . . . . . 21
3.4 Additional properties of the Barchi set . . . . . . . . . . 24
3.4.1 Down-complete poset (dcpo) structure . . . . . . . . . . 24
3.4.2 Algebraic properties . . . . . . . . . . 26
3.5 Sub-trees of $\mathbb{T}_p$ and $p$-bounded trees . . . . . . . . . . 28
3.5.1 Sub-trees and canonical isomorphism . . . . . . . . . . 29
3.5.2 Results over $p$-bounded trees . . . . . . . . . . 33
3.6 First applications . . . . . . . . . . 35
3.6.1 Polynomials and Hensel's Lemma . . . . . . . . . . 35
3.6.2 Compact languages . . . . . . . . . . 38

**References** . . . . . . . . . . 41

# Chapter 1
# Introduction

In this paper, we offer a unification of uniformly-bounded trees. A uniformly-bounded tree is a tree in which each node has no more than a constant $M$ number of children. By taking any prime number $p$ greater than $M$, we obtain a $p$-bounded tree. A tree in which each node has exactly $p$ children, which we will call the $\mathbb{Z}_p$ favoritism tree, has the property of including (modulo node values) every other $p$-bounded tree.

Imagine you are at a roundabout with exactly $p$ exits. Any of these exits will lead you to another roundabout with exactly $p$ exits (not including the path taken to this roundabout). Then driving along this infinite series of roundabouts defines a sequence $(a_k)_{k\geq 0}$ with values in $\{0, ..., p-1\}$, such that $a_k$ is the number of the exit taken at the $k$-th visited roundabout. Results on the $p$-adic integer ring $\mathbb{Z}_p$ allow us to see $(a_k)_{k\geq 0}$ as a $p$-adic number by considering $\sum_{k=0}^{\infty} a_k p^k$, and conversely. As such, any path in these infinite roundabouts can be identified to a unique $p$-adic integer. In this analogy, the roundabouts are the nodes of the $\mathbb{Z}_p$ favoritism tree, and we clearly see some link between node paths in $\mathbb{Z}_p$ and $p$-adic numbers. Furthermore, if we consider that the car can have only finite amounts of gas, then the path taken can be seen as a prefix of a $p$-adic number. Using this, we obtain a surjection from $\bar{\mathbb{N}} \times \mathbb{Z}_p$ to the set of paths of the $\mathbb{Z}_p$ favoritism tree, where $\bar{\mathbb{N}}$ is the set $\{0, 1, ..., \infty\}$ of non-negative integers and $\infty$. The quotient set for this surjection defines the Barchi set $\mathbb{T}_p$, which is the object of study in this paper.

Study of the Barchi set is justified by the previous remark stating that any $p$-bounded tree can be seen (at least, structurally) as a sub-tree of the $\mathbb{Z}_p$ favoritism tree, and we will show that any $p$-bounded tree can be structurally identified to a sub-tree of the Barchi set $\mathbb{T}_p$. We will define a natural metric over $\mathbb{T}_p$ which will make sub-trees compact, and we will be able to deduce properties of any $p$-bounded tree.

The Barchi set has various properties across multiple domains of mathematics. To name a few of these properties, it is a partially-ordered compact metric space, a dcpo, and we can define a sum and a multiplication over the Barchi set. While these operators do not grant a ring structure to the Barchi set, we will see that the restriction of these operators to certain subsets produces rings isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$ or $\mathbb{Z}_p$.

Applications of the Barchi set are not lacking. Trees can be found everywhere, and they are most often uniformly-bounded, meaning that they can be seen as sub-trees of $\mathbb{T}_p$ for some prime $p$. One application we will discuss is in the matter of formal languages, and we will manage to formally translate the compactness of the Barchi set to any language over a finite alphabet.

In the first chapter we will formally introduce the $\mathbb{Z}_p$ favoritism tree. Notably, we will introduce the notion of "favoritism" tree, which forces a total order on the children of any node, such that we can properly identify paths of this tree to a sequence of $\{0, ..., p-1\}$. We take the time to properly define trees, sub-trees, and node and structural paths in order to leave no confusion. The main definition is the one of favoritism tree, and the main result is the existence of a "generated sub-tree", i.e the smallest sub-tree containing a certain set of paths. The rest can be skimmed to the reader's desire.

In the second chapter, we introduce the Barchi set and explore different properties (topological, algebraic, and discrete). This is the heart of the paper, and we show in section 3.5 that $p$-bounded trees can be identified to sub-trees of the Barchi set, validating the study of the Barchi set.

It is recommended to be somewhat comfortable with the notion of $p$-adic integers before reading this paper. The most notable result we will be using is the *series representation* theorem:

**Theorem 1** *For any p-adic integer x, there exists a unique sequence* $(a_k)_{k\geq 0}$ *with values in* $\{0, ..., p-1\}$ *such that* $x = \sum_{k=0}^{\infty} a_k p^k$. *Subsequently:*

$$\mathbb{Z}_p = \left\{ \sum_{k=0}^{\infty} a_k p^k : \forall k \geq 0,\ \ a_k \in \{0, ..., p-1\} \right\}$$

We will often use the term *series representation* of a $p$-adic integer in order to refer to the associated sequence of *digits* $(a_k)_{k\geq 0}$. It is also recommended to have seen the definition of the $p$-adic absolute value and its ultrametric properties.

The reader may find themselves wondering at some point or another why we work with $p$-adic numbers for the construction of the Barchi set, rather than real numbers. It is well-known that any real number between 0 and 1 can be uniquely written as some infinite series $\sum_{k=1}^{\infty} a_k p^{-k}$, similarly to any $p$-adic number. As it happens, I have painfully attempted to make this work. While a lot of properties remain true and still function well, namely any "structural" property (tree structures, dcpo, compact metric space), the algebraic properties that we derive with ease from $\mathbb{Z}_p$ are far harder to obtain with real numbers. Notably, as we will discuss later, the sum over $\mathbb{T}_p$ is very easy to inherit from $\mathbb{Z}_p$, but this is mostly due to the ultrametric character of the $p$-adic absolute value. So we have decided to leave the real numbers behind, although it can be an interesting topic to explore in the future. Notably, if a sum can be properly defined for a "real Barchi set", then we could possibly derive some equivalent to Hensel's lemma for real polynomials.

*NOTE: The version of this paper you are reading is a draft. It is not suitable for distribution, and is only intended to be seen by curious eyes who seek to understand what I have been babbling on about for the past few months. As a result, a lot of commentary is missing, and most sections are nothing more than a succession of definitions, lemmas, and propositions, with little to no supporting text. Though most of the following statements have been proofread, some errors may still be present. If you find any, do not hesitate to let me know at nicolas.kress@ens-paris-saclay.fr*

*You have been warned.*

# Chapter 2
# The $\mathbb{Z}_p$ favoritism tree

As discussed in the introduction, we will be properly introducing the definition of trees, which we will allow to be infinite. In the analogy of infinite roundabouts, we are only allowed to identify a path to a sequence $(a_k)_{k\geq 0}$ when the exits are labeled $0, ..., p-1$. To guarantee this, we introduce the notion of "favoritism trees", which have a "favoritism function" imposing some numbering of the children of a node. We will additionally show some first properties that will be useful in the following chapters.

## 2.1 Definition of infinite trees

**Definition 1 (Graph)** A (directed) **graph** $G$ is a pair $(V, E)$, where $V$ is a non empty set of **nodes** and $E \subseteq \{(x, y) : x \neq y \in V\}$ is a set of **edges**.
The binary relation $(x, y) \in E$ over $V \times V$ is denoted $x \to y$.

**Definition 2 (Paths)** In a graph $G = (V, E)$, a **path** from a node $x$ to a node $y$ is a finite sequence of nodes $x = x_0, x_1, ..., x_n = y$, where for all $k \in \{0, \ldots, n-1\}$, $x_k \to x_{k+1}$. The integer $n \geq 0$ is called the **length** of the path. A path is denoted $x_0 \to x_1 \to \ ... \to x_n$.

**Definition 3 (Tree)** A **tree** $t$ is a triplet $(V, E, \mathcal{R}_t)$ where:

- $\mathcal{R}_t \in V$ is called the **root**.
- $(V, E)$ is a graph.
- For all $x \in V$, there exists a unique path from $\mathcal{R}_t$ to $x$ in $(V, E)$.

**Definition 4** Let $t = (V, E, \mathcal{R}_t)$ be a tree. We define the following operators for all $x \in V$:

- $\mathbf{ch}_t\,(x) := \{y : x \to y\}$ is the set of **children** of $x$ in $t$.
- When $x \neq \mathcal{R}_t$, $\mathbf{p}\,(x) \in V$ is the **parent** of $x$, defined as $x_{n-1}$ in the unique path $\mathcal{R}_t \to \ ... \ \to x_{n-1} \to x_n = x$.

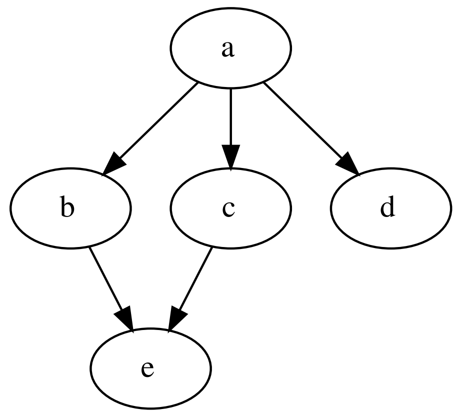



**Fig. 2.1** An example of a graph, per definition 1. Only directed graphs will be considered in this paper.

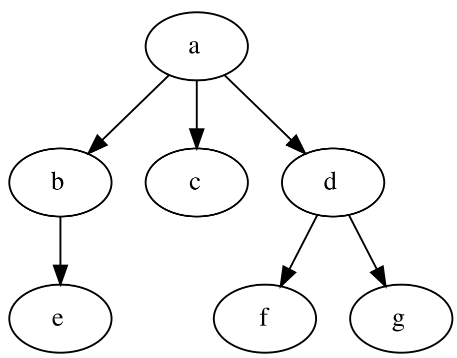



**Fig. 2.2** An example of a tree. Here, the root is $a$.

- $\mathbf{w}_t(x) \in \mathbb{N} \cup \{\infty\}$ is the **weight** of $x$, defined as the number of elements of $\mathbf{ch}_t(x)$.

When the context is clear, we will write $\mathbf{ch}(x)$ and $\mathbf{w}(x)$ instead of $\mathbf{ch}_t(x)$ and $\mathbf{w}_t(x)$ respectively.

*Example 1* In figure 2.2, $\mathbf{ch}(d) = \{g, h\}$, and $\mathbf{p}(c) = a$. For the weight function, $\mathbf{w}(c) = 0$ and $\mathbf{w}(b) = 1$.

**Definition 5 (Favoritism tree)** A **favoritism tree** is a quadruplet $t = (V, E, \mathcal{R}, \triangleleft)$ where:

- $(V, E, \mathcal{R})$ is a tree
- For all $x \in V$, $\mathbf{ch}(x)$ is finite.
- $\triangleleft : x \in V \mapsto \triangleleft_x$ associates to $x$ a total order over $\mathbf{ch}(x)$.
- For all $x \in V$, $\mathbf{ch}(x)$ has a smallest element for $\triangleleft_x$.

*Remark 1* The above definition is the first that diverges from the previously "classic" definitions. The conditions imposed on $\mathbf{ch}(x)$ allows us to consider the $n$-th smallest element of this set, which we denote $\mathbf{ch}(x).(n)$.

**Definition 6 (Node paths)** If $t = (V, E, \mathcal{R}_t, \triangleleft)$ is a favoritism tree, then we define the **set of node paths** of $t$, denoted $\mathrm{paths}(t)$, as the set of all node sequences $(n_k)_{N \geq k \geq 0}$ such that:

- $N \in \mathbb{N}^* \cup \{\infty\}$
- $n_0 = \mathcal{R}_t$
- $\forall k \in \{0, ..., N-1\}, \quad n_k = \mathbf{p}(n_{k+1})$

We furthermore define:

- $\mathrm{paths}_\infty(t) \subseteq \mathrm{paths}(t)$ is the set of infinite node paths (i.e $N = \infty$ in the above definition).
- $\mathrm{paths}_\mathrm{f}(t) \subseteq \mathrm{paths}(t)$ is the set of finite node paths (i.e $N < \infty$ in the above definition).

We trivially have the following partition:

$$\mathrm{paths}(t) = \mathrm{paths}_\infty(t) \sqcup \mathrm{paths}_\mathrm{f}(t)$$

*Remark 2* In the previous definition, we do not exploit the fact that $t$ is a favoritism tree, in contrast with a "classic" tree. In fact, the above notations might sometimes be used for trees, but the upcoming notion of **structural paths** requires the structure of a favoritism tree.

**Definition 7 (Structural path)** Let $t = (V, E, \mathcal{R}_t, \triangleleft)$ be a favoritism tree. For any node path $(n_k)_{N \geq k \geq 0} \in \mathrm{paths}(t)$, we define the associated **structural path** $(a_k)_{N-1 \geq k \geq 0}$ by:

$$\forall k \in \{0, ..., N-1\}, \ n_{k+1} = \mathbf{ch}(n_k).(a_k)$$

The set of all structural paths of $t$ is denoted $t^{\mathbb{N}}$.

*Example 2* In figure 2.2, the directional path associated to the node path $a, d, h$ is $2, 1$.

**Lemma 1** *In a favoritism tree $f = (V, E, \mathcal{R}_f, \triangleleft)$, there is a bijection between the set of node paths and the set of directional paths.*

*Proof.* The definition of a directional path gives a surjection from $\mathrm{paths}(f)$ to $f^{\mathbb{N}}$. Furthermore, we trivially have that two distinct node paths have distinct associated structural paths. □

## 2.2 Preliminary results on trees

**Definition 8** If $f = (V_f, E_f, \mathcal{R}_f, \triangleleft)$ and $g = (V_g, E_g, \mathcal{R}_g, \triangleleft)$ are two favoritism trees, then $g$ is called a:

- **structural sub-tree** of $f$ when $g^{\mathbb{N}} \subseteq f^{\mathbb{N}}$.
- **sub-tree** of $f$ when $V_g \subseteq V_f$, $E_g \subseteq E_f$, and $\mathcal{R}_f = \mathcal{R}_g$.

*Example 3* The figure 2.3 shows a tree $t_1$ for which the tree $t_2$ of figure 2.2 is a structural sub-tree. Note that the labeling of nodes doesn't matter in the sub-tree relationship. $t_2$ is not a sub-tree of $t_1$, because $a, d, g$ is a node path in $t_2$, but not in $t_1$.

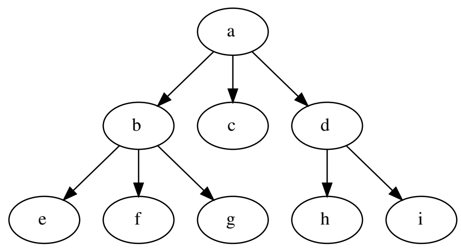



**Fig. 2.3** An example of a tree, for which the tree figure 2.2 is a structural sub-tree.

*Remark 3* A sub-tree is a structural sub-tree.

**Lemma 2** *If* $f = (V_f, E_f, \mathcal{R}_f, \lhd)$ *and* $g = (V_g, E_g, \mathcal{R}_g, \lhd)$ *are two favoritism trees, then* $g$ *is a sub-tree of* $f$ *if and only if* $\text{paths}\,(g) \subseteq \text{paths}\,(f)$.

*Proof.* Suppose $g$ is a sub-tree of $f$. Because all node paths of $g$ begin with $\mathcal{R}_g$, we must have $\mathcal{R}_g = \mathcal{R}_f$. Furthermore, if $x$ is a node of $g$ different from $\mathcal{R}_g$, then there exists a finite node path $(n_k)_{K \geq k \geq 0}$ from the root $\mathcal{R}_g$ to $x$ with $K \geq 1$. Because this node path also exists in $f$, we have $x \in V_f$. Furthermore, we have $n_{K-1} \rightarrow_g n_K = x$, meaning that $n_{K-1} \rightarrow_f n_K = x$, and consequently $(n_{K-1}, n_K) \in E_f$. This being true for all $x \in V_g \setminus \{\mathcal{R}_g\}$, we have $V_g \subseteq V_f$ and $E_g \subseteq E_f$. The converse is immediate. $\square$

In the following proposition, we will talk of a "smallest" tree $(V, E, \mathcal{R})$ verifying a property $\mathcal{P}$. What we mean by "smallest", is that for any tree $(V', E', \mathcal{R})$ (where $V$ and $V'$ are both subsets of a fixed set of nodes) that also verifies $\mathcal{P}$, we have $V \subseteq V'$ and $E \subseteq E'$.

**Proposition 1 (Generated tree)** *If* $V$ *is a set of nodes, if* $\mathcal{R}$ *is a node of* $V$, *and if* $S$ *is a non-empty set of node sequences* $(n_k)_{N \geq k \geq 0}$ *(of elements of* $V$*) that verify the following properties:*

- $n_0 = \mathcal{R}$ *(root start)*
- *For all* $i, j \in \{0, ..., N\}$, $(i \neq j) \implies (n_i \neq n_j)$ *(distinct nodes)*
- *For all* $i \in \{0, ..., N-1\}$, *and for all* $(m_j)_{K \geq j \geq 0} \in S$, *(rank respect)* *if there exists* $j \in \{0, ..., K\}$ *for which* $n_i = m_j$, *then* $i = j$ *and* $n_{i-1} = m_{j-1}$ *if* $i > 1$.

*then there exists* $V' \subseteq V$ *and* $E \subseteq \{(x,y) : x \neq y \in V'\}$ *for which* $t := (V', E, \mathcal{R})$ *is a tree such that* $\text{paths}(t) \supseteq S$.
*There exists a smallest such tree is called the* ***generated tree*** *of* $S$ *and is denoted* $\langle S \rangle$.

*Proof.* We can begin by defining

$$V' := \bigcup_{(n_k)_{N \geq k \geq 0} \in S} \{n_k : 0 \leq k \leq N\}$$

which is the set of all nodes passed over by at least one path. Because of the *(root start)* property, $\mathcal{R} \in V'$. We then define:

$$E := \bigcup_{(n_k)_{N \geq k \geq 0} \in S} \big\{(n_k, n_{k+1}) : k \in \{0, ..., N-1\}\big\}$$

By property *(distinct nodes)*, $E$ is a subset of $\{(x,y) : x, y \in V',\ x \neq y\}$. At this point, we've created a graph $G := (V', E)$. Let's show that $(V', E, \mathcal{R})$ is a tree: consider $x \in V'$ distinct from $\mathcal{R}$. By definition of $V'$, there exists $(n_k)_{N \geq k \geq 0} \in S$ and $K \in \{0, ..., N\}$ such that $x = n_K$. By property *(root start)*, we have $n_0 = \mathcal{R}$, and by definition of $E$:

$$\mathcal{R} = n_0 \to n_1 \to\ \ldots\ \to n_K = x$$

is a path of $G$. So a path from the root to $x$ exists. Let's show that it is unique: let's consider a path $\mathcal{R} = x_0 \to\ \ldots\ \to x_l = x$ in $G$. Without loss of generality, we can suppose $l \leq K$. By definition of $E$, there exists $(m^l_k)_{N_l \geq k \geq 0} \in S$ and $j \in \{1, ..., N_l\}$ such that $m^l_j = x_l = x$ and $m^l_{j-1} = x_{l-1}$. By property *(rank respect)*, we have $j = K$ and $m^l_{K-1} = n_{K-1}$, meaning $x_{l-1} = n_{K-1}$.
We can recursively iterate this process by taking $(m^i_k) \in S$ for all $i \in \{1, ..., l-1\}$, and we get that $n_{K-i} = x_{l-i}$. Finally, for $i = l$, we can consider $(m^0_k)_{N_0 \geq k \geq 0} \in S$ and $j \in \{1, ..., N_0\}$ such that $m^0_j = x_1$ and $m^0_{j-1} = x_0 = \mathcal{R}$. Consequently, by property *(rank respect)* applied to $m^0_j = x_1 = n_{K-l+1}$, we have $j = K - l + 1$ and $n_{K-l} = m^0_{j-1} = \mathcal{R}$. By property *(distinct nodes)*, we conclude that $K = l$, and we have shown that $n_i = x_i$ for all $i \in \{0, ..., K\}$. Consequently, $(V', E, \mathcal{R})$ is a tree.
The inclusion $S \subseteq \text{paths}(t)$ is immediate by the definition of $E$. Lastly, let's show that this tree is the smallest such tree for the sub-tree relation. Let's consider another tree $t' := (V'', E', \mathcal{R}_{t'})$ that satisfies $\text{paths}(t') \supseteq S$. Then every path of $S$ is a path of $t'$: as a result, all nodes passed over by sequences of $S$ are elements of $V''$. So, $V' \subseteq V''$. Furthermore, if we take $(n_k)_{N \geq k \geq 0} \in S$ and $k \in \{0, ..., N-1\}$, then we know that $(n_k, n_{k+1}) \in E'$, because $(n_k)$ is a path of $t'$. As a result, we have $E \subseteq E'$. Lastly, because $(n_k)$ is a path of $t'$, we necessarily have $\mathcal{R}_{t'} = \mathcal{R}$.
So $t$ is a sub-tree of $t'$, and we conclude that $\langle S \rangle := t$ is well defined. $\square$

*Remark 4* In the previous proposition, the set of hypotheses on $S$ isn't sufficient to guarantee paths $(t) = S$. For example, if we were to take the set of nodes $V = \{0,1\}^*$, set of finite words over the alphabet $\{0,1\}$, and as root $\varepsilon$, the empty word, then we can take the following set $S$:

$$S := \bigcup_{N\in\mathbb{N}} \{(\varepsilon, 0, 00, 000, ..., 0^N, 10^N, 110^N, ...)\}$$

where $0^N$ designates the word consisting of $N$ zeroes. Then in the generated tree $\langle S\rangle$, the infinite path $(0^n)_{n\in\mathbb{N}}$ exists, even though it is not a path of $S$.

**Lemma 3** *If $t = (V, E, \mathcal{R})$ is a tree, then any non-empty subset $S$ of* paths $(t)$ *verifies the properties of Proposition 1. This result holds true for favoritism trees.*

*Proof.* Because $S \subseteq \text{paths}\,(t)$, the *(root start)* property is verified by all node sequences of $S$. Let's consider a node path $(n_k)_{N\geq k\geq 0} \in S$ and distinct $i, j \in \{0, ..., N\}$. Without loss of generality, we can suppose $i < j$. The path from $\mathcal{R}$ to $n_j$ can be obtained by taking the one from $\mathcal{R}$ to $n_i$ and adding $n_{i+1}, ..., n_j$, which remains a node path in $t$. Because the path from the root to $n_j$ is unique, we necessarily have $n_i \neq n_j$, hence the *(distinct nodes)* property.

If $(m_j)_{K\geq j\geq 0} \in S$ is another node path, and if there exist $i \in \{0, ..., N\}$ and $j \in \{0, ..., K\}$ such that $n_i = m_j$, then we have $i = j$ by comparing path lengths. When $i > 1$, $n_i = m_j$ implies $n_{i-1} = \mathbf{p}\,(n_i) = \mathbf{p}\,(m_j) = m_{j-1}$, hence the *(rank respect)* property. □

**Lemma 4** *For any favoritism tree $t$, $\langle \text{paths}\,(t)\rangle = t$.*

*Proof.* Left to the reader. □

**Definition 9 (Morphism)** Given two favoritism trees $t = (V_t, E_t, \mathcal{R}_t, \triangleleft_t)$ and $s = (V_s, E_s, \mathcal{R}_s, \triangleleft_s)$, a function $\phi : V_t \to V_s$ is a **favoritism tree morphism** when:

- $\phi(\mathcal{R}_t) = \mathcal{R}_s$,
- For any $x, y \in V_t$, $(x, y) \in E_t$ iff $(\phi(x), \phi(y)) \in E_s$,
- For any $x \in V_t$ and any $y, z \in \mathbf{ch}_t\,(x)$, $\phi(y) \triangleleft_s \phi(z)$ iff $y \triangleleft_t z$.

In other words, $\phi$ must preserve the root, ancestry, and the favoritism function. $\phi$ is said to be a **favoritism tree isomorphism** whenever $\phi$ is a bijection.

**Lemma 5** *For any favoritism tree morphism $\phi : V_t \to V_s$ and any sub-tree $t'$, the image of $t'$ by $\phi$ is a sub-tree of $s$. Moreover, $t'$ is isomorphic to this sub-tree.*

*Proof.* Denote $V_{t'}$ the set of nodes of $t'$, and $V := \phi(V_{t'})$. The *image* of $t'$ by $\phi$ is explicitly $(V, E_s \cap (V \times V), \mathcal{R}_s, \triangleleft_s)$. Let's show that it is a favoritism tree.

Consider any $y \in V$, and consider $x \in V_{t'}$ such that $y = \phi(x)$. Then there exists a unique node path $\mathcal{R}_t \to x_1 \to \cdots \to x_n = x$ in $t$, and by induction over $n$, we have $\mathcal{R}_s \to \phi(x_1) \to \cdots \to \phi(x_n) = y$. Conversely, any node path from $\mathcal{R}_s$ to $y$ defines a node path from $\mathcal{R}_t$ to $x$, which grants us uniqueness. So $(V, E_s \cap (V \times V), \mathcal{R}_s, \triangleleft_s)$ is a favoritism tree. It is also trivially a sub-tree of $s$ by this definition.

Let's show that the restriction $\phi' : V_{t'} \to V$ of $\phi$ is an isomorphism. It is trivially onto, and for any $x, y \in V_{t'}$, if $\phi(x) = \phi(y)$ then the node paths from $\mathcal{R}$ to $x$ and to $y$ are equal. By considering the associated node paths in $t'$, we obtain that $x = y$, and so $\phi$ is one-to-one. So $t'$ and $(V, E_s \cap V \times V, \mathcal{R}_s, \triangleleft_s)$ are isomorphic. □

**Proposition 2** *For any favoritism tree $t = (V, E, \mathcal{R}, \triangleleft)$ and any favoritism function $\lhd$ over $t$, $t$ is isomorphic to $t_{\lhd} := (V, E, \mathcal{R}, \lhd)$.*

*Proof.* We inductively define $\phi$ over $V$ by:

- $\phi(\mathcal{R}) := \mathcal{R}$,
- If $\phi(x)$ has been defined, then for any $y = \mathbf{ch}\,(x)\,.(n)$, we define $\phi(y)$ as the $n$-th smallest element in the finite set $\mathbf{ch}\,(x)$.

Then $\phi$ is a bijection, and we can easily show that it is a morphism. □

## 2.3 The $\mathbb{Z}_p$ tree

In the construction of the $\mathbb{Z}_p$ tree, we will be using the following notations: if $\Sigma$ is a finite alphabet, then $\Sigma^*$ is the set of finite words over $\Sigma$, and $\varepsilon$ is the empty word. The concatenation of two words $u, v$ is denoted $u \cdot v$, and the length of a word $w$ is denoted $|w|$. The individual letters of a word $w$ of length $n$ are $w_1, ..., w_n$ such that $w = w_n \cdot ... \cdot w_1$.

**Definition 10 ($\mathbb{Z}_p$ favoritism tree)** The favoritism tree $(V, E_p, \varepsilon, \triangleleft)$ is called the $\mathbb{Z}_p$ favoritism tree, where:

- $V = \Sigma^*$, with $\Sigma = \{0, ..., p-1\}$
- $E_p = \{(x, y) : \exists a \in \Sigma,\ y = a \cdot x\}$
- The favoritism function $\triangleleft$ is defined over the children of a node $x \in V$ by $0 \cdot x \triangleleft 1 \cdot x \triangleleft\ ... \triangleleft\ (p-1) \cdot x$

The $\mathbb{Z}_p$ favoritism tree is denoted $Z_p$.

*Proof.* Let's verify that $Z_p$ is a well-defined favoritism tree.

To begin, $(V, E_p, \varepsilon)$ is a tree, since any word can be uniquely constructed through finite concatenation from the empty word $\varepsilon$. In other words, there is a unique path from the root to any node.

Furthermore, for all $x \in V$, $\mathbf{ch}\,(x)$ is finite, and $\triangleleft$ is a total order over $\mathbf{ch}\,(x)$. □

*Example 4* Figures 2.4 and 2.5 show partial representations of $Z_2$ and $Z_3$ respectively.

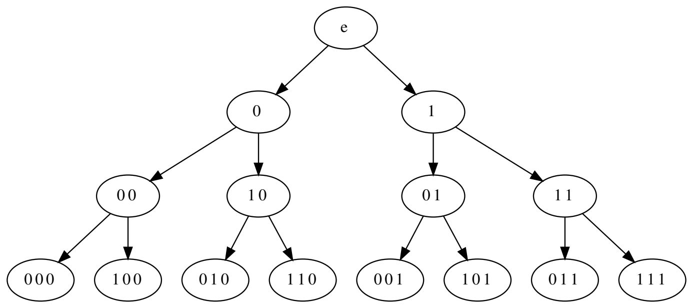



**Fig. 2.4** A sub-tree of $Z_2$.

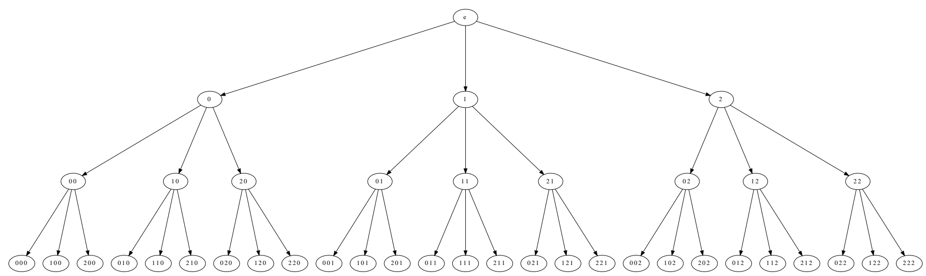


**Fig. 2.5** A sub-tree of $Z_3$.

**Definition 11** A favoritism tree is said to be $p$-bounded if each node has no more than $p$ children.

**Proposition 3** *Any $p$-bounded tree is isomorphic to a sub-tree of $Z_p$.*

*Proof.* Let $t = (V, E, \mathcal{R}, \lhd)$ be a $p$-bounded tree. For any $v \in V \setminus \{\mathcal{R}\}$, denote $v_0 := \mathcal{R} \to v_1 \to \cdots \to v_n := v$ the unique path from $\mathcal{R}$ to $v$ in $t$. Consider the structural path $(a_k)_{0 \leq k \leq n-1}$ associated to $(n_k)_{0 \leq k \leq n}$. Then $\phi(v)$ is the word $a_{n-1} \cdot a_{n-2} \cdot \ ... \ \cdot a_0 \in \{0, ..., p-1\}^*$. $\phi(\mathcal{R})$ is defined as the empty word $\varepsilon$.

Let's show that $\phi$ is a morphism. We have $\phi(\mathcal{R}) = \varepsilon$, and for any $x, y \in V$, if $(x, y) \in E$ then $x \to y$ and the path from $\mathcal{R}$ to $y$ can be written $\mathcal{R} \to \cdots \to x \to y$. By definition of $\phi$, this implies $(\phi(x), \phi(y)) \in E_p$. Conversely, if $(\phi(x), \phi(y)) \in E_p$, then by considering the associated structural paths, we see that $(x, y) \in E$. Lastly, if $x \in V$ and $y, z \in \mathbf{ch}_t(x)$, denote $y = \mathbf{ch}_t(x).(i)$ and $z = \mathbf{ch}_t(x).(j)$ with $i, j \in \{0, ..., p-1\}$. Then $i < j$ iff $y \lhd z$. But by definition of the favoritism function over $Z_p$, $i < j$ iff $\phi(y) \lhd \phi(z)$. So $\phi$ is a morphism.

By lemma 5, the image of $t$ by $\phi$ is a sub-tree of $Z_p$, and $t$ is isomorphic to this sub-tree. $\square$

# Chapter 3
# The Barchi set $\mathbb{T}_p$

In the first section, we will define the Barchi set $\mathbb{T}_p$ as a quotient of $\bar{\mathbb{N}} \times \mathbb{Z}_p$ by a surjection $\chi : \bar{\mathbb{N}} \times \mathbb{Z}_p \to \operatorname{paths}(Z_p)$. There then exists a one-to-one $\bar{\chi} : \mathbb{T}_p \to \operatorname{paths}(Z_p)$, called the Barchi bijection that is fundamental in the study of the Barchi set.

In section 3.2, we will introduce a natural partial order over $\mathbb{T}_p$, which corresponds to the usual ancestry relation in a tree. We will also define a metric over $\mathbb{T}_p$, which is both similar to an intuitive distance over a tree, but also similar to the $p$-adic absolute value.

In section 3.3, we discuss topological properties of $\mathbb{T}_p$. Notably, we will show some results on convergence that are different from the usual setting of real sequences or $p$-adic sequences. Moreover, using these results we show that $\mathbb{T}_p$ is complete and compact.

In section 3.4, we will show the previously-mentioned down-complete poset (dcpo) structure of $\mathbb{T}_p$, and we will obtain a first fixed-point theorem. We will also introduce sum and multiplication, before showing that $\mathbb{Z}_p$ and each $\mathbb{Z}/p^n\mathbb{Z}$ can be naturally seen as a ring included in $(\mathbb{T}_p, +, \cdot)$.

In section 3.5, we will formally introduce the notion of sub-tree of $\mathbb{T}_p$, before showing that it is exactly the same as a sub-tree of $Z_p$. Moreover, sub-trees of $\mathbb{T}_p$ are compact, and the set of sub-trees constitutes a complete lattice granting us a second fixed-point theorem. We will then introduce the notion of $p$-bounded trees, and justify the study of $\mathbb{T}_p$ by exhibiting multiple propositions over $p$-bounded trees.

In the last section, we will consider two different applications of the Barchi set. First, we will consider polynomials with coefficients in $\mathbb{T}_p$, and we will be able to prove Hensel's lemma using the Barchi set. To show how powerful the Barchi set is as a mathematical tool, we will also apply the compactness of $\mathbb{T}_p$ to languages, and prove that *Every Language is Compact*, an assertion we will give concrete meaning to.

## 3.1 The Barchi set $\mathbb{T}_p$

We denote $\bar{\mathbb{N}} = \{0, 1, 2, \dots, \infty\}$ the set of non-negative numbers and infinity.

**Definition 12** We define the function $\chi : \bar{\mathbb{N}} \times \mathbb{Z}_p \to \text{paths}\,(Z_p)$ by the following. For all $(n, x) \in \bar{\mathbb{N}} \times \mathbb{Z}_p$, denote $(x_k)_{k \geq 0} \in \{0, ..., p-1\}^{\mathbb{N}}$ the series representation of $x$.

- If $n < \infty$ then $\chi(n, x)$ is the (finite) node path defined by $n_0 = \mathcal{R}_{Z_p}$, and for all $0 \leq k \leq n-1$, $n_{k+1} = \mathbf{ch}\,(n_k)\,.(x_k)$,
- If $n = \infty$ then $\chi(n, x)$ is the (infinite) node path defined by $n_0 = \mathcal{R}_{Z_p}$, and for all $k \in \mathbb{N}$, $n_{k+1} = \mathbf{ch}\,(n_k)\,.(x_k)$.

**Lemma 6** *We have the following properties:*

1. *$\chi$ is a surjection.*
2. *For any $(n, x), (m, y) \in \bar{\mathbb{N}} \times \mathbb{Z}_p$, $\chi(n, x) = \chi(m, y)$ if and only if $n = m$ and $|x - y|_p \leq p^{-n}$.*

*Proof.* For the first point, consider a finite node path $N = (n_k)_{K \geq k \geq 0} \in \text{paths}\,(Z_p)$, and the associated structural path $(a_k)_{K-1 \geq k \geq 0} \in \{0, ..., p-1\}^K$. By taking the $p$-adic integer $x$ defined by:

$$x := \sum_{k=0}^{K-1} a_k p^k \in \mathbb{Z}_p$$

we have that $\chi(n, x) = N$. The same reasoning goes for infinite nodes.

Note that we have here chosen to extend the sequence $(a_k)$ by implicitly defining $a_k = 0$ for $k > n$. However, by the definition of $\chi$, we could have taken any other extension of $(a_k)$, granting us the following direction of the second point:

$$\left(n = m \text{ and } |x - y|_p \leq p^{-n}\right) \implies (\chi(n, x) = \chi(n, y))$$

Lastly, if we have $\chi(n, x) = \chi(m, y)$, then the lengths of the node paths are equal: $n = m$. It follows that the first $n \geq 1$ terms of the series representation of $x$ and $y$ are equal, and so $|x - y|_p \leq p^{-n}$. □

**Definition 13** We define the set $\mathbb{T}_p$ as the quotient of $\bar{\mathbb{N}} \times \mathbb{Z}_p$ by $\chi$, i.e by the equivalence relation $\sim$ defined by:

$$(n, x) \sim (m, y) \iff \chi(n, x) = \chi(m, y)$$

The class of an element $(n, x) \in \bar{\mathbb{N}} \times \mathbb{Z}_p$ is denoted $[n, x] \in \mathbb{T}_p$, except for the class of $(0, x) \in \bar{\mathbb{N}} \times \mathbb{Z}_p$, which is denoted $\mathbf{B}$. By the previous lemma, we know that the function $[n, x] \mapsto n$ is well-defined, and will henceforth be denoted $\omega$.

There is a bijection $\bar{\chi} : \mathbb{T}_p \to \text{paths}\,(Z_p)$, such that for any $(n, x) \in \bar{\mathbb{N}} \times \mathbb{Z}_p$, $\bar{\chi}([n, x]) = \chi(n, x)$. $\bar{\chi}$ is called the **Barchi** bijection.

## 3.2 First properties of $\mathbb{T}_p$

### 3.2.1 Partial order

**Definition 14 (Ancestry, Alignment)** For any $N = [n, x], M = [m, y] \in \mathbb{T}_p$, denote $\bar{\chi}(N) = (n_k)_{n \geq k \geq 0}$ and $\bar{\chi}(M) = (m_j)_{m \geq j \geq 0}$. The set $\{k \geq 0 : n_k = m_k\}$ contains 0 and has a maximum $r \geq 0$. We define the **nearest common ancestor** of $N$ and $M$, denoted $nca(N, M)$, as $\bar{\chi}^{-1}((n_k)_{r \geq k \geq 0})$.

We say that $N$ is an **ancestor** of $M$, denoted $N \leq M$, when $nca(N, M) = N$. Furthermore, we say that $N$ and $M$ are **aligned** when $N \leq M$ or $M \leq N$.

**Lemma 7** *$N = [n, x] \leq [m, y] = M$ iff $n \leq m$ and $N = [n, y]$.*

*Proof.* Suppose $N \leq M$. Consider the series representations $(x_k)_{k \geq 0}$ and $(y_k)_{k \geq 0}$ of $x$ and $y$ respectively. By definition of $\leq$, we have $nca(N, M) = N$. In terms of node paths of $Z_p$, we have that the node path $\bar{\chi}(M)$ coincides with that of $\bar{\chi}(N)$ up until the node $\bar{\chi}(N)$ is reached. This already grants us $n \leq m$. Moreover, the structural node paths associated to $\bar{\chi}(M)$ and $\bar{\chi}(N)$ coincide for the first $n$ terms. Subsequently, by definition of the Barchi bijection, we have:

$$\forall k \in \{0, ..., n-1\}, \quad x_k = y_k$$

which grants us $N = [n, y]$ by lemma 6.

Conversely, if $n \leq m$ and $N = [n, y]$, then the previous reasoning works in reverse, by comparing structural paths. □

**Lemma 8** *$\leq$ is a partial order over $\mathbb{T}_p$.*

*Proof.* $\leq$ verifies the following:

- (Reflexivity) For any $N \in \mathbb{T}_p$, $nca(N, N) = N$ and so $N \leq N$.
- (Transitivity) For any $A, B, C \in \mathbb{T}_p$ such that $A \leq B$ and $B \leq C$, we have $nca(A, B) = A$, and $nca(B, C) = B$. The rigorous proof that this implies $nca(A, C) = A$ is left to the reader, but can otherwise be seen on a diagram.
- (Antisymmetry) For any $N, M \in \mathbb{T}_p$ such that $N \leq M$ and $M \leq N$, we have $M = nca(N, M) = N$.

So $(\mathbb{T}_p, \leq)$ is partially ordered. □

**Lemma 9** *For any $A, B, C \in \mathbb{T}_p$, if $B \leq A$ and $C \leq A$ then $B$ and $C$ are comparable. In other words, the set of ancestors of $A$ is a totally ordered set with respect to $\leq$.*

*Proof.* We have $nca(A, B) = B$ and $nca(A, C) = C$. By lemma 7, if $A = [n, x]$ then $B = [\omega(B), x]$ and $C = [\omega(C), x]$. It follows by this same lemma that if $\omega(B) \leq \omega(C)$ then $B \leq C$, and if $\omega(B) \geq \omega(C)$ then $C \leq B$. □

### 3.2.2 Metric over $\mathbb{T}_p$

**Definition 15** For any two $N, M \in \mathbb{T}_p$, we define $d(N, M)$ by:

- When $N$ and $M$ are aligned, then

$$d(N, M) := \sum_{k=\min(\omega(N),\omega(M))+1}^{\max(\omega(M),\omega(N))} p^{-k}$$

- Otherwise,

$$d(N, M) := d(N, nca(N, M)) + d(nca(N, M), M)$$

  where $d(N, nca(N, M))$ falls into the previous case.

**Lemma 10** *For any $N, M \in \mathbb{T}_p$,*

$$d(N, M) = d(N, nca(N, M)) + d(nca(N, M), M)$$

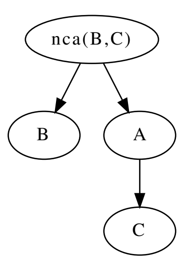
*Proof.* Whenever $N$ and $M$ are not aligned, this result is the definition. Whenever $N$ and $M$ are aligned, however, we have $nca(N, M) = N$ or $nca(N, M) = M$, meaning $d(N, nca(N, M)) = 0$ or $d(M, nca(N, M)) = 0$. Following the different cases, we obtain the above equality. □

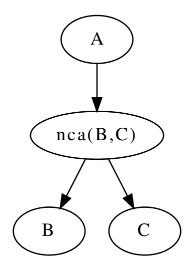
**Lemma 11** *For any $A, B, C \in \mathbb{T}_p$ such that $A \leq C$, we have:*

$$d(B, C) \leq d(B, A) + d(A, C)$$

*with equality if and only if $nca(B, C) \leq A$.*

The below figures illustrate lemma 11. The left figure shows the case $nca(B, C) \leq A$, when equality occurs, and the right figure shows the case $A \leq nca(B, C)$.




*Proof.* We denote $D := nca(B, C)$.

- When $D \leq A \leq C$, we show that $nca(A, B) = D$, which is left to the reader, but can be seen on the left-hand diagram above. We subsequently have:

$$d(A, B) = d(A, D) + d(D, B)$$

and

$$\begin{aligned}
d(A,C) &= \sum_{k=\omega(A)+1}^{\omega(C)} p^{-k} \\
&= \sum_{k=\omega(D)+1}^{\omega(C)} p^{-k} - \sum_{k=\omega(D)+1}^{\omega(A)} p^{-k} \\
&= d(C,D) - d(A,D)
\end{aligned}$$

By summing these equations, we have $d(A,B) + d(A,C) = d(D,B) + d(C,D) = d(B,C)$.

- When $A \leq D \leq C$, which corresponds to the right-hand diagram above, we show that $nca(A,B) = A$, and we have:

$$\begin{aligned}
d(A,B) &= \sum_{k=\omega(A)+1}^{\omega(B)} p^{-k} \\
&= \sum_{k=\omega(A)+1}^{\omega(D)} p^{-k} + \sum_{k=\omega(D)+1}^{\omega(B)} p^{-k} \\
&= d(A,D) + d(D,B)
\end{aligned}$$

and

$$d(A,C) = d(A,D) + d(D,C)$$

in the same way. By summing these results, we have:

$$d(B,C) = d(B,D) + d(D,C) = d(B,A) + d(A,C) - 2 \cdot d(D,A)$$

Given these results, and seeing that $d(D,A) = 0$ if and only if $A = D$ (which will be shown when $d$ is proven to be a distance), we have the inequality, and equality if and only if $D \leq A$. □

**Lemma 12** *For any $A, B, C \in \mathbb{T}_p$, we have:*

$$d(A,B) \leq d(A, nca(A,C)) + d(nca(A,C), B)$$

*Proof.* Immediate application of lemma 11, with $nca(A,C), B$ and $A$ instead of $A, B$ and $C$ respectively, seeing that $nca(A,C) \leq A$. □

**Proposition 4** *$d$ is a distance over $\mathbb{T}_p$.*

*Proof.* $d : \mathbb{T}_p \times \mathbb{T}_p \to \mathbb{Q}_+$ verifies:

- (Definite positivity) For any $N, M \in \mathbb{T}_p$ such that $d(N,M) = 0$, we have $d(N, nca(N,M)) = 0$. Because $N$ and $nca(N,M)$ are aligned, we have $\omega(N) = \omega(nca(N,M))$, and so $N = nca(N,M)$ by lemma 6. Similarly, $M = nca(N,M)$ and so $N = M$.

- (Symmetry) $d$ is symmetric by definition.
- (Triangle inequality) For any $A, B, C \in \mathbb{T}_p$, we have by lemma 12 that

  $$d(A,B) \leq d(A, nca(A,C)) + d(nca(A,C), B)$$

  and subsequently

  $$d(A,B) \leq d(A,C) + d(nca(A,C), B) - d(nca(A,C), C)$$

  We similarly have that

  $$d(B, nca(A,C)) \leq d(B, nca(B,C)) + d(nca(B,C), nca(A,C))$$

  by lemma 11, where we replaced $A, B$ and $C$ by $nca(B,C)$, $nca(A,C)$ and $B$ respectively. It follows that:

  $$\begin{aligned} d(A,B) \leq \quad & d(A,C) + d(B, nca(B,C)) \\ & + d(nca(B,C), nca(A,C)) \\ & - d(nca(A,C), C) \\ \\ = \quad & d(A,C) + d(C,B) \\ & + d(nca(B,C), nca(A,C)) \\ & - d(nca(A,C), C) - d(nca(B,C), C) \end{aligned}$$

  since $d(C,B) = d(B, nca(B,C)) + d(nca(B,C), C)$. But

  $$d(nca(B,C), nca(A,C)) \leq d(nca(A,C), C) + d(C, nca(B,C))$$

  Indeed, $nca(B,C)$ and $nca(A,C)$ are both ascendants of $C$, meaning that they are comparable for $\leq$. The previous inequality can be shown by expressing each term as a sum.
  We've therefore shown that:

  $$d(A,B) \leq d(A,C) + d(C,B)$$

So $(\mathbb{T}_p, d)$ is a metric space. □

## 3.3 Topology over $\mathbb{T}_p$

In the following paragraphs, we will denote $B(x,r)$ the open ball centered at $x \in \mathbb{T}_p$ of radius $r \in \mathbb{Q}_+$, for $d$, and $\bar{B}(x,r)$ the closed ball.

### 3.3.1 Results on convergence

**Lemma 13** *For any sequence $(N_k)_{k\geq 0}$ of elements of $\mathbb{T}_p$, $(N_k)_{k\geq 0}$ converges to $N \in \omega^{-1}(\{\infty\})$ if and only if $(nca(N_k, N))_{k\geq 0}$ converges to $N$.*

*Proof.* Suppose that $(nca(N_k, N))_{k\geq 0}$ converges to $N \in \omega^{-1}(\{\infty\})$. Because $M_k := nca(N_k, N)$ and $N$ are aligned for any $k \geq 0$, we have:

$$d(M_k, N) = \sum_{j=\omega(M_k)+1}^{\infty} p^{-j} \underset{k\to\infty}{\longrightarrow} 0$$

Moreover, $(\omega(M_k))_{k\geq 0}$ converges to $\infty$. But by lemma 12, $d(N, N_k) = d(N, M_k) + d(M_k, N_k)$ for all $k \geq 0$. We therefore need to show that $(d(M_k, N_k))_{k\geq 0}$ converges to 0. We have:

$$d(M_k, N_k) = \sum_{j=\omega(M_k)+1}^{\omega(N_k)} p^{-j} \leq \sum_{j=\omega(M_k)+1}^{\infty} p^{-j} = d(M_k, N) \underset{k\to\infty}{\longrightarrow} 0$$

and so $N_k \underset{k\to\infty}{\overset{d}{\longrightarrow}} N$.

Conversely, suppose that $(N_k)_{k\geq 0}$ converges to $N$. Then:

$$d(N_k, N) = d(N_k, M_k) + d(M_k, N) \underset{k\to\infty}{\longrightarrow} 0$$

Because $d$ is positive, we have $d(M_k, N) \underset{k\to\infty}{\longrightarrow} 0$, and so $(nca(N_k, N))_{k\geq 0}$ also converges to $N$. □

Note that this characterization falls short in the case of $\omega(N) < \infty$. For instance, if $K = [\infty, x]$ is a descendant of $N = [\omega(N), x]$, then $(nca(N, [k, x]))_{k\geq 0}$ converges to $N$ while $([k, x])_{k\geq 0}$ converges to $K \neq N$. However, if $(\omega(N_k))_{k\geq 0}$ is bounded by $\omega(N)$, then the characterization holds. However, if $(N_k)_{k\geq 0}$ converges to $N$, then $(nca(N_k, N))_{k\geq 0}$ also converges to $N$.

**Lemma 14** *For any sequence $(N_k)_{k\geq 0}$ and for any $N \in \mathbb{T}_p$ such that $\omega(N) \geq \omega(N_k)$ for all $k \geq 0$, the following assertions are equivalent:*

1. *$(N_k)_{k\geq 0}$ converges to $N$,*
2. *$(nca(N_k, N))_{k\geq 0}$ converges to $N$.*

*Regardless of the condition on $(\omega(N_k))_{k\geq 0}$, 1. $\implies$ 2. is always true.*

*Proof.* We denote $M_k := nca(N_k, N)$ for any $k \geq 0$. For any $k \geq 0$,

$$\begin{aligned} d(N_k, N) &= d(N_k, M_k) + d(M_k, N) \\ &= \sum_{j=\omega(M_k)+1}^{\omega(N_k)} p^{-j} + \sum_{j=\omega(M_k)+1}^{\omega(N)} p^{-j} \\ &\leq 2d(M_k, N) \end{aligned}$$

which establishes 2. $\implies$ 1. Moreover, the first equality (obtained by lemma 12) establishes 1. $\implies$ 2., and this did not require the condition over $(\omega(N_k))_{k\geq 0}$. To summarize, we have:

$$\forall k \geq 0,\ d(M_k, N) \leq d(N_k, N) \leq 2d(M_k, N)$$

and the rightmost inequality holds as long as $\omega(N_k) \leq \omega(N)$. $\square$

**Lemma 15** *If $N < M$ and $(M_k)_{k\geq 0}$ converges to $M$, then there exists $k \geq 0$ such that $N < nca(M_k, M) \leq M$.*

*Proof.* By the previous lemma, $(nca(M_k, M))_{k\geq 0}$ converges to $M$. We denote $K_k := nca(M_k, M)$ for all $k \geq 0$. Then $(\omega(K_k)) \underset{k\to\infty}{\longrightarrow} \omega(M)$, and because $\omega(N) < \omega(M)$, there exists $k \geq 0$ such that $\omega(N) < \omega(K_k) \leq \omega(M)$, but since $K_k$, $N$, and $M$ are aligned, this implies $N \leq K_k \leq M$. $\square$

**Lemma 16** *If $N$ is a finite node (i.e $\omega(N) < \infty$), then for any sequence $(N_k)_{k\geq 0}$ that converges to $N$, $N_k = N$ for any $k$ past a certain rank.*

*Proof.* By lemma 13, $(nca(N_k, N))_{k\geq 0}$ converges to $N$. Because $nca(N_k, N) \leq N$, this implies that $\omega(nca(N_k, N))$ converges to $\omega(N)$. So there exists $k_0 \geq 0$ such that for all $k \geq k_0$, $nca(N_k, N) = N$, i.e:

$$\forall k \geq k_0,\ N \leq N_k$$

Then:

$$\forall k \geq k_0,\ d(N_k, N) = \sum_{j=\omega(N)+1}^{\omega(N_k)} p^{-j}$$

and so $(\omega(N_k))_{k\geq 0}$ converges to $\omega(N)$. This implies the existence of $k_1 \geq k_0$ such that for all $k \geq k_1$, $\omega(N_k) = \omega(N)$, and so:

$$\forall k \geq k_1,\ N_k = N$$

$\square$

**Lemma 17** *For any $N \in \mathbb{T}_p$, $nca(N, \cdot)$ is continuous.*

*Proof.* Let $(M_k)_{k\geq 0}$ be a sequence of elements of $\mathbb{T}_p$ that converges to $M \in \mathbb{T}_p$.

If $N < M$, then by lemma 15 there exists $k_0$ such that for all $k \geq k_0$, $N < M_k \leq M$, and so $nca(N, M_k) = N$ for all $k \geq k_0$.

If $N = M$, then $\lim nca(M_k, N) = N$ by lemma 14.

If $M < N$, then there exists $k_0 \geq 0$ such that for all $k \geq k_0$, $M_k < N$ and so $\omega(M_k) < \omega(N)$. It follows that $\lim nca(N, M_k) = \lim M_k = M = nca(N, M)$. $\square$

**Proposition 5 (Squeeze theorem)** *For any sequences $(A_n), (B_n)$ and $(C_n)$ of elements of $\mathbb{T}_p$ such that*

$$\forall n \geq 0, \quad A_n \leq B_n \leq C_n$$

*then:*

1. *If* $L := \lim A_n = \lim C_n$*, then* $\lim B_n = L$*,*
2. *If* $(A_n)$ *and* $(B_n)$ *are convergent, then* $\lim A_n \leq \lim B_n$*,*
3. *If* $\lim \omega(A_n) = \infty$*, then* $\lim \omega(B_n) = \infty$*.*

*Proof.* 1. We have $nca(A_n, L) \leq nca(B_n, L) \leq nca(C_n, L)$ for all $n \geq 0$, and so:

$$\begin{aligned}\forall n \geq 0,\ d(B_n, L) &= d(B_n, nca(B_n, L)) + d(nca(B_n, L), L) \\ &\leq d(C_n, nca(B_n, L)) + d(nca(A_n, L), L) \\ &\leq d(C_n, nca(A_n, L)) + d(nca(A_n, L), L)\end{aligned}$$

By lemma 14, we know that $\lim nca(A_n, L) = L$, so $\lim d(C_n, nca(A_n, L)) = 0$. So $\lim d(B_n, L) = 0$.

2. Denote $L := \lim A_n$. We know that $\lim nca(A_n, L) = L$ by lemma 14. But $nca(A_n, L) \leq nca(B_n, L) \leq L$, and so by application of 1., we have $\lim nca(B_n, L) = L$. Subsequently, because $\lim nca(B_n, L) = nca(\lim B_n, L)$ by lemma 17, we have $L = \lim A_n \leq \lim B_n$.
3. We have $\omega(A_n) \leq \omega(B_n)$ for all $n \geq 0$.

□

### 3.3.2 Completeness and compactness

**Proposition 6** $\mathbb{T}_p$ *is complete for* $d$*.*

*Proof.* Consider some Cauchy sequence $(N_k)_{k\geq 0}$ of $\mathbb{T}_p$. We have two cases:

- Suppose that $(\omega(N_k))_{k\geq 0}$ is eventually constant, past some rank $n \in \mathbb{N}$. Then for some rank $N \geq n$, for any $k \geq N$, we have $d(N_k, N_N) = 0$, and so $N_k = N_N$. It follows that $(N_k)_{k\geq 0}$ converges to $N_N \in \mathbb{T}_p$.
- When $(\omega(N_k))_{k\geq 0}$ is not eventually constant, then we can extract $(N_{n_k})_{k\geq 0}$ such that for no $k \geq 0$ do we have $\omega(N_k) = \omega(N_{k+1})$. For any fixed $K \in \mathbb{N}$, there exists $\alpha_K \in \mathbb{N}$ such that

$$\forall k, l \geq \alpha_K, \quad d(N_{n_k}, N_{n_l}) \leq p^{-K}$$

It follows that:

$$\forall k \neq l \geq \alpha_K, \quad \sum_{n=\min(\omega(N_{n_k}),\omega(N_{n_l}))+1}^{\max(\omega(N_{n_k}),\omega(N_{n_l}))} p^{-n} \leq p^{-K}$$

and so for any $k \geq \alpha_K$, if we suppose $\omega(N_{n_{k+1}}) > \omega(N_{n_k})$, then:

$$\left(\omega(N_{n_{k+1}}) - \omega(N_{n_k})\right) p^{-\omega(N_{n_{k+1}})} \leq p^{-K}$$

Consequently, $\omega(N_{n_{k+1}}) \geq K$. Similarly, whenever $\omega(N_{n_k}) > \omega(N_{n_{k+1}})$, we have $\omega(N_{n_k}) \geq K$. Most notably, we have that for any $k \geq \alpha_K$,

$$\max\left(\omega(N_{n_k}), \omega(N_{n_{k+1}})\right) \geq K$$

We can therefore extract anew to obtain $(N_{m_k})_{k\geq 0}$ such that for any $k \geq 0$, $\omega(N_{m_k}) \geq k$. This sequence is still a Cauchy sequence, so consider a fixed $K \in \mathbb{N}$. There exists $\beta_K \in \mathbb{N}$ such that

$$\forall k, l \geq \beta_K, \quad d(N_{m_k}, N_{m_l}) \leq p^{-K}$$

If we were to write the distance as a sum (or eventually two sums), we would have that $\omega(nca(N_{m_k}, N_{m_l})) \geq K$ for any $k, l \geq \beta_k$. It follows that for any $k, l \geq \beta_K$, at least the first $K-1$ terms of any associated series representation of $N_{m_k}$ and $N_{m_l}$ coincide.
As we can find some such $\beta_K$ for any $K \in \mathbb{N}$, we can define $x \in \mathbb{Z}_p$ such that the first $K-1$ terms of the series representation of $x$ are exactly the first $K-1$ terms of any series representation of $N_{m_k}$ with $k \geq \beta_K$. Subsequently, we define $L := [\infty, x]$, and let's show that $(N_{m_k})_{k\geq 0}$ converges to $L$ (which will show that $(N_k)_{k\geq 0}$ converges to $L$). Let $\varepsilon > 0$ and consider $K \in \mathbb{N}$ such that $p^{-K} \leq \varepsilon \times \frac{p-1}{2p}$. Then, for any $k, l \geq \beta_K$, we have

$$d(N_{m_k}, N_{m_l}) \leq p^{-K} \leq \varepsilon \times \frac{p-1}{2p}$$

For any $k \geq \beta_K$, we have seen that if $N_{m_k} = [i, y]$, then $[K-1, y] \leq L$. It follows that

$$\begin{aligned} d(N_{m_k}, L) &\leq d(L, [K-1, y]) + d([K-1, y], [i, y]) \\ &= \sum_{j=K}^{\infty} p^{-j} + d([K-1, y], [i, y]) \\ &\leq p^{-K} \times \left( \sum_{j=0}^{\infty} p^{-j} + \sum_{j=0}^{i-K} p^{-j} \right) \\ &\leq p^{-K} \times \frac{2p}{p-1} \leq \varepsilon \end{aligned}$$

Subsequently, for any $k \geq \beta_K \in \mathbb{N}$,

$$d(N_{m_k}, L) \leq \varepsilon$$

And so $(N_k)_{k\geq 0}$ converges in $(\mathbb{T}_p, d)$. We have therefore shown that $(\mathbb{T}_p, d)$ is complete.

$\square$

**Lemma 18** *For any $x \in \mathbb{Z}_p$ and $k \in \bar{\mathbb{N}}$,*

$$[\infty, x] \in B\left([k, x], p^{-k} \times \frac{p}{p-1}\right)$$

*Proof.* All we need to show is that

$$d([\infty, x], [k, x]) < p^{-k} \times \frac{p}{p-1}$$

but since $[k, x] \leq [\infty, x]$, we have:

$$d([\infty, x], [k, x]) = \sum_{j=k+1}^{\infty} p^{-j} = \frac{p^{-k}}{p-1} < p^{-k} \times \frac{p}{p-1}$$

so $[\infty, x] \in B([k, x], p^{-k} \times \frac{p}{p-1})$. $\square$

**Proposition 7** *$\mathbb{T}_p$ is compact.*

*Proof.* Let's show that $\mathbb{T}_p$ is precompact. Let $\varepsilon > 0$, and consider $K \in \mathbb{N}$ such that

$$p^{-K} \times \frac{p}{p-1} \leq \varepsilon$$

By lemma 18,

$$\omega^{-1}(\{\infty\}) \subseteq \bigcup_{N \in \omega^{-1}(\{K\})} B(N, \varepsilon)$$

Given that this inclusion is also valid for any $j \geq K$ instead of $\infty$, we need only add balls to cover $\bigcup_{k<K} \omega^{-1}(\{k\})$. But this set is finite, and so:

$$\mathbb{T}_p \subseteq \bigcup_{k \leq K} \left( \bigcup_{N \in \omega^{-1}(\{k\})} B(N, \varepsilon) \right)$$

Where only a finite number of balls are on the right hand side.

We've now shown that $\mathbb{T}_p$ is precompact. Exploiting proposition 6, $\mathbb{T}_p$ is also complete. It follows that $\mathbb{T}_p$ is compact. $\square$

**Corollary 1** *$K \subseteq \mathbb{T}_p$ is compact if and only if it is closed.*

*Proof.* Immediate consequence of the previous proposition. $\square$

**Lemma 19** *For all $x, y \in \mathbb{Z}_p$,*

$$d([\infty, x], [\infty, y]) = \frac{2}{p-1} \left|x - y\right|_p$$

*Proof.* Denote $K = [k, x] = nca([\infty, x], [\infty, y])$. Then the first $k$ digits (i.e terms of the series representation) of $x$ coincide with the first $k$ digits of $y$. Because $K$ is the *nearest* common ancestor, we have that $k$ is maximal. It follows that $\left|x - y\right|_p = p^{-k}$. But:

$$d([\infty,x],[\infty,y]) = \sum_{j=k+1}^{\infty} p^{-j} + \sum_{j=k+1}^{\infty} p^{-j} = \frac{2p^{-k}}{p-1}$$

So $d([\infty,x],[\infty,y]) = \frac{2}{p-1}\left|x-y\right|_p$. □

## 3.4 Additional properties of the Barchi set

### 3.4.1 Down-complete poset (dcpo) structure

**Definition 16 (Downward directed set)** A **(downward) directed set** is a non-empty preordered set $A$ such that for any $a$ and $b$ in $A$ there exists $c$ in $A$ for which $c \leq a$ and $c \leq b$.

**Definition 17 (Down-complete poset (dcpo))** A partially ordered set $X$ is a **down-complete poset** (dcpo) if every downward directed subset has an infimum in $X$.

**Proposition 8** $\mathbb{T}_p$ *is a dcpo. Additionally, for any downward directed subset $F$, the infimum of $F$ is in $F$.*

*Proof.* Thanks to the Barchi bijection, it is sufficient to show that the set of node paths of $Z_p$ is a dcpo, where the considered order is implicitly defined relatively to $\leq$ over $\mathbb{T}_p$ with the Barchi bijection. Over paths$(Z_p)$, this relation equates to "being a prefix of".

Consider some downward directed set $F \subseteq \text{paths}(Z_p)$. Lemma 3 grants us the ability to consider the generated sub-tree $f = \langle F \rangle$ of $Z_p$ with nodes $V$. If $\mathcal{R}_{Z_p} \in V$ has *more* than one child in $f$, then there exists $a, b \in F$ such that $nca(a,b) = \mathcal{R}_{Z_p}$ and so $\inf F = \mathcal{R}_{Z_p}$. Furthermore, we have $\mathcal{R}_{Z_p} \in F$ because there exists $c \in F$ such that $c \leq nca(a,b) = \mathcal{R}_{Z_p}$, and so $\mathcal{R}_{Z_p} = c \in F$. On the other hand, if $\mathcal{R}_{Z_p}$ has only one child (if it has no children then $F = \{\mathcal{R}_{Z_p}\}$), then consider the smallest integer $k \geq 1$ such that at least one descendant of $\mathcal{R}_{Z_p}$ of depth $k$ has more than one child. If no such $k$ exists, then all the elements of $F$ are aligned, past which point $\inf F$ trivially exists (and is in $F$). Suppose now that $k$ exists. Then consider the only (by minimality of $k$) descendant $n \in V$ of $\mathcal{R}_{Z_p}$ with more than one child. We have two cases:

- If some ancestor $n'$ of $n$ is in $F$, then the smallest (w.r.t $\leq$) such ancestor is trivially equal to $\inf F$, and is in $F$.
- If no such ancestor exists, let's show that $n = \inf F$. Consider any element $a \in F$. We have $nca(n,a) \in V$ and $nca(n,a) \leq n$. Because $nca(n,a)$ is an ancestor of $n$, it either has only one child (if it is distinct from $n$), or it is equal to $n$. But if it has only one child, then this child is also a common ancestor of both $n$ and $a$, which is absurd by minimality of $nca(n,a)$. It follows that $nca(n,a) = n$, and so $n \leq a$.

For any $n' \in V$ such that $\forall a \in F, n' \leq a$, we can consider $a \in F$ and $b \in F$ such that $nca(a,b) = n$ because $n$ has at least two children. It follows that $n' \leq a$ and $n' \leq b$ so $n' \leq nca(a,b) = n$. Consequently, $n = \inf F$, and $n \in F$ because there exists $c \in F$ such that $c \leq nca(a,b) = n$, but no ancestor of $n$ is in $F$, so $n = c \in F$.

In either case, $\inf F$ exists in $V$. Furthermore, we have shown that $\inf F \in F$. Because the Barchi bijection is order-preserving (by definition of the order over paths of $Z_p$), the result holds for $\mathbb{T}_p$. □

**Definition 18 (Scott-continuity)** If $X$ and $Y$ are two dcpos, a function $f : X \to Y$ is said to be **Scott-continuous** if for any downward directed subset $F \subseteq X$,

$$\inf f(F) = f(\inf F)$$

**Proposition 9** *For any dcpo $X$, $f : \mathbb{T}_p \to X$ is order-preserving if and only if $f$ is Scott-continuous.*

*Proof.* Suppose that $f$ is increasing. Let $F$ be a downward directed subset of $\mathbb{T}_p$. By monotony, we have for all $x \in F$ that $f(x) \geq f(\inf F)$. Subsequently, $\inf f(F) \geq f(\inf F)$. We only need to show that $\inf f(F) \leq f(\inf F)$. By proposition 8, $x_0 := \inf F$ is an element of $F$, and so $f(\inf F) = f(x_0) \geq \inf f(F)$. Hence $f(\inf F) = \inf f(F)$.

If $f$ is Scott-continuous, then consider any $x, y \in \mathbb{T}_p$ with $x \leq y$. $F := \{x, y\}$ is a downward directed subset, and so $\inf f(F) = f(\inf F)$. In other words, $\min(f(x), f(y)) = f(x)$, so $f(x) \leq f(y)$. □

**Theorem 2** *Any increasing function $f : F \to \mathbb{T}_p$ over a downward directed subset $F$ of $\mathbb{T}_p$ has a minimum.*

*Proof.* If $f : F \to \mathbb{T}_p$ is increasing then it is Scott-continuous. It follows that $f(F)$ is also a downward directed subset of $\mathbb{T}_p$. By proposition 8, $\inf f(F)$ exists and is in $f(F)$. Therefore, there exists some $x_0 \in F$ such that:

$$\forall x \in F,\ f(x_0) \leq f(x)$$

□

**Corollary 2 (First fixed-point theorem)** *For any closed downward directed subset $A \subseteq \mathbb{T}_p$, any increasing function $f : A \to A$ has a least fixed point, denoted* $\mathrm{lfp}(f)$*.*

*Proof.* We know that $f$ has a minimum in $A$, say $x_0 \in A$ such that $f(x) \geq f(x_0)$ for all $x \in A$. Now consider the sequence $(f^n(x_0))_{n\geq 1}$. By compactness of $\mathbb{T}_p$, and therefore of $A$, $(f^n(x_0))_{n\geq 1}$ has a sub-sequence $(f^{n_k}(x_0))_{k\geq 1}$ that converges towards some $l \in A$. But $(f^n(x_0))_{n\geq 1}$ is increasing: by induction, if $n = 1$ then $f(f(x_0)) \geq f(x_0)$, and for any $n \geq 1$, if $f^{n+1}(x_0) \geq f^n(x_0)$ then:

$$f^{n+2}(x_0) \geq f^{n+1}(x_0)$$

So all of the $f^n(x_0)$, $n \geq 1$, are aligned, and furthermore $f^n(x_0) \leq l$ for any $n \geq 1$. The entire sequence therefore converges towards $l$, its supremum. Let's show that $f(l) = l$. Suppose that for some $n \geq 1$, $f^n(x_0) = l$. Then $f(l) = f^{n+1}(x_0) \geq l$, so $f(l) = l$. Suppose that for all $n \geq 1$, $f^n(x_0) < l$. Then because $f$ is increasing, $(\omega(f^n(x_0)))_{n\geq 1}$ is an increasing and non-stationary sequence of $\bar{\mathbb{N}}^*$. Consequently, because $\omega(l) \geq \omega(f^n(x_0))$ for all $n \geq 1$, $\omega(l) = \infty$. We can therefore write $l = [\infty, x]$. We also have $\omega(f(l)) = \infty$ since $f(l) \geq f^{n+1}(x_0)$ for all $n \geq 1$. We can therefore write $f(l) = [\infty, y]$. Because for all $n \geq 2$, $f^n(x_0) \leq nca(l, f(l))$, we know that the first $\omega(f^n(x_0))$ terms of the respective series representations of $x$ and $y$ coincide. In other words,

$$\forall n \geq 2, \quad |x-y|_p \leq p^{-\omega(f^n(x_0))}$$

As $\omega(f^n(x_0)) \to \infty$, this implies $x = y$ and consequently $f(l) = l$.

Consider another fixed point $x_1 \in A$. Then $f(x_0) \leq f(x_1) = x_1$ by minimality of $f(x_0)$. By iterating inductively, we get that $f^n(x_0) \leq x_1$ for any $n \geq 1$. Subsequently, $l = \sup(f^n(x_0)) \leq x_1$. Hence, $l$ is the least fixed point of $f$. □

We will later (section 3.5) work on *sub-trees* of $\mathbb{T}_p$, and these sub-trees will be closed downward directed subsets of $\mathbb{T}_p$. As such, we will be able to apply corollary 2 to any function $f : A \to A$ when $A$ is a sub-tree of $\mathbb{T}_p$. Moreover, note that the least fixed point is given by $\sup_n f^n(x_0)$ where $x_0 \in A$ minimizes $f$.

### 3.4.2 Algebraic properties

**Definition 19 (Sum and product of nodes)** We define the sum of $N = [n, x]$ and $M = [m, y]$ by:

$$N + M := [\min(n, m), x + y]$$

We also define the product by:

$$N \cdot M := [\min(n, m), xy]$$

**Lemma 20** *$+$ and $\cdot$ are well-defined over $\mathbb{T}_p$. Furthermore, $+$ and $\cdot$ are commutative and associative, and $\cdot$ is distributive over $+$.*

*Proof.* For any $[n', x']$ and $[m', y']$ such that $[n, x] = [n', x']$ and $[m, y] = [m', y']$, we have $n = n'$ and $m = m'$. The first $r := \min(n, m)$ terms of the series representations of $x$ and $y$ are respectively equal to the first $r$ terms of the series representations of $x'$ and $y'$. It follows that the first $r$ terms of the series representation of $x+y$ coincide with the first $r$ terms of the series representation of $x' + y'$. Subsequently, $[r, x' + y'] = [r, x + y]$. Same goes for the product.

$+$ and $\cdot$ are evidently commutative and associative. For any $A = [a, x], B = [b, y]$ and $C = [c, z]$ in $\mathbb{T}_p$, we have:

$$A \cdot (B + C) = A \cdot [\min(b, c), y + z] = [\min(a, b, c), xy + xz]$$

and

$$A \cdot B + A \cdot C = [\min(a, b), xy] + [\min(a, c), yz] = [\min(a, b, c), xy + yz]$$

so $\cdot$ is distributive over $+$. ◻

*Example 5* Let's consider a few calculations. For the sake of simplicity, we will also allow ourselves to write nodes (meaning elements of $\mathbb{T}_p$) as the just the last $\omega(N)$ digits of the associated $p$-adic number, meaning that we might write 1020 instead of $[4, ...1020]$.

- In $\mathbb{T}_3$, we have:
$$1020 + 121 = 211$$
- In $\mathbb{T}_2$, we have:
$$01 + 11 = 00$$
- In $\mathbb{T}_{11}$, we have:
$$420420 + 696867 = 007187$$

We also have the following products:

- In $\mathbb{T}_3$, $1020 \cdot 121 = 120$, and $1020 \cdot 0121 = 1120$.
- In $\mathbb{T}_2$, $01 \cdot 11 = 11$, and $1 \cdot 11 = 1$.

**Lemma 21 (Order compatibility)** *For any* $[a, \alpha], [b, \beta], [c, \gamma] \in \mathbb{T}_p$*, if* $[a, \alpha] \leq [b, \beta]$ *then:*

$$[a, \alpha] + [c, \gamma] \leq [b, \beta] + [c, \gamma]$$

*and*

$$[a, \alpha][c, \gamma] \leq [b, \beta][c, \gamma]$$

*Proof.* We have

$$\begin{aligned}
[a, \alpha] + [c, \gamma] &= [a, \beta] + [c, \gamma] \\
&= [\min(a, c), \beta + \gamma] \\
&\leq [\min(b, c), \beta + \gamma] \\
&= [b, \beta] + [c, \gamma]
\end{aligned}$$

and the same reasoning works for multiplication. ◻

**Proposition 10** $(\omega^{-1}(\{\infty\}), +, \cdot)$ *is a ring, isomorphic to* $\mathbb{Z}_p$*.*

*Proof.* The unit here is $[\infty, 1]$ and the zero is $[\infty, 0]$. It is easily verified that these are multiplicative and additive identities respectively.

To show the ring structure, it is sufficient to show that for any $N = [\infty, x]$ there exists some $M = [\infty, y]$ such that $N + M = [\infty, 0]$. This is immediate by taking $M = [\infty, -x]$. So $\omega^{-1}(\{\infty\})$ is a ring.

We define the ring homomorphism $\phi_\infty : \mathbb{Z}_p \to \omega^{-1}(\{\infty\})$ by:

$$\forall x \in \mathbb{Z}_p, \quad \phi_\infty(x) := [\infty, x]$$

$\phi_\infty$ is a ring homomorphism, as $\phi_\infty(1) = \mathbf{1}$, and for any $x, y \in \mathbb{Z}_p$, we have:

$$\phi_\infty(x+y) = [\infty, x+y] = [\infty, x] + [\infty, y] = \phi_\infty(x) + \phi_\infty(y)$$

and

$$\phi_\infty(xy) = [\infty, xy] = [\infty, x] \cdot [\infty, y] = \phi_\infty(x)\phi_\infty(y)$$

Furthermore, $\phi_\infty$ is a ring isomorphism from $\mathbb{Z}_p$ to $\omega^{-1}(\{\infty\})$ as $\phi_\infty$ is trivially a bijection. $\square$

**Proposition 11** *For any $k \geq 1$, $(\omega^{-1}(\{k\}), +, \cdot)$ is a ring, isomorphic to $\mathbb{Z}/p^k\mathbb{Z}$.*

*Proof.* The unit is $[k, 1]$ and the zero is $[k, 0]$.

We define $\phi_k : \mathbb{Z}/p^k\mathbb{Z} \to \omega^{-1}(\{k\})$ by:

$$\forall x \in (\mathbb{Z}/p^k\mathbb{Z}) \setminus \{0, 1\}, \quad \phi_k(x) := [k, \hat{x}]$$

where $\hat{x} \in \mathbb{Z}_p$ is obtained by considering the $k$ digits $x_1, ..., x_k \in \{0, ..., p-1\}$ of $x \in \mathbb{Z}/p^k\mathbb{Z}$, and taking $\hat{x} := \sum_{n=0}^{k} x_n p^n$. Then $\phi_k$ is a ring homomorphism as $\phi_k(1) = [k, 1]$ by definition and for any $x, y \in \mathbb{Z}/p^k\mathbb{Z}$, we have:

$$\phi_k(x+y) = [k, \widehat{x+y}] = [k, \hat{x} + \hat{y}] = [k, \hat{x}] + [k, \hat{y}] = \phi_k(x) + \phi_k(y)$$

Similarly,

$$\phi_k(xy) = [k, \widehat{xy}] = [k, \hat{x}\hat{y}] = [k, \hat{x}] \cdot [k, \hat{y}] = \phi_k(x)\phi_k(y)$$

So $\phi_k$ is a ring isomorphism from $\mathbb{Z}/p^k\mathbb{Z}$ to $\phi_k(\mathbb{Z}/p^k\mathbb{Z})$ as $\phi_k$ is trivially a bijection. $\square$

**Lemma 22** *For any $k \geq 1$, $\omega^{-1}(\{k\})$ is finite and has exactly $p^k$ elements.*

*Proof.* Immediate consequence of the previous proposition. $\square$

## 3.5 Sub-trees of $\mathbb{T}_p$ and $p$-bounded trees

In this section, we will introduce the notion of *sub-tree* of $\mathbb{T}_p$, which we will show to be exactly the same as the notion of sub-tree of $Z_p$. We also justify here the study of $\mathbb{T}_p$, by showing that any $p$-bounded tree (favoritism tree in which each node has no more than $p$ children) can be structurally identified to a sub-tree of $\mathbb{T}_p$. A first consequence of this is that any $p$-bounded tree is "compact" in the sense of the metric $d$ over $\mathbb{T}_p$.

As we will see in this section, results over $\mathbb{T}_p$ carry over to any $p$-bounded tree, which is not a particularly restrictive imposition over a tree: most often, a tree can be easily turned into a favoritism tree, and rarely is this tree not uniformly bound[1].

### 3.5.1 Sub-trees and canonical isomorphism

**Definition 20 (Sub-tree of $\mathbb{T}_p$)** A non-empty subset $A \subseteq \mathbb{T}_p$ is said to be a sub-tree of $\mathbb{T}_p$ when:

$$\bar{\chi}(A) = \text{paths}\left(\langle \bar{\chi}(A) \rangle\right)$$

**Lemma 23** *A non-empty subset $A \subseteq \mathbb{T}_p$ is a sub-tree of $\mathbb{T}_p$ if and only if both the following assertions are true:*

1. *For every $N \in A$, all ancestors of $N$ are in $A$,*
2. *Every increasing sequence of nodes of $A$ converges in $A$.*

*Proof.* Suppose 1. and 2. We immediately have the inclusion $\bar{\chi}(A) \subseteq \text{paths}\left(\langle \bar{\chi}(A) \rangle\right)$. Consider some node path $(n_k)_{0 \le k \le K} \in \text{paths}\left(\langle \bar{\chi}(A) \rangle\right)$, and consider $N := \bar{\chi}^{-1}((n_k)) \in \mathbb{T}_p$.

If $K$ is finite, then there exists some $(m_k)_{0 \le k \le K'} \in \bar{\chi}(A)$ such that:

- $K \le K'$,
- and $n_k = m_k$ for all $k \le K$.

Then if we denote $M := \bar{\chi}^{-1}((m_k)) \in A$, we have $N \le M$, and by 1., we have $N \in A$.

If $K = \infty$, then for every $k \ge 0$, there exists some $(m_{k,j})_{0 \le j \le K_k} \in \bar{\chi}^{-1}(A)$ such that:

- $k \le K_j \le \infty$,
- and $n_i = m_{k,i}$ for every $i \le k$.

Then, for every $k \ge 0$, if we denote $M_k := \bar{\chi}^{-1}((m_{k,j})_j) \in A$, we can define $L_k$ as an ancestor of $nca(M_k, N)$ such that $\omega(L_k) = k$. Then $(L_k)_{k \ge 0}$ is an increasing sequence of nodes of $A$ (by 1.), and so by 2., $(L_k)_{k \ge 0}$ converges to some $L \in A$. Because for all $k \ge 0$ we have $L_k \le nca(M_k, N)$, we also have (prop. 5) $nca(M_k, N) \underset{k \to \infty}{\longrightarrow} L$, and so $M_k \underset{k \to \infty}{\longrightarrow} L$ by lemma 13. It follows that $L = N$, and so $N \in A$.

So $\bar{\chi}(A) = \text{paths}\left(\langle \bar{\chi}(A) \rangle\right)$, and $A$ is a sub-tree of $\mathbb{T}_p$.

Conversely, suppose the set equality. Let's show 1. Consider some $N \in A$ and some ancestor $M \le N$. We denote

$$\bar{\chi}(M) =: (m_k)_{0 \le k \le K} \text{ and } \bar{\chi}(N) =: (n_k)_{0 \le k \le K'}$$

[1] A tree is uniformly bound when there exists some integer $M$ such that any node of the tree has no more than $M$ children. Trivially, any uniformly bound tree is $p$-bounded for some prime $p$ greater than $M$.

Because $M \leq N$, we have $K \leq K'$ and for all $k \leq K$, $m_k = n_k$. So for all $k \leq K$, $n_k \in \langle \bar{\chi}(A) \rangle$, and it follows that

$$\bar{\chi}(N) = (n_k)_{0 \leq k \leq K} \in \text{paths}\,(\langle \bar{\chi}(A) \rangle) = \bar{\chi}(A)$$

So $N \in A$. Let's now show 2. Let $(N_k)_{k \geq 0}$ be an increasing sequence of nodes of $A$. Then $(N_k)_{k \geq 0}$ converges to some $L \in \mathbb{T}_p$ such that $N_k \leq L$ for all $k \geq 0$. Consider the node paths $\bar{\chi}^{-1}(L) =: (l_k)_{0 \leq k \leq K} \in \text{paths}\,(Z_p)$ and $(n_{k,j})_{0 \leq j \leq K_k} := \bar{\chi}^{-1}(N_k) \in \text{paths}\,(\bar{\chi}(A))$ for all $k \geq 0$. Because $(N_k)$ is increasing, $(\omega(N_k))$ is increasing and so $(K_k)_{k \geq 0}$ is also increasing, and converges to $K$. Then for all $k \leq K$, we have $n_{k,i} = l_i$ for all $i \leq K_k$. It follows that for all $k \leq K$, we have that $l_k$ is a node of $\langle \bar{\chi}(A) \rangle$, and so $(l_k)_{0 \leq k \leq K}$ is in $\text{paths}\,(\langle \bar{\chi}(A) \rangle) = \bar{\chi}(A)$. So $L \in A$. □

**Lemma 24** *For any family of sub-trees $(A_i)_{i \in I}$ with $I \neq \emptyset$, $\bigcap_{i \in I} A_i$ is a sub-tree of $\mathbb{T}_p$. If $I$ is finite, then $\bigcup_{i \in I} A_i$ is a sub-tree of $\mathbb{T}_p$.*

*Proof.* Immediate by the previous lemma. The only verification needed is that the intersection is non-empty: this is immediate as $\mathbf{B} \in A_i$ for all $i$. □

Note however that $\bigcup_{i \in I} A_i$ is not always a sub-tree if $I$ is infinite. Consider for example some $x \in \mathbb{Z}_p$ and:

$$\forall i \geq 0, \quad A_i := \{[j, x] : 0 \leq j \leq i\}$$

Then $\bigcup_{i \geq 0} A_i$ is not a sub-tree of $\mathbb{T}_p$, as the increasing sequence $([j, x])_{j \geq 0}$ does not converge in $\bigcup_{i \geq 0} A_i$.

**Proposition 12** *Sub-trees of $\mathbb{T}_p$ are closed. Because $\mathbb{T}_p$ is compact, sub-trees are compact.*

*Proof.* Let $A$ be a sub-tree of $\mathbb{T}_p$. Consider a convergent sequence $(a_n)_{n \geq 0}$ of elements of $A$, that converges to some $a \in \mathbb{T}_p$. If $a$ is finite (i.e $\omega(a) < \infty$), then by lemma 16, $a_n = a$ for all $n$ past a certain rank, and so $a \in A$.

If $\omega(a) = \infty$, then $\omega(a_n) \leq \omega(a)$ for all $n \geq 0$. By lemma 14, the sequence defined by $b_n := nca(a_n, a) \in A$ for all $n \geq 0$ converges to $a$. Because $\omega(b_n) \longrightarrow \infty$, we can extract a strictly increasing subsequence $(b_{n_k})_{k \geq 0}$ (under the condition that $b_n \neq a$ for all $n$, in which case $a \in A$). Then every node path $\bar{\chi}(b_{n_k})$ is a prefix of the following one, and the length of these paths goes to infinity. We can therefore define an infinite path $\delta$ in $Z_p$ such that every $\bar{\chi}(b_{n_k})$ is a prefix of this path. By the Barchi bijection, $\bar{\chi}^{-1}(\delta) = a$. Notably,

$$\delta \in \text{paths}\,(\langle \bar{\chi}(A) \rangle)$$

Meaning that $a \in A$ by definition of a sub-tree. □

**Lemma 25** *For any sub-tree $A$ of $Z_p$, $\bar{\chi}^{-1}(\text{paths}\,(A))$ is a sub-tree of $\mathbb{T}_p$.*

*Proof.* We have:

$$A = \langle \text{paths}\,(A) \rangle$$

If we write $B := \bar{\chi}^{-1}(\text{paths}\,(A))$, this is exactly the same as writing:

$$\bar{\chi}(B) = \text{paths}\,(\langle \bar{\chi}(B) \rangle)$$

and so $B$ is a sub-tree. □

**Proposition 13** *There is a bijection between the set of sub-trees of $Z_p$ and the set of sub-trees of $\mathbb{T}_p$, and this bijection is $A \mapsto \bar{\chi}^{-1}(\text{paths}\,(A))$.*

*Proof.* We've seen that this function, here denoted $f$, is well-defined. Consider the function $g$, which associates to any sub-tree $B$ of $\mathbb{T}_p$, the sub-tree $\langle \bar{\chi}(B) \rangle$ of $Z_p$. Let's show that $f(g(B)) = B$. We have:

$$f(g(B)) = \bar{\chi}^{-1}\big(\text{paths}\,(g(B))\,\big) = \bar{\chi}^{-1}\big(\text{paths}\,(\langle \bar{\chi}(B) \rangle)\,\big) = \bar{\chi}^{-1}(\bar{\chi}(B)) = B$$

where the second-to-last equality stems from the definition of a sub-tree of $\mathbb{T}_p$. Conversely, if $A$ is a sub-tree of $Z_p$, then:

$$g(f(A)) = \big\langle \bar{\chi}\big(\bar{\chi}^{-1}(\text{paths}\,(A))\big) \big\rangle = A$$

So $f$ is a bijection, and its reciprocal is $\langle \bar{\chi}(\cdot) \rangle$. □

**Theorem 3 (Canonical isomorphism)** *For any $p$-bounded tree $t = (V, E, \mathcal{R}, \triangleleft)$, there exists a sub-tree $A$ of $\mathbb{T}_p$ and a bijection $\theta : \text{paths}\,(t) \to A$ such that:*

1. $\theta((\mathcal{R})) = \mathbf{B}$,
2. *For any $n = (n_i)_{0 \le i \le I},\ m = (m_j)_{0 \le j \le J} \in \text{paths}\,(t)$, we have:*

$$(I \le J \text{ and } \forall i \le I, n_i = m_i) \iff (\theta(n) \le \theta(m))$$

3. *For any $n = (n_i)_{0 \le i \le I},\ m = (m_j)_{0 \le j \le J} \in \text{paths}\,(t)$, if $1 \le I = J < \infty$ and $n_I, m_I \in \boldsymbol{ch}\,(n_{I-1})$, then $n_I \triangleleft m_I$ if and only if the $I$-th digit of any $p$-adic number associated to $\theta(n)$ is less than the $I$-th digit of any $p$-adic number associated to $\theta(m)$.*

*$\theta$ is called the **canonical isomorphism** of $t$ and $A$ the **translation** of $t$.*

*Proof.* By proposition 3, there exists a sub-tree $t'$ of $Z_p$, with nodes $V'$, and an isomorphism $\phi : V \to V'$. By proposition 13, $A := \bar{\chi}^{-1}(\text{paths}\,(t'))$ is a sub-tree of $\mathbb{T}_p$. For any node path $n = (n_k)_{0 \le k \le K} \in \text{paths}\,(t)$, we define:

$$\theta(n) := \bar{\chi}^{-1}\big(\ (\phi(n_k))_{k \ge 0}\ \big)$$

where $(\phi(n_k))_{k \ge 0} \in \text{paths}\,(Z_p)$ because $\phi$ is an isomorphism. Then we immediately have point 1. Consider any two node paths $n = (n_i)_{0 \le i \le I}$ and $m = (m_j)_{0 \le j \le J}$ in $t$. Then $\omega(\theta(n)) = I$. Suppose that $I \le J$ and $n_i = m_i$ for all $i \le I$. Then for all $i \le I$, $\phi(n_i) = \phi(m_i)$, and so by definition of the

Barchi bijection, the first $I$ digits of any $p$-adic integers associated to $\theta(n)$ and to $\theta(m)$ are equal. It follows that since $\omega(\theta(n)) \leq \omega(\theta(m))$, we have $\theta(n) \leq \theta(m)$. Conversely, suppose $\theta(n) \leq \theta(m)$. Then $I \leq J$, and similarly, $\phi(n_i) = \phi(m_i)$ for any $i \leq I$. It follows that $n_i = m_i$ for any $i \leq I$.

For the third point, suppose $1 \leq I = J < \infty$ and $n_I, m_I \in \mathbf{ch}\,(n_{I-1})$. Denote $x_I$ and $y_I$ the $I$-th digit of the $p$-adic integers associated to $\theta(n)$ and $\theta(m)$ respectively. The first $I-1$ digits are all equal since $n_I, m_i \in \mathbf{ch}\,(n_{I-1})$. We additionally have $x_I = \phi(n_I)$ and $\phi(m_I) = y_I$, so the equivalence is immediate. □

**Proposition 14** *The set of sub-trees of $\mathbb{T}_p$ is a complete lattice for $\subseteq$.*

*Proof.* Consider a family $(A_i)_{i\in I}$ of sub-trees of $\mathbb{T}_p$. If $I = \emptyset$ then $\sup_{i\in I} A_i = \{\mathbf{B}\}$ and $\inf_{i\in I} A_i = \mathbb{T}_p$. Suppose $I \neq \emptyset$. Then $\bigcap_{i\in I} A_i$ is a sub-tree of $\mathbb{T}_p$ by lemma 24, and it is the infimum. For the supremum, we have seen the the union of all of the $A_i$ is not generally a sub-tree. We define:

$$A := \bar{\chi}^{-1}\left(\mathrm{paths}\left(\left\langle \bigcup_{i\in I} \bar{\chi}(A_i)\right\rangle\right)\right)$$

Let's show that $A$ is a sub-tree of $\mathbb{T}_p$. We have:

$$\begin{aligned}
\mathrm{paths}\left(\langle \bar{\chi}(A)\rangle\right) &= \mathrm{paths}\left(\left\langle \mathrm{paths}\left(\left\langle \bigcup_{i\in I} \bar{\chi}(A_i)\right\rangle\right)\right\rangle\right) \\
&= \mathrm{paths}\left(\left\langle \bigcup_{i\in I} \bar{\chi}(A_i)\right\rangle\right) \\
&= \bar{\chi}(A)
\end{aligned}$$

So $A$ is a sub-tree of $\mathbb{T}_p$. Consider any $i \in I$. Let's show that $A_i \subseteq A$. We have:

$$\bar{\chi}(A_i) = \mathrm{paths}\left(\langle \bar{\chi}(A_i)\rangle\right) \subseteq \mathrm{paths}\left(\left\langle \bigcup_{j\in I} \bar{\chi}(A_j)\right\rangle\right) = \bar{\chi}(A)$$

Furthermore, consider any sub-tree $B$ of $\mathbb{T}_p$ such that for all $i \in I$, $A_i \subseteq B$. This implies that $\bigcup_{i\in I} A_i \subseteq B$, and so $\bigcup_{i\in I} \bar{\chi}(A_i) \subseteq \bar{\chi}(B)$. It follows that:

$$\bar{\chi}(A) = \mathrm{paths}\left(\left\langle \bigcup_{i\in I} \bar{\chi}(A_i)\right\rangle\right) \subseteq \mathrm{paths}\left(\langle \bar{\chi}(B)\rangle\right) = \bar{\chi}(B)$$

And so $A \subseteq B$. Subsequently, $A$ is the supremum of $(A_i)_{i\in I}$. We have thus shown:

$$\inf_{i\in I} A_i = \bigcap_{i\in I} A_i \text{ and } \sup_{i\in I} A_i = \bar{\chi}^{-1}\left(\mathrm{paths}\left(\left\langle \bigcup_{i\in I} \bar{\chi}(A_i)\right\rangle\right)\right)$$

□

**Proposition 15** *For any family $(A_i)_{i\in I}$ of sub-trees of $\mathbb{T}_p$, if $I \neq \emptyset$ then:*

$$\sup_{i\in I} A_i = \overline{\bigcup_{i\in I} A_i}$$

*where $\overline{A}$ designates the topological closure of $A \subseteq \mathbb{T}_p$ w.r.t. $d$.*

*Proof.* For the direct inclusion, let's show that $B := \overline{\bigcup_{i\in I} A_i}$ is a sub-tree of $\mathbb{T}_p$ that contains each $A_i$.

Consider any $N \in B$. If there exists $i \in I$ such that $N \in A_i$, then all ancestors of $N$ are in $A_i$ and subsequently all ancestors of $N$ are in $B$. Otherwise, consider some ancestor $M$ of $N$. Then consider the smallest $r \in \mathbb{Q}_+^*$ such that $M \in B(N, r)$. By definition of the closure, there exists $N' \in B(N, rp^{-1}) \cap \bigcup_{i\in I} A_i$. There exists $i \in I$ such that $N' \in A_i$. We then have:

$$d(N, N') = d(N, nca(N, N')) + d(nca(N, N'), N') \leq rp^{-1}$$

by which we conclude that $nca(N, N') \in B(N, rp^{-1})$. But $r = d(M, N) > d(nca(N, N'), N)$, and so $M \leq nca(N, N') \leq N$. So $M \leq N'$. Because $N' \in A_i$, and $A_i$ is a sub-tree, $M \in A_i \subseteq B$.

Moreover, any increasing sequence of elements of $B$ converges in $\mathbb{T}_p$, but since $B$ is closed, any increasing sequence of elements of $B$ converges in $B$.

So $B$ is a sub-tree of $\mathbb{T}_p$, and we immediately have $A_i \subseteq B$ for any $i \in I$. It follows that $\sup_{i\in I} A_i \subseteq B$.

Conversely, $\sup_{i\in I} A_i$ is a closed subset of $\mathbb{T}_p$ (prop. 12) which contains $\bigcup_{i\in I} A_i$. It follows that $B \subseteq \sup_{i\in I} A_i$, and we conclude that:

$$\sup_{i\in I} A_i = \bar{\chi}^{-1}\left(\text{paths}\left(\left\langle \bigcup_{i\in I} \bar{\chi}(A_i) \right\rangle\right)\right) = \overline{\bigcup_{i\in I} A_i}$$

□

**Corollary 3 (Second fixed-point theorem)** *If $\mathbb{S}_p$ is the set of sub-trees of $\mathbb{T}_p$, then for any monotonic function $f : \mathbb{S}_p \to \mathbb{S}_p$, the set of fixed points of $f$ is a complete lattice.*

*Proof.* Application of the Knaster-Tarski fixed point theorem of monotonic functions over complete lattices. □

### 3.5.2 Results over $p$-bounded trees

Theorem 3 justifies the study of $\mathbb{T}_p$ and its sub-trees. In the same way that the study of $\mathbb{R}^n$ extends to any real vector space with dimension $n$, the study of

$\mathbb{T}_p$ extends to any $p$-bounded tree. We will now see how previous results shown over $\mathbb{T}_p$ carry over to $p$-bounded trees.

Let's begin by considering proposition 12 "sub-trees of $\mathbb{T}_p$ are closed". This notably implies that sub-trees of $\mathbb{T}_p$ are compact. Consider any $p$-bounded tree $t = (V, E, \mathcal{R}, \lhd)$ and the canonical isomorphism $\theta : V \to A$, where $A$ is a sub-tree of $\mathbb{T}_p$. Then the smallest topology (over $\mathrm{paths}\,(t)$) making $\theta$ continuous makes $\mathrm{paths}\,(t)$ compact: this is the translation of proposition 12.

**Proposition 16 (The set of node paths is compact)** *For any $p$-bounded tree $t$ with canonical isomorphism $\theta$, $(\mathrm{paths}\,(t)\,, d_t)$ is a compact metric space, where $d_t$ is defined by:*

$$\forall n, m \in \mathrm{paths}\,(t)\,,\ d_t(n, m) := d(\theta(n), \theta(m))$$

Similarly, we can translate the first fixed-point theorem. Consider the partial order $\preceq$ defined by:

$$\forall n = (n_i)_{0 \le i \le I}, m = (m_j)_{0 \le j \le J} \in \mathrm{paths}\,(t)\,,\ n \preceq m \iff \theta(n) \le \theta(m)$$

The properties of $\theta$ given by theorem 3 gives us a more concrete understanding of this order as a "prefix" relationship between sequences. Then for any increasing function $f : \mathrm{paths}\,(t) \to \mathrm{paths}\,(t)$, $f$ has a minimum, say, in $n_0$, by theorem 2, and a least fixed-point $\mathrm{lfp}(f) = \sup_{n \ge 0} f^n(n_0)$ by corollary 2.

**Proposition 17 (First fixed-point theorem)** *For any $p$-bounded tree $t$ and any increasing $f : \mathrm{paths}\,(t) \to \mathrm{paths}\,(t)$ with respect to $\preceq$ defined above, $f$ has a minimum reached in $p_0 \in \mathrm{paths}\,(t)$ and a least fixed-point $\mathrm{lfp}(f) \in \mathrm{paths}\,(t)$ given by:*

$$\mathrm{lfp}(f) = \sup_{n \ge 1} f^n(p_0)$$

Let's continue with the second fixed-point theorem. Denote $\mathbb{S}(t)$ the set of sub-trees of $t$. Then for any $s \in \mathbb{S}(t)$, the canonical isomorphism associated to $s$ is nothing but the restriction of $\theta$ to $\mathrm{paths}\,(s)$ and to $\theta(\mathrm{paths}\,(s))$. Furthermore, $\theta(\mathrm{paths}\,(s))$ is the translation of $s$ and is a sub-tree of $\mathbb{T}_p$ that is included in $A$. As such, $\mathbb{S}(t)$ can be identified to the set $\mathit{\Gamma} := \{B \cap A : B \text{ is a sub-tree of } \mathbb{T}_p\}$. By proposition 14, the set of sub-trees of $\mathbb{T}_p$ is a complete lattice for $\subseteq$, and we can then show that $\mathit{\Gamma}$ is also a complete lattice. It follows that similarly, $\mathbb{S}(t)$ is a complete lattice for $\subseteq$[2]. Moreover, introducing anew the distance $d_t$ over $\mathrm{paths}\,(t)$, we have that for any family $(t_i)_{i \in I}$ $(I \neq \emptyset)$ of sub-trees of $\mathbb{S}(t)$,

$$\sup_{i \in I} A_i = \overline{\bigcup_{i \in I} A_i}$$

We also have access to the Knaster-Tarski fixed-point theorem, finally granting us the second fixed-point theorem.

[2] Lemma 2 allows us to identify any sub-tree to its set of paths, making sense of inclusion.

**Proposition 18 ($\mathbb{S}(t)$ is a complete lattice)** *For any $p$-bounded tree $t$, the set $\mathbb{S}(t)$ of sub-trees of $t$ is a complete lattice for inclusion. Moreover, with the previously-mentioned distance $d_t$ over* paths $(t)$*, we have for any non-empty family $(A_i)_{i\in I}$ of sub-trees of $t$,*

$$\sup_{i\in I} A_i = \overline{\bigcup_{i\in I} A_i}$$

*where we identify the sub-tree $A_i$ to* $\text{paths}\,(A_i) \subseteq \text{paths}\,(t)$ *by lemma 2.*

## 3.6 First applications

### 3.6.1 Polynomials and Hensel's Lemma

Although $\mathbb{T}_p$ is not a ring, consider the set $\mathbb{T}_p[X]$ of polynomials over $\mathbb{T}_p$.

**Definition 21** For any polynomial $P = \sum_{k=0}^{n} a_k X^k \in \mathbb{T}_p[X]$, where each $a_k$ is in $\mathbb{T}_p$, we define the **level** $l(P) \in \bar{\mathbb{N}}$ of $P$ by:

$$l(P) := \min\{\omega(a_k) : 0 \leq k \leq n\}$$

Moreover, for any $j \leq l(P)$, we denote $P_j \in \mathbb{T}[X]$ the polynomial defined by:

$$P_j := \sum_{k=0}^{n} [j, \alpha_k] X^k$$

where $a_k = [\omega(a_k), \alpha_k]$ for all $k$.

**Lemma 26** *1. For any $j \leq l(P)$ and $x \in \mathbb{T}_p$, $P_j(x) \leq P(x)$,*
*2. If $\deg P \geq 1$, then for any $x \in \mathbb{T}_p$, $\omega(P(x)) = \min(\omega(x), l(P))$,*
*3. $P$ is increasing w.r.t. $\leq$,*
*4. For any $x \in \mathbb{T}_p$, $P(x) = P_{l(P)}(x)$.*

*Proof.* 1. By induction over $\deg P$. If $P = [a, \alpha]X + [b, \beta]$ (i.e $\deg P \leq 1$) then $l(P) = \min(a, b)$. For any $[i, x] \in \mathbb{T}_p$ and $j \leq l(P)$, we have:

$$P_j(x) = [\min(i, j),\ \alpha x + \beta] \leq [\min(a, b, i),\ \alpha x + \beta] = P(x)$$

Suppose now that $P = \sum_{k=0}^{n} [a_k, \alpha_k] X^k$ where $n \geq 0$. Let $j \leq l(P)$ and $[i, x] \in \mathbb{T}_p$. Then:

$$P([i, x]) = [a_0, \alpha_0] + [i, x] \cdot \left( \underbrace{\sum_{k=0}^{n-1} [a_{k+1}, \alpha_{k+1}] \cdot [i, x]^k}_{=:Q([i,x])} \right)$$

By induction hypothesis, $Q_j([i,x]) \leq Q([i,x])$ since $j \leq l(P) \leq l(Q)$. It follows that:

$$P([i,x]) \geq [a_0, \alpha_0] + [i,x]Q_j([i,x]) \geq [j, \alpha_0] + [i,x]Q_j([i,x]) = P_j([i,x])$$

2. Immediate by definition of $+$ and $\cdot$.
3. This is an immediate consequence of lemma 21.
4. By 1., we have $P(x) \geq P_{l(P)}(x)$. By 2., we have $\omega(P(x)) = \min(l(P), \omega(x))$ and $\omega(P_{l(P)}(x)) = \min(l(P_{l(P)}), \omega(x)) = \min(l(P), \omega(x))$, so $P(x) = P_{l(P)}(x)$. $\square$

**Proposition 19** *Consider any $P \in \mathbb{T}_p[X]$ and any $[i,x] \in \mathbb{T}_p$ such that $0 \leq i \leq l(P)$. For any $\delta \in \{0, ..., p-1\}$,*

$$P([i, x + \delta p^i]) = P([i,x]) + [i, \delta p^i] \cdot P'([i,x])$$

*where $P' \in \mathbb{T}_p[X]$ denotes the (usual) derivative of $P$.*

*Proof.* For any $k \geq 1$,

$$\begin{aligned} [i, x + \delta p^i]^k &= \left([i,x] + [i, \delta p^i]\right)^k \\ &= \sum_{j=0}^{k} \binom{k}{j} \underbrace{[i, \delta p^i]^j}_{= [i,0] \text{ if } j \geq 2} [i,x]^{k-j} \\ &= [i,x]^k + [i, \delta p^i] \cdot k[i,x]^{k-1} \end{aligned}$$

It follows that:

$$\begin{aligned} P([i, x + \delta p^i]) &= \sum_{k=0}^{n} [a_k, \alpha_k][i,x]^k + [i, \delta p^i] \sum_{k=1}^{n} k[a_k, \alpha_k][i,x]^{k-1} \\ &= P([i,x]) + [i, \delta p^i] \cdot P'([i,x]) \end{aligned}$$

$\square$

**Theorem 4 (Hensel's Lemma, Barchean form)** *Consider any $P \in \mathbb{T}_p[X]$ and $[i,x] \in \mathbb{T}_p$ such that $2 \leq i+1 \leq l(P)$. If $P([i,x]) = [i,0]$ and $P'([i,x]) \not\geq [1,0]$ then there exists a unique $[i+1, y] \geq [i,x]$ such that $P([i+1,y]) = [i+1, 0]$ and $P'([i+1,y]) \not\geq [1,0]$.*

*Proof.* Suppose $P([i,x]) = [i,0]$ and $P'([i,x]) \not\geq [1,0]$. Without loss of generality, we can suppose that $x = \sum_{k=0}^{i} x_k p^k \in \mathbb{Z}_p$, meaning that all digits past rank $i$ are 0. Then, by the previous proposition, for all $\delta \in \{0, ..., p-1\}$,

$$P([i+1, x + \delta p^{i+1}]) = P([i+1, x]) + [i+1, \delta p^{i+1}]P'([i+1, x])$$

But

$$P([i+1, x]) \geq P_i([i+1, x]) = P([i,x]) = [i, 0]$$

and so we can write for some $\eta \in \{0, ..., p-1\}$ $P([i+1, x]) = [i+1, \eta p^{i+1}]$. Moreover, because $P'([i, x]) \ngeq [1, 0]$, we can write $P'([i+1, x]) = [i+1, z] \ngeq [1, 0]$, where $z = \sum_{k=0}^{i+1} z_k p^k \in \mathbb{Z}_p$ and $z_0 \neq 0$. Then for any $\delta \in \{0, ..., p-1\}$, we have

$$[i+1, \delta p^{i+1}]P'([i+1, x]) = [i+1, \delta p^{i+1}][i+1, z] = [i+1, \delta z_0 \cdot p^{i+1}]$$

Because $p$ is prime and $z_0 \neq 0$, there exists a value for $\delta \in \{0, ..., p-1\}$ such that $\delta z_0 = 1 \mod p$. Fix this value of $\delta$, and denote $\lambda \in \{0, ..., p-1\}$ such that $-\eta\delta = \lambda \mod p$. We have $z_0 \lambda = -\eta \mod p$, meaning:

$$[i+1, \lambda p^{i+1}]P'([i+1, x]) = [i+1, \lambda z_0 p^{i+1}] = [i+1, -\eta p^{i+1}]$$

So,

$$P([i+1, x + \lambda p^{i+1}]) = [i+1, \eta p^{i+1}] + [i+1, -\eta p^{i+1}] = [i+1, 0]$$

Moreover, we still have $P'([i+1, y]) \ngeq [1, 0]$, where:

$$y := x + \lambda p^{i+1} \in \mathbb{Z}_p$$

Let's show that $y$ is unique: consider any $y' \in \mathbb{Z}_p$ that satisfies $[i+1, y'] \geq [i, x]$ and $P([i+1, y']) = [i+1, 0]$. Because $[i+1, y'] \geq [i, x]$, we can write without loss of generality that $y' = x + \mu p^{i+1}$. Then:

$$P([i+1, x]) + [i+1, \mu p^{i+1}]P'([i+1, x]) = [i+1, 0]$$

and so

$$[i+1, \eta p^{i+1}] + [i+1, \mu p^{i+1}][i+1, z] = [i+1, 0]$$

As a result, $\mu z_0 = -\eta \mod p$, but since $z_0 \neq 0$ there exists only one such $\mu \in \{0, ..., p-1\}$, and it is $\lambda$, so $\mu = \lambda$ and subsequently $[i+1, y] = [i+1, y']$. □

**Corollary 4 (Hensel's lemma, usual form)** *For any $P \in \mathbb{Z}_p$, if there exits $\alpha_0 \in \mathbb{Z}_p$ such that*

$$P(\alpha_0) = 0 \mod p \quad \textit{and} \quad P'(\alpha_0) \neq 0 \mod p$$

*then there exists a unique $\alpha \in \mathbb{Z}_p$ such that $P(\alpha) = 0$ and $\alpha = \alpha_0 \mod p$.*

*Proof.* We are going to apply the previous theorem by induction, seeing that $P$ can be identified to a polynomial (still denoted $P$) of $\mathbb{T}_p[X]$ with all of its coefficients in $\omega^{-1}(\{\infty\})$, and so $l(P) = \infty$. The hypotheses for this corollary can be translated into:

$$P([1, \alpha_0]) = [1, 0] \text{ and } P'([1, \alpha_0]) \ngeq [1, 0]$$

By application of the previous theorem, there exists a unique $[2, \alpha_1] \in \mathbb{T}_p$ such that $P([2, \alpha_1]) = [2, 0]$ and $[2, \alpha_1] \geq [1, \alpha_0]$. Moreover, we still have $P'([2, \alpha_1]) \ngeq [1, 0]$. We construct a sequence $([k, \alpha_{k-1}])_{k \geq 1}$ of elements of $\mathbb{T}_p$ by iterating this process. For any $k \geq 1$, we have $P([k, \alpha_{k-1}]) = [k, 0]$ and $[k, \alpha_{k-1}] \leq [k+1, \alpha_k]$. This subsequently defines an increasing sequence that is convergent to some $[\infty, \alpha]$ for some $\alpha \in \mathbb{Z}_p$. Then $P([\infty, \alpha]) \geq P([k, \alpha_{k-1}]) = [k, 0]$ for all $k \geq 1$, meaning that $P([\infty, \alpha]) = [\infty, 0]$. Then $\alpha \in \mathbb{Z}_p$ is the root granted by Hensel's lemma for $P$ as a polynomial with coefficients in $\mathbb{Z}_p$.

Let's now show uniqueness. If $\beta \in \mathbb{Z}_p$ verifies $P([\infty, \beta]) = [\infty, 0]$ and $[\infty, \beta] \geq [1, \alpha_0]$ then we have $P([2, \beta]) = [2, 0]$. By uniqueness granted by theorem 4, we have $[2, \beta] = [2, \alpha_1]$. By iterating, we find that $[k, \beta] = [k, \alpha_{k-1}]$ for all $k \geq 1$, meaning then that $\beta = \alpha$. □

### 3.6.2 Compact languages

Moving on to a far less algebraic topic, we will now apply 16 to the tree generated by a fixed finite alphabet.

In this subsection, we will consider fixed a **non-empty finite** alphabet (i.e. set of distinct letters), denoted $\Sigma$. We have already introduced some vocabulary relative to formal languages during the construction of the $\mathbb{Z}_p$ favoritism tree $Z_p$, as the nodes were elements of the language $\{0, ..., p-1\}^*$. We will now denote $u \preceq v$ the assertion "$u$ is a (left) prefix of $v$", formally defined by:

$$u \preceq v \iff \exists w \in \Sigma^*,\ v = w \cdot u$$

Similarly to how we constructed $Z_p$, we can always construct a tree with words over $\Sigma$ as nodes, which we will do now. To be able to fully define this tree as a favoritism tree, however, we will suppose access to an (arbitrary) total order $\triangleleft$ over $\Sigma$. For example, for the usual latin alphabet, we could consider $a \triangleleft b \triangleleft c \triangleleft \cdots \triangleleft z$.

**Definition 22 ($\Sigma$ favoritism tree)** We define the favoritism tree $\sigma := (\Sigma^*, E, \varepsilon, \triangleleft)$, where:

$$E := \{(u, v) : \exists a \in \Sigma,\ v = a \cdot u\}$$

$\sigma$ is called the $\Sigma$ favoritism tree.

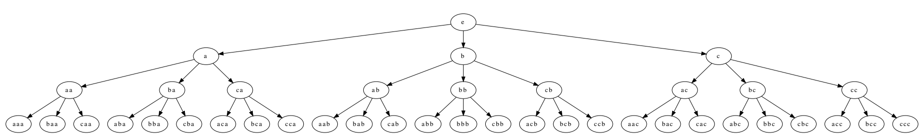



**Fig. 3.1** A sub-tree of the $\Sigma$ favoritism tree, where $\Sigma = \{a, b, c\}$ and arbitrarily, $a \triangleleft b \triangleleft c$.

Let $p$ be a prime number greater than the (finite) size of $\Sigma$. We will consider $p$ fixed in all that follows. Then the $\Sigma$ favoritism tree is trivially $p$-bounded. By proposition 16, the set paths$(\sigma)$ is compact for the right metric (imported from $\mathbb{T}_p$). Let's consider how this translates to words over $\Sigma$.

First, let's show some preliminary results. Denote $\theta$ : paths$(\sigma) \to A$ the canonical isomorphism (thm. 3) associated to $\sigma$, where $A$ is a sub-tree of $\mathbb{T}_p$. We will temporarily identify any finite word to its associated node path, so writing $\theta(w)$ for some word $w$ will be legible. For any $u, v \in \Sigma^*$, we have:

$$u \preceq v \iff \theta(u) \leq \theta(v)$$

This is an immediate consequence of the second property of $\theta$ announced by theorem 3.

Consider any sequence $(u_n)_{n\geq 0}$ of words over $\Sigma$. Because $A$ is compact (prop. 12), there exists some strictly increasing $\phi : \mathbb{N} \to \mathbb{N}$ and some element $U \in A$ such that $(\theta(u_{\phi(n)}))_{n\geq 0}$ converges to $U$.

Suppose that $U$ is finite, meaning that $\omega(U) < \infty$. Then $\theta^{-1}(U)$ is (formally, a *finite node path* which can be identified to a finite word) a finite word over $\Sigma$. Moreover, by lemma 16, there exists some rank $n_0 \geq 0$ such that:

$$\forall n \geq n_0,\ u_{\phi(n)} = \theta^{-1}(U) \in \Sigma^*$$

This is not a particularly interesting case. However, when $U$ is infinite, we know by lemma 14 that:

$$nca\left(\theta(u_{\phi(n)}), U\right) \underset{n\to\infty}{\longrightarrow} U$$

We can see $U$ as an infinite word over $\Sigma$. Indeed, $\theta^{-1}(U) =: (v_n)_{n\geq 0}$ is an infinite node path of $\sigma$, meaning that each $v_n$ is a word over $\Sigma$, and for all $n \geq 0$, there exists some letter $a_n \in \Sigma$ such that $v_{n+1} = a_n \cdot v_n$. Then the sequence $u := (a_n)_{n\geq 0}$ can be (informally) seen as an infinite word. Moreover, because $(nca(\theta(u_{\phi(n)}), U))_{n\geq 0}$ converges to $U$, we know that the sequence of longest common prefixes of $u$ and $u_{\phi(n)}$ "converges" to $u$ itself. While this explanation is not very formal, from evaluating $\theta$ on words instead of paths to using infinite words, it helps build the intuition behind the following definition.

**Definition 23 (Compact language)** A language $L \subseteq \Sigma^*$ is said to be **compact** if for any sequence $(u_n)_{n\geq 0}$ of words of $L$, there exists a strictly increasing $\phi : \mathbb{N} \to \mathbb{N}$ such that one of the two following alternatives is true:

1. $(u_{\phi(n)})_{n\geq 0}$ is constant.
2. There exists a strictly increasing (w.r.t. $\preceq$) sequence $(v_n)_{n\geq 0}$ of words of $\Sigma^*$ such that for any $n \geq 0$, $v_n \preceq u_{\phi(n)}$.

The above definition properly translates the "compactness" of paths$(\sigma)$ given by proposition 16. As such, we are going to see that $\Sigma^*$ is a compact language.

**Proposition 20** *$\Sigma^*$ is a compact language.*

*Proof.* Let $(u_n)_{n\geq 0}$ be a sequence of words over $\Sigma$, and denote $U_n \in \text{paths}\,(\sigma)$ the finite node path associated to $u_n$, for all $n \geq 0$. Then $(\theta(U_n))_{n\geq 0}$ has a convergent subsequence $(\theta(U_{\phi(n)}))_{n\geq 0}$. Denote $V \in A$ the limit of this subsequence.

Suppose first that $\omega(V) < \infty$. Then there exists $n_0 \geq 0$ such that $\theta(U_{\phi(n)}) = V$ for all $n \geq n_0$ by lemma 16. So, we can extract again to obtain a constant sequence $(\theta(u_{\psi(n)}))_{n\geq 0}$. Because $\theta$ is a bijection, we have that $(u_{\psi(n)})_{n\geq 0}$ is constant, which is the first alternative.

Suppose now that $\omega(V) = \infty$. By lemma 14, we know that the sequence defined for all $n \geq 0$ by

$$V_n := nca(\theta(U_{\phi(n)}), V)$$

converges to $V$. This implies that $(\omega(V_n))_{n\geq 0}$ goes to infinity, and so there exists a strictly increasing $\psi : \mathbb{N} \to \mathbb{N}$ such that $(\omega(V_{\psi(n)}))_{n\geq 0}$ is strictly increasing. Because each $V_{\psi(n)}$ is an ancestor of $V$, they are all comparable. Since their values by $\omega$ are strictly increasing, this means that $(V_{\psi(n)})_{n\geq 0}$ is strictly increasing.

We thus define $v_n$ as the node associated to the finite node path $\theta^{-1}(V_{\psi(n)})$ for all $n \geq 0$.

So $(v_n)_{n\geq 0}$ is strictly increasing with respect to $\preceq$, and we have:

$$\forall n \geq 0,\ v_n \preceq u_{\phi(\psi(n))}$$

Which establishes the second alternative. □

**Theorem 5 (Every language is compact)** *Every language over a finite alphabet is a compact language.*

*Proof.* Let $L$ be a language over $\Sigma$. Let $(u_n)_{n\geq 0}$ be a sequence of words of $L$. As a sequence of words of $\Sigma^*$, the previous proposition grants the existence of a strictly increasing $\phi : \mathbb{N} \to \mathbb{N}$ such that one of the two alternatives is verified. □

# References

1. S. C. Kleene, *Introduction to Metamathematics*, Éditions Jacques Gabay, Paris, 2015.
2. F. Q. Gouvêa, *p-adic Numbers: An Introduction*, Graduate Texts in Mathematics, vol. 199, Springer, 1997.