All Barchi No Bitey

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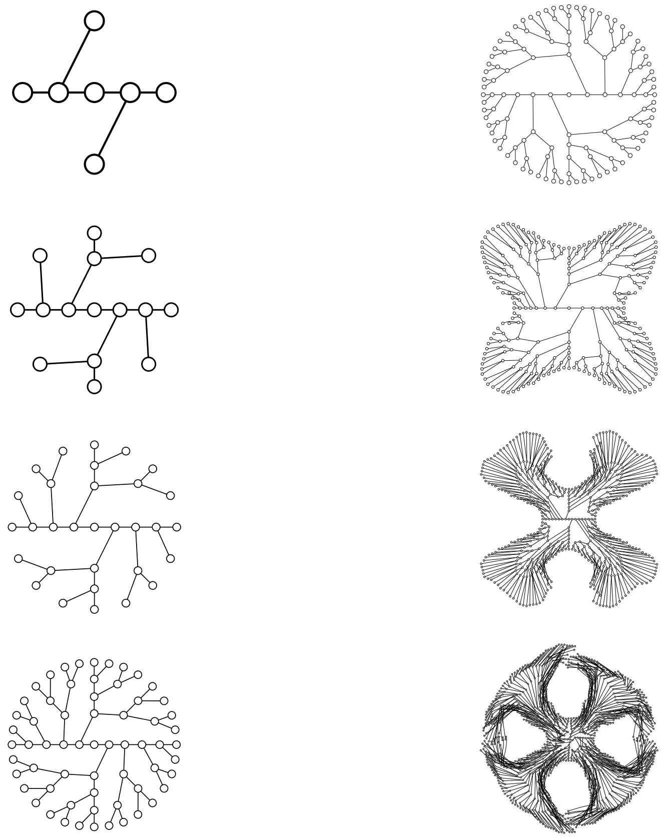
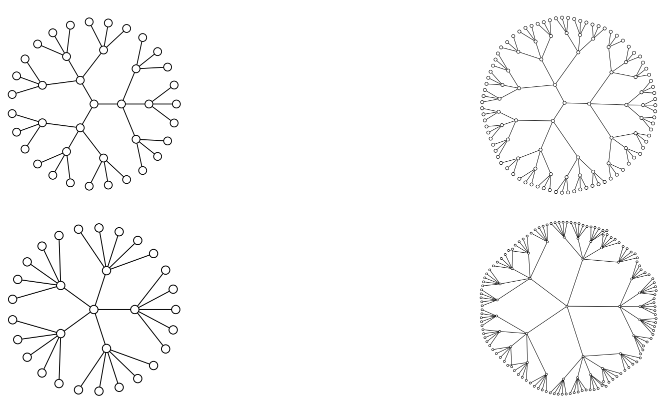


Fig. 1. Above are all sub-trees of $\mathbb{T}_2$, of increasing breadth. Below are sub-trees of $\mathbb{T}_3$ (first line) and $\mathbb{T}_5$ (second line). The last few sub-trees $\mathbb{T}_2$ had trouble compiling.



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Introduction

In this paper, we offer a unification of uniformly-bounded trees. A uniformly-bounded tree is a tree in which each node has no more than a constant M number of children. By taking any prime number p greater than M, we obtain a p-bounded tree. A tree in which each node has exactly p children, which we will call the $\mathbb{Z}_p$ favoritism tree, has the property of including (modulo node values) every other p-bounded tree.

Imagine you are at a roundabout with exactly p exits. Any of these exits will lead you to another roundabout with exactly p exits (not including the path taken to this roundabout). Then driving along this infinite series of roundabouts defines a sequence $(a_k)_{k\geq 0}$ with values in $\{0,...,p-1\}$, such that a_k is the number of the exit taken at the k-th visited roundabout. Results on the p-adic integer ring $\mathbb{Z}_p$ allow us to see $(a_k)_{k\geq 0}$ as a p-adic number by considering $\sum_{k=0}^{\infty} a_k p^k$, and conversely. As such, any path in these infinite roundabouts can be identified to a unique p-adic integer. In this analogy, the roundabouts are the nodes of the $\mathbb{Z}_p$ favoritism tree, and we clearly see some link between node paths in $\mathbb{Z}_p$ and p-adic numbers. Furthermore, if we consider that the car can have only finite amounts of gas, then the path taken can be seen as a prefix of a p-adic number. Using this, we obtain a surjection from $\bar{\mathbb{N}} \times \mathbb{Z}_p$ to the set of paths of the $\mathbb{Z}_p$ favoritism tree, where $\bar{\mathbb{N}}$ is the set $\{0, 1, ..., \infty\}$ of non-negative integers and ∞. The quotient set for this surjection defines the Barchi set $\mathbb{T}_p$, which is the object of study in this paper.

Study of the Barchi set is justified by the previous remark stating that any p-bounded tree can be seen (at least, structurally) as a sub-tree of the $\mathbb{Z}_p$ favoritism tree, and we will show that any p-bounded tree can be structurally identified to a sub-tree of the Barchi set $\mathbb{T}_p$. We will define a natural metric over $\mathbb{T}_p$ which will make sub-trees compact, and we will be able to deduce properties of any p-bounded tree. Through the Barchi set, we are able to exhibit properties of any p-bounded trees.

The Barchi set has various properties across multiple domains of mathematics. To name a few of these properties, it is a partially-ordered compact metric space, a dcpo, and we can define a sum and a multiplication over the Barchi set. While these operators do not grant a ring structure to the Barchi set, we will see that the restriction of these operators to certain subsets produces rings isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$ or $\mathbb{Z}_p$.

Applications of the Barchi set are not lacking. Trees can be found everywhere, and they are most often uniformly-bounded, meaning that they can be seen as sub-trees of $\mathbb{T}_p$ for some prime p. We will consider some applications across various domains in which trees can be found.

In the first chapter we will formally introduce the $\mathbb{Z}_p$ favoritism tree. Notably, we will introduce the notion of "favoritism" tree, which forces a total order on the children of any node, such that we can properly identify paths of this tree to a sequence of $\{0, ..., p-1\}$. We take the time to properly define trees, sub-trees, and node and structural paths in order to leave no confusion. The main definition is the one of favoritism tree, and the main result is the existence of a "generated sub-tree", i.e the smallest sub-tree containing a certain set of paths. The rest can be skimmed to the reader's desire.

In the second chapter, we introduce the Barchi set and explore different properties (topological, algebraic, and discrete). This is the heart of the paper, and we show in section 2.5 that p-bounded trees can be identified to sub-trees of the Barchi set, validating the study of the Barchi set.

In the third chapter, we explore different applications of the Barchi set.

It is recommended to be somewhat comfortable with the notion of p-adic integers before reading this paper. The most notable result we will be using is the *series representation* theorem:

Theorem 1. *For any p-adic integer x, there exists a unique sequence $(a_k)_{k\geq 0}$ with values in $\{0, ..., p-1\}$ such that $x = \sum_{k=0}^{\infty} a_k p^k$. Subsequently:*

$$\mathbb{Z}_p = \left\{ \sum_{k=0}^{\infty} a_k p^k : \forall k \geq 0,\ a_k \in \{0, ..., p-1\} \right\}$$

We will often use the term *series representation* of a p-adic integer in order to refer to the associated sequence of *digits* $(a_k)_{k\geq 0}$. It is also recommended to have seen the definition of the p-adic absolute value and its ultrametric properties.

The $\mathbb{Z}_p$ favoritism tree

Chapter I

As discussed in the introduction, we will be properly introducing the definition of trees, which we will allow to be infinite. In the analogy of infinite roundabouts, we are only allowed to identify a path to a sequence $(a_k)_{k\geq 0}$ when the exits are labeled $0, ..., p-1$. To guarantee this, we introduce the notion of "favoritism trees", which have a "favoritism function" imposing some numbering of the children of a node. We will additionally show some first properties that will be useful in the following chapters.

1.1 Definition of infinite trees

Definition 1 (Graph). *A (directed)* ***graph*** *G is a pair (V, E), where V is a non empty set of* ***nodes*** *and $E \subseteq \{(x, y) : x \neq y \in V\}$ is a set of* ***edges****.*
The binary relation $(x, y) \in E$ over $V \times V$ is denoted $x \to y$.

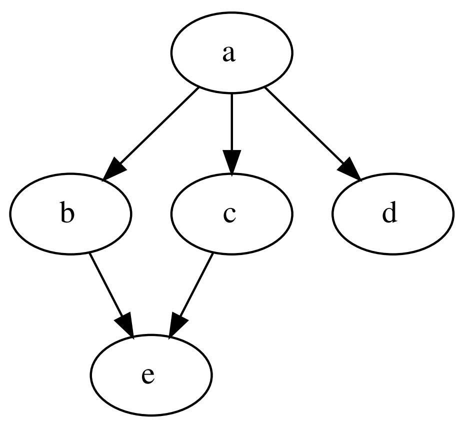


Fig. 2. An example of a graph, per definition 1. Only directed graphs will be considered in this paper.

Definition 2 (Paths). *In a graph $G = (V, E)$, a* ***path*** *from a node x to a node y is a finite sequence of nodes $x = x_0, x_1, ..., x_n = y$, where for all $k \in \{0, \ldots, n-1\}$, $x_k \to x_{k+1}$. The integer $n \geq 0$ is called the* ***length*** *of the path. A path is denoted $x_0 \to x_1 \to \ ... \to x_n$.*

Definition 3 (Tree). *A* ***tree*** *t is a triplet $(V, E, \mathcal{R}_t)$ where:*

- *$\mathcal{R}_t \in V$ is called the **root**.*
- *(V, E) is a graph.*
- *For all $x \in V$, there exists a unique path from $\mathcal{R}_t$ to x in (V, E).*

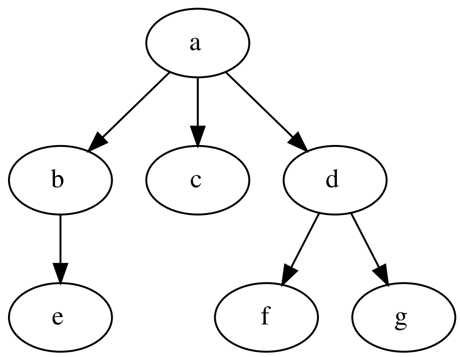


Fig. 3. An example of a tree. Here, the root is a.

Definition 4. *Let $t = (V, E, \mathcal{R}_t)$ be a tree. We define the following operators for all $x \in V$:*

- *$\boldsymbol{ch}_t(x) := \{y : x \to y\}$ is the set of **children** of x in t.*
- *When $x \neq \mathcal{R}_t$, $\mathbf{p}(x) \in V$ is the **parent** of x, defined as x_{n-1} in the unique path $\mathcal{R}_t \to \ \ldots \ \to x_{n-1} \to x_n = x$.*
- *$\boldsymbol{w}_t(x) \in \mathbb{N} \cup \{\infty\}$ is the **weight** of x, defined as the number of elements of $\boldsymbol{ch}_t(x)$.*

When the context is clear, we will write $\boldsymbol{ch}(x)$ and $\boldsymbol{w}(x)$ instead of $\boldsymbol{ch}_t(x)$ and $\boldsymbol{w}_t(x)$ respectively.

Example 1. In figure 3, $\mathbf{ch}(d) = \{g, h\}$, and $\mathbf{p}(c) = a$. For the weight function, $\mathbf{w}(c) = 0$ and $\mathbf{w}(b) = 1$.

Definition 5 (Favoritism tree). *A **favoritism tree** is a quadruplet $t = (V, E, \mathcal{R}, \triangleleft)$ where:*

- *$(V, E, \mathcal{R})$ is a tree*
- *For all $x \in V$, $\boldsymbol{ch}(x)$ is finite.*
- *$\triangleleft : x \in V \mapsto \triangleleft_x$ associates to x a total order over $\boldsymbol{ch}(x)$.*
- *For all $x \in V$, $\boldsymbol{ch}(x)$ has a smallest element for $\triangleleft_x$.*

Remark 1. The above definition is the first that diverges from the previously "classic" definitions. The conditions imposed on $\mathbf{ch}(x)$ allows us to consider the n-th smallest element of this set, which we denote $\mathbf{ch}(x).(n)$.

Definition 6 (Node paths). *If $t = (V, E, \mathcal{R}_t, \triangleleft)$ is a favoritism tree, then we define the **set of node paths** of t, denoted* $\text{paths}(t)$*, as the set of all node sequences $(n_k)_{N \geq k \geq 0}$ such that:*

- $N \in \mathbb{N}^* \cup \{\infty\}$
- $n_0 = \mathcal{R}_t$
- $\forall k \in \{0, ..., N-1\}, \quad n_k = \boldsymbol{p}(n_{k+1})$

We furthermore define:

- $\text{paths}_\infty(t) \subseteq \text{paths}(t)$ *is the set of infinite node paths (i.e $N = \infty$ in the above definition).*
- $\text{paths}_f(t) \subseteq \text{paths}(t)$ *is the set of finite node paths (i.e $N < \infty$ in the above definition).*

We trivially have the following partition:

$$\text{paths}(t) = \text{paths}_\infty(t) \sqcup \text{paths}_f(t)$$

Remark 2. In the previous definition, we do not exploit the fact that t is a favoritism tree, in contrast with a "classic" tree. In fact, the above notations might sometimes be used for trees, but the upcoming notion of **structural paths** requires the structure of a favoritism tree.

Definition 7 (Structural path). *Let $t = (V, E, \mathcal{R}_t, \triangleleft)$ be a favoritism tree. For any node path $(n_k)_{N \geq k \geq 0} \in \text{paths}(t)$, we define the associated **structural path** $(a_k)_{N-1 \geq k \geq 0}$ by:*

$$\forall k \in \{0, ..., N-1\}, \; n_{k+1} = \boldsymbol{ch}(n_k).(a_k)$$

The set of all structural paths of t is denoted $t^{\mathbb{N}}$.

Example 2. In figure 3, the directional path associated to the node path a, d, h is $2, 1$.

Lemma 1. *In a favoritism tree $f = (V, E, \mathcal{R}_f, \triangleleft)$, there is a bijection between the set of node paths and the set of directional paths.*

Proof. The definition of a directional path gives a surjection from $\text{paths}(f)$ to $f^{\mathbb{N}}$. Furthermore, we trivially have that two distinct node paths have distinct associated structural paths. $\square$

1.2 Preliminary results on trees

Definition 8. *If $f = (V_f, E_f, \mathcal{R}_f, \triangleleft)$ and $g = (V_g, E_g, \mathcal{R}_g, \triangleleft)$ are two favoritism trees, then g is called a:*

- ***structural sub-tree** of f when $g^{\mathbb{N}} \subseteq f^{\mathbb{N}}$.*
- ***sub-tree** of f when $V_g \subseteq V_f$, $E_g \subseteq E_f$, and $\mathcal{R}_f = \mathcal{R}_g$.*

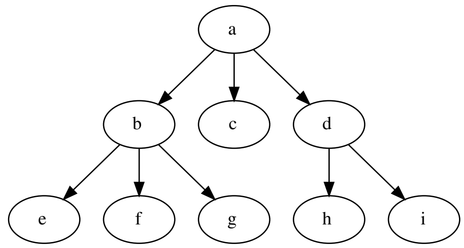


Fig. 4. An example of a tree, for which the tree figure 3 is a structural sub-tree.

Example 3. The figure 4 shows a tree t_1 for which the tree t_2 of figure 3 is a structural sub-tree. Note that the labeling of nodes doesn't matter in the sub-tree relationship. t_2 is not a sub-tree of t_1, because a, d, g is a node path in t_2, but not in t_1.

Remark 3. A sub-tree is a structural sub-tree.

Lemma 2. *If $f = (V_f, E_f, \mathcal{R}_f, \lhd)$ and $g = (V_g, E_g, \mathcal{R}_g, \lhd)$ are two favoritism trees, then g is a sub-tree of f if and only if* $\text{paths}\,(g) \subseteq \text{paths}\,(f)$.

Proof. Suppose g is a sub-tree of f. Because all node paths of g begin with $\mathcal{R}_g$, we must have $\mathcal{R}_g = \mathcal{R}_f$. Furthermore, if x is a node of g different from $\mathcal{R}_g$, then there exists a finite node path $(n_k)_{K \geq k \geq 0}$ from the root $\mathcal{R}_g$ to x with $K \geq 1$. Because this node path also exists in f, we have $x \in V_f$. Furthermore, we have $n_{K-1} \to_g n_K = x$, meaning that $n_{K-1} \to_f n_K = x$, and consequently $(n_{K-1}, n_K) \in E_f$. This being true for all $x \in V_g \setminus \{\mathcal{R}_g\}$, we have $V_g \subseteq V_f$ and $E_g \subseteq E_f$. The converse is immediate. $\square$

In the following proposition, we will talk of a "smallest" tree $(V, E, \mathcal{R})$ verifying a property $\mathcal{P}$. What we mean by "smallest", is that for any tree $(V', E', \mathcal{R})$ (where V and V' are both subsets of a fixed set of nodes) that also verifies $\mathcal{P}$, we have $V \subseteq V'$ and $E \subseteq E'$.

Proposition 1 (Generated tree). *If V is a set of nodes, if $\mathcal{R}$ is a node of V, and if S is a non-empty set of node sequences $(n_k)_{N \geq k \geq 0}$ (of elements of V) that verify the following properties:*

- $n_0 = \mathcal{R}$ *(root start)*
- *For all $i, j \in \{0, ..., N\}$, $(i \neq j) \implies (n_i \neq n_j)$* *(distinct nodes)*
- *For all $i \in \{0, ..., N-1\}$, and for all $(m_j)_{K \geq j \geq 0} \in S$,* *(rank respect)*
  if there exists $j \in \{0, ..., K\}$ for which $n_i = m_j$, then $i = j$ and $n_{i-1} = m_{j-1}$ if $i > 1$.

then there exists $V' \subseteq V$ and $E \subseteq \{(x, y) : x \neq y \in V'\}$ for which $t := (V', E, \mathcal{R})$ is a tree such that $\text{paths}\,(t) \supseteq S$.
There exists a smallest such tree is called the ***generated tree*** *of S and is denoted $\langle S \rangle$.*

Proof. We can begin by defining

$$V' := \bigcup_{(n_k)_{N \geq k \geq 0} \in S} \{n_k : 0 \leq k \leq N\}$$

which is the set of all nodes passed over by at least one path. Because of the *(root start)* property, $\mathcal{R} \in V'$. We then define:

$$E := \bigcup_{(n_k)_{N \geq k \geq 0} \in S} \big\{(n_k, n_{k+1}) : k \in \{0, ..., N-1\}\big\}$$

By property *(distinct nodes)*, E is a subset of $\{(x, y) : x, y \in V',\ x \neq y\}$. At this point, we've created a graph $G := (V', E)$. Let's show that $(V', E, \mathcal{R})$ is a tree: consider $x \in V'$ distinct from $\mathcal{R}$. By definition of V', there exists $(n_k)_{N \geq k \geq 0} \in S$ and $K \in \{0, ..., N\}$ such that $x = n_K$. By property *(root start)*, we have $n_0 = \mathcal{R}$, and by definition of E:

$$\mathcal{R} = n_0 \to n_1 \to \ \ldots\ \to n_K = x$$

is a path of G. So a path from the root to x exists. Let's show that it is unique: let's consider a path $\mathcal{R} = x_0 \to \ \ldots\ \to x_l = x$ in G. Without loss of generality, we can suppose $l \leq K$. By definition of E, there exists $(m^l_k)_{N_l \geq k \geq 0} \in S$ and $j \in \{1, ..., N_l\}$ such that $m^l_j = x_l = x$ and $m^l_{j-1} = x_{l-1}$. By property *(rank respect)*, we have $j = K$ and $m^l_{K-1} = n_{K-1}$, meaning $x_{l-1} = n_{K-1}$.
We can recursively iterate this process by taking $(m^i_k) \in S$ for all $i \in \{1, ..., l-1\}$, and we get that $n_{K-i} = x_{l-i}$. Finally, for $i = l$, we can consider $(m^0_k)_{N_0 \geq k \geq 0} \in S$ and $j \in \{1, ..., N_0\}$ such that $m^0_j = x_1$ and $m^0_{j-1} = x_0 = \mathcal{R}$. Consequently, by property *(rank respect)* applied to $m^0_j = x_1 = n_{K-l+1}$, we have $j = K - l + 1$ and $n_{K-l} = m^0_{j-1} = \mathcal{R}$. By property *(distinct nodes)*, we conclude that $K = l$, and we have shown that $n_i = x_i$ for all $i \in \{0, ..., K\}$. Consequently, $(V', E, \mathcal{R})$ is a tree.
The inclusion $S \subseteq \mathrm{paths}\,(t)$ is immediate by the definition of E. Lastly, let's show that this tree is the smallest such tree for the sub-tree relation. Let's consider another tree $t' := (V'', E', \mathcal{R}_{t'})$ that satisfies $\mathrm{paths}\,(t') \supseteq S$. Then every path of S is a path of t': as a result, all nodes passed over by sequences of S are elements of V''. So, $V' \subseteq V''$. Furthermore, if we take $(n_k)_{N \geq k \geq 0} \in S$ and $k \in \{0, ..., N-1\}$, then we know that $(n_k, n_{k+1}) \in E'$, because (n_k) is a path of t'. As a result, we have $E \subseteq E'$. Lastly, because (n_k) is a path of t', we necessarily have $\mathcal{R}_{t'} = \mathcal{R}$. So t is a sub-tree of t', and we conclude that $\langle S \rangle := t$ is well defined. $\square$

Remark 4. In the previous proposition, the set of hypotheses on S isn't sufficient to guarantee $\mathrm{paths}\,(t) = S$. For example, if we were to take the set of nodes $V = \{0, 1\}^*$, set of finite words over the alphabet $\{0, 1\}$, and as root ε, the empty word, then we can take the following set S:

$$S := \bigcup_{N \in \mathbb{N}} \{(\varepsilon, 0, 00, 000, ..., 0^N, 10^N, 110^N, ...)\}$$

where 0^N designates the word consisting of N zeroes. Then in the generated tree $\langle S \rangle$, the infinite path $(0^n)_{n\in\mathbb{N}}$ exists, even though it is not a path of S.

Lemma 3. *If $t = (V, E, \mathcal{R})$ is a tree, then any non-empty subset S of* paths (t) *verifies the properties of Proposition 1. This result holds true for favoritism trees.*

Proof. Because $S \subseteq$ paths (t), the *(root start)* property is verified by all node sequences of S. Let's consider a node path $(n_k)_{N\geq k\geq 0} \in S$ and distinct $i, j \in \{0, ..., N\}$. Without loss of generality, we can suppose $i < j$. The path from $\mathcal{R}$ to n_j can be obtained by taking the one from $\mathcal{R}$ to n_i and adding $n_{i+1}, ..., n_j$, which remains a node path in t. Because the path from the root to n_j is unique, we necessarily have $n_i \neq n_j$, hence the *(distinct nodes)* property.

If $(m_j)_{K\geq j\geq 0} \in S$ is another node path, and if there exist $i \in \{0, ..., N\}$ and $j \in \{0, ..., K\}$ such that $n_i = m_j$, then we have $i = j$ by comparing path lengths. When $i > 1$, $n_i = m_j$ implies $n_{i-1} = \mathbf{p}(n_i) = \mathbf{p}(m_j) = m_{j-1}$, hence the *(rank respect)* property. □

1.3 The $\mathbb{Z}_p$ tree

In the construction of the $\mathbb{Z}_p$ tree, we will be using the following notations: if Σ is a finite alphabet, then Σ^* is the set of finite words over Σ, and ε is the empty word. The concatenation of two words u, v is denoted $u \cdot v$, and the length of a word w is denoted $|w|$. The individual letters of a word w of length n are $w_1, ..., w_n$ such that $w = w_n \cdot ... \cdot w_1$.

Definition 9 ($\mathbb{Z}_p$ favoritism tree). *The favoritism tree $(V, E_p, \varepsilon, \triangleleft)$ is called the $\mathbb{Z}_p$ favoritism tree, where:*

- $V = \Sigma^*$, *with* $\Sigma = \{0, ..., p-1\}$
- $E_p = \{(x, y) : \exists a \in \Sigma,\ y = a \cdot x\}$
- *The favoritism function $\triangleleft$ is defined over the children of a node $x \in V$ by* $0 \cdot x \triangleleft 1 \cdot x \triangleleft\ ... \triangleleft\ (p-1) \cdot x$

The $\mathbb{Z}_p$ favoritism tree is denoted Z_p.

Proof. Let's verify that Z_p is a well-defined favoritism tree.

To begin, (V, E_p, ε) is a tree, since any word can be uniquely constructed through finite concatenation from the empty word ε. In other words, there is a unique path from the root to any node.

Furthermore, for all $x \in V$, $\mathbf{ch}(x)$ is finite, and $\triangleleft$ is a total order over $\mathbf{ch}(x)$. □

Example 4. Figures 5 and 6 show partial representations of Z_2 and Z_3 respectively. These representations place the root at the top, but other representations are possible. Figure 1 shows a circular representation of different sub-trees of Z_2, Z_3 and Z_5.

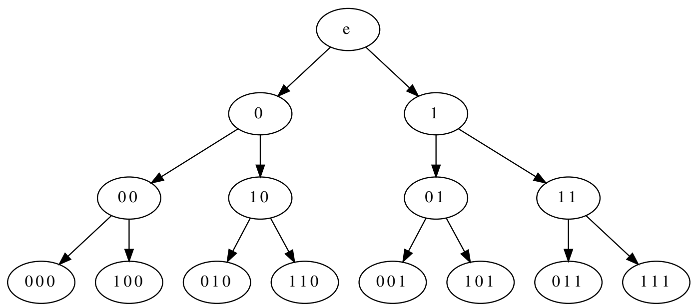


Fig. 5. A sub-tree of Z_2.

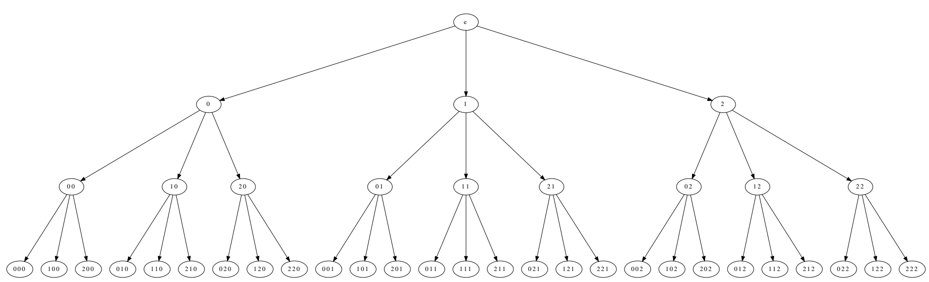


Fig. 6. A sub-tree of Z_3.

The Barchi set $\mathbb{T}_p$

Chapter II

In the first section, we will define the Barchi set $\mathbb{T}_p$ as a quotient of $\bar{\mathbb{N}} \times \mathbb{Z}_p$ by a surjection $\chi : \bar{\mathbb{N}} \times \mathbb{Z}_p \to \mathrm{paths}\,(Z_p)$. There then exists a one-to-one $\bar{\chi} : \mathbb{T}_p \to \mathrm{paths}\,(Z_p)$, called the Barchi bijection that is fundamental in the study of the Barchi set.

In section 2.2, we will introduce a natural partial order over $\mathbb{T}_p$, which corresponds to the usual ancestry relation in a tree. We will also define a metric over $\mathbb{T}_p$, which is both similar to an intuitive distance over a tree, but also similar to the p-adic absolute value.

In section 2.3, we discuss topological properties of $\mathbb{T}_p$. Notably, we will show some results on convergence that are different from the usual setting of real sequences or p-adic sequences. Moreover, using these results we show that $\mathbb{T}_p$ is complete and compact.

In section 2.4, we will show the previously-mentioned down-complete poset (dcpo) structure of $\mathbb{T}_p$, and we will obtain a first fixed-point theorem. We will also introduce sum and multiplication, before showing that $\mathbb{Z}_p$ and each $\mathbb{Z}/p^n\mathbb{Z}$ can be naturally seen as a ring included in $(\mathbb{T}_p, +, \cdot)$.

In section 2.5, we will formally introduce the notion of sub-tree of $\mathbb{T}_p$, before showing that it is exactly the same as a sub-tree of Z_p. Moreover, sub-trees of $\mathbb{T}_p$ are compact, and the set of sub-trees constitutes a complete lattice granting us a second fixed-point theorem. We will then introduce the notion of p-bounded trees, and justify the study of $\mathbb{T}_p$ by showing theorem 3, which states that "the set of structural paths of a p-bounded tree is compact", an assertion we will clarify.

2.1 The Barchi set $\mathbb{T}_p$

We denote $\bar{\mathbb{N}} = \{0, 1, 2, \ldots, \infty\}$ the set of non-negative numbers and infinity.

Definition 1. *We define the function $\chi : \bar{\mathbb{N}} \times \mathbb{Z}_p \to \mathrm{paths}\,(Z_p)$ by the following. For all $(n, x) \in \bar{\mathbb{N}} \times \mathbb{Z}_p$, denote $(x_k)_{k\geq 0} \in \{0, ..., p-1\}^{\mathbb{N}}$ the series representation of x.*

- *If $n < \infty$ then $\chi(n, x)$ is the (finite) node path defined by $n_0 = \mathcal{R}_{Z_p}$, and for all $0 \leq k \leq n-1$, $n_{k+1} = \boldsymbol{ch}\,(n_k)\,.(x_k)$,*
- *If $n = \infty$ then $\chi(n, x)$ is the (infinite) node path defined by $n_0 = \mathcal{R}_{Z_p}$, and for all $k \in \mathbb{N}$, $n_{k+1} = \boldsymbol{ch}\,(n_k)\,.(x_k)$.*

Lemma 1. *We have the following properties:*

1. *χ is a surjection.*

2. *For any* $(n,x),(m,y) \in \bar{\mathbb{N}} \times \mathbb{Z}_p$, $\chi(n,x) = \chi(m,y)$ *if and only if* $n = m$ *and* $|x-y|_p \leq p^{-n}$.

Proof. For the first point, consider a finite node path $N = (n_k)_{K \geq k \geq 0} \in \text{paths}\,(Z_p)$, and the associated structural path $(a_k)_{K-1 \geq k \geq 0} \in \{0, ..., p-1\}^K$. By taking the p-adic integer x defined by:

$$x := \sum_{k=0}^{K-1} a_k p^k \in \mathbb{Z}_p$$

we have that $\chi(n,x) = N$. The same reasoning goes for infinite nodes.

Note that we have here chosen to extend the sequence (a_k) by implicitly defining $a_k = 0$ for $k > n$. However, by the definition of χ, we could have taken any other extension of (a_k), granting us the following direction of the second point:

$$\left(n = m \text{ and } |x-y|_p \leq p^{-n}\right) \implies (\chi(n,x) = \chi(n,y))$$

Lastly, if we have $\chi(n,x) = \chi(m,y)$, then the lengths of the node paths are equal: $n = m$. It follows that the first $n \geq 1$ terms of the series representation of x and y are equal, and so $|x-y|_p \leq p^{-n}$. □

Definition 2. *We define the set* $\mathbb{T}_p$ *as the quotient of* $\bar{\mathbb{N}} \times \mathbb{Z}_p$ *by* χ, *i.e by the equivalence relation* $\sim$ *defined by:*

$$(n,x) \sim (m,y) \iff \chi(n,x) = \chi(m,y)$$

The class of an element $(n,x) \in \bar{\mathbb{N}} \times \mathbb{Z}_p$ *is denoted* $[n,x] \in \mathbb{T}_p$, *except for the class of* $(0,x) \in \bar{\mathbb{N}} \times \mathbb{Z}_p$, *which is denoted* $\mathbf{B}$. *By the previous lemma, we know that the function* $[n,x] \mapsto n$ *is well-defined, and will henceforth be denoted* ω.

There is a bijection $\bar{\chi} : \mathbb{T}_p \to \text{paths}\,(Z_p)$, *such that for any* $(n,x) \in \bar{\mathbb{N}} \times \mathbb{Z}_p$, $\bar{\chi}([n,x]) = \chi(n,x)$. $\bar{\chi}$ *is called the* ***Barchi*** *bijection.*

2.2 First properties of $\mathbb{T}_p$

2.2.1 Partial order

Definition 3 (Ancestry, Alignment). *For any* $N = [n,x], M = [m,y] \in \mathbb{T}_p$, *denote* $\bar{\chi}(N) = (n_k)_{n \geq k \geq 0}$ *and* $\bar{\chi}(M) = (m_j)_{m \geq j \geq 0}$. *The set* $\{k \geq 0 : n_k = m_k\}$ *contains* 0 *and has a maximum* $r \geq 0$. *We define the* ***nearest common ancestor*** *of* N *and* M, *denoted* $nca(N,M)$, *as* $\bar{\chi}^{-1}((n_k)_{r \geq k \geq 0})$.

We say that N *is an* ***ancestor*** *of* M, *denoted* $N \leq M$, *when* $nca(N,M) = N$. *Furthermore, we say that* N *and* M *are* ***aligned*** *when* $N \leq M$ *or* $M \leq N$.

Lemma 2. $N = [n,x] \leq [m,y] = M$ *iff* $n \leq m$ *and* $N = [n,y]$.

Proof. Suppose $N \leq M$. Consider the series representations $(x_k)_{k\geq 0}$ and $(y_k)_{k\geq 0}$ of x and y respectively. By definition of $\leq$, we have $nca(N, M) = N$. In terms of node paths of Z_p, we have that the node path $\bar{\chi}(M)$ coincides with that of $\bar{\chi}(N)$ up until the node $\bar{\chi}(N)$ is reached. This already grants us $n \leq m$. Moreover, the structural node paths associated to $\bar{\chi}(M)$ and $\bar{\chi}(N)$ coincide for the first n terms. Subsequently, by definition of the Barchi bijection, we have:

$$\forall k \in \{0, ..., n-1\}, \quad x_k = y_k$$

which grants us $N = [n, y]$ by lemma 1.

Conversely, if $n \leq m$ and $N = [n, y]$, then the previous reasoning works in reverse, by comparing structural paths. □

Lemma 3. *$\leq$ is a partial order over $\mathbb{T}_p$.*

Proof. $\leq$ verifies the following:

– (Reflexivity) For any $N \in \mathbb{T}_p$, $nca(N, N) = N$ and so $N \leq N$.
– (Transitivity) For any $A, B, C \in \mathbb{T}_p$ such that $A \leq B$ and $B \leq C$, we have $nca(A, B) = A$, and $nca(B, C) = B$. The rigorous proof that this implies $nca(A, C) = A$ is left to the reader, but can otherwise be seen on a diagram.
– (Antisymmetry) For any $N, M \in \mathbb{T}_p$ such that $N \leq M$ and $M \leq N$, we have $M = nca(N, M) = N$.

So $(\mathbb{T}_p, \leq)$ is partially ordered. □

Lemma 4. *For any $A, B, C \in \mathbb{T}_p$, if $B \leq A$ and $C \leq A$ then B and C are comparable. In other words, the set of ancestors of A is a totally ordered set with respect to $\leq$.*

Proof. We have $nca(A, B) = B$ and $nca(A, C) = C$. By lemma 2, if $A = [n, x]$ then $B = [\omega(B), x]$ and $C = [\omega(C), x]$. It follows by this same lemma that if $\omega(B) \leq \omega(C)$ then $B \leq C$, and if $\omega(B) \geq \omega(C)$ then $C \leq B$. □

2.2.2 Metric over $\mathbb{T}_p$

Definition 4. *For any two $N, M \in \mathbb{T}_p$, we define $d(N, M)$ by:*

– *When N and M are aligned, then*

$$d(N, M) := \sum_{k=\min(\omega(N),\omega(M))+1}^{\max(\omega(M),\omega(N))} p^{-k}$$

– *Otherwise,*

$$d(N, M) := d(N, nca(N, M)) + d(nca(N, M), M)$$

where $d(N, nca(N, M))$ falls into the previous case.

Lemma 5. *For any $N, M \in \mathbb{T}_p$,*

$$d(N,M) = d(N, nca(N,M)) + d(nca(N,M), M)$$

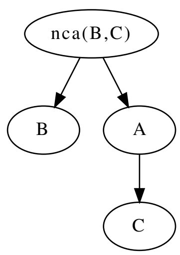
Proof. Whenever N and M are not aligned, this result is the definition. Whenever N and M are aligned, however, we have $nca(N,M) = N$ or $nca(N,M) = M$, meaning $d(N, nca(N,M)) = 0$ or $d(M, nca(N,M)) = 0$. Following the different cases, we obtain the above equality. □

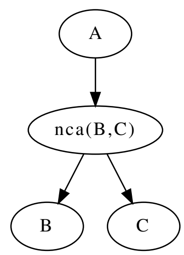
Lemma 6. *For any $A, B, C \in \mathbb{T}_p$ such that $A \leq C$, we have:*

$$d(B,C) \leq d(B,A) + d(A,C)$$

with equality if and only if $nca(B,C) \leq A$.

The below figures illustrate lemma 6. The left figure shows the case $nca(B,C) \leq A$, when equality occurs, and the right figure shows the case $A \leq nca(B,C)$.





Proof. We denote $D := nca(B,C)$.

- When $D \leq A \leq C$, we show that $nca(A,B) = D$, which is left to the reader, but can be seen on the left-hand diagram above. We subsequently have:

$$d(A,B) = d(A,D) + d(D,B)$$

and

$$\begin{aligned} d(A,C) &= \sum_{k=\omega(A)+1}^{\omega(C)} p^{-k} \\ &= \sum_{k=\omega(D)+1}^{\omega(C)} p^{-k} - \sum_{k=\omega(D)+1}^{\omega(A)} p^{-k} \\ &= d(C,D) - d(A,D) \end{aligned}$$

By summing these equations, we have $d(A,B)+d(A,C) = d(D,B)+d(C,D) = d(B,C)$.

- When $A \leq D \leq C$, which corresponds to the right-hand diagram above, we show that $nca(A, B) = A$, and we have:

$$\begin{aligned} d(A,B) &= \sum_{k=\omega(A)+1}^{\omega(B)} p^{-k} \\ &= \sum_{k=\omega(A)+1}^{\omega(D)} p^{-k} + \sum_{k=\omega(D)+1}^{\omega(B)} p^{-k} \\ &= d(A,D) + d(D,B) \end{aligned}$$

  and

$$d(A,C) = d(A,D) + d(D,C)$$

  in the same way. By summing these results, we have:

$$d(B,C) = d(B,D) + d(D,C) = d(B,A) + d(A,C) - 2 \cdot d(D,A)$$

Given these results, and seeing that $d(D, A) = 0$ if and only if $A = D$ (which will be shown when d is proven to be a distance), we have the inequality, and equality if and only if $D \leq A$. □

Lemma 7. *For any $A, B, C \in \mathbb{T}_p$, we have:*

$$d(A,B) \leq d(A, nca(A,C)) + d(nca(A,C), B)$$

Proof. Immediate application of lemma 6, with $nca(A, C), B$ and A instead of A, B and C respectively, seeing that $nca(A, C) \leq A$. □

Proposition 1. *d is a distance over $\mathbb{T}_p$.*

Proof. $d : \mathbb{T}_p \times \mathbb{T}_p \to \mathbb{Q}_+$ verifies:

- (Definite positivity) For any $N, M \in \mathbb{T}_p$ such that $d(N, M) = 0$, we have $d(N, nca(N, M)) = 0$. Because N and $nca(N, M)$ are aligned, we have $\omega(N) = \omega(nca(N, M))$, and so $N = nca(N, M)$ by lemma 1. Similarly, $M = nca(N, M)$ and so $N = M$.
- (Symmetry) d is symmetric by definition.
- (Triangle inequality) For any $A, B, C \in \mathbb{T}_p$, we have by lemma 7 that

$$d(A,B) \leq d(A, nca(A,C)) + d(nca(A,C), B)$$

  and subsequently

$$d(A,B) \leq d(A,C) + d(nca(A,C), B) - d(nca(A,C), C)$$

  We similarly have that

$$d(B, nca(A,C)) \leq d(B, nca(B,C)) + d(nca(B,C), nca(A,C))$$

by lemma 6, where we replaced A, B and C by $nca(B,C)$, $nca(A,C)$ and B respectively. It follows that:

$$\begin{aligned} d(A,B) \leq \quad & d(A,C) + d(B, nca(B,C)) \\ & + d(nca(B,C), nca(A,C)) \\ & - d(nca(A,C), C) \\ \\ = \quad & d(A,C) + d(C,B) \\ & + d(nca(B,C), nca(A,C)) \\ & - d(nca(A,C), C) - d(nca(B,C), C) \end{aligned}$$

since $d(C,B) = d(B, nca(B,C)) + d(nca(B,C), C)$. But

$$d(nca(B,C), nca(A,C)) \leq d(nca(A,C), C) + d(C, nca(B,C))$$

Indeed, $nca(B,C)$ and $nca(A,C)$ are both ascendants of C, meaning that they are comparable for $\leq$. The previous inequality can be shown by expressing each term as a sum.
We've therefore shown that:

$$d(A,B) \leq d(A,C) + d(C,B)$$

So $(\mathbb{T}_p, d)$ is a metric space. □

2.3 Topology over $\mathbb{T}_p$

In the following paragraphs, we will denote $B(x,r)$ the open ball centered at $x \in \mathbb{T}_p$ of radius $r \in \mathbb{Q}_+$, for d, and $\bar{B}(x,r)$ the closed ball.

2.3.1 Results on convergence

Lemma 8. *For any sequence $(N_k)_{k\geq 0}$ of elements of $\mathbb{T}_p$, $(N_k)_{k\geq 0}$ converges to $N \in \omega^{-1}(\{\infty\})$ if and only if $(nca(N_k, N))_{k\geq 0}$ converges to N.*

Proof. Suppose that $(nca(N_k, N))_{k\geq 0}$ converges to $N \in \omega^{-1}(\{\infty\})$. Because $M_k := nca(N_k, N)$ and N are aligned for any $k \geq 0$, we have:

$$d(M_k, N) = \sum_{j=\omega(M_k)+1}^{\infty} p^{-j} \underset{k\to\infty}{\longrightarrow} 0$$

Moreover, $(\omega(M_k))_{k\geq 0}$ converges to ∞. But by lemma 7, $d(N, N_k) = d(N, M_k) + d(M_k, N_k)$ for all $k \geq 0$. We therefore need to show that $(d(M_k, N_k))_{k\geq 0}$ converges to 0. We have:

$$d(M_k, N_k) = \sum_{j=\omega(M_k)+1}^{\omega(N_k)} p^{-j} \leq \sum_{j=\omega(M_k)+1}^{\infty} p^{-j} = d(M_k, N) \underset{k\to\infty}{\longrightarrow} 0$$

and so $N_k \xrightarrow[k\to\infty]{d} N$.

Conversely, suppose that $(N_k)_{k\geq 0}$ converges to N. Then:

$$d(N_k, N) = d(N_k, M_k) + d(M_k, N) \underset{k\to\infty}{\longrightarrow} 0$$

Because d is positive, we have $d(M_k, N) \underset{k\to\infty}{\longrightarrow} 0$, and so $(nca(N_k, N))_{k\geq 0}$ also converges to N. □

Note that this characterization falls short in the case of $\omega(N) < \infty$. For instance, if $K = [\infty, x]$ is a descendant of $N = [\omega(N), x]$, then $(nca(N, [k, x]))_{k\geq 0}$ converges to N while $([k, x])_{k\geq 0}$ converges to $K \neq N$. However, if $(\omega(N_k))_{k\geq 0}$ is bounded by $\omega(N)$, then the characterization holds. However, if $(N_k)_{k\geq 0}$ converges to N, then $(nca(N_k, N))_{k\geq 0}$ also converges to N.

Lemma 9. *For any sequence $(N_k)_{k\geq 0}$ and for any $N \in \mathbb{T}_p$ such that $\omega(N) \geq \omega(N_k)$ for all $k \geq 0$, the following assertions are equivalent:*

1. *$(N_k)_{k\geq 0}$ converges to N,*
2. *$(nca(N_k, N))_{k\geq 0}$ converges to N.*

Regardless of the condition on $(\omega(N_k))_{k\geq 0}$, 1. $\Longrightarrow$ 2. is always true.

Proof. We denote $M_k := nca(N_k, N)$ for any $k \geq 0$. For any $k \geq 0$,

$$\begin{aligned} d(N_k, N) &= d(N_k, M_k) + d(M_k, N) \\ &= \sum_{j=\omega(M_k)+1}^{\omega(N_k)} p^{-j} + \sum_{j=\omega(M_k)+1}^{\omega(N)} p^{-j} \\ &\leq 2d(M_k, N) \end{aligned}$$

which establishes 2. $\Longrightarrow$ 1. Moreover, the first equality (obtained by lemma 7) establishes 1. $\Longrightarrow$ 2., and this did not require the condition over $(\omega(N_k))_{k\geq 0}$. To summarize, we have:

$$\forall k \geq 0,\ d(M_k, N) \leq d(N_k, N) \leq 2d(M_k, N)$$

and the rightmost inequality holds as long as $\omega(N_k) \leq \omega(N)$. □

Lemma 10. *If $N < M$ and $(M_k)_{k\geq 0}$ converges to M, then there exists $k \geq 0$ such that $N < nca(M_k, M) \leq M$.*

Proof. By the previous lemma, $(nca(M_k, M))_{k\geq 0}$ converges to M. We denote $K_k := nca(M_k, M)$ for all $k \geq 0$. Then $(\omega(K_k)) \underset{k\to\infty}{\longrightarrow} \omega(M)$, and because $\omega(N) < \omega(M)$, there exists $k \geq 0$ such that $\omega(N) < \omega(K_k) \leq \omega(M)$, but since K_k, N, and M are aligned, this implies $N \leq K_k \leq M$. □

Lemma 11. *If N is a finite node (i.e $\omega(N) < \infty$), then for any sequence $(N_k)_{k\geq 0}$ that converges to N, $N_k = N$ for any k past a certain rank.*

Proof. By lemma 8, $(nca(N_k, N))_{k\geq 0}$ converges to N. Because $nca(N_k, N) \leq N$, this implies that $\omega(nca(N_k, N))$ converges to $\omega(N)$. So there exists $k_0 \geq 0$ such that for all $k \geq k_0$, $nca(N_k, N) = N$, i.e:

$$\forall k \geq k_0,\ N \leq N_k$$

Then:

$$\forall k \geq k_0,\ d(N_k, N) = \sum_{j=\omega(N)+1}^{\omega(N_k)} p^{-j}$$

and so $(\omega(N_k))_{k\geq 0}$ converges to $\omega(N)$. This implies the existence of $k_1 \geq k_0$ such that for all $k \geq k_1$, $\omega(N_k) = \omega(N)$, and so:

$$\forall k \geq k_1,\ N_k = N$$

□

Lemma 12. *For any $N \in \mathbb{T}_p$, $nca(N, \cdot)$ is continuous.*

Proof. Let $(M_k)_{k\geq 0}$ be a sequence of elements of $\mathbb{T}_p$ that converges to $M \in \mathbb{T}_p$.

If $N < M$, then by lemma 10 there exists k_0 such that for all $k \geq k_0$, $N < M_k \leq M$, and so $nca(N, M_k) = N$ for all $k \geq k_0$.

If $N = M$, then $\lim nca(M_k, N) = N$ by lemma 9.

If $M < N$, then there exists $k_0 \geq 0$ such that for all $k \geq k_0$, $M_k < N$ and so $\omega(M_k) < \omega(N)$. It follows that $\lim nca(N, M_k) = \lim M_k = M = nca(N, M)$.

□

Proposition 2 (Squeeze theorem). *For any sequences $(A_n), (B_n)$ and (C_n) of elements of $\mathbb{T}_p$ such that*

$$\forall n \geq 0, \quad A_n \leq B_n \leq C_n$$

then:

1. *If $L := \lim A_n = \lim C_n$, then $\lim B_n = L$,*
2. *If (A_n) and (B_n) are convergent, then $\lim A_n \leq \lim B_n$,*
3. *If $\lim \omega(A_n) = \infty$, then $\lim \omega(B_n) = \infty$.*

Proof. 1. We have $nca(A_n, L) \leq nca(B_n, L) \leq nca(C_n, L)$ for all $n \geq 0$, and so:

$$\begin{aligned}\forall n \geq 0,\ d(B_n, L) &= d(B_n, nca(B_n, L)) + d(nca(B_n, L), L)\\ &\leq d(C_n, nca(B_n, L)) + d(nca(A_n, L), L)\\ &\leq d(C_n, nca(A_n, L)) + d(nca(A_n, L), L)\end{aligned}$$

By lemma 9, we know that $\lim nca(A_n, L) = L$, so $\lim d(C_n, nca(A_n, L)) = 0$. So $\lim d(B_n, L) = 0$.

2. Denote $L := \lim A_n$. We know that $\lim nca(A_n, L) = L$ by lemma 9. But $nca(A_n, L) \leq nca(B_n, L) \leq L$, and so by application of 1., we have $\lim nca(B_n, L) = L$. Subsequently, because $\lim nca(B_n, L) = nca(\lim B_n, L)$ by lemma 12, we have $L = \lim A_n \leq \lim B_n$.
3. We have $\omega(A_n) \leq \omega(B_n)$ for all $n \geq 0$. □

2.3.2 Completeness and compactness

Proposition 3. *$\mathbb{T}_p$ is complete for d.*

Proof. Consider some Cauchy sequence $(N_k)_{k\geq 0}$ of $\mathbb{T}_p$. We have two cases:

- Suppose that $(\omega(N_k))_{k\geq 0}$ is eventually constant, past some rank $n \in \mathbb{N}$. Then for some rank $N \geq n$, for any $k \geq N$, we have $d(N_k, N_N) = 0$, and so $N_k = N_N$. It follows that $(N_k)_{k\geq 0}$ converges to $N_N \in \mathbb{T}_p$.
- When $(\omega(N_k))_{k\geq 0}$ is not eventually constant, then we can extract $(N_{n_k})_{k\geq 0}$ such that for no $k \geq 0$ do we have $\omega(N_k) = \omega(N_{k+1})$. For any fixed $K \in \mathbb{N}$, there exists $\alpha_K \in \mathbb{N}$ such that

  $$\forall k, l \geq \alpha_K, \quad d(N_{n_k}, N_{n_l}) \leq p^{-K}$$

  It follows that:

  $$\forall k \neq l \geq \alpha_K, \qquad \sum_{n=\min(\omega(N_{n_k}),\omega(N_{n_l}))+1}^{\max(\omega(N_{n_k}),\omega(N_{n_l}))} p^{-n} \leq p^{-K}$$

  and so for any $k \geq \alpha_K$, if we suppose $\omega(N_{n_{k+1}}) > \omega(N_{n_k})$, then:

  $$\left(\omega(N_{n_{k+1}}) - \omega(N_{n_k})\right) p^{-\omega(N_{n_{k+1}})} \leq p^{-K}$$

  Consequently, $\omega(N_{n_{k+1}}) \geq K$. Similarly, whenever $\omega(N_{n_k}) > \omega(N_{n_{k+1}})$, we have $\omega(N_{n_k}) \geq K$. Most notably, we have that for any $k \geq \alpha_K$,

  $$\max\left(\omega(N_{n_k}), \omega(N_{n_{k+1}})\right) \geq K$$

  We can therefore extract anew to obtain $(N_{m_k})_{k\geq 0}$ such that for any $k \geq 0$, $\omega(N_{m_k}) \geq k$. This sequence is still a Cauchy sequence, so consider a fixed $K \in \mathbb{N}$. There exists $\beta_K \in \mathbb{N}$ such that

  $$\forall k, l \geq \beta_K, \quad d(N_{m_k}, N_{m_l}) \leq p^{-K}$$

  If we were to write the distance as a sum (or eventually two sums), we would have that $\omega(nca(N_{m_k}, N_{m_l})) \geq K$ for any $k, l \geq \beta_k$. It follows that for any $k, l \geq \beta_K$, at least the first $K-1$ terms of any associated series representation of N_{m_k} and N_{m_l} coincide.
  As we can find some such β_K for any $K \in \mathbb{N}$, we can define $x \in \mathbb{Z}_p$ such that the first $K-1$ terms of the series representation of x are exactly the first $K-1$ terms of any series representation of N_{m_k} with $k \geq \beta_K$. Subsequently, we define $L := [\infty, x]$, and let's show that $(N_{m_k})_{k\geq 0}$ converges to L (which will show that $(N_k)_{k\geq 0}$ converges to L). Let $\varepsilon > 0$ and consider $K \in \mathbb{N}$ such that $p^{-K} \leq \varepsilon \times \frac{p-1}{2p}$. Then, for any $k, l \geq \beta_K$, we have

  $$d(N_{m_k}, N_{m_l}) \leq p^{-K} \leq \varepsilon \times \frac{p-1}{2p}$$

For any $k \geq \beta_K$, we have seen that if $N_{m_k} = [i, y]$, then $[K-1, y] \leq L$. It follows that

$$\begin{aligned} d(N_{m_k}, L) &\leq d(L, [K-1, y]) + d([K-1, y], [i, y]) \\ &= \sum_{j=K}^{\infty} p^{-j} + d([K-1, y], [i, y]) \\ &\leq p^{-K} \times \left(\sum_{j=0}^{\infty} p^{-j} + \sum_{j=0}^{i-K} p^{-j} \right) \\ &\leq p^{-K} \times \frac{2p}{p-1} \leq \varepsilon \end{aligned}$$

Subsequently, for any $k \geq \beta_K \in \mathbb{N}$,

$$d(N_{m_k}, L) \leq \varepsilon$$

And so $(N_k)_{k \geq 0}$ converges in $(\mathbb{T}_p, d)$. We have therefore shown that $(\mathbb{T}_p, d)$ is complete. □

Lemma 13. *For any $x \in \mathbb{Z}_p$ and $k \in \bar{\mathbb{N}}$,*

$$[\infty, x] \in B\left([k, x], p^{-k} \times \frac{p}{p-1}\right)$$

Proof. All we need to show is that

$$d([\infty, x], [k, x]) < p^{-k} \times \frac{p}{p-1}$$

but since $[k, x] \leq [\infty, x]$, we have:

$$d([\infty, x], [k, x]) = \sum_{j=k+1}^{\infty} p^{-j} = \frac{p^{-k}}{p-1} < p^{-k} \times \frac{p}{p-1}$$

so $[\infty, x] \in B([k, x], p^{-k} \times \frac{p}{p-1})$. □

Proposition 4. *$\mathbb{T}_p$ is compact.*

Proof. Let's show that $\mathbb{T}_p$ is precompact. Let $\varepsilon > 0$, and consider $K \in \mathbb{N}$ such that

$$p^{-K} \times \frac{p}{p-1} \leq \varepsilon$$

By lemma 13,

$$\omega^{-1}(\{\infty\}) \subseteq \bigcup_{N \in \omega^{-1}(\{K\})} B(N, \varepsilon)$$

Given that this inclusion is also valid for any $j \geq K$ instead of ∞, we need only add balls to cover $\bigcup_{k<K} \omega^{-1}(\{k\})$. But this set is finite, and so:

$$\mathbb{T}_p \subseteq \bigcup_{k \leq K} \left(\bigcup_{N \in \omega^{-1}(\{k\})} B(N, \varepsilon) \right)$$

Where only a finite number of balls are on the right hand side.

We've now shown that $\mathbb{T}_p$ is precompact. Exploiting proposition 3, $\mathbb{T}_p$ is also complete. It follows that $\mathbb{T}_p$ is compact. □

Corollary 1. *$K \subseteq \mathbb{T}_p$ is compact if and only if it is closed.*

Proof. Immediate consequence of the previous proposition. □

Lemma 14. *For all $x, y \in \mathbb{Z}_p$,*

$$d([\infty, x], [\infty, y]) = \frac{2}{p-1} |x - y|_p$$

Proof. Denote $K = [k, x] = nca([\infty, x], [\infty, y])$. Then the first k digits (i.e terms of the series representation) of x coincide with the first k digits of y. Because K is the *nearest* common ancestor, we have that k is maximal. It follows that $|x - y|_p = p^{-k}$. But:

$$d([\infty, x], [\infty, y]) = \sum_{j=k+1}^{\infty} p^{-j} + \sum_{j=k+1}^{\infty} p^{-j} = \frac{2p^{-k}}{p-1}$$

So $d([\infty, x], [\infty, y]) = \frac{2}{p-1} |x - y|_p$. □

2.4 Additional properties of the Barchi set

2.4.1 Down-complete poset (dcpo) structure

Definition 5 (Downward directed set). *A **(downward) directed set** is a non-empty preordered set A such that for any a and b in A there exists c in A for which $c \leq a$ and $c \leq b$.*

Definition 6 (Down-complete poset (dcpo)). *A partially ordered set X is a **down-complete poset** (dcpo) if every downward directed subset has an infimum in X.*

Proposition 5. *$\mathbb{T}_p$ is a dcpo. Additionally, for any downward directed subset F, the infimum of F is in F.*

Proof. Thanks to the Barchi bijection, it is sufficient to show that the set of node paths of Z_p is a dcpo, where the considered order is implicitly defined relatively to $\leq$ over $\mathbb{T}_p$ with the Barchi bijection. Over paths (Z_p), this relation equates to "being a prefix of".

Consider some downward directed set $F \subseteq \text{paths}\,(Z_p)$. Lemma 3 grants us the ability to consider the generated sub-tree $f = \langle F \rangle$ of Z_p with nodes V. If $\mathcal{R}_{Z_p} \in V$ has *more* than one child in f, then there exists $a, b \in F$ such that $nca(a, b) = \mathcal{R}_{Z_p}$ and so $\inf F = \mathcal{R}_{Z_p}$. Furthermore, we have $\mathcal{R}_{Z_p} \in F$ because there exists $c \in F$ such that $c \leq nca(a, b) = \mathcal{R}_{Z_p}$, and so $\mathcal{R}_{Z_p} = c \in F$. On the other hand, if $\mathcal{R}_{Z_p}$ has only one child (if it has no children then $F = \{\mathcal{R}_{Z_p}\}$), then consider the smallest integer $k \geq 1$ such that at least one descendant of $\mathcal{R}_{Z_p}$ of depth k has more than one child. If no such k exists, then all the elements of F are aligned, past which point $\inf F$ trivially exists (and is in F). Suppose now that k exists. Then consider the only (by minimality of k) descendant $n \in V$ of $\mathcal{R}_{Z_p}$ with more than one child. We have two cases:

- If some ancestor n' of n is in F, then the smallest (w.r.t $\leq$) such ancestor is trivially equal to $\inf F$, and is in F.
- If no such ancestor exists, let's show that $n = \inf F$. Consider any element $a \in F$. We have $nca(n, a) \in V$ and $nca(n, a) \leq n$. Because $nca(n, a)$ is an ancestor of n, it either has only one child (if it is distinct from n), or it is equal to n. But if it has only one child, then this child is also a common ancestor of both n and a, which is absurd by minimality of $nca(n, a)$. It follows that $nca(n, a) = n$, and so $n \leq a$.
  For any $n' \in V$ such that $\forall a \in F, n' \leq a$, we can consider $a \in F$ and $b \in F$ such that $nca(a, b) = n$ because n has at least two children. It follows that $n' \leq a$ and $n' \leq b$ so $n' \leq nca(a, b) = n$. Consequently, $n = \inf F$, and $n \in F$ because there exists $c \in F$ such that $c \leq nca(a, b) = n$, but no ancestor of n is in F, so $n = c \in F$.

In either case, $\inf F$ exists in V. Furthermore, we have shown that $\inf F \in F$. Because the Barchi bijection is order-preserving (by definition of the order over paths of Z_p), the result holds for $\mathbb{T}_p$. □

Definition 7 (Scott-continuity). *If X and Y are two dcpos, a function $f : X \to Y$ is said to be* ***Scott-continuous*** *if for any downward directed subset $F \subseteq X$,*

$$\inf f(F) = f(\inf F)$$

Proposition 6. *For any dcpo X, $f : \mathbb{T}_p \to X$ is order-preserving if and only if f is Scott-continuous.*

Proof. Suppose that f is increasing. Let F be a downward directed subset of $\mathbb{T}_p$. By monotony, we have for all $x \in F$ that $f(x) \geq f(\inf F)$. Subsequently, $\inf f(F) \geq f(\inf F)$. We only need to show that $\inf f(F) \leq f(\inf F)$. By proposition 5, $x_0 := \inf F$ is an element of F, and so $f(\inf F) = f(x_0) \geq \inf f(F)$. Hence $f(\inf F) = \inf f(F)$.

If f is Scott-continuous, then consider any $x, y \in \mathbb{T}_p$ with $x \leq y$. $F := \{x, y\}$ is a downward directed subset, and so $\inf f(F) = f(\inf F)$. In other words, $\min(f(x), f(y)) = f(x)$, so $f(x) \leq f(y)$. $\square$

Theorem 1. *Any increasing function $f : F \to \mathbb{T}_p$ over a downward directed subset F of $\mathbb{T}_p$ has a minimum.*

Proof. If $f : F \to \mathbb{T}_p$ is increasing then it is Scott-continuous. It follows that $f(F)$ is also a downward directed subset of $\mathbb{T}_p$. By proposition 5, $\inf f(F)$ exists and is in $f(F)$. Therefore, there exists some $x_0 \in F$ such that:

$$\forall x \in F,\ f(x_0) \leq f(x)$$

$\square$

Corollary 2 (First fixed-point theorem). *For any closed downward directed subset $A \subseteq \mathbb{T}_p$, any increasing function $f : A \to A$ has a least fixed point, denoted* $\mathrm{lfp}(f)$.

Proof. We know that f has a minimum in A, say $x_0 \in A$ such that $f(x) \geq f(x_0)$ for all $x \in A$. Now consider the sequence $(f^n(x_0))_{n \geq 1}$. By compactness of $\mathbb{T}_p$, and therefore of A, $(f^n(x_0))_{n \geq 1}$ has a sub-sequence $(f^{n_k}(x_0))_{k \geq 1}$ that converges towards some $l \in A$. But $(f^n(x_0))_{n \geq 1}$ is increasing: by induction, if $n = 1$ then $f(f(x_0)) \geq f(x_0)$, and for any $n \geq 1$, if $f^{n+1}(x_0) \geq f^n(x_0)$ then:

$$f^{n+2}(x_0) \geq f^{n+1}(x_0)$$

So all of the $f^n(x_0)$, $n \geq 1$, are aligned, and furthermore $f^n(x_0) \leq l$ for any $n \geq 1$. The entire sequence therefore converges towards l, its supremum. Let's show that $f(l) = l$. Suppose that for some $n \geq 1$, $f^n(x_0) = l$. Then $f(l) = f^{n+1}(x_0) \geq l$, so $f(l) = l$. Suppose that for all $n \geq 1$, $f^n(x_0) < l$. Then because f is increasing, $(\omega(f^n(x_0)))_{n \geq 1}$ is an increasing and non-stationary sequence of $\bar{\mathbb{N}}^*$. Consequently, because $\omega(l) \geq \omega(f^n(x_0))$ for all $n \geq 1$, $\omega(l) = \infty$. We can therefore write $l = [\infty, x]$. We also have $\omega(f(l)) = \infty$ since $f(l) \geq f^{n+1}(x_0)$ for all $n \geq 1$. We can therefore write $f(l) = [\infty, y]$. Because for all $n \geq 2$, $f^n(x_0) \leq nca(l, f(l))$, we know that the first $\omega(f^n(x_0))$ terms of the respective series representations of x and y coincide. In other words,

$$\forall n \geq 2, \quad |x - y|_p \leq p^{-\omega(f^n(x_0))}$$

As $\omega(f^n(x_0)) \to \infty$, this implies $x = y$ and consequently $f(l) = l$.

Consider another fixed point $x_1 \in A$. Then $f(x_0) \leq f(x_1) = x_1$ by minimality of $f(x_0)$. By iterating inductively, we get that $f^n(x_0) \leq x_1$ for any $n \geq 1$. Subsequently, $l = \sup(f^n(x_0)) \leq x_1$. Hence, l is the least fixed point of f. $\square$

We will later (section 2.5) work on *sub-trees* of $\mathbb{T}_p$, and these sub-trees will be closed downward directed subsets of $\mathbb{T}_p$. As such, we will be able to apply corollary 2 to any function $f : A \to A$ when A is a sub-tree of $\mathbb{T}_p$. Moreover, note that the least fixed point is given by $\sup_n f^n(x_0)$ where $x_0 \in A$ minimizes f.

2.4.2 Algebraic properties

Definition 8 (Sum and product of nodes). *We define the sum of* $N = [n, x]$ *and* $M = [m, y]$ *by:*

$$N + M := [\min(n, m), x + y]$$

We also define the product by:

$$N \cdot M := [\min(n, m), xy]$$

Lemma 15. $+$ *and* $\cdot$ *are well-defined over* $\mathbb{T}_p$*. Furthermore,* $+$ *and* $\cdot$ *are commutative and associative, and* $\cdot$ *is distributive over* $+$*.*

Proof. For any $[n', x']$ and $[m', y']$ such that $[n, x] = [n', x']$ and $[m, y] = [m', y']$, we have $n = n'$ and $m = m'$. The first $r := \min(n, m)$ terms of the series representations of x and y are respectively equal to the first r terms of the series representations of x' and y'. It follows that the first r terms of the series representation of $x+y$ coincide with the first r terms of the series representation of $x' + y'$. Subsequently, $[r, x' + y'] = [r, x + y]$. Same goes for the product.

$+$ and $\cdot$ are evidently commutative and associative. For any $A = [a, x], B = [b, y]$ and $C = [c, z]$ in $\mathbb{T}_p$, we have:

$$A \cdot (B + C) = A \cdot [\min(b, c), y + z] = [\min(a, b, c), xy + xz]$$

and

$$A \cdot B + A \cdot C = [\min(a, b), xy] + [\min(a, c), yz] = [\min(a, b, c), xy + yz]$$

so $\cdot$ is distributive over $+$. □

Example 1. Let's consider a few calculations. For the sake of simplicity, we will also allow ourselves to write nodes (meaning elements of $\mathbb{T}_p$) as the just the last $\omega(N)$ digits of the associated p-adic number, meaning that we might write 1020 instead of $[4, ...1020]$.

– In $\mathbb{T}_3$, we have:
$$1020 + 121 = 211$$
– In $\mathbb{T}_2$, we have:
$$01 + 11 = 00$$
– In $\mathbb{T}_{11}$, we have:
$$420420 + 696867 = 007187$$

We also have the following products:

– In $\mathbb{T}_3$, $1020 \cdot 121 = 120$, and $1020 \cdot 0121 = 1120$.
– In $\mathbb{T}_2$, $01 \cdot 11 = 11$, and $1 \cdot 11 = 1$.

Proposition 7. $(\omega^{-1}(\{\infty\}), +, \cdot)$ *is a ring, isomorphic to* $\mathbb{Z}_p$*.*

Proof. The unit here is $[\infty, 1]$ and the zero is $[\infty, 0]$. It is easily verified that these are multiplicative and additive identities respectively.

To show the ring structure, it is sufficient to show that for any $N = [\infty, x]$ there exists some $M = [\infty, y]$ such that $N + M = [\infty, 0]$. This is immediate by taking $M = [\infty, -x]$. So $\omega^{-1}(\{\infty\})$ is a ring.

We define the ring homomorphism $\phi_\infty : \mathbb{Z}_p \to \omega^{-1}(\{\infty\})$ by:

$$\forall x \in \mathbb{Z}_p, \quad \phi_\infty(x) := [\infty, x]$$

ϕ_∞ is a ring homomorphism, as $\phi_\infty(1) = \mathbf{1}$, and for any $x, y \in \mathbb{Z}_p$, we have:

$$\phi_\infty(x + y) = [\infty, x + y] = [\infty, x] + [\infty, y] = \phi_\infty(x) + \phi_\infty(y)$$

and

$$\phi_\infty(xy) = [\infty, xy] = [\infty, x] \cdot [\infty, y] = \phi_\infty(x)\phi_\infty(y)$$

Furthermore, ϕ_∞ is a ring isomorphism from $\mathbb{Z}_p$ to $\omega^{-1}(\{\infty\})$ as ϕ_∞ is trivially a bijection. □

Proposition 8. *For any $k \geq 1$, $(\omega^{-1}(\{k\}), +, \cdot)$ is a ring, isomorphic to $\mathbb{Z}/p^k\mathbb{Z}$.*

Proof. The unit is $[k, 1]$ and the zero is $[k, 0]$.

We define $\phi_k : \mathbb{Z}/p^k\mathbb{Z} \to \omega^{-1}(\{k\})$ by:

$$\forall x \in (\mathbb{Z}/p^k\mathbb{Z}) \setminus \{0, 1\}, \quad \phi_k(x) := [k, \hat{x}]$$

where $\hat{x} \in \mathbb{Z}_p$ is obtained by considering the k digits $x_1, ..., x_k \in \{0, ..., p-1\}$ of $x \in \mathbb{Z}/p^k\mathbb{Z}$, and taking $\hat{x} := \sum_{n=0}^{k} x_n p^n$. Then ϕ_k is a ring homomorphism as $\phi_k(1) = [k, 1]$ by definition and for any $x, y \in \mathbb{Z}/p^k\mathbb{Z}$, we have:

$$\phi_k(x + y) = [k, \widehat{x + y}] = [k, \hat{x} + \hat{y}] = [k, \hat{x}] + [k, \hat{y}] = \phi_k(x) + \phi_k(y)$$

Similarly,

$$\phi_k(xy) = [k, \widehat{xy}] = [k, \hat{x}\hat{y}] = [k, \hat{x}] \cdot [k, \hat{y}] = \phi_k(x)\phi_k(y)$$

So ϕ_k is a ring isomorphism from $\mathbb{Z}/p^k\mathbb{Z}$ to $\phi_k(\mathbb{Z}/p^k\mathbb{Z})$ as ϕ_k is trivially a bijection. □

Lemma 16. *For any $k \geq 1$, $\omega^{-1}(\{k\})$ is finite and has exactly p^k elements.*

Proof. Immediate consequence of the previous proposition. □

2.5 *p*-bounded trees

In this section, we will introduce the notion of *sub-tree* of $\mathbb{T}_p$, which we will show to be exactly the same as the notion of sub-tree of Z_p. We also justify here the study of $\mathbb{T}_p$, by showing that any p-bounded tree (favoritism tree in which each node has no more than p children) can be structurally identified to a sub-tree of $\mathbb{T}_p$. A first consequence of this is that any p-bounded tree is "compact" in the sense of the metric d over $\mathbb{T}_p$.

As we will see in this section, results over $\mathbb{T}_p$ carry over to any p-bounded tree, which is not a particularly restrictive imposition over a tree: most often, a tree can be easily turned into a favoritism tree, and rarely is this tree not uniformly bound[1].

2.5.1 Sub-trees

Definition 9 (Sub-tree of $\mathbb{T}_p$). *A non-empty subset $A \subseteq \mathbb{T}_p$ is said to be a sub-tree of $\mathbb{T}_p$ when:*

$$\bar{\chi}(A) = \text{paths}\left(\langle\bar{\chi}(A)\rangle\right)$$

Lemma 17. *A non-empty subset $A \subseteq \mathbb{T}_p$ is a sub-tree of $\mathbb{T}_p$ if and only if both the following assertions are true:*

1. *For every $N \in A$, all ancestors of N are in A,*
2. *Every increasing sequence of nodes of A converges in A.*

Proof. Suppose 1. and 2. We immediately have the inclusion $\bar{\chi}(A) \subseteq \text{paths}\left(\langle\bar{\chi}(A)\rangle\right)$. Consider some node path $(n_k)_{0\leq k\leq K} \in \text{paths}\left(\langle\bar{\chi}(A)\rangle\right)$, and consider $N := \bar{\chi}^{-1}((n_k)) \in \mathbb{T}_p$.

If K is finite, then there exists some $(m_k)_{0\leq k\leq K'} \in \bar{\chi}(A)$ such that:

- $K \leq K'$,
- and $n_k = m_k$ for all $k \leq K$.

Then if we denote $M := \bar{\chi}^{-1}((m_k)) \in A$, we have $N \leq M$, and by 1., we have $N \in A$.

If $K = \infty$, then for every $k \geq 0$, there exists some $(m_{k,j})_{0\leq j\leq K_k} \in \bar{\chi}^{-1}(A)$ such that:

- $k \leq K_j \leq \infty$,
- and $n_i = m_{k,i}$ for every $i \leq k$.

Then, for every $k \geq 0$, if we denote $M_k := \bar{\chi}^{-1}((m_{k,j})_j) \in A$, we can define L_k as an ancestor of $nca(M_k, N)$ such that $\omega(L_k) = k$. Then $(L_k)_{k\geq 0}$ is an increasing sequence of nodes of A (by 1.), and so by 2., $(L_k)_{k\geq 0}$ converges to some $L \in A$. Because for all $k \geq 0$ we have $L_k \leq nca(M_k, N)$, we also have (prop. 2) $nca(M_k, N) \underset{k\to\infty}{\longrightarrow} L$, and so $M_k \underset{k\to\infty}{\longrightarrow} L$ by lemma 8. It follows that $L = N$, and so $N \in A$.

So $\bar{\chi}(A) = \text{paths}\left(\langle\bar{\chi}(A)\rangle\right)$, and A is a sub-tree of $\mathbb{T}_p$.

Conversely, suppose the set equality. Let's show 1. Consider some $N \in A$ and some ancestor $M \leq N$. We denote

$$\bar{\chi}(M) =: (m_k)_{0\leq k\leq K} \text{ and } \bar{\chi}(N) =: (n_k)_{0\leq k\leq K'}$$

[1] A tree is uniformly bound when there exists some integer M such that any node of the tree has no more than M children. Trivially, any uniformly bound tree is p-bounded for some prime p greater than M.

Because $M \leq N$, we have $K \leq K'$ and for all $k \leq K$, $m_k = n_k$. So for all $k \leq K$, $n_k \in \langle \bar{\chi}(A) \rangle$, and it follows that

$$\bar{\chi}(N) = (n_k)_{0 \leq k \leq K} \in \operatorname{paths}(\langle \bar{\chi}(A) \rangle) = \bar{\chi}(A)$$

So $N \in A$. Let's now show 2. Let $(N_k)_{k \geq 0}$ be an increasing sequence of nodes of A. Then $(N_k)_{k \geq 0}$ converges to some $L \in \mathbb{T}_p$ such that $N_k \leq L$ for all $k \geq 0$. Consider the node paths $\bar{\chi}^{-1}(L) =: (l_k)_{0 \leq k \leq K} \in \operatorname{paths}(Z_p)$ and $(n_{k,j})_{0 \leq j \leq K_k} := \bar{\chi}^{-1}(N_k) \in \operatorname{paths}(\bar{\chi}(A))$ for all $k \geq 0$. Because (N_k) is increasing, $(\omega(N_k))$ is increasing and so $(K_k)_{k \geq 0}$ is also increasing, and converges to K. Then for all $k \leq K$, we have $n_{k,i} = l_i$ for all $i \leq K_k$. It follows that for all $k \leq K$, we have that l_k is a node of $\langle \bar{\chi}(A) \rangle$, and so $(l_k)_{0 \leq k \leq K}$ is in $\operatorname{paths}(\langle \bar{\chi}(A) \rangle) = \bar{\chi}(A)$. So $L \in A$. $\square$

Lemma 18. *For any family of sub-trees $(A_i)_{i \in I}$ with $I \neq \emptyset$, $\bigcap_{i \in I} A_i$ is a sub-tree of $\mathbb{T}_p$. If I is finite, then $\bigcup_{i \in I} A_i$ is a sub-tree of $\mathbb{T}_p$.*

Proof. Immediate by the previous lemma. The only verification needed is that the intersection is non-empty: this is immediate as $\mathbf{B} \in A_i$ for all i. $\square$

Note however that $\bigcup_{i \in I} A_i$ is not always a sub-tree if I is infinite. Consider for example some $x \in \mathbb{Z}_p$ and:

$$\forall i \geq 0, \quad A_i := \{[j, x] : 0 \leq j \leq i\}$$

Then $\bigcup_{i \geq 0} A_i$ is not a sub-tree of $\mathbb{T}_p$, as the increasing sequence $([j, x])_{j \geq 0}$ does not converge in $\bigcup_{i \geq 0} A_i$.

Proposition 9. *Sub-trees of $\mathbb{T}_p$ are closed. Because $\mathbb{T}_p$ is compact, sub-trees are compact.*

Proof. Let A be a sub-tree of $\mathbb{T}_p$. Consider a convergent sequence $(a_n)_{n \geq 0}$ of elements of A, that converges to some $a \in \mathbb{T}_p$. If a is finite (i.e $\omega(a) < \infty$), then by lemma 11, $a_n = a$ for all n past a certain rank, and so $a \in A$.

If $\omega(a) = \infty$, then $\omega(a_n) \leq \omega(a)$ for all $n \geq 0$. By lemma 9, the sequence defined by $b_n := nca(a_n, a) \in A$ for all $n \geq 0$ converges to a. Because $\omega(b_n) \longrightarrow \infty$, we can extract a strictly increasing subsequence $(b_{n_k})_{k \geq 0}$ (under the condition that $b_n \neq a$ for all n, in which case $a \in A$). Then every node path $\bar{\chi}(b_{n_k})$ is a prefix of the following one, and the length of these paths goes to infinity. We can therefore define an infinite path δ in Z_p such that every $\bar{\chi}(b_{n_k})$ is a prefix of this path. By the Barchi bijection, $\bar{\chi}^{-1}(\delta) = a$. Notably,

$$\delta \in \operatorname{paths}(\langle \bar{\chi}(A) \rangle)$$

Meaning that $a \in A$ by definition of a sub-tree. $\square$

Lemma 19. *For any sub-tree A of Z_p, $\bar{\chi}^{-1}(\operatorname{paths}(A))$ is a sub-tree of $\mathbb{T}_p$.*

Proof. We have:

$$A = \langle \text{paths}\,(A) \rangle$$

If we write $B := \bar{\chi}^{-1}(\text{paths}\,(A))$, this is exactly the same as writing:

$$\bar{\chi}(B) = \text{paths}\,(\langle \bar{\chi}(B) \rangle)$$

and so B is a sub-tree. □

Proposition 10. *There is a bijection between the set of sub-trees of Z_p and the set of sub-trees of $\mathbb{T}_p$, and this bijection is $A \mapsto \bar{\chi}^{-1}(\text{paths}\,(A))$.*

Proof. We've seen that this function, here denoted f, is well-defined. Consider the function g, which associates to any sub-tree B of $\mathbb{T}_p$, the sub-tree $\langle \bar{\chi}(B) \rangle$ of Z_p. Let's show that $f(g(B)) = B$. We have:

$$f(g(B)) = \bar{\chi}^{-1}\big(\text{paths}\,(g(B))\,\big) = \bar{\chi}^{-1}\big(\text{paths}\,(\langle \bar{\chi}(B) \rangle)\,\big) = \bar{\chi}^{-1}(\bar{\chi}(B)) = B$$

where the last equality stems from the definition of a sub-tree of $\mathbb{T}_p$. Conversely, if A is a sub-tree of Z_p, then:

$$g(f(A)) = \big\langle \bar{\chi}\big(\bar{\chi}^{-1}(\text{paths}\,(A))\big) \big\rangle = A$$

So f is a bijection, and its reciprocal is $\langle \bar{\chi}(\cdot) \rangle$. □

Proposition 11. *The set of sub-trees of $\mathbb{T}_p$ is a complete lattice for $\subseteq$.*

Proof. Consider a family $(A_i)_{i \in I}$ of sub-trees of $\mathbb{T}_p$. If $I = \emptyset$ then $\sup_{i \in I} A_i = \{\mathbf{B}\}$ and $\inf_{i \in I} A_i = \mathbb{T}_p$. Suppose $I \neq \emptyset$. Then $\bigcap_{i \in I} A_i$ is a sub-tree of $\mathbb{T}_p$ by lemma 18, and it is the infimum. For the supremum, we have seen the the union of all of the A_i is not generally a sub-tree. We define:

$$A := \bar{\chi}^{-1}\left(\text{paths}\left(\left\langle \bigcup_{i \in I} \bar{\chi}(A_i) \right\rangle\right)\right)$$

Let's show that A is a sub-tree of $\mathbb{T}_p$. We have:

$$\begin{aligned}
\text{paths}\,(\langle \bar{\chi}(A) \rangle) &= \text{paths}\left(\left\langle \text{paths}\left(\left\langle \bigcup_{i \in I} \bar{\chi}(A_i) \right\rangle\right) \right\rangle\right) \\
&= \text{paths}\left(\left\langle \bigcup_{i \in I} \bar{\chi}(A_i) \right\rangle\right) \\
&= \bar{\chi}(A)
\end{aligned}$$

So A is a sub-tree of $\mathbb{T}_p$. Consider any $i \in I$. Let's show that $A_i \subseteq A$. We have:

$$\bar{\chi}(A_i) = \text{paths}\,(\langle \bar{\chi}(A_i) \rangle) \subseteq \text{paths}\left(\left\langle \bigcup_{j \in I} \bar{\chi}(A_j) \right\rangle\right) = \bar{\chi}(A)$$

Furthermore, consider any sub-tree B of $\mathbb{T}_p$ such that for all $i \in I$, $A_i \subseteq B$. This implies that $\bigcup_{i\in I} A_i \subseteq B$, and so $\bigcup_{i\in I} \bar{\chi}(A_i) \subseteq \bar{\chi}(B)$. It follows that:

$$\bar{\chi}(A) = \text{paths}\left(\left\langle \bigcup_{i\in I} \bar{\chi}(A_i) \right\rangle\right) \subseteq \text{paths}\left(\langle \bar{\chi}(B) \rangle\right) = \bar{\chi}(B)$$

And so $A \subseteq B$. Subsequently, A is the supremum of $(A_i)_{i\in I}$. We have thus shown:

$$\inf_{i\in I} A_i = \bigcap_{i\in I} A_i \text{ and } \sup_{i\in I} A_i = \bar{\chi}^{-1}\left(\text{paths}\left(\left\langle \bigcup_{i\in I} \bar{\chi}(A_i) \right\rangle\right)\right)$$

□

Proposition 12. *For any family $(A_i)_{i\in I}$ of sub-trees of $\mathbb{T}_p$, if $I \neq \emptyset$ then:*

$$\sup_{i\in I} A_i = \overline{\bigcup_{i\in I} A_i}$$

where $\overline{A}$ designates the topological closure of $A \subseteq \mathbb{T}_p$ w.r.t. d.

Proof. For the direct inclusion, let's show that $B := \overline{\bigcup_{i\in I} A_i}$ is a sub-tree of $\mathbb{T}_p$ that contains each A_i.

Consider any $N \in B$. If there exists $i \in I$ such that $N \in A_i$, then all ancestors of N are in A_i and subsequently all ancestors of N are in B. Otherwise, consider some ancestor M of N. Then consider the smallest $r \in \mathbb{Q}_+^*$ such that $M \in B(N, r)$. By definition of the closure, there exists $N' \in B(N, rp^{-1}) \cap \bigcup_{i\in I} A_i$. There exists $i \in I$ such that $N' \in A_i$. We then have:

$$d(N, N') = d(N, nca(N, N')) + d(nca(N, N'), N') \leq rp^{-1}$$

by which we conclude that $nca(N, N') \in B(N, rp^{-1})$. But $r = d(M, N) > d(nca(N, N'), N)$, and so $M \leq nca(N, N') \leq N$. So $M \leq N'$. Because $N' \in A_i$, and A_i is a sub-tree, $M \in A_i \subseteq B$.

Moreover, any increasing sequence of elements of B converges in $\mathbb{T}_p$, but since B is closed, any increasing sequence of elements of B converges in B.

So B is a sub-tree of $\mathbb{T}_p$, and we immediately have $A_i \subseteq B$ for any $i \in I$. It follows that $\sup_{i\in I} A_i \subseteq B$.

Conversely, $\sup_{i\in I} A_i$ is a closed subset of $\mathbb{T}_p$ (prop. 9) which contains $\bigcup_{i\in I} A_i$. It follows that $B \subseteq \sup_{i\in I} A_i$, and we conclude that:

$$\sup_{i\in I} A_i = \bar{\chi}^{-1}\left(\text{paths}\left(\left\langle \bigcup_{i\in I} \bar{\chi}(A_i) \right\rangle\right)\right) = \overline{\bigcup_{i\in I} A_i}$$

□

Corollary 3 (Second fixed-point theorem). *If $\mathbb{S}_p$ is the set of sub-trees of $\mathbb{T}_p$, then for any monotonic function $f : \mathbb{S}_p \to \mathbb{S}_p$, the set of fixed points of f is a complete lattice.*

Proof. Application of the Knaster-Tarski fixed point theorem of monotonic functions over complete lattices. □

2.5.2 p-bounded trees

Definition 10 (p-bounded tree). *A favoritism tree is said to be p-bounded when every node has at most p children. For any given set X, the set of p-bounded trees with nodes in X is denoted $A_p(X)$.*

Remark 1. A p-bounded tree for which all nodes have exactly p children is structurally equal to the $\mathbb{Z}_p$ favoritism tree Z_p (i.e modulo node values). Moreover, any *uniformly bound* favoritism tree, meaning for which there exists some (positive) integer M such that all nodes have at most M children, is a p-bounded tree, by taking the smallest prime integer p greater than M.

Definition 11. *For some fixed set X, we define the* ***set of structural trees labeled by*** *X, denoted $\mathcal{A}_p(X)$ as the quotient of $A_p(X)$ by $f \mapsto f^{\mathbb{N}}$, which associates $f \in A_p(X)$ to the set of structural paths in f.*

Theorem 2 (Translation to $\mathbb{T}_p$). *For any fixed set X, there is an injection from $\mathcal{A}_p(X)$ to the set of sub-trees of $\mathbb{T}_p$.*

Moreover, for any $f = (V, E, \mathcal{R}, \triangleleft) \in \mathcal{A}_p(X)$, there exists an isomorphism[2] *$\theta : V \to A$ where $A = \theta(V)$ is a sub-tree of $\mathbb{T}_p$.*

Proof. There exists a canonical $\rho : \mathcal{A}_p(X) \to \mathcal{P}(Z_p^{\mathbb{N}})$ such that for any $f \in A_p(X)$,

$$\rho(\bar{f}) = f^{\mathbb{N}} \subseteq Z_p^{\mathbb{N}}$$

where $\bar{f}$ is the class of f in $\mathcal{A}_p(X)$. A structural sub-tree of Z_p can be uniquely identified to a sub-tree of Z_p. In turn, by proposition 10, this sub-tree can be uniquely identified to a sub-tree of $\mathbb{T}_p$.

Moving on to the isomorphism, we have a canonical injection $\eta : V \to \text{paths}\,(Z_p)$ by identifying a node to its structural path in f, and by then identifying the structural path in f to a node path in Z_p. By then considering $A := \bar{\chi}^{-1} \circ \eta(V)$, we obtain a sub-tree of $\mathbb{T}_p$, and we can verify that $\bar{\chi}^{-1} \circ \eta : V \to A$ is isomorphic. $\square$

The path from $A_p(X)$ to the set $\mathbb{S}_p$ of sub-trees (s.t.) of $\mathbb{T}_p$ is summarized by the following diagram:

$$A_p(X) \twoheadrightarrow \mathcal{A}_p(X) \hookrightarrow \text{struct.s.t. of } Z_p \xrightarrow{\sim} \text{sub-trees of } Z_p \xrightarrow{\sim} \mathbb{S}_p$$

where $\twoheadrightarrow$ indicates a surjection, $\hookrightarrow$ indicates an injection, and $\xrightarrow{\sim}$ indicates a bijection. The notation *s.s.t.* means "structural sub-tree". In terms of elements, we can consider the following sequence (in the same order):

$$p\text{-bd.t.} \to p\text{-bd.struct.t.} \to \text{struct.s.t. of } Z_p \to \text{s.t. of } Z_p \to \text{s.t. of } \mathbb{T}_p$$

where "p-bd" means "p-bounded".

[2] Isomorphism here signifies that for any $(x, y) \in E$, $\theta(x)$ is the immediate ancestor of $\theta(y)$. In other words, $\theta(x) < \theta(y)$, and for every $z \in A$ such that $z < \theta(y)$, $z \leq \theta(x)$.

Let's make some sense of the phrase "The set of paths in a p-bounded tree is compact". We will temporarily denote $\theta : \mathcal{A}_p(X) \to \mathbb{T}_p$ the injection granted by the above diagram.

For any p-bounded tree f, we consider a node sequence $(n_k)_{k\geq 0}$. For every $k \geq 0$, n_k can be identified to the (unique) node path leading to it, which we denote $p_k \in \operatorname{paths}(f)$. Then p_k can be identified to a structural path $\mathbf{p}_k \in f^{\mathbb{N}}$. These paths can then be identified to paths in Z_p, and subsequently to elements of $\mathbb{T}_p$. These elements (denoted $(x_k)_{k\geq 0}$) live in the sub-tree (of $\mathbb{T}_p$) $\theta(\bar{f})$, and so there is a convergent sub-sequence. Denote $x \in \theta(\bar{f})$ the limit, and consider the strictly increasing $\phi : \mathbb{N} \to \mathbb{N}$ associated to the sub-sequence.

Then x is the image of some structural path in f, and can subsequently be identified to some node path $p \in \operatorname{paths}(f)$. If $\omega(x) < \infty$, then this node path is finite, and can therefore be identified to some node of f: an infinite number of terms of the sequence (n_k) are equal to this node, which is how we will translate "convergence" in this case. If $\omega(x) = \infty$ however, we cannot write p as a node. However, for any node n of the node path p, there exists some k such that n_k is a descendant of n, which is how we define convergence in this case.

More formally (but not very mathematically rigorous),

1. The node path p is finite if and only if $\omega(x) < \infty$. Then, by lemma 11, an infinite number of x_k are equal to x. In other words, if p is finite, then denote n the last node of p: for an infinite number of $k \geq 0$, $n_k = n$.
2. The node path p is infinite if and only if $\omega(x) = \infty$. Consider any node n in this path: this node can be identified to a strict ancestor y of x in $\theta(\bar{f}) \subseteq \mathbb{T}_p$, i.e $y < x$. By lemma 10, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $y < x_{\phi(k)}$. It follows that for all $k \geq k_0$, $n_{\phi(k)}$ is a descendant of n.

This gives the following theorem:

Theorem 3 (The set of structural paths is compact). *For any p-bounded tree f, the set of node paths of f is compact relative to d.* [3]

For any node sequence $(n_k)_{k\geq 0}$ in f, the sequence $(\mathbf{p}_k)_{k\geq 0}$ of structural node paths associated to each n_k has a convergent sub-sequence, for the distance d. In other words, at least one of the following assertions is true:

1. *Either there exists a node n such that for an infinite number of k, $n_k = n$,*
2. *Or there exists an infinite node path (def.6) $p = (m_k)_{k\geq 0}$ for which:*

$$\forall k \geq 0,\ \forall j \geq 0,\ \exists j \geq j_0,\ m_k \text{ is an ancestor of } n_j$$

There are multiple reformulations of assertion 2. All of the following are equivalent to 2.:

[3] Note that the distance d over $\mathbb{T}_p$ can be extended to $\operatorname{paths}(Z_p)$ (by the Barchi bijection), and so can be defined over $Z_p^{\mathbb{N}}$. It follows that d can be defined over $f^{\mathbb{N}} \subseteq Z_p^{\mathbb{N}}$, and subsequently can be defined over $\operatorname{paths}(f)$, which is the distance used in this theorem.

– There exists of a subsequence $(n_{k_j})_{j\geq 0}$ such that for every $j \geq 0$, n_{k_j} is a strict ancestor of $n_{k_{j+1}}$,
– If we denote $\leq$ the ancestry relation (i.e $n \leq m$ if and only if n is an ancestor of m), then there exists a strictly increasing subsequence of $(n_k)_{k\geq 0}$,
– Denoting p_k the node path associated to n_k for all $k \geq 0$, the sub-tree $\langle\{p_k : k \geq 0\}\rangle$ is infinite,
– The smallest sub-tree containing each n_k is infinite.

Note that 1. and 2. are not mutually exclusive. This theorem is not particularly surprising, and can be proved without using the Barchi set. Given the sequence (n_k), we can take the sub-tree $g := \langle\{p_k : k \geq 0\}\rangle$ generated by the paths (p_k) of each (n_k). Then (n_k) has values in g. If g is finite, then (n_k) has values in a finite set: 1. is subsequently trivial. If g is infinite, then for any infinite path p, we have 2.

This theorem, and how we got to it, is interesting from another point of view, however. Through considerations on the Barchi set, we've deduced a property true for any p-bounded tree. In the same sense that proving properties of the vector space $\mathbb{R}^n$ grants information on any vector space of finite dimension n, proving properties of $\mathbb{T}_p$ grants information on any p-bounded tree. The generality of these considerations will be shown in the following chapters, where we will consider proof trees in certain logic systems (see chapter), but also another point of view on the Collatz conjecture.

Let's also consider how corollary 2 (First fixed-point theorem) applies to p-bounded trees. Because this theorem does not involve any form of metric, we can apply it freely.

Theorem 4 (First fixed-point theorem). *We here denote $\to^*$ the transitive reflexive closure of the binary relation $\to$ in a tree. It is a partial order over the set of nodes.*

For any finite *p-bounded tree $t = (V, E, \mathcal{R}, \triangleleft)$, any function $f : V \to V$ that is increasing with respect to $\to^*$ has a least fixed point.*

Proof. By results on p-bounded trees, we can translate t to a sub-tree A of $\mathbb{T}_p$ via a bijection $\theta : V \to A$. A sub-tree is evidently a downward-directed subset of $\mathbb{T}_p$, and the canonical translation θ verifies:

$$\forall x \leq y \in A,\ g(x) \leq g(y)$$

where $g := \theta \circ f \circ \theta^{-1}$. Application of corollary 2 to the increasing $g : A \to A$ grants the existence of a least fixed point $\mathrm{lfp}(g) \in A$. Then:

$$f(\theta(\mathrm{lfp}(g))) = \theta(g(\mathrm{lfp}(g))) = \theta(\mathrm{lfp}(g))$$

So f has a least fixed point, given by:

$$\theta\Big(\mathrm{lfp}\big(\theta \circ f \circ \theta^{-1}\big)\Big) \in V$$

$\square$

A more applicable result is the translation of the proof of corollary 2, where we explicitly define the least fixed-point. Using the same notations as the previous theorem, we have:

$$\mathrm{lfp}(f) = \sup_{n \geq 1} f^n(v_0)$$

where $v_0 := \theta^{-1}(x_0)$, and $x_0 \in A$ minimizes $g : A \to A$, as granted by theorem 1.

We can also translate the second-fixed point theorem (corollary 3). Consider any fixed p-bounded tree $t = (V, E, \mathcal{R}, \lhd)$. Given the canonical $\theta : V \to A$ where $A \in \mathbb{S}_p$ is a sub-tree of $\mathbb{T}_p$, any sub-tree (as defined by def. 8) of t has an image $B \in \mathbb{S}_p$ by θ such that $B \subseteq A$.

Theorem 5 (Second fixed-point theorem). *Consider a p-bounded tree $t = (V, E, \mathcal{R}, \lhd)$. Then the set $\mathbb{S}(t)$ of sub-trees of t is a complete lattice for $\subseteq$. The set of fixed points of any monotonic function $f : \mathbb{S}(t) \to \mathbb{S}(t)$ is a complete lattice.*

Proof. Fix the bijection $\theta : V \to A$ for a certain $A \in \mathbb{S}_p$. Consider any family $(V_i)_{i \in I}$ of sub-trees (characterized by their node set) of t. If $I = \emptyset$, then $\sup_{i \in I} V_i = \{\mathcal{R}\}$ and $\inf_{i \in I} V_i = V$. Otherwise, consider the family $(A_i)_{i \in I}$ defined for all $i \in I$ by $A_i := \theta(V_i)$. Each A_i is a sub-tree of $\mathbb{T}_p$ and is included in A.

Then $\sup_{i \in I} A_i \subseteq A$ and $\inf_{i \in I} A_i \subseteq A$ exist in $\mathbb{T}_p$. Let's show that $\inf_{i \in I} V_i = \theta^{-1}(\inf_{i \in I} A_i)$. First, we trivially have $\theta^{-1}(\inf_{i \in I} A_i) \subseteq \inf_{i \in I} V_i$. Conversely, we have $\theta(\inf_{i \in I} V_i) \subseteq \inf_{i \in I} A_i$ by definition of $\inf_{i \in I} A_i$ as a greatest lower bound. As such, we have equality. We reason similarly to show the equality for the supremum.

So inf and sup are well-defined over $\mathbb{S}(t)$. It follows that $(\mathbb{S}(t), \subseteq)$ is a complete lattice, and the fixed-point theorem results from application of the Knaster-Tarski fixed point theorem. $\square$

2.5.3 Algorithmic applications of the fixed-point theorems

The previous theorems are a useful tool in showing the correction of certain tree algorithms. Consider for example algorithm 1, which is a sketch of a tree traversal.

Algorithm 1 Tree traversal sketch

Require: A tree $t = (V, E, \mathcal{R})$, a node $v_0 \in V$, and $f : V \to V$
  $v \leftarrow v_0$
  while $f(v) \neq v$ **do**
    $v \leftarrow f(v)$
  end while

For a more concrete example, consider algorithm 2. It has the same overall structure as algorithm 1, except for the end conditions. It has the same effect as the first algorithm with parameters t, $v_0 = \mathcal{R}$, and $f_x : V \to V$, where:

$$f_x(v) := \begin{cases} \mathbf{ch}(v).(1) & \text{if } x > v \text{ and } |\mathbf{ch}(v)| = 2 \\ \mathbf{ch}(v).(0) & \text{if } x < v \text{ and } |\mathbf{ch}(v)| \geq 1 \\ v & \text{if } x \neq v \text{ and } \mathbf{ch}(v) = \emptyset \\ x & \text{if } x = v \end{cases}$$

Algorithm 2 Binary Search

```
Require: A binary search tree t = (V, E, R), a target integer x.
  v ← R
  while v ≠ x and ch(v) ≠ ∅ do
      if x < v then
          v ← ch(v).(0)
      else
          v ← ch(v).(1)
      end if
  end while
  if v = x then
      Return true
  end if
  Return true
```

We invite the reader to verify that f_x is increasing, and that $\mathcal{R}$ minimizes f_x. By the first fixed-point theorem, $(f_x^n(\mathcal{R}))_{n \geq 1}$ converges to $\text{lfp}(f_x)$. To show the correction of algorithm 2, it is therefore sufficient to show that:

$$\text{lfp}(f_x) = x \iff x \in V$$

Assuming the tree t is finite, then there exists some $n \geq 1$ such that $f_x^n(\mathcal{R}) = \text{lfp}(f_x)$. Note furthermore that by definition of f_x, this implies that the loop condition fails once the least fixed point is reached. Subsequently, the algorithm terminates and the final value of v is $\text{lfp}(f_x)$. If the above equivalence is shown, then correction is established.

Returning to algorithm 1, if v_0 minimizes f, and if f is increasing, then v converges to $\text{lfp}(f)$. Concretely, we have shown the following result.

Theorem 6 (Tree traversal convergence). *Consider any algorithm of the same form of algorithm 1. If each of the following is true:*

1. *t is finite,*
2. *v_0 minimizes f,*
3. *f is increasing,*

Then the algorithm terminates and the final value of v is $\text{lfp}(f)$.

Because everything we showed in $\mathbb{T}_p$ was concretely over a set of paths, we cannot conclude that $\mathrm{lfp}(f)$ exists when t is infinite. However, if we see f as a partial function $\mathrm{paths}(t) \to \mathrm{paths}(t)$ by identifying any $v \in V$ to its associated node path, then $\mathrm{lfp}(f)$ always exists, albeit sometimes as an element of $\mathrm{paths}_\infty(t)$.

Bibliography

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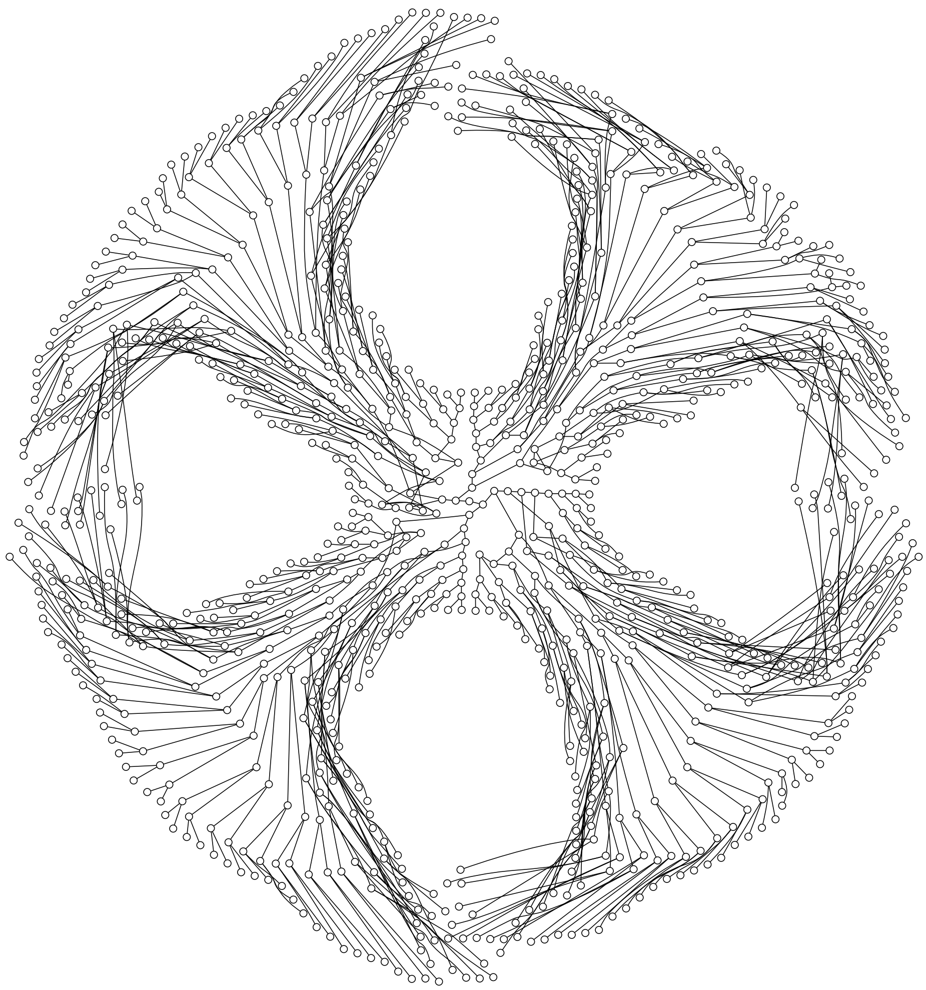

Fig. 7. One of the figures of figure 1. The node-placing algorithm had trouble handling so many nodes and created this monstrosity.