All Barchi No Bitey

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ENS Paris-Saclay

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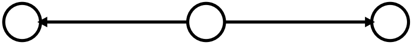
Introduction

- Roundabout
- Exits 0 or 1



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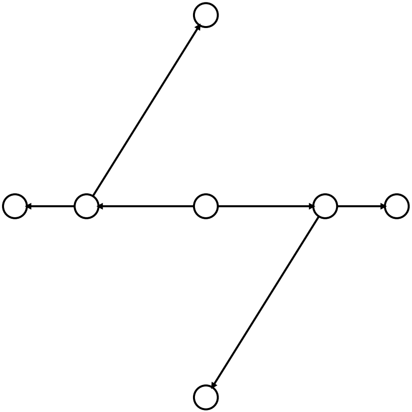
- Exits 00, 10, 01, 11



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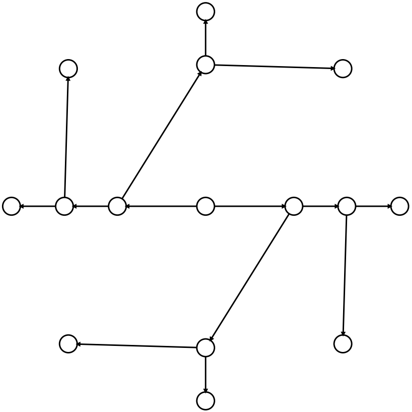

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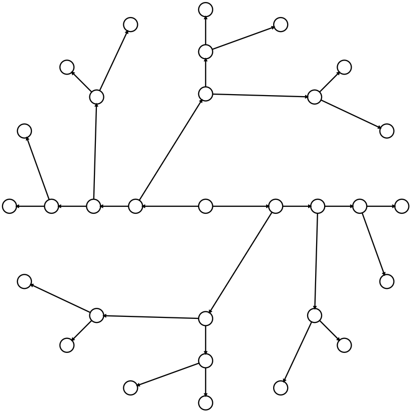
- etc...



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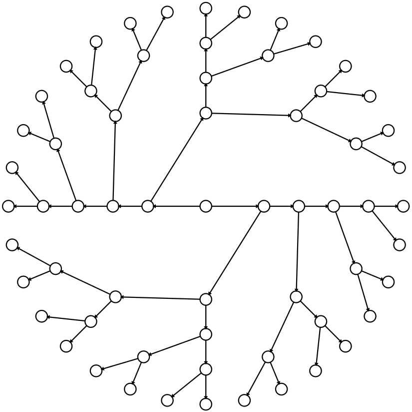
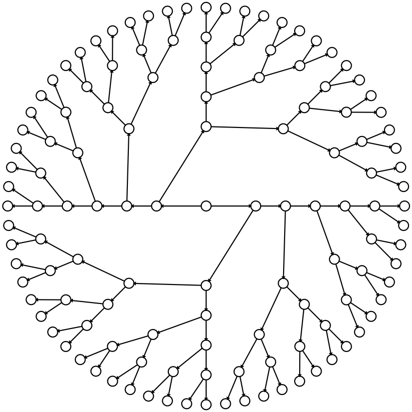




Introduction

- Prime p “exits”
- Infinite sequence
- Elements in $\{0, \ldots, p-1\}$
- p-adic numbers

p-adic integers

$\mathbb{Z}_p$, commutative ring

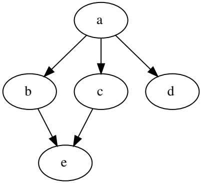
$$\mathbb{Z}_p = \left\{ \sum_{k=0}^{\infty} a_k p^{-k} \right\}$$

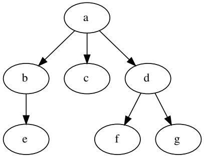
where $a_k \in \{0, \ldots, p-1\}$

Table of Contents

Trees

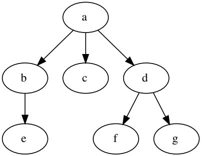
- (Directed) Graph
  - Couple $G = (V, E)$
  - $v \in V$ **node**
  - $(x, y) \in E$ **edge**, $x \to y$
- Tree
  - Triple $t = (V, E, \mathcal{R})$,
  - (V, E) graph
  - $\mathcal{R} \in V$ **root**
  - $\forall x \in V$, $\mathcal{R} \to^* x$ unique

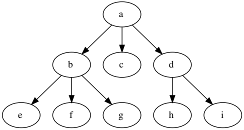




Favoritism trees

- Tree
  - **ch**(x) set of **children**
- Favoritism tree
  - Favoritism function $\lhd$
  - **ch**(x) is totally ordered
  - **ch**$(x).(k)$, k-th smallest child for $\lhd$



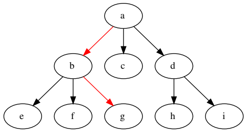


Node paths

- Node path $(n_k)_{K \geq k \geq 0}$
  - $n_0 = \mathcal{R}$
  - $0 \leq k \leq K - 1$, $n_{k+1} \in \mathbf{ch}(n_k)$
  - Set of node paths paths(t)
- Structural node path $(a_k)_{K \geq k \geq 0}$
  - Associated to a node path
  - $0 \leq k \leq K - 1$, $n_{k+1} = \mathbf{ch}(n_k).(a_k)$
  - Set of structural node paths $t^{\mathbb{N}}$

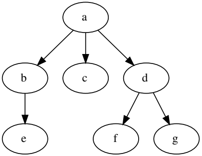
$$a \to b \to g$$

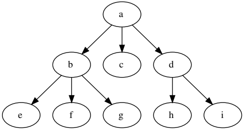
$$0, \; 2$$



Sub-trees

- Sub-tree
  - t sub-tree of s
  - $\text{paths}(t) \subseteq \text{paths}(s)$





The $\mathbb{Z}_p$ favoritism tree

- Favoritism tree Z_p
- $V = \{0, \dots, p-1\}^* = \Sigma^*$
- $u \to v$ iff $v = a \cdot u$ for some $a \in \Sigma$
- $\mathcal{R} = \varepsilon$
- $0 \cdot u \triangleleft 1 \cdot u \triangleleft \cdots \triangleleft (p-1) \cdot u$

p-bounded trees

p-bounded tree

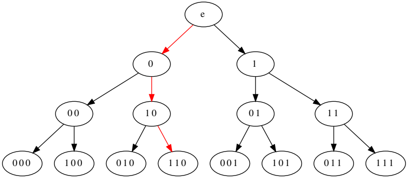
A favoritism tree, where $|\mathbf{ch}(x)| \leq p$ for all $x \in V$

p-bounded trees and Z_p

Any p-bounded tree is (isomorphic to) a sub-tree of Z_p

The χ function

- $n \in \bar{\mathbb{N}} = \{0, 1, \dots, \} \cup \{\infty\}$
- x p-adic integer, $\sum a_k p^{-k}$
- $\chi(n, x) \in \text{paths}(Z_p)$
- $n_0 := \varepsilon$
- $\forall k \geq 0,\ n_{k+1} = a_k \cdot n_k$



$\chi(3, 110)$

The Barchi bijection

Lemma

$\chi(n,x) = \chi(m,y)$ iff $n = m$ and $|x-y|_p \leq p^{-n}$

The Barchi bijection

$$\mathbb{T}_p = (\bar{\mathbb{N}} \times \mathbb{Z}_p)/(\chi)$$

Bijection $\bar{\chi} : \mathbb{T}_p \to \text{paths}(Z_p)$. Classes denoted $[n,x]$, and $\omega([n,x]) = n$ well-defined function.

Partial order

- $[n, x]$ and $[m, y]$
- $\bar{\chi}([n, x])$ and $\bar{\chi}([m, y])$ coincide for $k \leq \min(n, m)$ terms
- Define $nca([n, x], [m, y]) = [k, x] = [k, y]$

Partial order

$$[n, x] \leq [m, y] \iff nca([n, x], [m, y]) = [n, x]$$

Defines a partial order.

Mathematical interpretation

- Information
- $[n, x]$ knowledge of the first n digits
- $X \leq Y$ if Y holds more information
- Numerical representation

$$
\begin{aligned}
[2, x] \quad &\longleftrightarrow \quad x_1 x_0 \\
[4, x] \quad &\longleftrightarrow \quad x_3 x_2 x_1 x_0 \\
[\infty, x] \quad &\longleftrightarrow \quad x \text{ is known}
\end{aligned}
$$

Metric

- If $X \leq Y$,

$$d(X, Y) = \sum_{k=\omega(X)+1}^{\omega(Y)} p^{-k}$$

- Otherwise,

$$d(X, Y) = d(X, nca(X, Y)) + d(nca(X, Y), Y)$$

Metric

$(\mathbb{T}_p, d)$ is a metric space

Convergence

Convergence by nearest common ancestor

$$(X_n \xrightarrow[n\to\infty]{} X) \implies (nca(X_n, X) \xrightarrow[n\to\infty]{} X)$$

The converse is true whenever $\omega(X) \geq \omega(X_n)$ for all $n \geq 0$.

Convergence to a finite element

If $\omega(X) < \infty$,

$$(X_n \xrightarrow[n\to\infty]{} X) \iff (\forall n \geq n_0,\ X_n = X)$$

Completeness and compactness

Completeness

$(\mathbb{T}_p, d)$ is complete.

Compactness

$(\mathbb{T}_p, d)$ is compact.

This means any p-bounded tree is compact for d (informally).

Dcpo structure

Downward directed set

$(S, \preceq)$ is a downward directed set if

$$\forall x, y \in S,\ \exists z \in S,\ z \preceq x \text{ and } z \preceq y$$

Down-complete poset (dcpo)

$(S, \preceq)$ is a dcpo if every downward directed subset has an inf in S.

Dcpo structure

$\mathbb{T}_p$ is a dcpo. Moreover, inf F is in F.

First fixed-point theorem

Scott-continuity

$f : S \to T$, S, T dcpos, is Scott-continuous if monotonous and

$$\inf f(F) = f(\inf F)$$

for any directed subset F.

Scott-continuity in $\mathbb{T}_p$

$f : \mathbb{T}_p \to \mathbb{T}_p$ is Scott-continuous iff f is increasing.

First fixed-point theorem

Any increasing $f : \mathbb{T}_p \to \mathbb{T}_p$ has a minimum (in X_0) and a least fixed point given by $\mathrm{lfp}(f) = \sup f^n(X_0)$.

Algebraic structure

Sum

$$[n, x] + [m, y] := [\min(n, m),\ x + y]$$

Well-defined, commutative and associative.

Product

$$[n, x] \times [m, y] := [\min(n, m),\ x \times y]$$

Well-defined, commutative, associative, and distributive over $+$.

Algebraic structure (continued)

Sub-"rings"

For any $k \geq 1$, $(\omega^{-1}(\{k\}), +, \times)$ is a ring, isomorphic to $\mathbb{Z}/p^k\mathbb{Z}$.
$(\omega^{-1}(\{\infty\}), +, \times)$ is a ring, isomorphic to $\mathbb{Z}_p$.

- $\mathbb{T}_p$ is not a ring...
- but contains infinitely many!
- Projective limit

Sub-trees of $\mathbb{T}_p$

- $A \subseteq \mathbb{T}_p$
- $\forall X \in A$, ancestors of X are in A
- Any $(X_n) \uparrow$ in A converges in A

Sub-trees

$$\text{sub-trees of } Z_p \approx \text{ sub-trees of } \mathbb{T}_p$$

Any p-bounded tree can be seen as a sub-tree of $\mathbb{T}_p$.

Second fixed-point theorem

Complete lattice

The set $\mathbb{S}_p$ of sub-trees of $\mathbb{T}_p$ is a complete lattice w.r.t. $\subseteq$.

Second fixed-point theorem

For any increasing $f : \mathbb{S}_p \to \mathbb{S}_p$, $\mathrm{Fix}(f)$ is a complete lattice. (Knaster-Tarski theorem)

Application to p-bounded trees

First fixed-point theorem

Any increasing $f : \text{paths}(t) \to \text{paths}(t)$ has a minimum (in X_0) and a least fixed-point given by $\text{lfp}(f) = \sup f^n(X_0)$.

Second fixed-point theorem

The set of sub-trees of t is a complete lattice for inclusion.

The set of node paths is compact

$\text{paths}(t)$ is compact (for d).

Hensel's lemma

For $P \in \mathbb{T}_p[X]$, $[i, x] \in \mathbb{T}_p$ and $\delta \in \{0, \dots, p-1\}$,

$$P([i, x + \delta p^i]) = P([i, x]) + [i, \delta p^i] \cdot P'([i, x])$$

Theorem 4 (Hensel's Lemma, Barchean form) *Consider any $P \in \mathbb{T}_p[X]$ and $[i, x] \in \mathbb{T}_p$ such that $2 \leq i + 1 \leq l(P)$. If $P([i, x]) = [i, 0]$ and $P'([i, x]) \not\geq [1, 0]$ then there exists a unique $[i+1, y] \geq [i, x]$ such that $P([i+1, y]) = [i+1, 0]$ and $P'([i+1, y]) \not\geq [1, 0]$.*

And we can obtain the "usual" form of Hensel's lemma.

Compact formal languages

Σ finite alphabet, $\preceq$ the suffix relationship, $u \preceq v$ iff $\exists w, v = w \cdot u$

Every language is compact

$L \subseteq \Sigma^*$ is compact in the following sense. For any sequence $(u_n)_{n\geq 0}$ of words of L, there exists a strictly increasing $\phi : \mathbb{N} \to \mathbb{N}$ such that either 1. or 2. is verified.

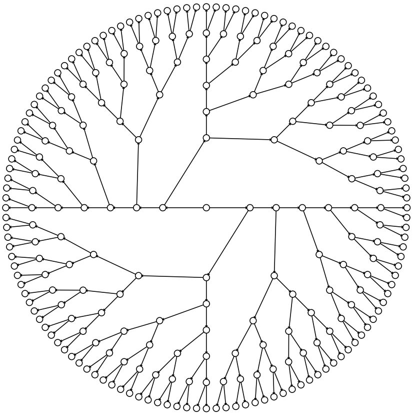
1. $(u_{\phi(n)})_{n\geq 0}$ is constant
2. There exists a strictly increasing $(v_n)_{n\geq 0}$ of words of Σ^* such that for all $n \geq 0$, $v_n \preceq u_{\phi(n)}$

Informally, there is a convergent subsequence, although it can converge towards an "infinite word".

Conclusion

If nothing else, remember this:

- *p*-adic numbers and *p*-bounded trees are linked
- The Barchi set is a tool
- Applications across various domains



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