Iterated Belief Change as Learning (supplementary material)

Paper #5810

This supplementary material contains the proofs of Lemma 1 and Proposition 2, whose statements are recalled below:

Lemma 1. *For each dataset* $\mathsf{D} \in \mathcal{D}$ *and each instance description* $\omega_{\mathbf{X}} \in \Omega_{\mathbf{X}}$*,*

$$d_H(\omega_{\mathbf{X}}^{\mathsf{D}}, \mathsf{D}) \leq d_H(\omega_{\mathbf{X}}, \mathsf{D})$$

Proposition 2. *For each* TOCF $\Psi = (\mathsf{D}, \kappa, \tau)$ *and each instance description* $\omega_{\mathbf{X}} \in \Omega_{\mathbf{X}}$*, we have that*

$$\kappa(\omega_{\mathbf{X}}) = d_H(\omega_{\mathbf{X}}, \mathsf{D}) - d_H(\omega_{\mathbf{X}}^{\mathsf{D}}, \mathsf{D})$$

We start with proving an intermediate lemma:

Lemma 2. *For each dataset* $\mathsf{D} \in \mathcal{D}$ *and each instance description* $\omega^*_{\mathbf{X}} \in \Omega_{\mathbf{X}}$*, we have that*

$$d_H(\omega^*_{\mathbf{X}}, \mathsf{D}) = \sum_{x_i \in P_{\mathbf{X}}} |\{\varphi_\omega \in \mathsf{D} \mid \varphi_\omega \models \mathbf{y} \leftrightarrow \omega^*_{\mathbf{X}}(x_i) \neq \omega_{\mathbf{X}}(x_i)\}|$$

Proof of Lemma 2. Let $\mathsf{D} \in \mathcal{D}$ be any dataset and $\omega^*_{\mathbf{X}} \in \Omega_{\mathbf{X}}$ be any instance description. By definition of $d_H(\omega^*_{\mathbf{X}}, \mathsf{D})$, we know that

$$d_H(\omega^*_{\mathbf{X}}, \mathsf{D}) = \sum_{\varphi_\omega \in \mathsf{D}} d_H(\omega^*_{\mathbf{X}}, \varphi_\omega)$$

This can be equivalently written as

$$d_H(\omega^*_{\mathbf{X}}, \mathsf{D}) = \sum_{\varphi_\omega \in \mathsf{D}, \varphi_\omega \models \mathbf{y}} d_H(\omega^*_{\mathbf{X}}, \varphi_\omega) + \sum_{\varphi_\omega \in \mathsf{D}, \varphi_\omega \models \neg\mathbf{y}} d_H(\omega^*_{\mathbf{X}}, \varphi_\omega)$$

Then, by definition of $d_H(\omega^*_{\mathbf{X}}, \varphi_\omega)$, this is equivalent to

$$d_H(\omega^*_{\mathbf{X}}, \mathsf{D}) = \sum_{\varphi_\omega \in \mathsf{D}, \varphi_\omega \models \mathbf{y}} d_H(\omega^*_{\mathbf{X}}, \omega_{\mathbf{X}}) + \sum_{\varphi_\omega \in \mathsf{D}, \varphi_\omega \models \neg\mathbf{y}} d_H(\omega^*_{\mathbf{X}}, \overline{\omega_{\mathbf{X}}})$$

Yet for each $\omega_{\mathbf{X}} \in \Omega_{\mathbf{X}}$, we know by definition of $d_H(\omega^*_{\mathbf{X}}, \omega_{\mathbf{X}})$ that

$$d_H(\omega^*_{\mathbf{X}}, \omega_{\mathbf{X}}) = |\{x_i \in P_{\mathbf{X}} \mid \omega^*_{\mathbf{X}}(x_i) \neq \omega_{\mathbf{X}}(x_i)\}|$$

So, on the one hand, we have that

$$\begin{aligned} \textstyle\sum_{\varphi_\omega \in \mathsf{D}, \varphi_\omega \models \mathbf{y}} d_H(\omega^*_{\mathbf{X}}, \omega_{\mathbf{X}}) &= \textstyle\sum_{\varphi_\omega \in \mathsf{D}, \varphi_\omega \models \mathbf{y}} |\{x_i \in P_{\mathbf{X}} \mid \omega^*_{\mathbf{X}}(x_i) \neq \omega_{\mathbf{X}}(x_i)\}| \\ &= \textstyle\sum_{x_i \in P_{\mathbf{X}}} |\varphi_\omega \in \mathsf{D} \mid \varphi_\omega \models \mathbf{y}, \omega^*_{\mathbf{X}}(x_i) \neq \omega_{\mathbf{X}}(x_i)| \end{aligned}$$

And, on the other hand, we have that

$$\begin{aligned} \textstyle\sum_{\varphi_\omega \in \mathsf{D}, \varphi_\omega \models \neg\mathbf{y}} d_H(\omega^*_{\mathbf{X}}, \overline{\omega_{\mathbf{X}}}) &= \textstyle\sum_{\varphi_\omega \in \mathsf{D}, \varphi_\omega \models \neg\mathbf{y}} |\{x_i \in P_{\mathbf{X}} \mid \omega^*_{\mathbf{X}}(x_i) \neq \overline{\omega_{\mathbf{X}}}(x_i)\}| \\ &= \textstyle\sum_{x_i \in P_{\mathbf{X}}} |\varphi_\omega \in \mathsf{D} \mid \varphi_\omega \models \neg\mathbf{y}, \omega^*_{\mathbf{X}}(x_i) \neq \overline{\omega_{\mathbf{X}}}(x_i)| \\ &= \textstyle\sum_{x_i \in P_{\mathbf{X}}} |\varphi_\omega \in \mathsf{D} \mid \varphi_\omega \models \neg\mathbf{y}, \omega^*_{\mathbf{X}}(x_i) = \omega_{\mathbf{X}}(x_i)| \end{aligned}$$

We got that

$$d_H(\omega^*_{\mathbf{X}}, \mathsf{D}) = \sum_{x_i \in P_{\mathbf{X}}} |\varphi_\omega \in \mathsf{D} \mid \varphi_\omega \models \mathbf{y}, \omega^*_{\mathbf{X}}(x_i) \neq \omega_{\mathbf{X}}(x_i)| + \sum_{x_i \in P_{\mathbf{X}}} |\varphi_\omega \in \mathsf{D} \mid \varphi_\omega \models \neg\mathbf{y}, \omega^*_{\mathbf{X}}(x_i) = \omega_{\mathbf{X}}(x_i)|$$

But this can be equivalently written as

$$d_H(\omega^*_{\mathbf{X}}, \mathsf{D}) = \sum_{x_i \in P_{\mathbf{X}}} |\{\varphi_\omega \in \mathsf{D} \mid \varphi_\omega \models \mathbf{y} \leftrightarrow \omega^*_{\mathbf{X}}(x_i) \neq \omega_{\mathbf{X}}(x_i)\}|$$

This concludes the proof of the lemma. □

Let us now prove Lemma 1.

Lemma 1. *For each dataset* $\mathsf{D} \in \mathcal{D}$ *and each instance description* $\omega_{\mathbf{X}} \in \Omega_{\mathbf{X}}$,

$$d_H(\omega^{\mathsf{D}}_{\mathbf{X}}, \mathsf{D}) \leq d_H(\omega_{\mathbf{X}}, \mathsf{D})$$

Proof of Lemma 1. Let $\mathsf{D} \in \mathcal{D}$ be any dataset.

First, recall that $\omega^{\mathsf{D}}_{\mathbf{X}}$ is defined for each $x_i \in P_{\mathbf{X}}$ as:

$$\omega^{\mathsf{D}}_{\mathbf{X}}(x_i) = \begin{cases} 1 & \text{if } \mathsf{D}^1(x_i) \geq \mathsf{D}^0(x_i) \\ 0 & \text{otherwise,} \end{cases}$$

where:

$$\mathsf{D}^1(x_i) = |\{\varphi_\omega \in \mathsf{D} \mid \varphi_\omega \models \mathbf{y} \leftrightarrow \omega(x_i) = 1\}|$$
$$\mathsf{D}^0(x_i) = |\{\varphi_\omega \in \mathsf{D} \mid \varphi_\omega \models \mathbf{y} \leftrightarrow \omega(x_i) = 0\}|$$

Second, by Lemma 2, we know that for each instance description $\omega^*_{\mathbf{X}} \in \Omega_{\mathbf{X}}$,

$$d_H(\omega^*_{\mathbf{X}}, \mathsf{D}) = \sum_{x_i \in P_{\mathbf{X}}} d^i_H(\omega^*_{\mathbf{X}}, \mathsf{D}),$$

where for each $x_i \in P_{\mathbf{X}}$, $d^i_H(\omega^*_{\mathbf{X}}, \mathsf{D}) = |\{\varphi_\omega \in \mathsf{D} \mid \varphi_\omega \models \mathbf{y} \leftrightarrow \omega^*_{\mathbf{X}}(x_i) \neq \omega_{\mathbf{X}}(x_i)\}|$.

Clearly, for each instance description $\omega^*_{\mathbf{X}} \in \Omega_{\mathbf{X}}$ and each $x_i \in P_{\mathbf{X}}$, we have that

$$d^i_H(\omega^*_{\mathbf{X}}, \mathsf{D}) = \begin{cases} \mathsf{D}^1(x_i) & \text{if } \omega^*_{\mathbf{X}}(x_i) = 0 \\ \mathsf{D}^0(x_i) & \text{if } \omega^*_{\mathbf{X}}(x_i) = 1 \end{cases}$$

Thus,

$$d^i_H(\omega^*_{\mathbf{X}}, \mathsf{D}) \in \{\mathsf{D}^1(x_i), \mathsf{D}^0(x_i)\}$$

And by definition of $\omega^{\mathsf{D}}_{\mathbf{X}}$, for each $x_i \in P_{\mathbf{X}}$, we have that

$$d^i_H(\omega^{\mathsf{D}}_{\mathbf{X}}, \mathsf{D}) = \min(\mathsf{D}^1(x_i), \mathsf{D}^0(x_i))$$

Thus, we got that for each instance description $\omega^*_{\mathbf{X}} \in \Omega_{\mathbf{X}}$ and for each $x_i \in P_{\mathbf{X}}$ that

$$d^i_H(\omega^{\mathsf{D}}_{\mathbf{X}}, \mathsf{D}) \leq d^i_H(\omega_{\mathbf{X}}, \mathsf{D})$$

Hence, for each instance description $\omega^*_{\mathbf{X}} \in \Omega_{\mathbf{X}}$,

$$d_H(\omega^{\mathsf{D}}_{\mathbf{X}}, \mathsf{D}) \leq d_H(\omega^*_{\mathbf{X}}, \mathsf{D})$$

This concludes the proof of the lemma. □

We now prove Proposition 2.

Proposition 2. *For each* TOCF $\Psi = (\mathsf{D}, \kappa, \tau)$ *and each instance description* $\omega_{\mathbf{X}} \in \Omega_{\mathbf{X}}$*, we have that*

$$\kappa(\omega_{\mathbf{X}}) = d_H(\omega_{\mathbf{X}}, \mathsf{D}) - d_H(\omega^{\mathsf{D}}_{\mathbf{X}}, \mathsf{D})$$

Proof of Proposition 2. Let us first prove that there is a mapping $f : E_{tocf} \to \mathbb{N}$ such that for each TOCF $\Psi = (\mathsf{D}, \kappa, \tau)$ and each instance description $\omega^*_{\mathbf{X}} \in \Omega_{\mathbf{X}}$, $\kappa(\omega^*_{\mathbf{X}}) = d_H(\omega^*_{\mathbf{X}}, \mathsf{D}) - f(\Psi)$. We prove it by induction on the size of the dataset D in each TOCF $\Psi = (\mathsf{D}, \kappa, \tau)$.

Base case ($|\mathsf{D}| = 0$): in this case, $\Psi = \Psi^{\top}_{*} = (\emptyset, \kappa_{\top}, 0)$, with $\kappa_{\top}(\omega^*_{\mathbf{X}}) = 0$ for each $\omega^*_{\mathbf{X}} \in \Omega_{\mathbf{X}}$. Then, let f be any mapping $f : E_{tocf} \to \mathbb{N}$ satisfying $f(\Psi_{\top}) = 0$. We get that $d_H(\omega_{\mathbf{X}}, \emptyset) - f(\Psi_{\top}) = 0$, thus $\kappa_{\top}(\omega^*_{\mathbf{X}}) = d_H(\omega^*_{\mathbf{X}}, \emptyset) - f(\Psi_{\top})$. This shows the statement for the base case where $|\mathsf{D}| = 0$.

Induction hypothesis: let $k \geq 0$, and assume that there is a mapping $f : E_{tocf} \to \mathbb{N}$ such that for each TOCF $\Psi = (\mathsf{D}, \kappa, \tau)$ such that $|\mathsf{D}| \leq k$ and each instance description $\omega^*_{\mathbf{X}} \in \Omega_{\mathbf{X}}$, $\kappa(\omega^*_{\mathbf{X}}) = d_H(\omega^*_{\mathbf{X}}, \mathsf{D}) - f(\Psi)$. Let $\mathsf{D}' \in \mathcal{D}$ such that $|\mathsf{D}'| = k + 1$. By definition of the TOCF $\Psi' = (\mathsf{D}', \kappa', \tau')$, $\kappa' = \kappa \bullet \varphi_{\omega}$, with $\Psi = (\kappa, \mathsf{D}, \tau)$ and $\mathsf{D}' = \mathsf{D} \sqcup (\varphi_{\omega})$. Yet it can be seen by construction of κ' that for each instance description $\omega^*_{\mathbf{X}} \in \Omega_{\mathbf{X}}$, $\kappa'(\omega^*_{\mathbf{X}}) = (\kappa \bullet \varphi_{\omega})(\omega^*_{\mathbf{X}}) = \kappa(\omega^*_{\mathbf{X}}) + d_H(\omega^*_{\mathbf{X}}, \varphi_{\omega}) - \alpha_{\kappa}$, where α_{κ} is a shift value depending on κ only, i.e., independent of $\omega^*_{\mathbf{X}}$. This means that any mapping $f' : E_{tocf} \to \mathbb{N}$ satisfying $f'((\mathsf{D}, \kappa, \tau)) = f'((\mathsf{D}, \kappa, \tau))$ for each $(\mathsf{D}, \kappa, \tau)$ such that $|\mathsf{D}| \leq n$, and $f'((\mathsf{D}', \kappa', \tau')) = \alpha_{\kappa}$, is such that for each instance description $\omega^*_{\mathbf{X}} \in \Omega_{\mathbf{X}}$, $\kappa'(\omega^*_{\mathbf{X}}) = d_H(\omega^*_{\mathbf{X}}, \mathsf{D}') - f(\Psi')$.

This concludes the proof by induction that there is a mapping $f : E_{tocf} \to \mathbb{N}$ such that for each TOCF $\Psi = (\mathsf{D}, \kappa, \tau)$ and each instance description $\omega^*_{\mathbf{X}} \in \Omega_{\mathbf{X}}$,

$$\kappa(\omega^*_{\mathbf{X}}) = d_H(\omega^*_{\mathbf{X}}, \mathsf{D}) - f(\Psi) \tag{1}$$

But now, we know that for each TOCF $\Psi = (\mathsf{D}, \kappa, \tau)$, κ is an OCF, which means that $\min(\{\kappa(\omega^*_{\mathbf{X}}) \mid \omega^*_{\mathbf{X}} \in \Omega_{\mathbf{X}}\}) = 0$. Stated equivalently, for each instance description $\omega'_{\mathbf{X}} \in \Omega_{\mathbf{X}}$,

$$\kappa(\omega'_{\mathbf{X}}) = 0 \quad \text{if and only if} \quad \forall \omega^*_{\mathbf{X}} \in \Omega_{\mathbf{X}} \;\; \kappa(\omega'_{\mathbf{X}}) \leq \kappa(\omega^*_{\mathbf{X}}) \tag{2}$$

Yet $\kappa(\omega'_{\mathbf{X}}) \leq \kappa(\omega^*_{\mathbf{X}})$ if and only if (from Equation 1) $d_H(\omega'_{\mathbf{X}}, \mathsf{D}) - f(\Psi) \leq d_H(\omega^*_{\mathbf{X}}, \mathsf{D}) - f(\Psi)$ if and only if $d_H(\omega'_{\mathbf{X}}, \mathsf{D}) \leq d_H(\omega^*_{\mathbf{X}}, \mathsf{D})$, and since from Lemma 1 $d_H(\omega^{\mathsf{D}}_{\mathbf{X}}, \mathsf{D}) \leq d_H(\omega^*_{\mathbf{X}}, \mathsf{D})$ for each $\omega^*_{\mathbf{X}} \in \Omega_{\mathbf{X}}$, we get that $\kappa(\omega^{\mathsf{D}}_{\mathbf{X}}) \leq \kappa(\omega^*_{\mathbf{X}})$ for each $\omega^*_{\mathbf{X}} \in \Omega_{\mathbf{X}}$.

Using Equation 2, we get that for each $\kappa(\omega^{\mathsf{D}}_{\mathbf{X}}) = 0$, which together with Equation 1 gives $d_H(\omega^{\mathsf{D}}_{\mathbf{X}}, \mathsf{D}) - f(\Psi) = 0$, i.e., $f(\Psi) = d_H(\omega^{\mathsf{D}}_{\mathbf{X}}, \mathsf{D})$.

Using Equation 1 again, we get for each TOCF $\Psi = (\mathsf{D}, \kappa, \tau)$ and each instance description $\omega^*_{\mathbf{X}} \in \Omega_{\mathbf{X}}$ that

$$\kappa(\omega^*_{\mathbf{X}}) = d_H(\omega^*_{\mathbf{X}}, \mathsf{D}) - d_H(\omega^{\mathsf{D}}_{\mathbf{X}}, \mathsf{D})$$

This concludes the proof. □