Context-Based Belief Revision (Proofs)

Paper #242

This supplementary material contains all proofs of lemmata and propositions. It also comprises all necessary definitions for reference.

Definition 1 (KM revision operator (Katsuno and Mendelzon 1991))**.** *A belief change operator* $\circ$ *is called a* KM revision operator *if for all formulae* φ, ψ, α, $\beta \in \mathcal{L}$,

(R1) $\varphi \circ \alpha \models \alpha$

(R2) *If* $\varphi \wedge \alpha \not\models \bot$ *then* $\varphi \circ \alpha \equiv \varphi \wedge \alpha$

(R3) *If* $\alpha \not\models \bot$ *then* $\varphi \circ \alpha \not\models \bot$

(R4) *If* $\varphi \equiv \psi$ *and* $\alpha \equiv \beta$ *then* $\varphi \circ \alpha \equiv \psi \circ \beta$

(R5) $(\varphi \circ \alpha) \wedge \beta \models \varphi \circ (\alpha \wedge \beta)$

(R6) *If* $(\varphi \circ \alpha) \wedge \beta \not\models \bot$ *then* $\varphi \circ (\alpha \wedge \beta) \models (\varphi \circ \alpha) \wedge \beta$

Definition 2 (Faithful Assignment)**.** *A* faithful assignment *is a mapping associating each formula* $\varphi \in \mathcal{L}$ *with a total preorder* $\preceq_\varphi$ *such that the following conditions are satisfied for all* φ, $\psi \in \mathcal{L}_*$:

1. $[\varphi] = \min(\Omega, \preceq_\varphi)$
2. *If* $\varphi \equiv \psi$, *then* $\preceq_\varphi = \preceq_\psi$

Theorem 1 ((Katsuno and Mendelzon 1991))**.** *A belief change operator* $\circ$ *is a KM revision operator if and only if there exists a faithful assignment* $\varphi \mapsto \preceq_\varphi$ *such that for all* $\varphi, \alpha \in \mathcal{L}$:

$$[\varphi \circ \alpha] = \min([\alpha], \preceq_\varphi)$$

Definition 3 (Credibility-limited revision operator (Booth et al. 2012))**.** *A belief change operator* $\circ$ *is called a* credibility-limited (CL) revision operator *if for all formulae* $\varphi \in \mathcal{L}_*$, ψ, α, $\beta \in \mathcal{L}$,

(P1) $\varphi \circ \alpha \models \alpha$ *or* $\varphi \circ \alpha \equiv \varphi$

(P2) *If* $\varphi \wedge \alpha \not\models \bot$ *then* $\varphi \circ \alpha \equiv \varphi \wedge \alpha$

(P3) $\varphi \circ \alpha \not\models \bot$

(P4) *If* $\varphi \equiv \psi$ *and* $\alpha \equiv \beta$ *then* $\varphi \circ \alpha \equiv \psi \circ \beta$

(P5) *If* $\varphi \circ \alpha \models \alpha$ *and* $\alpha \models \beta$, *then* $\varphi \circ \beta \models \beta$

(P6) $\varphi \circ (\alpha \vee \beta) \equiv \begin{cases} \varphi \circ \alpha, & \textit{or} \\ \varphi \circ \beta, & \textit{or} \\ (\varphi \circ \alpha) \vee (\varphi \circ \beta) & \end{cases}$

Definition 4 (CL Faithful Assignment)**.** *A* CL faithful assignment *is a mapping associating each formula* $\varphi \in \mathcal{L}_*$ *with a pair* $(C_\varphi, \preceq_\varphi)$, *where* $[\varphi] \subseteq C_\varphi \subseteq \Omega$, $\preceq_\varphi$ *is a total preorder on* C_φ, *and such that the following conditions are satisfied for all* φ, $\psi \in \mathcal{L}_*$:

1. $[\varphi] = \min(C_\varphi, \preceq_\varphi)$
2. *If* $\varphi \equiv \psi$, *then* $(C_\varphi, \preceq_\varphi) = (C_\psi, \preceq_\psi)$

Theorem 2 ((Booth et al. 2012))**.** *A belief change operator* $\circ$ *is a CL revision operator if and only if there exists a CL faithful assignment* $\varphi \mapsto (C_\varphi, \preceq_\varphi)$ *such that for all* $\varphi \in \mathcal{L}_*$, $\alpha \in \mathcal{L}$:

$$[\varphi \circ \alpha] = \begin{cases} \min([\alpha], \preceq_\varphi) & \textit{if } [\alpha] \cap C_\varphi \neq \emptyset \\ [\varphi] & \textit{otherwise} \end{cases}$$

Theorem 3 ((Booth et al. 2012))**.** *If CL revision operator* $\circ$ *satisfies (R1) , then* $\circ$ *is a KM revision operator.*

Definition 5 (Context-Based Credibility)**.** *Let* $\circ$ *be a belief change operator and* $\varphi \in \mathcal{L}_*$. *A formula* $\alpha \in \mathcal{L}$ *is:*

- φ-absolutely credible for $\circ$ *if for each* $\beta \in \mathcal{L}$, $\varphi \circ (\alpha \vee \beta) \models \alpha \vee \beta$
- φ-absolutely non-credible for $\circ$ *if for each* $\beta \in \mathcal{L}$, $\varphi \circ (\alpha \wedge \beta) \not\models \alpha \wedge \beta$
- φ-contextually credible for $\circ$ *if* $\varphi \circ \alpha \models \alpha$ *and there exists* $\beta \in \mathcal{L}$ *such that* $\varphi \circ (\alpha \vee \beta) \not\models \alpha \vee \beta$
- φ-contextually non-credible for $\circ$ *if* $\varphi \circ \alpha \not\models \alpha$ *and there exists* $\beta \in \mathcal{L}$ *such that* $\varphi \circ (\alpha \wedge \beta) \models \alpha \wedge \beta$

Proposition 1. *If a belief change operator* $\circ$ *satisfies (P6), then for each* $\varphi \in \mathcal{L}_*$ *and each* $\alpha \in \mathcal{L}$, *if* $\varphi \circ \alpha \not\models \alpha$, *then there is a complete formula* $\phi_\omega \models \alpha$ *such that* $\varphi \circ \phi_\omega \not\models \phi_\omega$.

Proof. Assume that $\circ$ is a belief change operator that satisfies (P6). Let us prove the contrapositive statement. Let $\varphi \in \mathcal{L}_*$, $\alpha \in \mathcal{L}$, and assume that $\varphi \circ \phi_\omega \models \phi_\omega$ for each world $\omega \in [\alpha]$. We prove that $\varphi \circ \alpha \models \alpha$ by induction on the number of models of α. The base case is when $|[\alpha]| = 1$, and in this case the prove is trivial. Now, let $k \geq 1$, and assume now that for each formula α such that $|[\alpha]| = k$, if $\varphi \circ \phi_\omega \models \phi_\omega$ for each world $\omega \in [\alpha]$, then $\varphi \circ \phi_\omega \models \phi_\omega$. Let $\beta \in \mathcal{L}$ such that $|[\beta]| = k + 1$, and assume that for each world $\omega \in [\alpha]$, $\varphi \circ \phi_\omega \models \phi_\omega$. Then β is such that $\beta \equiv \alpha \vee \phi_\omega$ for some formula $\alpha \in \mathcal{L}$ such that $|[\alpha]| = k$. Then by (P6),

$$\varphi \circ \beta \equiv \begin{cases} \varphi \circ \alpha & \text{or} \\ \varphi \circ \phi_\omega & \text{or} \\ (\varphi \circ \alpha) \vee (\varphi \circ \phi_\omega) & \end{cases}$$

Since $\varphi \circ \alpha \models \alpha$ and $\varphi \circ \phi_\omega \models \phi_\omega$ by the induction hypothesis, it can be seen that in every case, $\varphi \circ \beta \models \beta$. This concludes the proof by induction that for each $\varphi \in \mathcal{L}_*$ and each $\alpha \in \mathcal{L}$, if $\varphi \circ \phi_\omega \models \phi_\omega$ for each world $\omega \in [\alpha]$, then $\varphi \circ \alpha \models \alpha$. □

Proposition 2. *If $\circ$ is a belief change operator satisfying (P1), (P2) and (P3), then $\circ$ satisfies (RT) if and only if it satisfies (CR56).*

Proof. (Only if part) Let $\circ$ be a belief change operator satisfying (P1), (P2), (P3) and (RT), and let us show that $\circ$ satisfies (CR56). So let $\varphi \in \mathcal{L}_*$, $\alpha, \beta \in \mathcal{L}$ and assume that (i) $\varphi \circ \alpha \models \alpha$ and (ii) $(\varphi \circ \alpha) \wedge \beta \not\models \bot$. We need to show that $\varphi \circ (\alpha \wedge \beta) \equiv (\varphi \circ \alpha) \wedge \beta$. In the case where $\varphi \wedge \alpha \not\models \bot$, then by (P2) $\varphi \circ \alpha \equiv \varphi \wedge \alpha$, so by condition (ii) $(\varphi \wedge \alpha) \wedge \beta \not\models \bot$, i.e., $\varphi \wedge (\alpha \wedge \beta) \not\models \bot$, which by (P2) means that $\varphi \circ (\alpha \wedge \beta) \equiv \varphi \wedge (\alpha \wedge \beta)$, so we get that $(\varphi \circ \alpha) \wedge \beta \equiv \varphi \circ (\alpha \wedge \beta)$. Then assume that (iii) $\varphi \wedge \alpha \models \bot$. By (P1), note that either (iv-1) $\varphi \circ (\alpha \wedge \neg\beta) \models \alpha \wedge \neg\beta$ or (iv-2) $\varphi \circ (\alpha \wedge \neg\beta) \equiv \varphi$. Note also that (v) $(\varphi \circ \alpha) \wedge \beta \not\equiv \varphi \wedge \beta$, since conditions (i) and (ii) together imply that $(\varphi \circ \alpha) \wedge \beta \wedge \alpha \not\models \bot$, whereas condition (iii) implies that $\varphi \wedge \beta \wedge \alpha \models \bot$. Now, since $\varphi \circ \alpha \models \alpha$ by (RT) we get that:

$$\varphi \circ \alpha \equiv \begin{cases} \varphi \circ (\alpha \wedge \beta) & \text{(case 1)} \quad \text{or} \\ \varphi \circ (\alpha \wedge \neg\beta) & \text{(case 2)} \quad \text{or} \\ (\varphi \circ (\alpha \wedge \beta)) \vee (\varphi \circ (\alpha \wedge \neg\beta)) & \text{(case 3)} \end{cases}$$

Assume first case 2, i.e., $\varphi \circ \alpha \equiv \varphi \circ (\alpha \wedge \neg\beta)$, and let us show that this leads to a contradiction. Since $\varphi \circ \alpha \equiv \varphi \circ (\alpha \wedge \neg\beta)$, we get that $(\varphi \circ \alpha) \wedge \beta \equiv (\varphi \circ (\alpha \wedge \neg\beta)) \wedge \beta$. If we fall into case (iv-1) above, i.e., $\varphi \circ (\alpha \wedge \neg\beta) \models \alpha \wedge \neg\beta$, then we get that $(\varphi \circ (\alpha \wedge \neg\beta)) \wedge \beta \models \bot$, so $(\varphi \circ \alpha) \wedge \beta \models \bot$, which directly contradicts (ii). If we fall instead into case (iv-2), i.e., $\varphi \circ (\alpha \wedge \neg\beta) \equiv \varphi$, then we get that $(\varphi \circ \alpha) \wedge \beta \equiv \varphi \wedge \beta$, which directly contradicts (v). Then, we fall into case 1 or case 3. We intend to show that both cases lead to the same conclusion that $(\varphi \circ \alpha) \wedge \beta \equiv \varphi \circ (\alpha \wedge \beta)$. Assume we are in case 1, i.e., $\varphi \circ \alpha \equiv \varphi \circ (\alpha \wedge \beta)$. Since by (i), $(\varphi \circ \alpha) \wedge \beta \models \alpha \wedge \beta$, we get that $\varphi \circ (\alpha \wedge \beta) \models \alpha \wedge \beta$, and then $\varphi \circ (\alpha \wedge \beta) \wedge \beta \equiv \varphi \circ (\alpha \wedge \beta)$, thus $(\varphi \circ \alpha) \wedge \beta \equiv \varphi \circ (\alpha \wedge \beta)$. Assume then case 3, i.e., $\varphi \circ \alpha \equiv (\varphi \circ (\alpha \wedge \beta)) \vee (\varphi \circ (\alpha \wedge \neg\beta))$. If we fall into case (iv-1) above, i.e., $\varphi \circ (\alpha \wedge \neg\beta) \models \alpha \wedge \neg\beta$, then the proof that $(\varphi \circ \alpha) \wedge \beta \equiv \varphi \circ (\alpha \wedge \beta)$ is identical to case 1 since in this case $(\varphi \circ \alpha) \wedge \beta \equiv \varphi \circ (\alpha \wedge \beta) \wedge \beta$. If we are in case (iv-2), i.e., $\varphi \circ (\alpha \wedge \neg\beta) \equiv \varphi$, then we get that $\varphi \models \varphi \circ \alpha$, which contradicts (i) and (iii). We got that $(\varphi \circ \alpha) \wedge \beta \equiv \varphi \circ (\alpha \wedge \beta)$ in every case, which concludes the only if part of the proof.

(If part) Let $\circ$ be a belief change operator satisfying (P1), (P2), (P3) and (CR56), and let us show that $\circ$ satisfies (RT). So let $\varphi \in \mathcal{L}_*$, $\alpha, \beta \in \mathcal{L}$ and assume that $\varphi \circ (\alpha \vee \beta) \models \alpha \vee \beta$. Then (i) $\varphi \circ (\alpha \vee \beta) \equiv ((\varphi \circ (\alpha \vee \beta)) \wedge \alpha) \vee ((\varphi \circ (\alpha \vee \beta)) \wedge \beta)$. We need to show that:

$$\varphi \circ (\alpha \vee \beta) \equiv \begin{cases} \varphi \circ \alpha & \text{or} \\ \varphi \circ \beta & \text{or} \\ (\varphi \circ \alpha) \vee (\varphi \circ \beta) \end{cases}$$

Let us remark by (CR56) that (ii) if $(\varphi \circ (\alpha \vee \beta)) \wedge \alpha \not\models \bot$, then $(\varphi \circ (\alpha \vee \beta)) \wedge \alpha \equiv \varphi \circ ((\alpha \vee \beta) \wedge \alpha) \equiv \varphi \circ \alpha$; and similarly (iii) if $(\varphi \circ (\alpha \vee \beta)) \wedge \beta \not\models \bot$, then $(\varphi \circ (\alpha \vee \beta)) \wedge \beta \equiv \varphi \circ \beta$. Since $\varphi \circ (\alpha \vee \beta) \models \alpha \vee \beta$, by (P3) we know that $\alpha \vee \beta \not\models \bot$, thus $(\varphi \circ (\alpha \vee \beta)) \wedge \alpha \not\models \bot$ or $(\varphi \circ (\alpha \vee \beta)) \wedge \beta \not\models \bot$. Then we fall into one of the following three cases (1) $(\varphi \circ (\alpha \vee \beta)) \wedge \alpha \not\models \bot$ and $(\varphi \circ (\alpha \vee \beta)) \wedge \beta \models \bot$; (2) $(\varphi \circ (\alpha \vee \beta)) \wedge \alpha \models \bot$ and $(\varphi \circ (\alpha \vee \beta)) \wedge \beta \not\models \bot$; or (3) $(\varphi \circ (\alpha \vee \beta)) \wedge \alpha \not\models \bot$ and $(\varphi \circ (\alpha \vee \beta)) \wedge \beta \not\models \bot$. Yet in case (1), we get from (i) and (ii) above that $\varphi \circ (\alpha \vee \beta) \equiv \varphi \circ \alpha$; in case (2), we get from (i) and (iii) above that $\varphi \circ (\alpha \vee \beta) \equiv \varphi \circ \beta$; and in case (3), we get from (i), (ii) and (iii) above that $\varphi \circ (\alpha \vee \beta) \equiv (\varphi \circ \alpha) \vee (\varphi \circ \beta)$. This shows that (RT) is satisfied. □

Definition 6. *A belief change operator $\circ$ is called a* Context-Based (CB) revision operator *if it satisfies (P1-P4), (RT), (CC) and (NCP).*

Proposition 3. *A belief change operator is a CL revision operator if and only if it is a CB revision operator that satisfies (P5).*

Proof. (Only if part) Let $\circ$ be a CL revision operator, i.e., $\circ$ satisfies (P1-P6). We need to show that $\circ$ is satisfies (RT), (NCP) and (CC). Yet (RT) is a direct consequence of (P6); (NCP) is satisfied by vacuity, since by (P5) we cannot have that $\varphi \circ \alpha \models \alpha$ and $\varphi \circ (\alpha \vee \beta) \not\models \alpha \vee \beta$; and (CC) is a direct consequence of (P5).

(If part) Let $\circ$ be a CB revision operator satisfying (P5). We only need to show that $\circ$ satisfies (P6). In the case when $\varphi \circ (\alpha \vee \beta) \models \alpha \vee \beta$, then the conclusion of (P6) is a direct consequence of (RT). So assume that $\varphi \circ (\alpha \vee \beta) \not\models \alpha \vee \beta$, which by (P1) means that $\varphi \circ (\alpha \vee \beta) \equiv \varphi$. Yet since $\varphi \circ (\alpha \vee \beta) \not\models \alpha \vee \beta$, by (P5) we get that $\varphi \circ \alpha \not\models \alpha$ and $\varphi \circ \beta \not\models \beta$, which by (P1) again means that $\varphi \circ \alpha \equiv \varphi$ and $\varphi \circ \beta \equiv \varphi$. We got that $\varphi \circ (\alpha \vee \beta) \equiv \varphi$ and that $\varphi \circ \alpha \equiv \varphi \circ \beta \equiv \varphi \circ (\alpha \vee \beta) \equiv \varphi$ which shows that (P6) is satisfied. □

Proposition 4. *For each $\mu, \sigma \in \mathcal{L}$, $\bullet_{\mu,\sigma}$ is a CB revision operator.*

Proof. The fact that $\bullet_{\mu,\sigma}$ satisfies (P1), (P2), (P3) and (P4) is direct by definition.

To show that (RT) is satisfied, it is enough by Prop. 2 to show (CR56) is satisfied. So let $\varphi \in \mathcal{L}_*$, $\alpha, \beta \in \mathcal{L}$ and assume that (i) $\varphi \bullet_{\mu,\sigma} \alpha \models \alpha$ and (ii) $(\varphi \bullet_{\mu,\sigma} \alpha) \wedge \beta \not\models \bot$. We must show that $\varphi \bullet_{\mu,\sigma} (\alpha \wedge \beta) \equiv (\varphi \bullet_{\mu,\sigma} \alpha) \wedge \beta$. We fall into one of the following two cases:
(Case 1) Assume first that $\varphi \wedge (\alpha \wedge \beta) \not\models \bot$. Then by definition, $\varphi \bullet_{\mu,\sigma} (\alpha \wedge \beta) \equiv \varphi \wedge (\alpha \wedge \beta)$. And since $\varphi \wedge (\alpha \wedge \beta) \not\models \bot$, we also have that $\varphi \wedge \alpha \not\models \bot$, so by definition, $\varphi \bullet_{\mu,\sigma} \alpha \equiv \varphi \wedge \alpha$, thus $(\varphi \bullet_{\mu,\sigma} \alpha) \wedge \beta \equiv (\varphi \wedge \alpha) \wedge \beta$. This shows that $\varphi \bullet_{\mu,\sigma} (\alpha \wedge \beta) \equiv (\varphi \bullet_{\mu,\sigma} \alpha) \wedge \beta$.
(Case 2) Assume now that (iii) $\varphi \wedge (\alpha \wedge \beta) \models \bot$. This together with conditions (ii) and (P2) means that (iv) $\varphi \wedge \alpha \models \bot$. By (i), (iv) and Lemma 6, we get that (v) $\varphi \bullet_{\mu,\sigma} \alpha \not\equiv \varphi$, which together with (iv) and by definition of $\varphi \bullet_{\mu,\sigma} \alpha$ means that (vi) $\varphi \bullet_{\mu,\sigma} \alpha = \mu \wedge \alpha$, thus (vii) $(\varphi \bullet_{\mu,\sigma} \alpha) \wedge \beta \equiv$

$\mu \wedge \alpha \wedge \beta$. Assume now toward a contradiction that (viii-1) $\mu \wedge (\alpha \wedge \beta) \models \bot$ or (viii-2) $\alpha \wedge \beta \not\models \sigma$. By (ii) and (vi), $\mu \wedge (\alpha \wedge \beta) \not\models \bot$, so this violates (viii-1). And (viii-2) implies that $\alpha \not\models \sigma$, which by definition of $\varphi \bullet_{\mu,\sigma} \alpha$ means that $\varphi \bullet_{\mu,\sigma} \alpha = \varphi$, which contradicts (v). Then none of the conditions (viii-1) and (viii-2) holds, thus $\mu \wedge (\alpha \wedge \beta) \not\models \bot$ and $\alpha \wedge \beta \models \sigma$. Then together with (iii) and by definition of $\varphi \bullet_{\mu,\sigma} (\alpha \wedge \beta)$, we get that (ix) $\varphi \bullet_{\mu,\sigma} (\alpha \wedge \beta) \equiv \mu \wedge (\alpha \wedge \beta)$. And conditions (vii) and (ix) together show that $\varphi \bullet_{\mu,\sigma} (\alpha \wedge \beta) \equiv (\varphi \bullet_{\mu,\sigma} \alpha) \wedge \beta$.

This concludes the proof that $\bullet_{\mu,\sigma}$ satisfies (CR56), or equivalently, that $\bullet_{\mu,\sigma}$ satisfies (RT).

Let us show that $\bullet_{\mu,\sigma}$ satisfies (NCP). Let $\varphi \in \mathcal{L}_*$, $\alpha, \beta, \gamma, \delta \in \mathcal{L}$, and assume that (i) $\varphi \bullet_{\mu,\sigma} \alpha \models \alpha$, (ii) $\varphi \bullet_{\mu,\sigma} (\alpha \vee \beta) \not\models \alpha \vee \beta$, and (iii) $\varphi \bullet_{\mu,\sigma} (\alpha \wedge \gamma) \not\models \alpha$. We need to show that $\varphi \bullet_{\mu,\sigma} (\alpha \wedge \gamma \wedge \delta) \not\models \alpha$. Let us show that $\varphi \wedge \alpha \wedge \gamma \wedge \delta \models \bot$ and $\mu \wedge \alpha \wedge \gamma \wedge \delta \models \bot$. On the one hand, by (ii) and (P2), we get that (iv) $\varphi \wedge (\alpha \vee \beta) \models \bot$, thus (v) $\varphi \wedge \alpha \wedge \gamma \wedge \delta \models \bot$. On the other hand, by (iv), $\varphi \wedge (\alpha \wedge \gamma) \models \bot$ so by (iii) and by definition of $\varphi \bullet_{\mu,\sigma} (\alpha \wedge \gamma)$ we get that $\mu \wedge \alpha \wedge \gamma \models \bot$, thus (vi) $\mu \wedge \alpha \wedge \gamma \wedge \delta \models \bot$. Conditions (v) and (vi) together imply by definition of $\varphi \bullet_{\mu,\sigma} (\alpha \wedge \gamma \wedge \delta)$ that (vii) $\varphi \bullet_{\mu,\sigma} (\alpha \wedge \gamma \wedge \delta) \equiv \varphi$. Yet $\varphi \not\models \bot$, so by (iv) we know that $\varphi \not\models \alpha$. Hence, by (vii), $\varphi \bullet_{\mu,\sigma} (\alpha \wedge \gamma \wedge \delta) \not\models \alpha$. This concludes the proof that $\bullet_{\mu,\sigma}$ satisfies (NCP).

We now show that $\bullet_{\mu,\sigma}$ satisfies (CC). Let $\varphi \in \mathcal{L}_*$, $\alpha, \beta \in \mathcal{L}$, and assume that (i) $\varphi \bullet_{\mu,\sigma} \alpha \models \alpha$, (ii) $\varphi \bullet_{\mu,\sigma} \beta \models \beta$, and (iii) $\varphi \bullet_{\mu,\sigma} (\alpha \wedge \beta) \models \alpha \wedge \beta$. We need to show that $\varphi \bullet_{\mu,\sigma} (\alpha \vee \beta) \models \alpha \vee \beta$. If $\varphi \wedge \alpha \not\models \bot$ or $\varphi \wedge \beta \not\models \bot$, then $\varphi \wedge (\alpha \vee \beta) \not\models \bot$ and then by definition of $\varphi \bullet_{\mu,\sigma} (\alpha \vee \beta)$ we get that $\varphi \bullet_{\mu,\sigma} (\alpha \vee \beta) = \varphi \wedge (\alpha \vee \beta)$, thus $\varphi \bullet_{\mu,\sigma} (\alpha \vee \beta) \models \alpha \vee \beta$. So assume that (iv) $\varphi \wedge \alpha \models \bot$ and (v) $\varphi \wedge \beta \models \bot$. By (i), (iv) and Lemma 6, we get that $\varphi \bullet_{\mu,\sigma} \alpha \not\equiv \varphi$. Similarly, by (ii), (v) and Lemma 6, we get that $\varphi \bullet_{\mu,\sigma} \beta \not\equiv \varphi$. Hence, by definition of $\varphi \bullet_{\mu,\sigma} \alpha$ and $\varphi \bullet_{\mu,\sigma} \beta$ and together with conditions (iv) and (v), we get that (vi) $\mu \wedge \alpha \not\models \bot$, (vii) $\alpha \models \sigma$, (viii) $\mu \wedge \beta \not\models \bot$, and (ix) $\beta \models \sigma$. Yet any of the conditions (vi) or (viii) implies $\mu \wedge (\alpha \vee \beta) \not\models \bot$, and (vii) and (ix) together imply that $\alpha \vee \beta \models \sigma$, which by definition of $\varphi \bullet_{\mu,\sigma} (\alpha \vee \beta)$ means that $\varphi \bullet_{\mu,\sigma} (\alpha \vee \beta) \models \alpha \vee \beta$. This concludes the proof that $\bullet_{\mu,\sigma}$ satisfies (CC). ☐

Proposition 5. *For each $\mu, \sigma \in \mathcal{L}$, the operator $\bullet_{\mu,\sigma}$ is:*

1. *a CL revision operator iff (σ is valid or $\mu \wedge \sigma \models \bot$)*
2. *a KM revision operator iff both μ and σ are valid*

Proof. Point 1: (If part) Assume that (i-1) σ is valid or (i-2) $\mu \wedge \sigma \models \bot$, and let us show that $\bullet_{\mu,\sigma}$ is a CL revision operator. First note that if σ is valid, then $\varphi \bullet_{\mu,\sigma} \alpha$ is defined for each $\varphi \in \mathcal{L}_*$, $\alpha \in \mathcal{L}$ as:

$$\varphi \bullet_{\mu,\sigma} \alpha = \begin{cases} \varphi \wedge \alpha & \text{if } \varphi \wedge \alpha \not\models \bot \\ \mu \wedge \alpha & \text{if } \varphi \wedge \alpha \models \bot \text{ and } \mu \wedge \alpha \not\models \bot, \\ \varphi & \text{in the remaining case} \end{cases} \tag{1}$$

And if $\mu \wedge \sigma \models \bot$, then for each $\alpha \in \mathcal{L}$, we necessarily have that $\alpha \wedge \mu \models \bot$ or $\alpha \not\models \sigma$, so $\varphi \bullet_{\mu,\sigma} \alpha$ is defined for each $\varphi \in \mathcal{L}_*$, $\alpha \in \mathcal{L}$ as:

$$\varphi \bullet_{\mu,\sigma} \alpha = \begin{cases} \varphi \wedge \alpha & \text{if } \varphi \wedge \alpha \not\models \bot \\ \varphi & \text{otherwise} \end{cases} \tag{2}$$

Since $\bullet_{\mu,\sigma}$ is a CB revision operator, by Prop. 3 we only need to show that $\bullet_{\mu,\sigma}$ satisfies (P5). So, let $\varphi \in \mathcal{L}_*$, $\alpha, \beta \in \mathcal{L}$, assume that (ii) $\varphi \bullet_{\mu,\sigma} \alpha \models \alpha$ and (iii) $\alpha \models \beta$, and let us show that $\varphi \bullet_{\mu,\sigma} \beta \models \beta$. In the case when $\varphi \wedge \beta \not\models \bot$, then by definition $\varphi \bullet_{\mu,\sigma} \beta = \varphi \wedge \beta$, and so we directly get that $\varphi \bullet_{\mu,\sigma} \beta \models \beta$. So, assume that (iv) $\varphi \wedge \beta \models \bot$. By (ii), (iv) and Lemma 6, we know that (v) $\varphi \bullet_{\mu,\sigma} \beta \not\equiv \varphi$, which contradicts Equation 2 above and then means that $\mu \wedge \sigma \not\models \bot$, that is, condition (i-2) does not hold and thus condition (i-1) holds, i.e., σ is valid. In this case, from Equation 1 above, and by (iv) and (v), we get that $\varphi \bullet_{\mu,\sigma} \beta = \mu \wedge \beta$, thus $\varphi \bullet_{\mu,\sigma} \beta \models \beta$. This shows that $\bullet_{\mu,\sigma}$ satisfies (P5), thus $\bullet_{\mu,\sigma}$ is a CL revision operator, which concludes the (if) part of the proof.

(Only if part) Assume that $\bullet_{\mu,\sigma}$ is a CL revision operator, and assume toward a contradiction that σ is not valid and $\mu \wedge \sigma \not\models \bot$. Then let ω, ω' be two worlds such that $\omega \in [\mu \wedge \sigma]$ and $\omega' \in [\neg\sigma]$. Without loss of generality we can assume that $\mathcal{L}$ is generated from at least two propositional variables, which means that there exists a third world ω'' distinct from both ω and ω'. This means we can find a formula φ be a formula such that $\omega'' \in [\varphi]$, i.e., $\varphi \in \mathcal{L}_*$, and such that $\{\omega, \omega'\} \cap [\varphi] = \emptyset$. Then on the one hand, we can easily verify that $\varphi \wedge \phi_\omega \models \bot$, $\mu \wedge \phi_\omega \mu \not\models \bot$, and $\phi_\omega \models \sigma$, so by definition, $\varphi \bullet_{\mu,\sigma} \phi_\omega = \mu \wedge \phi_\omega$, thus (i) $\varphi \bullet_{\mu,\sigma} \phi_\omega \models \phi_\omega$. And on the other hand, we can also verify that $\varphi \wedge \phi_{\omega,\omega'} \models \bot$ and $\phi_{\omega,\omega'} \not\models \sigma$, so by definition, $\varphi \bullet_{\mu,\sigma} \phi_{\omega,\omega'} = \varphi$, and since $\{\omega, \omega'\} \cap [\varphi] = \emptyset$, this implies that (ii) $\varphi \bullet_{\mu,\sigma} \phi_{\omega,\omega'} \not\models \phi_{\omega,\omega'}$. Conditions (i) and (ii) together contradicts (P5) and the fact that $\bullet_{\mu,\sigma}$ is a CL revision operator. This shows that if $\bullet_{\mu,\sigma}$ is a CL revision operator, then σ is valid or $\mu \wedge \sigma \models \bot$, which concludes the (only if) part of the proof.

Point 2: (If part) Assume that both μ and σ are valid, and let us show that $\bullet_{\mu,\sigma}$ is a KM revision operator. First remark that since μ and σ are valid, $\varphi \bullet_{\mu,\sigma} \alpha$ is defined for each $\varphi \in \mathcal{L}_*$, $\alpha \in \mathcal{L}$ as:

$$\varphi \bullet_{\mu,\sigma} \alpha = \begin{cases} \varphi \wedge \alpha & \text{if } \varphi \wedge \alpha \not\models \bot \\ \alpha & \text{otherwise} \end{cases} \tag{3}$$

It is then clear that for each $\varphi \in \mathcal{L}_*$, $\alpha \in \mathcal{L}$, we have that $\varphi \bullet_{\mu,\sigma} \alpha \models \alpha$, which means that $\bullet_{\mu,\sigma}$ satisfies (R1). Then from Th. 3, $\bullet_{\mu,\sigma}$ is a KM revision operator.

(Only if part) Assume that μ is not valid or σ is not valid, i.e., $\neg\mu \vee \neg\sigma \not\models \bot$, and let us show that $\bullet_{\mu,\sigma}$ is not a KM revision operator. It is enough to show $\bullet_{\mu,\sigma}$ does not satisfy (R1). Since $\neg\mu \vee \neg\sigma \not\models \bot$, there exists a world ω such that $\omega \in [\neg\mu \vee \neg\sigma]$. Let ω' be a world distinct from ω, and let φ be a formula such that $[\varphi] = \{\omega'\}$. Then $\varphi \wedge \phi_\omega \models \bot$. Moreover, if $\omega \in [\neg\mu]$, then $\phi_\omega \wedge \mu \not\models \bot$, and if $\omega \in [\neg\sigma]$, then $\phi_\omega \models \sigma$. In both cases, $\varphi \bullet_{\mu,\sigma} \phi_\omega \equiv \varphi$ by definition. Then, by Lemma 6, $\varphi \bullet_{\mu,\sigma} \phi_\omega \not\models \phi_\omega$, so $\bullet_{\mu,\sigma}$ does not satisfy

(R1). This shows that $\bullet_{\mu,\sigma}$ is not a KM revision operator and concludes the proof. $\square$

Definition 7 (CB faithful Assignment)**.** *A* CB faithful assignment *(CB assignment for short) Ψ is a mapping associating each formula $\varphi \in \mathcal{L}_*$ with a context structure space $\mathcal{S}_\varphi = \{\mathcal{C}^0, \ldots, \mathcal{C}^k\}$ where $k \geq 0$ and such that the following conditions are satisfied for all φ, $\psi \in \mathcal{L}_*$:*

1. $[\varphi] = \min(Cred^0_\varphi, \preceq^0_\varphi)$
2. *If $\varphi \equiv \psi$, then $\mathcal{S}_\varphi = \mathcal{S}_\psi$*

Definition 8. *Given a context structure $\mathcal{C} = (Cont, Cred, \preceq)$, we say that a formula $\alpha \in \mathcal{L}$ is* $\mathcal{C}$-credible *if $[\alpha] \cap Cred \neq \emptyset$ and $[\alpha] \subseteq Cont$.*

Lemma 1. *Let $\mathcal{S} = \{\mathcal{C}^0, \ldots, \mathcal{C}^k\}$ be a context structure space and α be a formula. If α is $\mathcal{C}^i$-credible for some $i \in \{1, \ldots, k\}$, then for all $j \in \{1, \ldots, k\}$, if $i \neq j$ then $[\alpha] \cap Cred^j_\varphi = \emptyset$ and $[\alpha] \not\subseteq Cont^j_\varphi$.*

Proof. Direct by definition of a context structure space $\mathcal{S} = \{\mathcal{C}^0, \ldots, \mathcal{C}^k\}$ which requires that for all $i, j \in \{1, \ldots, k\}$, if $i \neq j$ then $Cred^i_\varphi \cap Cont^j_\varphi = \emptyset$, and by definition of a context, i.e., $Cred^i_\varphi \subseteq Cont^i_\varphi$ for each context $\mathcal{C}^i_\varphi$. $\square$

Definition 9. *Let Ψ be a CB assignment $\varphi \mapsto \mathcal{S}_\varphi$ with $\mathcal{S}_\varphi = \{\mathcal{C}^0_\varphi, \ldots, \mathcal{C}^k_\varphi\}$, $k \geq 0$, and let $\circ$ be a belief change operator. We say that* $\circ$ is represented by Ψ *if for all $\varphi \in \mathcal{L}_*$, $\alpha \in \mathcal{L}$,*

$$[\varphi \circ \alpha] = \begin{cases} \min([\alpha], \preceq^0_\varphi) & \text{if } \alpha \text{ is } \mathcal{C}^0_\varphi\text{- credible} \\ \min([\alpha], \preceq^{i_\alpha}_\varphi) & \text{if } \alpha \text{ is not } \mathcal{C}^0_\varphi\text{- credible} \\ & \text{and } \alpha \text{ is } \mathcal{C}^{i_\alpha}_\varphi\text{- credible} \\ [\varphi] & \text{in the remaining case} \end{cases}$$

Proposition 6. *If a belief change operator $\circ$ is represented by a CB assignment, then $\circ$ is a CB revision operator.*

Lemma 2. *If a belief change operator $\circ$ is represented by a CB assignment $\varphi \mapsto \{\mathcal{C}^0_\varphi, \ldots, \mathcal{C}^k_\varphi\}$, $k \geq 0$, then for each $\varphi \in \mathcal{L}_*$, $\alpha \in \mathcal{L}$, if α is φ-contextually credible then α is not $\mathcal{C}^0_\varphi$-credible and α is $\mathcal{C}^{i_\alpha}_\varphi$-credible.*

Proof. Let $\circ$ be a belief change operator represented by a CB assignment $\varphi \mapsto \{\mathcal{C}^0_\varphi, \ldots, \mathcal{C}^k_\varphi\}$, $k \geq 0$, let $\varphi \in \mathcal{L}_*$, $\alpha \in \mathcal{L}$, and assume that α is φ-contextually credible for $\circ$, which means that (i) $\varphi \circ \alpha \models \alpha$ and there exists a formula $\beta \in \mathcal{L}$ such that (ii) $\varphi \circ (\alpha \vee \beta) \not\models \alpha \vee \beta$. We need to prove that α is not $\mathcal{C}^0_\varphi$-credible and α is $\mathcal{C}^{i_\alpha}_\varphi$-credible.

Let us first prove that α is not $\mathcal{C}^0_\varphi$-credible By condition (ii) above we know that $[\varphi \circ (\alpha \vee \beta)] \neq \min([\alpha \vee \beta], \preceq^0_\varphi)$ so by definition of $[\varphi \circ (\alpha \vee \beta)]$ we get that $\alpha \vee \beta$ is not $\mathcal{C}^0_\varphi$-credible, i.e., since trivially $[\alpha \vee \beta] \subseteq Cont^0_\varphi = \Omega$, we get that $[\alpha \vee \beta] \cap Cont^0_\varphi = \emptyset$. Then $[\alpha] \cap Cont^0_\varphi = \emptyset$, which means that α is not $\mathcal{C}^0_\varphi$-credible.

Let us now prove that α is $\mathcal{C}^{i_\alpha}_\varphi$-credible. Assume toward a contradiction that for each $i \in \{1, \ldots, k\}$, α is not $\mathcal{C}^i_\varphi$-credible. Then by definition $[\varphi \circ \alpha] = [\varphi]$. Yet from condition (i) above, we get that $\varphi \models \alpha$. So $\varphi \wedge (\alpha \vee \beta) \equiv \varphi \not\models \bot$, so by (P2) $\varphi \circ (\alpha \vee \beta) \equiv \varphi \wedge (\alpha \vee \beta) \equiv \varphi$, but since $\varphi \models \alpha$, we also have that $\varphi \models \alpha \vee \beta$, and then $\varphi \circ (\alpha \vee \beta) \models \alpha \vee \beta$, which contradicts condition (ii) above. We got that there exists $i \in \{1, \ldots, k\}$ such that α is $\mathcal{C}^i_\varphi$-credible, i.e., α is $\mathcal{C}^{i_\alpha}_\varphi$-credible and this concludes the proof. $\square$

Proof. [Proof of Prop. 6] Let Ψ be an assignment $\varphi \mapsto \mathcal{S}_\varphi$ with $\mathcal{S}_\varphi = \{\mathcal{C}^0_\varphi, \ldots, \mathcal{C}^k_\varphi\}$, $k \geq 0$, $\circ$ be a belief change operator, and assume that $\circ$ is represented by Ψ. We want to prove that $\circ$ is a CB revision operator, i.e., it satisfies (P1-P4), (RT), (NCP) and (CC).

In the following, α, β, γ denote any formulae from $\mathcal{L}$. Let us prove that all CB postulates are satisfied:

(P1): direct by definition of $[\varphi \circ \alpha]$.

(P2): if $\varphi \wedge \alpha \not\models \bot$, then since $[\varphi] \subseteq Cred^0_\varphi$ by condition 1 of a CB assignment, we get that $[\alpha] \cap Cred^0_\varphi \neq \emptyset$, and then α is $\mathcal{C}^0_\varphi$-credible, thus (i) $[\varphi \circ \alpha] = \min([\alpha], \preceq^0_\varphi)$, by definition of $[\varphi \circ \alpha]$. On the other hand, $[\varphi \wedge \alpha] = [\varphi] \cap [\alpha]$; yet again by condition 1 of a CB assignment, $[\varphi] = \min(Cred^0_\varphi, \preceq^0_\varphi)$, so $[\varphi] \cap [\alpha] = \min(Cred^0_\varphi, \preceq^0_\varphi) \cap [\alpha]$; and since $\min(Cred^0_\varphi, \preceq^0_\varphi) \cap [\alpha] \neq \emptyset$, $\min(Cred^0_\varphi, \preceq^0_\varphi) \cap [\alpha] = \min([\alpha], \preceq^0_\varphi)$; hence, (ii) $[\varphi \wedge \alpha] = \min([\alpha], \preceq^0_\varphi)$. We got from (i) and (ii) that $[\varphi \circ \alpha] = [\varphi \wedge \alpha]$, i.e., $\varphi \circ \alpha \equiv \varphi \wedge \alpha$.

(P3): if α is $\mathcal{C}^0_\varphi$-credible, then $[\alpha] \cap Cred^0_\varphi \neq \emptyset$ and $[\varphi \circ \alpha]$ is defined as $[\varphi \circ \alpha] = \min([\alpha], \preceq^0_\varphi)$, so $\min([\alpha], \preceq^0_\varphi) \neq \emptyset$, thus $[\varphi \circ \alpha] \neq \emptyset$. Otherwise, if α is $\mathcal{C}^{i_\alpha}_\varphi$-credible, then $[\varphi \circ \alpha] = \min([\alpha], \preceq^{i_\alpha}_\varphi)$ and $[\alpha] \cap Cred^{i_\alpha}_\varphi \neq \emptyset$, thus $[\varphi \circ \alpha] \neq \emptyset$. In the remaining case, $[\varphi \circ \alpha] = [\varphi]$ by definition of $[\varphi \circ \alpha]$, and we also get that $[\varphi \circ \alpha] \neq \emptyset$ since φ is consistent. In all cases, we got that $\varphi \circ \alpha \not\models \bot$.

(P4): direct by condition 2 of a CB assignment and by definition of $[\varphi \circ \alpha]$.

(RT): it is enough to show from Prop. 2 that (CR56) is satisfied. So assume that $\varphi \circ \alpha \models \alpha$ and that $(\varphi \circ \alpha) \wedge \beta \not\models \bot$. We must prove that $(\varphi \circ \alpha) \wedge \beta \equiv \varphi \circ (\alpha \wedge \beta)$. We fall into one of the following three cases, which correspond to the cases used to define $[\varphi \in \alpha]$ in Def. 9:

(Case 1) Assume first that α is $\mathcal{C}^0_\varphi$-credible. Then (i) $[\varphi \circ \alpha] = \min([\alpha], \preceq^0_\varphi)$, by definition of $[\varphi \circ \alpha]$. Since $(\varphi \circ \alpha) \wedge \beta \not\models \bot$, we know that $\min([\alpha], \preceq^0_\varphi) \cap [\beta] \neq \emptyset$, thus $[\alpha \wedge \beta] \cap Cred^0_\varphi \neq \emptyset$, which means that $\alpha \wedge \beta$ is $\mathcal{C}^0_\varphi$-credible as well, and thus (ii) $[\varphi \circ (\alpha \wedge \beta)] = \min([\alpha \wedge \beta], \preceq^0_\varphi)$, by definition of $[\varphi \circ (\alpha \wedge \beta)]$. But since $\min([\alpha], \preceq^0_\varphi) \cap [\beta] \neq \emptyset$, $\min([\alpha], \preceq^0_\varphi) \cap [\beta] = \min([\alpha \wedge \beta], \preceq^0_\varphi)$, so from (i) and (ii) we get that $(\varphi \circ \alpha) \wedge \beta \equiv \varphi \circ (\alpha \wedge \beta)$.

(Case 2) Assume now that α is not $\mathcal{C}^0_\varphi$-credible and $\mathcal{C}^{i_\alpha}_\varphi$-credible. Then (i) $[\alpha] \cap Cred^0_\varphi = \emptyset$, (ii) $[\alpha] \subseteq Cont^{i_\alpha}_\varphi$, and (iii) $[\varphi \circ \alpha] = \min([\alpha], \preceq^{i_\alpha}_\varphi)$, by definition of $[\varphi \circ \alpha]$. By (i), $[\alpha \wedge \beta] \cap Cred^0_\varphi = \emptyset$, and trivially $[\alpha \wedge \beta] \subseteq Cont^0_\varphi = \Omega$, so $\alpha \wedge \beta$ is not $\mathcal{C}^0_\varphi$-credible. And since $(\varphi \circ \alpha) \wedge \beta \not\models \bot$, we know that $\min([\alpha], \preceq^{i_\alpha}_\varphi) \cap [\beta] \neq \emptyset$, thus $[\alpha \wedge \beta] \cap Cred^{i_\alpha}_\varphi \neq \emptyset$; and by (ii), $[\alpha \wedge \beta] \subseteq Cont_{\varphi}^{i_\alpha}$; so this means that $\alpha \wedge \beta$ is $\mathcal{C}_{\varphi}^{i_\alpha}$-credible as well. But we learned from Lemma 1 that each formula is $\mathcal{C}^i$-credible for at most one non-absolute context $\mathcal{C}^i \in \mathcal{S}$, so we can write that $\alpha \wedge \beta$ is $\mathcal{C}^{i_{\alpha\wedge\beta}}$-credible, with $i_{\alpha\wedge\beta} = i_\alpha$. And by definition, $[\varphi \circ (\alpha \wedge \beta)] = \min([\alpha \wedge \beta], \preceq_{\varphi}^{i_{\alpha\wedge\beta}})$, thus (iv) $[\varphi \circ (\alpha \wedge \beta)] = \min([\alpha \wedge \beta], \preceq_{\varphi}^{i_\alpha})$. But since $\min([\alpha], \preceq_{\varphi}^{0}) \cap [\beta] \neq \emptyset$, $\min([\alpha], \preceq_{\varphi}^{i_\alpha}) \cap [\beta] = \min([\alpha \wedge \beta], \preceq_{\varphi}^{i_\alpha})$, so from (iii) and (iv) we get that $(\varphi \circ \alpha) \wedge \beta \equiv \varphi \circ (\alpha \wedge \beta)$.

(Case 3) The remaining case is when α is not $\mathcal{C}_{\varphi}^{i}$-credible for each $i \in \{0, \ldots, k\}$. This case corresponds to the remaining case in the definition of $[\varphi \circ \alpha]$, i.e., (i) $[\varphi \circ \alpha] = [\varphi]$. Since $\varphi \circ \alpha \models \alpha$, (ii) $\varphi \models \alpha$. On the one hand, by (i), we get that (iii) $(\varphi \circ \alpha) \wedge \beta \equiv \varphi \wedge \beta$. On the other hand, $(\varphi \circ \alpha) \wedge \beta \not\models \bot$, so by (i), $\varphi \circ \beta \not\models \bot$; and then by (ii), $\varphi \wedge \alpha \wedge \beta \equiv \varphi \wedge \beta \not\models \bot$; so by (P2), we get that (iv) $\varphi \circ (\alpha \wedge \beta) \equiv \varphi \wedge \alpha \wedge \beta \equiv \varphi \wedge \beta$. By (iii) and (iv), we get that $(\varphi \circ \alpha) \wedge \beta \equiv \varphi \circ (\alpha \wedge \beta)$.

We have showed that if $\varphi \circ \alpha \models \alpha$ and $(\varphi \circ \alpha) \wedge \beta \not\models \bot$, then $(\varphi \circ \alpha) \wedge \beta \equiv \varphi \circ (\alpha \wedge \beta)$ in every possible case, which proves that $\circ$ satisfies (CR56), or equivalently from Prop. 2, that $\circ$ satisfies (RT).

(NCP): assume that (i) $\varphi \circ \alpha \models \alpha$, (ii) $\varphi \circ (\alpha \vee \beta) \not\models \alpha \vee \beta$, and (iii) $\varphi \circ (\alpha \wedge \gamma) \not\models \alpha$. We need to show that for all formulae δ, $\varphi \circ (\alpha \wedge \gamma \wedge \delta) \not\models \alpha$. Let us prove that for all $i \in \{0, \ldots, k\}$, $[\alpha \wedge \gamma] \cap Cred_{\varphi}^{i} = \emptyset$. From condition (iii) and the definition of $[\varphi \circ (\alpha \wedge \gamma)]$, we already know that $\alpha \wedge \gamma$ is not $\mathcal{C}_{\varphi}^{0}$-credible, so $[\alpha \wedge \gamma] \cap Cred_{\varphi}^{0} = \emptyset$. So what remains to be proved is that for all $i \in \{1, \ldots, k\}$, $[\alpha \wedge \gamma] \cap Cred_{\varphi}^{i} = \emptyset$. Conditions (i) and (ii) tell us that α is φ-contextually credible for $\circ$. By Lemma 2, this means that α is $\mathcal{C}_{\varphi}^{i_\alpha}$-credible (with $i_\alpha \in \{1, \ldots, k\}$), which in particular implies that (iv) $[\alpha] \subseteq Cont_{\varphi}^{i_\alpha}$ and by Lemma 1 that (v) for all $j \in \{1, \ldots, k\}$, if $j \neq i_\alpha$ then $[\alpha] \cap Cred_{\varphi}^{j} = \emptyset$. Now, by condition (iii) and by definition of $[\varphi \circ (\alpha \wedge \gamma)]$, we know that for all $i \in \{1, \ldots, k\}$, $\alpha \wedge \gamma$ is not $\mathcal{C}_{\varphi}^{i}$-credible, so in particular (vii) $\alpha \wedge \gamma$ is not $\mathcal{C}_{\varphi}^{i_\alpha}$-credible. Yet on the one hand, from condition (iv) we get that $[\alpha \wedge \gamma] \subseteq Cont_{\varphi}^{i_\alpha}$; in conjunction with condition (vii), we get that (viii) $[\alpha \wedge \gamma] \cap Cred_{\varphi}^{i_\alpha} = \emptyset$. On the other hand, from condition (v) we get that (ix) for all $j \in \{1, \ldots, k\}$, if $j \neq i_\alpha$ then $[\alpha \wedge \gamma] \cap Cred_{\varphi}^{j} = \emptyset$. Then conditions (viii) and (ix) together precisely tell us that for all $i \in \{1, \ldots, k\}$, $[\alpha \wedge \gamma] \cap Cred_{\varphi}^{i} = \emptyset$. We got that for all $i \in \{0, \ldots, k\}$, $[\alpha \wedge \gamma] \cap Cred_{\varphi}^{i} = \emptyset$. Then for each formula $\delta \in \mathcal{L}$, for all $i \in \{0, \ldots, k\}$, $[\alpha \wedge \gamma \wedge \delta] \cap Cred_{\varphi}^{i} = \emptyset$, so $\alpha \wedge \gamma \wedge \delta$ is not $\mathcal{C}_{\varphi}^{i}$-credible, and then by definition $[\varphi \circ (\alpha \wedge \gamma \wedge \delta)] = [\varphi]$. Now assume toward a contradiction that there exists a formula $\delta \in \mathcal{L}$ such that $\varphi \circ (\alpha \wedge \gamma \wedge \delta) \models \alpha$. Then $\varphi \models \alpha$. So $\varphi \wedge (\alpha \vee \beta) \equiv \varphi \not\models \bot$, so by (P2) $\varphi \circ (\alpha \vee \beta) \equiv \varphi \wedge (\alpha \vee \beta) \equiv \varphi$, but since $\varphi \models \alpha$, we also have that $\varphi \models \alpha \vee \beta$, and then $\varphi \circ (\alpha \vee \beta) \models \alpha \vee \beta$, which contradicts (ii). We got that $\varphi \circ (\alpha \wedge \gamma \wedge \delta) \not\models \alpha$, which concludes the proof that $\circ$ satisfies (NCP).

(CC): assume that $\varphi \circ \alpha \models \alpha$, $\varphi \circ \beta \models \beta$, and $\varphi \circ (\alpha \wedge \beta) \models \alpha \wedge \beta$. We must prove that $\varphi \circ (\alpha \vee \beta) \models \alpha \vee \beta$. The proof is trivial if any of α, β or $\alpha \vee \beta$ is φ-absolutely credible for $\circ$, so assume otherwise, which implies from Lemma 2 that α, β and $\alpha \vee \beta$ are respectively $Cred_{\varphi}^{i_\alpha}$-credible, $Cred_{\varphi}^{i_\beta}$-credible, and $Cred_{\varphi}^{i_\alpha \vee \ss_\beta}$-credible, thus (i) $[\alpha] \cap Cred_{\varphi}^{i_\alpha} \neq \emptyset$, (ii) $[\beta] \cap Cred_{\varphi}^{i_\beta} \neq \emptyset$, (iii) $[\alpha \wedge \beta] \cap Cred_{\varphi}^{i_{\alpha\wedge\beta}} \neq \emptyset$, (iv) $[\alpha] \subseteq Cont_{\varphi}^{i_\alpha}$, and (v) $[\beta] \subseteq Cont_{\varphi}^{i_\beta}$. Yet from condition (iii), $[\alpha] \cap Cred_{\varphi}^{i_{\alpha\wedge\beta}} \neq \emptyset$ so by condition (i) and Lemma 1 we get that $i_\alpha = i_{\alpha\wedge\beta}$; and similarly from condition (iii), $[\beta] \cap Cred_{\varphi}^{i_{\alpha\wedge\beta}} \neq \emptyset$ so by condition (ii) and Lemma 1 we get that $i_\beta = i_{\alpha\wedge\beta}$. That is, $i_\alpha = i_\beta = i_{\alpha\wedge\beta}$, and we denote this integer by i_*. Then, any of the conditions (i)-(iii) implies that $[\alpha \vee \beta] \cap Cred_{\varphi}^{i_*} \neq \emptyset$, and conditions (iv) and (v) together imply that $[\alpha \vee \beta] \subseteq Cont_{\varphi}^{i_*}$. This means that $\alpha \vee \beta$ is $\mathcal{C}_{\varphi}^{i_{\alpha\vee\beta}}$-credible, with $i_{\alpha\vee\beta} = i_*$, and thus $\varphi \circ (\alpha \vee \beta) \models \alpha \vee \beta$ by definition of $[\varphi \circ (\alpha \vee \beta)]$, which concludes the proof that $\circ$ satisfies (CC). ☐

Definition 10 (Assignment corresponding to $\circ$)**.** *Given a belief change operator $\circ$, the* assignment corresponding to $\circ$*, denoted by $\Psi_\circ$, is an assignment associating each formula $\varphi \in \mathcal{L}_*$ with a set $\mathcal{S}_\varphi = \{\mathcal{C}_{\varphi}^{0}, \ldots, \mathcal{C}_{\varphi}^{k}\}$, $k \geq 0$, where for each $i \in \{0, \ldots, k\}$, $\mathcal{C}_{\varphi}^{i} = (Cont_{\varphi}^{i}, Cred_{\varphi}^{i}, \preceq_{\varphi}^{i})$, $Cred_{\varphi}^{i}, Cont_{\varphi}^{i} \subseteq \Omega$ and $\preceq_{\varphi}^{i}$ is a binary relation on $Cred_{\varphi}^{i}$, all of which are defined from $\circ$ as follows, for each $\varphi \in \mathcal{L}_*$:*

- $Cont_{\varphi}^{0} = \Omega$
- $Cred_{\varphi}^{0} = \{\omega \in \Omega \mid \forall \omega' \in \Omega \quad \varphi \circ \phi_{\omega,\omega'} \models \phi_{\omega,\omega'}\}$
- $\{Cont_{\varphi}^{i} \mid 1 \leq i \leq k\} = \max(\mathcal{E}, \subseteq)$
- $\forall i \in \{1, \ldots, k\}$, $Cred_{\varphi}^{i} = \{\omega \in Cont_{\varphi}^{i} \mid \varphi \circ \phi_\omega \models \phi_\omega\}$
- $\forall i \in \{0, \ldots, k\}$ $\forall \omega, \omega' \in Cred_{\varphi}^{i} \quad \omega \preceq_{\varphi}^{i} \omega'$ *iff* $\omega \in [\varphi \circ \phi_{\omega,\omega'}]$

where $\max(\mathcal{E}, \subseteq) = \{W \in \mathcal{E} \mid \forall W' \in \mathcal{E} \;\; W \not\subset W'\}$, and $\mathcal{E} = \{W \subseteq (\Omega \setminus Cred_{\varphi}^{0}) \mid \exists \omega \in W : \forall \omega' \in W \; \varphi \circ \phi_{\omega,\omega'} \models \phi_{\omega,\omega'}\}$.

Definition 11 (Assignment $\Psi_{\bullet_{\mu,\sigma}}$)**.** *For each $\mu, \sigma \in \mathcal{L}$, let $\Psi_{\bullet_{\mu,\sigma}} : \varphi \mapsto \mathcal{S}_\varphi = \{\mathcal{C}_{\varphi}^{0}, \ldots, \mathcal{C}_{\varphi}^{k}\}$ be assignment defined for each $\varphi \in \mathcal{L}_*$ as:*

- *if $\neg\sigma \models \varphi$ or $\mu \wedge \sigma \models \varphi$, then $\mathcal{S}_\varphi = (\mathcal{C}_{\varphi}^{0})$, with $Cred_{\varphi}^{0} = [\varphi \vee (\mu \wedge \sigma)]$ and $\forall \omega, \omega' \in Cred_{\varphi}^{0}$, $\omega \preceq_{\varphi}^{0} \omega'$ iff $\omega \in [\varphi]$ or $\omega' \in [\neg\varphi]$*
- *otherwise, $\mathcal{S}_\varphi = (\mathcal{C}_{\varphi}^{0}, \mathcal{C}_{\varphi}^{1})$, with $Cred_{\varphi}^{0} = [\varphi]$, $Cont_{\varphi}^{1} = [\neg\varphi \wedge \sigma]$, $Cred_{\varphi}^{1} = [\neg\varphi \wedge \mu \wedge \sigma]$, and $\forall i \in \{0, 1\}$, $\preceq_{\varphi}^{i} = Cred_{\varphi}^{i} \times Cred_{\varphi}^{i}$.*

Proposition 7. *For each μ, σ, the assignment $\Psi_{\bullet_{\mu,\sigma}}$ corresponds to $\bullet_{\mu,\sigma}$.*

Proof. Let $\mu, \sigma \in \mathcal{L}$, and assume first that (i-1) $\neg\sigma \models \varphi$ or (i-2) $\mu \wedge \sigma \models \varphi$. We need to prove that $\mathcal{S}_\varphi = (\mathcal{C}_{\varphi}^{0})$, with $Cred_{\varphi}^{0} = [\varphi \vee (\mu \wedge \sigma)]$ and $\forall \omega, \omega' \in Cred_{\varphi}^{0}$, $\omega \preceq_{\varphi}^{0} \omega'$ iff $\omega \in [\varphi]$ or $\omega' \in [\neg\varphi]$.

Let us prove that $Cred^0_\varphi = [\varphi \vee (\mu \wedge \sigma)]$. We first prove the inclusion $Cred^0_\varphi \subseteq [\varphi \vee (\mu \wedge \sigma)]$. Let $\omega \in \Omega \setminus [\varphi \vee (\mu \wedge \sigma)]$, we need to show that $\omega \notin Cred^0_\varphi$. Since $\omega \notin [\varphi]$ and ($\omega \notin [\mu]$ or $\omega \notin [\sigma]$), we know that $\varphi \wedge \phi_\omega \models \bot$ and ($\mu \wedge \phi_\omega \models \bot$ or $\phi_\omega \not\models \sigma$), and then by definition of $\bullet_{\mu,\sigma}$ we get that $\varphi \bullet_{\mu,\sigma} \phi_\omega \equiv \varphi$. By Lemma 6, we get that $\varphi \bullet_{\mu,\sigma} \phi_\omega \not\models \phi_\omega$. And by definition of $Cred^0_\varphi$ in Def. 10, we get that $\omega \notin Cred^0_\varphi$. This proves that $Cred^0_\varphi \subseteq [\varphi \vee (\mu \wedge \sigma)]$. We now prove the other inclusion $[\varphi \vee (\mu \wedge \sigma)] \subseteq Cred^0_\varphi$. Let (ii) $\omega \in [\varphi \vee (\mu \wedge \sigma)]$, we need to show that $\omega \in Cred^0_\varphi$, i.e., by definition of $Cred^0_\varphi$ in Def. 10, we need to show that for each world $\omega' \in \Omega$, $\varphi \bullet_{\mu,\sigma} \phi_{\omega,\omega'} \models \phi_{\omega,\omega'}$. Then, let $\omega' \in \Omega$. If $\omega \in [\varphi]$ or $\omega' \in [\varphi]$, then $\varphi \wedge \phi_{\omega,\omega'} \not\models \bot$, so we directly get that $\varphi \bullet_{\mu,\sigma} \phi_{\omega,\omega'} \equiv \varphi \wedge \phi_{\omega,\omega'} \models \phi_{\omega,\omega'}$. Assume otherwise, i.e., (iii) $\omega, \omega' \in [\neg\varphi]$. Together with (ii), we get that (iv) $\omega \in [\mu \wedge \sigma]$, so (v) $\mu \wedge \phi_{\omega,\omega'} \not\models \bot$. Yet conditions (iii) and (iv) together contradict our initial condition (i-2), thus condition (i-1) holds, i.e., $\neg\sigma \models \varphi$, and by condition (iii) we get that (vi) $\phi_{\omega,\omega'} \models \sigma$. Condition (iii) implies that $\varphi \wedge \phi_{\omega,\omega'} \models \bot$, which together with conditions (v) and (vi) means that $\varphi \bullet_{\mu,\sigma} \phi_{\omega,\omega'} \models \phi_{\omega,\omega'}$, by definition of $\bullet_{\mu,\sigma}$. We got that the world $\omega \in [\varphi \vee (\mu \wedge \sigma)]$ is such that for each world $\omega' \in \Omega$, $\varphi \bullet_{\mu,\sigma} \phi_{\omega,\omega'} \models \phi_{\omega,\omega'}$, which by definition of $Cred^0_\varphi$ in Def. 10 means that $\omega \in Cont^0_\varphi$. This proves the other inclusion $[\varphi \vee (\mu \wedge \sigma)] \subseteq Cred^0_\varphi$. We got that $Cred^0_\varphi \subseteq [\varphi \vee (\mu \wedge \sigma)]$ and $[\varphi \vee (\mu \wedge \sigma)] \subseteq Cred^0_\varphi$, thus $Cred^0_\varphi = [\varphi \vee (\mu \wedge \sigma)]$.

Let us now show that $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi)$, i.e., that $\{Cont^i_\varphi \mid 1 \leq i \leq k\} = \emptyset$. This is true precisely when $\mathcal{E} = \emptyset$, which we intend to prove now. Let $\omega \in \Omega \setminus Cred^0_\varphi$. Since $Cred^0_\varphi = [\varphi \vee (\mu \wedge \sigma)]$, we get that $\omega \in [\neg\varphi \wedge (\neg\mu \vee \neg\sigma)]$. Then $\varphi \wedge \phi_\omega \models \bot$ and ($\mu \wedge \phi_\omega \models \bot$ or $\phi_\omega \not\models \sigma$), so $\varphi \bullet_{\mu,\sigma} \phi_\omega \equiv \varphi$ by definition, and by Lemma 6 we get that $\varphi \bullet_{\mu,\sigma} \phi_\omega \not\models \phi_\omega$. We have just proved that for each $\omega \in \Omega \setminus Cred^0_\varphi$, $\varphi \bullet_{\mu,\sigma} \phi_\omega \not\models \phi_\omega$. This means that for each set of worlds $W \subseteq \Omega \setminus Cred^0_\varphi$, there is no world $\omega \in W$ such that for all worlds $\omega' \in W$, $\varphi \bullet_{\mu,\sigma} \phi_{\omega,\omega'} \models \phi_{\omega,\omega'}$ (since $\varphi \bullet_{\mu,\sigma} \phi_{\omega,\omega'} \not\models \phi_{\omega,\omega'}$ holds in the particular case when $\omega' = \omega$). Hence, for each set of worlds $W \subseteq \Omega \setminus Cred^0_\varphi$, we get that $W \notin \mathcal{E}$. By definition of $\mathcal{E}$ in Def. 10, we get that $\mathcal{E} = \emptyset$. This means that $\{Cont^i_\varphi \mid 1 \leq i \leq k\} = \emptyset$, thus $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi)$.

Let us now show that for all worlds $\omega, \omega' \in Cred^0_\varphi$, $\omega \preceq^0_\varphi \omega'$ iff $\omega \in [\varphi]$ or $\omega' \in [\neg\varphi]$. Let $\omega, \omega' \in Cred^0_\varphi$, and let us first prove the (if) part of the statement. So, assume that $\omega \in [\varphi]$ or $\omega' \in [\neg\varphi]$, which equivalently means that $\omega \in [\varphi]$ or $\omega, \omega' \in [\neg\varphi]$. If $\omega \in [\varphi]$, then $\varphi \wedge \phi_{\omega,\omega'} \not\models \bot$, so by definition, $\varphi \bullet_{\mu,\sigma} \phi_{\omega,\omega'} \equiv \varphi \wedge \phi_{\omega,\omega'}$, so $\omega \in [\varphi \bullet_{\mu,\sigma} \phi_{\omega,\omega'}]$, which by definition of $\preceq^0_\varphi$ in Def. 10 means that $\omega \preceq^0_\varphi \omega'$. If $\omega, \omega' \in [\neg\varphi]$, then (ii) $\varphi \wedge \phi_{\omega,\omega'} \models \bot$. Yet $\omega \in Cred^0_\varphi = [\varphi \vee (\mu \wedge \sigma)]$, so $\omega \in [\mu \wedge \sigma]$, and then (iii) $\mu \wedge \phi_{\omega,\omega'} \not\models \bot$. Since $\omega, \omega' \in [\neg\varphi]$ and $\omega \in [\mu \wedge \sigma]$, in particular our initial assumption (i-2) cannot hold, thus condition (i-1) holds, i.e., $\neg\sigma \models \varphi$, and then $\omega, \omega' \in \sigma$, from which we get that (iv) $\phi_{\omega,\omega'} \models \sigma$. Conditions (ii-iv) together mean that $\varphi \circ \phi_{\omega,\omega'} \equiv \mu \wedge \phi_{\omega,\omega'}$, thus $\omega \in [\varphi \circ \phi_{\omega,\omega'}]$, which by definition of $\preceq^0_\varphi$ in Def. 10 means that $\omega \preceq^0_\varphi \omega'$. This concludes the proof for the (if) part of the statement. To prove the (only if) part, it is enough to assume that $\omega \in [\neg\varphi]$ and $\omega' \in [\varphi]$, and show that $\omega \not\preceq^0_\varphi \omega'$. Yet since $\omega \in [\neg\varphi]$ and $\omega' \in [\varphi]$, we get that $\varphi \wedge \phi_{\omega,\omega'} \not\models \bot$, and then by definition of $\bullet_{\mu,\sigma}$ that $\varphi \bullet_{\mu,\sigma} \phi_{\omega,\omega'} \equiv \varphi \wedge \phi_{\omega,\omega'}$. Yet $[\varphi \wedge \phi_{\omega,\omega'}] = \{\omega'\}$, so $\omega \notin [\varphi \bullet_{\mu,\sigma} \phi_{\omega,\omega'}]$, which by definition of $\preceq^0_\varphi$ in Def. 10 means that $\omega \not\preceq^0_\varphi \omega'$. This concludes the proof for the (only if) part of the statement. Hence, we got that for all worlds $\omega, \omega' \in Cred^0_\varphi$, $\omega \preceq^0_\varphi \omega'$ iff $\omega \in [\varphi]$ or $\omega' \in [\neg\varphi]$.

This concludes the proof that if $\neg\sigma \models \varphi$ or $\mu \wedge \sigma \models \varphi$, then $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi)$, with $Cred^0_\varphi = [\varphi \vee (\mu \wedge \sigma)]$ and $\forall \omega, \omega' \in Cred^0_\varphi, \omega \preceq^0_\varphi \omega'$ iff $\omega \in [\varphi]$ or $\omega' \in [\neg\varphi]$.

Let us now consider the other case, i.e., assume that (i) $\neg\sigma \not\models \varphi$ and (ii) $\mu \wedge \sigma \not\models \varphi$. We need to prove that $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi, \mathcal{C}^1_\varphi)$, with $Cred^0_\varphi = [\varphi]$, $Cont^1_\varphi = [\neg\varphi \wedge \sigma]$, $Cred^1_\varphi = [\neg\varphi \wedge \mu \wedge \sigma]$, and $\forall i \in \{0, 1\}$, $\preceq^i_\varphi = Cred^i_\varphi \times Cred^i_\varphi$.

Let us prove that $Cred^0_\varphi = [\varphi]$. We first prove the inclusion $Cred^0_\varphi \subseteq [\varphi]$. Let $\omega \in [\neg\varphi]$, we need to show that $\omega \notin Cred^0_\varphi$. By our assumption (i) above, we know that there exists a world $\omega' \in [\neg\varphi \wedge \neg\sigma]$. So we get that $\varphi \wedge \phi_{\omega,\omega'} \models \bot$ and $\phi_{\omega,\omega'} \not\models \sigma$, so by definition of $\bullet_{\mu,\sigma}$, we know that $\varphi \bullet_{\mu,\sigma} \phi_{\omega,\omega'} \equiv \varphi$; together with Lemma 6, we get that $\varphi \bullet_{\mu,\sigma} \phi_{\omega,\omega'} \not\models \phi_{\omega,\omega'}$. by definition of $Cred^0_\varphi$ in Def. 10, we get that $\omega \notin Cred^0_\varphi$. This proves that $Cred^0_\varphi \subseteq [\varphi]$. We now prove the other inclusion $[\varphi] \subseteq Cred^0_\varphi$. Let $\omega \in [\varphi]$, then for each world $\omega' \in \Omega$, $\varphi \wedge \phi_{\omega,\omega'} \not\models \bot$, so by definition $\varphi \bullet_{\mu,\sigma} \phi_{\omega,\omega'} \equiv \varphi \wedge \phi_{\omega,\omega'} \models \varphi$, and then by definition of $Cred^0_\varphi$ in Def. 10, we get that $\omega \in Cred^0_\varphi$. This proves the other inclusion $[\varphi] \subseteq Cred^0_\varphi$. We got that $Cred^0_\varphi \subseteq [\varphi]$ and $[\varphi] \subseteq Cred^0_\varphi$, thus $Cred^0_\varphi = [\varphi]$.

Let us now show that the set $[\neg\varphi \wedge \sigma]$ is the unique set of worlds that belongs to $\max(\mathcal{E}, \subseteq)$. Let us first show that $[\neg\varphi \wedge \sigma] \in \mathcal{E}$. By our assumption (ii), we know that $\neg\varphi \wedge \mu \wedge \sigma \not\models \bot$, so this means that there exists a world $\omega \in [\neg\varphi \wedge \mu \wedge \sigma]$. Now, take any world $\omega' \in [\neg\varphi \wedge \sigma]$. We can see that $\varphi \wedge \phi_{\omega,\omega'} \models \bot$, $\mu \wedge \phi_{\omega,\omega'} \not\models \bot$, and $\phi_{\omega,\omega'} \models \sigma$, so by definition of $\bullet_{\mu,\sigma}$ we get that $\varphi \bullet_{\mu,\sigma} \phi_{\omega,\omega'} \equiv \mu \wedge \phi_{\omega,\omega'} \models \phi_{\omega,\omega'}$. This shows that the set of worlds $W = [\neg\varphi \wedge \sigma]$ satisfies the property that $W \subseteq \Omega \setminus Cred^0_\varphi$, and that there exists a world $\omega \in W$ such that for all worlds $\omega' \in W$, $\varphi \bullet_{\mu,\sigma} \models \phi_{\omega,\omega'}$. Hence, by definition of $\mathcal{E}$ in Def. 10, we get that $[\neg\varphi \wedge \sigma] \in \mathcal{E}$. Let us now show that $\max(\mathcal{E}, \subseteq) = \{[\neg\varphi \wedge \sigma]\}$. To do so, we intend to prove that for each world $\omega \in \Omega \setminus (Cred^0_\varphi \cup [\neg\varphi \wedge \sigma])$, we have that ω does not belong to any set of worlds $W \in \mathcal{E}$. So, let $\omega \in \Omega \setminus (Cred^0_\varphi \cup [\neg\varphi \wedge \sigma])$, i.e., $\omega \in [\neg\varphi \wedge \neg\sigma]$. We can see that for all worlds $\omega' \in \Omega \setminus Cred^0_\varphi$, we have that $\varphi \wedge \phi_{\omega,\omega'} \models \bot$ and $\phi_{\omega,\omega'} \not\models \sigma$, which by definition of $\bullet_{\mu,\sigma}$ means that $\varphi \bullet_{\mu,\sigma} \phi_{\omega,\omega'} \equiv \varphi$, and by Lemma 6, $\varphi \bullet_{\mu,\sigma} \phi_{\omega,\omega'} \not\models \phi_{\omega,\omega'}$. This means by definition of $\mathcal{E}$ in Def. 10 that for each $W \in \mathcal{E}$, $\omega \notin W$. So this shows that $\max(\mathcal{E}, \subseteq) = \{[\neg\varphi \wedge \sigma]\}$. We can then say that $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi, \mathcal{C}^1_\varphi)$, with $Cont^1_\varphi = [\neg\varphi \wedge \sigma]$.

Let us now prove that $Cred^1_\varphi = [\neg\varphi \wedge \mu \wedge \sigma]$. We know by definition of $Cred^1_\varphi$ in Def. 10 that $Cred^1_\varphi \subseteq Cont^1_\varphi$. And for each world $\omega \in Cont^1_\varphi$ (which means that $\varphi \wedge \phi_\omega \models \bot$ and $\phi_\omega \models \sigma$), we have that $\omega \in [\mu]$ iff $\mu \wedge \phi_\omega \not\models \bot$ iff $\varphi \bullet_{\mu,\sigma} \phi_\omega \models \phi_\omega$, which by definition of $Cred^1_\varphi$ in Def. 10 means that $Cred^1_\varphi = [\neg\varphi \wedge \mu \wedge \sigma]$.

What remains to be showed is that $\forall i \in \{0,1\}$, $\preceq^i_\varphi = Cred^i_\varphi \times Cred^i_\varphi$. For $\preceq^0_\varphi$, since $Cred^0_\varphi = [\varphi]$, it is easy to see that for all worlds $\omega, \omega' \in Cred^0_\varphi = [\varphi]$, $\varphi \bullet_{\mu,\sigma} \phi_{\omega,\omega'} \equiv \varphi \wedge \phi_{\omega,\omega'} \equiv \phi_{\omega,\omega'}$, i.e., $\omega, \omega' \in [\varphi \bullet_{\mu,\sigma} \phi_{\omega,\omega'}]$, and so $\omega \preceq^0_\varphi \omega'$ and $\omega' \preceq^0_\varphi \omega$. For $\preceq^1_\varphi$, since $Cred^1_\varphi = [\neg\varphi \wedge \mu \wedge \sigma]$, we can also see that for all worlds $\omega, \omega' \in Cred^1_\varphi = [\varphi]$, $\varphi \wedge \phi_{\omega,\omega'} \models \bot$, $\mu \wedge \phi_{\omega,\omega'} \not\models \bot$ and $\phi_{\omega,\omega'} \models \sigma$, and so $\varphi \bullet_{\mu,\sigma} \phi_{\omega,\omega'} \equiv \phi_{\omega,\omega'}$, i.e., $\omega, \omega' \in [\varphi \bullet_{\mu,\sigma} \phi_{\omega,\omega'}]$, and so we also get that $\omega \preceq^1_\varphi \omega'$ and $\omega' \preceq^1_\varphi \omega$. This concludes the proof. □

Proposition 8. *If $\circ$ is a CB operator, then its corresponding assignment $\Psi_\circ$ is a CB assignment.*

Lemma 3. *Let $\circ$ be a CB revision operator. Then the assignment $\varphi \mapsto \mathcal{S}_\varphi$ that represents $\circ$ satisfies the following conditions, for each $\varphi \in \mathcal{L}_*$, all $i, j \in \{0, \ldots, k\}$, all $\omega, \omega' \in \Omega$ and each $W \subseteq \Omega$:*

1. $[\varphi] \subseteq Cred^0_\varphi$
2. $\emptyset \neq Cred^i_\varphi \subseteq Cont^i_\varphi$
3. *if* $W \cap Cred^i_\varphi \neq \emptyset$ *and* $W \subseteq Cont^i_\varphi$, *then* $\varphi \circ \phi_W \models \phi_W$
4. *if* $j \geq 1$ *and* $i \neq j$, *then* $Cred^i_\varphi \cap Cont^j_\varphi = \emptyset$

Proof. Point 1: if $\omega \in [\varphi]$, then for each $\omega' \in \Omega$, $\varphi \wedge \phi_{\omega,\omega'} \not\models \bot$, so by (P2) $\varphi \circ \phi_{\omega,\omega'} \equiv \varphi \wedge \phi_{\omega,\omega'}$, thus $\varphi \circ \phi_{\omega,\omega'} \models \phi_{\omega,\omega'}$. Hence, $\omega \in Cred^0_\varphi$ by definition of $Cred^0_\varphi$. This shows that $[\varphi] \subseteq Cred^0_\varphi$.

Point 2: we trivially have $Cred^0_\varphi \subseteq Cont^0_\varphi$ since $Cont^0_\varphi = \Omega$ by definition, and the fact that $Cred^i_\varphi \subseteq Cont^i_\varphi$ when $i \geq 1$ is direct by definition of $Cred^i_\varphi$ when $i \geq 1$. Let us prove that $Cred^i_\varphi \neq \emptyset$. The fact that $Cred^0_\varphi \neq \emptyset$ is a direct consequence of point 1 above and the fact that φ is consistent, so assume $i \geq 1$. By definition of $Cont^i_\varphi$ (through the definition of $\mathcal{E}$), $Cont^i_\varphi \neq \emptyset$, and there exists a world $\omega_* \in Cont^i_\varphi$ such that $\varphi \circ \phi_{\omega_*} \models \phi_{\omega_*}$; then by definition of $Cred^i_\varphi$ we also have that $\omega_* \in Cred^i_\varphi$, thus $Cred^i_\varphi \neq \emptyset$.

Point 3: let $i \in \{0, \ldots, k\}$ and W be a set of worlds such that $W \cap Cred^i_\varphi \neq \emptyset$ and $W \subseteq Cont^i_\varphi$. We prove that $\varphi \circ \phi_W \models \phi_W$ by induction on the size of W. We start with the base case where $|W| \leq 2$, i.e., $W = \{\omega, \omega'\}$ (with possibly $\omega = \omega'$). Assuming that $\varphi \circ \phi_{\omega,\omega'} \not\models \phi_{\omega,\omega'}$ would contradict the definition of W as an element of $\mathcal{E}$, since we would get that for each world $\omega_* \in \{\omega, \omega'\}$, there is a world $\omega'_* \in W$ ($\omega'_* = \omega'$ if $\omega_* = \omega$, or $\omega'_* = \omega$ if $\omega_* = \omega'$) such that $\varphi \circ \phi_{\omega_*,\omega'_*} \not\models \phi_{\omega_*,\omega'_*}$. Then, $\varphi \circ \phi_{\omega,\omega'} \models \phi_{\omega,\omega'}$, and this proves the claim that $\varphi \circ \phi_W \models \phi_W$ for $|W| \leq 2$. Now, let $n \in \{2, \ldots, |Cont^i_\varphi| - 1\}$ and assume by induction that $\varphi \circ \phi_W \models \phi_W$ for each $W \subseteq \Omega$ such that $W \cap Cred^i_\varphi \neq \emptyset$, $W \subseteq Cont^i_\varphi$ and $|W| \leq n$. We must prove that $\varphi \circ \phi_{W'} \models \phi_{W'}$ for each $W' \subseteq \Omega$ such that $W' \cap Cred^i_\varphi \neq \emptyset$, $W' \subseteq Cont^i_\varphi$ and $|W'| = |W| + 1$. Equivalently, we need to prove that $\varphi \circ \phi_{W \cup \{\omega\}} \models \phi_{W \cup \{\omega\}}$ for each world $\omega \in Cont^i_\varphi \setminus W$. Then let $\omega \in Cont^i_\varphi \setminus W$, and let us show that $\varphi \circ \phi_{W \cup \{\omega\}} \models \phi_{W \cup \{\omega\}}$. Let $\omega' \in W \cap Cred^i_\varphi$. We know by the induction hypothesis that $\varphi \circ \phi_{\omega,\omega'} \models \phi_{\omega,\omega'}$, $\varphi \circ \phi_W \models \phi_W$, and $\varphi \circ \phi_{\omega'} \models \phi_{\omega'}$. Then, by (CC) we get that $\varphi \circ \phi_{W \cup \{\omega\}} \models \phi_{W \cup \{\omega\}}$. This concludes the proof by induction that $\varphi \circ \phi_W \models \phi_W$, for each $W \subseteq \Omega$ such that $W \cap Cred^i_\varphi \neq \emptyset$ and $W \subseteq Cont^i_\varphi$.

Point 4: if $i = 0$, then trivially $Cred^0_\varphi \cap Cont^j_\varphi = \emptyset$ since $Cont^j_\varphi$ is an element of $\mathcal{E}$ which by definition is a set of worlds $W \subseteq (\Omega \setminus Cred^0_\varphi)$. Then, let $i, j \in \{1, \ldots, k\}$, and assume toward a contradiction that $i \neq j$ and $Cred^i_\varphi \cap Cont^j_\varphi \neq \emptyset$. We intend to show that $Cont^i_\varphi \cup Cont^j_\varphi$ is an element of the set $\mathcal{E}$. Let $\omega \in Cred^i_\varphi \cap Cont^j_\varphi$. Note since $\omega \in Cred^i_\varphi$ and from point 2 above that $\omega \in Cont^i_\varphi$; and since $\omega \in Cred^i_\varphi$, $\varphi \circ \phi_\omega \models \phi_\omega$, yet $\omega \in Cont^j_\varphi$, so by definition of $Cred^j_\varphi$ we also have that $\omega \in Cred^j_\varphi$. Hence, $\omega \in Cred^i_\varphi \cap Cred^j_\varphi \cap Cont^i_\varphi \cap Cont^j_\varphi$. Now, let ω' be any world from $Cont^i_\varphi \cup Cont^j_\varphi$. If $\omega' \in Cont^i_\varphi$, then $\{\omega, \omega'\} \cap Cred^i_\varphi \neq \emptyset$ and $\{\omega, \omega'\} \subseteq Cont^i_\varphi$; if $\omega' \in Cont^j_\varphi$, then $\{\omega, \omega'\} \subseteq Cont^j_\varphi$ and $\{\omega, \omega'\} \cap Cred^j_\varphi \neq \emptyset$; in both cases, by point 3 above we get that $\varphi \circ \phi_{\omega,\omega'} \models \phi_{\omega,\omega'}$. This shows that there is a world $\omega \in Cont^i_\varphi \cup Cont^j_\varphi$ such that for each $\omega' \in Cont^i_\varphi \cup Cont^j_\varphi$, we have that $\varphi \circ \phi_{\omega,\omega'} \models \phi_{\omega,\omega'}$. This means that $Cont^i_\varphi \cup Cont^j_\varphi \in \mathcal{E}$ and since $i \neq j$ this contradicts the $\subseteq$-maximality of $Cont^i_\varphi$ and $Cont^j_\varphi$.

□

Proof. [Proof of Prop. 8] Let $\circ$ be a CB revision operator, and $\Psi_\circ$ be its corresponding assignment $\varphi \mapsto \mathcal{S}_\varphi = \{\mathcal{C}^0_\varphi, \ldots, \mathcal{C}^k_\varphi\}$, $k \geq 0$. To show that $\Psi_\circ$ is a CB assignment, we first need to show that for each $\varphi \in \mathcal{L}_*$, $\mathcal{S}_\varphi$ is a context structure space, i.e., that (1) for each $i \in \{0, \ldots, k\}$, $\emptyset \neq Cred^i_\varphi \subseteq Cont^i_\varphi$; (2) for all $i, j \in \{1, \ldots, k\}$, if $i \neq j$ then $Cred^i_\varphi \cap Cont^j_\varphi = \emptyset$; and (3) for each $i \in \{0, \ldots, k\}$, $\preceq^i_\varphi$ is a total preorder on $Cred^i_\varphi$. Yet (1) and (2) are direct consequences of Lemma 3.2 and Lemma 3.4, so we only need to show (3). Let $i \in \{0, \ldots, k\}$. Recall that $\preceq^i_\varphi$ is defined for all worlds $\omega, \omega' \in Cred^i_\varphi$ as $\omega \preceq^i_\varphi \omega'$ iff $\omega \in [\varphi \circ \phi_{\omega,\omega'}]$. We need to show that $\preceq^i_\varphi$ is a total and transitive relation.

(Totality) Let $\omega, \omega' \in Cred^i_\varphi$. By Lemma 3.2, $\{\omega, \omega'\} \subseteq Cont^i_\varphi$ so by Lemma 3.3, $\varphi \circ \phi_{\omega,\omega'} \models \phi_{\omega,\omega'}$, i.e., $[\varphi \circ \phi_{\omega,\omega'}] \subseteq \{\omega, \omega'\}$. Then by (P3), $\omega \in [\varphi \circ \phi_{\omega,\omega'}]$ or $\omega' \in [\varphi \circ \phi_{\omega,\omega'}]$, thus $\omega \preceq^i_\varphi \omega'$ or $\omega' \preceq^i_\varphi \omega$.

(Transitivity) Let $\omega, \omega', \omega'' \in Cred^i_\varphi$. Assume toward a contradiction that (i) $\omega \preceq^i_\varphi \omega'$, (ii) $\omega' \preceq^i_\varphi \omega''$, and (iii)

$\omega \not\preceq^i_\varphi \omega''$. Recall from Lemma 3.2 that $\{\omega, \omega', \omega''\} \subseteq Cont^i_\varphi$, so by Lemma 3.3, for each $W \subseteq \{\omega, \omega', \omega''\}$, we have that $\varphi \circ \phi_W \models \phi_W$; and from (P3), $\varphi \circ \phi_W \not\models \bot$. By (iii) and since $\preceq^i_\varphi$ is total, we get that $\omega'' \preceq^i_\varphi \omega$, and since $\varphi \circ \phi_{\omega,\omega''} \models \phi_{\omega,\omega''}$, by definition of $\preceq^i_\varphi$ we get that (iv) $[\varphi \circ \phi_{\omega,\omega''}] = \{\omega''\}$. On the other hand, by (i) and (ii) we also get by definition of $\preceq^i_\varphi$ that (v) $\omega \in [\varphi \circ \phi_{\omega,\omega'}]$ and (vi) $\omega' \in [\varphi \circ \phi_{\omega',\omega''}]$.

Assume first that $\varphi \circ \phi_{\omega,\omega',\omega''} \wedge \phi_{\omega,\omega''} \models \bot$. Then (vii) $[\varphi \circ \phi_{\omega,\omega',\omega''}] = \{\omega'\}$. Note that by Prop. 2, $\circ$ satisfies (CR56), and since $(\varphi \circ \phi_{\omega,\omega',\omega''}) \wedge \phi_{\omega,\omega'} \not\models \bot$, by (CR56), $(\varphi \circ \phi_{\omega,\omega',\omega''}) \wedge \phi_{\omega,\omega'} \equiv \varphi \circ \phi_{\omega,\omega'}$. Then by (vii), we get that $[\varphi \circ \phi_{\omega,\omega'}] = \{\omega'\}$, which contradicts (v).

Since assuming $(\varphi \circ \phi_{\omega,\omega',\omega''}) \wedge \phi_{\omega,\omega''} \models \bot$ leads to a contradiction, we get that $(\varphi \circ \phi_{\omega,\omega',\omega''}) \wedge \phi_{\omega,\omega''} \not\models \bot$. By (CR56), $(\varphi \circ \phi_{\omega,\omega',\omega''}) \wedge \phi_{\omega,\omega''} \equiv \varphi \circ \phi_{\omega,\omega''}$, and by (iv) we get that (viii) $\omega \notin [\varphi \circ \phi_{\omega,\omega',\omega''}]$. This means we fall into one of the two remaining cases: (case 1) $\omega' \in [\varphi \circ \phi_{\omega,\omega',\omega''}]$, or (case 2) $[\varphi \circ \phi_{\omega,\omega',\omega''}] = \{\omega''\}$. In case 1, $(\varphi \circ \phi_{\omega,\omega',\omega''}) \wedge \phi_{\omega,\omega'} \not\models \bot$, so by (CR56) $(\varphi \circ \phi_{\omega,\omega',\omega''}) \wedge \phi_{\omega,\omega'} \equiv \varphi \circ \phi_{\omega,\omega'}$. Then by (viii), $\omega \notin [\varphi \circ \phi_{\omega,\omega'}]$, which directly contradicts condition (v). In case 2, $(\varphi \circ \phi_{\omega,\omega',\omega''}) \wedge \phi_{\omega',\omega''} \not\models \bot$, so by (CR56) $(\varphi \circ \phi_{\omega,\omega',\omega''}) \wedge \phi_{\omega',\omega''} \equiv \varphi \circ \phi_{\omega',\omega''}$. Then $[\phi_{\omega',\omega''}] = \{\omega''\}$, which contradicts condition (vi).

All cases lead to a contradiction, which shows that for all worlds $\omega, \omega', \omega'' \in Cred^i_\varphi$, if $\omega \preceq^i_\varphi \omega'$ and $\omega' \preceq^i_\varphi \omega''$, then $\omega \preceq^i_\varphi \omega''$. This concludes the proof that $\preceq^i_\varphi$ is transitive and that for each $i \in \{0, \ldots, k\}$, $\preceq^i_\varphi$ is a total preorder on $Cred^i_\varphi$.

At this stage, we have showed that for each $\varphi \in \mathcal{L}_*$, $\mathcal{S}_\varphi$ is a context structure space. What remains to be proved is that conditions 1 and 2 of a CB assignment are satisfied by $\Psi_\circ$ (cf. Def. 7). Condition 2 is a direct consequence of the definition of $\Psi_\circ$ and (P4). For condition 1, we want to prove that $[\varphi] = \min(Cred^0_\varphi, \preceq^0_\varphi)$. By Lemma 3.1, we already know that $[\varphi] \subseteq Cred^0_\varphi$. Let $\omega, \omega' \in Cred^0_\varphi$. Assume that $\omega \in [\varphi]$, then it is enough to prove that if $\omega' \in [\varphi]$, then $\omega \simeq^0_\varphi \omega'$, and that if $\omega' \notin [\varphi]$, then $\omega \prec^0_\varphi \omega'$. Assume first that $\omega' \in [\varphi]$. Since $\varphi \wedge \phi_{\omega,\omega'} \not\models \bot$, by (P2) we get that $[\varphi \circ \phi_{\omega,\omega'}] = [\varphi \wedge \phi_{\omega,\omega'}] = \{\omega, \omega'\}$. Then by definition of $\preceq^0_\varphi$, $\omega \preceq^0_\varphi \omega'$ and $\omega' \preceq^0_\varphi \omega$, i.e., $\omega \simeq^0_\varphi \omega'$. Assume now that $\omega' \notin [\varphi]$. Again, $\varphi \wedge \phi_{\omega,\omega'} \not\models \bot$, so by (P2) we get that $[\varphi \circ \phi_{\omega,\omega'}] = [\varphi \wedge \phi_{\omega,\omega'}] = \{\omega\}$. Then by definition of $\preceq^0_\varphi$, $\omega \preceq^0_\varphi \omega'$ and $\omega' \not\preceq^0_\varphi \omega$, i.e., $\omega \prec^0_\varphi \omega'$. This shows that $[\varphi] = \min(Cred^0_\varphi, \preceq^0_\varphi)$, and concludes the proof. □

Proposition 9. *If a belief change operator $\circ$ is a CB revision operator, then it is represented by $\Psi_\circ$.*

Lemma 4. *Let $\circ$ be a CB revision operator and $\Psi_\circ$ be its corresponding CB assignment. Then for all $\varphi \in \mathcal{L}_*$, $\alpha \in \mathcal{L}$,*

1. *$\forall i \in \{0, \ldots, k\}$, if α is $\mathcal{C}^i_\varphi$-credible, then $[\varphi \circ \alpha] \subseteq Cred^i_\varphi$*
2. *if $\varphi \circ \alpha \models \alpha$, then $\exists i \in \{0, \ldots, k\}$ s.t. α is $\mathcal{C}^i_\varphi$-credible*

Proof. Point 1: let $\varphi \in \mathcal{L}_*$, $\alpha \in \mathcal{L}$, $i \in \{0, \ldots, k\}$, and assume toward a contradiction that α is $\mathcal{C}^i_\varphi$-credible, i.e., (i) $[\alpha] \cap Cred^i_\varphi \neq \emptyset$ and (ii) $[\alpha] \subseteq Cont^i_\varphi$, and that (iii) $[\varphi \circ \alpha] \not\subseteq Cred^i_\varphi$. From Lemma 3.3, conditions (i) and (ii) imply that (iv) $\varphi \circ \alpha \models \alpha$. Then conditions (ii) and (iv) imply that (v) $[\varphi \circ \alpha] \subseteq Cont^i_\varphi$. And conditions (iii) and (v) implies that there exists a world $\omega \in [\varphi \circ \alpha]$ such that $\omega \in Cont^i_\varphi \setminus Cred^i_\varphi$. By (CR56), the fact that $\omega \in [\varphi \circ \alpha]$ together with condition (iv) means that $\varphi \circ \phi_\omega \models \phi_\omega$, which by definition of $Cred^i_\varphi$ contradicts the fact that $\omega \notin Cred^i_\varphi$.

Point 2: let $\varphi \in \mathcal{L}_*$, $\alpha \in \mathcal{L}$, and assume that $\varphi \circ \alpha \models \alpha$. Let $\omega \in [\varphi \circ \alpha]$. In the case when $[\alpha] \cap Cred^0_\varphi \neq \emptyset$, since trivially $[\alpha] \subseteq Cont^0_\varphi = \Omega$, we directly get that α is $\mathcal{C}^0_\varphi$-credible, so assume that $[\alpha] \cap Cred^0_\varphi = \emptyset$. Since $\varphi \circ \alpha \models \alpha$ and $\omega \in [\varphi \circ \alpha]$, we know that $\omega \in [\alpha]$, so for all $\omega' \in [\alpha]$ we get that $(\varphi \circ \phi_{\omega,\omega'}) \wedge \phi_{\omega,\omega'} \equiv \phi_{\omega,\omega'}$, which also means that $(\varphi \circ \phi_{\omega,\omega'}) \wedge \phi_{\omega,\omega'} \not\models \bot$, and by by (CR56) we get that $\varphi \circ \phi_{\omega,\omega'} \equiv \phi_{\omega,\omega'}$, so in particular $\varphi \circ \phi_{\omega,\omega'} \models \phi_{\omega,\omega'}$. We got for each world $\omega' \in [\alpha]$ that $\varphi \circ \phi_{\omega,\omega'} \models \phi_{\omega,\omega'}$, which means that $[\alpha] \in \mathcal{E}$ by definition of $\mathcal{E}$. And then, $[\alpha] \subseteq W$ for some set of worlds $W \in \max(\mathcal{E}, \subseteq)$, i.e., $[\alpha] \subseteq Cont^{i_*}_\varphi$ for some $i_* \in \{1, \ldots, k\}$. On the other hand, by (CR56) again we get that $\varphi \circ \phi_\omega \models \phi_\omega$, yet $\omega \in [\alpha] \subseteq Cont^{i_*}_\varphi$, thus $\omega \in Cred^{i_*}_\varphi$ by definition of $Cred^{i_*}_\varphi$. We got that $[\alpha] \cap Cred^{i_*}_\varphi \neq \emptyset$ and $[\alpha] \subseteq Cont^{i_*}_\varphi$, which means that α is $\mathcal{C}^{i_*}_\varphi$-credible. This shows that there exists $i \in \{0, \ldots, k\}$ such that α is $\mathcal{C}^i_\varphi$-credible, and concludes the proof. □

Proof. [Proof of Prop. 9] Let $\circ$ be a CB revision operator, and $\Psi_\circ$ be its corresponding assignment $\varphi \mapsto \mathcal{S}_\varphi = \{\mathcal{C}^0_\varphi, \ldots, \mathcal{C}^k_\varphi\}$, $k \geq 0$. By Prop. 8 we know that $\Psi_\circ$ is a CB assignment, which means that $\mathcal{S}_\varphi$ is a context structure space and conditions 1 and 2 of a CB assignment are satisfied. Let $\varphi \in \mathcal{L}_*$, $\alpha \in \mathcal{L}$, we want to show that $[\varphi \circ \alpha]$ satisfies the conditions stated in Def. 9.

Let us first show an intermediate result, i.e., if α is $\mathcal{C}^i_\varphi$-credible for some $i \in \{0, \ldots, k\}$, then $[\varphi \circ \alpha] = \min([\alpha], \preceq^i_\varphi)$. So let $i \in \{0, \ldots, k\}$ and suppose α is $\mathcal{C}^i_\varphi$-credible. Observe that since α is $\mathcal{C}^i_\varphi$-credible, $[\alpha] \cap Cred^i_\varphi \neq \emptyset$ and $[\alpha] \subseteq Cont^i_\varphi$, so by Lemma 3.3, $\varphi \circ \alpha \models \alpha$.

Let us show the first inclusion $[\varphi \circ \alpha] \subseteq \min([\alpha], \preceq^i_\varphi)$. So let $\omega \in [\varphi \circ \alpha]$ and assume toward a contradiction that $\omega \notin \min([\alpha], \preceq^i_\varphi)$. First, since $\varphi \circ \alpha \models \alpha$ and $\omega \in [\varphi \circ \alpha]$, we know that $\omega \in [\alpha]$. Second, since α is $\mathcal{C}^i_\varphi$-credible, by Lemma 4.1, $\omega \in Cred^i_\varphi$. Thus, $\omega \in [\alpha] \cap Cred^i_\varphi$. Now, since $\omega \notin \min([\alpha], \preceq^i_\varphi)$ we know that there exists a world $\omega' \in [\alpha] \cap Cred^i_\varphi$ such that $\omega' \prec^i_\varphi \omega$, which by definition of $\preceq^i_\varphi$ means that $\omega \notin [\varphi \circ \phi_{\omega,\omega'}]$. Yet since $\varphi \circ \alpha \models \alpha$ and $\{\omega, \omega'\} \subseteq [\alpha]$, by (CR56) we get that $\varphi \circ \alpha \equiv \varphi \circ \phi_{\omega,\omega'}$, which means that $\omega \notin [\varphi \circ \alpha]$ and leads to a contradiction.

We now show the other inclusion $\min([\alpha], \preceq^i_\varphi) \subseteq [\varphi \circ \alpha]$. Let $\omega \in \min([\alpha], \preceq^i_\varphi)$ and let us show that $\omega \in [\varphi \circ \alpha]$. Since $\omega \in \min([\alpha], \preceq^i_\varphi)$, for each $\omega' \in [\alpha] \cap Cred^i_\varphi$ we know that $\omega \preceq^i_\varphi \omega'$, and since $\varphi \circ \alpha \models \alpha$ and $\{\omega, \omega'\} \subseteq [\alpha]$, by (CR56) we get that $\varphi \circ \alpha \equiv \varphi \circ \phi_{\omega,\omega'}$, which means that $\omega \in [\varphi \circ \alpha]$.

This concludes the proof of our intermediate result, i.e., if α is $\mathcal{C}^i_\varphi$-credible for some $i \in \{0,\dots,k\}$, then $[\varphi \circ \alpha] = \min([\alpha], \preceq^i_\varphi)$.

Now, we want to show that $[\varphi \circ \alpha]$ satisfies the conditions stated in Def. 9. We fall into three cases: (case 1) α is $\mathcal{C}^0_\varphi$-credible; (case 2) α is not $\mathcal{C}^0_\varphi$-credible and α is $\mathcal{C}^{i_\alpha}_\varphi$-credible; (case 3) the remaining case. Yet in case 1, our intermediate result above implies that $[\varphi \circ \alpha] = \min([\alpha], \preceq^0_\varphi)$; and similarly in case 2, we get that $[\varphi \circ \alpha] = \min([\alpha], \preceq^{i_\alpha}_\varphi)$. What is left to be shown is that in case 3, $[\varphi \circ \alpha] = [\varphi]$. Yet in case 3, we have that for each $i \in \{0,\dots,k\}$, α in not $\mathcal{C}^i_\varphi$-credible, so by by Lemma 4.2, this means that $\varphi \circ \alpha \not\models \alpha$. Then by (P1), $\varphi \circ \alpha \equiv \varphi$, i.e., $[\varphi \circ \alpha] = [\varphi]$. And this concludes the proof that $\circ$ is represented by $\Psi_\circ$. □

Theorem 4. *A belief change operator $\circ$ is a CB revision operator if and only if there exists a CB assignment Ψ associating each formula $\varphi \in \mathcal{L}_*$ with a context structure space $\mathcal{S}_\varphi = \{\mathcal{C}^0, \dots, \mathcal{C}^k\}$ where $k \geq 0$, and such that for all $\varphi \in \mathcal{L}_*$, $\alpha \in \mathcal{L}$,*

$$[\varphi \circ \alpha] = \begin{cases} \min([\alpha], \preceq^0_\varphi) & \textit{if } \alpha \textit{ is } \mathcal{C}^0_\varphi\textit{- credible} \\ \min([\alpha], \preceq^{i_\alpha}_\varphi) & \textit{if } \alpha \textit{ is not } \mathcal{C}^0_\varphi\textit{- credible} \\ & \textit{and } \alpha \textit{ is } \mathcal{C}^{i_\alpha}_\varphi\textit{- credible} \\ [\varphi] & \textit{in the remaining case} \end{cases}$$

Proposition 10. *Let $\circ$ be a CB revision operator and $\Psi_\circ : \varphi \mapsto \mathcal{S}_\varphi = \{\mathcal{C}^0_\varphi, \dots, \mathcal{C}^k_\varphi\}$ be its corresponding CB assignment. Then:*

1. *$\circ$ is a CL revision operator iff $\forall \varphi \in \mathcal{L}_*$, $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi)$*
2. *$\circ$ is a KM revision operator iff $\forall \varphi \in \mathcal{L}_*$, $Cred^0_\varphi = \Omega$*
3. *$\circ$ is satisfies (P6) iff $\forall \varphi \in \mathcal{L}_*$, $k \leq 1$*

Proof. Let $\circ$ be a CB revision operator and $\Psi_\circ : \varphi \mapsto \mathcal{S}_\varphi = \{\mathcal{C}^0_\varphi, \dots, \mathcal{C}^k_\varphi\}$ be its corresponding CB assignment.

Point 1: (If part) Assume that for each $\varphi \in \mathcal{L}_*$, $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi)$. We need to prove that $\circ$ is a CL revision operator. By Prop. 3, it is enough to show that $\circ$ satisfies (P5). So let $\varphi \in \mathcal{L}_*$, $\alpha, \beta \in \mathcal{L}$, and assume that $\varphi \circ \alpha \models \alpha$ and $\alpha \models \beta$. We need to show that $\varphi \circ \beta \models \beta$. Since $\varphi \circ \alpha \models \alpha$ and $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi)$, by Lemma 4.2, we know that $\varphi \circ \alpha \models \alpha$ is $\mathcal{C}^0_\varphi$-credible. Then $[\alpha] \cap Cred^0_\varphi \neq \emptyset$, so since $\alpha \models \beta$, $[\beta] \cap Cred^0_\varphi \neq \emptyset$, thus β is $Cred^0_\varphi$-credible. By definition of $\varphi \circ \beta$ through $\Psi_\circ$ (cf. Def. 9), we get that $[\varphi \circ \beta] = \min([\beta], \preceq^0_\varphi)$, so $\varphi \circ \beta \models \beta$. This shows that $\circ$ satisfies (P5), hence $\circ$ is a CL revision operator.

(Only if part) Assume that $\circ$ is a CL revision operator, and let us show that for each $\varphi \in \mathcal{L}_*$, $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi)$. Toward a contradiction, assume that $\mathcal{S}_\varphi = \{\mathcal{C}^0_\varphi, \dots, \mathcal{C}^k_\varphi\}$ with $k \geq 1$. By Lemma 3.2, $\emptyset \neq Cred^1_\varphi \subseteq Cont^1_\varphi$. And by Lemma 7, $Cont^1_\varphi \subsetneq \Omega \setminus Cred^0_\varphi$. Then let $\omega \in Cred^1_\varphi$ and $\omega' \in \Omega \setminus Cont^1_\varphi$. We get that $\varphi \circ \phi_\omega \models \phi_\omega$ by definition of $Cred^1_\varphi$, and $\varphi \circ \phi_{\omega,\omega'} \not\models \phi_{\omega,\omega'}$ by definition of $Cont^1_\varphi$, which directly contradicts (P5) and that $\circ$ is a CL revision operator. This shows that for each $\varphi \in \mathcal{L}_*$, $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi)$.

Point 2: (If part) Assume that for each $\varphi \in \mathcal{L}_*$, $Cred^0_\varphi = \Omega$. We need to prove that $\circ$ is a CL revision operator. First, we can see that since $Cred^0_\varphi = \Omega$, $\Omega \setminus Cont^0_\varphi = \emptyset$, so $\mathcal{E} = \emptyset$ and so $\{Cont^i_\varphi \mid 1 \leq i \leq k\} = \emptyset$ by definition, hence $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi)$. By point 1 of this lemma, we already know that $\circ$ is a CL revision. Then by Th. 3, we only need to show that $\circ$ satisfies (R1) (restricted to consistent inputs). Yet since $Cred^0_\varphi = \Omega$, every consistent formula α is such that $[\alpha] \cap Cred^0_\varphi \neq \emptyset$, so by definition of $\varphi \circ \beta$ through $\Psi_\circ$ (cf. Def. 9), we get that $[\varphi \circ \alpha] = \min([\alpha], \preceq^0_\varphi)$, so $\varphi \circ \alpha \models \alpha$. This shows that $\circ$ satisfies (R1) (for consistent inputs α), hence $\circ$ is a KM revision operator.

(Only if part) Assume that $\circ$ is a KM revision operator, and let us show that for each $\varphi \in \mathcal{L}_*$, $Cred^0_\varphi = \Omega$. It is enough to show that $\Omega \subseteq Cred^0_\varphi$, so let $\omega \in \Omega$, by (R1) we know that for all worlds $\omega' \in \Omega$, $\varphi \circ \phi_{\omega,\omega'} \models \phi_{\omega,\omega'}$; thus $\omega \in Cred^0_\varphi$. This shows that $Cred^0_\varphi = \Omega$ and concludes the proof.

Point 3: (If part) Assume that $\Psi_\circ : \varphi \mapsto \mathcal{S}_\varphi = \{\mathcal{C}^0_\varphi, \dots, \mathcal{C}^k_\varphi\}$ is such that for each $\varphi \in \mathcal{L}_*$, $k \leq 1$. This means that for each $\varphi \in \mathcal{L}_*$, $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi)$ or $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi, \mathcal{C}^1_\varphi)$. We need to prove that $\circ$ satisfies (P6). So, let $\varphi \in \mathcal{L}_*$, $\alpha, \beta \in \mathcal{L}$, and let us show that:

$$\varphi \circ (\alpha \vee \beta) \equiv \begin{cases} \varphi \circ \alpha & \text{or} \\ \varphi \circ \beta & \text{or} \\ (\varphi \circ \alpha) \vee (\varphi \circ \beta) \end{cases}$$

Since $\circ$ is a CB operator, we already know that $\circ$ satisfies (RT), so if $\varphi \circ (\alpha \vee \beta) \models \alpha \vee \beta$, the proof is direct from (RT). Then, assume that (i) $\varphi \circ (\alpha \vee \beta) \not\models \alpha \vee \beta$. Assume toward a contradiction that $\varphi \circ \alpha \models \alpha$ and $\varphi \circ \beta \models \beta$. Since $\varphi \circ (\alpha \vee \beta) \not\models \alpha \vee \beta$, the fact that $\varphi \circ \alpha \models \alpha$ and $\varphi \circ \beta \models \beta$ means that both α and β are φ-contextually credible. By Lemma 2, we get that both α and β are not $\mathcal{C}^0_\varphi$-credible, that α is $\mathcal{C}^{i_\alpha}_\varphi$-credible, and that β is $\mathcal{C}^{i_\beta}_\varphi$-credible. This means that $k \geq 1$, i.e., $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi, \mathcal{C}^1_\varphi)$, and then we get that both α and β are $\mathcal{C}^1_\varphi$-credible. We got that (ii) $[\alpha] \cap Cred^1_\varphi \neq \emptyset$, (iii) $[\beta] \cap Cred^1_\varphi \neq \emptyset$, (iv) $[\alpha] \subseteq Cont^1_\varphi$, (v) $[\beta] \subseteq Cont^1_\varphi$, (vi) $[\alpha] \cap Cred^0_\varphi = \emptyset$, and (vii) $[\beta] \cap Cred^0_\varphi = \emptyset$. From (ii) and (iii), we get that (viii) $[\alpha \vee \beta] \cap Cred^1_\varphi \neq \emptyset$. From (iv) and (v) we get that (ix) $[\alpha \vee \beta] \subseteq Cont^1_\varphi$. And from (vi) and (vii), we get that (x) $[\alpha \vee \beta] \cap Cred^0_\varphi = \emptyset$. Then from (viii) and (ix), $\alpha \vee \beta$ is $\mathcal{C}^1_\varphi$-credible, and from (x), $\alpha \vee \beta$ is not $\mathcal{C}^0_\varphi$-credible. By definition of $\varphi \circ (\alpha \vee \beta)$ through $\Psi_\circ$ (cf. Def. 9), we get that $[\varphi \circ (\alpha \vee \beta)] = \min([\alpha \vee \beta], \preceq^1_\varphi)$, which contradicts our assumption (i). Thus, we must have that $\varphi \circ \alpha \not\models \alpha$ or $\varphi \circ \beta \not\models \beta$, which by (P1) means that $\varphi \circ \alpha \equiv \varphi$ or $\varphi \circ \alpha \equiv \varphi$. Yet by (P1) again, condition (i) implies that $\varphi \circ (\alpha \vee \beta) \equiv \varphi$. This shows that:

$$\varphi \circ (\alpha \vee \beta) \equiv \begin{cases} \varphi \circ \alpha & \text{or} \\ \varphi \circ \beta \end{cases}$$

Hence, $\circ$ satisfies (P6).

(Only if part) Assume that $\circ$ satisfies (P6), and let us show that $\Psi_\circ : \varphi \mapsto \mathcal{S}_\varphi = \{\mathcal{C}^0_\varphi, \ldots, \mathcal{C}^k_\varphi\}$ is such that for each $\varphi \in \mathcal{L}_*$, $k \leq 1$. So, let $\varphi \in \mathcal{L}_*$, and assume toward a contradiction that $k \geq 2$. This means that there exist $i, j \in \{1, \ldots, k\}$ such that $i \neq j$. By Lemma 3.2, $Cred^i_\varphi \neq \emptyset$ and $Cred^j_\varphi \neq \emptyset$, so let $\omega, \omega' \in \Omega$ such that $\omega \in Cred^i_\varphi$ and $\omega' \in Cred^j_\varphi$. By definition of $Cred^i_\varphi$ and $Cred^j_\varphi$, we know that $\varphi \circ \phi_\omega \models \phi_\omega$ and $\varphi \circ \phi_{\omega'} \models \phi_{\omega'}$. By Lemma 3.4, $Cred^j_\varphi \cap Cont^i_\varphi = \emptyset$, which means that $\{\omega, \omega'\} \not\subseteq Cont^i_\varphi$ (since $\omega' \in Cred^j_\varphi$). And since $\{\omega, \omega'\} \cap Cred^i_\varphi \neq \emptyset$, by Lemma 1, for each $l \in \{1, ldots, k\}$ such that $l \neq i$, we have that $\{\omega, \omega'\} \not\subseteq Cont^l_\varphi$. Then we got that for each $l \in \{1, ldots, k\}$ (including the case when $l = i$), $\{\omega, \omega'\} \not\subseteq Cont^l_\varphi$, thus $\{\omega, \omega'\}$ fails to satisfy the condition required to be an element of the set $\mathcal{E}$, which since $\varphi \circ \phi_\omega \models \phi_\omega$ means that $\varphi \circ \phi_{\omega,\omega'} \not\models \phi_{\omega,\omega'}$. Yet the fact that $\varphi \circ \phi_\omega \models \phi_\omega$ and $\varphi \circ \phi_{\omega'} \models \phi_{\omega'}$ means by (P6) that $\varphi \circ \phi_{\omega,\omega'} \models \phi_{\omega,\omega'}$, which leads to a contradiction. Assuming that $k \geq 2$ led to a contradiction, which concludes the proof that $k \leq 1$. □

Definition 12 (Context-Exclusive revision operator)**.** *A CB revision operator $\circ$ is called a* Context-Exclusive (CE) operator *if it satisfies the following condition:*

(CE) *If $\varphi \circ \alpha \models \alpha$, $\varphi \circ (\alpha \vee \beta) \not\models \alpha \vee \beta$ and $\alpha \wedge \gamma \not\models \bot$, then $\varphi \circ (\alpha \wedge \gamma) \models \alpha$*

Proposition 11. *Every CL revision operator is a CE revision operator.*

Proof. Assume that $\circ$ is a CL operator. We already know that it is a CB operator by Prop. 3. Then by definition of a CE operator, we only need to show that $\circ$ satisfies (CE). Yet since $\circ$ is a CL operator, then it satisfies (P5), so for each $\varphi \in \mathcal{L}_*$, $\alpha, \beta \in \mathcal{L}$, if $\varphi \circ \alpha \models \alpha$ and $\alpha \models \beta$, then $\varphi \circ (\alpha \vee \beta) \models \alpha \vee \beta$. Then the precondition of (CE) is never satisfied, thus $\circ$ satisfies (CE), and this shows that $\circ$ is a CE operator. □

Proposition 12. *Let $\circ$ be a CB revision operator and $\Psi_\circ : \varphi \mapsto \mathcal{S}_\varphi = \{\mathcal{C}^0_\varphi, \ldots, \mathcal{C}^k_\varphi\}$ be its corresponding CB assignment. Then $\circ$ is a CE revision operator iff $\forall \varphi \in \mathcal{L}_*$, $\forall i \in \{1, \ldots, k\}$, $Cont^i_\varphi = Cred^i_\varphi$.*

Proof. Let $\circ$ be a CB revision operator and $\Psi_\circ : \varphi \mapsto \mathcal{S}_\varphi = \{\mathcal{C}^0_\varphi, \ldots, \mathcal{C}^k_\varphi\}$ be its corresponding CB assignment.

(If part) Assume that for each $\varphi \in \mathcal{L}_*$ and each $i \in \{1, \ldots, k\}$, (i) $Cont^i_\varphi = Cred^i_\varphi$, and let us show that $\circ$ satisfies (CE). So let $\alpha, \beta, \gamma \in \mathcal{L}$, and assume that (ii) $\varphi \circ \alpha \models \alpha$, (iii) $\varphi \circ (\alpha \vee \beta) \not\models \alpha \vee \beta$, and (iv) $\alpha \wedge \gamma \not\models \bot$. We need to show that $\varphi \circ (\alpha \wedge \gamma) \models \alpha$. If $\alpha \wedge \gamma$ is $\mathcal{C}^0_\varphi$-credible, then by definition of $\circ$ through $\Psi_\circ$ (cf. Def. 9), we get that $[\varphi \circ (\alpha \wedge \gamma)] = \min([\alpha \wedge \gamma], \preceq^0_\varphi)$, so we directly get that $\varphi \circ (\alpha \wedge \gamma) \models \alpha$. Then, assume that $\alpha \wedge \gamma$ is not $\mathcal{C}^0_\varphi$-credible, and let us show that $\alpha \wedge \gamma$ is $\mathcal{C}^i_\varphi$-credible for some $i \in \{1, \ldots, k\}$. Since $\alpha \wedge \gamma$ is not $\mathcal{C}^0_\varphi$-credible, we know that (v) $[\alpha \wedge \gamma] \cap Cred^0_\varphi = \emptyset$. By conditions (ii) and (iii), α is φ-contextually credible, so by Lemma 2, α is not $\mathcal{C}^0_\varphi$-credible and α is $\mathcal{C}^{i_\alpha}_\varphi$-credible. Then $[\alpha] \cap Cred^{i_\alpha}_\varphi \neq \emptyset$ and $[\alpha] \subseteq Cont^{i_\alpha}_\varphi$. Yet by condition (i), $Cont^{i_\alpha}_\varphi = Cred^{i_\alpha}_\varphi$, so $[\alpha] \subseteq Cred^{i_\alpha}_\varphi$, thus $[\alpha \wedge \gamma] \subseteq Cred^{i_\alpha}_\varphi$, which by condition (iv) means that $[\alpha \wedge \gamma] \cap Cred^{i_\alpha}_\varphi \neq \emptyset$. This means that $\alpha \wedge \gamma$ is $\mathcal{C}^{i_{\alpha\wedge\gamma}}_\varphi$-credible, with $i_{\alpha\wedge\gamma} = i_\alpha$. By definition of $\circ$ through $\Psi_\circ$ (cf. Def. 9), we get that $[\varphi \circ (\alpha \wedge \gamma)] = \min([\alpha \wedge \gamma], \preceq^{\alpha\wedge\gamma}_\varphi)$, so we get that $\varphi \circ (\alpha \wedge \gamma) \models \alpha$. This shows that $\circ$ satisfies (CE).

(Only if part) Assume that $\circ$ is a CE revision operator, and let us show that for each $\varphi \in \mathcal{L}_*$ and each $i \in \{1, \ldots, k\}$, $Cont^i_\varphi = Cred^i_\varphi$. Let $\varphi \in \mathcal{L}_*$ and assume toward a contradiction that there exists $i \in \{1, \ldots, k\}$ such that $Cont^i_\varphi \neq Cred^i_\varphi$. By Lemma 3.2, we already know that $Cred^i_\varphi \subseteq Cont^i_\varphi$, so this means that (i) $Cred^i_\varphi \subsetneq Cont^i_\varphi$. By Lemma 7, we know that (ii) $Cont^i_\varphi \subsetneq \Omega \setminus Cred^0_\varphi$. And by Lemma 3.2, (iii) $Cred^i_\varphi \neq \emptyset$. Thus, conditions (i-iii) together mean that there are three worlds $\omega, \omega', \omega'' \in \Omega$ such that (iv) $\omega \in Cred^i_\varphi$, (v) $\omega' \in Cont^i_\varphi \setminus Cred^i_\varphi$, and (vi) $\omega'' \in \Omega \setminus (Cont^i_\varphi \cup Cred^0_\varphi)$. In particular, note from (iv-vi) and Lemma 3.2 and 3.4 that $\{\omega, \omega', \omega''\} \cap Cred^0_\varphi = \emptyset$, so that (vii) $\phi_{\omega,\omega',\omega''}$ is not $\mathcal{C}^0_\varphi$-credible. On the one hand, (iv) and (vi) together imply that $\{\omega, \omega', \omega''\} \not\subseteq Cont^i_\varphi$, so (viii) $\phi_{\omega,\omega',\omega''}$ is not $\mathcal{C}^i_\varphi$-credible. Yet (iv) implies that $\{\omega, \omega', \omega''\} \cap Cred^i_\varphi \neq \emptyset$, so if $\phi_{\omega,\omega',\omega''}$ was $\mathcal{C}^{i_{\phi_{\omega,\omega',\omega''}}}_\varphi$-credible (that would hold for some $i_{\phi_{\omega,\omega',\omega''}} \in \{1, \ldots, k\}$ with $i_{\phi_{\omega,\omega',\omega''}} \neq i$ by (vii) and (viii)), this would mean from Lemma 1 that $\{\omega, \omega', \omega''\} \cap Cred^i_\varphi = \emptyset$, contradicting (iv), thus (ix) $\phi_{\omega,\omega',\omega''}$ is not $\mathcal{C}^{i_{\phi_{\omega,\omega',\omega''}}}_\varphi$-credible. By Lemma 4.2, (vii) and (ix), we get that (x) $\varphi \circ \phi_{\omega,\omega',\omega''} \not\models \phi_{\omega,\omega',\omega''}$. On the other hand, by Lemma 4.3, (i) and (ii), we get that (xi) $\varphi \circ \phi_{\omega,\omega'} \models \phi_{\omega,\omega'}$. And by (v) and by definition of $Cred^i_\varphi$, we get that (xii) $\varphi \circ \phi_{\omega'} \not\models \phi_{\omega'}$. Conditions (x-xii) together contradict (CE), which concludes the proof that for each $\varphi \in \mathcal{L}_*$ and each $i \in \{1, \ldots, k\}$, $Cont^i_\varphi = Cred^i_\varphi$. □

Proposition 13. *For each $\mu, \sigma \in \mathcal{L}$, the operator $\bullet_{\mu,\sigma}$ is a CE operator iff σ is valid or $\mu \wedge \sigma \models \bot$ or $\sigma \models \mu$.*

Proof. (If part) If σ is valid or $\mu \wedge \sigma \models \bot$, then by Prop. 5.1 $\bullet_{\mu,\sigma}$ is a CL operator, and then by Prop. 11 $\bullet_{\mu,\sigma}$ is a CE operator. Then, assume that $\sigma \models \mu$. To conclude the proof that $\bullet_{\mu,\sigma}$ is a CE operator, by Prop. 12 it is enough to show that the assignment $\Psi_{\bullet_{\mu,\sigma}} : \varphi \mapsto \mathcal{S}_\varphi = \{\mathcal{C}^0_\varphi, \ldots, \mathcal{C}^k_\varphi\}$ corresponding to $\bullet_{\mu,\sigma}$ (cf. Def. 11 and Prop. 7) is such that for each $\varphi \in \mathcal{L}_*$ and each $i \in \{1, \ldots, k\}$, $Cont^i_\varphi = Cred^i_\varphi$. Yet since $\sigma \models \mu$, it is easy to verify by definition of this assignment $\Psi_{\bullet_{\mu,\sigma}}$ that for each $\varphi \in \mathcal{L}_*$, either $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi)$, or $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi, \mathcal{C}^1_\varphi)$ with $Cont^1_\varphi = Cred^1_\varphi = [\neg\varphi \wedge \sigma]$. This shows that $Cont^i_\varphi = Cred^i_\varphi$ for each $i \in \{1, \ldots, k\}$ in both cases, thus $\bullet_{\mu,\sigma}$ is a CE operator.

(Only if part) We show the contrapositive statement, i.e., let $\mu, \sigma \in \mathcal{L}$ and assume that (i) $\neg\sigma \not\models \bot$, (ii) $\mu \wedge \sigma \not\models \bot$, and (iii) $\sigma \not\models \mu$, and let us show that $\bullet_{\mu,\sigma}$ is not a CE operator.

Conditions (i-iii) mean that there exist three worlds $\omega, \omega', \omega''$ such that $\omega \in [\neg\sigma]$ (by (i)), $\omega' \in [\mu \wedge \sigma]$ (by (ii)), and $\omega'' \in [\neg\mu \wedge \sigma]$ (by (iii)). Let ω^3 be another world from Ω distinct from those three worlds, i.e., (iv) $\omega^3 \notin \{\omega, \omega', \omega''\}$. Then by definition of $\bullet_{\mu,\sigma}$, we get that (v) $\phi_{\omega^3} \bullet_{\mu,\sigma} \phi_{\omega'} \not\models \phi_{\omega'}$ (since by (iv), $\phi_{\omega^3} \wedge \phi_{\omega'} \models \bot$ and by (iii), $\mu \wedge \phi_{\omega'} \models \bot$); (vi) $\phi_{\omega^3} \bullet_{\mu,\sigma} \phi_{\omega',\omega''} \models \phi_{\omega',\omega''}$ (since by (iv), $\phi_{\omega^3} \wedge \phi_{\omega',\omega''} \models \bot$, by (ii), $\mu \wedge \phi_{\omega',\omega''} \models \bot$ and by (ii) and (iii), $\phi_{\omega',\omega''} \models \sigma$); and (vii) $\phi_{\omega^3} \bullet_{\mu,\sigma} \phi_{\omega,\omega',\omega''} \not\models \phi_{\omega,\omega',\omega''}$ (since by (iv), $\phi_{\omega^3} \wedge \phi_{\omega,\omega',\omega''} \models \bot$ and by (i), $\phi_{\omega,\omega',\omega''} \not\models \sigma$). Then conditions (v-vii) together contradict (CE), showing that $\bullet_{\mu,\sigma}$ is not a CE operator, which concludes the proof. □

Definition 13 (Context-Sensitive revision operator)**.** *A CB operator $\circ$ is called a* Context-Sensitive (CS) operator *if it satisfies the following condition:*

(CS) *If $\varphi \circ \alpha \models \alpha$ and $\forall \gamma \models \beta$ $\varphi \circ \gamma \not\models \gamma$, then $\varphi \circ (\alpha \vee \beta) \models \alpha \vee \beta$*

Proposition 14. *Every CL revision operator is a CS revision operator.*

Proof. Assume that $\circ$ is a CL operator. We already know that it is a CB operator by Prop. 3. Then by definition of a CS operator, we only need to show that $\circ$ satisfies (CS). Yet since $\circ$ is a CL operator, then it satisfies (P5), and it can be seen that (CS) is a direct consequence of (P5). This shows that $\circ$ is a CS operator. □

Lemma 5. *Let $\circ$ be a CS revision operator and $\Psi_\circ : \varphi \mapsto \mathcal{S}_\varphi = \{\mathcal{C}^0_\varphi, \ldots, \mathcal{C}^k_\varphi\}$ be its corresponding CB assignment. Then for each $\varphi \in \mathcal{L}_*$, $\{\mathcal{C}^0_\varphi, \ldots, \mathcal{C}^k_\varphi\}$ is such that $k \neq 1$.*

Proof. Let $\circ$ be a CS revision operator and $\Psi_\circ : \varphi \mapsto \mathcal{S}_\varphi = \{\mathcal{C}^0_\varphi, \ldots, \mathcal{C}^k_\varphi\}$ be its corresponding CB assignment, and assume toward a contradiction that there exists $\varphi \in \mathcal{L}_*$ such that $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi, \mathcal{C}^1_\varphi)$ (that is, $k = 1$). By Lemma 3.2, $Cred^1_\varphi \neq \emptyset$, so let $\omega \in Cred^1_\varphi$. By Lemma 7, we also know that $Cont^1_\varphi \subsetneq \Omega \setminus Cred^0_\varphi$, so let $\omega' \in \Omega \setminus (Cred^0_\varphi \cup Cont^1_\varphi)$. Remark that $\{\omega'\} \cap Cred^0_\varphi$ means that $\phi_{\omega'}$ is not $\mathcal{C}^0_\varphi$-credible, and $\{\omega'\} \not\subseteq Cont^1_\varphi$ means that $\phi_{\omega'}$ is not $\mathcal{C}^1_\varphi$-credible; so by Lemma 4.2, we get that (i) $\varphi \circ \phi_{\omega'} \not\models \phi_{\omega'}$. Yet by definition of $Cred^1_\varphi$, we have that (ii) $\varphi \circ \phi_\omega \models \phi_\omega$, and by definition of $Cont^1_\varphi$, we also know that (iii) $\varphi \circ \phi_{\omega,\omega'} \not\models \phi_{\omega,\omega'}$. Then conditions (i-iii) together contradict (CS), showing that $\bullet_{\mu,\sigma}$ is not a CS operator, which concludes the proof. □

Proposition 15. *A belief change operator is CL operator iff it a CS operator satisfying (P6).*

Proof. The (only if) part of the proof is direct from the fact that CL operators satisfy (CS) (cf. Prop. 14) and since CL operators satisfy (P6) by definition. As to the (if) part, assume that $\circ$ is a CS operator satisfying (P6). By Lemma 5 and Prop. 10.3, we get that the assignment $\Psi_\circ$ corresponding to $\circ$ is such that for each $\varphi \in \mathcal{L}_*$, $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi)$. By Prop. 10.1, this means that $\circ$ is a CL operator. □

Proposition 16. *Let $\circ$ be a CB operator and $\Psi_\circ : \varphi \mapsto \mathcal{S}_\varphi = \{\mathcal{C}^0_\varphi, \ldots, \mathcal{C}^k_\varphi\}$ be its corresponding CB assignment. Then $\circ$ is a CS revision operator iff $\forall \varphi \in \mathcal{L}_*$, $\forall i \in \{1, \ldots, k\}$, $Cont^i_\varphi = \Omega \setminus \bigcup\{Cred^j_\varphi \mid j \in \{0, \ldots, k\}, j \neq i\}$*

Proof. Let $\circ$ be a CB operator and $\Psi_\circ : \varphi \mapsto \mathcal{S}_\varphi = \{\mathcal{C}^0_\varphi, \ldots, \mathcal{C}^k_\varphi\}$ be its corresponding CB assignment.

(If part) Assume that for each $\varphi \in \mathcal{L}_*$ and each $i \in \{1, \ldots, k\}$, (i) $Cont^i_\varphi = \Omega \setminus \bigcup\{Cred^j_\varphi \mid j \in \{0, \ldots, k\}, j \neq i\}$, and let us show that $\circ$ is a CS operator. Assume toward a contradiction that $\circ$ does not satisfy (CS). This implies in particular that there exist $\varphi \in \mathcal{L}_*$, $\omega \in \Omega$ and $\alpha \in \mathcal{L}$ such that (ii) $\varphi \circ \alpha \models \alpha$, (iii), $\varphi \circ \phi_\omega \not\models \phi_\omega$ and (iv) $\varphi \circ (\alpha \vee \phi_\omega) \not\models \alpha \vee \phi_\omega$. By Lemma 2 and conditions (ii) and (iv), we know that α is $\mathcal{C}^{i_\alpha}_\varphi$-credible (with $i_\alpha \in \{1, \ldots, k\}$), that is, (v) $[\alpha] \cap Cred^{i_\alpha}_\varphi \neq \emptyset$ and (vi) $[\alpha] \subseteq Cont^{i_\alpha}_\varphi$. Yet by condition (v), we also have that (vii) $[\alpha \vee \phi_\omega] \cap Cred^{i_\alpha}_\varphi \neq \emptyset$. Then by Lemma 3.3 and condition (iv) and (vii), we get that (viii) $[\alpha \vee \phi_\omega] \not\subseteq Cont^{i_\alpha}_\varphi$. From (vi) and (viii), we get that $\omega \notin Cont^{i_\alpha}_\varphi$, which together with the initial assumption (i) means that $\omega \in Cred^i_\varphi$ for some $i \in \{0, \ldots, k\}$ with $i \neq j$. But by definition of $Cred^i_\varphi$, we get that $\varphi \circ \phi_\omega \models \phi_\omega$, which contradicts condition (iii). This shows that $\circ$ satisfies (CS), so $\circ$ is a CS operator.

(Only if part) Assume that $\circ$ is a CS revision operator, and let us show that for each $\varphi \in \mathcal{L}_*$ and each $i \in \{1, \ldots, k\}$, $Cont^i_\varphi = \Omega \setminus \bigcup\{Cred^j_\varphi \mid j \in \{0, \ldots, k\}, j \neq i\}$. Then, let $\varphi \in \mathcal{L}_*$ and $i \in \{1, \ldots, k\}$. By Lemma 3.4, we already got that $Cont^i_\varphi \subseteq \Omega \setminus \bigcup\{Cred^j_\varphi \mid j \in \{0, \ldots, k\}, j \neq i\}$. So we only need to show the other inclusion $\Omega \setminus \bigcup\{Cred^j_\varphi \mid j \in \{0, \ldots, k\}, j \neq i\} \subseteq Cont^i_\varphi$, i.e., let $\omega \in \Omega \setminus \bigcup\{Cred^j_\varphi \mid j \in \{0, \ldots, k\}, j \neq i\}$, and let us show that $\omega \in Cont^i_\varphi$. Assume toward a contradiction that (i) $\omega \notin Cont^i_\varphi$. Now, by Lemma 3.2, $\emptyset \neq Cred^i_\varphi \subseteq Cont^i_\varphi$, so there exists a world $\omega' \in \Omega$ such that (ii) $\omega' \in Cred^i_\varphi$, which in particular also means that (iii) $\omega' \in Cont^i_\varphi$. By conditions (i) and (iii) and by definition of $Cont^i_\varphi$, we get that (iv) $\varphi \circ \phi_{\omega,\omega'} \not\models \phi_{\omega,\omega'}$. and by condition (ii) and by definition of $Cred^i_\varphi$, we also know that (v) $\varphi \circ \phi_\omega \models \phi_\omega$. Yet by condition (i), we know that (vi) $\omega \notin Cred^i_\varphi$ since Lemma 3.2 tells us that $Cred^i_\varphi \subseteq Cont^i_\varphi$. Then conditions (iv-vi) together contradict (CS), which concludes the proof that for each $\varphi \in \mathcal{L}_*$ and each $i \in \{1, \ldots, k\}$, $Cont^i_\varphi = \Omega \setminus \bigcup\{Cred^j_\varphi \mid j \in \{0, \ldots, k\}, j \neq i\}$. □

Proposition 17. *For each $\mu, \sigma \in \mathcal{L}$, the operator $\bullet_{\mu,\sigma}$ is a CS operator iff it is a CL operator.*

Proof. The (if) part of the proof is a direct consequence of Prop. 14, so let us prove the (only if) part. Let $\mu, \sigma \in \mathcal{L}$, and assume that the operator $\bullet_{\mu,\sigma}$ is a CS operator. Remark that by Lemma 5, the assignment $\Psi_{\bullet_{\mu,\sigma}} = \varphi \mapsto \mathcal{S}_\varphi = \{\mathcal{C}^0_\varphi, \ldots, \mathcal{C}^k_\varphi\}$ corresponding to $\bullet_{\mu,\sigma}$ is such that for each $\varphi \in \mathcal{L}_*$, $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi)$, i.e., since the case when $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi, \mathcal{C}^1_\varphi)$

would mean that $k = 1$, contradicting Lemma 5. Then the fact that $\bullet_{\mu,\sigma}$ is a CL operator is a direct consequence of Prop. 10.1. This concludes the proof. □

Proposition 18. *For each $\mu, \sigma \in \mathcal{L}$, the operator $\bullet^{CS}_{\mu,\sigma}$ is a CS operator.*

Proof. Let $\mu, \sigma \in \mathcal{L}$. The proof that $\bullet^{CS}_{\mu,\sigma}$ is a CS operator goes as follows. We will first prove that $\bullet^{CS}_{\mu,\sigma}$ is a CB operator. This can be done by showing that $\bullet^{CS}_{\mu,\sigma}$ satisfies all required CB postulates, but it is easier to show that its corresponding assignment $\Psi_{\bullet^{CS}_{\mu,\sigma}}$ (cf. Lemma. 8) is a CB assignment, and that $\bullet^{CS}_{\mu,\sigma}$ is represented by $\Psi_{\bullet^{CS}_{\mu,\sigma}}$ according to Def. 9; according to Prop. 6, this will show that $\bullet^{CS}_{\mu,\sigma}$ is a CB operator. Then, we will show that $\Psi_{\bullet^{CS}_{\mu,\sigma}}$ satisfies the additional condition that for each $\varphi \in \mathcal{L}_*$ and each $i \in \{1, \ldots, k\}$, $Cont^i_\varphi = \Omega \setminus \bigcup\{Cred^j_\varphi \mid j \in \{0, \ldots, k\}, j \neq i\}$, which by Prop. 16 will show that $\bullet^{CS}_{\mu,\sigma}$ is a CS operator.

The fact that $\Psi_{\bullet^{CS}_{\mu,\sigma}}$ is a CB assignment is not difficult to verify. By definition of $\Psi_{\bullet^{CS}_{\mu,\sigma}}$ (cf. Lemma. 8), we have for each $\varphi \in \mathcal{L}_*$ that $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi)$ or $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi, \mathcal{C}^1_\varphi, \mathcal{C}^2_\varphi)$ and that in both cases all conditions defined a context structure space are required by $\mathcal{S}_\varphi$: indeed, in both cases $Cred^0_\varphi \neq \emptyset$ and $\preceq^0_\varphi$ is a total preorder on $Cred^0_\varphi$; and in the second case $\emptyset \neq Cred^i_\varphi \subseteq Cont^i_\varphi$ and $\preceq^i_\varphi$ is a total preorder for each $i \in \{1, 2\}$, and $Cred^1_\varphi \cap Cont^2_\varphi = Cred^2_\varphi \cap Cont^1_\varphi$, all being sufficient conditions for $\mathcal{S}_\varphi$ to be a context structure space. The remaining conditions 1 and 2 of a CB assignment, i.e., that $[\varphi] = \min(Cred^0_\varphi, \preceq^0_\varphi)$ and that if $\varphi \equiv \psi$, then $\mathcal{S}_\varphi = \mathcal{S}_\psi$, are also direct to verify by definition of $\Psi_{\bullet^{CS}_{\mu,\sigma}}$.

Let us now show that $\bullet^{CS}_{\mu,\sigma}$ is represented by $\Psi_{\bullet^{CS}_{\mu,\sigma}}$ according to Def. 9. That is, we need to show that for each $\varphi \in \mathcal{L}_*$, $\alpha \in \mathcal{L}$,

$$[\varphi \circ \alpha] = \begin{cases} \min([\alpha], \preceq^0_\varphi) & \text{if } \alpha \text{ is } \mathcal{C}^0_\varphi\text{- credible} \\ \min([\alpha], \preceq^{i_\alpha}_\varphi) & \text{if } \alpha \text{ is not } \mathcal{C}^0_\varphi\text{- credible} \\ & \text{and } \alpha \text{ is } \mathcal{C}^{i_\alpha}_\varphi\text{- credible} \\ [\varphi] & \text{in the remaining case} \end{cases}$$

Let $\varphi \in \mathcal{L}_*$, $\alpha \in \mathcal{L}$.

Assume first that α is $\mathcal{C}^0_\varphi$- credible. We must show that $[\varphi \bullet^{CS}_{\mu,\sigma} \alpha] = \min([\alpha], \preceq^0_\varphi)$. The fact that that α is $\mathcal{C}^0_\varphi$-credible means that $[\alpha] \cap Cred^0_\varphi \neq \emptyset$. In the case when $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi, \mathcal{C}^1_\varphi, \mathcal{C}^2_\varphi)$, $Cred^0_\varphi = [\varphi]$, so $\varphi \wedge \alpha \not\models \bot$, and $\varphi \bullet^{CS}_{\mu,\sigma} \alpha = \varphi \wedge \alpha$ by definition; yet we know by condition 1 of a CB assignment that $[\varphi] = \min(Cred^0_\varphi, \preceq^0_\varphi)$, so $[\varphi \wedge \alpha] = \min([\alpha], \preceq^0_\varphi)$, i.e., $[\varphi \bullet^{CS}_{\mu,\sigma} \alpha] = \min([\alpha], \preceq^0_\varphi)$. In the other case when $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi)$, $Cred^0_\varphi = [\varphi \vee \mu]$. If $\varphi \wedge \alpha \not\models \bot$, the proof is identical to the case above when $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi, \mathcal{C}^1_\varphi, \mathcal{C}^2_\varphi)$, so assume that (i) $\varphi \wedge \alpha \models \bot$. Then $[\alpha] \cap [\mu] \neq \emptyset$, i.e., (ii) $\mu \wedge \alpha \not\models \bot$. And since $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi)$, we have that $\sigma \models \varphi$ or $\neg\sigma \models \varphi$ or $\mu \models \varphi$, yet by (i) and (ii), $\mu \not\models \varphi$, and by (i), we get that (iii) $\alpha \models \sigma$ or $\alpha \models \neg\sigma$. By conditions (i-iii) we get that $\varphi \bullet^{CS}_{\mu,\sigma} \alpha = \mu \wedge \alpha$ by definition. Yet by definition of $\preceq^0_\varphi$, $\min([\neg\varphi], \preceq^0_\varphi) = [\neg\varphi \wedge \mu]$, so by (i), $\min([\alpha], \preceq^0_\varphi) = [\alpha \wedge \mu]$, from which we get that $[\varphi \bullet^{CS}_{\mu,\sigma} \alpha] = \min([\alpha], \preceq^0_\varphi)$. This proves that in every case where α is $\mathcal{C}^0_\varphi$- credible, we get that $[\varphi \bullet^{CS}_{\mu,\sigma} \alpha] = \min([\alpha], \preceq^0_\varphi)$, as expected.

Assume now that α is not $\mathcal{C}^0_\varphi$- credible and is $\mathcal{C}^{i_\alpha}_\varphi$-credible (i.e., with $i_\alpha \geq 1$). We must show that $[\varphi \bullet^{CS}_{\mu,\sigma} \alpha] = \min([\alpha], \preceq^{i_\alpha}_\varphi)$. The fact that α is $\mathcal{C}^{i_\alpha}_\varphi$-credible with $i_\alpha \geq 1$ means that we are in the case where $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi, \mathcal{C}^1_\varphi, \mathcal{C}^2_\varphi)$. Assume that α is $\mathcal{C}^1_\varphi$-credible (the case where α is $\mathcal{C}^2_\varphi$-credible can be proved similarly). Then, we have that $[\alpha] \cap Cred^1_\varphi \neq \emptyset$ and $[\alpha] \subseteq Cont^1_\varphi$. Then it is easy to verify that by definition of $Cred^1_\varphi$ and $Cont^1_\varphi$ that $\varphi \wedge \alpha \models \bot$, $\mu \wedge \alpha \not\models \bot$, and $\alpha \models \sigma$. Then $\varphi \bullet^{CS}_{\mu,\sigma} \alpha = \mu \wedge \alpha$ $(\equiv \neg\varphi \wedge \mu \wedge \alpha \wedge \sigma)$ by definition. Yet by definition of $\preceq^1_\varphi$, $\min(Cred^1_\varphi, \preceq^1_\varphi) = Cred^1_\varphi = [\neg\varphi \wedge \mu \wedge \sigma]$ by definition of $Cred^1_\varphi$. So, $\min(Cred^1_\varphi, \preceq^1_\varphi) = \min([\alpha], \preceq^1_\varphi) = [\neg\varphi \wedge \mu \wedge \alpha \wedge \sigma]$, and we got that $[\varphi \bullet^{CS}_{\mu,\sigma} \alpha] = \min([\alpha], \preceq^1_\varphi)$. The proof for the case where α is $\mathcal{C}^2_\varphi$-credible is similar, this shows that when α is not $\mathcal{C}^0_\varphi$- credible and is $\mathcal{C}^{i_\alpha}_\varphi$-credible, $[\varphi \bullet^{CS}_{\mu,\sigma} \alpha] = \min([\alpha], \preceq^{i_\alpha}_\varphi)$, as expected.

The remaining case is when α is not $\mathcal{C}^i_\varphi$- credible for each $i \in \{0, \ldots, k\}$. We must prove that $[\varphi \bullet^{CS}_{\mu,\sigma} \alpha] = [\varphi]$. In the case when $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi)$, the fact that α is not $\mathcal{C}^0_\varphi$- credible means that $[\alpha] \cap Cred^0_\varphi = \emptyset$, thus $\alpha \models \neg\varphi \wedge \neg\mu$, i.e., $\varphi \wedge \alpha \models \bot$ and $\varphi \wedge \mu \models \bot$, so $\varphi \bullet^{CS}_{\mu,\sigma} \alpha = \varphi$ by definition. In the other case, i.e., when $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi, \mathcal{C}^1_\varphi, \mathcal{C}^2_\varphi)$, the fact that α is not $\mathcal{C}^i_\varphi$- credible for each $i \in \{1, 2\}$ implies that $[\alpha] \cap Cred^1_\varphi = \emptyset$ and $[\alpha] \cap Cred^2_\varphi = \emptyset$. It is then easy to verify by definition of $Cred^1_\varphi$ and $Cred^2_\varphi$ that $\varphi \wedge \alpha \models \bot$ and $\mu \wedge \alpha \models \bot$, so $\varphi \bullet^{CS}_{\mu,\sigma} \alpha = \varphi$ by definition. In both cases we got that $\varphi \bullet^{CS}_{\mu,\sigma} \alpha = \varphi$, i.e., $[\varphi \bullet^{CS}_{\mu,\sigma} \alpha] = [\varphi]$, as required.

This concludes the proof that $\bullet^{CS}_{\mu,\sigma}$ is represented by $\Psi_{\bullet^{CS}_{\mu,\sigma}}$ according to Def. 9. Hence, by Prop. 6, $\Psi_{\bullet^{CS}_{\mu,\sigma}}$ is a CB operator.

To show that $\Psi_{\bullet^{CS}_{\mu,\sigma}}$ is a CS operator, what remains to be showed is that $\Psi_{\bullet^{CS}_{\mu,\sigma}}$ satisfies the additional condition that for each $\varphi \in \mathcal{L}_*$ and each $i \in \{1, \ldots, k\}$, $Cont^i_\varphi = \Omega \setminus \bigcup\{Cred^j_\varphi \mid j \in \{0, \ldots, k\}, j \neq i\}$. This is trivially true in the case when $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi)$. In the other case, i.e., when $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi, \mathcal{C}^1_\varphi, \mathcal{C}^2_\varphi)$, it is easy to verify by definition of $Cred^0_\varphi$, $Cred^1_\varphi$, $Cont^1_\varphi$, $Cred^2_\varphi$ and $Cont^2_\varphi$ that $Cont^1_\varphi = \Omega \setminus (Cred^0_\varphi \cup Cred^2_\varphi)$ (indeed, $\neg\varphi \wedge (\neg\mu \vee \sigma) \equiv \neg(\varphi \vee (\neg\varphi \wedge \mu \wedge \neg\sigma))$) and $Cont^2_\varphi = \Omega \setminus (Cred^0_\varphi \cup Cred^1_\varphi)$ (indeed, $\neg\varphi \wedge (\neg\mu \vee \neg\sigma) \equiv \neg(\varphi \vee (\neg\varphi \wedge \mu \wedge \sigma))$). This shows that for each $\varphi \in \mathcal{L}_*$ and each $i \in \{1, \ldots, k\}$, $Cont^i_\varphi = \Omega \setminus \bigcup\{Cred^j_\varphi \mid j \in \{0, \ldots, k\}, j \neq i\}$, and from Prop. 16 this shows that $\Psi_{\bullet^{CS}_{\mu,\sigma}}$ is a CS operator. □

Proposition 19. *For each $\mu, \sigma \in \mathcal{L}$, the operator $\bullet^{CS}_{\mu,\sigma}$ is a CL operator iff $\sigma \models \bot$ or σ is valid or $\mu \models \bot$*

Proof. (If part) let $\mu, \sigma \in \mathcal{L}$. If $\sigma \models \bot$ or σ is valid or $\mu \models \bot$, then by definition of the assignment $\Psi_{\bullet^{CS}_{\mu,\sigma}}$ corresponding to $\bullet^{CS}_{\mu,\sigma}$ (cf. Lemma. 8) we directly get that for each $\varphi \in \mathcal{L}_*$, $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi)$, which from Prop. 10.1 means that $\bullet^{CS}_{\mu,\sigma}$ is a CL operator.

(Only if part) We show the contrapositive statement, i.e., assume that (i) $\sigma \not\models \bot$, (ii) $\neg\sigma \not\models \bot$ and (iii) $\mu \not\models \bot$, and let us show that $\bullet^{CS}_{\mu,\sigma}$ is a not a CL operator. By conditions (i-iii), we know that there exist worlds $\omega, \omega', \omega'' \in \Omega$ such that $\omega \in [\sigma]$, $\omega' \in [\neg\sigma]$, and $\omega'' \in [\mu]$. Without loss of generality we can assume that $\mathcal{L}$ is generated from at least two propositional variables, which means that there exists a world $\omega^3 \in \Omega$ distinct from both ω, ω' and ω'', i.e., $\omega^3 \notin \{\omega, \omega', \omega''\}$. Then let $\varphi \in \mathcal{L}_*$ such that $[\varphi] = \{\omega^3\}$. We get that $\sigma \not\models \varphi$, $\neg\sigma \not\models \varphi$ and $\mu \not\models \varphi$, which by definition of of the assignment $\Psi_{\bullet^{CS}_{\mu,\sigma}}$ corresponding to $\bullet^{CS}_{\mu,\sigma}$ (cf. Lemma. 8) means that $\Psi_{\bullet^{CS}_{\mu,\sigma}}$ is such that $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi, \mathcal{C}^1_\varphi, \mathcal{C}^2_\varphi)$, and we get from Prop. 10.1 that $\bullet^{CS}_{\mu,\sigma}$ is a not a CL operator. $\square$

Corollary 1. *Let $\circ$ be a CB revision operator and $\Psi_\circ : \varphi \mapsto \mathcal{S}_\varphi = \{\mathcal{C}^0_\varphi, \ldots, \mathcal{C}^k_\varphi\}$ be its corresponding CB assignment. Then $\circ$ is a CES operator iff $\forall \varphi \in \mathcal{L}_*$, $\bigcup\{Cred^i_\varphi \mid i \in \{0, \ldots, k\}\} = \Omega$*

Corollary 2. *A belief change operator $\circ$ is a CES operator iff it satisfies:*

(CES) *If α is a complete formula, then $\varphi \circ \alpha \models \alpha$*

Proposition 20. *A belief change operator is KM operator iff it is a CES operator that satisfies (P6).*

Proof. The (only if) part of the proof is direct from the fact that CL operators satisfy (CS) (cf. Prop. 11), (CS) (cf. Prop. 14) and (P6), and since KM operators are CL ones (cf. Th. 3). As to the (if) part, assume that $\circ$ is a CB operator satisfying (CE), (CS), and (P6). From Prop. 15, we know that $\circ$ is a CL operator. Then by Prop. 10.1, the assignment $\Psi_\circ$ corresponding to $\circ$ is such that for each $\varphi \in \mathcal{L}_*$, $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi)$. And by Cor. 1, $Cred^0_\varphi = \Omega$. Hence, from Prop. 10.2, $\circ$ is a KM operator. $\square$

Proposition 21. *For each $\mu, \sigma \in \mathcal{L}$, the CS operator $\bullet^{CS}_{\mu,\sigma}$ is a CE operator and not a CL operator iff $\sigma \not\models \bot$, $\neg\sigma \not\models \bot$, and μ is valid.*

Proof. (If part) let $\mu, \sigma \in \mathcal{L}$, and assume that (i) $\sigma \not\models \bot$, (ii) $\neg\sigma \not\models \bot$, and (iii) μ is valid, and let us prove that $\bullet^{CS}_{\mu,\sigma}$ is a CE operator and not a CL operator. The fact that $\bullet^{CS}_{\mu,\sigma}$ is not a CL operator is a direct consequence of Prop. 18 and conditions (i-iii). So, let us show $\bullet^{CS}_{\mu,\sigma}$ is a CE operator. From Prop. 12, it is enough to show that the assignment $\Psi_{\bullet^{CS}_{\mu,\sigma}}$ corresponding to $\bullet^{CS}_{\mu,\sigma}$ (cf. Lemma. 8) is such that for each $\varphi \in \mathcal{L}_*$ and each $i \in \{1, \ldots, k\}$, $Cont^i_\varphi = Cred^i_\varphi$. So, let $\varphi \in \mathcal{L}_*$. The proof that for each $i \in \{1, \ldots, k\}$, $Cont^i_\varphi = Cred^i_\varphi$ is trivial when $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi)$, so by definition of $\Psi_{\bullet^{CS}_{\mu,\sigma}}$ we fall in the case where $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi, \mathcal{C}^1_\varphi, \mathcal{C}^2_\varphi)$ with $Cont^1_\varphi = [\neg\varphi \wedge (\neg\mu \vee \sigma)]$, $Cred^1_\varphi = [\neg\varphi \wedge \mu \wedge \sigma]$, $Cont^2_\varphi = [\neg\varphi \wedge (\neg\mu \vee \neg\sigma)]$, and $Cred^2_\varphi = [\neg\varphi \wedge \mu \wedge \neg\sigma]$. Yet by condition (iii), i.e., μ is valid, it is easy to verify that $Cont^1_\varphi = Cred^1_\varphi$ and $Cont^2_\varphi = Cred^2_\varphi$. This proves that for each $\varphi \in \mathcal{L}_*$ and each $i \in \{1, \ldots, k\}$, $Cont^i_\varphi = Cred^i_\varphi$, which shows by Prop. 12 that $\bullet^{CS}_{\mu,\sigma}$ is a CE operator and concludes the (if) part of the proof.

(Only if part) We prove the contrapositive statement, that is, assume that (i-1) $\sigma \models \bot$ or (i-2) σ is valid or (i-3) $\neg\mu \not\models \bot$, and assume that (ii) $\bullet^{CS}_{\mu,\sigma}$ is not a CL operator, and let us show that $\bullet^{CS}_{\mu,\sigma}$ is not a CE operator. By Prop. 19, assuming (i-1) or (i-2) holds would contradict (ii), so assume that (i-3) holds, i.e., $\neg\mu \not\models \bot$. Then, on the one hand, let $\omega \in [\neg\mu]$. On the other hand, by Prop. 19 and condition (ii) we also get that (iii) $\mu \not\models \bot$, so let $\omega' \in [\mu]$. It can then be seen that $\phi_{\omega'} \wedge \phi_\omega \models \bot$ and $\mu \wedge \phi_\omega \models \bot$, which by definition of $\bullet^{CS}_{\mu,\sigma}$ implies that $\phi_{\omega'} \bullet^{CS}_{\mu,\sigma} \phi_\omega \equiv \varphi$, and then by Lemma 6 we get that $\phi_{\omega'} \bullet^{CS}_{\mu,\sigma} \phi_\omega \not\models \phi_\omega$. This contradicts (CES), which by Cor. 2 means that $\phi_{\omega'} \bullet^{CS}_{\mu,\sigma}$ is not a CE operator, and this concludes the proof. $\square$

Lemma 6. *Let $\circ$ be a belief change operator. Then for each $\varphi \in \mathcal{L}_*$, $\alpha \in L$, if $\varphi \circ \alpha \models \alpha$ and $\varphi \wedge \alpha \models \bot$, then $\varphi \circ \alpha \not\equiv \varphi$.*

Proof. Let $\circ$ be a belief change operator satisfying (P1) and (P2), let $\varphi \in \mathcal{L}_*$, $\alpha \in L$, assume that $\varphi \circ \alpha \models \alpha$ and $\varphi \circ \alpha \equiv \varphi$, and let us show that $\varphi \wedge \alpha \not\models \bot$. Since $\varphi \circ \alpha \models \alpha$, we get that $\varphi \models \alpha$, thus $\varphi \wedge \alpha \equiv \varphi$. And since $\varphi \not\models \bot$, we get that $\varphi \wedge \alpha \not\models \bot$, which concludes the proof. $\square$

Lemma 7. *Let $\circ$ be a CB revision operator and $\Psi_\circ : \varphi \mapsto \mathcal{S}_\varphi = \{\mathcal{C}^0_\varphi, \ldots, \mathcal{C}^k_\varphi\}$ be its corresponding CB assignment. Then for each $\varphi \in \mathcal{L}_*$ and each $i \in \{1, \ldots, k\}$, $Cont^i_\varphi \subsetneq \Omega \setminus Cred^0_\varphi$.*

Proof. Let $\circ$ be a CB revision operator, $\Psi_\circ : \varphi \mapsto \mathcal{S}_\varphi = \{\mathcal{C}^0_\varphi, \ldots, \mathcal{C}^k_\varphi\}$ be its corresponding CB assignment, $\varphi \in \mathcal{L}_*$ and $i \in \{1, \ldots, k\}$. We must prove that $Cont^i_\varphi \subsetneq \Omega \setminus Cred^0_\varphi$. We already know by definition of $Cont^i_\varphi$ that $Cont^i_\varphi \subseteq \Omega \setminus Cred^0_\varphi$, so it is enough to show that $Cont^i_\varphi \neq \Omega \setminus Cred^0_\varphi$ Assume toward a contradiction that $Cont^i_\varphi = \Omega \setminus Cred^0_\varphi$. We then have that (i) $Cont^i_\varphi \cap Cred^0_\varphi = \emptyset$ and (ii) $Cont^i_\varphi \cup Cred^0_\varphi = \Omega$. By definition of $Cont^i_\varphi$, there exists a world $\omega \in Cont^i_\varphi$ such that for all worlds $\omega' \in Cont^i_\varphi$, $\varphi \circ \phi_{\omega,\omega'} \models \phi_{\omega,\omega'}$. And by definition of $Cred^0_\varphi$, for all worlds $\omega' \in Cont^i_\varphi$, $\varphi \circ \phi_{\omega,\omega'} \models \phi_{\omega,\omega'}$. Then by condition (ii) above, we got that $\omega \in Cont^i_\varphi$ and for all worlds $\omega' \in \Omega$, $\varphi \circ \phi_{\omega,\omega'} \models \phi_{\omega,\omega'}$. This means by definition of $Cred^0_\varphi$ that $\omega \in Cred^0_\varphi$. We got that $\omega \in Cont^i_\varphi \cap Cred^0_\varphi$, which contradicts condition (i) above and concludes the proof. $\square$

Lemma 8. *The assignment $\Psi_{\bullet^{CS}_{\mu,\sigma}}$ corresponding to $\bullet^{CS}_{\mu,\sigma}$ is defined as $\Psi_{\bullet^{CS}_{\mu,\sigma}} : \varphi \mapsto \mathcal{S}_\varphi = \{\mathcal{C}^0_\varphi, \ldots, \mathcal{C}^k_\varphi\}$, where for each $\varphi \in \mathcal{L}_*$:*

- *if $\sigma \models \varphi$ or $\neg\sigma \models \varphi$ or $\mu \models \varphi$, then $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi)$, with $Cred^0_\varphi = [\varphi \vee \mu]$ and $\forall \omega, \omega' \in Cred^0_\varphi$, $\omega \preceq^0_\varphi \omega'$ iff $\omega \in [\varphi]$ or $\omega' \in [\neg\varphi]$*

- *otherwise,* $\mathcal{S}_\varphi = (\mathcal{C}^0_\varphi, \mathcal{C}^1_\varphi, \mathcal{C}^2_\varphi)$*, with* $Cred^0_\varphi = [\varphi]$, $Cont^1_\varphi = [\neg\varphi \wedge (\neg\mu \vee \sigma)]$, $Cred^1_\varphi = [\neg\varphi \wedge \mu \wedge \sigma]$, $Cont^2_\varphi = [\neg\varphi \wedge (\neg\mu \vee \neg\sigma)]$, $Cred^2_\varphi = [\neg\varphi \wedge \mu \wedge \neg\sigma]$, *and* $\forall i \in \{0, 1, 2\}$, $\preceq^i_\varphi = Cred^i_\varphi \times Cred^i_\varphi$.

Proof. The verification is direct as it goes in the same vein as the one of Prop. 7, by showing that the context structure spaces given in the lemma match the definition of $\Psi_{\bullet^{CS}_{\mu,\sigma}}$. $\square$

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