Credible Models of Belief Update (Remaining Proofs)

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This supplementary material contains the full proof of Proposition 4.

Proposition 4. *If $\diamond$ is a CL update operator, then $\diamond^d$ is a KM update operator, and for all formulae φ, α, we have that $\varphi \diamond \alpha \vdash \varphi \diamond^d \alpha$.*

Proof. Let $\diamond$ be a Credibility-Limited update operator, and let us show that $\diamond^d$ is a KM update operator. By Theorem 1, there exists a credible faithful assignment $\omega_i \mapsto (\leq_{\omega_i}, \mathcal{C}_i)$ such that for all formulae φ, α, $[\![\varphi \diamond \alpha]\!] = \bigcup_{\omega_i \in [\![\varphi]\!]} \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$. For each world ω_i, let $\leq_i^\Omega$ be the binary relation over all worlds from Ω defined as $\leq_i^\Omega = \leq_{\omega_i} \cup E_i$ where $E_i = \bigcup\{\{(\omega,\omega'),(\omega',\omega')\} \mid \omega \in \mathcal{C}_i, \omega' \in \Omega \setminus \mathcal{C}_i\}$. We first intend to show that the assignment $\omega_i \mapsto \leq_i^\Omega$ is a faithful assignment.

Let $\omega_i \in \Omega$. Let us first observe that for all $\omega, \omega' \in \Omega$:

$$\text{if } (\omega \notin \mathcal{C}_i \text{ and } \omega \leq_i^\Omega \omega'), \text{ then } \omega = \omega'. \tag{2}$$

Let us now verify that $\leq_i^\Omega$ is a (partial) order.

(Reflexivity) By definition of $\leq_i^\Omega$, if $\omega \in \mathcal{C}_i$, then $\omega \leq_i^\Omega \omega$ holds since $(\omega,\omega) \in \leq_{\omega_i}$ and $\leq_{\omega_i}$ is reflexive, otherwise $(\omega,\omega) \in E_i$ and thus $\omega \leq_i^\Omega \omega$ holds trivially.

(Antisymmetry) Let $\omega \leq_i^\Omega \omega'$, $\omega \neq \omega'$, and let us show that $\omega' \not\leq_i^\Omega \omega$.

Assume first that $\omega, \omega' \in \mathcal{C}_i$. Then $(\omega,\omega') \in \leq_{\omega_i}$. Since $\leq_{\omega_i}$ is antisymmetric, we know that $(\omega',\omega) \notin \leq_{\omega_i}$. Yet we also have that $(\omega',\omega) \notin E_i$ since $\omega \in \mathcal{C}_i$, thus $\omega' \not\leq_i^\Omega \omega$. So assume now that $\omega \notin \mathcal{C}_i$ or $\omega' \notin \mathcal{C}_i$. The case where $\omega \notin \mathcal{C}_i$ leads to a contradiction: in this case, $\omega \notin \mathcal{C}_i$ and $\omega \leq_i^\Omega \omega'$ leads to $\omega = \omega'$ by Equation 2, which contradicts $\omega \neq \omega'$. In the remaining case, we have that $\omega' \notin \mathcal{C}_i$. Here, assuming that $\omega' \leq_i^\Omega \omega$ also leads to a contradiction: since $\omega' \notin \mathcal{C}_i$ and $\omega' \leq_i^\Omega \omega$, we get that $\omega = \omega'$ by Equation 2, which contradicts $\omega = \omega'$. This concludes the proof of antisymmetry of $\leq_i^\Omega$.

(Transitivity) Let ω, ω', $\omega'' \in \Omega$, and assume that $\omega \leq_i^\Omega \omega'$ and $\omega' \leq_i^\Omega \omega''$. We must show that $\omega \leq_i^\Omega \omega''$. We fall into one of the following cases:
Case 1: $\omega \notin \mathcal{C}_i$. Then by Equation 2, we get that $\omega = \omega'$, and using Equation 2 again, we get that $\omega' = \omega''$. Thus $\omega = \omega''$, and the fact that $\omega \leq_i^\Omega \omega''$ directly follows from the reflexivity of $\leq_i^\Omega$.
Case 2: $\omega \in \mathcal{C}_i$ and $\omega' \notin \mathcal{C}_i$. Since $\omega' \notin \mathcal{C}_i$, by Equation 2 we get that $\omega' = \omega''$, so $\omega'' \notin \mathcal{C}_i$. And since $\omega \in \mathcal{C}_i$ and $\omega'' \notin \mathcal{C}_i$, we directly get that $(\omega,\omega'') \in E_i$, thus $\omega \leq_i^\Omega \omega''$.
Case 3: $\omega, \omega' \in \mathcal{C}_i$ and $\omega'' \notin \mathcal{C}_i$. Since $\omega \in \mathcal{C}_i$ and $\omega'' \notin \mathcal{C}_i$, we directly get that $(\omega,\omega'') \in E_i$, thus $\omega \leq_i^\Omega \omega''$.
Case 4: $\omega, \omega', \omega'' \in \mathcal{C}_i$. Then $\omega \leq_{\omega_i} \omega'$ and $\omega' \leq_{\omega_i} \omega''$, so $\omega \leq_{\omega_i} \omega''$ since $\leq_{\omega_i}$ is transitive. Hence, $\omega \leq_i^\Omega \omega''$.

We have shown in every case that $\omega \leq_i^\Omega \omega''$, thus $\leq_i^\Omega$ is transitive.

At this point, we have shown that for each world ω_i, the relation $\leq_i^\Omega$ is a partial order. To show that the assignment $\omega_i \mapsto \leq_i^\Omega$ is faithful, we only need to show that for all $\omega_i, \omega \in \Omega$, if $\omega \neq \omega_i$ then $\omega_i <_i^\Omega \omega$. Let us first verify that $\omega_i \leq_i^\Omega \omega$. By definition of a credible assignment, $\omega_i \in \mathcal{C}_i$. If $\omega \in \mathcal{C}_i$, then we also know by definition of a credible assignment that $\omega_i \leq_{\omega_i} \omega$, so $\omega_i \leq_i^\Omega \omega$; and if $\omega \notin \mathcal{C}_i$, we directly get that $(\omega,\omega') \in E_i$, so $\omega_i \leq_i^\Omega \omega$. We now need to show that $\omega \not\leq_i^\Omega \omega_i$. Since $\omega_i \in \mathcal{C}_i$, $(\omega,\omega_i) \notin E_i$; and by definition of a credible assignment, $\omega \not\leq_{\omega_i} \omega_i$. Hence, $\omega \not\leq_i^\Omega \omega_i$. This shows that if $\omega \neq \omega_i$ then $\omega_i <_i^\Omega \omega$, and completes the proof that the assignment $\omega_i \mapsto \leq_i^\Omega$ is faithful.

Now, we intend to show that for all formulae φ, α, $[\![\varphi \diamond^d \alpha]\!] = \bigcup_{\omega_i \in [\![\varphi]\!]} \min([\![\alpha]\!], \leq_i^\Omega)$. Let φ, α be two formulae. We consider two cases:
Case 1: assume first that there exists a formula $\psi \vdash \varphi$, $\psi \nvdash \bot$, $\psi \diamond \varphi \vdash \bot$. We need to show that $[\![\varphi \diamond^d \alpha]\!] = [\![\alpha]\!]$. We know from Theorem 1 that $[\![\psi \diamond \alpha]\!] = \bigcup_{\omega_i \in [\![\psi]\!]} \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$. Yet $[\![\psi \diamond \alpha]\!] = \emptyset$, which means that for each world $\omega_i \in [\![\psi]\!]$, we have that $[\![\alpha]\!] \cap \mathcal{C}_i = \emptyset$, or stated equivalently, that for each world $\omega \in [\![\alpha]\!]$, $\omega \notin \mathcal{C}_i$. Let $\omega_i \in [\![\psi]\!]$. Since $\omega_i \mapsto \leq_i^\Omega$ is a faithful assignment, $\omega_i <_i^\Omega \omega$. And from Equation 2, we can see that for all $\omega, \omega' \notin \mathcal{C}_i$, if $\omega \neq \omega'$ then $\omega \not\leq_i^\Omega \omega'$, i.e., all non-credible worlds w.r.t. ω_i are pairwise incomparable. Together with the fact that $[\![\alpha]\!] \cap \mathcal{C}_i = \emptyset$, this means that $\min([\![\alpha]\!], \leq_i^\Omega) = [\![\alpha]\!]$. Hence, $\bigcup_{\omega_i \in [\![\varphi]\!]} \min([\![\alpha]\!], \leq_i^\Omega) = [\![\alpha]\!]$, and so $[\![\varphi \diamond^d \alpha]\!] = \bigcup_{\omega_i \in [\![\varphi]\!]} \min([\![\alpha]\!], \leq_i^\Omega)$.
Case 2: assume that case 1 does not hold. In particular, this means that for each *complete* formula $\psi = \gamma_{\omega_i}$ such that $\gamma_{\omega_i} \vdash \varphi$, we have that $\gamma_{\omega_i} \diamond \varphi \nvdash \bot$. Yet we know from Theorem 1 that $[\![\gamma_{\omega_i} \diamond \alpha]\!] = \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$, so this means that (i) for each world $\omega_i \in \varphi$, $[\![\alpha]\!] \cap \mathcal{C}_i \neq \emptyset$. Now for each world $\omega_i \in \varphi$, by definition of $\leq_i^\Omega$, for each world $\omega \in \mathcal{C}_i$,

we can easily see that:

$$\begin{array}{lll} \text{(ii)} & \omega \leq_i^\Omega \omega' \text{ iff } \omega \leq_{\omega_i} \omega' & \text{if } \omega' \in \mathcal{C}_i, \\ \text{(iii)} & \omega <_i^\Omega \omega', & \text{otherwise.} \end{array}$$

For each $\omega_i \in [\![\varphi]\!]$, we got that $\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) = \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_i^\Omega)$ (from (ii)), and $\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_i^\Omega) = \min([\![\alpha]\!], \leq_i^\Omega)$ (from (i) and (iii)), so $\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) = \min([\![\alpha]\!], \leq_i^\Omega)$. Hence, $[\![\varphi \diamond^d \alpha]\!] = [\![\varphi \diamond \alpha]\!]$.

We have shown that $\omega_i \mapsto \leq_i^\Omega$ is a fatihful assignment and that for all formulae φ, α, $[\![\varphi \diamond^d \alpha]\!] = \bigcup_{\omega_i \in [\![\varphi]\!]} \min([\![\alpha]\!], \leq_i^\Omega)$. From Proposition 1, this means that $\diamond^d$ is a KM update operator.

The fact that $\varphi \diamond \alpha \vdash \varphi \diamond^d \alpha$ for all formulae φ, α, is direct by definition of $\diamond^d$ and since $\diamond^d$ satisfies **(U1)**. This concludes the proof. ■