Iteration of Iterated Belief Revision (Remaining Proofs)

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This supplementary material contains the full proofs of Propositions 3 and 10.

Proposition 3.

1. $\circ_L$ *satisfies* ***(CE1-CE5)***
2. $\circ_N$ *satisfies* ***(CE3-CE5)****, but not* ***(CE1-CE2)***
3. $\circ_R$ *satisfies* ***(CE1)*** *and* ***(CE3-CE5)****, but not* ***(CE2)****.*

The proof uses the following lemmata:

Lemma 3. *Let $\Psi \mapsto \preceq_\Psi$ be a DP assignment, α, μ be two formulae and ω be a world.*

1. $(\omega \models \mu$ *and* $\omega \in \min([\alpha], \preceq_\Psi)) \Rightarrow \omega \in \min([\alpha], \preceq_{\Psi\circ\mu})$
2. $(\omega \not\models \mu$ *and* $\omega \in \min([\alpha], \preceq_{\Psi\circ\mu}) \Rightarrow \omega \in \min([\alpha], \preceq_\Psi)$

Proof of Lemma 3. 1. Since $\omega \in \min([\alpha], \preceq_\Psi))$, we have for each world $\omega' \models \alpha$ that $\omega \preceq_\Psi \omega'$. Let $\omega' \models \alpha$. If $\omega' \models \mu$, by *(CR1-CR2)* we get that $\omega \preceq_{\Psi\circ\mu} \omega'$. And if $\omega' \not\models \mu$, then by *(CR4)* we also get that $\omega \preceq_{\Psi\circ\mu} \omega'$. Hence, $\omega \in \min([\alpha], \preceq_{\Psi\circ\mu})$.

2. Since $\omega \in \min([\alpha], \preceq_{\Psi\circ\mu}))$, we have for each world $\omega' \models \alpha$ that $\omega \preceq_\Psi \omega'$. Let $\omega' \models \alpha$. If $\omega' \not\models \mu$, by *(C1-C2)* we get that $\omega \preceq_\Psi \omega'$. And if $\omega' \models \mu$, then by *(C3)* we also get that $\omega \preceq_\Psi \omega'$. Hence, $\omega \in \min([\alpha], \preceq_\Psi)$. □

Lemma 4. *Let $\omega \mapsto \preceq_\Psi$ be a DP assignment, α, μ be two formulae, ω, ω' be two worlds, and assume that $\omega \models \alpha \Leftrightarrow \omega' \models \alpha$. Then:*

1. $(\omega \models \mu \Leftrightarrow \omega' \models \mu) \Rightarrow (\omega \preceq_{\Psi\circ\alpha} \omega' \Leftrightarrow \omega \preceq_{\Psi\circ\mu\circ\alpha} \omega')$
2. $(\omega \models \mu$ *and* $\omega' \not\models \mu) \Rightarrow (\omega \prec_{\Psi\circ\alpha} \omega' \Rightarrow \omega \prec_{\Psi\circ\mu\circ\alpha} \omega')$
3. $(\omega \models \mu$ *and* $\omega' \not\models \mu) \Rightarrow (\omega \preceq_{\Psi\circ\alpha} \omega' \Rightarrow \omega \preceq_{\Psi\circ\mu\circ\alpha} \omega')$

Proof of Lemma 4. 1. By *(CR1-CR2)*, $\omega \preceq_{\Psi\circ\alpha} \omega'$ iff $\omega \preceq_\Psi \omega'$ iff $\omega \preceq_{\Psi\circ\mu} \omega'$ iff $\omega \preceq_{\Psi\circ\mu\circ\alpha} \omega'$.

2. If $\omega \prec_{\Psi\circ\alpha} \omega'$, then $\omega \prec_\Psi \omega'$ (by *(CR1-CR2)*), thus $\omega \prec_{\Psi\circ\mu} \omega'$ (by *(CR3)*), so $\omega \prec_{\Psi\circ\mu\circ\alpha} \omega'$ (by *(CR1-CR2)*).

3. If $\omega \preceq_{\Psi\circ\alpha} \omega'$, then $\omega \preceq_\Psi \omega'$ (by *(CR1-CR2)*), thus $\omega \preceq_{\Psi\circ\mu} \omega'$ (by *(CR4)*), so $\omega \preceq_{\Psi\circ\mu\circ\alpha} \omega'$ (by *(CR1-CR2)*). □

We now prove Proposition 3.

Proof of Proposition 3. Using Proposition 2 it is enough to prove that $\Psi \mapsto \preceq^L_\Psi$ satisfies *(CRE1-CRE5)*, that $\Psi \mapsto \preceq^N_\Psi$ satisfies *(CRE3)* and *(CRE4)*, and that $\Psi \mapsto \preceq^R_\Psi$ satisfies *(CRE1)*, *(CRE3)* and *(CRE4)*, where each assignment $\Psi \mapsto \preceq^N_\Psi$, $\Psi \mapsto \preceq^L_\Psi$ and $\Psi \mapsto \preceq^R_\Psi$ denotes the DP assignment corresponding to $\circ_N$, $\circ_L$ and $\circ_R$, respectively. Let Ψ be any epistemic state, μ, α be two formulae, and ω, ω' be two worlds.

• Proof that $\Psi \mapsto \preceq^L_\Psi$ satisfies *(CRE1-CRE5)*:
(CRE1): let $\omega, \omega' \models \mu$. The case when $\omega \models \alpha \Leftrightarrow \omega' \models \alpha$ is direct using Lemma 4.1. If $\omega \models \alpha$ and $\omega' \not\models \alpha$, then by *(Lex)* we have that $\omega \prec^L_{\Psi\circ_L\mu\circ_L\alpha}$ and $\omega \prec^L_{\Psi\circ_L\alpha}$, and *(CRE1)* directly follows. The proof for the case when $\omega \not\models \alpha$ and $\omega' \models \alpha$ is identical.
(CRE2): the proof is identical to the case for *(CRE1)*, assuming instead that $\omega, \omega' \not\models \mu$.
(CRE3): the proof is identical to the two previous cases, assuming instead that $\omega \models \mu$ and $\omega' \not\models \mu$, and using Lemma 4.2 instead of Lemma 4.1.
(CRE4): the proof is identical to the case for *(CRE3)*, using Lemma 4.3 instead of Lemma 4.2.
(CRE5): follows directly from *(Lex)* by setting $\alpha_1 = \alpha_2$.

• Proof that $\Psi \mapsto \preceq^N_\Psi$ satisfies *(CRE3-CRE5)*:
(CRE3): let $\omega \models \mu$ and $\omega' \not\models \mu$, and assume that $\omega \prec^N_{\Psi\circ_N\alpha} \omega'$. We need to prove that $\omega \prec^N_{\Psi\circ_N\mu\circ_N\alpha} \omega'$. The case when $\omega \models \alpha \Leftrightarrow \omega' \models \alpha$ is direct using Lemma 4.2. Assume that $\omega \models \alpha$, $\omega' \not\models \alpha$. If $\omega \notin \min([\alpha], \preceq^N_\Psi)$, since $\omega \prec^N_{\Psi\circ_N\alpha} \omega'$, by *(Nat)* and since $\omega, \omega' \notin \min([\alpha], \preceq^N_{\Psi\circ\mu})$ we get that $\omega \prec^N_\Psi \omega'$, by *(CR3)* we get that $\omega \prec^N_{\Psi\circ_N\mu} \omega'$, and by *(CR3)* again we get that $\omega \prec^N_{\Psi\circ_N\mu\circ_N\alpha} \omega'$. If $\omega \in \min([\alpha], \preceq^N_\Psi)$, by Lemma 3.1 we get that $\omega \in \min([\alpha], \preceq^N_{\Psi\circ_N\mu})$, and since $\omega' \not\models \alpha$, we get that $\omega \prec^N_{\Psi\circ_N\mu\circ_N\alpha} \omega'$ by Theorem 1. Assume now that $\omega \not\models \alpha$, $\omega' \models \alpha$. Then $\omega' \notin \min([\alpha], \preceq^N_\Psi)$ by Theorem 1. By Lemma 3.2, we get that $\omega' \notin \min([\alpha], \preceq^N_{\Psi\circ\mu}$). Since $\omega \prec^N_{\Psi\circ_N\alpha} \omega'$, by *(CR4)* we get that $\omega \prec^N_\Psi \omega'$, by *(CR3)* we get that $\omega \prec^N_{\Psi\circ_N\mu} \omega'$, and by *(Nat)* and since $\omega, \omega' \notin \min([\alpha], \preceq^N_{\Psi\circ\mu})$, we get that $\omega \prec^N_{\Psi\circ_N\mu\circ_N\alpha} \omega'$.
(CRE4): let $\omega \models \mu$ and $\omega' \not\models \mu$, and assume that $\omega \preceq^N_{\Psi\circ_N\alpha} \omega'$. We need to prove that $\omega \preceq^N_{\Psi\circ_N\mu\circ_N\alpha} \omega'$. The case when $\omega \models \alpha \Leftrightarrow \omega' \models \alpha$ is direct using Lemma 4.3. Assume that

$\omega \models \alpha$, $\omega' \not\models \alpha$. If $\omega \notin \min([\alpha], \preceq^N_\Psi)$, since $\omega \preceq^N_{\Psi\circ_N\alpha} \omega'$, by *(Nat)* and since $\omega, \omega' \notin \min([\alpha], \preceq^N_{\Psi\circ\mu})$ we get that $\omega \preceq^N_\Psi \omega'$, by *(CR4)* we get that $\omega \preceq^N_{\Psi\circ_N\mu} \omega'$, and by *(CR4)* again we get that $\omega \preceq^N_{\Psi\circ_N\mu\circ_N\alpha} \omega'$. If $\omega \in \min([\alpha], \preceq^N_\Psi)$, by Lemma 3.1 we get that $\omega \in \min([\alpha], \preceq^N_{\Psi\circ_N\mu})$, and since $\omega' \not\models \alpha$, we get that $\omega \preceq^N_{\Psi\circ_N\mu\circ_N\alpha} \omega'$ by Theorem 1. Assume now that $\omega \not\models \alpha$, $\omega' \models \alpha$. Since $\omega \preceq^N_{\Psi\circ_N\alpha} \omega'$, we know that $\omega' \notin \min([\alpha], \preceq^N_\Psi)$ by Theorem 1 and since $\omega \not\models \alpha$. So by Lemma 3.2, we get that $\omega' \notin \min([\alpha], \preceq^N_{\Psi\circ_N\mu})$. Since $\omega \preceq^N_{\Psi\circ_N\alpha} \omega'$, by *(CR3)* we get that $\omega \preceq^N_\Psi \omega'$, by *(CR4)* we get that $\omega \preceq^N_{\Psi\circ_N\mu} \omega'$, and by *(Nat)* and since $\omega, \omega' \notin \min([\alpha], \preceq^N_{\Psi\circ\mu})$, we get that $\omega \preceq^N_{\Psi\circ_N\mu\circ_N\alpha} \omega'$.
(CRE5): follows directly from *(Nat)* by setting $\alpha_1 = \alpha_2$.

• Proof that $\Psi \mapsto \preceq^R_\Psi$ satisfies *(CRE1)* and *(CRE3-CRE5)*:
(CRE1): Let $\omega, \omega' \models \mu$. We need to prove that $\omega \preceq^R_{\Psi\circ_R\alpha} \omega' \Leftrightarrow \omega \preceq^R_{\Psi\circ_R\mu\circ_R\alpha} \omega'$. The case when $\omega \models \alpha \Leftrightarrow \omega' \models \alpha$ is direct using Lemma 4.1. Assume that $\omega \models \alpha$, $\omega' \not\models \alpha$. If $\omega \in \min([\alpha], \preceq^R_\Psi)$, by Lemma 3.1 we get that $\omega \in \min([\alpha], \preceq^R_{\Psi\circ_R\mu})$. Since $\omega' \not\models \alpha$, we get that $\omega \prec^R_{\Psi\circ_R\alpha} \omega'$ and $\omega \prec^R_{\Psi\circ_R\mu\circ_R\alpha} \omega'$. Thus $\omega \preceq^R_{\Psi\circ_R\alpha} \omega' \Leftrightarrow \omega \preceq^R_{\Psi\circ_R\mu\circ_R\alpha} \omega'$. If $\omega \notin \min([\alpha], \preceq^R_\Psi)$, we fall into one of the following three cases: (i) $\omega \simeq^R_{\Psi\circ_R\alpha} \omega'$, (ii) $\omega \prec^R_{\Psi\circ_R\alpha} \omega'$, or (iii) $\omega' \prec^R_{\Psi\circ_R\alpha} \omega$. But case (i) leads to contradiction: indeed, $\omega \simeq^R_{\Psi\circ_R\alpha} \omega'$ implies from *(PR)* that $\omega \prec^R_\Psi \omega'$, and by *(DR)* we get that $\omega \prec^R_{\Psi\circ_R\alpha} \omega'$. In case (ii), since $\omega \prec^R_{\Psi\circ_R\alpha} \omega'$, by *(DR)* we get that $\omega \preceq^R_\Psi \omega'$, by *(C1)* we get that $\omega \preceq^R_{\Psi\circ_R\mu} \omega'$, and by *(PR)* we get that $\omega \prec^R_{\Psi\circ_R\mu\circ_R\alpha} \omega'$. And in case (iii), since $\omega' \prec^R_{\Psi\circ_R\alpha} \omega$, by *(C4)* we get that $\omega' \prec^R_\Psi \omega$, by *(C1)* we get that $\omega' \prec^R_{\Psi\circ_R\mu} \omega$, and by *(DR)* we get that $\omega' \prec^R_{\Psi\circ_R\mu\circ_R\alpha} \omega$. In all three cases (i-iii), we got that $\omega \preceq^R_{\Psi\circ_R\alpha} \omega' \Leftrightarrow \omega \preceq^R_{\Psi\circ_R\mu\circ_R\alpha} \omega'$. The proof when $\omega \not\models \alpha$, $\omega' \models \alpha$ is identical since ω and ω' play symmetrical roles.
(CRE3): let $\omega \models \mu$ and $\omega' \not\models \mu$, and assume that $\omega \prec^R_{\Psi\circ_R\alpha} \omega'$. We need to prove that $\omega \prec^R_{\Psi\circ_R\mu\circ_R\alpha} \omega'$. The case when $\omega \models \alpha \Leftrightarrow \omega' \models \alpha$ is direct using Lemma 4.2. Assume that $\omega \models \alpha$, $\omega' \not\models \alpha$. If $\omega \notin \min([\alpha], \preceq^R_\Psi)$, since $\omega \prec^R_{\Psi\circ_R\alpha} \omega'$, by *(DR)* and since $\omega, \omega' \notin \min([\alpha], \preceq^R_{\Psi\circ\mu})$ we get that $\omega \preceq^R_\Psi \omega'$, by *(PR)* we get that $\omega \prec^R_{\Psi\circ_R\mu} \omega'$, and by *(CR3)* we get that $\omega \prec^R_{\Psi\circ_R\mu\circ_R\alpha} \omega'$. If $\omega \in \min([\alpha], \preceq^R_\Psi)$, by Lemma 3.1 we get that $\omega \in \min([\alpha], \preceq^R_{\Psi\circ_R\mu})$, and since $\omega' \not\models \alpha$, we get that $\omega \prec^R_{\Psi\circ_R\mu\circ_R\alpha} \omega'$ by Theorem 1. Assume now that $\omega \not\models \alpha$, $\omega' \models \alpha$. Then $\omega' \notin \min([\alpha], \preceq^R_\Psi)$ by Theorem 1. By Lemma 3.2, we get that $\omega' \notin \min([\alpha], \preceq^R_{\Psi\circ\mu})$. Since $\omega \prec^R_{\Psi\circ_R\alpha} \omega'$, by *(CR4)* we get that $\omega \prec^R_\Psi \omega'$, by *(CR3)* we get that $\omega \prec^R_{\Psi\circ_R\mu} \omega'$, and by *(DR)* and since $\omega, \omega' \notin \min([\alpha], \preceq^R_{\Psi\circ\mu})$, we get that $\omega \prec^R_{\Psi\circ_R\mu\circ_R\alpha} \omega'$.
(CRE4): let $\omega \models \mu$ and $\omega' \not\models \mu$, and assume that $\omega \preceq^R_{\Psi\circ_R\alpha} \omega'$. We need to prove that $\omega \preceq^R_{\Psi\circ_R\mu\circ_R\alpha} \omega'$. The case when $\omega \models \alpha \Leftrightarrow \omega' \models \alpha$ is direct using Lemma 4.3. Assume that $\omega \models \alpha$, $\omega' \not\models \alpha$. If $\omega \notin \min([\alpha], \preceq^R_\Psi)$, since $\omega \preceq^R_{\Psi\circ_R\alpha} \omega'$, by *(DR)* and since $\omega, \omega' \notin \min([\alpha], \preceq^R_{\Psi\circ\mu})$ we get that $\omega \preceq^R_\Psi \omega'$, by *(CR4)* we get that $\omega \preceq^R_{\Psi\circ_R\mu} \omega'$, and by *(CR4)* again we get that $\omega \preceq^R_{\Psi\circ_R\mu\circ_R\alpha} \omega'$. If $\omega \in \min([\alpha], \preceq^R_\Psi)$, by Lemma 3.1 we get that $\omega \in \min([\alpha], \preceq^R_{\Psi\circ_R\mu})$, and since $\omega' \not\models \alpha$, we get that $\omega \preceq^R_{\Psi\circ_R\mu\circ_R\alpha} \omega'$ by Theorem 1. Assume now that $\omega \not\models \alpha$, $\omega' \models \alpha$. Since $\omega \preceq^R_{\Psi\circ_R\alpha} \omega'$, we know that $\omega' \notin \min([\alpha], \preceq^R_\Psi)$ by Theorem 1 and since $\omega \not\models \alpha$. So by Lemma 3.2, we get that $\omega' \notin \min([\alpha], \preceq^R_{\Psi\circ_R\mu})$. Since $\omega \preceq^R_{\Psi\circ_R\alpha} \omega'$, by *(CR3)* we get that $\omega \preceq^R_\Psi \omega'$, by *(CR4)* we get that $\omega \preceq^R_{\Psi\circ_R\mu} \omega'$, and by *(DR)* and since $\omega, \omega' \notin \min([\alpha], \preceq^R_{\Psi\circ\mu})$, we get that $\omega \prec^R_{\Psi\circ_R\mu\circ_R\alpha} \omega'$, so $\omega \preceq^R_{\Psi\circ_R\mu\circ_R\alpha} \omega'$.
(CRE5): follows directly from *(PR)* and *(DR)* by setting $\alpha_1 = \alpha_2$.

To show that $\circ_N$ and $\circ_R$ do not satisfy **(CE2)** and that $\circ_N$ does not satisfy **(CE1)**, it is enough from Proposition 2 to show that their corresponding assignments do not satisfy the semantic counterparts of **(CE1)** and **(CE2)**. We do so by proving a counter-example in each case.

• Proof that $\Psi \mapsto \preceq^N_\Psi$ and $\Psi \mapsto \preceq^R_\Psi$ do not satisfy *(CRE2)*:
Let $\star \in \{N, R\}$. Let ω_1, ω_2, ω_3 be three worlds, μ, α be two formulae such that $[\mu] = \{\omega_3\}$ and $[\alpha] = \{\omega_2, \omega_3\}$, and Ψ be any TPO where $\omega_1 \prec^\star_\Psi \omega_2 \prec^\star_\Psi \omega_3$. Note that $\omega_1, \omega_2 \not\models \mu$. On the one hand, we get that $\omega_2 \prec^\star_{\Psi\circ_\star\alpha} \omega_1$. On the other hand, we get that $\omega_3 \prec^\star_{\Psi\circ_\star\mu} \omega_1 \prec^\star_{\Psi\circ_\star\mu} \omega_2$, and thus $\omega_1 \prec^\star_{\Psi\circ_\star\mu\circ_\star\alpha} \omega_2$. So both assignments $\Psi \mapsto \preceq^N_\Psi$ and $\Psi \mapsto \preceq^R_\Psi$ do not satisfy *(CRE2)*.

• Proof that $\Psi \mapsto \preceq^N_\Psi$ does not satisfy *(CRE1)*:
Let ω_1, ω_2, ω_3 be three worlds, μ, α be two formulae such that $[\mu] = \{\omega_1, \omega_2\}$ and $[\alpha] = \{\omega_2, \omega_3\}$, and Ψ be any TPO where $\omega_3 \prec^N_\Psi \omega_1 \simeq_\Psi \omega_2$. Note that $\omega_1, \omega_2 \models \mu$. On the one hand, we get that $\omega_1 \simeq^N_{\Psi\circ_N\alpha} \omega_2$. On the other hand, we get that $\omega_1 \simeq^N_{\Psi\circ_N\mu} \omega_2 \prec^N_{\Psi\circ_N\mu} \omega_3$, and thus $\omega_2 \prec^N_{\Psi\circ_N\mu\circ_N\alpha} \omega_1$. So the assignment $\Psi \mapsto \preceq^N_\Psi$ does not satisfy *(CRE1)*.

□

Proposition 6. *For each $i \in \{1, 2\}$, a DP revision operator $\circ$ satisfies (CEiw) if and only if its corresponding DP assignment satisfies (CREiw).*

Proof. The proof is similar to the part of the proof of Proposition 2 showing the correspondence between **(CEi)** and *(CREi)*, for each $i \in \{1, 2\}$. Let $\circ$ be a DP revision operator, $\Psi \mapsto \preceq_\Psi$ be its corresponding DP assignment. Let Ψ be an epistemic state and μ, α be two formulae.
(Only if part) Let ω, ω' be two worlds. Assume that $\circ$ satisfies **(CE1w)**, and assume that ω, $\omega' \models \mu$ and $\omega, \omega' \notin \min([\alpha], \preceq_\Psi) \cup \min([\alpha], \preceq_{\Psi\circ\mu})$. So by Theorem 1 we know that $Bel(\Psi \circ \alpha) \models \neg\gamma_{\{\omega,\omega'\}}$ and $Bel(\Psi \circ (\mu, \alpha)) \models \neg\gamma_{\{\omega,\omega'\}}$. Then $\omega \preceq_{\Psi\circ\mu\circ\alpha} \omega'$ iff $\omega \models Bel(\Psi \circ \mu \circ \alpha \circ \gamma_{\{\omega,\omega'\}})$ (by Theorem 1) iff $\omega \models Bel(\Psi \circ \alpha \circ \gamma_{\{\omega,\omega'\}})$ (by **(CE1w)**) iff $\omega \preceq_{\Psi\circ\alpha} \omega'$ (by Theorem 1). So $\omega \preceq_{\Psi\circ\mu\circ\alpha} \omega' \Leftrightarrow \omega \preceq_{\Psi\circ\alpha} \omega'$, thus $\Psi \mapsto \preceq_\Psi$ satisfies *(CRE1w)*.

Assume that $\circ$ satisfies **(CE2w)**, and assume that $\omega, \omega' \not\models \mu$ and $\omega, \omega' \notin \min([\alpha], \preceq_\Psi) \cup \min([\alpha], \preceq_{\Psi\circ\mu})$. So by Theorem 1 we know that $Bel(\Psi \circ \alpha) \models \neg\gamma_{\{\omega,\omega'\}}$ and $Bel(\Psi \circ (\mu, \alpha)) \models \neg\gamma_{\{\omega,\omega'\}}$. Then $\omega \preceq_{\Psi\circ\mu\circ\alpha} \omega'$ iff $\omega \models Bel(\Psi \circ \mu \circ \alpha \circ \gamma_{\{\omega,\omega'\}})$ (by Theorem 1) iff $\omega \models Bel(\Psi \circ \alpha \circ \gamma_{\{\omega,\omega'\}})$ (by **(CE2w)**) iff $\omega \preceq_{\Psi\circ\alpha} \omega'$ (by Theorem 1). So $\omega \preceq_{\Psi\circ\mu\circ\alpha} \omega' \Leftrightarrow \omega \preceq_{\Psi\circ\alpha} \omega'$, thus $\Psi \mapsto \preceq_\Psi$ satisfies *(CRE2w)*.

(If part) Let β be a formula. Assume that $\Psi \mapsto \preceq_\Psi$ satisfies *(CRE1w)*, and that $\beta \models \mu$, $Bel(\Psi \circ \alpha) \models \neg\beta$ and $Bel(\Psi \circ (\mu, \alpha)) \models \neg\beta$. By Theorem 1, for all worlds $\omega, \omega' \in \beta$, we know that $\omega, \omega' \notin \min([\alpha], \preceq_\Psi) \cup \min([\alpha], \preceq_{\Psi\circ\mu})$. By *(CRE1w)*, since those worlds ω, ω' are models of μ, we get that $\omega \preceq_{\Psi\circ\mu\circ\alpha} \omega' \Leftrightarrow \omega \preceq_{\Psi\circ\alpha} \omega'$. This means that $\min([\beta], \preceq_{\Psi\circ\alpha}) = \min([\beta], \preceq_{\Psi\circ\mu\circ\alpha})$, so $Bel(\Psi \circ \alpha \circ \beta) \equiv Bel(\Psi \circ \alpha \circ \beta)$. Hence, $\circ$ satisfies **(CE1w)**.

Assume that $\Psi \mapsto \preceq_\Psi$ satisfies *(CRE2w)*, and that $\beta \models \neg\mu$, $Bel(\Psi \circ \alpha) \models \neg\beta$ and $Bel(\Psi \circ (\mu, \alpha)) \models \neg\beta$. By Theorem 1, for all worlds $\omega, \omega' \in \beta$, we know that $\omega, \omega' \notin \min([\alpha], \preceq_\Psi) \cup \min([\alpha], \preceq_{\Psi\circ\mu})$ By *(CRE2w)*, since those worlds ω, ω' are not models of μ, we get that $\omega \preceq_{\Psi\circ\mu\circ\alpha} \omega' \Leftrightarrow \omega \preceq_{\Psi\circ\alpha} \omega'$. This means that $\min([\beta], \preceq_{\Psi\circ\alpha}) = \min([\beta], \preceq_{\Psi\circ\mu\circ\alpha})$, so $Bel(\Psi \circ \alpha \circ \beta) \equiv Bel(\Psi \circ \alpha \circ \beta)$. Hence, $\circ$ satisfies **(CE2w)**. ☐

Proposition 10. *A DP assignment satisfies (FunWA) and (CRE5) if and only if it satisfies (FunWPred).*

Proof. The (if) part of the proof is direct by setting $\alpha = \beta$ to prove *(FunWA)*, and by setting $\omega = \omega^2$, $\omega' = \omega^3$ and $\Psi_1 = \Psi_2$ to prove *(CRE5)*. Let us show the (only if) part.

Let $\Psi \mapsto \preceq_\Psi$ be a DP assignment satisfying *(FunWA)* and *(CRE5)*. Let Ψ_1, Ψ_2 be two epistemic states, α, β be two formulas, and ω, ω', ω^2, ω^3 be four worlds such that $\omega \models \alpha \Leftrightarrow \omega^2 \models \beta$, $\omega' \models \alpha \Leftrightarrow \omega^3 \models \beta$, $\omega, \omega' \notin \min([\alpha], \preceq_{\Psi_1})$, $\omega^2, \omega^3 \notin \min([\beta], \preceq_{\Psi_2})$, and $\omega \preceq_{\Psi_1} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi_2} \omega^3$. We must prove that $\omega \preceq_{\Psi_1\circ\alpha} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi_2\circ\beta} \omega^3$.

We first provide the proof in the case when all four worlds ω, ω', ω^2, ω^3 are pairwise distinct. Assume first that $\omega, \omega' \models \alpha$. Then, $\omega^2, \omega^3 \models \beta$. From *(CR1)* we get that $\omega \preceq_{\Psi_1} \omega' \Leftrightarrow \omega \preceq_{\Psi_1\circ\alpha} \omega'$ and $\omega^2 \preceq_{\Psi_2} \omega^3 \Leftrightarrow \omega^2 \preceq_{\Psi_2\circ\beta} \omega^3$. Yet $\omega \preceq_{\Psi_1} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi_2} \omega^3$, so $\omega \preceq_{\Psi_1\circ\alpha} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi_2\circ\beta} \omega^3$. The case when $\omega, \omega' \not\models \alpha$ is proved similarly by using *(CR2)* instead of *(CR1)*. So, assume now that $\omega \models \alpha$ and $\omega' \not\models \alpha$. Then, $\omega^2 \models \beta$ and $\omega^3 \not\models \beta$. Since all worlds ω, ω', ω^2, ω^3 are pairwise distinct, there exists a formula γ such that $[\gamma] = \{\omega, \omega^2\} \cup \min([\alpha], \preceq_{\Psi_1}) \cup \min([\beta], \preceq_{\Psi_2})$. Clearly, we have that $\omega \models \alpha \wedge \gamma$, $\omega' \models \neg\alpha \wedge \neg\gamma$, and $\omega \notin \min([\alpha], \preceq_{\Psi_1}) \cup \min([\gamma], \preceq_{\Psi_1})$. So by *(CRE5)*, we get that (i) $\omega \preceq_{\Psi_1\circ\alpha} \omega' \Leftrightarrow \omega \preceq_{\Psi_1\circ\gamma} \omega'$. Likewise, since $\omega^2 \models \beta \wedge \gamma$, $\omega^3 \models \neg\beta \wedge \neg\gamma$, and and $\omega^2 \notin \min([\beta], \preceq_{\Psi_2}) \cup \min([\gamma], \preceq_{\Psi_2})$, by *(CRE5)* again we get that (ii) $\omega^2 \preceq_{\Psi_2\circ\beta} \omega^3 \Leftrightarrow \omega^2 \preceq_{\Psi_2\circ\gamma} \omega^3$. Lastly, since $\omega, \omega^2 \models \gamma$, $\omega', \omega^3 \not\models \gamma$, $\omega, \omega' \notin \min([\gamma], \preceq_{\Psi_1})$, $\omega^2, \omega^3 \notin \min([\gamma], \preceq_{\Psi_2})$ and $\omega \preceq_{\Psi_1} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi_2} \omega^3$, by *(FunWA)* we get that (iii) $\omega \preceq_{\Psi_1\circ\gamma} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi_2\circ\gamma} \omega^3$. Hence, from (i-iii) we get that $\omega \preceq_{\Psi_1\circ\alpha} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi_2\circ\beta} \omega^3$,

At this point, we have shown that the postcondition of *(FunWPred)* is true in the case when all four worlds ω, ω', ω^2, ω^3 are pairwise distinct. So, let us now consider the general case when this assumption does not necessarily holds.

Let us consider two epistemic states Ψ'_1, Ψ'_2, two formulae α', β' and four distinct, fresh worlds e, e', e^2, e^3 (i.e., the worlds e, e', e^2, e^3 are pairwise distinct and are distinct from the worlds ω, ω', ω^2, ω^3), such that the following sets of conditions are satisfied:

$$
\begin{array}{ll}
\text{Set (i):} & e \simeq_{\Psi'_1} \omega, e' \simeq_{\Psi'_1} \omega', \\
& e^2 \simeq_{\Psi'_2} \omega^2, e^3 \simeq_{\Psi'_2} \omega^3 \\
\text{Set (ii):} & \omega \models \alpha \Leftrightarrow \omega \models \alpha' \Leftrightarrow e \models \alpha', \\
& \omega' \models \alpha \Leftrightarrow \omega' \models \alpha' \Leftrightarrow e' \models \alpha', \\
& \omega^2 \models \beta \Leftrightarrow \omega^2 \models \beta' \Leftrightarrow e^2 \models \beta', \\
& \omega^3 \models \beta \Leftrightarrow \omega^3 \models \beta' \Leftrightarrow e^3 \models \beta' \\
\text{Set (iii):} & \omega \preceq_{\Psi_1} \omega' \Leftrightarrow \omega \preceq_{\Psi'_1} \omega', \\
& \omega^2 \preceq_{\Psi_2} \omega^3 \Leftrightarrow \omega^2 \preceq_{\Psi'_2} \omega^3 \\
\text{Set (iv):} & e, e' \notin \min([\alpha'], \preceq_{\Psi'_1}), \\
& e^2, e^3 \notin \min([\beta'], \preceq_{\Psi'_2}), \\
& \omega, \omega' \notin \min([\alpha'], \preceq_{\Psi_1}), \\
& \omega^2, \omega^3 \notin \min([\alpha'], \preceq_{\Psi_2})
\end{array}
$$

Roughly speaking, we created an instance $(\Psi'_1, \Psi'_2, \alpha', \beta', e, e', e^2, e^3)$ which is a "copy" of the instance $(\Psi_1, \Psi_2, \alpha, \beta, \omega, \omega', \omega^2, \omega^3)$, i.e., satisfying the preconditions of *(FunWPred)*, yet ensuring that the four worlds e, e', e^2, e^3 are pairwise distinct. Let us see how the preconditions of *(FunWPred)* are satisfied for this new instance.

First, we have that $e \models \alpha' \Leftrightarrow e^2 \models \beta'$. This is because $e \models \alpha' \Leftrightarrow \omega \models \alpha$ (from set (ii)), $\omega \models \alpha \Leftrightarrow \omega^2 \models \beta$ (from our initial assumption, i.e., one of the preconditions of *(FunWPred)*), and $\omega^2 \models \beta \Leftrightarrow e^2 \models \beta'$ (from set (ii)). Similarly, we have that $e' \models \alpha' \Leftrightarrow e^3 \models \beta'$.

Second, we have that $e, e' \notin \min([\alpha'], \preceq_{\Psi'_1})$ and $e^2, e^3 \notin \min([\beta'], \preceq_{\Psi'_2})$, which is directly expressed in set (iv).

Lastly, we have that $e \preceq_{\Psi'_1} e' \Leftrightarrow e^2 \preceq_{\Psi'_2} e^3$. This is because $e \preceq_{\Psi'_1} e' \Leftrightarrow \omega \preceq_{\Psi'_1} \omega'$ (from set (i)), $\omega \preceq_{\Psi'_1} \omega' \Leftrightarrow \omega \preceq_{\Psi_1} \omega'$ (from set (iii)), $\omega \preceq_{\Psi_1} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi_2} \omega^3$ (from our initial assumption, i.e., one of the preconditions of *(FunWPred)*), $\omega^2 \preceq_{\Psi_2} \omega^3 \Leftrightarrow \omega^2 \preceq_{\Psi'_2} \omega^3$ (from set (iii)), and $\omega^2 \preceq_{\Psi'_2} \omega^3 \Leftrightarrow e^2 \preceq_{\Psi'_2} e^3$ (from set (i)).

Since all preconditions of *(FunWPred)* are satisfied for the instance $(\Psi'_1, \Psi'_2, \alpha', \beta', e, e', e^2, e^3)$ and e, e', e^2, e^3 are pairwise distinct, we can conclude that:

$$e \preceq_{\Psi'_1\circ\alpha'} e' \Leftrightarrow e^2 \preceq_{\Psi'_2\circ\beta'} e^3 \qquad (1)$$

Recall that we need to prove that the postcondition of *(FunWPred)* is true for our initial instance $(\Psi_1, \Psi_2, \alpha, \beta, \omega, \omega', \omega^2, \omega^3)$, that is, we need to prove that $\omega \preceq_{\Psi_1\circ\alpha} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi_2\circ\beta} \omega^3$. We do so by proving the following chain of seven equivalences: $\omega \preceq_{\Psi_1\circ\alpha} \omega' \Leftrightarrow \omega \preceq_{\Psi_1\circ\alpha'} \omega' \Leftrightarrow \omega \preceq_{\Psi'_1\circ\alpha'} \omega' \Leftrightarrow e \preceq_{\Psi'_1\circ\alpha'} e' \Leftrightarrow e^2 \preceq_{\Psi'_2\circ\beta'} e^3 \Leftrightarrow \omega^2 \preceq_{\Psi'_2\circ\beta'} \omega^3 \Leftrightarrow \omega^2 \preceq_{\Psi_2\circ\beta'} \omega^3 \Leftrightarrow \omega^2 \preceq_{\Psi_2\circ\beta} \omega^3$.

(Equivalence 1) $\omega \preceq_{\Psi_1\circ\alpha} \omega' \Leftrightarrow \omega \preceq_{\Psi_1\circ\alpha'} \omega'$. Case a: $\omega, \omega' \models \alpha$. Then by set (ii), we have that $\omega, \omega' \models \alpha'$. So, $\omega \preceq_{\Psi_1\circ\alpha} \omega'$ iff $\omega \preceq_{\Psi_1} \omega'$ (by *(CR1)*) iff $\omega \preceq_{\Psi_1\circ\alpha'} \omega'$ (by *(CR1)* again). Case b: $\omega, \omega' \not\models \alpha$. Then by set (ii), we have that $\omega, \omega' \not\models \alpha'$. So, $\omega \preceq_{\Psi_1 \circ \alpha} \omega'$ iff $\omega \preceq_{\Psi_1} \omega'$ (by *(CR2)*) iff $\omega \preceq_{\Psi_1 \circ \alpha'} \omega'$ (by *(CR2)* again). Case c: $\omega \models \alpha$ and $\omega' \not\models \alpha$. Then by set (ii), we have that $\omega \models \alpha'$ and $\omega' \not\models \alpha'$. So, $\omega \models \alpha \wedge \alpha'$ and $\omega' \models \neg\alpha \wedge \neg\alpha'$. Then by *(CRE5)* and since $\omega \notin \min([\alpha \wedge \alpha'], \preceq_{\Psi_1})$ (from our initial assumption), we get that $\omega \preceq_{\Psi_1 \circ \alpha} \omega' \Leftrightarrow \omega \preceq_{\Psi_1 \circ \alpha'} \omega'$. Case d: $\omega \not\models \alpha$ and $\omega' \models \alpha$ is proved similarly as case c since ω and ω' play symmetrical roles.

(Equivalence 2) $\omega \preceq_{\Psi_1 \circ \alpha'} \omega' \Leftrightarrow \omega \preceq_{\Psi'_1 \circ \alpha'} \omega'$. We know that $\omega, \omega' \notin \min([\alpha'], \preceq_{\Psi_1})$ (set (iv)), $\omega, \omega' \notin \min([\alpha'], \preceq_{\Psi'_1})$ (from sets (i) and (iv)), and $\omega \preceq_{\Psi_1} \omega' \Leftrightarrow \omega \preceq_{\Psi'_1} \omega'$ (set (iii)). Then, from *(FunW)* we get that $\omega \preceq_{\Psi_1 \circ \alpha'} \omega' \Leftrightarrow \omega \preceq_{\Psi'_1 \circ \alpha'} \omega'$.

(Equivalence 3) $\omega \preceq_{\Psi'_1 \circ \alpha'} \omega' \Leftrightarrow e \preceq_{\Psi'_1 \circ \alpha'} e'$. We know that $\omega \models \alpha' \Leftrightarrow e \models \alpha'$ (set (ii)), $\omega' \models \alpha' \Leftrightarrow e' \models \alpha'$ (set (ii)), $\omega, \omega' \notin \min([\alpha'], \preceq_{\Psi'_1})$ (since $e, e' \notin \min([\alpha'], \preceq_{\Psi'_1})$ from set (iv) and $e \simeq_{\Psi'_1} \omega$ and $e' \simeq_{\Psi'_1} \omega'$ from set (i)), $e, e' \notin \min([\alpha'], \preceq_{\Psi'_1})$ (set (iv)), and $\omega \preceq_{\Psi'_1} \omega' \Leftrightarrow e \preceq_{\Psi'_1} e'$ (from set (i)). Then, from *(FunWA)* we get that $\omega \preceq_{\Psi'_1 \circ \alpha'} \omega' \Leftrightarrow e \preceq_{\Psi'_1 \circ \alpha'} e'$.

(Equivalence 4) $e \preceq_{\Psi'_1 \circ \alpha'} e' \Leftrightarrow e^2 \preceq_{\Psi'_2 \circ \beta'} e^3$. This results from Equation 1.

(Equivalence 5) $e^2 \preceq_{\Psi'_2 \circ \beta'} e^3 \Leftrightarrow \omega^2 \preceq_{\Psi'_2 \circ \beta'} \omega^3$. This is proved similarly to Equivalence 3.

(Equivalence 6) $\omega^2 \preceq_{\Psi'_2 \circ \beta'} \omega^3 \Leftrightarrow \omega^2 \preceq_{\Psi_2 \circ \beta'} \omega^3$. This is proved similarly to Equivalence 2.

(Equivalence 7) $\omega^2 \preceq_{\Psi_2 \circ \beta'} \omega^3 \Leftrightarrow \omega^2 \preceq_{\Psi_2 \circ \beta} \omega^3$. This is proved similarly to Equivalence 1.

We have proved that the seven equivalences above hold, which means that $\omega \preceq_{\Psi_1 \circ \alpha} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi_2 \circ \beta} \omega^3$. This shows that the postcondition of *(FunWPred)* is satisfied in every case.

We have shown that $\Psi \mapsto \preceq_\Psi$ satisfies *(FunWPred)*, which concludes the proof. $\square$