Targeting in Multi-Criteria Decision Making (Including Proofs)

Nicolas Schwind[1], **Patricia Everaere**[2], **Sébastien Konieczny**[3], **Emmanuel Lonca**[3]

[1]National Institute of Advanced Industrial Science and Technology, Tokyo, Japan
[2]Univ. Lille, CNRS, Centrale Lille, UMR 9189 CRIStAL, F-59000 Lille, France
[3]CRIL, CNRS – Université d'Artois, France
nicolas-schwind@aist.go.jp, patricia.everaere-caillier@univ-lille.fr, konieczny@cril.fr, lonca@cril.fr

Abstract

In this work, we introduce the notion of targeting for multi-criteria decision making. The problem consists in selecting the best alternatives related to one particular alternative, called the target. We use an axiomatic approach to this problem by establishing properties that any targeting method should satisfy. We present a representation theorem and show that satisfying the main properties of targeting requires aggregating the evaluations of the alternatives related to the target. We propose various candidate targeting methods and examine the properties satisfied by each method.

Introduction

In this work, we tackle the following problem: suppose you have a set of objects or alternatives, each evaluated according to multiple criteria. Traditional multi-criteria decision-making approaches (Greco, Ehrgott, and Figueira 2016; Bouyssou et al. 2000) typically aim to identify the best options among all alternatives. However, our focus is different. Instead of seeking the overall "best" alternatives, we aim to identify the best alternatives relative to a specific, targeted alternative, which we simply call the *target*. This target has a distinctive evaluation across the criteria, and our goal is to find the best alternatives most relevant to it. We term this problem *targeting*.

For example, imagine Agathe, a user, browsing through a selection of laptops and choosing one to use as a reference or target based on her personal preferences. This target laptop might perform well in terms of battery life (c_1), average in connectivity (c_2), but poorly in terms of processing speed (c_3). Agathe may not be explicitly aware of these last two criteria, but she selects the laptop because she values its long battery life, which aligns with her specific needs (e.g., frequent travel). She is not looking for the "best" laptop overall; rather, she is asking Brian, the shop staff, to recommend alternatives that are similar to her chosen laptop, particularly those with strong battery performance. Agathe's target laptop serves as a point of reference, and Brian's goal is to find the best alternatives in the same direction, i.e., laptops with improved battery life, as selecting laptops based on high processing speed would be irrelevant to Agathe. Our aim is to approach this problem systematically by providing a set of principles to guide the targeting process.

Previous work related to the selection of solutions has explored selecting subsets of nondominated solutions, often to avoid overwhelming users with too many options, which can impede effective decision-making. One common approach is to select k diverse solutions, i.e., a set of k nondominated solutions that are as far apart as possible in the solution space, maximizing pairwise diversity (Hebrard et al. 2005; Masin and Bukchin 2008; Petit and Trapp 2015). Another approach focuses on selecting a representative subset of the solution space (Vaz et al. 2015; Schwind et al. 2016; Demirović and Schwind 2020). These studies primarily aim to reduce the output size or computational effort but do not consider methods tailored to a specific target. More closely related to targeting is the work by Schwind *et al.* (Schwind et al. 2014), which introduced solution concepts aimed at identifying the best point in the direction of a given ideal point, that is, any arbitrarily chosen point that dominates the utopia point. However, their approach applies specifically to multiobjective constraint optimization problems, with the ideal point always lying outside the solution space. In contrast, our approach considers a target within the solution space itself. This is particularly practical when a user wishes to improve upon a specific alternative, especially when the set of objectives is not fully understood. Our contribution is also novel in that we formalize the targeting process through a set of properties, providing a principled framework for selecting alternatives. This allows one to better understand the behaviour of the different methods and to compare them more easily. This also allows us to show that the methods that satisfy the set of main properties we proposed have to use an aggregation function on the evaluations of the alternatives.

In the following, we begin by introducing necessary preliminaries before defining a set of principles and properties relevant to the targeting problem. We then characterize these principles in terms of preference orderings over alternatives, showing that functions satisfying these properties must perform an aggregation of the evaluations across the criteria. Building on this, we propose specific choice functions, analyze their properties, and illustrate their behavior through an example.

All proofs are given in an appendix.

Preliminaries

We are given a finite set of objects (also called alternatives) $\Omega = \{o_1, \ldots, o_n\}$, which we will refer to by the letters p, q, r, and $\mathbf{a}$ is a specific object from Ω called *target*. We are also given a finite set of criteria $\mathcal{C} = \{c_1, \ldots, c_m\}$. In this work, each criterion c_i is a function $c_i : \Omega \longrightarrow \mathbb{R}^+$ that assigns a non-negative value to each object with respect to the i-th criterion. The higher the value of a criterion for an object, the better the object is considered for that criterion. We will denote by $\gamma(p)$ the vector of evaluations of object $p \in \Omega$ with respect to the set of criteria $\mathcal{C}$, i.e., $\gamma(p) = (c_1(p), \ldots, c_m(p))$.

A subset S of $\mathcal{C}$ is called a *support*. Intuitively, a support is a set of criteria relevant for evaluating and comparing objects in a given context. We denote by $\overline{\mathsf{S}}$ the complement of S in $\mathcal{C}$, i.e., $\overline{\mathsf{S}} = \mathcal{C} \setminus \mathsf{S}$. Let $\mathsf{S} \subseteq \mathcal{C}$ and $p, q \in \Omega$. We say that p *Pareto dominates* q *on* S, denoted by $p \succeq^{\mathsf{S}}_{\star} q$, whenever $c_i(q) \leq c_i(p)$ for each criterion $c_i \in \mathsf{S}$. In other words, p Pareto dominates q on S if p is "at least as good as" q on each criterion in the support S. Note that $\succeq^{\mathsf{S}}_{\star}$ defines a partial ordering on Ω, and we denote by $\succ^{\mathsf{S}}_{\star}$ the strict part of $\succeq^{\mathsf{S}}_{\star}$ (i.e., $p \succ^{\mathsf{S}}_{\star} q$ iff $p \succeq^{\mathsf{S}}_{\star} q$ and $q \not\succeq^{\mathsf{S}}_{\star} p$), and by $\simeq^{\mathsf{S}}_{\star}$ its symmetric part (i.e., $p \simeq^{\mathsf{S}}_{\star} q$ iff $p \succeq^{\mathsf{S}}_{\star} q$ and $q \succeq^{\mathsf{S}}_{\star} p$). Given a subset of objects $O \subseteq \Omega$, the *Pareto front of* O *restricted to* S is the set $PF_{\mathsf{S}}(O) = \max(O, \succeq^{\mathsf{S}}_{\star}) = \{p \in O \mid \forall q \in O, q \not\succ^{\mathsf{S}}_{\star} p\}$, i.e., $PF_{\mathsf{S}}(O)$ consists of the objects in O that are not strictly Pareto dominated on S by any other object in O.

Given an object $\mathbf{a} \in \Omega$, the *support of* $\mathbf{a}$, denoted by $\mathsf{S}(\mathbf{a})$, is the set of criteria on which $\mathbf{a}$ has a strictly positive evaluation[1], i.e., $\mathsf{S}(\mathbf{a}) = \{c_i \in \mathcal{C} \mid c_i(\mathbf{a}) > 0\}$. Given a support $\mathsf{S} \subseteq \mathcal{C}$ and an object p, let $\gamma_{\mathsf{S}}(p)$ be the evaluation of p restricted to S, i.e., $\gamma_{\mathsf{S}}(p) = (c_{i_1}(p), \ldots, c_{i_k}(p))$ where $\{i_1, \ldots, i_k\} = \mathsf{S}$ and $i_j < i_{j'}$ iff $j < j'$. In particular, note that $\gamma_{\mathcal{C}}(p) = \gamma(p)$, and $p \simeq^{\mathsf{S}}_{\star} q$ iff $\gamma_{\mathsf{S}}(p) = \gamma_{\mathsf{S}}(q)$.

A *targeting method* is a mapping $\beta : 2^{\Omega} \times \Omega \to 2^{\Omega}$ that associates every subset of objects $O \subseteq \Omega$ and a *target* $\mathbf{a} \in \Omega$ with a subset of objects, denoted by $\beta_{\mathbf{a}}(O)$, such that $\beta_{\mathbf{a}}(O) \subseteq O$. Our goal is to specify a set of properties that an appropriate targeting method should satisfy. This will be the focus of the next section.

Properties on Targeting Methods

Let us introduce a set of properties that a targeting method β should satisfy. We divide these properties into two groups: the first group consists of the main properties that we expect every targeting method to satisfy, while the second group contains optional properties that may be relevant for specific subclasses of targeting methods.

For simplicity in presenting the properties, in the remainder of this section, we assume the following variables to be universally quantified: $\mathbf{a}$, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ are any objects from Ω; O and O' are any subsets of Ω; p, q are any objects from O; S is any support with $\mathsf{S} \subseteq \mathcal{C}$; c_i, c_j are any criteria from $\mathcal{C}$; x is any number in $\mathbb{R}^+$; μ is any number in $\mathbb{Q}^*$; and π is any permutation of $\{1, \ldots, m\}$.

[1] Note that it is also possible to choose another value than 0 as threshold if one wants to be more selective on the relevant criteria.

Main Properties

We begin by introducing the key properties for targeting methods.

The first requirement is that the method always returns a result when the input set is non-empty:

S1 If $O \neq \emptyset$ then $\beta_{\mathbf{a}}(O) \neq \emptyset$ — Non Vacuity

The second requirement is that the method must guarantee that two targets with the same evaluations lead to the same best elements. This property ensures that there is no other information than the evaluations of the target that is taken into account for selecting the results.

S2 If $\gamma(\mathbf{a}_1) = \gamma(\mathbf{a}_2)$, then $\beta_{\mathbf{a}_1}(O) = \beta_{\mathbf{a}_2}(O)$ — Neutrality

Next, the property of (Contribution) requires that only the evaluation of objects is used to determine the best elements, with only the relevant criteria – the support of the target – being considered:

S3 If $p \simeq^{\mathsf{S}(\mathbf{a})}_{\star} q$ and $p \in \beta_{\mathbf{a}}(O)$, then $q \in \beta_{\mathbf{a}}(O)$ — Contribution

Note that (Contribution) implies a property of "Anonymity," stating that two objects with identical evaluation vectors are treated equivalently:

If $\gamma(p) = \gamma(q)$ and $p \in \beta_{\mathbf{a}}(O)$, then $q \in \beta_{\mathbf{a}}(O)$ — Anonymity

Indeed, $p \simeq^{\mathsf{S}(\mathbf{a})}_{\star} q$ always holds whenever $\gamma(p) = \gamma(q)$ holds, irrespective of the target's support $\mathsf{S}(\mathbf{a})$.

The next property ensures that all criteria that are relevant according to the target are considered during evaluation. Specifically, if one object strictly outperforms another on those criteria, the underperforming object cannot be among the best:

S4 If $p \succ^{\mathsf{S}(\mathbf{a})}_{\star} q$, then $q \notin \beta_{\mathbf{a}}(O)$ — Responsiveness

This implies that the best objects selected by β are always within the Pareto front of O restricted to the target's support. Formally, if β satisfies (Responsiveness), then $\beta_{\mathbf{a}}(O) \subseteq PF_{\mathsf{S}(\mathbf{a})}(O)$.

The next property ensures consistency when restricting the evaluation to a subset of objects. If we consider a subset of O that includes some of the best objects from the original set, these objects should remain the best in the subset:

S5 If $O' \subseteq O$ and $\beta_{\mathbf{a}}(O) \cap O' \neq \emptyset$, then $\beta_{\mathbf{a}}(O') = \beta_{\mathbf{a}}(O) \cap O'$ — Contraction

This guarantees that the selection process is robust to focusing on smaller subsets.

The following property ensures that criteria on which the target performs identically have exactly the same impact on the evaluation of the objects. Specifically, if switching an object's evaluation on two such criteria produces an evaluation that matches another object, the two objects should be considered equivalently good. This property guarantees that some criteria are not intrinsically (arbitrarily) more important than others. This also emphasizes the fact that we consider a commensurability between the criteria. If for an application this is not the case from the scales used for the different criteria, then a first homogenization/normalization step will be required to make the scales commensurable.

S6 If $c_i(\mathbf{a}) = c_j(\mathbf{a})$, $p \simeq_\star^{\mathsf{S}(\mathbf{a})\setminus\{c_i,c_j\}} q$, $c_i(p) = c_j(q)$, $c_j(p) = c_i(q)$ and $p \in \beta_\mathbf{a}(O)$, then $q \in \beta_\mathbf{a}(O)$ Equivalence

So, this property ensures that the evaluation respects a form of symmetry when criteria are evaluated identically by the target.

The next property captures the intuition that criteria where the target performs better should have a stronger influence on object evaluation. If one criterion is more important for the target, and objects differ only on these criteria, the object with better performance on the more important criterion should be preferred:

S7 If $c_i(\mathbf{a}) > c_j(\mathbf{a})$, $p \simeq_\star^{\mathsf{S}(\mathbf{a})\setminus\{c_i,c_j\}} q$, $c_i(p) > c_j(p)$, $c_j(p) = c_i(q)$ and $c_i(p) = c_j(q)$, then $q \notin \beta_\mathbf{a}(O)$ Relevance

The next property reflects the idea that if an object is the best for two disjoint sets of criteria, it should remain the best when these criteria are combined. Formally:

S8 If $\mathsf{S}(\mathbf{a}_1) \cap \mathsf{S}(\mathbf{a}_2) = \emptyset$ and $\gamma(\mathbf{a}_3) = \gamma(\mathbf{a}_1) + \gamma(\mathbf{a}_2)$, then $\beta_{\mathbf{a}_1}(O) \cap \beta_{\mathbf{a}_2}(O) \subseteq \beta_{\mathbf{a}_3}(O)$ Composition

This property ensures that the targeting method respects the independence of unrelated sets of criteria.

Finally, the (Decomposition) property complements (Composition) by ensuring that the best objects for a union of independent criteria sets are precisely those that are best for both individual sets:

S9 If $\mathsf{S}(\mathbf{a}_1) \cap \mathsf{S}(\mathbf{a}_2) = \emptyset$, $\gamma(\mathbf{a}_3) = \gamma(\mathbf{a}_1) + \gamma(\mathbf{a}_2)$, and $\beta_{\mathbf{a}_1}(O) \cap \beta_{\mathbf{a}_2}(O) \neq \emptyset$, then $\beta_{\mathbf{a}_3}(O) \subseteq \beta_{\mathbf{a}_1}(O) \cap \beta_{\mathbf{a}_2}(O)$ Decomposition

So we are seeking for targeting methods that satisfy this set of nine properties.

Optional Properties

Let us now introduce additional properties that may define sensible subclasses of targeting methods. While not mandatory, these properties offer desirable behavior in specific contexts, depending on the goals and constraints of the targeting method.

The first property, (Dominance), ensures that the selected best objects always Pareto-dominate the target.

O1 If $p \in \beta_\mathbf{a}(O)$ and $\mathbf{a} \in O$, then $p \succeq_\star^{\mathsf{S}(\mathbf{a})} \mathbf{a}$ Dominance

This property is particularly relevant when the goal is to ensure that the best objects are never worse than the target. It is useful in applications where maintaining or exceeding the target's performance across all relevant criteria is essential.

Next, the (Compensation) property assumes that the evaluation scales of all criteria are the same. This means that if two criteria are equally valued by the target, increasing one while decreasing the other by the same amount results in an equally good object:

O2 If $p \simeq_\star^{\mathsf{S}(\mathbf{a})\setminus\{c_i,c_j\}} q$, $c_i(\mathbf{a}) = c_j(\mathbf{a})$, $c_i(q) = c_i(p) - x$, $c_j(q) = c_j(p) + x$ and $p \in \beta_\mathbf{a}(O)$, then $q \in \beta_\mathbf{a}(O)$ Compensation

This property is valuable when trade-offs between criteria are meaningful and the criteria are expressed on comparable scales. It is particularly suitable for scenarios where flexibility in balancing trade-offs is necessary.

Note that this property is a strengthening of the (Equivalence) property:

Proposition 1 *If β satisfies* **O2**, *then it satisfies* **S6**.

We can also show that the (Dominance) and (Compensation) properties are fundamentally incompatible when combined with the required (Non-Vacuity) property (**S1**). This incompatibility arises because (Dominance) disallows trade-offs that (Compensation) requires:

Proposition 2 *If β satisfies* **S1**, *then it cannot satisfy both* **O1** *and* **O2**.

This highlights the need to carefully choose between these properties depending on whether guaranteeing that the best objects are at least as good as the target or allowing trade-off flexibility is more important.

The following (Optimality) property requires that if the target belongs to the Pareto front of the object set (restricted to its support), it must also be selected as one of the best objects:

O3 If $\mathbf{a} \in PF_{\mathsf{S}(\mathbf{a})}(O)$, then $\mathbf{a} \in \beta_\mathbf{a}(O)$ Optimality

This property is most relevant when the target is not Pareto-dominated within its support, ensuring that it is recognized as a reasonable candidate for selection. It applies in contexts where preserving a balance among criteria, as represented by the target, is essential.

There is a direct relationship between this (Optimality) and (Dominance):

Proposition 3 *If β satisfies* **S1**, **S3** *and* **O1**, *then it satisfies* **O3**.

A direct consequence of Proposition 3 is that since targeting methods satisfying (Dominance) also satisfy (Optimality) in the presence of (Non-Vacuity) and (Contribution), it follows that when the target is in the Pareto front according to its own support, the best objects are precisely those that share the same evaluation as the target on its support:

Corollary 1 *If β satisfies* **S1**, **S3** *and* **O1**, *and* $\mathbf{a} \in PF_{\mathsf{S}(\mathbf{a})}(O)$, *then for each $p \in O$, we have that $p \in \beta_\mathbf{a}(O)$ if and only if $p \simeq_\star^{\mathsf{S}(\mathbf{a})} \mathbf{a}$*.

The next property ensures that the result of a targeting method depends only on the relative importance of the criteria in the target's support. It remains unaffected by linear transformations of the target's evaluation:

O4 If $\gamma(\mathbf{a}_1) = \mu \times \gamma(\mathbf{a}_2)$, then $\beta_{\mathbf{a}_1}(O) = \beta_{\mathbf{a}_2}(O)$ Proportionality

This property is interesting when relative importance matters more than absolute values, such as in normalized or scale-invariant evaluations.

Last, the (Symmetry) property states that the result should remain consistent under permutations of the criteria. If both the target and the evaluations of objects are permuted identically, the result should remain unchanged. Let us denote $\gamma(O) = \{\gamma(p) \mid p \in O\}$, and for a set of m-vectors V, $\pi(V) = \{\pi(v) \mid v \in V\}$:

O5 If $\gamma(\mathbf{a}_2) = \pi(\gamma(\mathbf{a}_1))$, $\gamma(O') = \pi(\gamma(O))$, $p' \in O'$, $\gamma(p') = \pi(\gamma(p))$, and $p \in \beta_{\mathbf{a}_1}(O)$, then $p' \in \beta_{\mathbf{a}_2}(O')$ Symmetry

This property is essential in scenarios where criteria are interchangeable, meaning their order or labeling does not affect the evaluation. It ensures fairness in contexts where the criteria are inherently equivalent. Since not satisfying it allows for instance to have some criteria that are inherently (i.e. independently of the target) more important than others. This can make sense in some situations.

Characterization

This section characterizes targeting methods that satisfy the main and optional properties by associating them with a construction involving preference orderings over alternatives.

Specifically, we can show that every function $\beta_{\mathbf{a}}$ can be characterized by a total preorder $\succeq_{\mathbf{a}}$ on alternatives Ω, and then computing $\beta_{\mathbf{a}}(O)$ consists in selecting the alternatives from O that are maximal w.r.t $\succeq_{\mathbf{a}}$. This is formalized through the concept of "targeting assignment":

Definition 1 *A* targeting assignment *is a mapping associating every $\mathbf{a} \in \Omega$ with a total preorder $\succeq_{\mathbf{a}}$ over elements of Ω.*[2]

Proposition 4 *β satisfies* **S1** *and* **S5** *iff there is a targeting assignment $\mathbf{a} \mapsto \succeq_{\mathbf{a}}$ such that for each $\mathbf{a} \in \Omega$ and each $O \subseteq \Omega$, $\beta_{\mathbf{a}}(O) = \max(O, \succeq_{\mathbf{a}})$.*

The (if) part of the proof of Proposition 4 consists in defining a targeting method β from a targeting assignment $\mathbf{a} \mapsto \succeq_{\mathbf{a}}$ as $\beta_{\mathbf{a}}(O) = \max(O, \succeq_{\mathbf{a}})$, for each $\mathbf{a} \in \Omega$ and each $O \subseteq \Omega$, and verify that β satisfies **S1** and **S5**. The (only if) part of the proof involves the construction of a targeting assignment $\mathbf{a} \mapsto \succeq_{\mathbf{a}}$ from β defined for all $\mathbf{a}, p, q \in \Omega$ as $p \succeq_{\mathbf{a}} q$ iff $p \in \beta_{\mathbf{a}}(\{p, q\})$. We say that such a targeting assignment *corresponds to* β, and we can show how properties of a targeting method β correspond to properties of its corresponding targeting assignment:

Proposition 5 *Let β satisfies* **S1** *and* **S5** *and $\mathbf{a} \mapsto \succeq_{\mathbf{a}}$ be its corresponding targeting assignment. Then for each $i \in \{2, 3, 4, 6, 7, 8, 9\}$, $j \in \{1, 2, 3, 4, 5\}$, β satisfies* **Si** *(resp.* **Oj***) iff $\mathbf{a} \mapsto \succeq_{\mathbf{a}}$ satisfies the following condition (SRi) (resp. (ORj)):*

(SR2) *If $\gamma(\mathbf{a}_1) = \gamma(\mathbf{a}_2)$, then $\succeq_{\mathbf{a}_1} = \succeq_{\mathbf{a}_2}$*

(SR3) *If $p \simeq_\star^{\mathsf{S}(\mathbf{a})} q$, then $p \simeq_{\mathbf{a}} q$*

(SR4) *If $p \succ_\star^{\mathsf{S}(\mathbf{a})} q$, then $p \succ_{\mathbf{a}} q$*

(SR6) *If $c_i(\mathbf{a}) = c_j(\mathbf{a})$, $p \simeq_\star^{\mathsf{S}(\mathbf{a}) \setminus \{c_i, c_j\}} q$, $c_i(p) = c_j(q)$ and $c_j(p) = c_i(q)$, then $p \simeq_{\mathbf{a}} q$*

(SR7) *If $c_i(\mathbf{a}) > c_j(\mathbf{a})$, $p \simeq_\star^{\mathsf{S}(\mathbf{a}) \setminus \{c_i, c_j\}} q$, $c_i(p) > c_j(q)$, $c_j(p) = c_i(q)$ and $c_i(p) = c_j(q)$, then $p \succ_{\mathbf{a}} q$*

(SR8) *If $\mathsf{S}(\mathbf{a}_1) \cap \mathsf{S}(\mathbf{a}_2) = \emptyset$, $\gamma(\mathbf{a}_3) = \gamma(\mathbf{a}_1) + \gamma(\mathbf{a}_2)$, $p \succeq_{\mathbf{a}_1} q$ and $p \succeq_{\mathbf{a}_2} q$, then $p \succeq_{\mathbf{a}_3} q$*

(SR9) *If $\mathsf{S}(\mathbf{a}_1) \cap \mathsf{S}(\mathbf{a}_2) = \emptyset$, $\gamma(\mathbf{a}_3) = \gamma(\mathbf{a}_1) + \gamma(\mathbf{a}_2)$, $p \succeq_{\mathbf{a}_1} q$ and $p \succ_{\mathbf{a}_2} q$, then $p \succ_{\mathbf{a}_3} q$*

(OR1) *If $p \not\succeq_\star^{\mathsf{S}(\mathbf{a})} \mathbf{a}$, then $\mathbf{a} \succ_{\mathbf{a}} p$*

(OR2) *If $p \simeq_\star^{\mathsf{S}(\mathbf{a}) \setminus \{c_i, c_j\}} q$, $c_i(\mathbf{a}) = c_j(\mathbf{a})$, $c_i(q) = c_i(p) - x$ and $c_j(q) = c_j(p) + x$, then $p \simeq_{\mathbf{a}} q$*

(OR3) *If $p \not\succ_\star^{\mathsf{S}(\mathbf{a})} \mathbf{a}$, then $\mathbf{a} \succeq_{\mathbf{a}} p$*

(OR4) *If $\gamma(\mathbf{a}_1) = \mu \times \gamma(\mathbf{a}_2)$, then $\succeq_{\mathbf{a}_1} = \succeq_{\mathbf{a}_2}$*

(OR5) *If $\gamma(\mathbf{a}_2) = \pi(\gamma(\mathbf{a}_1))$, $\gamma(p') = \pi(\gamma(p))$, $\gamma(q') = \pi(\gamma(q))$ and $p \succeq_{\mathbf{a}_1} q$, then $p' \succeq_{\mathbf{a}_2} q'$*

(SR2) states that two targets with the same evaluation will generate exactly the same preference order on objects. *(SR3)* states that two objects that are equivalent on the support of the target will be indifferent for the preference order. *(SR4)* states that if an object performs better than (Pareto dominates) another one, then this object will be strictly preferred. *(SR6)* states that if two criteria are identical for the target, then two objects that are identical on all criteria except on these two ones, where their evaluation is swapped, have to be indifferent for the preference order. *(SR7)* states that if a criterion is more important for the target than another one, then two objects that are identical on all criteria except on these two ones, where their evaluation is swapped, are ranked so that the preferred object is the one that performs best on the more important criterion. *(SR8)* states that if an object is at least as good than another one for two sets of disjoints criteria, then it will also be better for (a target whose support is) the union of the criteria. *(SR9)* states that if an object is at least as good than another one for a set of criteria and strictly better for a disjoint set of criteria, then it will be strictly better for (a target whose support is) the union of the criteria.

(OR1) states that if an object does not Pareto-dominate the target on its support, then the target will be strictly better. *(OR2)* states that if two criteria are identical for the target, then two objects that are identical on all criteria except on these two ones, where their evaluation is reversed but the difference identical, then they have to be indifferent for the preference order. *(OR3)* states that if an object does not strictly Pareto-dominate the target on its support, then it can not be strictly better than the target. *(OR4)* states that if a target has a vector of evaluations that is proportional to the vector of another target, then they will generate exactly the same preference order on objects. *(OR5)* states that if a target vector of evaluations is a permutation of the vector of another target, and if two objects related to the first target have a vector of evaluations that are the permutations of two objects related to the second target, then the relation between the two objects of the first target is the same than the relation between the two objects of the second target.

Note that this kind of link between choice functions and preferences (preorder) can be related to results that exist for social choice (Arrow 1959) or belief change (Gärdenfors 1988; Fermé and Hansson 2011). But the properties on the

[2] For a given total preorder $\succeq$, the symbols $\succ$ and $\simeq$ denote respectively the strict part and the indifference part of $\succeq$.

preorder allow to better understand the operators. The relationship established by Proposition 5 provides a different rationale for the properties of targeting methods (**S2-S9**) and (**O1-O5**), as these properties naturally arise from a preference ordering over alternatives. For example, the way the pairs of properties **S3/S4**, **S6/S7**, and **S8/S9** complement each other becomes evident from their counterparts *(SR3)/(SR4)*, *(SR6)/(SR7)*, and *(SR8)/(SR9)*, within this ordering structure.

In particular, Propositions 4 and 5 imply that for targeting methods satisfying the first five conditions of **S1-S5**, selecting the best alternatives according to a given target is equivalent to aggregating the evaluations of the alternatives:

Corollary 2 *β satisfies* **S1-S5** *iff there exists a set of strictly monotone aggregation functions*[3] $\{f_\beta^{\gamma_{\mathsf{S}(\mathbf{a})}(\mathbf{a})} \mid \mathbf{a} \in \Omega\}$ *such that for each $O \subseteq \Omega$*

$$\beta_{\mathbf{a}}(O) = \arg\max_{p \in O} \{f_\beta^{\gamma_{\mathsf{S}(\mathbf{a})}(\mathbf{a})}(\gamma_{\mathsf{S}(\mathbf{a})}(p))\}$$

This result provides an interesting representation of targeting functions, since it shows that targeting functions satisfying conditions **S1-S5** correspond to targeting functions defined from an aggregation of the criteria that are relevant for the target.

Targeting Functions

Let us now consider some functions that seem to be valuable candidates as targeting methods.

Most of these functions will use a weight function for the criteria given a target $\mathbf{a}$, $w(\mathbf{a}) = (w_1(\mathbf{a}), \ldots, w_{|\mathcal{C}|}(\mathbf{a}))$, which is defined for each $c_i \in \mathcal{C}$ as:

$$w_i(\mathbf{a}) = c_i(\mathbf{a}) / \sum_{x \in \mathsf{S}(\mathbf{a})} c_x(\mathbf{a})$$

Roughly speaking, this function defines the weight of a criterion for a target $\mathbf{a}$ as the importance of this criterion for the target relative to the (sum of the) other ones (note that other functions can also be used to define this weight, leading to variations of the proposed methods below—this possibility is not further investigated in this paper).

In the rest of this section, O denotes any set of objects $O \subseteq \Omega$, and $\mathbf{a}$ any target $\mathbf{a} \in \Omega$ such that $\mathsf{S}(\mathbf{a}) \neq \emptyset$.

The first targeting function simply computes the sum of the evaluation of the criteria weighted by their relative importance for the target.

Definition 2 (Weighted Sum)

$$\beta_{\mathbf{a}}^{ws}(O) = \arg\max_{p \in O} \mathsf{BWS}_{\mathbf{a}}(p)$$

where $\mathsf{BWS}_{\mathbf{a}}(p) = \sum_i w_i(\mathbf{a}) \times c_i(p)$.

[3] A strictly monotone aggregation function is a mapping f associating a vector of values $(v_1, \ldots, v_k) \in \mathbb{R}^+ \times \ldots \times \mathbb{R}^+$ with a single value $v \in \mathbb{R}^+$ such that if $\forall i, v_i \geq v'_i$ then $f((v_1, \ldots, v_k)) \geq f((v'_1, \ldots, v'_k))$, and if additionally $\exists j : v_j > v'_j$, then $f((v_1, \ldots, v_k)) > f((v'_1, \ldots, v'_k))$.

This method focuses on the numerical evaluation on each criterion, which assumes some kind of commensurability between the criteria. That is not always the case in all multi-criteria applications. One possibility, then, is not to focus on the evaluation of each element, but on its rank with respect to each criterion. Given any object $p \in \Omega$ and any criterion $c_i \in \mathcal{C}$, let $r_{c_i}(p)$ denote the *rank* of p w.r.t. c_i, defined as:

- $p \geq_{c_i} q$ iff $c_i(p) \geq c_i(q)$;
- best elements b_c for a criterion $\geq_c$ have first rank: $r_{c_i}(b_c) = 1$;
- for any element p, its rank $r_c(p)$ is one plus the length of the shortest path $b_c >_c \ldots >_c p$.

Then the Ordinal Weighted Sum selection function chooses the objects that have the best (weighted) rank:

Definition 3 (Ordinal Weighted Sum)

$$\beta_{\mathbf{a}}^{ows}(O) = \arg\min_{p \in O} \mathsf{OWS}_{\mathbf{a}}(p)$$

where $\mathsf{OWS}_{\mathbf{a}}(p) = \sum_i w_i(\mathbf{a}) \times r_{c_i}(p)$

For this function, we turn to an ordinal evaluation of the elements, but for the weights we keep the evaluation of the target. Another possibility is to also use an ordinal weight of the criteria:

Definition 4 (Purely Ordinal Weighted Sum) *Define the rank of each criterion c with respect to* $\mathbf{a}$ *:*

- $c_i \geq_{\mathbf{a}} c_j$ *iff* $c_i(\mathbf{a}) \geq c_j(\mathbf{a})$*;*
- *best criteria* $c_{\mathbf{a}}$ *for* $\geq_{\mathbf{a}}$ *have first rank:* $r_{\mathbf{a}}(c_{\mathbf{a}}) = 1$*;*
- *for any criterion c such that* $c(\mathbf{a}) \neq 0$ *its rank* $r_{\mathbf{a}}(c)$ *is one plus the length of the shortest path* $c_{\mathbf{a}} > \ldots > c$*;*
- *all criteria c such that* $c(\mathbf{a}) = 0$ *are not ranked.*

Then we will use a lexicographical comparison with respect to this criteria ranking: $\vec{\mathcal{C}^{\mathbf{a}}} = \langle \{c \text{ s.t. } r_{\mathbf{a}}(c) = 1\}, \{c \text{ s.t. } r_{\mathbf{a}}(c) = 2\}, \ldots, \{c \text{ s.t. } r_{\mathbf{a}}(c) = m\} \rangle$.

$$\beta_{\mathbf{a}}^{pws}(O) = \arg\min_{p \in O} (\mathsf{PWS}_{\mathbf{a}}(p), \geq_{lex})$$

where $\mathsf{PWS}_{\mathbf{a}}(p) = \langle \sum_{c \in \vec{\mathcal{C}}_1^{\mathbf{a}}} r_c(p), \ldots, \sum_{c \in \vec{\mathcal{C}}_m^{\mathbf{a}}} r_c(p) \rangle$*, and* $\geq_{lex}$ *is defined for two vectors* $x = \langle x_1, \ldots, x_n \rangle$ *and* $y = \langle y_1, \ldots, y_n \rangle$ *as* $x >_{lex} y$ *if* $\exists i$ *s.t.* $\forall j < i$ $x_j = y_j$ *and* $x_i > y_j$*, and* $x \simeq_{lex} y$ *iff* $\forall i$ $x_i = y_i$.

One function that often allows obtaining results that are more balanced with respect to criteria is the product (just as the Nash welfare function (Nash 1953; Luce and Raiffa 1957; Thomson 1994)). Here, as we consider weights, we use the power function to weight the evaluation of each criterion.

Definition 5 (Weighted Geometric Mean)

$$\beta_{\mathbf{a}}^{wgm}(O) = \arg\max_{p \in O} \mathsf{WGM}_{\mathbf{a}}(p)$$

where $\mathsf{WGM}_{\mathbf{a}}(p) = \prod_{c_i \in \mathcal{C}} [1 + c_i(p)]^{w_i(\mathbf{a})}$

Note that we have to shift the evaluation of each criterion by 1 to avoid the problem of the absorbing element 0.

All functions given so far start from the origin point and intend to find the best elements "towards" the target. One alternative is not to focus on the origin (the worst potential element), but to start from the best potential element that we call the ideal element. This ideal element is the (hypothetical) element whose evaluation for each criterion is equal to the maximal evaluation among all elements in Ω. In the general case, this element does not belong to O, so we look at the closest element of O from this point "towards" the target (i.e., by using a distance that is parameterized by the target element). This idea comes from a method that is used in cooperative game theory for the bargaining problem called the Kalai-Smorodinsky solution (Kalai and Smorodinsky 1975; Thomson 1994).

More formally, let us define the coordinates of the ideal point:

Definition 6 (Ideal Point Distance) *The ideal point* b^O *is defined as* $b^O = \langle \max_{p\in O} c_1(p), \ldots, \max_{p\in O} c_n(p)\rangle$.

$$\beta^{ipd}_{\mathbf{a}}(O) = \arg\min_{p\in O} \mathsf{IPD}_{\mathbf{a}}(p)$$

where $\mathsf{IPD}_{\mathbf{a}}(p) = \sqrt{\sum_{c_i\in \mathcal{C}} [(b^O_i - c_i(p)) * w_i(\mathbf{a})]^2}$

The next method, Weighted LexiMin, ensures the evaluation of the best solutions to be "proportional" to that of the target. The idea is similar to the one introduced in (Schwind et al. 2014) in the context of multi-objective constraint optimization, where it is called "weighted egalitarian method."

Given a vector of numbers v, v^i denotes the i^{th} component of v. The vector $v_{\le}$ (resp. $v_{\ge}$) denotes the vector composed of each element of v rearranged in a non-decreasing order (resp. non-increasing order). For instance, if $v = (3, 2, 5, 3)$, then $v_{\le} = (2, 3, 3, 5)$ and $v_{\ge} = (5, 3, 3, 2)$. Let v, w be two vectors of numbers of the same size, i.e., $v, w \in \mathbb{R}^k$ for some $k \ge 0$. We say that v lexically succeeds w when there exists $i \in \{1, \ldots, m\}$ such that for each $j \in \{1, \ldots, i-1\}$, $v^j = w^j$ and $v^i \ge w^i$. Additionally, when $w^i > 0$ for each $i \in \{1, \ldots, k\}$, we denote by (v/w) the vector $(v^1/w^1, \ldots, v^k/w^k)$, and we denote by $(v * w)$ the vector $(v^1 * w^1, \ldots, v^k * w^k)$. Let us also denote by $w_{\mathsf{S}(\mathbf{a})}(\mathbf{a})$ the vector $(w_{i_1}(\mathbf{a}), \ldots, w_{i_{|\mathsf{S}(\mathbf{a})|}}(\mathbf{a}))$ where $i_j < i_{j'}$ iff $j < j'$.

Given $p, \mathbf{a} \in \Omega$, we denote by $\mathsf{WLMin}_{\mathbf{a}}(p)$ the vector $\mathsf{WLMin}_{\mathbf{a}}(p) = (\gamma_{\mathsf{S}(\mathbf{a})}(p)/w_{\mathsf{S}(\mathbf{a})}(\mathbf{a}))_{\le}$. Then let $\ge^{\mathbf{a}}_{leximin}$ be the relation over Ω defined for all $p, q \in \Omega$ as $p \ge^{\mathbf{a}}_{leximin} q$ iff $\mathsf{WLMin}_{\mathbf{a}}(p)$ lexically succeeds $\mathsf{WLMin}_{\mathbf{a}}(q)$. It is easy to see that $\ge^{\mathbf{a}}_{leximin}$ is a total ordering. Then:

Definition 7 (Weighted LexiMin)

$$\beta^{wlmin}_{\mathbf{a}}(O) = \arg\max(O, \ge^{\mathbf{a}}_{leximin})$$

The last method, called Weighted LexiMax, gives more priority to the criteria having a high value for both the candidates and the target. It is defined similarly to the Weighted LexiMin as follows. Given $p, \mathbf{a} \in \Omega$, we denote by $\mathsf{WLMax}_{\mathbf{a}}(p)$ the vector $\mathsf{WLMax}_{\mathbf{a}}(p) = (\gamma_{\mathsf{S}(\mathbf{a})}(p) * w(\mathbf{a}))_{\ge}$.

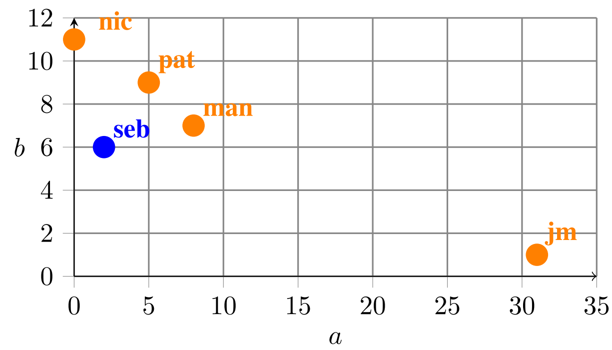

Figure 1: Representation of the alternatives on criteria a and b

Let $\ge^{\mathbf{a}}_{leximax}$ be the relation over Ω defined for all $p, q \in \Omega$ as $p \ge^{\mathbf{a}}_{leximax} q$ iff $\mathsf{WLMax}_{\mathbf{a}}(p)$ lexically succeeds $\mathsf{WLMax}_{\mathbf{a}}(q)$. Similarly, $\ge^{\mathbf{a}}_{leximax}$ is a strict total ordering, so let us denote by $\ge^{\mathbf{a}}_{leximax}$ the corresponding total ordering. Then:

Definition 8 (Weighted LexiMax)

$$\beta^{wlmax}_{\mathbf{a}}(O) = \arg\max(O, \ge^{\mathbf{a}}_{leximax})$$

Let us now consider the following example to illustrate the behaviour of all the methods defined in this section. Let $\mathcal{C} = \{a, b, c, d\}$ and $\Omega = O = \{seb, pat, jm, man, nic\}$. The evaluation of the criteria for the participants are as follows: $\gamma(seb) = \{2, 6, 0, 0\}$; $\gamma(jm) = \{31, 1, 7, 1\}$; $\gamma(pat) = \{5, 9, 0, 11\}$; $\gamma(man) = \{8, 7, 2, 1\}$; $\gamma(nic) = \{0, 11, 12, 8\}$. We focus on the target seb. Since the support of seb contains only two criteria, we can represent all the alternatives projected onto this support, as shown in Figure 1.

Let us examine the outcomes of the proposed methods on this example (see Table 2, where the result of each method appears in blue). The methods yield noticeably different selections. β^{ws} selects jm because he has very high evaluation on criterion a. β^{ows} selects nic, who has the best rank on the most important criterion b. β^{pws} selects pat and man, since the relative ranks of criteria a and b are identical for the target, and these two candidates are well ranked on both. β^{wgm} selects pat, who has a strong average evaluation across all relevant criteria. β^{ipd} selects man, the alternative closest to the ideal point relative to the target seb. β^{wlmin} selects pat, whose evaluations are proportional to those of seb. Finally, β^{wlmax} selects nic, as both seb and nic perform well on criterion b.

Properties of the targeting functions

In the previous section, we saw that the behaviours of these functions are quite different on the illustrative example. Let us now study their behaviour more systematically by examining which properties are satisfied by each method.

Proposition 6 *The Weighted Sum method* β^{ws} *satisfies* **S1, S2, S3, S4, S5, S6, S7, S8, S9, O2, O4, O5**. *It does not satisfy* **O1** *and* **O3**.

Proposition 7 *The Ordinal Weighted Sum method* β^{ows} *satisfies* **S1, S2, S3, S4, S8, S9, O4, O5**. *It does not satisfy* **S5, S6, S7, O1, O2, O3**.

	S1	S2	S3	S4	S5	S6	S7	S8	S9	O1	O2	O3	O4	O5
Weighted Sum	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✗	✓	✓
Ordinal Weighted Sum	✓	✓	✓	✓	✗	✗	✗	✓	✓	✗	✗	✗	✓	✓
Purely Ordinal Weighted Sum	✓	✓	✓	✓	✗	✗	✗	✓	✓	✗	✗	✗	✓	✓
Weigthed Geometric Mean	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✗	✗	✓	✓
Ideal Point Distance	✓	✓	✓	✓	✗	✗	✗	✓	✓	✗	✗	✗	✓	✓
Weighted LexiMin	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓
Weighted LexiMax	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✗	✗	✓	✓

Table 1: Properties of targeting functions

β^{ws}	$\mathsf{BWS}_{seb}(jm) = 8.5$ $\mathsf{BWS}_{seb}(man) = 7.25$	$\mathsf{BWS}_{seb}(seb) = 5$ $\mathsf{BWS}_{seb}(pat) = 8$ $\mathsf{BWS}_{seb}(nic) = 8.25$
β^{ows}	$\mathsf{OWS}_{seb}(jm) = 4$ $\mathsf{OWS}_{seb}(man) = 2.75$	$\mathsf{OWS}_{seb}(seb) = 4$ $\mathsf{OWS}_{seb}(pat) = 2.25$ $\mathsf{OWS}_{seb}(nic) = 2$
β^{pws}	$\mathsf{PWS}_{seb}(jm) = (6)$ $\mathsf{PWS}_{seb}(man) = (5)$	$\mathsf{PWS}_{seb}(seb) = (8)$ $\mathsf{PWS}_{seb}(pat) = (5)$ $\mathsf{PWS}_{seb}(nic) = (6)$
β^{wgm}	$\mathsf{WGM}_{seb}(jm) = 4$ $\mathsf{WGM}_{seb}(man) = 8.24$	$\mathsf{WGM}_{seb}(seb) = 5.66$ $\mathsf{WGM}_{seb}(pat) = 8.8$ $\mathsf{WGM}_{seb}(nic) = 6.45$
β^{ipd}	$\mathsf{IPD}_{seb}(jm) = 7.5$ $\mathsf{IPD}_{seb}(man) = 6.49$	$\mathsf{IPD}_{seb}(seb) = 8.16$ $\mathsf{IPD}_{seb}(pat) = 6.67$ $\mathsf{IPD}_{seb}(nic) = 7.75$
β^{wlmin}	$\mathsf{WLMin}_{seb}(jm) = (1.33, 124)$ $\mathsf{WLMin}_{seb}(man) = (9.3, 32)$	$\mathsf{WLMin}_{seb}(seb) = (8, 8)$ $\mathsf{WLMin}_{seb}(pat) = (12, 20)$ $\mathsf{WLMin}_{seb}(nic) = (0, 14.6)$
β^{wlmax}	$\mathsf{WLMax}_{seb}(jm) = (7.75, 0.75)$ $\mathsf{WLMax}_{seb}(man) = (5.25, 2)$	$\mathsf{WLMax}_{seb}(seb) = (4.5, 0.5)$ $\mathsf{WLMax}_{seb}(pat) = (6.75, 1.25)$ $\mathsf{WLMax}_{seb}(nic) = (8.25, 0)$

Table 2: Results of the targeting methods

Proposition 8 *The Purely Ordinal Weighted Sum method β^{pws} satisfies* **S1, S2, S3, S4, S8, S9, O4, O5**. *It does not satisfy* **S5, S6, S7, O1, O2** *and* **O3**.

Proposition 9 *The Weighted Geometric Mean method β^{wgm} satisfies* **S1, S2, S3, S4, S5, S6, S7, S8, S9, O4, O5**. *It does not satisfy* **O1, O2** *and* **O3**.

Proposition 10 *The Ideal Point Distance method β^{ipd} satisfies* **S1, S2, S3, S4, S8, S9, O4, O5**. *It does not satisfy* **S5, S6, S7, O1, O2** *and* **O3**.

Proposition 11 *The Weighted LexiMin method β^{wlmin} satisfies* **S1, S2, S3, S4, S5, S6, S7, S8, S9, O1, O3, O4, O5**. *It does not satisfy* **O2**.

Proposition 12 *The Weighted LexiMax method β^{wlmax} satisfies* **S1, S2, S3, S4, S5, S6, S7, S8, S9, O4, O5**. *It does not satisfy* **O1, O2, O3**.

These results are summarized in Table 1 (see the appendix for the proofs).

The Weighted Sum, Weighted Geometric Mean, Weighted LexiMin and Weighted LexiMax methods satisfy all the main properties. The more "ordinal" methods (Ordinal Weighted Sum and Pure Ordinal methods) must give up some of these properties. The Ideal Point Distance method, which operates differently by referring to a hypothetical best element, also fails to satisfy several main properties. Notably, Weighted LexiMin is the only method that satisfies the optional properties **O1** and **O3**.

Finally, note that **O4** is satisfied by all the proposed targeting functions. We do not include this property among the main ones, however, because its satisfaction follows directly from the normalization of the weights in our functions. Sensible targeting functions without weight normalization may exist, which is why we regard **O4** as an optional property.

Conclusion

In this paper, we introduced a novel multi-criteria decision-making problem, which we call *targeting*, where the objective is to identify the best elements of a set with respect to a given target element. We proposed a set of fundamental axioms that any sensible targeting method should satisfy, together with additional axioms that characterize specific subclasses of such methods. We then presented several candidate targeting methods and analyzed their formal properties.

For targeting to be meaningful, it should be applied in contexts where the criteria are relatively independent, so that focusing on the support of the target (and ignoring irrelevant criteria) is justified. Many practical applications, such as the laptop-configuration scenario presented in the introduction, naturally satisfy this requirement.

Among the methods we studied, four satisfy all mandatory properties: Weighted Sum, Weighted Geometric Mean, Weighted LexiMin, and Weighted LexiMax.

Our attempts to adopt more ordinal approaches, i.e., methods relying primarily on rankings rather than numerical values, were less successful, as the ordinal variants fail to satisfy three of the mandatory postulates. Designing a fully ordinal targeting method remains an interesting open problem.

Finally, we implemented these methods in the *Profile Expert Finder* tool (https://coscinus.org/pef/) of coscinus, which is designed to identify experts corresponding to a given computer science author.

Acknowledgements

This work was supported by the AI Chair BE4musIA (ANR-20-CHIA-0028); the ANR THEMIS (ANR-20-CE23-0018), funded by the French National Research Agency; and by the JSPS KAKENHI Grant Number JP25K00375.

References

Arrow, K. J. 1959. Rational Choice Functions and Orderings. *Economica*, 26(102): 121–127.

Bouyssou, D.; Marchant, T.; Pirlot, M.; Perny, P.; Tsoukiàs, A.; and Vincke, P. 2000. *Evaluation and Decision Models: A Critical Perspective*, volume 32 of *International Series in Operations Research and Management Science*. Kluwer Academic Publishers.

Demirović, E.; and Schwind, N. 2020. Representative Solutions for Bi-Objective Optimisation. In *Proceedings of the 24th AAAI Conference on Artificial Intelligence (AAAI'20)*, 1436–1443.

Fermé, E.; and Hansson, S. O. 2011. AGM 25 Years: Twenty-Five Years of Research in Belief Change. *Journal of Philosophical Logic*, 40(2): 295–331.

Gärdenfors, P. 1988. *Knowledge in flux*. MIT Press.

Greco, S.; Ehrgott, M.; and Figueira, J. 2016. *Multiple criteria decision analysis: state of the art surveys. 2nd edition*. Springer-Verlag.

Hebrard, E.; Hnich, B.; O'Sullivan, B.; and Walsh, T. 2005. Finding Diverse and Similar Solutions in Constraint Programming. In *Proceedings of the 20th AAAI Conference on Artificial Intelligence (AAAI'05)*, 372–377.

Kalai, E.; and Smorodinsky, M. 1975. Other Solutions to Nash's Bargaining Problem. *Econometrica*, 43(3): 513–518.

Luce, R. D.; and Raiffa, H. 1957. *Games and Decisions*. New York: Wiley.

Masin, M.; and Bukchin, Y. 2008. Diversity Maximization Approach for Multiobjective Optimization. *Operational Research*, 56(2): 411–424.

Nash, J. 1953. Two-Person Cooperative Games. *Econometrica*, 21(1): 128–140.

Petit, T.; and Trapp, A. C. 2015. Finding Diverse Solutions of High Quality to Constraint Optimization Problems. In *Proceedings of the 24th International Joint Conference on Artificial Intelligence (IJCAI'15)*, 260–267.

Schwind, N.; Okimoto, T.; Clement, M.; and Inoue, K. 2016. Representative Solutions for Multi-Objective Constraint Optimization Problems. In *Proceedings of the 15th International Conference on Principles of Knowledge Representation and Reasoning (KR'16)*, 601–604.

Schwind, N.; Okimoto, T.; Konieczny, S.; Wack, M.; and Inoue, K. 2014. Utilitarian and Egalitarian Solutions for Multi-objective Constraint Optimization. In *Proceedings of the 2014 IEEE 26th International Conference on Tools with Artificial Intelligence (ICTAI'14)*, 170–177.

Thomson, W. 1994. Cooperative models of bargaining. volume 2 of *Handbook of Game Theory with Economic Applications*, 1237–1284. Elsevier.

Vaz, D.; Paquete, L.; Fonseca, C. M.; Klamroth, K.; and Stiglmayr, M. 2015. Representation of the non-dominated set in biobjective discrete optimization. *Computers & Operations Research*, 63: 172–186.

Appendix: Proofs

In these proofs, we will note $\gamma_{[c_i(o)\to x]}(o) = \{c_1(o), \ldots, c_{i-1}(o), \quad x \quad, c_{i+1}(o), \quad \ldots, \quad c_m(o)\}$, i.e the vector where we replace the evaluation $c_i(o)$ by the value x (similarly $\gamma_{[c_i(o)\to x, c_j(o)\to y]}(o) = \{c_1(o), \ldots, c_{i-1}(o), \quad x \quad, c_{i+1}(o), \quad \ldots, c_{j-1}(o), y, c_{j+1}(o), \ldots, c_m(o)\}$).

Proposition 1 *If β satisfies* **S1**, *then it cannot satisfy both* **O1** *and* **O2**.

Proof: Let $\mathcal{C} = \{c_1, c_2\}$ and Ω be a set of objects containing $O = \{p, \mathbf{a}\}$, with $\gamma(p) = (1, 3)$ and $\gamma(\mathbf{a}) = (2, 2)$. Let β be a targeting method satisfying **S1**, and assume toward a contradiction that β satisfies **O1** and **O2**. Since $p \not\succeq_\star^{\mathsf{S}(\mathbf{a})} \mathbf{a}$ and $\mathbf{a} \in O$, from **O1** we get that $p \notin \beta_\mathbf{a}(O)$. Yet $\beta_\mathbf{a}(O) \subseteq O$ and **S1** requires that $\beta_\mathbf{a}(O) \neq \emptyset$, thus $\beta_\mathbf{a}(O) = \{\mathbf{a}\}$. On the other hand, with $i = 1$, $j = 2$ and $x = 1$, one can easily see that the preconditions of **O2** are satisfied: indeed, since $\mathsf{S}(\mathbf{a}) \setminus \{c_1, c_2\} = \emptyset$, we trivially get that $p \simeq_\star^{\mathsf{S}(\mathbf{a})\setminus\{c_1,c_2\}} \mathbf{a}$; we also have that $c_1(\mathbf{a}) = c_2(\mathbf{a})(= 2)$, that $c_1(p) = c_1(\mathbf{a}) - 1$, and that $c_2(p) = c_2(\mathbf{a}) + 1$. Yet we have shown that $p \notin \beta_\mathbf{a}(O)$, which by **O2** means that $\mathbf{a} \notin \beta_\mathbf{a}(O)$. But since $\beta_\mathbf{a}(O) \subseteq O$, this means that $\beta_\mathbf{a}(O) = \emptyset$, which contradicts **S1**. This shows that if β satisfy **S1**, then it cannot satisfy both **O1** and **O2**. □

Proposition 2 *If β satisfies* **O2**, *then it satisfies* **S6**.

Proof: Suppose $c_i(\mathbf{a}) = c_j(\mathbf{a})$, $p \simeq_\star^{\mathsf{S}(\mathbf{a})\setminus\{c_i,c_j\}} q$, $c_i(p) = c_j(q)$, $c_j(p) = c_i(q)$ and $p \in \beta_\mathbf{a}(O)$.

Then now let us note $x = c_i(p) - c_i(q) = c_j(q) - c_j(p)$.

Rewriting these equalities we have $c_i(q) = c_i(p) - x$ and $c_j(q) = c_j(p) + x$, that are the hypotheses of **O2**. Then by **O2** we obtain $q \in \beta_\mathbf{a}(O)$, that is also the conclusion we seek for **S6**. □

Proposition 3 *If β satisfies* **S1**, **S3** *and* **O1**, *then it satisfies* **O3**.

Proof: Assume that β satisfies **S1**, **S3** and **O1**. Let $\mathbf{a} \in PF_{\mathsf{S}(\mathbf{a})}(O)$ for some $O \subseteq \Omega$, we want to show that $\mathbf{a} \in \beta_\mathbf{a}(O)$. Note that $\mathbf{a} \in O$, so that $O \neq \emptyset$. The fact that $\mathbf{a} \in PF_{\mathsf{S}(\mathbf{a})}(O)$ precisely means that $p \not\succ_\star^{\mathsf{S}(\mathbf{a})} \mathbf{a}$ for every $p \in O$. By **S1**, $\beta_\mathbf{a}(O) \neq \emptyset$, so we know that there exists $p \in O$ such that $p \in \beta_\mathbf{a}(O)$. Then by **O1**, we get that $p \succeq_\star^{\mathsf{S}(\mathbf{a})} \mathbf{a}$. Together with the fact that $p \not\succ_\star^{\mathsf{S}(\mathbf{a})} \mathbf{a}$, we get that $p \simeq_\star^{\mathsf{S}(\mathbf{a})} \mathbf{a}$, and from **S3** we get that $\mathbf{a} \in \beta_\mathbf{a}(O)$, which concludes the proof that β satisfies **O3**. □

Proposition 4 *β satisfies* **S1** *and* **S5** *iff there is a targeting assignment $\mathbf{a} \mapsto \succeq_\mathbf{a}$ such that for each $\mathbf{a} \in \Omega$ and each $O \subseteq \Omega$, $\beta_\mathbf{a}(O) = \max(O, \succeq_\mathbf{a})$.*

Proof: (If part) Let β be defined for each $\mathbf{a} \in \Omega$ and each $O \subseteq \Omega$ as $\beta_\mathbf{a}(O) = \max(O, \succeq_\mathbf{a})$, where $\mathbf{a} \mapsto \preceq_\mathbf{a}$ is a targeting assignment. The proof that β satisfies **S1** is direct from the fact that Ω is a finite set. To prove that β satisfies **S5**, consider $O' \subseteq O$ such that $\beta_\mathbf{a}(O) \cap O' \neq \emptyset$. Let us first prove that $\beta_\mathbf{a}(O) \cap O' \subseteq \beta_\mathbf{a}(O')$. Let $p \in \beta_\mathbf{a}(O) \cap O'$. By definition of $\beta_\mathbf{a}(O)$, we have that $p \succeq_\mathbf{a} q$ for each $q \in O$, and in particular for each $q \in O'$ since $O' \subseteq O$. This means that $p \in \beta_\mathbf{a}(O')$. Hence, $\beta_\mathbf{a}(O) \cap O' \subseteq \beta_\mathbf{a}(O')$. Let us now prove that $\beta_\mathbf{a}(O') \subseteq \beta_\mathbf{a}(O) \cap O'$. Let $p \in \beta_\mathbf{a}(O')$ and assume toward a contradiction that $p \notin \beta_\mathbf{a}(O) \cap O'$, which means that $p \notin \beta_\mathbf{a}(O)$ since $p \in \beta_\mathbf{a}(O') \subseteq O'$. Now, we initially assumed that $\beta_\mathbf{a}(O) \cap O' \neq \emptyset$, so there exists $q \in \beta_\mathbf{a}(O) \cap O'$, which means by definition of $\beta_\mathbf{a}(O)$ that $q \succ_\mathbf{a} p$, and since $q \in O'$ this contradicts the fact that $p \in \max(O', \succeq_\mathbf{a})$, i.e., that $p \in \beta_\mathbf{a}(O')$. Hence, $\beta_\mathbf{a}(O') \subseteq \beta_\mathbf{a}(O) \cap O'$ and this concludes the proof that $\beta_\mathbf{a}(O) \cap O' = \beta_\mathbf{a}(O')$, i.e., that β satisfies **S5**.

(Only if part) Let β satisfy **S1** and **S5**, and from β let us define a mapping $\mathbf{a} \mapsto \succeq_\mathbf{a}$ associating every target $\mathbf{a} \in \Omega$ with a binary relation $\succeq_\mathbf{a} \subseteq \Omega \times \Omega$ as $p \succeq_\mathbf{a} q$ iff $p \in \beta_\mathbf{a}(\{p, q\})$, for all $p, q \in \Omega$. We first show that $\succeq_\mathbf{a}$ is a total preorder. Since $\beta_\mathbf{a}(\{p, q\}) \subseteq \{p, q\}$ and since from **S1** $\beta_\mathbf{a}(\{p, q\}) \neq \emptyset$, we get that $p \in \beta_\mathbf{a}(\{p, q\})$ or $q \in \beta_\mathbf{a}(\{p, q\})$, i.e., $p \succeq_\mathbf{a} q$ or $q \succeq_\mathbf{a} p$, which shows that $\succeq_\mathbf{a}$ is a total relation. It remains to be shown that $\succeq_\mathbf{a}$ is transitive. So let $p, q, r \in \Omega$, and assume toward a contradiction that $p \succeq_\mathbf{a} q$, $q \succeq_\mathbf{a} r$ and $p \not\succeq_\mathbf{a} r$. By definition of $\succeq_\mathbf{a}$, $\beta_\mathbf{a}(\{p, r\}) = \{r\}$. By **S5**, $\beta_\mathbf{a}(\{p, q, r\}) \cap \{p, r\} \subseteq \beta_\mathbf{a}(\{p, r\})$, and since $p \notin \beta_\mathbf{a}(\{p, r\})$ we get that $p \notin \beta_\mathbf{a}(\{p, q, r\})$. Note also that $\beta_\mathbf{a}(\{p, q, r\}) \subseteq \{p, q, r\}$. Assume first that $q \in \beta_\mathbf{a}(\{p, q, r\})$. By **S1**, $\beta_\mathbf{a}(\{p, q, r\}) \neq \emptyset$, so $\beta_\mathbf{a}(\{p, q, r\}) \cap \{p, q\} = \{q\}$, and by **S5** $\beta_\mathbf{a}(\{p, q, r\}) \cap \{p, q\} = \beta_\mathbf{a}(\{p, q\})$, so $\beta_\mathbf{a}(\{p, q\}) = \{q\}$, which means by definition of $\succeq_\mathbf{a}$ that $p \not\succeq_\mathbf{a} q$ and leads to a contradiction. So assume now that $q \notin \beta_\mathbf{a}(\{p, q, r\})$. Then by **S1** and since $\beta_\mathbf{a}(\{p, q, r\}) \cap \{p, q\} = \emptyset$, we get that $\beta_\mathbf{a}(\{p, q, r\}) = \{r\}$. But then by **S5** again $\beta_\mathbf{a}(\{p, q, r\}) \cap \{q, r\} = \beta_\mathbf{a}(\{q, r\})$, so $\beta_\mathbf{a}(\{q, r\}) = \{r\}$, which means by definition of $\succeq_\mathbf{a}$ that $q \not\succeq_\mathbf{a} r$ and also leads to a contradiction. Both cases lead to a contradiction, which means that $\succeq_\mathbf{a}$ is transitive.

We now want to show that $\beta_\mathbf{a}(O) = \max(O, \succeq_\mathbf{a})$. We first show the first inclusion $\beta_\mathbf{a}(O) \subseteq \max(O, \succeq_\mathbf{a})$. Let $p \in \beta_\mathbf{a}(O)$, assume toward a contradiction that $p \notin \max(O, \succeq_\mathbf{a})$. Then there exists $q \in O$ such that $q \succ_\mathbf{a} p$, i.e., $\beta_\mathbf{a}(\{p, q\}) = \{q\}$ by definition of $\succeq_\mathbf{a}$. Yet by **S5** $\beta_\mathbf{a}(O) \cap \{p, q\} = \beta_\mathbf{a}(\{p, q\}) = \{q\}$, which means that $p \notin \beta_\mathbf{a}(O)$ and leads to a contradiction. Hence, $\beta_\mathbf{a}(O) \subseteq \max(O, \succeq_\mathbf{a})$. Let us now prove the other inclusion $\max(O, \succeq_\mathbf{a}) \subseteq \beta_\mathbf{a}(O)$. Let $p \in \max(O, \succeq_\mathbf{a})$. Then for each $q \in O$, $p \succeq_\mathbf{a} q$, i.e., $p \in \beta_\mathbf{a}(\{p, q\})$ by definition of $\succeq_\mathbf{a}$. This is true in particular when $q \in \beta_\mathbf{a}(O)$ since $\beta_\mathbf{a}(O) \subseteq O$. But then by **S5**, $\beta_\mathbf{a}(O) \cap \{p, q\} = \beta_\mathbf{a}(\{p, q\})$, so $p \in \beta_\mathbf{a}(O)$. Hence, $\beta_\mathbf{a}(O) = \max(O, \succeq_\mathbf{a})$, which concludes the proof. □

Proposition 5 *Let β satisfies* **S1** *and* **S5** *and $\mathbf{a} \mapsto \succeq_\mathbf{a}$ be its corresponding targeting assignment. Then for each $i \in \{2, 3, 4, 6, 7, 8, 9\}$, $j \in \{1, 2, 3, 4, 5\}$, β satisfies* **Si** *(resp.* **Oj***) iff $\mathbf{a} \mapsto \succeq_\mathbf{a}$ satisfies the following condition (SRi) (resp. (ORj)):*

- *(SR2) If $\gamma(\mathbf{a}_1) = \gamma(\mathbf{a}_2)$, then $\succeq_{\mathbf{a}_1} = \succeq_{\mathbf{a}_2}$*
- *(SR3) If $p \simeq_\star^{\mathsf{S}(\mathbf{a})} q$, then $p \simeq_\mathbf{a} q$*
- *(SR4) If $p \succ_\star^{\mathsf{S}(\mathbf{a})} q$, then $p \succ_\mathbf{a} q$*
- *(SR6) If $c_i(\mathbf{a}) = c_j(\mathbf{a})$, $p \simeq_\star^{\mathsf{S}(\mathbf{a})\setminus\{c_i,c_j\}} q$, $c_i(p) = c_j(q)$ and $c_j(p) = c_i(q)$, then $p \simeq_\mathbf{a} q$*

(SR7) *If* $c_i(\mathbf{a}) > c_j(\mathbf{a})$, $p \simeq_\star^{\mathsf{S}(\mathbf{a})\setminus\{c_i,c_j\}} q$, $c_i(p) > c_j(q)$, $c_j(p) = c_i(q)$ *and* $c_i(p) = c_j(q)$, *then* $p \succ_\mathbf{a} q$

(SR8) *If* $\mathsf{S}(\mathbf{a}_1) \cap \mathsf{S}(\mathbf{a}_2) = \emptyset$, $\gamma(\mathbf{a}_3) = \gamma(\mathbf{a}_1) + \gamma(\mathbf{a}_2)$, $p \succeq_{\mathbf{a}_1} q$ *and* $p \succeq_{\mathbf{a}_2} q$, *then* $p \succeq_{\mathbf{a}_3} q$

(SR9) *If* $\mathsf{S}(\mathbf{a}_1) \cap \mathsf{S}(\mathbf{a}_2) = \emptyset$, $\gamma(\mathbf{a}_3) = \gamma(\mathbf{a}_1) + \gamma(\mathbf{a}_2)$, $p \succeq_{\mathbf{a}_1} q$ *and* $p \succ_{\mathbf{a}_2} q$, *then* $p \succ_{\mathbf{a}_3} q$

(OR1) *If* $p \not\succeq_\star^{\mathsf{S}(\mathbf{a})} \mathbf{a}$, *then* $\mathbf{a} \succ_\mathbf{a} p$

(OR2) *If* $p \simeq_\star^{\mathsf{S}(\mathbf{a})\setminus\{c_i,c_j\}} q$, $c_i(\mathbf{a}) = c_j(\mathbf{a})$, $c_i(q) = c_i(p) - x$ *and* $c_j(q) = c_j(p) + x$, *then* $p \simeq_\mathbf{a} q$

(OR3) *If* $p \not\succ_\star^{\mathsf{S}(\mathbf{a})} \mathbf{a}$, *then* $\mathbf{a} \succeq_\mathbf{a} p$

(OR4) *If* $\gamma(\mathbf{a}_1) = \mu \times \gamma(\mathbf{a}_2)$, *then* $\succeq_{\mathbf{a}_1} = \succeq_{\mathbf{a}_2}$

(OR5) *If* $\gamma(\mathbf{a}_2) = \pi(\gamma(\mathbf{a}_1))$, $\gamma(p') = \pi(\gamma(p))$, $\gamma(q') = \pi(\gamma(q))$ *and* $p \succeq_{\mathbf{a}_1} q$, *then* $p' \succeq_{\mathbf{a}_2} q'$

Proof: Let β satisfy **S1** and **S5** and $\mathbf{a} \mapsto \succeq_\mathbf{a}$ be its corresponding targeting assignment.
Recall that $\mathbf{a} \mapsto \succeq_\mathbf{a}$ is defined for all $\mathbf{a}, p, q \in \Omega$ as $p \succeq_\mathbf{a} q$ iff $p \in \beta_\mathbf{a}(\{p, q\})$.

(If part) Recall from proposition 4 that β is characterized from $\mathbf{a} \mapsto \succeq_\mathbf{a}$ as $\beta_\mathbf{a}(O) = \max(O, \succeq_\mathbf{a})$, for each $\mathbf{a} \in \Omega$ and each $O \subseteq \Omega$. We want to show that for each $i \in \{2, 3, 4, 6, 7, 8, 9\}$, $j \in \{1, 2, 3, 4, 5\}$, if $\mathbf{a} \mapsto \succeq_\mathbf{a}$ satisfies the condition *(SRi)* (resp. *(ORj)*), then β satisfies **Si** (resp. **Oj**).

Assume $\mathbf{a} \mapsto \succeq_\mathbf{a}$ satisfies *(SR2)*, and let us show that β satisfies **S2**. Let $\mathbf{a}_1, \mathbf{a}_2 \in \Omega$, and assume that $\gamma(\mathbf{a}_1) = \gamma(\mathbf{a}_2)$. Let $O \subseteq \Omega$. By *(SR2)*, $\succeq_{\mathbf{a}_1} = \succeq_{\mathbf{a}_2}$, so $\max(O, \succeq_{\mathbf{a}_1}) = \max(O, \succeq_{\mathbf{a}_2})$, thus by definition of β, we get that $\beta_{\mathbf{a}_1}(O) = \beta_{\mathbf{a}_2}(O)$. Hence, β satisfies **S2**.

Assume $\mathbf{a} \mapsto \succeq_\mathbf{a}$ satisfies *(SR3)*, and let us show that β satisfies **S3**. Let $\mathbf{a} \in \Omega$, $O \subseteq \Omega$, $p, q \in O$, and assume that $p \simeq_\star^{\mathsf{S}(\mathbf{a})} q$ and $p \in \beta_\mathbf{a}(O)$. By definition of β, $p \in \max(O, \succeq_\mathbf{a})$, and since by *(SR3)*, $p \simeq_\mathbf{a} q$, we also get that $q \in \max(O, \succeq_\mathbf{a})$. Thus, by definition of β again, $q \in \beta_\mathbf{a}(O)$. Hence, β satisfies **S3**.

Assume $\mathbf{a} \mapsto \succeq_\mathbf{a}$ satisfies *(SR4)*, and let us show that β satisfies **S4**. Let $\mathbf{a} \in \Omega$, $O \subseteq \Omega$, $p, q \in O$, and assume that $p \succ_\star^{\mathsf{S}(\mathbf{a})} q$. By *(SR4)*, $p \succ_\mathbf{a} q$, so by definition of β we get that $q \notin \max(O, \succeq_\mathbf{a})$. Thus, by definition of β again, $q \notin \beta_\mathbf{a}(O)$. Hence, β satisfies **S4**.

Assume $\mathbf{a} \mapsto \succeq_\mathbf{a}$ satisfies *(SR6)*, and let us show that β satisfies **S6**. Let $\mathbf{a} \in \Omega$, $O \subseteq \Omega$, $p, q \in O$, $c_i, c_j \in \mathcal{C}$, and assume that $c_i(\mathbf{a}) = c_j(\mathbf{a})$, $p \simeq_\star^{\mathsf{S}(\mathbf{a})\setminus\{c_i,c_j\}} q$, $c_i(p) = c_j(q)$, $c_j(p) = c_i(q)$, and $p \in \beta_\mathbf{a}(O)$. By definition of β, $p \in \max(O, \succeq_\mathbf{a})$, and since by *(SR6)*, $p \simeq_\mathbf{a} q$, we also get that $q \in \max(O, \succeq_\mathbf{a})$. Thus, by definition of β again, $q \in \beta_\mathbf{a}(O)$. Hence, β satisfies **S6**.

Assume $\mathbf{a} \mapsto \succeq_\mathbf{a}$ satisfies *(SR7)*, and let us show that β satisfies **S7**. Let $\mathbf{a} \in \Omega$, $O \subseteq \Omega$, $p, q \in O$, $c_i, c_j \in \mathcal{C}$, and assume that $c_i(\mathbf{a}) > c_j(\mathbf{a})$, $p \simeq_\star^{\mathsf{S}(\mathbf{a})\setminus\{c_i,c_j\}} q$, $c_i(p) > c_j(p)$, $c_j(p) = c_i(q)$, and $c_i(p) = c_j(q)$. By *(SR7)*, $p \succ_\mathbf{a} q$, so by definition of β we get that $q \notin \max(O, \succeq_\mathbf{a})$. Thus, by definition of β again, $q \notin \beta_\mathbf{a}(O)$. Hence, β satisfies **S7**.

Assume $\mathbf{a} \mapsto \succeq_\mathbf{a}$ satisfies *(SR8)*, and let us show that β satisfies **S8**. Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \Omega$, and assume that $\mathsf{S}(\mathbf{a}_1) \cap \mathsf{S}(\mathbf{a}_2) = \emptyset$ and $\gamma(\mathbf{a}_3) = \gamma(\mathbf{a}_1) + \gamma(\mathbf{a}_2)$. Let $O \subseteq \Omega$, we want to show that $\beta_{\mathbf{a}_1}(O) \cap \beta_{\mathbf{a}_2}(O) \subseteq \beta_{\mathbf{a}_3}(O)$. So, let $p \in \beta_{\mathbf{a}_1}(O) \cap \beta_{\mathbf{a}_2}(O)$, and let us show that $p \in \beta_{\mathbf{a}_3}(O)$. By definition of β and since $p \in \beta_{\mathbf{a}_1}(O) \cap \beta_{\mathbf{a}_2}(O)$, we know that $p \in \max(O, \succeq_{\mathbf{a}_1})$ and $p \in \max(O, \succeq_{\mathbf{a}_2})$, which means that for each $q \in O$, $p \succeq_{\mathbf{a}_1} q$ and $p \succeq_{\mathbf{a}_2} q$. So, by *(SR8)*, we get for each $q \in O$ that $p \succeq_{\mathbf{a}_3} q$. This means that $p \in \max(O, \succeq_{\mathbf{a}_3})$, and by definition of β again, that $p \in \beta_{\mathbf{a}_3}(O)$. This shows that $\beta_{\mathbf{a}_1}(O) \cap \beta_{\mathbf{a}_2}(O) \subseteq \beta_{\mathbf{a}_3}(O)$, i.e., that β satisfies **S8**.

Assume $\mathbf{a} \mapsto \succeq_\mathbf{a}$ satisfies *(SR9)*, and let us show that β satisfies **S9**. Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \Omega$, and assume that $\mathsf{S}(\mathbf{a}_1) \cap \mathsf{S}(\mathbf{a}_2) = \emptyset$ and $\gamma(\mathbf{a}_3) = \gamma(\mathbf{a}_1) + \gamma(\mathbf{a}_2)$. Let $O \subseteq \Omega$, assume that $\beta_{\mathbf{a}_1}(O) \cap \beta_{\mathbf{a}_2}(O) \neq \emptyset$. This means that there exists an object $p \in O$ such that $p \in \beta_{\mathbf{a}_1}(O) \cap \beta_{\mathbf{a}_2}(O)$, i.e., by definition of β, that $p \in \max(O, \succeq_{\mathbf{a}_1})$ and $p \in \max(O, \succeq_{\mathbf{a}_2})$. Now, we want to show that $\beta_{\mathbf{a}_3}(O) \subseteq \beta_{\mathbf{a}_1}(O) \cap \beta_{\mathbf{a}_2}(O)$. So, let $q \notin \beta_{\mathbf{a}_1}(O) \cap \beta_{\mathbf{a}_2}(O)$ and let us show that $q \notin \beta_{\mathbf{a}_3}(O)$. , and let us show that $p \in \beta_{\mathbf{a}_1}(O) \cap \beta_{\mathbf{a}_2}(O)$. Since $q \notin \beta_{\mathbf{a}_1}(O) \cap \beta_{\mathbf{a}_2}(O)$, this means that $q \notin \beta_{\mathbf{a}_1}(O)$ or $q \notin \beta_{\mathbf{a}_2}(O)$. Let us assume that $q \notin \beta_{\mathbf{a}_2}(O)$ (the other case when $q \notin \beta_{\mathbf{a}_1}(O)$ can be proved in a similar way since $\mathbf{a}_1$ and $\mathbf{a}_2$ play symmetrical roles.) So, on the one hand, by definition of β, $q \notin \max(O, \succeq_{\mathbf{a}_2})$. Yet $p \in \max(O, \succeq_{\mathbf{a}_2})$, which means that $p \succ_{\mathbf{a}_2} q$. On the other hand, $p \in \max(O, \succeq_{\mathbf{a}_1})$, $p \succeq_{\mathbf{a}_2} q$. Then by *(SR9)*, we know that $p \succ_{\mathbf{a}_3} q$. By definition of β again, this means that $q \notin \max(O, \succeq_{\mathbf{a}_3})$, and concludes the proof that $\beta_{\mathbf{a}_3}(O) \subseteq \beta_{\mathbf{a}_1}(O) \cap \beta_{\mathbf{a}_2}(O)$. Hence, β satisfies **S9**.

Assume $\mathbf{a} \mapsto \succeq_\mathbf{a}$ satisfies *(OR1)*, and let us show that β satisfies **O1**. Let $\mathbf{a} \in \Omega$, $O \subseteq \Omega$, $p \in O$, and assume that $p \in \beta_\mathbf{a}(O)$ and $\mathbf{a} \in O$. By definition of β, we know that $p \in \max(O, \succeq_\mathbf{a})$. This means that $\mathbf{a} \not\succ_\mathbf{a} p$. Then the contrapositive statement of *(OR1)* tells us that $p \succeq_\star^{\mathsf{S}(\mathbf{a})} \mathbf{a}$. Hence, β satisfies **O1**.

Assume $\mathbf{a} \mapsto \succeq_\mathbf{a}$ satisfies *(OR2)*, and let us show that β satisfies **O2**. Let $\mathbf{a} \in \Omega$, $O \subseteq \Omega$, $c_i, c_j \in C$, $x \in \mathbb{R}$, $p, q \in O$, and assume that $p \simeq_\star^{\mathsf{S}(\mathbf{a})\setminus\{c_i,c_j\}} q$, $c_i(\mathbf{a}) = c_j(\mathbf{a})$, $c_i(q) = c_i(p) - x$, $c_j(q) = c_j(p) + x$ and $p \in \beta_\mathbf{a}(O)$. By definition of β, $p \in \max(O, \succeq_\mathbf{a})$, and since by *(OR2)*, $p \simeq_\mathbf{a} q$, we also get that $q \in \max(O, \succeq_\mathbf{a})$. Thus, by definition of β again, $q \in \beta_\mathbf{a}(O)$. Hence, β satisfies **O2**.

Assume $\mathbf{a} \mapsto \succeq_\mathbf{a}$ satisfies *(OR3)*, and let us show that β satisfies **O3**. Let $\mathbf{a} \in \Omega$, $O \subseteq \Omega$, and assume that $\mathbf{a} \in PF_{\mathsf{S}(\mathbf{a})}(O)$. Then, for each object $p \in O$, $p \not\succ_\star^{\mathsf{S}(\mathbf{a})} \mathbf{a}$. By *(OR3)*, we get that for each object $p \in O$, $\mathbf{a} \succeq_\mathbf{a} p$, i.e., $\mathbf{a} \in \max(O, \succeq_\mathbf{a})$. By definition of β, this means that $\mathbf{a} \in \beta_\mathbf{a}(O)$. Hence, β satisfies **O3**.

Assume $\mathbf{a} \mapsto \succeq_\mathbf{a}$ satisfies *(OR4)*, and let us show that β satisfies **O4**. Let $\mathbf{a}_1, \mathbf{a}_2 \in \Omega$, $\mu \in \mathbb{Q}^*$, and assume that $\gamma(\mathbf{a}_1) = \mu \times \gamma(\mathbf{a}_2)$. By *(OR4)*, we get that $\succeq_{\mathbf{a}_1} = \succeq_{\mathbf{a}_2}$. This means that for each $O \subseteq \Omega$, we have that $\max(O, \succeq_{\mathbf{a}_1}) = \max(O, \succeq_{\mathbf{a}_2})$, i.e., $\succeq_{\mathbf{a}_1} = \succeq_{\mathbf{a}_2}$. Hence, β satisfies **O4**.

Assume $\mathbf{a} \mapsto \succeq_\mathbf{a}$ satisfies *(OR5)*, and let us show that β satisfies **O5**. Let $\mathbf{a}_1, \mathbf{a}_2 \in \Omega$, $O, O' \subseteq \Omega$, $p \in O$, $p' \in$

O', π be a permutation of $\{1,\ldots,m\}$, and assume that $\gamma(\mathbf{a}_2)=\pi(\gamma(\mathbf{a}_1))$, $\gamma(O')=\pi(\gamma(O))$, $\gamma(p')=\pi(\gamma(p))$ and $p\in\beta_{\mathbf{a}_1}(O)$. By definition of β, $p\in\max(O,\succeq_{\mathbf{a}_1})$, and thus $p\succeq_{\mathbf{a}_1} q$, for each $q\in O$. Then by *(OR5)*, we get that $p'\succeq_{\mathbf{a}_2} q'$, for each object $q'\in\Omega$ such that $\gamma(q')=\pi(\gamma(q))$. Yet by definition of O', such objects q' are such that $q\in O'$. So we got that $p'\succeq_{\mathbf{a}_2} q'$, for each object $q'\in O'$. Since $p'\in O'$, this means that $p'\in\max(O',\succeq_{\mathbf{a}_2})$, and by definition of β that $p'\in\beta_{\mathbf{a}_2}(O')$. Hence, β satisfies **O5**.

This concludes the (if) part of the proof.

(Only if part) Since the targeting assignment $\mathbf{a}\mapsto\succeq_{\mathbf{a}}$ corresponds to β, recall that it is defined for all $\mathbf{a},p,q\in\Omega$ as $p\succeq_{\mathbf{a}} q$ iff $p\in\beta_{\mathbf{a}}(\{p,q\})$. We want to show that for each $i\in\{2,3,4,6,7,8,9\}$, $j\in\{1,2,3,4,5\}$, if β satisfies **Si** (resp. **Oj**), then $\mathbf{a}\mapsto\succeq_{\mathbf{a}}$ satisfies the condition *(SRi)* (resp. *(ORj)*).

Assume β satisfies **S2**, and let us show that $\mathbf{a}\mapsto\succeq_{\mathbf{a}}$ satisfies *(SR2)*. Let $\mathbf{a}_1,\mathbf{a}_2\in\Omega$, and assume that $\gamma(\mathbf{a}_1)=\gamma(\mathbf{a}_2)$. We need to show that $\succeq_{\mathbf{a}_1}=\succeq_{\mathbf{a}_2}$, i.e., that for all $p,q\in\Omega$, $p\succeq_{\mathbf{a}_1} q$ iff $p\succeq_{\mathbf{a}_2} q$. So, let $p,q\in\Omega$. By **S2**, $\beta_{\mathbf{a}_1}(\{p,q\})=\beta_{\mathbf{a}_2}(\{p,q\})$. Yet by definition of $\succeq_{\mathbf{a}_1}$ and $\succeq_{\mathbf{a}_2}$, we know that $p\succeq_{\mathbf{a}_1} q$ iff $p\in\beta_{\mathbf{a}_1}(\{p,q\})$ and $p\succeq_{\mathbf{a}_2} q$ iff $p\in\beta_{\mathbf{a}_2}(\{p,q\})$. Thus, we got that $p\succeq_{\mathbf{a}_1} q$ iff $p\succeq_{\mathbf{a}_2} q$. Hence, $\mathbf{a}\mapsto\succeq_{\mathbf{a}}$ satisfies *(SR2)*.

Assume β satisfies **S3**, and let us show that $\mathbf{a}\mapsto\succeq_{\mathbf{a}}$ satisfies *(SR3)*. Let $\mathbf{a},p,q\in\Omega$, and assume that $p\simeq_\star^{\mathsf{S}(\mathbf{a})} q$. By **S3**, if $p\in\beta_{\mathbf{a}}(\{p,q\})$, then $q\in\beta_{\mathbf{a}}(\{p,q\})$. And since p and q play symmetrical roles, we also get that if $q\in\beta_{\mathbf{a}}(\{p,q\})$, then $p\in\beta_{\mathbf{a}}(\{p,q\})$. That is, $p\in\beta_{\mathbf{a}}(\{p,q\})$ iff $q\in\beta_{\mathbf{a}}(\{p,q\})$. Yet by definition of $\succeq_{\mathbf{a}}$, $p\succeq_{\mathbf{a}} q$ iff $p\in\beta_{\mathbf{a}}(\{p,q\})$, and $q\succeq_{\mathbf{a}} p$ iff $q\in\beta_{\mathbf{a}}(\{p,q\})$. Thus, we get that $p\succeq_{\mathbf{a}} q$ iff $q\succeq_{\mathbf{a}} p$. Yet $\succeq$ is a total relation, which means that $p\succeq_{\mathbf{a}} q$ and $q\succeq_{\mathbf{a}} p$, i.e., $p\simeq_{\mathbf{a}} q$. Hence, $\mathbf{a}\mapsto\succeq_{\mathbf{a}}$ satisfies *(SR3)*.

Assume β satisfies **S4**, and let us show that $\mathbf{a}\mapsto\succeq_{\mathbf{a}}$ satisfies *(SR4)*. Let $\mathbf{a},p,q\in\Omega$, and assume that $p\succ_\star^{\mathsf{S}(\mathbf{a})} q$. By **S4**, $q\notin\beta_{\mathbf{a}}(\{p,q\})$. By definition of $\succeq_{\mathbf{a}}$, this means that $q\not\succeq_{\mathbf{a}} p$, which means that $p\succ_{\mathbf{a}} q$ since $\succeq_{\mathbf{a}}$ is a total relation. Hence, $\mathbf{a}\mapsto\succeq_{\mathbf{a}}$ satisfies *(SR4)*.

Assume β satisfies **S6**, and let us show that $\mathbf{a}\mapsto\succeq_{\mathbf{a}}$ satisfies *(SR6)*. Let $\mathbf{a},p,q\in\Omega$, $c_i,c_j\in\mathcal{C}$, and assume that $c_i(\mathbf{a})=c_j(\mathbf{a})$, $p\simeq_\star^{\mathsf{S}(\mathbf{a})\setminus\{c_i,c_j\}} q$, $c_i(p)=c_j(q)$, and $c_j(p)=c_i(q)$. By **S6**, if $p\in\beta_{\mathbf{a}}(\{p,q\})$, then $q\in\beta_{\mathbf{a}}(\{p,q\})$. And since p and q play symmetrical roles, we also get that if $q\in\beta_{\mathbf{a}}(\{p,q\})$, then $p\in\beta_{\mathbf{a}}(\{p,q\})$. That is, $p\in\beta_{\mathbf{a}}(\{p,q\})$ iff $q\in\beta_{\mathbf{a}}(\{p,q\})$. Yet by definition of $\succeq_{\mathbf{a}}$, $p\succeq_{\mathbf{a}} q$ iff $p\in\beta_{\mathbf{a}}(\{p,q\})$, and $q\succeq_{\mathbf{a}} p$ iff $q\in\beta_{\mathbf{a}}(\{p,q\})$. Thus, we get that $p\succeq_{\mathbf{a}} q$ iff $q\succeq_{\mathbf{a}} p$. Yet $\succeq$ is a total relation, which means that $p\succeq_{\mathbf{a}} q$ and $q\succeq_{\mathbf{a}} p$, i.e., $p\simeq_{\mathbf{a}} q$. Hence, $\mathbf{a}\mapsto\succeq_{\mathbf{a}}$ satisfies *(SR6)*.

Assume β satisfies **S7**, and let us show that $\mathbf{a}\mapsto\succeq_{\mathbf{a}}$ satisfies *(SR7)*. Let $\mathbf{a},p,q\in\Omega$, $c_i,c_j\in\mathcal{C}$, and assume that $c_i(\mathbf{a})>c_j(\mathbf{a})$, $p\simeq_\star^{\mathsf{S}(\mathbf{a})\setminus\{c_i,c_j\}} q$, $c_i(p)>c_j(q)$, $c_j(p)=c_i(q)$ and $c_i(p)=c_j(q)$. By **S7**, $q\notin\beta_{\mathbf{a}}(\{p,q\})$. By definition of $\succeq_{\mathbf{a}}$, this means that $q\not\succeq_{\mathbf{a}} p$, which means that $p\succ_{\mathbf{a}} q$ since $\succeq_{\mathbf{a}}$ is a total relation. Hence, $\mathbf{a}\mapsto\succeq_{\mathbf{a}}$ satisfies *(SR7)*.

Assume β satisfies **S8**, and let us show that $\mathbf{a}\mapsto\succeq_{\mathbf{a}}$ satisfies *(SR8)*. Let $\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3\in\Omega$, and assume that $\mathsf{S}(\mathbf{a}_1)\cap\mathsf{S}(\mathbf{a}_2)=\emptyset$ and $\gamma(\mathbf{a}_3)=\gamma(\mathbf{a}_1)+\gamma(\mathbf{a}_2)$. Let $p,q\in\Omega$ and assume that $p\succeq_{\mathbf{a}_1} q$ and $p\succeq_{\mathbf{a}_2} q$. We want to show that $p\succeq_{\mathbf{a}_3} q$. Yet by definition of $\succeq_{\mathbf{a}_1}$ and $\succeq_{\mathbf{a}_2}$, $p\succeq_{\mathbf{a}_1} q$ and $p\succeq_{\mathbf{a}_2} q$ imply that $p\in\beta_{\mathbf{a}_1}(\{p,q\})$ and $p\in\beta_{\mathbf{a}_2}(\{p,q\})$. Then by **S8**, we get that $p\in\beta_{\mathbf{a}_3}(\{p,q\})$. And by definition of $\succeq_{\mathbf{a}_3}$, this means that $p\succeq_{\mathbf{a}_3} q$. Hence, $\mathbf{a}\mapsto\succeq_{\mathbf{a}}$ satisfies *(SR8)*.

Assume β satisfies **S9**, and let us show that $\mathbf{a}\mapsto\succeq_{\mathbf{a}}$ satisfies *(SR9)*. Let $\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3\in\Omega$, and assume that $\mathsf{S}(\mathbf{a}_1)\cap\mathsf{S}(\mathbf{a}_2)=\emptyset$ and $\gamma(\mathbf{a}_3)=\gamma(\mathbf{a}_1)+\gamma(\mathbf{a}_2)$. Let $p,q\in\Omega$ and assume that $p\succeq_{\mathbf{a}_1} q$ and $p\succ_{\mathbf{a}_2} q$. We want to show that $p\succ_{\mathbf{a}_3} q$. B definition of $\succeq_{\mathbf{a}_1}$ and $\succeq_{\mathbf{a}_2}$, $p\succeq_{\mathbf{a}_1} q$ and $p\succ_{\mathbf{a}_2} q$ imply that (i) $p\in\beta_{\mathbf{a}_1}(\{p,q\})$, (ii) $p\in\beta_{\mathbf{a}_2}(\{p,q\})$, and (iii) $q\notin\beta_{\mathbf{a}_2}(\{p,q\})$. In particular, (i) and (ii) mean that $\beta_{\mathbf{a}_1}(\{p,q\})\cap\beta_{\mathbf{a}_2}(\{p,q\})\neq\emptyset$. Then by **S9**, we get that (iv) $\beta_{\mathbf{a}_3}(\{p,q\})\subseteq\beta_{\mathbf{a}_1}(\{p,q\})\cap\beta_{\mathbf{a}_2}(\{p,q\})$. Recall that we want to show that $p\succ_{\mathbf{a}_3} q$, so Assume toward a contradiction that $q\succeq_{\mathbf{a}_3} p$. By definition of $\succeq_{\mathbf{a}_3}$, this would means that $q\in\beta_{\mathbf{a}_3}(\{p,q\})$, and then by (iv) that $q\in\beta_{\mathbf{a}_2}(\{p,q\})$, which contradicts (iii). Thus, $p\succ_{\mathbf{a}_3} q$. Hence, $\mathbf{a}\mapsto\succeq_{\mathbf{a}}$ satisfies *(SR9)*.

Assume β satisfies **O1**, and let us show that $\mathbf{a}\mapsto\succeq_{\mathbf{a}}$ satisfies *(OR1)*. Let $\mathbf{a},p\in\Omega$, and assume that $p\not\succeq_\star^{\mathsf{S}(\mathbf{a})}\mathbf{a}$. Using the contrapositive statement of **O1**, we get that $p\notin\beta_{\mathbf{a}}(\{\mathbf{a},p\})$. By definition of $\succeq_{\mathbf{a}}$, this means that $p\not\succeq_{\mathbf{a}}\mathbf{a}$, i.e., $\mathbf{a}\succ_{\mathbf{a}} p$. Hence, $\mathbf{a}\mapsto\succeq_{\mathbf{a}}$ satisfies *(OR1)*.

Assume β satisfies **O2**, and let us show that $\mathbf{a}\mapsto\succeq_{\mathbf{a}}$ satisfies *(OR2)*. Let $\mathbf{a},p,q\in\Omega$, $c_i,c_j\in\mathcal{C}$, $x\in\mathbb{R}$, and assume that $p\simeq_\star^{\mathsf{S}(\mathbf{a})\setminus\{c_i,c_j\}} q$, $c_i(\mathbf{a})=c_j(\mathbf{a})$, $c_i(q)=c_i(p)-x$ and $c_j(q)=c_j(p)+x$. By **S1**, $\beta_{\mathbf{a}}(\{p,q\})\neq\emptyset$, thus $p\in\beta_{\mathbf{a}}(\{p,q\})$ or $q\in\beta_{\mathbf{a}}(\{p,q\})$. Assume the first case, i.e., (i) $p\in\beta_{\mathbf{a}}(\{p,q\})$ (the other case where $q\in\beta_{\mathbf{a}}(\{p,q\})$ can be proved similarly since p and q play symmetrical roles.) By definition of $\succeq_{\mathbf{a}}$, we know that (ii) $p\succeq_{\mathbf{a}} q$. And by (i) and **O2**, we also have that $q\in\beta_{\mathbf{a}}(\{p,q\})$, i.e., (iii) $q\succeq_{\mathbf{a}} p$ by definition of $\succeq_{\mathbf{a}}$. From (ii) and (iii), we get that $p\simeq_{\mathbf{a}} q$. Hence, $\mathbf{a}\mapsto\succeq_{\mathbf{a}}$ satisfies *(OR2)*.

Assume β satisfies **O3**, and let us show that $\mathbf{a}\mapsto\succeq_{\mathbf{a}}$ satisfies *(OR3)*. Let $\mathbf{a},p\in\Omega$, and assume that $p\not\succ_\star^{\mathsf{S}(\mathbf{a})}\mathbf{a}$. This precisely means that $\mathbf{a}\in PF_{\mathsf{S}(\mathbf{a})}(O)$, so by **O3**, we get that $\mathbf{a}\in\beta_{\mathbf{a}}(\{\mathbf{a},p\})$. And by definition of $\succeq_{\mathbf{a}}$, this means that $\mathbf{a}\succeq_{\mathbf{a}} p$. Hence, $\mathbf{a}\mapsto\succeq_{\mathbf{a}}$ satisfies *(OR3)*.

Assume β satisfies **O4**, and let us show that $\mathbf{a}\mapsto\succeq_{\mathbf{a}}$ satisfies *(OR4)*. Let $\mathbf{a}_1,\mathbf{a}_2\in\Omega$, $\mu\in\mathbb{Q}^*$, and assume that $\gamma(\mathbf{a}_1)=\mu\times\gamma(\mathbf{a}_2)$. We want to show that $\succeq_{\mathbf{a}_1}=\succeq_{\mathbf{a}_2}$, i.e., that for each $p,q\in\Omega$, $p\succeq_{\mathbf{a}_1} q$ if and only if $p\succeq_{\mathbf{a}_2} q$. Yet for each $p,q\in\Omega$, we know that $p\succeq_{\mathbf{a}_1} q$ if and only if $p\in\beta_{\mathbf{a}_1}(\{p,q\})$ (by definition of $\succeq_{\mathbf{a}_1}$) if and only if $q\in\beta_{\mathbf{a}_2}(\{p,q\})$ (by **O4**) if and only if $p\succeq_{\mathbf{a}_2} q$. (by definition of $\succeq_{\mathbf{a}_2}$). This shows that $\succeq_{\mathbf{a}_1}=\succeq_{\mathbf{a}_2}$. Hence, $\mathbf{a}\mapsto\succeq_{\mathbf{a}}$ satisfies *(OR4)*.

Assume β satisfies **O5**, and let us show that $\mathbf{a}\mapsto\succeq_{\mathbf{a}}$ satisfies *(OR5)*. Let $\mathbf{a}_1,\mathbf{a}_2\in\Omega$, $p,p',q,q'\in\Omega$, π be a permutation of $\{1, \ldots, m\}$, and assume that (i) $\gamma(\mathbf{a}_2) = \pi(\gamma(\mathbf{a}_1))$, (ii) $\gamma(p') = \pi(\gamma(p))$, (iii) $\gamma(q') = \pi(\gamma(q))$ and (iv) $p \succeq_{\mathbf{a}_1} q$. Note that on the one hand, by (ii) and (iii), we get that $\gamma(\{p', q'\}) = \pi(\gamma(\{p, q\}))$. On the other hand, by definition of $\succeq_{\mathbf{a}_1}$ we get that $p \in \beta_{\mathbf{a}_1}(\{p, q\}$. Then by **O5**, we get that $p' \in \beta_{\mathbf{a}_2}(\{p', q'\}$, which by definition of $\succeq_{\mathbf{a}_2}$ means that $p' \succeq_{\mathbf{a}_2} q'$. Hence, $\mathbf{a} \mapsto \succeq_{\mathbf{a}}$ satisfies *(OR5)*.

This concludes the (only if) part of the proof, and the proof of the proposition. All correspondences between (Si) and (SRi) for each $i \in \{2, 3, 4, 5, 7, 8\}$ and between (Oj) and (ORj) for each $j \in \{2, 3\}$ are direct from the fact that $\beta_{\mathbf{a}}(O) = \max(O, \succeq_{\mathbf{a}})$, for each $O \subseteq \Omega$ and each $\mathbf{a} \in \Omega$ (cf. Proposition 4). The correspondence between **O1** and *(OR1)* can be better understood by considering the contrapositive of *(OR1)*: if $p \succeq_{\mathbf{a}} \mathbf{a}$, then $p \succeq_{\star}^{\mathsf{S}(\mathbf{a})} \mathbf{a}$. $\square$

For the following proofs, let stated a lemma that will be helpful for the property **O4**:

Lemma 1 *Suppose that* $\gamma(\mathbf{a}_1) = \mu \times \gamma(\mathbf{a}_2)$ *for* $\mu \in \mathbb{Q}^*$. *Then* $w_i(\mathbf{a}_1) = w_i(\mathbf{a}_2)$

Proof: First note that if $\mu \times \gamma(\mathbf{a}_2)$, $\mathsf{S}(\mathbf{a}_1) = \mathsf{S}(\mathbf{a}_2)$.

$$w_i(\mathbf{a}_1) = c_i(\mathbf{a}_1) / \sum_{x \in \mathsf{S}(\mathbf{a}_1)} c_x(\mathbf{a}_1)$$

$$w_i(\mathbf{a}_1) = \mu \times c_i(\mathbf{a}_2) / \sum_{x \in \mathsf{S}(\mathbf{a}_2)} \mu \times c_x(\mathbf{a}_2)$$

$$w_i(\mathbf{a}_1) = \mu \times c_i(\mathbf{a}_2) / \mu \times \sum_{x \in \mathsf{S}(\mathbf{a}_2)} c_x(\mathbf{a}_2)$$

$$w_i(\mathbf{a}_1) = c_i(\mathbf{a}_2) / \sum_{x \in \mathsf{S}(\mathbf{a}_2)} c_x(\mathbf{a}_2) = w_i(\mathbf{a}_2)$$

$\square$

Proposition 6 *The Weighted Sum satisfies* **S1, S2, S3, S4, S5, S6, S7, S8, S9, O2, O4, O5**. *It does not satisfy* **O1** *or* **O3**.

Proof: **S1, S2, S5** and **O5** are straightforward.

For **S3**, if $c_i \notin \mathsf{S}(\mathbf{a}), c_i(\mathbf{a}) = 0$. So, if $p \simeq_{\star}^{\mathsf{S}(\mathbf{a})} q$, $\sum_i w_i(\mathbf{a}) \times c_i(p) = \sum_i w_i(\mathbf{a}) \times c_i(q)$ and $p \in \beta_{\mathbf{a}}^{ws}(O)$ implies $q \in \beta_{\mathbf{a}}^{ws}(O)$.

For **S4**, suppose $p \succ_{\star}^{\mathsf{S}(\mathbf{a})} q$. If $c \notin \mathsf{S}(\mathbf{a})$, $c(\mathbf{a}) = 0$ we have to consider only the criteria in $\mathsf{S}(\mathbf{a})$. As $p \succ_{\star}^{\mathsf{S}(\mathbf{a})} q$, $\forall c_i \in \mathsf{S}(\mathbf{a})$, $c_i(p) \leq c_i(q)$ and $\exists c \in \mathsf{S}(\mathbf{a})$ s.t. $c(p) < c(q)$. So $\sum_j w_j(\mathbf{a}) \times c_j(p) < \sum_j w_j(\mathbf{a}) \times c_j(q)$ and $p \notin \beta_{\mathbf{a}}^{ws}(O)$.

For **S6**, suppose $c_i(a) = c_j(a)$, $p \simeq_{\star}^{\mathsf{S}(\mathbf{a}) \setminus \{c_i, c_j\}} q$, $c_i(p) = c_j(q)$ and $c_j(p) = c_i(q)$. As $c_i(\mathbf{a}) = c_j(\mathbf{a})$, $w_i(\mathbf{a}) = w_j(\mathbf{a})$.

$\mathsf{BWS}_{\mathbf{a}}(p) = \sum_k w_k(\mathbf{a}) \times c_k(p)$

$\mathsf{BWS}_{\mathbf{a}}(p) = \sum_{k \neq i,j} w_k(\mathbf{a}) \times c_k(p) + w_i(\mathbf{a}) \times c_i(p) + w_j(\mathbf{a}) \times c_j(p)$

$\mathsf{BWS}_{\mathbf{a}}(p) = \sum_{k \neq i,j} w_k(\mathbf{a}) \times c_k(p) + w_j(\mathbf{a}) \times c_i(p) + w_i(\mathbf{a}) \times c_j(p)$ as $w_i(\mathbf{a}) = w_j(\mathbf{a})$.

$\mathsf{BWS}_{\mathbf{a}}(p) = \sum_{k \neq i,j} w_k(\mathbf{a}) \times c_k(p) + w_j(\mathbf{a}) \times c_j(q) + w_i(\mathbf{a}) \times c_i(q)$ as $c_i(p) = c_j(q)$ and $c_j(p) = c_i(q)$.

$\mathsf{BWS}_{\mathbf{a}}(p) = \sum_{k \neq i,j} w_k(\mathbf{a}) \times c_k(q) + w_j(\mathbf{a}) \times c_j(q) + w_i(\mathbf{a}) \times c_i(q)$ as $p \simeq_{\star}^{\mathsf{S}(\mathbf{a}) \setminus \{c_i, c_j\}} q$.

So $\mathsf{BWS}_{\mathbf{a}}(p) = \mathsf{BWS}_{\mathbf{a}}(q)$ and the result holds.

For **S7**, suppose $c_i(a) > c_j(a)$, $c_j(p) = c_i(q)$, $c_i(q) = c_j(q)$ and $p \simeq_{\star}^{\mathsf{S}(\mathbf{a}) \setminus \{c_i, c_j\}} q$, then $\sum_{k \neq i,j} w_k(\mathbf{a}) \times c_k(p) = \sum_{k \neq i,j} w_k(\mathbf{a}) \times c_k(q)$. We only need to compare $w_i(\mathbf{a}) \times c_i(p) + w_j(\mathbf{a}) \times c_j(p)$ and $w_i(\mathbf{a}) \times c_i(q) + w_j(\mathbf{a}) \times c_j(q)$:

$$w_i(\mathbf{a}) \times c_i(p) + w_j(\mathbf{a}) \times c_j(p) - [w_i(\mathbf{a}) \times c_i(q) + w_j(\mathbf{a}) \times c_j(q)] =$$

$$w_i(\mathbf{a}) \times [c_i(p) - c_i(q)] + w_j(\mathbf{a}) \times [c_j(p) - c_j(q)] =$$

$$w_i(\mathbf{a}) \times [c_i(p) - c_j(p)] + w_j(\mathbf{a}) \times [c_j(p) - c_i(p)] \quad (1)$$

as $c_i(p) = c_j(q)$ and $c_j(p) = c_i(q)$.

Equality (1) is then equivalent to:

$$(c_i(p) - c_j(p)) \times [w_i(\mathbf{a}) - w_j(\mathbf{a})]$$

As $c_i(p) > c_j(p)$, $c_i(p) - c_j(p) > 0$.

As $c_i(\mathbf{a}) > c_j(\mathbf{a})$, $w_i(\mathbf{a}) > w_j(\mathbf{a})$ and $w_i(\mathbf{a}) - w_j(\mathbf{a}) > 0$.

So $(c_i(p) - c_j(p)) \times [w_i(\mathbf{a}) - w_j(\mathbf{a})] > 0$ and $w_i(\mathbf{a}) \times c_i(p) + w_j(\mathbf{a}) \times c_j(p) > w_i(\mathbf{a}) \times c_i(q) + w_j(\mathbf{a}) \times c_j(q)$. As a consequence, $q \notin \beta_{\mathbf{a}}^{ws}(O)$.

For **S8**, let $\mathsf{S}(\mathbf{a}_1) \cap \mathsf{S}(\mathbf{a}_2) = \emptyset$, $\gamma(\mathbf{a}_3) = \gamma(\mathbf{a}_1) + \gamma(\mathbf{a}_2)$, $p \in \beta_{\mathbf{a}_1}^{ws}(O)$ and $p \in \beta_{\mathbf{a}_2}^{ws}(O)$. Let $q \in O$:

$\mathsf{BWS}_{\mathbf{a}_3}(q) = \sum_k w_k(\mathbf{a}_3) \times c_k(q)$

As $\mathsf{S}(\mathbf{a}_1) \cap \mathsf{S}(\mathbf{a}_2) = \emptyset$ and $\gamma(\mathbf{a}_3) = \gamma(\mathbf{a}_1) + \gamma(\mathbf{a}_2)$, we have:

$\mathsf{BWS}_{\mathbf{a}_3}(q) = \sum_k w_k(\mathbf{a}_1) \times c_k(q) + \sum_k w_k(\mathbf{a}_2) \times c_k(q)$

As $p \in \beta_{\mathbf{a}_1}^{ws}(O)$ and $p \in \beta_{\mathbf{a}_2}^{ws}(O)$, this sum is maximal for the object p among all object in O and $p \in \beta_{\mathbf{a}_3}^{ws}(O)$.

For **S9**, let $\mathsf{S}(\mathbf{a}_1) \cap \mathsf{S}(\mathbf{a}_2) = \emptyset$, $\gamma(\mathbf{a}_3) = \gamma(\mathbf{a}_1) + \gamma(\mathbf{a}_2)$, and $\beta_{\mathbf{a}_1}^{ws}(O) \cap \beta_{\mathbf{a}_2}^{ws}(O) \neq \emptyset$.

We already know that $\beta_{\mathbf{a}_1}^{ws}(O) \cap \beta_{\mathbf{a}_2}^{ws}(O) \subseteq \beta_{\mathbf{a}_3}^{ws}(O)$

As stated in the proof of **S8**, we have $\forall q \in O$:

$\mathsf{BWS}_{\mathbf{a}_3}(q) = \mathsf{BWS}_{\mathbf{a}_1}(q) + \mathsf{BWS}_{\mathbf{a}_2}(q)$ Suppose $q \in \beta_{\mathbf{a}_3}^{ws}(O)$ and $q \notin \beta_{\mathbf{a}_1}^{ws}(O) \cap \beta_{\mathbf{a}_2}^{ws}(O)$. Let $p \in \beta_{\mathbf{a}_1}^{ws}(O) \cap \beta_{\mathbf{a}_2}^{ws}(O)$. Then either $\mathsf{BWS}_{\mathbf{a}_1}(q) < \mathsf{BWS}_{\mathbf{a}_1}(p)$ or $\mathsf{BWS}_{\mathbf{a}_2}(q) < \mathsf{BWS}_{\mathbf{a}_2}(p)$. As furthermore $\forall \mathsf{BWS}_{\mathbf{a}_1}(q) \leq \mathsf{BWS}_{\mathbf{a}_1}(p)$ and $\mathsf{BWS}_{\mathbf{a}_2}(q) \leq \mathsf{BWS}_{\mathbf{a}_2}(p)$, we get $\mathsf{BWS}_{\mathbf{a}_1}(q) + \mathsf{BWS}_{\mathbf{a}_2}(q) < \mathsf{BWS}_{\mathbf{a}_1}(p) + \mathsf{BWS}_{\mathbf{a}_2}(p)$ and $q \notin \beta_{\mathbf{a}_3}^{ws}(O)$: contradiction and **S9** is satisfied.

For **O2**, let $p \simeq_{\star}^{\mathsf{S}(\mathbf{a} \setminus \{c_i, c_j\})} q$ and $c_i(\mathbf{a}) = c_j(\mathbf{a})$, $c_i(q) = c_i(p) - x$, $c_j(q) = c_j(p) + x$ and $p \in\in \beta_{\mathbf{a}}^{ws}(O)$.

As $p \in \beta_{\mathbf{a}}^{ws}(O)$, $\forall r \in O$, $\sum_{c_k \in S(\mathbf{a})} w_k(\mathbf{a}) \times c_k(p) \geq \sum_{c_k \in S(\mathbf{a})} w_k(\mathbf{a}) \times c_k(r)$.

We know that $\gamma(q) = \gamma_{[c_i(p) \to (c_i(p) - x), c_j(p) \to (c_j(p) + x)]}(p)$, so $\sum_{c_k \in S(\mathbf{a}), k \neq i,j} w_k(\mathbf{a}) \times c_k(p) = \sum_{c_k \in S(\mathbf{a}), k \neq i,j} w_k(\mathbf{a}) \times c_k(q)$. As $c_i(q) = c_i(p) - x$ and $c_j(q) = c_j(p) + x$, we have $c_i(q) \times w_i(\mathbf{a}) + c_j(q) \times w_j(\mathbf{a}) = (c_i(p) - x) \times w_i(\mathbf{a}) + (c_j(p) + x) \times w_j(a)$. As $c_i(\mathbf{a}) = c_j(\mathbf{a})$, $w_i(\mathbf{a}) = w_j(\mathbf{a})$. Then $(c_i(p) - x) \times w_i(\mathbf{a}) + (c_j(p) + x) \times w_j(a) = c_i(p) \times w_i(\mathbf{a}) + c_j(p) \times w_j(a) - x \times w_j(a) + x \times w_j(a) = c_i(p) \times w_i(\mathbf{a}) + c_j(p) \times w_j(a)$. Then $\sum_{c_k \in S(\mathbf{a})} w_k(\mathbf{a}) \times c_k(p) = \sum_{c_k \in S(\mathbf{a})} w_k(\mathbf{a}) \times c_k(q)$, and $\forall r \in O$, $\sum_{c_k \in \mathsf{S}(\mathbf{a})} w_k(\mathbf{a}) \times c_k(q) \geq \sum_{c_k \in \mathsf{S}(\mathbf{a})} w_k(\mathbf{a}) \times c_k(r)$: $q \in \beta_{\mathbf{a}}^{ws}(O)$.

For **O1**, consider $O = \{\mathbf{a}, p\}$ with $\gamma(\mathbf{a}) = (2, 2)$ and $\gamma(p) = (10, 1)$. $w_1(\mathbf{a}) \times c_1(\mathbf{a}) + w_2(\mathbf{a}) \times c_2(\mathbf{a}) = \frac{1}{2} \times 2 + \frac{1}{2} \times 2 < \frac{1}{2} \times 10 + \frac{1}{2} \times 1 = w_1(\mathbf{a}) \times c_1(p) + w_2(\mathbf{a}) \times c_2(p)$ so $p \in \beta^{ws}_{\mathbf{a}}(O)$ but $c_2(p) < c_2(\mathbf{a})$: $p \not\succeq^{\mathsf{S}(\mathbf{a})}_{\star}$ $\mathbf{a}$. **O1** is not satisfied.

For **O3**, consider $O = \{\mathbf{a}, p\}$ with $\gamma(\mathbf{a}) = (2, 2)$ and $\gamma(p) = (10, 1)$. $\mathbf{a}$ is in the Pareto Front on $\mathsf{S}(\mathbf{a})$, but $w_1(\mathbf{a}) \times c_1(\mathbf{a}) + w_2(\mathbf{a}) \times c_2(\mathbf{a}) = \frac{1}{2} \times 2 + \frac{1}{2} \times 2 < \frac{1}{2} \times 10 + \frac{1}{2} \times 1 = w_1(\mathbf{a}) \times c_1(p) + w_2(\mathbf{a}) \times c_2(p)$: $\mathbf{a} \notin \beta^{ws}_{\mathbf{a}}(O)$. **O3** is not satisfied. As **O1**, if **S1** and **S3** are satisfied, entails **O3**, **O1** is not satisfied.

For **O4**, the proof is straightforward from lemma 1. □

Proposition 7 *β^{ows} satisfies* **S1, S2, S3, S4, S8, S9, O4, O5**. *It does not satisfy* **S5, S6, S7, O1, O2, O3**.

Proof: **S1**, **S2** and **O5** are straightforward.

For **S3**, if $c_i \notin \mathsf{S}(\mathbf{a})$, $w_i(\mathbf{a}) = 0$. So the evaluation of p and q with respect to $\mathsf{OWS}_{\mathbf{a}}$ are the same and the conclusion holds.

For **S4**, suppose $p \succ^{\mathsf{S}(\mathbf{a})}_{\star} q$. If $c \notin \mathsf{S}(\mathbf{a})$, $c(\mathbf{a}) = 0$ we have to consider only the criteria in $\mathsf{S}(\mathbf{a})$. As $p \succ^{\mathsf{S}(\mathbf{a})}_{\star} q$, $\forall c_i \in \mathsf{S}(\mathbf{a})$, $c_i(p) \le c_i(q)$ and $\exists c \in \mathsf{S}(\mathbf{a})$ s.t. $c(p) < c(q)$. Then $\forall c_i \in \mathsf{S}(\mathbf{a})$, $r_{c_i}(p) \ge r_{c_i}(q)$ and $\exists c \in \mathsf{S}(\mathbf{a})$ s.t. $r_c(p) > r_c(q)$. As a consequence, $\mathsf{OWS}_{\mathbf{a}}(q) < \mathsf{OWS}_{\mathbf{a}}(p)$: $p \notin \beta^{ows}_{\mathbf{a}}(O)$.

For **S5**, consider the counter-example: $O = \{\mathbf{a}, p, q, r_1, r_2\}$, $\gamma(\mathbf{a}) = (1, 1, 1)$, $\gamma(p) = (1, 2, 3)$, $\gamma(q) = (3, 2, 1)$, $\gamma(r_1) = (2, 2, 1)$ and $\gamma(r_2) = (1, 1, 2)$.
$\mathsf{OWS}_{\mathbf{a}}(\mathbf{a}) = \sum_i w_i(\mathbf{a}) \times r_{c_i}(\mathbf{a}) = 3 \times \frac{1}{3} + 3 \times \frac{1}{3} + 3 \times \frac{1}{3} = 3$.
$\mathsf{OWS}_{\mathbf{a}}(p) = \sum_i w_i(\mathbf{a}) \times r_{c_i}(p) = 3 \times \frac{1}{3} + 2 \times \frac{1}{3} + 1 \times \frac{1}{3} = 2$.
$\mathsf{OWS}_{\mathbf{a}}(q) = \sum_i w_i(\mathbf{a}) \times r_{c_i}(q) = 1 \times \frac{1}{3} + 2 \times \frac{1}{3} + 3 \times \frac{1}{3} = 2$.
$\mathsf{OWS}_{\mathbf{a}}(r_1) = \sum_i w_i(\mathbf{a}) \times r_{c_i}(r_1) = 2 \times \frac{1}{3} + 2 \times \frac{1}{3} + 3 \times \frac{1}{3} = \frac{7}{3}$.
$\mathsf{OWS}_{\mathbf{a}}(r_2) = \sum_i w_i(\mathbf{a}) \times r_{c_i}(r_2) = 3 \times \frac{1}{3} + 3 \times \frac{1}{3} + 2 \times \frac{1}{3} = \frac{8}{3}$.
$p \in \beta^{ows}_{\mathbf{a}}(O)$.

Consider $O' = \{a, p, q, r_1\} \subseteq O$. $\beta_{\mathbf{a}}(O) \cap O' \neq \emptyset$, more precisely $p \in \beta^{ows}_{\mathbf{a}}(O) \cap O'$. We have:
$\mathsf{OWS}_{\mathbf{a}}(\mathbf{a}) = \sum_i w_i(\mathbf{a}) \times r_{c_i}(\mathbf{a}) = 3 \times \frac{1}{3} + 3 \times \frac{1}{3} + 2 \times \frac{1}{3} = \frac{8}{3}$.
$\mathsf{OWS}_{\mathbf{a}}(p) = \sum_i w_i(\mathbf{a}) \times r_{c_i}(p) = 3 \times \frac{1}{3} + 2 \times \frac{1}{3} + 1 \times \frac{1}{3} = 2$.
$\mathsf{OWS}_{\mathbf{a}}(q) = \sum_i w_i(\mathbf{a}) \times r_{c_i}(q) = 1 \times \frac{1}{3} + 2 \times \frac{1}{3} + 2 \times \frac{1}{3} = \frac{5}{3}$.
$\mathsf{OWS}_{\mathbf{a}}(r_1) = \sum_i w_i(\mathbf{a}) \times r_{c_i}(r_1) = 2 \times \frac{1}{3} + 2 \times \frac{1}{3} + 2 \times \frac{1}{3} = 2$.
$\mathsf{OWS}_{\mathbf{a}}(O') = \{q\}$ and $p \notin \beta^{ows}_{\mathbf{a}}(O')$. **S5** is not satisfied.

For **S6**, consider the counter-example: $O = \{\mathbf{a}, p, q, r\}$, $\gamma(\mathbf{a}) = (2, 1, 2)$, $\gamma(p) = (2, 3, 3)$, $\gamma(q) = (3, 3, 2)$ and $\gamma(r) = (4, 0, 2.5)$.

On this example, $c_1(\mathbf{a}) = c_3(\mathbf{a})$, $p \simeq^{\mathsf{S}(\mathbf{a} \setminus \{c_1, c_3\})}_{\star} p$, $c_1(p) = c_3(q)$ and $c_3(p) = c_1(q)$.
$\mathsf{OWS}_{\mathbf{a}}(\mathbf{a}) = \sum_i w_i(\mathbf{a}) \times r_{c_i}(\mathbf{a}) = 3 \times \frac{2}{5} + 2 \times \frac{1}{5} + 3 \times \frac{2}{5} = \frac{14}{5}$.
$\mathsf{OWS}_{\mathbf{a}}(p) = \sum_i w_i(\mathbf{a}) \times r_{c_i}(p) = 3 \times \frac{2}{5} + 1 \times \frac{1}{5} + 1 \times \frac{2}{5} = \frac{9}{5}$.
$\mathsf{OWS}_{\mathbf{a}}(q) = \sum_i w_i(\mathbf{a}) \times r_{c_i}(q) = 2 \times \frac{2}{5} + 1 \times \frac{1}{5} + 3 \times \frac{2}{5} = \frac{11}{5}$.
$\mathsf{OWS}_{\mathbf{a}}(r) = \sum_i w_i(\mathbf{a}) \times r_{c_i}(r) = 1 \times \frac{2}{5} + 3 \times \frac{1}{5} + 2 \times \frac{2}{5} = \frac{9}{5}$.
$p \in \beta^{ows}_{\mathbf{a}}(O)$, but $q \notin \beta^{ows}_{\mathbf{a}}(O)$, so **S6** is not satisfied.

For **S7**, consider the counter-example: $O = \{\mathbf{a}, p, q, r\}$, $\gamma(\mathbf{a}) = (2, 1, 2)$, $\gamma(p) = (3, 2, 3)$, $\gamma(q) = (2, 3, 3)$ and $\gamma(r) = (4, 2.5, 1)$. On this example, $c_1(\mathbf{a}) = c_2(\mathbf{a})$, $p \simeq^{\mathsf{S}(\mathbf{a} \setminus \{c_1, c_2\})}_{\star} q$, $c_1(p) > c_2(p)$, $c_2(p) = c_1(q)$ and $c_1(p) = c_2(q)$. $\mathsf{OWS}_{\mathbf{a}}(\mathbf{a}) = \sum_i w_i(\mathbf{a}) \times r_{c_i}(\mathbf{a}) = 3 \times \frac{2}{5} + 4 \times \frac{1}{5} + 2 \times \frac{2}{5} = \frac{14}{5}$. $\mathsf{OWS}_{\mathbf{a}}(p) = \sum_i w_i(\mathbf{a}) \times r_{c_i}(p) = 2 \times \frac{2}{5} + 3 \times \frac{1}{5} + 1 \times \frac{2}{5} = \frac{9}{5}$. $\mathsf{OWS}_{\mathbf{a}}(q) = \sum_i w_i(\mathbf{a}) \times r_{c_i}(q) = 3 \times \frac{2}{5} + 1 \times \frac{1}{5} + 1 \times \frac{2}{5} = \frac{9}{5}$. $\mathsf{OWS}_{\mathbf{a}}(r) = \sum_i w_i(\mathbf{a}) \times r_{c_i}(r) = 1 \times \frac{2}{5} + 2 \times \frac{1}{5} + 3 \times \frac{2}{5} = \frac{10}{5}$.

$c_1(\mathbf{a}) > c_2(\mathbf{a})$, $p \simeq^{\mathsf{S}(\mathbf{a}) \setminus \{c_1, c_2\}}_{\star} q$, $c_1(p) > c_2(p)$, $c_2(p) = c_1(q)$ and $c_1(p) = c_2(q)$. Nevertheless, $q \in \beta^{ows}_{\mathbf{a}}(O)$, so **S7** is not satisfied.

For **S8**, let $\mathsf{S}(\mathbf{a}_1) \cap \mathsf{S}(\mathbf{a}_2) = \emptyset$, $\gamma(\mathbf{a}_3) = \gamma(\mathbf{a}_1) + \gamma(\mathbf{a}_2)$, $p \in \beta^{ows}_{\mathbf{a}_1}(O)$ and $p \in \beta^{ows}_{\mathbf{a}_2}(O)$. Let $q \in O$:
$\mathsf{OWS}_{\mathbf{a}_3}(q) = \sum_k w_k(\mathbf{a}_3) \times r_{c_k}(q)$

As $\mathsf{S}(\mathbf{a}_1) \cap \mathsf{S}(\mathbf{a}_2) = \emptyset$ and $\gamma(\mathbf{a}_3) = \gamma(\mathbf{a}_1) + \gamma(\mathbf{a}_2)$, we have:
$\mathsf{OWS}_{\mathbf{a}_3}(q) = \sum_k w_k(\mathbf{a}_1) \times r_{c_k}(q) + \sum_k w_k(\mathbf{a}_2) \times r_{c_k}(q)$
As $p \in \beta^{ows}_{\mathbf{a}_1}(O)$ and $p \in \beta^{ows}_{\mathbf{a}_2}(O)$, this sum is maximal for the object p among all object in O and $p \in \beta^{ows}_{\mathbf{a}_3}(O)$.

For **S9**, let $\mathsf{S}(\mathbf{a}_1) \cap \mathsf{S}(\mathbf{a}_2) = \emptyset$, $\gamma(\mathbf{a}_3) = \gamma(\mathbf{a}_1) + \gamma(\mathbf{a}_2)$, and $\beta^{ows}_{\mathbf{a}_1}(O) \cap \beta^{ows}_{\mathbf{a}_2}(O) \neq \emptyset$.

We already know that $\beta^{ows}_{\mathbf{a}_1}(O) \cap \beta^{ows}_{\mathbf{a}_2}(O) \subseteq \beta^{ows}_{\mathbf{a}_3}(O)$

As stated in the proof of **S8**, we have $\forall q \in O$:

$\mathsf{OWS}_{\mathbf{a}_3}(q) = \mathsf{OWS}_{\mathbf{a}_1}(q) + \mathsf{OWS}_{\mathbf{a}_2}(q)$ Suppose $q \in \beta^{ows}_{\mathbf{a}_3}(O)$ and $q \notin \beta^{ows}_{\mathbf{a}_1}(O) \cap \beta^{ows}_{\mathbf{a}_2}(O)$. Let $p \in \beta^{ows}_{\mathbf{a}_1}(O) \cap \beta^{ows}_{\mathbf{a}_2}(O)$. Then either $\mathsf{OWS}_{\mathbf{a}_1}(q) < \mathsf{OWS}_{\mathbf{a}_1}(p)$ or $\mathsf{OWS}_{\mathbf{a}_2}(q) < \mathsf{OWS}_{\mathbf{a}_2}(p)$. As furthermore $\forall \mathsf{OWS}_{\mathbf{a}_1}(q) \le \mathsf{OWS}_{\mathbf{a}_1}(p)$ and $\mathsf{OWS}_{\mathbf{a}_2}(q) \le \mathsf{OWS}_{\mathbf{a}_2}(p)$, we get $\mathsf{OWS}_{\mathbf{a}_1}(q) + \mathsf{OWS}_{\mathbf{a}_2}(q) < \mathsf{OWS}_{\mathbf{a}_1}(p) + \mathsf{OWS}_{\mathbf{a}_2}(p)$ and $q \notin \beta^{ows}_{\mathbf{a}_3}(O)$: contradiction and **S9** is satisfied. For **O1**, consider the counter-example: $O = \{\mathbf{a}, p\}$, $\gamma(\mathbf{a}) = (2, 2)$ and $\gamma(p) = (3, 1)$: $p \in \beta^{ows}_{\mathbf{a}}(O)$ but $c_2(p) < c_2(\mathbf{a})$: $p \not\succeq^{\mathsf{S}(\mathbf{a})}_{\star}$ $\mathbf{a}$.

For **O2**, let $O = \{\mathbf{a}, p, q, r\}$ with $\gamma(\mathbf{a}) = (2, 2)$, $\gamma(p) = (4, 1)$, $\gamma(q) = (1, 4)$ and $\gamma(r) = (3, 2)$. $p \in \beta^{ows}_{\mathbf{a}}(O)$, $\gamma(q) = \gamma_{[c_1(p) \to c_1(p) - 3, c_2(p) \to c_2(p) + 3]}(p)$ but $q \notin \beta^{ows}_{\mathbf{a}}(O)$.

For **O3**, consider $\mathbf{a}$ with $\gamma(\mathbf{a}) = (3, 2)$ and p s.t. $\gamma(p) = (4, 1)$. $\mathbf{a} \in PF_{\mathsf{S}(\mathbf{a})}(O)$ but $\beta^{ows}_{\mathbf{a}}(O) = \{p\}$ and $\mathbf{a} \notin \beta^{ows}_{\mathbf{a}}(O)$. As **S1** and **S3** are satisfied and not **O3**, **O1** is not satisfied.

For **O4**, from lemma 1, we know that $w_i(\mathbf{a}_1) = w_i(\mathbf{a}_2)$. So clearly, $\forall p \in O, \mathsf{OWS}_{\mathbf{a}_1}(p) = \mathsf{OWS}_{\mathbf{a}_2}(p)$, $\beta^{ows}_{\mathbf{a}_1}(O) = \beta^{ows}_{\mathbf{a}_2}(O)$ and **O4** is satisfied. □

Proposition 8 *β^{pws} satisfies* **S1, S2, S3, S4, S8, S9, O4, O5**. *It does not satisfy* **S5, S6, S7, O1, O2, O3**.

Proof: **S1** and **O5** are straightforward.

For **S2**, if $\gamma(\mathbf{a}_1) = \gamma(\mathbf{a}_2)$, then the ranks of the criteria c_i are the same for $\mathbf{a}_1$ and $\mathbf{a}_2$, and so the evaluation of the elements of O for β^{pws}. Then $\beta^{pws}_{\mathbf{a}_1}(O) = \beta^{pws}_{\mathbf{a}_2}(O)$.

For **S3**, if $p \simeq^{\mathsf{S}(\mathbf{a})}_{\star} q$, the ranks vectors of p and q are identical on the support of $\mathbf{a}$. $p \in \beta^{pws}_{\mathbf{a}}(O)$ entails $q \in \beta^{pws}_{\mathbf{a}}(O)$ and **S3** is satisfied.

For **S4**, suppose $p \succ^{\mathsf{S}(\mathbf{a})}_{\star} q$. Only the criteria of $\mathsf{S}(\mathbf{a})$ are considered. $\forall c_j \in \mathsf{S}(\mathbf{a}), c_j(p) \le c_j(q)$ and $r_{c_j}(p) \ge r_{c_j}(q)$. $\exists c_i \in \mathsf{S}(\mathbf{a}), c_j(p) < c_j(q)$ and $r_{c_j}(p) > r_{c_j}(q)$. Then $\mathsf{PWS}_{\mathbf{a}}(p) >_{lex} \mathsf{PWS}_{\mathbf{a}}(q)$ and $p \notin \beta^{pws}_{\mathbf{a}}(O)$: **S4** is satisfied.

For **S5**, consider the set $O = \{\mathbf{a}, p, q, r_1, r_2\}$ and $O' = \{\mathbf{a}, p, q, r_2\}$ with $\gamma(\mathbf{a}) = (1,1,1)$, $\gamma(p) = (1,2,3)$, $\gamma(q) = (3,2,1)$, $\gamma(r_1) = (2,2,1)$ and $\gamma(r_2) = (1,1,2)$. On O, $r_{c_1}(\mathbf{a}) = r_{c_2}(\mathbf{a}) = r_{c_3}(\mathbf{a})$. On O, the ranks of $\mathbf{a}$ are $(3,2,3)$ so $\mathsf{PWS}_{\mathbf{a}}(\mathbf{a}) = (8)$. The ranks of p are $(3,1,1)$ so $\mathsf{PWS}_{\mathbf{a}}(p) = (5)$. The ranks of q are $(1,1,3)$ so $\mathsf{PWS}_{\mathbf{a}}(q) = (5)$. The ranks of r_1 are $(2,1,3)$ so $\mathsf{PWS}_{\mathbf{a}}(r_1) = (6)$ and the ranks of r_2 are $(3,2,2)$ so $\mathsf{PWS}_{\mathbf{a}}(r_2) = (7)$. Then $\beta^{pws}_{\mathbf{a}}(O) = \{p, q\}$ and $\beta^{pws}_{\mathbf{a}}(O) \cup O' \neq \emptyset$. For $\mathbf{a}$ on O', $r_{c_1}(\mathbf{a}) = r_{c_2}(\mathbf{a}) < r_{c_3}(\mathbf{a})$ On O', the ranks of $\mathbf{a}$ are $(2,2,3)$ so $\mathsf{PWS}_{\mathbf{a}}(\mathbf{a}) = (4,3)$. The ranks of p are $(2,1,1)$ so $\mathsf{PWS}_{\mathbf{a}}(p) = (3,1)$. The ranks of q are $(1,1,3)$ so $\mathsf{PWS}_{\mathbf{a}}(q) = (2,3)$ and the ranks of r_2 are $(2,2,2)$ so $\mathsf{PWS}_{\mathbf{a}}(r_2) = (4,2)$. $\beta^{pws}_{\mathbf{a}}(O') = \{q\}$ and $\beta^{pws}_{\mathbf{a}}(O) \cup O' \neq \beta^{pws}_{\mathbf{a}}(O')$: **S5** is not satisfied.

For **S6**, consider the counter-example: $O = \{\mathbf{a}, p, q, r\}$, $\gamma(\mathbf{a}) = (2,2,2)$, $\gamma(p) = (2,3,3)$, $\gamma(q) = (3,3,2)$ and $\gamma(r) = (4,0,2.5)$.

On this example, $c_1(\mathbf{a}) = c_2(\mathbf{a}) = c_3(\mathbf{a})$, $p \simeq^{\mathsf{S}(\mathbf{a} \setminus \{c_1,c_3\})}_{\star} p$, $c_1(p) = c_3(q)$ and $c_3(p) = c_1(q)$. The three criteria c_1, c_2 and c_3 are first for the target $\mathbf{a}$. $\mathsf{PWS}_{\mathbf{a}}(\mathbf{a}) = (3+2+3) = (8)$.

$\mathsf{PWS}_{\mathbf{a}}(p) = (3+1+1) = (5)$.

$\mathsf{PWS}_{\mathbf{a}}(q) = (2+1+3) = (6)$.

$\mathsf{PWS}_{\mathbf{a}}(r) = (1+3+2) = (6)$.

$p \in \beta^{pws}_{\mathbf{a}}(O)$, but $q \notin \beta^{pws}_{\mathbf{a}}(O)$, so **S6** is not satisfied.

For **S7**, consider $O = \{\mathbf{a}, p, q\}$ with $\gamma(\mathbf{a}) = (3,2)$, $\gamma(p) = (5,4)$ and $\gamma(q) = (4,5)$. On O, $r_{c_1}(\mathbf{a}) = r_{c_2}(\mathbf{a})$. On O, the ranks of $\mathbf{a}$ are $(3,3)$ so $\mathsf{PWS}_{\mathbf{a}}(\mathbf{a}) = (6)$. The ranks of p are $(1,2)$ so $\mathsf{PWS}_{\mathbf{a}}(p) = (3)$. The ranks of q are $(2,1)$ so $\mathsf{PWS}_{\mathbf{a}}(q) = (3)$: $q \in \beta^{pws}_{\mathbf{a}}(O)$ and **S7** is not satisfied.

For **S8**, suppose $\mathsf{S}(\mathbf{a}_1) \cap \mathsf{S}(\mathbf{a}_2) = \emptyset$, $\gamma(\mathbf{a}_3) = \gamma(\mathbf{a}_1) + \gamma(\mathbf{a}_2)$, $p \in \beta^{pws}_{\mathbf{a}_1}(O)$ and $p \in \beta^{pws}_{\mathbf{a}_2}(O)$.

Suppose $p \notin \beta^{pws}_{\mathbf{a}_3}(O)$. There is $q \in O$ s.t. $\mathsf{PWS}_{\mathbf{a}_3}(q) < \mathsf{PWS}_{\mathbf{a}_3}(p)$ and i be the minimal index in $\vec{\mathcal{C}^{\mathbf{a}}}$ s.t $\Sigma_{c \in \mathcal{C}^{\mathbf{a}_3}_i} r_{\mathbf{a}_3}(q) < \Sigma_{c \in \mathcal{C}^{\mathbf{a}_3}_i} r_{\mathbf{a}_3}(p)$. The criteria $c \in \mathcal{C}^{\mathbf{a}_3}_i$ belong either to $\mathcal{C}^{\mathbf{a}_1}_{i_1}$ or $\mathcal{C}^{\mathbf{a}_2}_{i_2}$, but not to both because $\mathsf{S}(\mathbf{a}_1) \cap \mathsf{S}(\mathbf{a}_2) = \emptyset$. We denote $\mathcal{C}^{\mathbf{a}_1}_{i_1} = \{c \in \mathcal{C}^{\mathbf{a}_3}_i \cap \mathsf{S}(()\mathbf{a}_1)\}$ and $\mathcal{C}^{\mathbf{a}_2}_{i_2} = \{c \in \mathcal{C}^{\mathbf{a}_3}_i \cap \mathsf{S}(()\mathbf{a}_2)\}$. $\mathcal{C}^{\mathbf{a}_1}_{i_1}$ and $\mathcal{C}^{\mathbf{a}_2}_{i_2}$ is a partition of $\mathcal{C}^{\mathbf{a}_3}_i$. As a consequence, $\Sigma_{c \in \mathcal{C}^{\mathbf{a}_3}_i} r_{\mathbf{a}_3}(q) = \Sigma_{c \in \mathcal{C}^{\mathbf{a}_1}_{i_1}} r_{\mathbf{a}_1}(q) + \Sigma_{c \in \mathcal{C}^{\mathbf{a}_2}_{i_2}} r_{\mathbf{a}_2}(q)$.

So $\Sigma_{c \in \mathcal{C}^{\mathbf{a}_1}_{i_1}} r_{\mathbf{a}_1}(q) + \Sigma_{c \in \mathcal{C}^{\mathbf{a}_2}_{i_2}} r_{\mathbf{a}_2}(q) < \Sigma_{c \in \mathcal{C}^{\mathbf{a}_1}_{i_1}} r_{\mathbf{a}_1}(p) + \Sigma_{c \in \mathcal{C}^{\mathbf{a}_2}_{i_2}} r_{\mathbf{a}_2}(p)$.

Either $\Sigma_{c \in \mathcal{C}^{\mathbf{a}_1}_{i_1}} r_{\mathbf{a}_1}(q) < \Sigma_{c \in \mathcal{C}^{\mathbf{a}_1}_{i_1}} r_{\mathbf{a}_1}(p)$ or $\Sigma_{c \in \mathcal{C}^{\mathbf{a}_2}_{i_2}} r_{\mathbf{a}_2}(q) < \Sigma_{c \in \mathcal{C}^{\mathbf{a}_2}_{i_2}} r_{\mathbf{a}_2}(p)$.

For example, suppose that $\Sigma_{c \in \mathcal{C}^{\mathbf{a}_1}_{i_1}} r_{\mathbf{a}_1}(q) < \Sigma_{c \in \mathcal{C}^{\mathbf{a}_1}_{i_1}} r_{\mathbf{a}_1}(p)$ (the other case is symmetrical).

As $\Sigma_{c \in \mathcal{C}^{\mathbf{a}_1}_{i_1}} r_{\mathbf{a}_1}(q) < \Sigma_{c \in \mathcal{C}^{\mathbf{a}_1}_{i_1}} r_{\mathbf{a}_1}(p)$ and $p \in \beta^{pws}_{\mathbf{a}_1}(O)$, there is a minimal index $j_1 < i_1$ s.t. $\Sigma_{c \in \mathcal{C}^{\mathbf{a}_1}_{i_1}} r_{\mathbf{a}_1}(q) > \Sigma_{c \in \mathcal{C}^{\mathbf{a}_1}_{i_1}} r_{\mathbf{a}_1}(p)$. Let j the index corresponding to j_1 in $\mathcal{C}^{\mathbf{a}_3}$ and j_2 the index corresponding to j_1 in $\mathcal{C}^{\mathbf{a}_2}$. As $j_1 < i_1$, $j < i$ and $j_2 < i_2$.

$\Sigma_{c \in \mathcal{C}^{\mathbf{a}_3}_j} r_{\mathbf{a}_3}(q) = \Sigma_{c \in \mathcal{C}^{\mathbf{a}_1}_{j_1}} r_{\mathbf{a}_1}(q) + \Sigma_{c \in \mathcal{C}^{\mathbf{a}_2}_{j_2}} r_{\mathbf{a}_2}(q)$

As $j < i$, we know that $\Sigma_{c \in \mathcal{C}^{\mathbf{a}_3}_j} r_{\mathbf{a}_3}(q) = \Sigma_{c \in \mathcal{C}^{\mathbf{a}_3}_j} r_{\mathbf{a}_3}(p)$ and that $\Sigma_{\mathcal{C}^{\mathbf{a}_1}_{j_1}} r_{\mathbf{a}_1}(q) > \Sigma_{c \in \mathcal{C}^{\mathbf{a}_1}_{j_1}} r_{\mathbf{a}_1}(p)$: it means that $\Sigma_{\mathcal{C}^{\mathbf{a}_2}_{j_2}} r_{\mathbf{a}_2}(q) < \Sigma_{c \in \mathcal{C}^{\mathbf{a}_1}_{j_2}} r_{\mathbf{a}_2}(p)$. As $p \in \beta^{pws}_{\mathbf{a}_2}(O)$, it entails that there is an index $j'_2 < j_2$ s.t. $\Sigma_{\mathcal{C}^{\mathbf{a}_2}_{j_2}} r_{\mathbf{a}_2}(q) > \Sigma_{c \in \mathcal{C}^{\mathbf{a}_1}_{j_2}} r_{\mathbf{a}_2}(p)$, and with the same reasoning, that there is an index $j'_1 < j_1$ s.t. $\Sigma_{c \in \mathcal{C}^{\mathbf{a}_1}_{j'_1}} r_{\mathbf{a}_1}(q) > \Sigma_{c \in \mathcal{C}^{\mathbf{a}_1}_{j'_1}} r_{\mathbf{a}_1}(p)$: contradicts the minimality of j_1.

S8 is satisfied.

For **S9**, suppose $\mathsf{S}(\mathbf{a}_1) \cap \mathsf{S}(\mathbf{a}_2) = \emptyset$, $\gamma(\mathbf{a}_3) = \gamma(\mathbf{a}_1) + \gamma(\mathbf{a}_2)$, and $\beta_{\mathbf{a}_1}(O) \cap \beta_{\mathbf{a}_2}(O) \neq \emptyset$.

Let $q \in \beta_{\mathbf{a}_3}(O)$. Suppose toward a contradiction that $q \notin \beta_{\mathbf{a}_1}(O) \cap \beta_{\mathbf{a}_2}(O)$. It means that $q \notin \beta_{\mathbf{a}_1}(O)$ or $q \notin \beta_{\mathbf{a}_2}(O)$. Suppose for example that $q \notin \beta_{\mathbf{a}_1}(O)$. Consider $p \in \beta_{\mathbf{a}_1}(O) \cap \beta_{\mathbf{a}_2}(O)$. We know that $\mathsf{PWS}_{\mathbf{a}_1}(p) < \mathsf{PWS}_{\mathbf{a}_1}(q)$.

Let i_1 be the minimal index s.t. $\Sigma_{c \in \mathcal{C}^{\mathbf{a}_1}_{i_1}} r_{\mathbf{a}_1}(p) < \Sigma_{c \in \mathcal{C}^{\mathbf{a}_1}_{i_1}} r_{\mathbf{a}_1}(q)$. Note i and i_2 be the corresponding indexes respectively for $\mathbf{a}_3$ and $\mathbf{a}_2$. As $q \in \beta_{\mathbf{a}_3}(O)$, it entails that

$\Sigma_{c \in \mathcal{C}^{\mathbf{a}_3}_i} r_{\mathbf{a}_3}(q) = \Sigma_{c \in \mathcal{C}^{\mathbf{a}_1}_{i_1}} r_{\mathbf{a}_1}(q) + \Sigma_{c \in \mathcal{C}^{\mathbf{a}_2}_{i_2}} r_{\mathbf{a}_2}(q) \leq \Sigma_{c \in \mathcal{C}^{\mathbf{a}_1}_{i_1}} r_{\mathbf{a}_1}(p) + \Sigma_{c \in \mathcal{C}^{\mathbf{a}_2}_{i_2}} r_{\mathbf{a}_2}(q)$.

As $\Sigma_{c \in \mathcal{C}^{\mathbf{a}_1}_{i_1}} r_{\mathbf{a}_1}(p) < \Sigma_{c \in \mathcal{C}^{\mathbf{a}_1}_{i_1}} r_{\mathbf{a}_1}(q)$, we get $\Sigma_{c \in \mathcal{C}^{\mathbf{a}_2}_{i_2}} r_{\mathbf{a}_2}(p) > \Sigma_{c \in \mathcal{C}^{\mathbf{a}_2}_{i_2}} r_{\mathbf{a}_2}(q)$. But $p \in \beta_{\mathbf{a}_2}(O)$. So there is $i'_2 < i_2$ s.t. $\Sigma_{c \in \mathcal{C}^{\mathbf{a}_2}_{i'_2}} r_{\mathbf{a}_2}(p) < \Sigma_{c \in \mathcal{C}^{\mathbf{a}_2}_{i'_2}} r_{\mathbf{a}_2}(q)$. Due to the optimality of q for $\mathbf{a}_3$, this condition leads to the existence of $i'_1 < i_1$ s.t. $\Sigma_{c \in \mathcal{C}^{\mathbf{a}_1}_{i'_1}} r_{\mathbf{a}_1}(p) > \Sigma_{c \in \mathcal{C}^{\mathbf{a}_1}_{i'_1}} r_{\mathbf{a}_1}(q)$: contradicts the minimality of i_1. **S9** is satisfied.

For **O2**, let $O = \{\mathbf{a}, p, q, r\}$ with $\gamma(\mathbf{a}) = (2,2)$, $\gamma(p) = (4,1)$, $\gamma(q) = (1,4)$ and $\gamma(r) = (3,2)$. $p \in \beta^{pws}_{\mathbf{a}}(O)$, $\gamma(q) = \gamma_{[c_1(p) \to c_1(p)-3, c_2(p) \to c_2(p)+3]}(p)$ but $q \notin \beta^{pws}_{\mathbf{a}}(O)$.

For **O3**, consider $\mathbf{a}$ with $\gamma(\mathbf{a}) = (3,2)$ and p s.t. $\gamma(p) = (4,1)$. $\mathbf{a} \in PF_{\mathsf{S}(\mathbf{a})}(O)$ but $\beta^{pws}_{\mathbf{a}}(O) = \{p\}$ and $\mathbf{a} \notin \beta^{pws}_{\mathbf{a}}(O)$. As **S1** and **S3** are satisfied and not **O3**, **O1** is not satisfied.

For **O4**, if $\gamma(\mathbf{a}_1) = \mu \times \gamma(\mathbf{a}_2)$, the ranks of the criteria for $\mathbf{a}_1$ are the same than the ranks for $\mathbf{a}_2$. As a consequence, $\forall p \in O$, $\mathsf{PWS}_{\mathbf{a}_1}(p) = \mathsf{PWS}_{\mathbf{a}_2}(p)$ and **O4** is satisfied.

$\square$

Proposition 9 *The Weighted Geometric Mean method satisfies* **S1, S2, S3, S4, S5, S6, S7, S8, S9, O4, O5**. *It does not satisfy* **O1, O2** *nor* **O3**.

Proof: **S1, S2, S5** and **O5** are straightforward.

For **S3**, if $c_i \notin (\mathbf{a})$, $w_i(\mathbf{a}) = 0$ and $\forall p \in O, [1 + c_i(p)]^{w_i(\mathbf{a})} = 1$. Then $\mathsf{WGM}_{\mathbf{a}}(p) = \prod_{c_i \in \mathsf{S}(\mathbf{a})} [1 + c_i(p)]^{w_i(\mathbf{a})}$ and $\mathsf{WGM}_{\mathbf{a}}(q) = \prod_{c_i \in \mathsf{S}(\mathbf{a})} [1 + c_i(q)]^{w_i(\mathbf{a})}$. So if $p \simeq^{\mathsf{S}(\mathbf{a})}_{\star} q$, $\mathsf{WGM}_{\mathbf{a}}(p) = \mathsf{WGM}_{\mathbf{a}}(q)$ and **S3** is satisfied.

For **S4**, suppose p and q s.t. $p \succ^{\mathsf{S}(\mathbf{a})}_{\star} q$. So, $\forall c_i \in \mathsf{S}(\mathbf{a}), c_i(p) \leq c_i(q)$ and $\exists c_j \in \mathsf{S}(\mathbf{a})$ s.t. $c_i(p) < c_i(q)$. Furthermore, $\forall c_i \notin \mathsf{S}(()\mathbf{a})$, $c(\mathbf{a}) = 0$ and $(1 + c(p))^{w_i(\mathbf{a})} = (1 + c(q))^{w_i(\mathbf{a})}$. Then $\prod_{c_j \in \mathcal{C}} [1 + c_j(p)]^{w_j(\mathbf{a})} < prod_{c_j \in \mathcal{C}} [1 + c_j(q)]^{w_j(\mathbf{a})}$. Then $p \notin \beta^{wgm}_{\mathbf{a}}(O)$: **S4** is satisfied.

For **S6**, suppose $c_i(a) = c_j(a)$, $p \simeq^{\mathsf{S}(\mathbf{a}) \setminus \{c_i, c_j\}}_{\star} q$, $c_i(p) = c_j(q)$ and $c_j(p) = c_i(q)$. As $c_i(\mathbf{a}) = c_j(\mathbf{a})$, $w_i(\mathbf{a}) = w_j(\mathbf{a})$.

$\mathsf{WGM}_{\mathbf{a}}(p) = \prod_{c_k \in \mathsf{S}(\mathbf{a})} [1 + c_k(p)]^{w_k(\mathbf{a})}$

$\mathsf{WGM}_{\mathbf{a}}(p) = prod_{k\neq i,j}[1 + c_k(p)]^{w_k(\mathbf{a})} \times [1 + c_i(p)]^{w_i(\mathbf{a})} \times [1 + c_j(p)]^{w_j(\mathbf{a})}$

$\mathsf{WGM}_{\mathbf{a}}(p) = prod_{k\neq i,j}[1 + c_k(p)]^{w_k(\mathbf{a})} \times [1 + c_i(p)]^{w_j(\mathbf{a})} \times [1 + c_i(p)]^{w_i(\mathbf{a})}$ as $w_i(\mathbf{a}) = w_j(\mathbf{a})$.

$\mathsf{WGM}_{\mathbf{a}}(p) = prod_{k\neq i,j}[1 + c_k(p)]^{w_k(\mathbf{a})} \times [1 + c_j(q)]^{w_j(\mathbf{a})} \times [1 + c_i(q)]^{w_i(\mathbf{a})}$ as $c_i(p) = c_j(q)$ and $c_j(p) = c_i(q)$.

$\mathsf{WGM}_{\mathbf{a}}(p) = prod_{k\neq i,j}[1 + c_k(q)]^{w_k(\mathbf{a})} \times [1 + c_j(q)]^{w_j(\mathbf{a})} \times [1 + c_i(q)]^{w_i(\mathbf{a})}$ as $p \simeq_\star^{\mathsf{S}(\mathbf{a})\setminus\{c_i,c_j\}} q$.

So $\mathsf{WGM}_{\mathbf{a}}(p) = \mathsf{WGM}_{\mathbf{a}}(q)$ and the result holds.

For **S7**, suppose $p, q \in O$, $c_i(\mathbf{a}) > c_j(\mathbf{a})$, with $p \simeq_\star^{\mathsf{S}(\mathbf{a})\setminus\{c_i,c_j\}} q$, $c_i(p) > c_j(p)$, $c_j(p) = c_i(q)$ and $c_i(p) = c_j(q)$. So $\prod_{c_k\in\mathcal{C},k\neq i,j}[1 + c_k(p)]^{w_k(\mathbf{a})} = \prod_{c_k\in\mathcal{C},k\neq i,j}[1 + c_k(q)]^{w_k(\mathbf{a})}$. It stays to compare $(1 + c_i(p))^{w_i(\mathbf{a})} \times (1 + c_j(p))^{w_j(\mathbf{a})}$ and $(1 + c_i(q))^{w_i(\mathbf{a})} \times (1 + c_j(q))^{w_j(\mathbf{a})}$.

$$(1 + c_i(p))^{w_i(\mathbf{a})} \times (1 + c_j(p))^{w_j(\mathbf{a})} = (1 + c_j(q))^{w_i(\mathbf{a})} \times (1 + c_i(q))^{w_j(\mathbf{a})} \quad (2)$$

as $c_j(p) = c_i(q)$ and $c_i(p) = c_j(q)$.

$(1 + c_j(q))^{w_i(\mathbf{a})} \times (1 + c_i(q))^{w_j(\mathbf{a})} = (1 + c_j(q))^{w_j(\mathbf{a})} \times (1 + c_j(q))^{w_i(\mathbf{a})-w_j(\mathbf{a})} \times (1 + c_i(q))^{w_j(\mathbf{a})}$

as $w_i(\mathbf{a}) > w_j(\mathbf{a})$.

$$(1 + c_j(q))^{w_j(\mathbf{a})} \times (1 + c_j(q))^{w_i(\mathbf{a})-w_j(\mathbf{a})} \times (1 + c_i(q))^{w_j(\mathbf{a})} > (1 + c_j(q))^{w_j(\mathbf{a})} \times (1 + c_i(q))^{w_i(\mathbf{a})-w_j(\mathbf{a})} \times (1 + c_i(q))^{w_j(\mathbf{a})} \quad (3)$$

because $c_j(q) > c_i(q)$.

Finally, from 2 and 3,

$$(1 + c_i(p))^{w_i(\mathbf{a})} \times (1 + c_j(p))^{w_j(\mathbf{a})} > (1 + c_i(q))^{w_i(\mathbf{a})} \times (1 + c_j(q))^{w_j(\mathbf{a})}$$

Then $q \notin \beta_{\mathbf{a}}^{wgm}(O)$ and **S7** is satisfied.

For **S8**, let $\mathsf{S}(\mathbf{a}_1) \cap \mathsf{S}(\mathbf{a}_2) = \emptyset$, $\gamma(\mathbf{a}_3) = \gamma(\mathbf{a}_1) + \gamma(\mathbf{a}_2)$, $p \in \beta_{\mathbf{a}_1}^{wgm}(O)$ and $p \in \beta_{\mathbf{a}_2}^{wgm}(O)$. Let $q \in O$:

$\mathsf{WGM}_{\mathbf{a}_3}(q) = \prod_{c_k\in\mathsf{S}(\mathbf{a}_3)}[1 + c_k(q)]^{w_k(\mathbf{a}_3)}$

As $\mathsf{S}(\mathbf{a}_1) \cap \mathsf{S}(\mathbf{a}_2) = \emptyset$ and $\gamma(\mathbf{a}_3) = \gamma(\mathbf{a}_1) + \gamma(\mathbf{a}_2)$, we have:

$\mathsf{WGM}_{\mathbf{a}_3}(q) = \prod_{c_k\in\mathsf{S}(\mathbf{a}_1)}[1 + c_k(q)]^{w_k(\mathbf{a}_1)} \times \prod_{c_k\in\mathsf{S}(\mathbf{a}_2)}[1 + c_k(q)]^{w_k(\mathbf{a}_2)}$

$\mathsf{WGM}_{\mathbf{a}_3}(q) = \mathsf{WGM}_{\mathbf{a}_1}(q) \times \mathsf{WGM}_{\mathbf{a}_2}(q)$.

As $p \in \beta_{\mathbf{a}_1}^{wgm}(O)$ and $p \in \beta_{\mathbf{a}_2}^{wgm}(O)$, this product is maximal for the object p among all object in O and $p \in \beta_{\mathbf{a}_3}^{wgm}(O)$. **S8** is satisfied.

For **S9**, let $\mathsf{S}(\mathbf{a}_1) \cap \mathsf{S}(\mathbf{a}_2) = \emptyset$, $\gamma(\mathbf{a}_3) = \gamma(\mathbf{a}_1) + \gamma(\mathbf{a}_2)$, and $\beta_{\mathbf{a}_1}^{wgm}(O) \cap \beta_{\mathbf{a}_2}^{wgm}(O) \neq \emptyset$.

We already know that $\beta_{\mathbf{a}_1}^{wgm}(O) \cap \beta_{\mathbf{a}_2}^{wgm}(O) \subseteq \beta_{\mathbf{a}_3}^{wgm}(O)$.

As stated in the proof of **S8**, we have $\forall q \in O$:

$\mathsf{WGM}_{\mathbf{a}_3}(q) = \mathsf{WGM}_{\mathbf{a}_1}(q) \times \mathsf{WGM}_{\mathbf{a}_2}(q)$

Suppose $q \in \beta_{\mathbf{a}_3}^{wgm}(O)$ and $q \notin \beta_{\mathbf{a}_1}^{wgm}(O) \cap \beta_{\mathbf{a}_2}^{wgm}(O)$. Let $p \in \beta_{\mathbf{a}_1}^{wgm}(O) \cap \beta_{\mathbf{a}_2}^{wgm}(O)$. Then either $\mathsf{WGM}_{\mathbf{a}_1}(q) < \mathsf{WGM}_{\mathbf{a}_1}(p)$ or $\mathsf{WGM}_{\mathbf{a}_2}(q) < \mathsf{WGM}_{\mathbf{a}_2}(p)$. As furthermore $\forall q \in O, \mathsf{WGM}_{\mathbf{a}_1}(q) \leq \mathsf{WGM}_{\mathbf{a}_1}(p)$ and $\mathsf{WGM}_{\mathbf{a}_2}(q) \leq \mathsf{WGM}_{\mathbf{a}_2}(p)$, we get $\mathsf{WGM}_{\mathbf{a}_1}(q) \times \mathsf{WGM}_{\mathbf{a}_2}(q) < \mathsf{WGM}_{\mathbf{a}_1}(p) \times \mathsf{WGM}_{\mathbf{a}_2}(p)$ and $q \notin \beta_{\mathbf{a}_3}^{wgm}(O)$: contradiction and **S9** is satisfied.

For **O2**, let $p \simeq_\star^{\mathsf{S}(\mathbf{a}\setminus\{c_i,c_j\})} q$ and $c_i(\mathbf{a}) = c_j(\mathbf{a})$, $c_i(q) = c_i(p) - x$, $c_j(q) = c_j(p) + x$ and $p \in\in \beta_{\mathbf{a}}^{ws}(O)$.

As $p \in \beta_{\mathbf{a}}^{ws}(O)$, $\forall r \in O$, $\sum_{c_k\in S(\mathbf{a})} w_k(\mathbf{a}) \times c_k(p) \geq \sum_{c_k\in S(\mathbf{a})} w_k(\mathbf{a}) \times c_k(r)$.

We know that $\gamma(q) = \gamma_{[c_i(p)\to(c_i(p)-x),c_j(p)\to(c_j(p)+x)]}(p)$, so $\sum_{c_k\in S(\mathbf{a}),k\neq i,j} w_k(\mathbf{a}) \times c_k(p) = \sum_{c_k\in S(\mathbf{a}),k\neq i,j} w_k(\mathbf{a}) \times c_k(q)$. As $c_i(q) = c_i(p) - x$ and $c_j(q) = c_j(p) + x$, we have $c_i(q) \times w_i(\mathbf{a}) + c_j(q) \times w_j(\mathbf{a}) = (c_i(p) - x) \times w_i(\mathbf{a}) + (c_j(p) + x) \times w_j(a)$. As $c_i(\mathbf{a}) = c_j(\mathbf{a})$, $w_i(\mathbf{a}) = w_j(\mathbf{a})$. Then $(c_i(p) - x) \times w_i(\mathbf{a}) + (c_j(p) + x) \times w_j(a) = c_i(p) \times w_i(\mathbf{a}) + c_j(p) \times w_j(a) - x \times w_j(a) + x \times w_j(a) = c_i(p) \times w_i(\mathbf{a}) + c_j(p) \times w_j(a)$. Then $\sum_{c_k\in S(\mathbf{a})} w_k(\mathbf{a}) \times c_k(p) = \sum_{c_k\in S(\mathbf{a})} w_k(\mathbf{a}) \times c_k(q)$, and $\forall r \in O$, $\sum_{c_k\in\mathsf{S}(\mathbf{a})} w_k(\mathbf{a}) \times c_k(q) \geq \sum_{c_k\in\mathsf{S}(\mathbf{a})} w_k(\mathbf{a}) \times c_k(r)$: $q \in \beta_{\mathbf{a}}^{ws}(O)$.

For **O1** and **O3**, consider $O = \{\mathbf{a}, p\}$ with $\gamma(\mathbf{a}) = (2, 2)$ and $\gamma(p) = (1, 10)$. $p \in \beta_{\mathbf{a}}^{wgm}(O)$ but $c_1(p) < c_1(\mathbf{a})$ so **O1** is not satisfied. Furthermore, $\mathbf{a}$ is in the Pareto Front but $\mathbf{a} \notin \beta_{\mathbf{a}}^{wgm}(O)$.

For **O2**, consider $O = \{\mathbf{a}, p, q\}$ with $\gamma(\mathbf{a}) = (2, 2)$, $\gamma(p) = (2, 4)$ and $\gamma(q) = (1, 5)$. $p \in \beta_{\mathbf{a}}^{wgm}(O)$ but $q \notin \beta_{\mathbf{a}}^{wgm}(O)$.

For **O4**, the result is straightforward as $\forall i, w_i(\mathbf{a}_1) = w_i(\mathbf{a}_2)$ so $\mathsf{WGM}_{\mathbf{a}_1}(O) = \mathsf{WGM}_{\mathbf{a}_2}(O)$. $\square$

Proposition 10 *The Ideal Point Distance method satisfies* **S1, S2, S3, S4, S8, S9, O4, O5**. *It does not satisfied* **S5, S6, S7, O1, O2** *nor* **O3**.

Proof: S1, S2 and **O5** are straightforward.

For **S3**, if $c_i \notin \mathsf{S}(\mathbf{a})$, $w_i(a) = 0$. In this case, if $p \simeq_\star^{\mathsf{S}(\mathbf{a})} q$, $\mathsf{IPD}_{\mathbf{a}}(p) = \mathsf{IPD}_{\mathbf{a}}(q)$.

For **S4**, suppose $p \succeq_\star^{\mathsf{S}(\mathbf{a})} q$. Then $\forall c_j \in \mathsf{S}(\mathbf{a})$, $c_j(p) \leq c_j(q)$ and $\exists c_i \in \mathsf{S}(\mathbf{a})$, $c_i(p) < c_i(q)$. Note that we have:

$\sqrt{\sum_{c_i\in\mathcal{C}}[(b_i^O - c_i(p)) * w_i(\mathbf{a})]^2} = \sqrt{\sum_{c_i\in\mathsf{S}(\mathbf{a})}[(b_i^O - c_i(p)) * w_i(\mathbf{a})]^2}$.

$\forall c_j \in \mathsf{S}(\mathbf{a})$, $c_j(p) \leq c_j(q)$, so $|b_i^O - c_i(p)| \geq |b_i^O - c_i(q)|$. As $c_i(p) < c_i(q)$, $|b_i^O - c_i(p)| > |b_i^O - c_i(q)|$. Then $\sqrt{\sum_{c_i\in\mathsf{S}(\mathbf{a})}[(b_i^O - c_i(p)) * w_i(\mathbf{a})]^2} > \sqrt{\sum_{c_i\in\mathsf{S}(\mathbf{a})}[(b_i^O - c_i(q)) * w_i(\mathbf{a})]^2}$.

So $p \notin \beta_{\mathbf{a}}^{ipd}(O)$ and **S4** is satisfied.

For **S5**, consider $O = \{\mathbf{a}, p, q, r\}$ and $O' = \{\mathbf{a}, p, r\}$ with $\gamma(\mathbf{a}) = (1, 1, 1)$, $\gamma(p) = (10, 10, 8)$, $\gamma(q) = (1, 10, 10)$ and $\gamma(r) = (9, 9, 9)$. Then $\gamma(b^O) = (10, 10, 10)$, $\beta_{\mathbf{a}}^{ipd}(O) = \{r\}$ and $\beta_{\mathbf{a}}^{ipd}(O) \cap O' \neq \emptyset$. But $\gamma(b^{O'}) = (10, 10, 9)$ and $\beta_{\mathbf{a}}^{ipd}(O') = \{p\}$: **S5** is not satisfied.

For **S6**, consider $O = \{\mathbf{a}, p, q, r\}$ with $\gamma(\mathbf{a}) = (1, 1)$, $\gamma(p) = (15, 14)$, $\gamma(q) = (14, 15)$ and $\gamma(r) = (0, 20)$. $c_1(\mathbf{a}) = c_2(\mathbf{a})$, $c_1(p) = c_2(q)$, $c_2(p) = c_1(q)$ and $p \simeq_\star^{\mathsf{S}(\mathbf{a})\setminus\{c_1,c_2\}} q$.

The ideal point b^O is such that $\gamma(b^O) = (15, 20)$.

$\mathsf{IPD}_{\mathbf{a}}(\mathbf{a}) = \sqrt{\frac{1}{2} \times 14^2 + \frac{1}{2} \times 19^2} \approx 16,69$

$\mathsf{IPD}_{\mathbf{a}}(p) = \sqrt{\frac{1}{2} \times 36} \approx 4,24$

$\mathsf{IPD}_{\mathbf{a}}(q) = \sqrt{\frac{1}{2} \times 1 + \frac{1}{2} \times 25} \approx 3,6$

$\mathsf{IPD}_{\mathbf{a}}(r) = \sqrt{\frac{1}{2} \times 225} \approx 10,61$

$q \in \beta^{ipd}_{\mathbf{a}}$ but $p \notin \beta^{ipd}_{\mathbf{a}}$: **S6** is not satisfied.

For **S7**, consider $O = \{\mathbf{a}, p, q, r\}$ with $\gamma(\mathbf{a}) = (2,1)$, $\gamma(p) = (15,14)$, $\gamma(q) = (14,15)$ and $\gamma(r) = (0,20)$. $c_1(\mathbf{a}) > c_2(\mathbf{a})$, $c_1(p) = c_2(q)$, $c_2(p) = c_1(q)$ and $p \simeq_\star^{\mathsf{S}(\mathbf{a})\setminus\{c_1,c_2\}} q$.

The ideal point b^O is such that $\gamma(b^O) = (15,20)$.

$\mathsf{IPD}_{\mathbf{a}}(\mathbf{a}) = \sqrt{\frac{2}{3} \times 13^2 + \frac{1}{3} \times 19^2} \approx 15,26$

$\mathsf{IPD}_{\mathbf{a}}(p) = \sqrt{\frac{1}{3} \times 36} \approx 3,46$

$\mathsf{IPD}_{\mathbf{a}}(q) = \sqrt{\frac{2}{3} \times 1 + \frac{1}{3} \times 25} = 3$

$\mathsf{IPD}_{\mathbf{a}}(r) = \sqrt{\frac{2}{3} \times 225} \approx 12,25$

$q \in \beta^{ipd}_{\mathbf{a}}$ and **S7** is not satisfied.

For **S8**, let $\mathsf{S}(\mathbf{a}_1) \cap \mathsf{S}(\mathbf{a}_2) = \emptyset$, $\gamma(\mathbf{a}_3) = \gamma(\mathbf{a}_1) + \gamma(\mathbf{a}_2)$, $p \in \beta^{ipd}_{\mathbf{a}_1}(O)$ and $p \in \beta^{ipd}_{\mathbf{a}_2}(O)$. The ideal point is the same whatever the target. Let $q \in O$:

$\mathsf{IPD}_{\mathbf{a}_3}(q) = \sqrt{\sum_{c_i \in \mathcal{C}} [(b^O_i - c_i(p)) * w_i(\mathbf{a}_3)]^2}$

As $\mathsf{S}(\mathbf{a}_1) \cap \mathsf{S}(\mathbf{a}_2) = \emptyset$ and $\gamma(\mathbf{a}_3) = \gamma(\mathbf{a}_1) + \gamma(\mathbf{a}_2)$, we have:

$\mathsf{IPD}_{\mathbf{a}_3}(q) = \sqrt{\sum_{c_i \in \mathcal{C}} [(b^O_i - c_i(p)) * w_i(\mathbf{a}_1)]^2 + \sum_{c_i \in \mathcal{C}} [(b^O_i - c_i(p)) * w_i(\mathbf{a}_2)]^2}$

$\mathsf{IPD}_{\mathbf{a}_3}(q)^2 = \mathsf{IPD}_{\mathbf{a}_1}(q)^2 + \mathsf{IPD}_{\mathbf{a}_2}(q)^2$

As $p \in \beta^{ipd}_{\mathbf{a}_1}(O)$ and $p \in \beta^{ipd}_{\mathbf{a}_2}(O)$, this sum is minimal for the object p among all object in O and $p \in \beta^{ipd}_{\mathbf{a}_3}(O)$. **S8** is satisfied.

For **S9**, let $\mathsf{S}(\mathbf{a}_1) \cap \mathsf{S}(\mathbf{a}_2) = \emptyset$, $\gamma(\mathbf{a}_3) = \gamma(\mathbf{a}_1) + \gamma(\mathbf{a}_2)$, and $\beta^{ipd}_{\mathbf{a}_1}(O) \cap \beta^{ipd}_{\mathbf{a}_2}(O) \neq \emptyset$.

We already know that $\beta^{ipd}_{\mathbf{a}_1}(O) \cap \beta^{ipd}_{\mathbf{a}_2}(O) \subseteq \beta^{ipd}_{\mathbf{a}_3}(O)$.

As stated in the proof of **S8**, we have $\forall q \in O$:

$\mathsf{IPD}_{\mathbf{a}_3}(q)^2 = \mathsf{IPD}_{\mathbf{a}_1}(q)^2 + \mathsf{IPD}_{\mathbf{a}_2}(q)^2$

Suppose $q \in \beta^{ipd}_{\mathbf{a}_3}(O)$ and $q \notin \beta^{ipd}_{\mathbf{a}_1}(O) \cap \beta^{ipd}_{\mathbf{a}_2}(O)$. Let $p \in \beta^{ipd}_{\mathbf{a}_1}(O) \cap \beta^{ipd}_{\mathbf{a}_2}(O)$. Then either $\mathsf{IPD}_{\mathbf{a}_1}(q) > \mathsf{IPD}_{\mathbf{a}_1}(p)$ or $\mathsf{IPD}_{\mathbf{a}_2}(q) > \mathsf{IPD}_{\mathbf{a}_2}(p)$. As furthermore $\forall q \in O, \mathsf{IPD}_{\mathbf{a}_1}(q) \geq \mathsf{IPD}_{\mathbf{a}_1}(p)$ and $\mathsf{IPD}_{\mathbf{a}_2}(q) \geq \mathsf{IPD}_{\mathbf{a}_2}(p)$, we get $\mathsf{IPD}_{\mathbf{a}_1}(q)^2 + \mathsf{IPD}_{\mathbf{a}_2}(q)^2 > \mathsf{IPD}_{\mathbf{a}_1}(p)^2 + \mathsf{IPD}_{\mathbf{a}_2}(p)^2$ and $q \notin \beta^{ipd}_{\mathbf{a}_3}(O)$: contradiction and **S9** is satisfied.

For **O1**, consider $\gamma(\mathbf{a}) = (2,2)$ and $\gamma(p) = (1,100)$. Then $\gamma(b^O) = (2,100)$. $p \in \beta^{ipd}_{\mathbf{a}}(O)$ but $c_1(p) < c_1(\mathbf{a})$.

For **O2**, consider $\gamma(\mathbf{a}) = (1,1)$, $\gamma(p) = (1,5)$, $\gamma(q) = (5,1)$ and $\gamma(r) = (10,-6)$, $\gamma(s) = (0,12)$. $\gamma(b^O) = (10,12)$. $p \in \beta^{ipd}_{\mathbf{a}}(O)$ and not q: **O2** is not satisfied.

For **O3**, $\gamma(\mathbf{a}) = (2,2)$ and $\gamma(p) = (1,100)$: $\mathbf{a} \in PF_{\mathsf{S}(\mathbf{a})}(O)$ but $\mathbf{a} \notin \beta^{ipd}_{\mathbf{a}}(O)$. **O3** is not satisfied and **O1** is also not satisfied (**S1** and **S3** are satisfied).

For **O4**, as $w_i(\mathbf{a}_1) = w_i(\mathbf{a}_2)$, $\forall p \in O, \mathsf{IPD}_{\mathbf{a}_1}(p) = \mathsf{IPD}_{\mathbf{a}_2}(p)$ and **O4** is satisfied. □

Proposition 11 β^{wlmin} *satisfies* **S1, S2, S3, S4, S5, S6, S7, S8, S9, O1, O2, O4, O5**. *It does not satisfy* **O3**.

Proof: It is easy to see from Proposition 4 that the mapping $\mathbf{a} \mapsto \succeq_{\mathbf{a}}$ associating each target $\mathbf{a}$ with a total preorder over Ω defined for all $p, q \in \Omega$ as $p \succeq_{\mathbf{a}} q$ iff $p \geq^{\mathbf{a}}_{leximin} q$ is a targeting assignment corresponding to β^{wlmin}, which shows that β^{wlmin} satisfies **S1** and **S5**. **O5** is straightforward. To show the remaining properties, it is enough to show that the corresponding conditions on the targeting assignment are satisfied (cf. Proposition 5), and the proof is direct for *(SR3)* and *(SR4)* by definition of $\geq^{\mathbf{a}}_{leximin}$.

For *(SR6)*, suppose $c_i(\mathbf{a}) = c_j(\mathbf{a})$, $p \simeq_\star^{\mathsf{S}(\mathbf{a})\setminus\{c_i,c_j\}} q$, $c_i(p) = c_j(p)$ and $c_j(p) = c_i(q)$. From $p \simeq_\star^{\mathsf{S}(\mathbf{a})\setminus\{c_i,c_j\}} q$, we know that $\gamma_{\mathsf{S}(\mathbf{a})\setminus\{c_i,c_j\}}(p) = \gamma_{\mathsf{S}(\mathbf{a})\setminus\{c_i,c_j\}}(q)$. As $c_i(p) = c_j(p)$, $c_j(p) = c_i(q)$ and $c_i(\mathbf{a}) = c_j(\mathbf{a})$, we get $c_i(p)/c_i(\mathbf{a}) = c_j(q)/c_j(\mathbf{a})$ and $c_j(p)/c_j(\mathbf{a}) = c_i(q)/c_i(\mathbf{a})$. So $\mathsf{WLMin}_{\mathbf{a}}(p) = (\gamma_{\mathsf{S}(\mathbf{a})}(p)/w(\mathbf{a}))_\leq = (\gamma_{\mathsf{S}(\mathbf{a})}(q)/w(\mathbf{a}))_\leq = \mathsf{WLMin}_{\mathbf{a}}(q)$ and $p \simeq_{\mathbf{a}} q$.

For *(SR7)*, assume that (i) $c_i(\mathbf{a}) > c_j(\mathbf{a})$, (ii) $p \simeq_\star^{\mathsf{S}(\mathbf{a})\setminus\{c_i,c_j\}} q$, (iii) $c_i(p) > c_j(p)$, (iv) $c_j(p) = c_i(q)$, and (v) $c_i(p) = c_j(q)$. We need to prove that $p \succ_{\mathbf{a}} q$, i.e., that $p >^{\mathbf{a}}_{leximin} q$. Yet by (ii), $\gamma_{\mathsf{S}(\mathbf{a})\setminus\{c_i,c_j\}}(p) = \gamma_{\mathsf{S}(\mathbf{a})\setminus\{c_i,c_j\}}(q)$, which means that $c_k(p) = c_k(q)$ for each $c_k \in \mathsf{S}(\mathbf{a}) \setminus \{c_i, c_j\}$. Then to prove that $p >^{\mathbf{a}}_{leximin} q$, it is enough to prove that $(c_i(p)/w_i(\mathbf{a}), c_j(p)/w_j(\mathbf{a}))_\leq$ lexically succeeds $(c_i(q)/w_i(\mathbf{a}), c_j(q)/w_j(\mathbf{a}))_\leq$. By (i), we get that (vi) $w_i(\mathbf{a}) > w_j(\mathbf{a})$. By (iii-v), $c_i(q) < c_j(q)$, and then by (vi), we get that (vii) $c_i(q)/w_i(\mathbf{a}) < c_j(q)/w_j(\mathbf{a})$. Since we need to prove that $(c_i(p)/w_i(\mathbf{a}), c_j(p)/w_j(\mathbf{a}))_\leq$ lexically succeeds $(c_i(q)/w_i(\mathbf{a}),\ c_j(q)/w_j(\mathbf{a}))_\leq$, by (vii) it is enough to prove that $\min(c_i(p)/w_i(\mathbf{a}),\ c_j(p)/w_j(\mathbf{a})) > c_i(q)/w_i(\mathbf{a})$. Yet on the one hand, by (iii) and (iv), $c_i(p) > c_i(q)$, so $c_i(p)/w_i(\mathbf{a}) > c_i(q)/w_i(\mathbf{a})$; and on the other hand, by (vi) we get that $c_i(q)/w_j(\mathbf{a}) > c_i(q)/w_i(\mathbf{a})$, which by (iv) can be rewritten as $c_j(p)/w_j(\mathbf{a}) > c_i(q)/w_i(\mathbf{a})$. We have shown that $\min(c_i(p)/w_i(\mathbf{a}), c_j(p)/w_j(\mathbf{a})) > c_i(q)/w_i(\mathbf{a})$, which by (vii) means that $(c_i(p)/w_i(\mathbf{a}), c_j(p)/w_j(\mathbf{a}))_\leq$ lexically succeeds $(c_i(q)/w_i(\mathbf{a}),\ c_j(q)/w_j(\mathbf{a}))_\leq$, so that $p >^{\mathbf{a}}_{leximin} q$, and thus $p \succ_{\mathbf{a}} q$, which concludes the proof that $\mathbf{a} \mapsto \succeq_{\mathbf{a}}$ satisfies *(SR7)*.

For *(SR8)*, suppose $\mathsf{S}(\mathbf{a}_1) \cap \mathsf{S}(\mathbf{a}_2) = \emptyset$, $\gamma(\mathbf{a}_3) = \gamma(\mathbf{a}_1) + \gamma(\mathbf{a}_2)$, $p \succeq_{\mathbf{a}_1} q$ and $p \succeq_{\mathbf{a}_2} q$.

As $\mathsf{S}(\mathbf{a}_1) \cap \mathsf{S}(\mathbf{a}_2) = \emptyset$ and $\gamma(\mathbf{a}_3) = \gamma(\mathbf{a}_1) + \gamma(\mathbf{a}_2)$, the vector $(\gamma_{\mathsf{S}(\mathbf{a}_3)}(p)/w(\mathbf{a}_3))_\leq$ is obtained by ordaining in an increasing order the values of the vector $(\gamma_{\mathsf{S}(\mathbf{a}_1)}(p)/w(\mathbf{a}_1))_\leq$ and the values of the vector $(\gamma_{\mathsf{S}(\mathbf{a}_2)}(p)/w(\mathbf{a}_2))_\leq$. As $p \succeq_{\mathbf{a}_1} q$ and $p \succeq_{\mathbf{a}_2} q$, it is clear (rules of composition of ordered vectors) that $p \succeq_{\mathbf{a}_3} q$ and *(SR8)* is satisfied.

For *(SR9)*, following the precedent proof with the condition $p \succ_{\mathbf{a}_2} q$, we get $p \succ_{\mathbf{a}_3} q$ and *(SR9)* is satisfied.

For *(OR1)*, assume that $p \not\succeq_\star^{\mathsf{S}(\mathbf{a})} \mathbf{a}$, and let us prove that $\mathbf{a} \succ_{\mathbf{a}} p$, that is, we need to prove that $\mathbf{a} >^{\mathbf{a}}_{leximin} p$. First, observe that for each $c_i \in \mathsf{S}(\mathbf{a})$, $c_i(\mathbf{a})/w_i(\mathbf{a}) = \sum_{x \in \mathsf{S}(\mathbf{a})} c_x(\mathbf{a})$, that is, $\mathsf{WLMin}_{\mathbf{a}}(\mathbf{a})$ is a constant vector. and thus by definition of $\mathsf{WLMin}_{\mathbf{a}}(\mathbf{a})$ we get that $\mathsf{WLMin}_{\mathbf{a}}(\mathbf{a}) = (\sum_{x \in \mathsf{S}(\mathbf{a})} c_x(\mathbf{a}), \ldots, \sum_{x \in \mathsf{S}(\mathbf{a})} c_x(\mathbf{a}))$. (this is indeed the case for every object $\mathbf{a} \in \Omega$). Then to prove that $\mathbf{a} >^{\mathbf{a}}_{leximin} p$, we need to prove that $\mathsf{WLMin}_{\mathbf{a}}(\mathbf{a})$ lexically succeeds $\mathsf{WLMin}_{\mathbf{a}}(p)$, and to do this, it is enough to find a criterion $c_i \in \mathsf{S}(\mathbf{a})$ such that $c_i(\mathbf{a})/w_i(\mathbf{a}) > c_i(p)/w_i(\mathbf{a})$. i.e., such that $\sum_{x \in \mathsf{S}(\mathbf{a})} c_x(\mathbf{a}) > c_i(p)/w_i(\mathbf{a})$. Yet $p \not\succeq^{\mathsf{S}(\mathbf{a})}_{\star} \mathbf{a}$, which means that exists $c_i \in \mathsf{S}(\mathbf{a})$ such that $c_i(\mathbf{a}) > c_i(p)$, which can precisely rewritten as $c_i(\mathbf{a})/w_i(\mathbf{a}) > c_i(p)/w_i(\mathbf{a})$. This concludes the proof for *(OR1)*. $\sum_{x \in \mathsf{S}(\mathbf{a})} c_x(\mathbf{a}) > c_i(p)/w_i(\mathbf{a})$ by dividing each side of the equation by $w_i(\mathbf{a})$.

At this stage, using Proposition 5 we have proved that β^{wlmin} satisfies **S1, S2, S3, S4, S5, S7, S8, O1**.

The fact that β^{wlmin} satisfies **O3** is a direct consequence of Proposition 3 and that β^{wlmin} satisfies **S1**, **S3** and **O1**.

Lastly, the fact that β^{wlmin} does not satisfy **O2** is a direct consequence of Proposition 2 and that β^{wlmin} satisfies **O1**.

For *(OR4)*, if $\gamma(\mathbf{a}_1) = \mu \times \gamma(\mathbf{a}_2)$, $\forall p \in O$, the vector $\gamma_{\mathsf{S}(\mathbf{a}_1)}(p)/w(\mathbf{a}_1))_{\leq} = \mu \times \gamma_{\mathsf{S}(\mathbf{a}_2)}(p)/w(\mathbf{a}_2))_{\leq}$ (as $\mu > 0$, the order is conserved). As a consequence, $\succeq_{\mathbf{a}_1} = \succeq_{\mathbf{a}_2}$ and *(OR4)* is satisfied.

This concludes the proof. □

Proposition 12 *β^{wlmax} satisfies* **S1, S2, S3, S4, S5, S6, S7, S8, S9, O4, O5**. *It does not satisfy* **O1, O2, O3**.

Proof: The proof that β^{wlmax} satisfies **S1, S2, S3, S4, S5, S7, S6, S8, S9, O4, O5** is similar to the one of Proposition 11 for β^{wlmin}, except for **S7**. We provide below the proof for all properties for the sake of completeness. First, it is easy to see from Proposition 4 that the mapping $\mathbf{a} \mapsto \succeq_{\mathbf{a}}$ associating each target $\mathbf{a}$ with a total preorder over Ω defined for all $p, q \in \Omega$ as $p \succeq_{\mathbf{a}} q$ iff $p \geq^{\mathbf{a}}_{leximax} q$ is a targeting assignment corresponding to β^{wlmax}, which shows that β^{wlmax} satisfies **S1** and **S5**. To show the remaining properties, it is enough to show that the corresponding conditions on the targeting assignment are satisfied (cf. Proposition 5), and the proof is direct for *(SR3)* and *(SR4)* by definition of $\geq^{\mathbf{a}}_{leximax}$.

For *(SR6)*, the proof is almost identical to the one of Proposition 11. With the assumptions, it is easy to see that So $\mathsf{WLMax}_{\mathbf{a}}(p) = (\gamma_{\mathsf{S}(\mathbf{a})}(p) * w(\mathbf{a}))_{\geq} = (\gamma_{\mathsf{S}(\mathbf{a})}(q) * w(\mathbf{a}))_{\geq} = \mathsf{WLMax}_{\mathbf{a}}(q)$ and $p \simeq_{\mathbf{a}} q$.

For *(SR7)*, assume that (i) $c_i(\mathbf{a}) > c_j(\mathbf{a})$, (ii) $p \simeq^{\mathsf{S}(\mathbf{a}) \setminus \{c_i, c_j\}}_{\star} q$, (iii) $c_i(p) > c_j(p)$, (iv) $c_j(p) = c_i(q)$, and (v) $c_i(p) = c_j(q)$. We need to prove that $p \succ_{\mathbf{a}} q$, i.e., that $p >^{\mathbf{a}}_{leximax} q$. Yet by (ii), $\gamma_{\mathsf{S}(\mathbf{a}) \setminus \{c_i, c_j\}}(p) = \gamma_{\mathsf{S}(\mathbf{a}) \setminus \{c_i, c_j\}}(q)$, which means that $c_k(p) = c_k(q)$ for each $c_k \in \mathsf{S}(\mathbf{a}) \setminus \{c_i, c_j\}$. Then to prove that $p >^{\mathbf{a}}_{leximax} q$, we need to prove that $(c_i(p) \cdot w_i(\mathbf{a}), c_j(p) \cdot w_j(\mathbf{a}))_{\geq}$ lexically succeeds $(c_i(q) \cdot w_i(\mathbf{a}), c_j(q) \cdot w_j(\mathbf{a}))_{\geq}$, and to do that it is enough to prove that $c_i(p) \cdot w_i(\mathbf{a}) > \max(c_i(q) \cdot w_i(\mathbf{a}), c_j(q) \cdot w_j(\mathbf{a}))$. Yet on the one hand, by (iii) and (iv), $c_i(p) > c_i(q)$, so $c_i(p) \cdot w_i(\mathbf{a}) > c_i(q) \cdot w_i(\mathbf{a})$; and on the other hand, by (i), $w_i(\mathbf{a}) > w_j(\mathbf{a})$, so $c_i(p) \cdot w_i(\mathbf{a}) > c_i(p) \cdot w_j(\mathbf{a})$, and since by (v) $c_i(p) = c_j(q)$ so $c_i(p) \cdot w_i(\mathbf{a}) > c_j(q) \cdot w_j(\mathbf{a})$. We got that $c_i(p) \cdot w_i(\mathbf{a}) > \max(c_i(q) \cdot w_i(\mathbf{a}), c_j(q) \cdot w_j(\mathbf{a}))$, so $(c_i(p) \cdot w_i(\mathbf{a}), c_j(p) \cdot w_j(\mathbf{a}))_{\geq}$ lexically succeeds $(c_i(q) \cdot w_i(\mathbf{a}), c_j(q) \cdot w_j(\mathbf{a}))_{\geq}$, which shows that $p >^{\mathbf{a}}_{leximax} q$, i.e., $p \succ_{\mathbf{a}} q$, which concludes the proof that $\mathbf{a} \mapsto \succeq_{\mathbf{a}}$ satisfies *(SR7)*.

For *(SR8)* and *(SR9)*, the proofs are similar to the ones of β^{wlmin}, based on the composition and decomposition of ordered vectors.

At this stage, using Proposition 5 we have proved that β^{wlmax} satisfies **S1, S2, S3, S4, S5, S7, S8**.

Let us now show that $\mathbf{a} \mapsto \succeq_{\mathbf{a}}$ does not satisfy *(OR3)*. Let $\mathcal{C} = \{c_1, c_2\}$ and Ω be such that $p, \mathbf{a} \in \Omega$ with $\gamma(\mathbf{a}) = (1, 1)$ and $\gamma(p) = (2, 0)$. Then $p \not\succeq^{\mathsf{S}(\mathbf{a})}_{\star} \mathbf{a}$ and $p \succ_{\mathbf{a}} \mathbf{a}$ since $\mathsf{WLMax}(p) = (2, 0)$ lexically succeeds $\mathsf{WLMax}(\mathbf{a}) = (1, 1)$. Hence, $\mathbf{a} \mapsto \succeq_{\mathbf{a}}$ does not satisfy *(OR3)*, and by Proposition 5 this means that β^{wlmax} does not satisfy **O3**.

Then, the fact that β^{wlmin} does not satisfy **O1** is a direct consequence of Proposition 3 and that β^{wlmin} satisfies **S1** and **S3** but not **O3**.

Lastly, let us show that $\mathbf{a} \mapsto \succeq_{\mathbf{a}}$ does not satisfy *(OR2)*. Let $\mathcal{C} = \{c_1, c_2\}$ and Ω be such that $p, q, \mathbf{a} \in \Omega$ with $\gamma(\mathbf{a}) = \gamma(q) = (1, 1)$ and $\gamma(p) = (2, 0)$. Then trivially $p \simeq^{\mathsf{S}(\mathbf{a}) \setminus \{c_1, c_2\}}_{\star} q$ since $\mathsf{S}(\mathbf{a}) \setminus \{c_1, c_2\} = \emptyset$; $c_1(\mathbf{a}) = c_2(\mathbf{a})$; $c_1(q) = c_1(p) - 1$; and $c_2(q) = c_2(p) + 1$. Yet *(OR2)* would require that $p \simeq_{\mathbf{a}} q$, but we have that $p \succ_{\mathbf{a}} q$ since $\mathsf{WLMax}(p) = (2, 0)$ lexically succeeds $\mathsf{WLMax}(q) = (1, 1)$. Hence, $\mathbf{a} \mapsto \succeq_{\mathbf{a}}$ does not satisfy **OR2**, and by Proposition 5 this means that β^{wlmax} does not satisfy **O2**.

For *(OR4)*, we obtain easily that if $\gamma(\mathbf{a}_1) = \mu \times \gamma(\mathbf{a}_2)$, $\forall p \in O$, the vector $\gamma_{\mathsf{S}(\mathbf{a}_1)}(p) * w(\mathbf{a}_1))_{\leq} = \mu * \gamma_{\mathsf{S}(\mathbf{a}_2)}(p) * w(\mathbf{a}_2))_{\leq}$ (as $\mu > 0$, the order is conserved). As a consequence, $\succeq_{\mathbf{a}_1} = \succeq_{\mathbf{a}_2}$ and *(OR4)* is satisfied.

This concludes the proof. □