Characterizing Consensuses in Belief Flow Networks (with Proofs in an Appendix)

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ABSTRACT

The BeliefFlow framework models how logical beliefs spread in networks of interacting agents. In a Belief Flow Network (BFN), agents hold epistemic states capturing current and conditional beliefs and revise them asynchronously, taking into account the beliefs of those that influence them as specified by an acquaintance graph, using an improvement operator, a rational form of iterated belief change. Earlier work showed that in strongly connected BFNs, all agents always converge to a global consensus, regardless of initial beliefs, revision policies, or the stochastic order of communications. This paper examines the nature of such consensuses. While past results proved *that* consensus is reached, we characterize *which* formulas may emerge. Given a BFN scheme defined by its acquaintance graph and the agents' initial beliefs, we provide necessary and sufficient conditions for a formula to be realizable as a consensus outcome. A key outcome of our study is that deciding whether a formula is a possible consensus for a given scheme can be done in polynomial time with polynomially many calls to an NP oracle. This matches the complexity of inference for single iterated belief change operators, showing that consensus characterization in BFNs is no harder than reasoning about belief change itself.

KEYWORDS

Belief Diffusion, Iterated Belief Change, Social Networks, Chacterizing Consensuses

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1 INTRODUCTION

The diffusion of beliefs in networks of interacting agents has been widely studied in multi-agent systems and artificial intelligence. Understanding how individual beliefs evolve and propagate through communication channels is essential to model various forms of collective reasoning. Seminal works such as the DeGroot model [9] have inspired a wide range of models for consensus formation and information dynamics, including both probabilistic and logic-based approaches [3, 7, 10, 11].

Recently, Belief Flow Networks (BFNs) have been introduced as a formal framework for studying how logically complex beliefs propagate asynchronously through a network of rational agents [21]. In a BFN, agents hold epistemic states, which are abstract objects from which their current beliefs (a propositional formula) can be extracted, and which also encode conditional information that governs how belief change is performed [8]. Agents revise their beliefs using improvement operators [14, 15, 17], which provide rational models for iterated belief change. Communication proceeds as a stochastic process over a directed acquaintance graph: at each step, an agent receives a propositional formula from a neighbor and updates her beliefs accordingly. The model captures realistic dynamics such as asynchronous influence, individualized change reluctance, and the logical structure of beliefs.

A central result in [21] is that BFNs always reach a global consensus after finitely many steps, provided the communication graph is strongly connected. This leaves open the key question of *which* formulas may emerge as consensuses, given the initial network and belief settings. Our paper addresses this problem.

We investigate the consensus characterization problem in BFNs: given a BFN *scheme*, simply consisting of an influence graph and the agents' initial propositional beliefs, what are the formulas that can arise as final consensuses through the belief evolution process? To address this, we define BFN schemes as abstractions that omit epistemic states and improvement operators, and we distinguish two cases. In the consistent case, where the conjunction of all agents' initial beliefs is itself consistent, we show that the only possible consensus is precisely this conjunction. In the inconsistent case, where conflicting initial beliefs are present, the analysis becomes more intricate. Our main result is a complete characterization of the formulas that may be consensuses, based on logical entailment and structural properties of the graph. We also show that deciding whether a formula can be a consensus belongs to the class P^{NP}, matching the complexity of reasoning in iterated belief change [16].

2 PRELIMINARIES

2.1 Basic Notions and Iterated Belief Change

We work over a propositional language $\mathcal{L}$ built from a finite set of variables $\mathcal{P}$ and the usual connectives. A world maps $\mathcal{P}$ to $\{0,1\}$. We write $[\alpha]$ for the models of a formula $\alpha \in \mathcal{L}$, i.e., the set of worlds satisfying α, $\models$ for entailment between formulas ($\varphi \models \psi$ iff $[\varphi] \subseteq [\psi]$), $\equiv$ for equivalence ($\varphi \equiv \psi$ iff $\varphi \models \psi$ and $\psi \models \varphi$), and Ω for the set of all worlds. A total preorder $\preceq$ on worlds (tpo) is a reflexive, transitive relation on Ω. For a tpo $\preceq$ and $W \subseteq \Omega$, $\min(W, \preceq)$ denotes the $\preceq$-minimal worlds in W, i.e., $\min(W, \preceq) = \{\omega \in W \mid \forall \omega' \in W,\ \omega \preceq \omega'\}$.

Iterated belief change models how a rational agent updates her beliefs when receiving successive inputs from influencing sources. An *epistemic state* Ψ is an abstract object representing an agent's belief state from which both the agent's current beliefs (a propositional formula $Bel(\Psi) \in \mathcal{L}$) can be extracted, but that also comprises some conditional information that governs how change is made [8]. An *epistemic space* refers to the collection of all such states [23].

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One of the most standard epistemic spaces is the *tpo-based epistemic space.* In this epistemic space, each epistemic state is represented as a tpo Ψ, and $Bel(\Psi) = \psi$ where $[\psi] = \min(\Omega, \Psi)$.

Given an epistemic space, a (iterated belief) change operator $\circ$ maps a state Ψ and formula α to a new state $\Psi \circ \alpha$. Operators are meant to satisfy rationality requirements for incorporating new information. A widely used class is *improvement operators*, characterized by nine postulates (I1)–(I9) [14, 15, 17]. For $\circ$, define $\Psi \circ^1 \alpha = \Psi \circ \alpha$ and $\Psi \circ^k \alpha = (\Psi \circ^{k-1} \alpha) \circ \alpha$ for $k > 1$. Then set $\Psi \star_\circ \alpha = \Psi \circ^k \alpha$, where k is the least index with $Bel(\Psi \circ^k \alpha) \models \alpha$ (guaranteed by (I1), see below). Then, an improvement operator is a change operator satisfying the following postulates [14, 15, 17]:

(I1) $\exists k \in \mathbb{N}_*$ s.t. $Bel(\Psi \circ^k \alpha) \models \alpha$
(I2) If $Bel(\Psi) \wedge \alpha \not\models \bot$, then $Bel(\Psi \star_\circ \alpha) \equiv Bel(\Psi) \wedge \alpha$
(I3) If $\alpha \not\models \bot$, then $Bel(\Psi \circ \alpha) \not\models \bot$
(I4) if $\alpha_i \equiv \beta_i$ for all $i \in \{1, \ldots, m\}$,
then $Bel(\Psi \circ \alpha_1 \circ \ldots \circ \alpha_m) \equiv Bel(\Psi \circ \beta_1 \circ \ldots \circ \beta_m)$
(I5) $Bel(\Psi \star_\circ \alpha) \wedge \beta \models Bel(\Psi \star_\circ (\alpha \wedge \beta))$
(I6) If $Bel(\Psi \star_\circ \alpha) \wedge \beta \not\models \bot$,
then $Bel(\Psi \star_\circ (\alpha \wedge \beta)) \models Bel(\Psi \star_\circ \alpha) \wedge \beta$
(I7) If $\alpha \models \beta$, then $Bel((\Psi \circ \beta) \star_\circ \alpha) \equiv Bel(\Psi \star_\circ \alpha)$
(I8) If $\alpha \models \neg\beta$, then $Bel((\Psi \circ \beta) \star_\circ \alpha) \equiv Bel(\Psi \star_\circ \alpha)$
(I9) If $Bel(\Psi \star_\circ \alpha) \not\models \neg\beta$, then $Bel((\Psi \circ \beta) \star_\circ \alpha) \models \beta$

Among these nine postulates, (I1)–(I6) are the core conditions; (I7)–(I9) govern iteration.

Improvement operators admit a semantic account via plausibility orderings over worlds (the lower the more plausible in those orderings). A mapping $\Psi \mapsto \preceq_\Psi$ assigning a tpo $\preceq_\Psi$ to each state Ψ is called a *gradual assignment* if it satisfies:

1. If $\omega, \omega' \models Bel(\Psi)$ then $\omega \simeq_\Psi \omega'$
2. If $\omega \models Bel(\Psi)$ and $\omega' \not\models Bel(\Psi)$ then $\omega \prec_\Psi \omega'$
3. For any $n > 0$, if $\alpha_i \equiv \beta_i$ for any $i \leq n$, then $\preceq_{\Psi \circ \alpha_1 \circ \ldots \circ \alpha_n} = \preceq_{\Psi \circ \beta_1 \circ \ldots \circ \beta_n}$
4. If $\omega, \omega' \models \alpha$, then $\omega \preceq_\Psi \omega' \Leftrightarrow \omega \preceq_{\Psi \circ \alpha} \omega'$
5. If $\omega, \omega' \models \neg\alpha$, then $\omega \preceq_\Psi \omega' \Leftrightarrow \omega \preceq_{\Psi \circ \alpha} \omega'$
6. If $\omega \models \alpha$ and $\omega' \models \neg\alpha$, then $\omega \preceq_\Psi \omega' \Rightarrow \omega \prec_{\Psi \circ \alpha} \omega'$

Proposition 1 ([14]). *An operator $\circ$ is an improvement operator iff there exists a gradual assignment mapping each Ψ to a plausibility ordering, a tpo $\preceq_\Psi$ such that, for every formula α, $[Bel(\Psi \star_\circ \alpha)] = \min([\alpha], \preceq_\Psi)$.*

Operators satisfying the DP success postulate (R*1) ($Bel(\Psi \circ \alpha) \models \alpha$) [8] are called DP revision operators [1, 8], and then $\star_\circ = \circ$.

We next recall two well-known examples of improvement operators on the tpo-epistemic space: Nayak's lexicographic operator $\circ_L$ [18] and the one-improvement operator $\circ_I$ [15]. For these operators $\circ \in \{\circ_L, \circ_I\}$, each epistemic state Ψ (a tpo) can actually be identified through $\circ$'s gradual assignment with Ψ's plausibility ordering $\preceq_\Psi$, so relative plausibility is encoded directly in each state. Nayak's lexicographic operator $\circ_L$ produces $\Psi' = \Psi \circ_L \alpha$ that strictly prefers every α-world to every non-α-world and leaves all other pairwise comparisons unchanged [18]. The one-improvement operator $\circ_I$ moves all α-worlds one layer toward the bottom in Ψ; in $\Psi' = \Psi \circ_I \alpha$ it creates a new bottom layer for $\min([\alpha], \preceq_\Psi)$ when this set is included in the original bottom layer (see [15] for a formal definition). Figure 1 illustrates both behaviors.

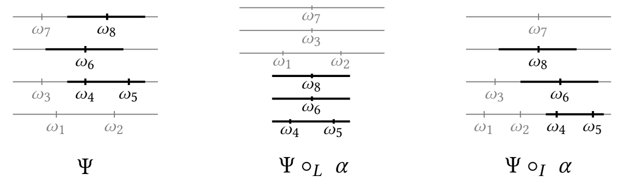

Figure 1: Illustration of the behavior of the improvement operators $\circ_L$ and $\circ_I$, with $[\alpha] = \{\omega_4, \omega_5, \omega_6, \omega_8\}$.

Note that the lexicographic operator $\circ_L$ satisfies (R*1) and is maximally change-inclined among improvement operators [19]: upon receiving α, it always considers α-worlds as strictly more plausible than $\neg\alpha$-worlds after revision. The one-improvement operator $\circ_I$, on the other hand, does not satisfy (R*1) and models a more change-reluctant agent that may not accept a new input into her beliefs immediately.

2.2 Belief Flow Networks

Belief Flow Networks (BFNs) model how agents change their epistemic states and transmit their beliefs through a network of directed influences [21]. Each agent holds an epistemic state and revises it asynchronously by incorporating the beliefs of its influencers according to her own improvement operator. Formally, a BFN is a tuple $\mathcal{B} = \langle (V, A), \vec{\Psi}, \vec{\circ}, \mathcal{S} \rangle$ with four components.

(1) Acquaintance graph. $G = (V, A)$ (also called an *influence graph*) where $V = \{1, \ldots, n\}$ is the set of agents and $A \subseteq V \times V$ is an irreflexive set of directed edges. An edge $(i, j) \in A$ means that information can flow from agent i to agent j.

(2) Initial epistemic state profile. $\vec{\Psi} = \langle \Psi^1, \ldots, \Psi^n \rangle$ gives the initial epistemic state of each agent, with $Bel(\Psi^i)$ assumed consistent for all $i \in V$.

(3) Change policy profile. $\vec{\circ} = \langle \circ_1, \ldots, \circ_n \rangle$ specifies each agent's change policy. Every $\circ_i$ is an improvement operator and is assumed to satisfy a mild strengthening of (I1), called (I1*) $\exists k \in \mathbb{N}_*$ s.t. $\forall \Psi \in \mathcal{E}, \forall \alpha \in \mathcal{L}$, $Bel(\Psi \circ^k \alpha) \models \alpha$. That is, the number of steps required for every agent $i \in V$ to fully accept any input formula is fixed and independent of its current epistemic state. This condition is satisfied by most of the improvement operators introduced in the literature, in particular by improvement operators defined over finite epistemic spaces like the tpo-epistemic space (thus, the two operators $\circ_L$ and $\circ_I$ introduced previously both satisfy (I1*)).

(4) Communication process. $\mathcal{S} = (A_s)_{s \in \mathbb{N}}$ is a stochastic process on A describing which communication link is triggered at each step. It is assumed to be a *chain with complete connections* [13]: there exists $\delta > 0$ such that for any s and any $e_0, \ldots, e_s \in A$, $\Pr(A_s = e_s \mid A_{s-1} = e_{s-1}, \ldots, A_0 = e_0) \geq \delta$. This ensures that every edge has a nonzero probability of occurring. Such a definition covers both Markov chains (where the next edge depends on the previous one) and Bernoulli schemes (independent draws) [25].

Runs and epistemic evolution. A $\mathcal{B}$*-run* $\sigma = (\sigma_s)_{s \in \mathbb{N}}$ is an infinite sequence generated by $\mathcal{S}$, with $\sigma_s \in A$ indicating the edge activated at step s. Each agent i has an epistemic state $\Psi^i_{\sigma_s}$ at step s in run σ, with $\Psi^i_{\sigma_0} = \Psi^i$. When $\sigma_s = (j, i)$, agent j's current beliefs $Bel(\Psi^j_{\sigma_s})$ are received by i, who updates her epistemic state via her policy $\circ_i$; for each step $s > 0$:

$$\Psi^i_{\sigma_{s+1}} = \Psi^i_{\sigma_s} \ \circ_i \ Bel(\Psi^j_{\sigma_s})$$

If i is not the target of the triggered edge, her state remains unchanged ($\Psi^i_{\sigma_{s+1}} = \Psi^i_{\sigma_s}$). The sequence of all agents' states at each step forms the *epistemic profile sequence* $\vec{\Psi}_\sigma = (\vec{\Psi}_{\sigma_s})_{s\in\mathbb{N}}$, where for each $s \in \mathbb{N}$, $\vec{\Psi}_{\sigma_s} = \langle \Psi^1_{\sigma_s}, \ldots, \Psi^n_{\sigma_s} \rangle$.

Outcomes, stability, and consensus. For a given agent i in run σ, the *belief sequence* $Seq_\sigma(i) = (Bel(\Psi^i_{\sigma_s}))_{s\in\mathbb{N}}$ records the successive beliefs of i. Among its subsequences, the *σ-outcome sequence* $Seq^*_\sigma(i)$ is the earliest one where every belief formula appears infinitely often (up to equivalence). The *σ-outcome* of i, written $Out_\sigma(i)$, is the weakest formula entailing all formulas in $Seq^*_\sigma(i)$. Agent i is *stable* in σ if all formulas in $Seq^*_\sigma(i)$ are equivalent to $Out_\sigma(i)$, that is, her beliefs eventually stop changing. A set of agents $V' \subseteq V$ is stable in σ if each of its members is stable.

A BFN $\mathcal{B}$ *reaches a consensus* in a $\mathcal{B}$-run σ if V is stable in σ and there exists a formula α such that $Out_\sigma(i) \equiv \alpha$ for every $i \in V$. We denote this formula by $Out_\sigma(\mathcal{B})$ and call it the *consensus of $\mathcal{B}$ in σ*. A BFN $\mathcal{B}$ is *strongly consensual* if it reaches a consensus in every $\mathcal{B}$-run.

Basic properties and strong consensus. BFNs satisfy several appealing properties that guarantee consistent and coherent belief propagation [21]. For every $\mathcal{B}$-run σ, step s, and formula φ:

(CP) $\forall i \in V \ Bel(\Psi^i_{\sigma_s}) \not\models \bot$
(AP) $(\forall i \in V \ \varphi \models Bel(\Psi^i)) \Rightarrow (\forall i \in V \ \varphi \models Bel(\Psi^i_{\sigma_s}))$
(UP) $(\forall i \in V \ Bel(\Psi^i) \models \varphi) \Rightarrow (\forall i \in V \ Bel(\Psi^i_{\sigma_s}) \models \varphi)$
(DR) $\forall (j, i) \in A, \exists s' \geq s$ s.t. $Bel(\Psi^i_{\sigma_{s'}}) \wedge Bel(\Psi^j_{\sigma_{s'}}) \not\models \bot$

Intuitively, (CP) (Consistency Preservation) states that agents always maintain consistent beliefs at every stage of the process. (AP) (Agreement Preservation) ensures that if all agents initially agree on a set of possible worlds satisfying a formula φ, this agreement is preserved throughout all future steps. (UP) (Unanimity Preservation), the dual of (AP), guarantees that if all agents initially believe φ, they continue to do so regardless of the communication dynamics. Finally, (DR) (Delayed Responsiveness) captures the idea that when one agent j can influence another agent i, there will always exist a future moment in the interaction where their beliefs become mutually consistent.

Proposition 2. *[21] Every BFN satisfies (CP), (AP), (UP), and (DR).*

Finally, it has been showed in [21] that under strong connectivity of the underlying graph, the set of all agents is guaranteed to be strongly consensual. Formally, given a graph $G = (V, A)$ and two agents $i, j \in V$, $i \rightsquigarrow_G j$ denotes the fact that there exists a directed path from i to j in G. Then G is said to be *strongly connected* if $i \rightsquigarrow_G j$ holds for all distinct $i, j \in V$. We naturally lift this notion to BFNs, saying that a BFN $\mathcal{B} = \langle (V, A), \vec{\Psi}, \vec{\circ}, \mathcal{S} \rangle$ is strongly connected if its underlying graph is.

Proposition 3. *[21] Every strongly connected BFN is strongly consensual.*

This is a central result: when the influence structure is strongly connected, the network inevitably converges to a common belief, regardless of the stochastic communication process or the agents' individual epistemic states and change policies. In other words, as long as every agent can eventually reach every other through some sequence of communications, their beliefs are guaranteed to align after a finite number of steps.

Building on this result, we focus in the rest of this paper on strongly connected BFNs, as they provide the structural basis for all subsequent results.

3 EXAMPLE AND PROBLEM DEFINITION

Let us start this section with a concrete, simple example of a BFN.

Example 1. Consider three agents, Alice (agent 1), Bob (agent 2), and Charles (agent 3), engaged in an internal discussion about the development of a new prototype. Alice is an engineer working on system development, Bob is a project manager coordinating communication, and Charles is an external quality auditor temporarily added to the project channel. Each agent forms and changes beliefs about the project's progress, based on their own interpretation of available information and on the posts they read from others on the company's internal communication platform. Agents can only see posts from those they follow. Assume Bob serves as the central communication hub: Alice and Bob follow each other (so they can mutually influence one another), and Bob and Charles also follow each other, but Alice and Charles do not have a direct connection. We have $V = \{1, 2, 3\}$ and $A = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$.

Now, let $\mathcal{P}$ be the set of propositional variables $\mathcal{P} = \{p, q, r\}$, where p stands for "The prototype is close to meeting the safety requirements", q stands for "The performance benchmarks meet the required standard", and r stands for "The client appears inclined to continue the project."

Initially, Alice believes $p \wedge q$: from the engineer's perspective, both the safety and performance aspects of the prototype are adequate. Bob believes r: as project manager, he is confident that the client is likely to continue the project. Charles believes $\neg p \wedge \neg q$: based on audit reports, the prototype seems to fail to meet the safety and performance standards. Thus, Alice and Charles hold opposite beliefs about the technical aspects (p, q), while Bob's belief (r) concerns an independent managerial expectation.

Let $\circ_1$, $\circ_2$ and $\circ_3$ be Alice, Bob and Charles' change policies, respectively, arbitrarily chosen DP revision operators. Recall that DP revision operators are, accordingly, improvement operators, but they additionally satisfying the success postulate (R*1), i.e., $Bel(\Psi \circ_i \alpha) \models \alpha$ for each $i \in \{1, 2, 3\}$ and for all Ψ, α (this choice is made for simplicity in this example). Let Ψ^1, Ψ^2, Ψ^3 be any epistemic states such that the beliefs associated with those states reflect the example above, i.e., $Bel(\Psi^1) = p \wedge q$, $Bel(\Psi^2) = r$, and $Bel(\Psi^3) = \neg p \wedge \neg q$. And let $\mathcal{S}$ be any stochastic process. This fully specifies a BFN $\mathcal{B} = \langle (V, A), \vec{\Psi}, \vec{\circ}, \mathcal{S} \rangle$, where $\vec{\Psi} = \langle \Psi^1, \Psi^2, \Psi^3 \rangle$ and $\vec{\circ} = \langle \circ_1, \circ_2, \circ_3 \rangle$.

Consider now any $\mathcal{B}$-run σ starting with the A-sequence $\delta = ((2, 1), (1, 2), (2, 3))$. We have that $\sigma_0 = (2, 1)$, $\sigma_1 = (1, 2)$, and $\sigma_2 = (2, 3)$. It can then be verified that, given the above specifications of $\mathcal{B}$, the beliefs of the agents will evolve in the $\mathcal{B}$-run σ according to Table 1. For instance, since the first triggered edge is $\sigma_0 = (2, 1)$, we get that $Bel(\Psi^1_{\sigma_1}) \equiv Bel(\Psi^1_{\sigma_0} \circ_1 Bel(\Psi^2_{\sigma_0})) \equiv Bel(\Psi^1_{\sigma_0}) \wedge Bel(\Psi^2_{\sigma_0})$ (by

Step s	**Trig. edge** σ_s	$\mathbf{Bel}(\Psi^1_{\sigma_s})$	$\mathbf{Bel}(\Psi^2_{\sigma_s})$	$\mathbf{Bel}(\Psi^3_{\sigma_s})$
0	–	$p \land q$	r	$\neg p \land \neg q$
1	$(2, 1)$	$\mathbf{p \land q \land r}$	r	$\neg p \land \neg q$
2	$(1, 2)$	$p \land q \land r$	$\mathbf{p \land q \land r}$	$\neg p \land \neg q$
3	$(2, 3)$	$p \land q \land r$	$p \land q \land r$	$\mathbf{p \land q \land r}$
≥ 4	any $(i, j) \in A$	$p \land q \land r$	$p \land q \land r$	$p \land q \land r$

Table 1: Evolution of the agents' beliefs in the $\mathcal{B}$-run σ.

(I2), since $\star_{\circ_1} = \circ_1$ and since $Bel(\Psi^1_{\sigma_0}) \land Bel(\Psi^2_{\sigma_0}) \equiv (p \land q) \land r \not\models \bot$), thus $Bel(\Psi^1_{\sigma_1}) \equiv p \land q \land r$. A consensus is reached from step 3 in σ, with $Out_\sigma(\mathcal{B}) \equiv p \land q \land r$.

It can be easily seen in this example that different $\mathcal{B}$-runs may lead to different consensuses. For instance, switching agents 1 and 3 in σ would, by the symmetric roles played by agents 1 and 3, lead to $Out_\sigma(\mathcal{B}) \equiv \neg p \land \neg q \land r$. This non-deterministic behavior reflects a natural situation: the order in which communications are triggered affects the consensus.

Since this BFN is strongly connected, Proposition 3 already guarantees that every $\mathcal{B}$-run reaches a consensus. The question we address here is *which* consensuses may arise. Given a BFN $\mathcal{B}$, we denote by $Out(\mathcal{B})$ this set of all possible consensuses in the BFN $\mathcal{B}$, simply called the *outcome set of* $\mathcal{B}$. Yet our goal is not to compute the outcome set of a single, fully specified BFN, but the outcome set of a *BFN scheme* that specifies only some parameters of a BFN.

Our first point (Proposition 4) is that the choice of the stochastic process $\mathcal{S}$ is *irrelevant* to the outcome set:

Proposition 4. *For all strongly connected BFNs $\mathcal{B} = \langle (V, A), \vec{\Psi}, \vec{\circ}, \mathcal{S} \rangle$ and $\mathcal{B}' = \langle (V, A), \vec{\Psi}, \vec{\circ}, \mathcal{S}' \rangle$, we have that $Out(\mathcal{B}) = Out(\mathcal{B}')$.*

That is, for all strongly connected BFNs with the same graph and epistemic state / change policy profiles, $Out(\mathcal{B})$ is the same. Hence, $\mathcal{S}$ can be taken *out* of the parameters when studying possible consensuses.

Our second point is that it is not reasonable to assume we are given full epistemic states and change policies. Indeed, on the one hand, epistemic states are abstract objects whose representation is irrelevant to change outcomes. The change behavior is driven by the improvement operator, which associates through a gradual assignment each epistemic state Ψ with a total preorder over worlds (cf. Proposition 1). From Ψ itself, the only usable datum is its belief $Bel(\Psi)$ (see the preliminaries). Moreover, it is known that epistemic spaces can be uncountable in the general case, so epistemic states may not be finitely representable [23]. On the other hand, change policies are too heavy to specify directly. Giving each agent's improvement operator explicitly amounts to providing a labeled transition system over epistemic states (with transitions $\Psi, \varphi \mapsto \Psi \circ \varphi$), which is typically very large or even infinite, and thus impractical as input.

For these reasons, the input we adopt is a *BFN scheme*: it keeps the influence graph (unchanged) and replaces the epistemic profile by a *belief profile* only, which is a vector of propositional formulas specifying the initial beliefs of each agent. We still assume agents' (unknown) change policies are improvement operators satisfying the postulates from the preliminaries. By Proposition 4, we also do not need to specify $\mathcal{S}$.

Definition 2 (BFN scheme). A BFN scheme is a tuple $\mathrm{B} = \langle (V, A), \vec{B} \rangle$, where (V, A) is a strongly connected acquaintance graph and $\vec{B} = \langle B^1, \ldots, B^n \rangle$ is a *belief profile*, that is, B^i is a propositional formula for each $i \in V$.

A BFN $\mathcal{B} = \langle (V, A), \vec{\Psi}, \vec{\circ}, \mathcal{S} \rangle$ *satisfies* a scheme $\mathrm{B} = \langle (V', A'), \vec{B} \rangle$, written $\mathcal{B} \models \mathrm{B}$, iff $(V, A) = (V', A')$ and, for each agent $i \in V$, $Bel(\Psi^i) \equiv B^i$. We lift the outcome notion to BFN schemes by:

$$Out(\mathrm{B}) = \bigcup \{ Out(\mathcal{B}) \mid \mathcal{B} \models \mathrm{B} \}$$

Interpreted this way, answering the question for a scheme B and formula φ directly answers: *"Does there exist a BFN satisfying* B *for which φ is a possible consensus?"* This is a natural decision problem, given that only the graph and initial beliefs are typically available. In the rest of the paper, for any BFN scheme $\mathrm{B} = \langle (V, A), \vec{B} \rangle$ and formula φ, we aim to characterize whether $\varphi \in Out(\mathrm{B})$.

4 CONSISTENT BFN SCHEMES

We start with a simple but important case, where the conjunction of all agents' initial beliefs is consistent:

Definition 3 (Consistent BFN scheme). A BFN scheme $\mathrm{B} = \langle (V, A), \vec{B} \rangle$ is said to be *consistent* if $\bigwedge_{i \in V} B^i \not\models \bot$. Otherwise, B is *inconsistent.*

Proposition 5. *Let $\mathrm{B} = \langle (V, A), \vec{B} \rangle$ be a BFN scheme. If* B *is consistent, then $Out(\mathrm{B}) = \{ \bigwedge_{i \in V} B^i \}$.*

The proof idea is straightforward. From (AP), the models of every agent's belief are never lost. By Proposition 3, the system must reach a consensus after finitely many steps. Then, any consensus weaker than $\bigwedge_{i \in V} B^i$ would imply some agent to weaken her beliefs to include a world that was not part of the joint initial beliefs, which would require revising by information inconsistent with her own beliefs, contradicting again (AP). Thus, the only possible consensus in every BFN satisfying such a scheme is this conjunction $\bigwedge_{i \in V} B^i$. Notably, this outcome does not depend on specific epistemic states or change policies.

This result is quite intuitive: in a strongly connected group where all agents start from compatible beliefs, every exchange only reinforces existing agreement. Since no conflicts arise, all agents eventually adopt the conjunction of their initial beliefs, i.e., the group's collective belief.

This proposition also strengthens a related result from the framework of Belief Revision Games [20], where the same conclusion was obtained only under stronger assumptions, that is, complete communication graphs and synchronous updates. Here, the same behavior emerges in any strongly connected graph and under asynchronous communication.

5 INCONSISTENT BFN SCHEMES

When the BFN scheme is *inconsistent*, the agents cannot jointly hold the conjunction of their initial beliefs. Still, by Proposition 3, every run reaches *some* consensus. Our example from Section 3 already shows that different runs may lead to different consensuses.

In this section we state and prove a characterization of when a given formula φ can be a consensus for an inconsistent BFN scheme. The statement relies on two simple conditions and a few graph-based notions. We first introduce the notions, the conditions, formally state the characterization theorem, and then show the conditions are necessary and sufficient.

Throughout, let $\mathsf{B} = \langle (V, A), \vec{B} \rangle$ be an arbitrary inconsistent BFN scheme with $\vec{B} = \langle B^1, \ldots, B^n \rangle$, and let φ be any consistent formula.

5.1 The Characterization Statement

We start with the introduction of a set lightweight notions that depend only on the BFN scheme (graph and belief profile).

Observable neutrality in a scheme. For $i \in V$, we say that i is *observably neutral on φ in the scheme* B if and only if

$$\varphi \models B^i \quad \text{or} \quad B^i \wedge \varphi \models \bot$$

Equivalently, i is either fully compatible with φ (all models of φ are models of B^i) or conflicts with it.

This notion lifts verbatim to BFNs $\mathcal{B}$ and $\mathcal{B}$-runs: in a BFN $\mathcal{B} \models \mathsf{B}$, i is observably neutral on φ (at step s in a run σ) by replacing B^i with $Bel(\Psi^i)$ (respectively, $Bel(\Psi^i_{\sigma_s})$). Note that since observable neutrality depends only on the belief profile $\vec{B} = \langle B^1, \ldots, B^n \rangle$, i is observably neutral on φ in B if and only if i is observably neutral on φ in *every* $\mathcal{B} \models \mathsf{B}$.

Support and restricted graph. The *support of* φ is the set of agents whose beliefs are consistent with φ:

$$V_{\downarrow\varphi} = \{i \in V \mid B^i \wedge \varphi \not\models \bot\}$$

We write $G_{\downarrow\varphi}$ for the subgraph of G induced by $V_{\downarrow\varphi}$, i.e., $G_{\downarrow\varphi} = G[V_{\downarrow\varphi}] = (V_{\downarrow\varphi}, A_{\downarrow\varphi})$ with $A_{\downarrow\varphi} = A \cap (V_{\downarrow\varphi} \times V_{\downarrow\varphi})$.

In-neighborhood and upstream reach. For a given agent $i \in V$, its *φ-in-neighborhood* is

$$N^-_\varphi(i) = \{j \in V_{\downarrow\varphi} \mid (j, i) \in A\}$$

For $i, j \in V_{\downarrow\varphi}$, let $i \rightsquigarrow^*_{G_{\downarrow\varphi}} j$ mean that there is a (possibly empty) directed path from i to j in $G_{\downarrow\varphi}$. For a subset $V'_{\downarrow\varphi} \subseteq V_{\downarrow\varphi}$, define its *$\varphi$-upstream reach set* by

$$Reach^-_\varphi(V'_{\downarrow\varphi}) = \{i \in V_{\downarrow\varphi} \mid \exists j \in V'_{\downarrow\varphi} : i \rightsquigarrow^*_{G_{\downarrow\varphi}} j\}$$

In particular, note that $V'_{\downarrow\varphi} \subseteq Reach^-_\varphi(V'_{\downarrow\varphi})$.

Backward cone. For a given agent $i \in V$, its *φ-backward cone* is

$$C_\varphi(i) = \{i\} \cup Reach^-_\varphi(N^-_\varphi(i))$$

Intuitively, the set of agents $C_\varphi(i)$ gathers every supporter of φ that can indrectly influence i through $G_{\downarrow\varphi}$, plus i itself (which may or may not support φ).

Lastly, for any $i \in V$, define its *φ-belief closure* as:

$$BC_\varphi(i) = \bigwedge \{B^j \mid j \in C_\varphi(i)\}$$

We are now ready to state the characterization result:

Theorem 4. *$\varphi \in Out(\mathsf{B})$ if and only if the two following conditions are satisfied:*

I. for each agent $i \in V$, i is observably neutral on φ in B

II. there exists an agent $i_* \in V$ such that $BC_\varphi(i_*) \models \bot$

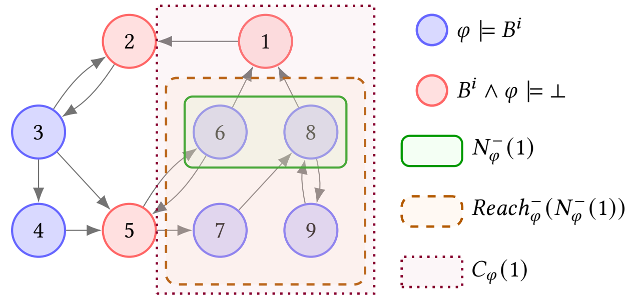

Figure 2: Illustration of a BFN scheme satisfying condition I.

Figure 2 shows a BFN scheme with nine agents that satisfies condition I for some formula φ. It also highlights $N^-_\varphi(1)$, $Reach^-_\varphi(N^-_\varphi(1))$, and $C_\varphi(1)$. Checking condition II for agent 1 amounts to testing whether $B^1 \wedge B^6 \wedge B^7 \wedge B^8 \wedge B^9 \models \bot$.

Note that, under condition I, condition II can only hold for agents outside the support of φ (e.g., $i \in \{1, 2, 5\}$ in the figure). If $i \in V_{\downarrow\varphi}$, then $C_\varphi(i) \subseteq V_{\downarrow\varphi}$, and then by condition I, $B^j \models \varphi$ for each $j \in C_\varphi(i)$, so $\bigwedge\{B^j \mid j \in C_\varphi(i)\}$ is trivially consistent.

The remainder of the section proves Theorem 4: we first show that conditions I and II are both necessary, then that together they are sufficient.

5.2 Necessity of Condition I

To prove that condition I is necessary for φ to be a possible consensus of B, we argue a bit stronger: in any BFN $\mathcal{B} \models \mathsf{B}$, every agent must be "conditionally neutral" on φ. Given a BFN $\mathcal{B} = \langle (V, A), \vec{\Psi}, \vec{\circ}, \mathcal{S} \rangle$, an agent $i \in V$, a $\mathcal{B}$-run σ, and a step $s \in \mathbb{N}$, we say that i is *conditionally neutral on φ in σ at step s* if $Bel(\Psi^i_{\sigma_s} \star_{\circ_i} \varphi) \equiv \varphi$. We simply say that i is conditionally neutral on φ in $\mathcal{B}$ if $Bel(\Psi^i \star_{\circ_i} \varphi) \equiv \varphi$. Conditional neutrality means that, were i to revise by φ, the result would be exactly φ, neither stronger nor weaker. This occurs in two basic situations: (i) φ already entails i's belief, so revision acts as conjunction (postulate (I2)): $Bel(\Psi \star_{\circ_i} \varphi) \equiv Bel(\Psi) \wedge \varphi \equiv \varphi$; or (ii) φ contradicts i's belief, and revision yields acceptance of φ and nothing else. In both cases, this means i treats all φ-worlds as equally plausible. This is made clear by Lemma 1 below. Recall that $\Psi^i \mapsto \preceq^i_{\Psi^i}$ denotes the gradual assignment corresponding to $\circ_i$ (Proposition 1):

Lemma 1. *Let $i \in V$, σ be a $\mathcal{B}$-run and $s \in \mathbb{N}$. Then $i \in V$ is conditionally neutral on φ in σ at s if and only if for all worlds ω, $\omega' \models \varphi$, we have that $\omega \simeq^i_{\Psi^i_{\sigma_s}} \omega'$.*

Notably, conditional neutrality implies observable neutrality:

Lemma 2. *Let $i \in V$, σ be a $\mathcal{B}$-run and $s \in \mathbb{N}$. If i is conditionally neutral on φ in σ at s, then i is observably neutral on φ in σ at s.*

The next lemma shows that conditional neutrality for all agents is a "global" property: if it holds at some step in a given run, then it holds at every step of every run, including in the initial state.

Lemma 3. *The following statements are equivalent:*

1. *for each $i \in V$, i is conditionally neutral on φ in $\mathcal{B}$*
2. *for each $\mathcal{B}$-run σ, each step $s \in \mathbb{N}$ and each $i \in V$, i is conditionally neutral on φ in σ at step s*

3. *there exists a $\mathcal{B}$-run σ and a step $s \in \mathbb{N}$ such that for each $i \in V$, i is conditionally neutral on φ in σ at step s*

Using Lemmas 1 and 3, we obtain:

Lemma 4. *Let $\mathcal{B} \models$ B. If $\varphi \in Out(\mathcal{B})$, then for each agent $i \in V$, i is conditionally neutral on φ in $\mathcal{B}$.*

The proof of Lemma 4 builds on the following key idea. By contrapositive of (3⟹1) in Lemma 3, if some agent is not conditionally neutral initially, then in every run and at every step there remains (by Lemma 1 and conditions 1 and 2 of gradual assignments) an agent whose beliefs are not equivalent to φ; hence φ cannot be a consensus.

We can now state necessity of condition I, which relies on Lemmas 2 and 4:

Proposition 6. *If $\varphi \in Out(\mathrm{B})$, then for each $i \in V$, i is observably neutral on φ in* B.

5.3 Necessity of Condition II

We now show that condition II is also a necessary condition for φ to be a possible consensus of B.

Recall that the support of φ in a scheme B is the set $V_{\downarrow\varphi} = \{i \in V \mid B^i \wedge \varphi \not\models \bot\}$. We extend this notion dynamically to any BFN $\mathcal{B} \models$ B, run σ, and step $s \in \mathbb{N}$:

$$Sup_{\mathcal{B}}(\varphi, \sigma, s) = \{i \in V \mid Bel(\Psi^i_{\sigma_s}) \wedge \varphi \not\models \bot\}$$

Combining Lemmas 2, 3 (1⟹2), and 4 yields:

Lemma 5. *Let $\mathcal{B} \models$ B. If $\varphi \in Out(\mathcal{B})$, then for each $\mathcal{B}$-run σ and step $s \in \mathbb{N}$, $Sup_{\mathcal{B}}(\varphi, \sigma, s) = \{i \in V \mid \varphi \models Bel(\Psi^i_{\sigma_s})\}$.*

That is, if φ is a possible consensus, then all agents in the support of φ at any step are exactly those whose beliefs are entailed by φ.

From this, we note that for φ to be a possible consensus of B, there must exist at least one agent initially inconsistent with φ:

Lemma 6. *If $\varphi \in Out(\mathrm{B})$, then $V \setminus V_{\downarrow\varphi} \neq \emptyset$.*

The proof directly follows from Lemma 5 together with the fact that B is inconsistent.

The next lemma gives the key intuition behind condition II. If the φ-belief closure of every agent is consistent, then any agent initially rejecting φ will keep rejecting it forever, no matter how communication unfolds.

Lemma 7. *Assume that for each agent $i \in V \setminus V_{\downarrow\varphi}$, we have that $BC_\varphi(i) \not\models \bot$. Then for every BFN $\mathcal{B} \models$ B, $\mathcal{B}$-run σ and step $s \in \mathbb{N}$, we have for each agent $i \in V \setminus V_{\downarrow\varphi}$ that $i \in V \setminus Sup_{\mathrm{B}}(\varphi, \sigma, s)$.*

The proof of Lemma 7 roughly proceeds as follows. Toward a contradiction, assume that each belief hull $BC_\varphi(i)$ is consistent. For every agent i initially rejecting φ, one can then associate a world ω_i that satisfies both the beliefs of i and of all agents in its φ-upstream reach set $Reach^-_\varphi(N^-_\varphi(i))$, while falsifying φ. By induction on the steps of any run, this world ω_i remains at least as plausible as any φ-world for all those agents, which prevents i from ever fully accepting φ. Thus, any agent that starts outside the support of φ remains outside it at every step of every run.

Combining Lemmas 6 and 7, we conclude:

Proposition 7. *If $\varphi \in Out(\mathrm{B})$, then there exists an agent $i_* \in V$ such that $BC_\varphi(i_*) \models \bot$.*

5.4 Sufficiency of Conditions I and II

We now show that conditions I and II together are sufficient for φ to be a possible consensus of B. The proof proceeds constructively: assuming that B satisfies both conditions I and II, we exhibit one BFN $\mathcal{B} \models$ B and one $\mathcal{B}$-run in which all agents ultimately converge to the belief φ. The argument unfolds through three essential components, captured by Lemmas 8–10 below, each corresponding to a key phase in the construction of such a run.

Because of condition II, there exists an agent i_* whose belief closure $BC_\varphi(i_*)$ is inconsistent. This implies that i_* initially rejects φ, that is, $i_* \in V \setminus V_{\downarrow\varphi}$. To initiate the construction, we consider communication paths ending in the agents of i_*'s φ-in-neighborhood $N^-_\varphi(i_*)$. Starting from the most distant agents in i_*'s φ-upstream reach set $Reach^-_\varphi(i_*)$, we successively trigger these paths so that the beliefs of the agents in $N^-_\varphi(i_*)$ become as close as possible to φ while remaining in the support of φ, leaving all other agents' epistemic states unchanged. Intuitively, this procedure propagates toward the direct φ-supporting predecessors of i_* the "closest approximation" of φ.

This preliminary stage rests on the following lemma, which guarantees that whenever a sequence of agents linked by a path currently holds beliefs that are jointly consistent, the run can always be extended so that the last agent in that path comes to believe exactly the conjunction of all beliefs along it:

Lemma 8. *Let $\mathcal{B} = \langle (V, A), \vec{\Psi}, \vec{\circ}, \mathcal{S} \rangle$ be a BFN, $i_* \in V$, $p = (i_0, \ldots, i_m)$, $m > 0$, be a path in (V, A) such that $i_* = i_m$, σ be a $\mathcal{B}$-run, $s_* \in \mathbb{N}$, and assume that $\bigwedge\{Bel(\Psi^{i_k}_{\sigma_{s_*}}) \mid k \in \{0, \ldots, m\}\} \not\models \bot$. Then there exist a $\mathcal{B}$-run σ' and a step t such that:*

- *for each $i \in V \setminus \{i_1, \ldots, i_m\}$, $\Psi^i_{\sigma'_t} = \Psi^i_{\sigma_{s_*}}$, and*
- *$Bel(\Psi^{i_*}_{\sigma'_t}) \equiv \bigwedge\{Bel(\Psi^{i_k}_{\sigma_{s_*}}) \mid k \in \{0, \ldots, m\}\}$.*

Once such a configuration is reached, attention turns to the interactions between i_* and its φ-in-neighbors $N^-_\varphi(i_*)$. By construction and by condition II, the joint beliefs of i_*'s φ-in-neighbors are inconsistent with those of i_* at that stage. The goal is to extend the run so that i_*'s beliefs eventually coincide with φ itself. The idea is to assume that, among all worlds not compatible with i_*'s current beliefs, those satisfying φ are the most plausible for i_*. If i_* uses a receptive revision policy (such as Nayak's lexicographic revision operator $\circ_L$), successive communications from all its φ-in-neighbors make φ-worlds in i_*' state increasingly more plausible than any alternative worlds not constantly supported by these neighbors. Since the neighbors' joint beliefs are inconsistent with i_*'s own beliefs, no other worlds remain equally plausible, and after sufficient interactions, i_*'s beliefs become exactly φ.

Formally, given a total preorder $\preceq$ over worlds and $k \in \mathbb{N}$, let $lvl(\preceq, k)$ denote the set of worlds at the k-th layer of $\preceq$, defined as $lvl(\preceq, 0) = \min(\Omega, \preceq)$ and, for each $k \geq 1$, $lvl(\preceq, k) = \min(\Omega \setminus \bigcup_{k' < k} lvl(\preceq, k'), \preceq)$. The following lemma captures the preceding reasoning:

Lemma 9. *Let $\alpha_1, \ldots, \alpha_m$ be m propositional formulas, $m \geq 1$, such that $\varphi \models \bigwedge_{1 \leq i \leq m} \alpha_i$. Let $\circ_L$ be Nayak's lexicographic revision operator on total preorders over worlds. Let Ψ be a total preorder over worlds such that $Bel(\Psi) \wedge \bigwedge_{1 \leq i \leq m} \alpha_i \models \bot$ and $lvl(\Psi, 1) = [\varphi]$. Then $Bel(((\Psi \circ_L \alpha_1) \circ_L \ldots) \circ_L \alpha_m) \equiv \varphi$.*

At this point, the run has reached a configuration in which agent i_*'s beliefs are equivalent to φ. From there, the run can be further extended by triggering communication paths that emanate from i_* and gradually cover all agents in V. Assuming that all agents in the graph are conditionally neutral on φ (an assumption consistent with both conditions I and II, and with the properties so far established for i_*'s epistemic state and revision policy), any agent that receives φ as input will, upon revision, adopt beliefs equivalent to φ. As the influence of i_* propagates throughout the network, each agent in turn aligns her beliefs with φ, and the entire system ultimately converges to a consensus on φ:

Lemma 10. *Let $\mathcal{B} \models \mathrm{B}$, σ be a $\mathcal{B}$-run, $s_* \in \mathbb{N}$, and assume that for each agent $i \in V$, i is conditionally neutral on φ in σ at s_*. If there exists an agent $i_* \in V$ such that $Bel(\Psi^{i_*}_{\sigma_{s_*}}) \equiv \varphi$, then $\varphi \in Out(\mathcal{B})$.*

Combining Lemmas 8, 9, and 10 yields the desired result: whenever both conditions I and II hold for B, there exists a BFN $\mathcal{B} \models \mathrm{B}$ and a corresponding run leading all agents to converge to φ:

Proposition 8. *If:*

I. for each agent $i \in V$, i is observably neutral on φ in B*, and*
II. there exists an agent $i_* \in V$ such that $BC_\varphi(i_*) \models \bot$,

then $\varphi \in Out(\mathrm{B})$.

This completes the proof of Theorem 4.

6 COMPUTATIONAL COMPLEXITY

We now investigate the computational complexity of the consensus membership decision problem:

Definition 5 (CM). The Consensus Membership (CM) decision problem is defined as follows:

- **Input:** BFN scheme $\mathrm{B} = \langle (V, A), \vec{B} \rangle$, propositional formula φ
- **Question:** Does $\varphi \in Out(\mathrm{B})$ hold?

The results from the previous sections directly yield an algorithmic procedure for CM. Algorithm 1 makes this procedure explicit, showing how all conditions defining consensus membership can be verified in polynomial time with polynomially many calls to an NP oracle.

Given as input a BFN scheme $\mathrm{B} = \langle (V, A), \vec{B} \rangle$ and a formula φ, the algorithm first checks whether B is consistent (line 1). If so, Proposition 5 ensures that a unique consensus is possible, namely $\bigwedge_{i \in V} B^i$, and the algorithm returns **true** if φ is equivalent to this conjunction, **false** otherwise (line 2). If the BFN scheme is inconsistent, the algorithm proceeds to check conditions I and II. Lines 3–8 compute the support of φ and verify condition I. For each agent $i \in V$, if $\varphi \models B^i$, i is added to the support of φ (lines 5–6); if instead $B^i \wedge \varphi \not\models \bot$, i is not observably neutral on φ, i.e., condition I is not satisfied, and the algorithm returns **false** (line 8). Reaching line 9 therefore guarantees that all agents are observably neutral on φ, and that the support $V_{\downarrow\varphi}$ has been correctly computed, i.e., that $Support(\mathrm{B}, \varphi) = V_{\downarrow\varphi}$. The algorithm then checks condition II (lines 9–13). For each agent i not in the support of φ, it computes the φ-backward cone $C_\varphi(i)$ (line 10), which can be done in polynomial time given the graph structure. It then constructs the corresponding φ-belief closure $BC_\varphi(i)$ and checks its consistency (line 11). If for some i, $BC_\varphi(i) \models \bot$, condition II is satisfied and the algorithm

Algorithm 1: Deciding whether $\varphi \in Out(\mathrm{B})$

Input: BFN scheme $\mathcal{B} = \langle (V, A), \vec{B} \rangle$, prop. formula φ
Output: **true** if $\varphi \in Out(\mathrm{B})$, **false** otherwise

1 **if** $\bigwedge_{i \in V} B^i \not\models \bot$ **then**
2   **return** $\bigwedge_{i \in V} B^i \equiv \varphi$
3 $Support(\mathrm{B}, \varphi) \leftarrow \emptyset$
4 **for** $i \in V$ **do**
5   **if** $\varphi \models B^i$ **then**
6         $Support(\mathrm{B}, \varphi) \leftarrow Support(\mathrm{B}, \varphi) \cup \{i\}$
7   **else if** $B^i \wedge \varphi \not\models \bot$ **then**
8   **return** false
9 **for** $i \in V \setminus Support(\mathrm{B}, \varphi)$ **do**
10   BackwardCone$(\mathrm{B}, i, \varphi) \leftarrow C_\varphi(i)$
11   **if** $\bigwedge \{B^i \mid i \in$ *BackwardCone*$(\mathrm{B}, i, \varphi)\} \models \bot$ **then**
12   **return** true
13 **return** false

returns **true** (line 12); otherwise, if no such agent exists, it returns **false** (line 13).

Algorithm 1 thus implements a procedure that decides CM. It runs in polynomial time, with polynomially many calls to an NP oracle (in lines 1, 2, 5, 7, and 11). As a direct consequence:

Proposition 9. $\mathsf{CM} \in \mathsf{P}^{\mathsf{NP}}$.

This result places the consensus membership problem at the same computational level as classical inference under iterated belief change [16, 22]. Indeed, it has been shown that the inference problem (*Given an epistemic state Ψ, a sequence of revision formulas $\alpha_1, \ldots, \alpha_n$, and a formula β, does $((\Psi \circ \alpha_1) \circ \ldots) \circ \alpha_n \models \beta$ hold?*) is P^{NP}-complete in general [16], and $\mathsf{P}^{\mathsf{NP}}[O(\log n)]$-complete for specific representations and single-step revision cases ($n = 1$) [22]. Hence, verifying whether a formula can emerge as a consensus given a BFN scheme (without simulating runs explicitly but using only initial beliefs and the graph structure) is computationally on par with reasoning about iterated belief change itself.

7 BACK TO THE EXAMPLE

We now return to the Alice–Bob–Charles example introduced in Section 3, focusing on the underlying BFN scheme $\mathrm{B} = \langle (V, A), \vec{B} \rangle$ defined by $V = \{1, 2, 3\}$, $A = \{(1, 2), (2, 1), (2, 3), (3, 2)\}$, and the belief profile $\vec{B} = \langle B^1, B^2, B^3 \rangle$, where $B^1 = p \wedge q$, $B^2 = r$, and $B^3 = \neg p \wedge \neg q$.

We test whether each of the following five formulas

$$\varphi_1 = p \wedge q \wedge \neg r, \qquad \varphi_2 = p \wedge q \wedge r,$$
$$\varphi_3 = (p \Leftrightarrow \neg q) \wedge r, \qquad \varphi_4 = \neg p \wedge \neg q \wedge r,$$
$$\varphi_5 = \neg p \wedge \neg q \wedge \neg r$$

belong to $Out(\mathrm{B})$, and let $\mathcal{F} = \{\varphi_1, \ldots, \varphi_5\}$.

Since B is an inconsistent BFN scheme, the two conditions of Theorem 4 must be checked. It can be easily verified that for each $\varphi_k \in \mathcal{F}$ and each agent $i \in \{1, 2, 3\}$, either $\varphi_k \models B^i$ or $B^i \wedge \varphi_k \models \bot$. Hence, all agents are observably neutral on every φ_k, and only condition II remains to be tested: for each $\varphi_k \in \mathcal{F}$, we must check whether there exists an agent $i_* \in \{1, 2, 3\}$ such that $BC_{\varphi_k}(i_*) \models \bot$.

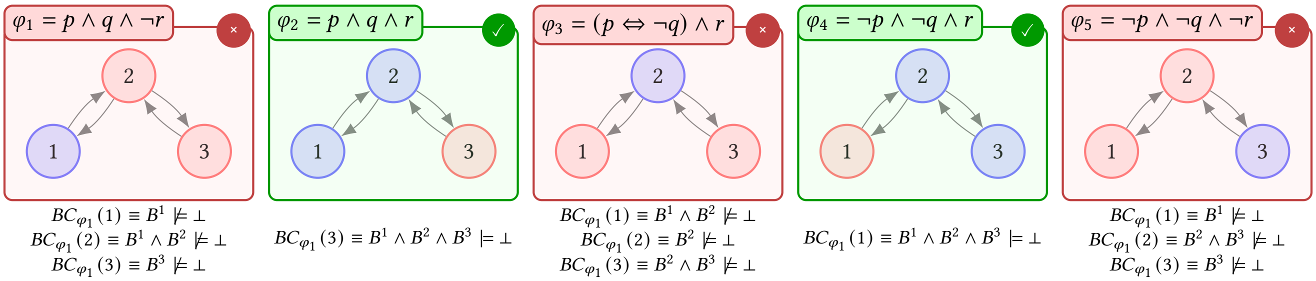

Figure 3: Checking whether each formula $\varphi_k \in \mathcal{F}$ satisfies condition II.

Figure 3 illustrates, for each $\varphi_k \in \mathcal{F}$, the graph annotated with its φ_k-support (following the legend of Figure 2) and the corresponding verification of condition II. The result is that $\varphi_2, \varphi_4 \in Out(\mathrm{B})$, while $\varphi_1, \varphi_3, \varphi_5 \notin Out(\mathrm{B})$. This outcome also leads to a complete description of the set $Out(\mathrm{B})$. Specifically, one can verify that:

- if ψ is a consistent formula such that $\psi \models \varphi_k$ and φ_k satisfies conditions I and II, then ψ also satisfies conditions I and II;
- if ψ is such that $\psi \models \varphi_k \not\models \bot$ and $\psi \models \neg\varphi_k \not\models \bot$ for some φ_k, then ψ does not satisfy condition I.

Since every consistent formula satisfies one of these two cases, it is not difficult to verify that we obtain:

$$Out(\mathrm{B}) = \{\psi \in \mathcal{L}_{\mathcal{P}} \mid \psi \models \varphi_2 \text{ or } \psi \models \varphi_4\}$$

In words, every consensus entails r (which is uncontested by any agent) and either $p \wedge q$ or $\neg p \wedge \neg q$, reflecting that the final consensus coincides with the initial belief of either Alice or Charles, depending on the run, epistemic states, and agents' policies.

As a variant, consider the same BFN scheme but with a complete influence graph, i.e., $A = V \times V$. In this case, similar reasoning shows that:

$$Out(\mathrm{B}) = \{\psi \in \mathcal{L}_{\mathcal{P}} \mid \psi \models \varphi_k \text{ for some } k \in \{1, 2, 4, 5\}\}$$

Here, possible consensuses include formulas entailing $\neg r$. This is because, with direct communication between Alice and Charles, revising one's epistemic state by the other's beliefs can make $\neg r$ prevail as a more entrenched proposition than r in their initial *conditional* beliefs.

8 DISCUSSION

This paper presents a characterization of the formulas that may arise as consensuses in Belief Flow Networks (BFNs). Given a BFN scheme, that is, an influence graph and the agents' initial propositional beliefs, we provide a decision procedure to determine whether a formula is a possible consensus. The characterization is purely logical and graph-based, independent of the agents' internal epistemic states, change policies, and the underlying stochastic dynamics. We show that the problem is solvable in P^{NP}, keeping it computationally feasible despite the expressiveness of the framework.

Our contribution strengthens the theoretical understanding of BFNs beyond their convergence guarantees [21], enabling a precise analysis of which beliefs may emerge as final consensuses. This lays the foundation for planning, verification, and control in logic-based models of belief diffusion, where outcomes need to be both predictable and interpretable.

Several logical approaches to belief change in networks have pursued similar goals. Models such as those by Gallo et al. [11], Cholvy [7], and Vicol et al. [26] use graph-based interactions and propositional representations, but rely on deterministic, often synchronous update schemes. These frameworks focus on properties like local consistency or disagreement minimization, without guaranteeing convergence or characterizing the full space of reachable consensuses. In contrast, BFNs employ asynchronous stochastic dynamics along with formal convergence guarantees, now complemented by a logical characterization of outcomes. Within the belief merging literature, Schwind and Marquis [24] propose an axiomatic account of consensus formation. Like other belief merging models, their framework lacks iterative dynamics and influence graphs, but it shares the goal of capturing rational group agreement under logical constraints. Our results extend this perspective into dynamic, multi-step settings where belief evolution emerges from agent interactions. Other models have addressed opinion diffusion over more structured domains. Brill et al. [5] and Botan et al. [4] analyze convergence under aggregation rules applied to rankings or Boolean vectors with constraints. These works also aim to characterize stable outcomes, though not in logical languages. Aranda et al. [2], building on DeGroot-style models, study fairness and consensus in asynchronous value-based settings. Despite the numerical nature of their opinion space, the asynchronous communication regime and convergence objectives make their model conceptually close to BFNs. Strategic variants of belief diffusion, such as those by Caragiannis et al. [6] and Grandi et al. [12], study equilibrium formation in game-theoretic settings. While they consider stability under rational manipulation, BFNs focus on spontaneous, uncoordinated dynamics under iterated belief change.

A natural direction for future work is to characterize the complete set of reachable consensus formulas, a task that is central to understanding the global behavior of BFNs. Since this set may be prohibitively large to compute in general, a closely related and more tractable alternative is to focus on inference: determining whether a given formula is entailed by all agents in every possible consensus. This question connects directly to belief monitoring and predictive reasoning, for instance, evaluating whether a proposed policy will necessarily lead all agents to conclude that it implies reduced emissions. Both directions aim to deepen our understanding of how beliefs evolve and stabilize in logical diffusion frameworks, and remain largely unexplored within the BFN setting.

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APPENDIX: PROOFS

We start with a couple of lemmata regarding improvement operators, that will be used in subsequent proofs:

Lemma 11. *Let $\circ$ be an improvement operator. Then for each epistemic state Ψ and each formula α, if $Bel(\Psi) \wedge \alpha \not\models \bot$, then $Bel(\Psi \circ \alpha) \equiv Bel(\Psi) \wedge \alpha$.*

Proof. Assume that $Bel(\Psi) \wedge \alpha \not\models \bot$.

Let us first prove that $Bel(\Psi) \wedge \alpha \models Bel(\Psi \circ \alpha)$. Assume toward a contradiction that $Bel(\Psi) \wedge \alpha \not\models Bel(\Psi \circ \alpha)$. This means that there exists a world $\omega \in \Omega$ such that (i) $\omega \models Bel(\Psi)$, (ii) $\omega \models \alpha$ and (iii) $\omega \not\models Bel(\Psi \circ \alpha)$. Since $\alpha \not\models \bot$, by (I3) we get that $Bel(\Psi \circ \alpha) \not\models \bot$, so there exists a world $\omega' \in \Omega$ such that (iv) $\omega' \models Bel(\Psi \circ \alpha)$. By (iii), (iv) and condition (2) of a gradual assignment, we get that (v) $\omega' \prec_{\Psi \circ \alpha} \omega$. On the other hand, by (i) and condition (1-2) of a gradual assignment, we get that (vi) $\omega \preceq_{\Psi} \omega'$. Now, by (ii), (v), (vi) and condition 6 of a gradual assignment, we get that (vii) $\omega' \models \alpha$. Then by (ii), (vi), (vii) and condition 4 of a gradual assignment, we get that $\omega \preceq_{\Psi \circ \alpha} \omega'$, which contradicts (v). This shows that $Bel(\Psi) \wedge \alpha \models Bel(\Psi \circ \alpha)$.

Let us now prove that $Bel(\Psi \circ \alpha) \models Bel(\Psi) \wedge \alpha$. Assume toward a contradiction that $Bel(\Psi \circ \alpha) \not\models Bel(\Psi) \wedge \alpha$. This means that there exists a world $\omega \in \Omega$ such that (i) $\omega \models Bel(\Psi \circ \alpha)$ and (ii) $\omega \not\models Bel(\Psi) \wedge \alpha$. Yet $Bel(\Psi) \wedge \alpha \not\models \bot$, so there exists a world $\omega' \in \Omega$ such that (iii) $\omega' \models Bel(\Psi)$ and (iv) $\omega' \models \alpha$. We fall into one of the following two cases:
Case A: $\omega \not\models Bel(\Psi)$. By (iii) and condition (1-2) of a gradual assignment, we get that (A-v) $\omega' \prec_{\Psi} \omega$. By (iv), we get that (A-vi) $\omega' \prec_{\Psi} \omega$, using condition 4 (resp. 6) of a gradual assignment if $\omega \models \alpha$ (resp. if $\omega \not\models \alpha$). Then by (iv), (A-vi) and condition (1) of a gradual assignment, we get that $\omega \not\models Bel(\Psi \circ \alpha)$, which contradicts (i).
Case B: $\omega \models Bel(\Psi)$. By (ii), we get that (B-v) $\omega \not\models \alpha$. And since $\omega \models Bel(\Psi)$, by (iii) and condition (1) of a gradual assignment, we get that (B-vi) $\omega \simeq_{\Psi} \omega'$. Then by (iv), (B-v), (B-vi) and condition 6 of a gradual assignment, we get that (B-vii) $\omega' \prec_{\Psi \circ \alpha} \omega$. Yet by (i) and condition (1-2) of a gradual assignment, we know that $\omega \preceq_{\Psi \circ \alpha} \omega'$, which contradicts (B-vii).
We have shown that both cases A and B lead to contradiction, that shows that $Bel(\Psi \circ \alpha) \models Bel(\Psi) \wedge \alpha$.

This shows that $Bel(\Psi \circ \alpha) \equiv Bel(\Psi) \wedge \alpha$ and concludes the proof. □

Lemma 12. *Let $\circ$ be an improvement operator. Then for each epistemic state Ψ and each formula α, $Bel(\Psi \circ \alpha) \models Bel(\Psi) \vee \alpha$.*

Proof. Assume toward a contradiction that $Bel(\Psi \circ \alpha) \not\models Bel(\Psi) \vee \alpha$. Then there exists a world $\omega_* \in \Omega$ such that $\omega_* \models Bel(\Psi \circ \alpha)$ and $\omega_* \not\models Bel(\Psi) \vee \alpha$. Since $\omega_* \not\models Bel(\Psi)$, by conditions 1 and 2 of a gradual assignment there exists a world $\omega \in \Omega$ such that $\omega \prec_{\Psi} \omega_*$. Since $\omega_* \not\models \alpha$, we get that $\omega \prec_{\Psi \circ \alpha} \omega_*$ using condition 6 (resp. condition 5) of a gradual assignment if $\omega \models \alpha$ (resp. if $\omega \not\models \alpha$). By conditions 1 and 2 of a gradual assignment, this means that $\omega_* \not\models Bel(\Psi \circ \alpha)$ and leads to a contradiction. This concludes the proof that $Bel(\Psi \circ \alpha) \models Bel(\Psi) \vee \alpha$. □

Lemma 13. *Let $\mathcal{B} = \langle (V, A), \vec{\Psi}, \vec{\circ}, \mathcal{S}\rangle$ be a BFN, and δ be any finite A-sequence $\delta = (e_s)_{s\in\{0,\dots,k\}}$, with $k \geq 0$. Then there exists a $\mathcal{B}$-run σ such that for each step $s \in \{0, \dots, k\}$, $\sigma_s = \delta_s$.*

Proof. Let $\mathcal{B} = \langle (V, A), \vec{\Psi}, \vec{\circ}, \mathcal{S}\rangle$ be a BFN, and δ be any finite A-sequence $\delta = (e_s)_{s\in\{0,\dots,k\}}$, with $k \geq 0$. By definition, the stochastic process $\mathcal{S}$ is such that $\mathcal{S} = (A_s)_{s\in\mathbb{N}}$, where all random variables A_s have the domain set A. Since $\mathcal{S}$ is a chain with complete connections, we know in particular that for each step $s \in \mathbb{N}$ and each edge $(i, j) \in A$, $Pr(A_s = (i, j)) > 0$. This means that $Pr(\bigcap\{A_s = \delta_s \mid s \leq k\}) > 0$. Thus, there exists a $\mathcal{B}$-run σ with prefix δ, i.e., such that for each step $s \in \{0, \dots, k\}$, $\sigma_s = \delta_s$. □

Proposition 4. *For all strongly connected BFNs $\mathcal{B} = \langle (V, A), \vec{\Psi}, \vec{\circ}, \mathcal{S}\rangle$ and $\mathcal{B}' = \langle (V, A), \vec{\Psi}, \vec{\circ}, \mathcal{S}'\rangle$, we have that $Out(\mathcal{B}) = Out(\mathcal{B}')$.*

Proof. It is enough to prove that $Out(\mathcal{B}) \subseteq Out(\mathcal{B}')$. Let $\varphi \in Out(\mathcal{B})$, and let σ a $\mathcal{B}$-run such that $Out_\sigma(\mathcal{B}) = \varphi$. By definition of $Out_\sigma(\mathcal{B})$, we have that $Out_\sigma(i) \equiv \varphi$, which means that for each agent $i \in V$, i is stable in σ and every formula in $Seq^*_\sigma(i)$ is equivalent to $Out_\sigma(i) \equiv \varphi$ (recall that $Seq^*_\sigma(i)$ is the σ-outcome sequence of i in σ). Let us denote by s^i the step such that $Seq^*_\sigma(i) = (Bel(\Psi^i_{\sigma_{s^i}}), Bel(\Psi^i_{\sigma_{s^i+1}}), \dots)$ Then, take $s^* = \min_{i\in V}\{s^i \in \mathbb{N} \mid i \in V\}$. Clearly, for each $i \in V$, $Bel(\Psi^i_{\sigma_{s^*}}) \equiv \varphi$. Let δ be the prefix of σ up to step s^*, i.e., $\delta = (e_s)_{s\in\{0,\dots,s^*\}}$ be defined for each $s \in \{0, \dots, s^*\}$ as $\delta_s = \sigma_s$. Since $\mathcal{B}$ and $\mathcal{B}'$ have the same influence graph (V, A), by Lemma 13, there exists a $\mathcal{B}'$-run σ' with prefix δ, i.e., for each $s \in \{0, \dots, s^*\}$, $\delta_s = \sigma_s$. And since $\mathcal{B}$ and $\mathcal{B}'$ have the same epistemic state profile $\vec{\Psi}$ and change policy profile $\vec{\circ}$, we get for each $i \in V$ that $Bel(\Psi^i_{\sigma'_{s^*}}) \equiv Bel(\Psi^i_{\sigma_{s^*}}) \equiv \varphi$, from which we get that $Out_{\sigma'}(i) \equiv \varphi$, i.e., $Out_\sigma(\mathcal{B}') \equiv \varphi$, thus $\varphi \in Out(\mathcal{B}')$. □

Proposition 5. *Let $\mathrm{B} = \langle (V, A), \vec{B}\rangle$ be a BFN scheme. If B is consistent, then $Out(\mathrm{B}) = \{\bigwedge_{i\in V} B^i\}$.*

Proof. Let $\mathrm{B} = \langle (V, A), \vec{B}\rangle$. We want to show that for each BFN $\mathcal{B}$ such that $\mathcal{B} \models \mathrm{B}$, we have that $Out(\mathcal{B}) = \{\bigwedge_{i\in V} Bel(\Psi^i)\}$. So, let $\mathcal{B} \models \mathrm{B}$. Since B is consistent, we know that $\bigwedge_{i\in V} Bel(\Psi^i) \not\models \bot$. Let σ be any $\mathcal{B}$-run, and let us show that $Out_\sigma(\mathcal{B}) \equiv \bigwedge_{i\in V} Bel(\Psi^i)$. By (AP) we directly get that $\bigwedge_{i\in V} Bel(\Psi^i) \models Out_\sigma(\mathcal{B})$. What remains to be showed is that $Out_\sigma(\mathcal{B}) \models \bigwedge_{i\in V} Bel(\Psi^i)$. Assume toward a contradiction that $Out_\sigma(\mathcal{B}) \not\models \bigwedge_{i\in V} Bel(\Psi^i)$. Recall that $Out_\sigma(\mathcal{B}) \equiv Out_\sigma(i)$ for each agent $i \in V$. So, there is an agent $j \in V$ such that $Out_\sigma(j) \not\models Bel(\Psi^j)$. By definition of $Out_\sigma(j)$ and induction on $Seq_\sigma(j)$ (the σ-outcome sequence of j), there must exist a step $s \in \mathbb{N}$ such that $Bel(\Psi^j_{\sigma_{s+1}}) \not\models Bel(\Psi^j_{\sigma_s})$. Let $(k, l) \in A$ be the edge triggered at step s in σ, i.e., $\sigma_s = (k, l)$. By definition of $Bel(\Psi^j_{\sigma_{s+1}})$ and since $Bel(\Psi^j_{\sigma_s}) \not\equiv Bel(\Psi^j_{\sigma_{s+1}})$, we must have that $l = j$, i.e., $\sigma_s = (k, j)$, and $\Psi^j_{\sigma_{s+1}} = \Psi^j_{\sigma_s} \circ_j Bel(\Psi^k_{\sigma_s})$. Recall that $Bel(\Psi^j_{\sigma_{s+1}}) \not\models Bel(\Psi^j_{\sigma_s})$, which is then equivalently written as $Bel(\Psi^j_{\sigma_s} \circ_j Bel(\Psi^k_{\sigma_s})) \not\models Bel(\Psi^j_{\sigma_s})$. By Lemma 11, we get that $Bel(\Psi^j_{\sigma_s}) \wedge Bel(\Psi^k_{\sigma_s}) \models \bot$. But since $\bigwedge_{i\in V} Bel(\Psi^i) \not\models \bot$, (AP) requires that $Bel(\Psi^j_{\sigma_s}) \wedge Bel(\Psi^k_{\sigma_s}) \not\models \bot$. This leads to a contradiction, from which we can conclude that $Out_\sigma(\mathcal{B}) \models \bigwedge_{i\in V} Bel(\Psi^i)$. We got that $Out_\sigma(\mathcal{B}) \equiv \bigwedge_{i\in V} Bel(\Psi^i)$ for each $\mathcal{B}$-run σ. Hence, $Out(\mathcal{B}) = \{\bigwedge_{i\in V} Bel(\Psi^i)\}$. Therefore, $Out(\mathrm{B}) = \{\bigwedge_{i\in V} B^i\}$. □

Lemma 1. *Let $i \in V$, σ be a $\mathcal{B}$-run and $s \in \mathbb{N}$. Then $i \in V$ is conditionally neutral on φ in σ at s if and only if for all worlds ω, $\omega' \models \varphi$, we have that $\omega \simeq^i_{\Psi^i_{\sigma_s}} \omega'$.*

Proof. Let ω, $\omega' \models \varphi$. We have that $i \in V$ is conditionally neutral on φ in σ at s if and only if $Bel(\Psi^i_{\sigma_s} \star_{\circ_i} \varphi) \equiv \varphi$, if and only if $\omega \simeq^i_{\Psi^i_{\sigma_s} \star_{\circ_i} \varphi} \omega'$ (using conditions 1 and 2 of a gradual assignment), if and only if $\omega \simeq^i_{\Psi^i_{\sigma_s}} \omega'$ (using condition 4 of a gradual assignment iteratively). □

Lemma 2. *Let $i \in V$, σ be a $\mathcal{B}$-run and $s \in \mathbb{N}$. If i is conditionally neutral on φ in σ at s, then i is observably neutral on φ in σ at s.*

Proof. Let φ be a formula, $i \in V$, σ be a $\mathcal{B}$-run and $s \in \mathbb{N}$. Assume that i is conditionally neutral on φ in σ at step s. We need to prove that i is observably neutral on φ in σ at step s, which by definition boils down to prove that if $Bel(\Psi^i_{\sigma_s}) \wedge \varphi \not\models \bot$, then $\varphi \models Bel(\Psi^i_{\sigma_s})$. So, assume that $Bel(\Psi^i_{\sigma_s}) \wedge \varphi \not\models \bot$. Since $\circ_i$ satisfies postulate (I2), we get that $Bel(\Psi^i_{\sigma_s} \star_{\circ_i} \varphi) \equiv Bel(\Psi^i_{\sigma_s}) \wedge \varphi$, so $Bel(\Psi^i_{\sigma_s}) \wedge \varphi \equiv \varphi$, which implies that $\varphi \models Bel(\Psi^i_{\sigma_s})$ and concludes the proof. □

Lemma 3. *The following statements are equivalent:*

1. *for each $i \in V$, i is conditionally neutral on φ in $\mathcal{B}$*
2. *for each $\mathcal{B}$-run σ, each step $s \in \mathbb{N}$ and each $i \in V$, i is conditionally neutral on φ in σ at step s*
3. *there exists a $\mathcal{B}$-run σ and a step $s \in \mathbb{N}$ such that for each $i \in V$, i is conditionally neutral on φ in σ at step s*

Proof. ($2 \Rightarrow 3$) Trivial.

($3 \Rightarrow 1$) Let us prove the contrapositive statement. That is, assume the negation of condition 1, i.e., that there is an agent $i \in V$ that is not conditionally neutral on φ in $\mathcal{B}$, and let us prove that the negation of condition 3 holds. Since i is not conditionally neutral on φ in $\mathcal{B}$, $Bel(\Psi^i \star_{\circ_i} \varphi) \not\equiv \varphi$. By Lemma 1 we know there exist two worlds $\omega, \omega' \models \varphi$ such that $\omega \prec^i_{\Psi^i} \omega'$.

Let us prove that for every $\mathcal{B}$-run σ, at each step $s \in \mathbb{N}$ there is an agent $j \in V$ such that $\omega \not\simeq^j_{\Psi^j_{\sigma_s}} \omega'$. We prove it by induction on the step s. The base case when $s = 0$ is direct with $j = i$ since we already have that $\omega \prec^i_{\Psi^i} \omega'$. Now, let $s \geq 0$ and assume there exists $j \in V$ such that $\omega \not\simeq^j_{\Psi^j_{\sigma_s}} \omega'$. We want to prove there exists $k \in V$ such that $\omega \not\simeq^k_{\Psi^k_{\sigma_{s+1}}} \omega'$. Let $(l, m) \in A$ be the edge triggered at step s in σ, i.e., $\sigma_s = (l, m)$. If $m \neq j$, then $\Psi^j_{\sigma_{s+1}} = \Psi^j_{\sigma_s}$ by definition, and the claim directly holds with $k = j$. Then assume $m = j$. Then $\sigma_s = (l, j)$ and $\Psi^j_{\sigma_{s+1}} = \Psi^j_{\sigma_s} \circ_j Bel(\Psi^l_{\sigma_s})$ by definition. The case when $\omega \not\simeq^j_{\Psi^j_{\sigma_{s+1}}} \omega'$ directly proves the claim with $k = j$, so assume we fall in the other case, i.e., $\omega \simeq^j_{\Psi^j_{\sigma_{s+1}}} \omega'$. Recall that $\omega \not\simeq^j_{\Psi^j_{\sigma_s}} \omega'$, so (i) $\omega \prec^j_{\Psi^j_{\sigma_s}} \omega'$ or (ii) $\omega' \prec^j_{\Psi^j_{\sigma_s}} \omega$. Since $\Psi^j_{\sigma_{s+1}} = \Psi^j_{\sigma_s} \circ_j Bel(\Psi^l_{\sigma_s})$, by conditions 4-6 of a gradual assignment, in case (i) we get that $\omega \models Bel(\Psi^l_{\sigma_s})$ and $\omega' \not\models Bel(\Psi^l_{\sigma_s})$, and in case (ii) that $\omega' \models Bel(\Psi^l_{\sigma_s})$ and $\omega \not\models Bel(\Psi^l_{\sigma_s})$, which means by condition 2 of a gradual assignment that $\omega \prec^l_{\Psi^l_{\sigma_s}} \omega'$ or $\omega' \prec^l_{\Psi^l_{\sigma_s}} \omega$, i.e., $\omega \not\simeq^l_{\Psi^l_{\sigma_s}} \omega'$. Yet $\sigma_s = (l, j)$, so $l \neq j$ (recall that

A is irreflexive). Then $\Psi^l_{\sigma_{s+1}} = \Psi^l_{\sigma_s}$ by definition, so $\omega \not\simeq^l_{\Psi^l_{\sigma_{s+1}}} \omega'$, which proves the claim with $k = l$.

At this stage, we proved that for every $\mathcal{B}$-run σ and each step $s \in \mathbb{N}$, there exists $j \in V$ and two worlds $\omega, \omega' \models \varphi$ such that $\omega \not\simeq^j_{\Psi^j_{\sigma_s}} \omega'$, which by Lemma 1 means that j is not conditionally neutral on φ in σ at s. This means that the negation of condition 3 holds, which concludes the proof that the negation of condition 1 implies the negation of condition 3, or equivalently, that $1 \Rightarrow 3$.

($1 \Rightarrow 2$) Assume that for each $i \in V$, i is conditionally neutral on φ in $\mathcal{B}$. We prove that for each $\mathcal{B}$-run σ, each step $s \in \mathbb{N}$ and each $i \in V$, i is conditionally neutral on φ in σ at step s by induction on s. The base case when $s = 0$ is direct, since by definition, an agent i is conditionally neutral on φ in $\mathcal{B}$ precisely when i is conditionally neutral on φ in σ at step 0, for any $\mathcal{B}$-run σ. Then, let σ be a $\mathcal{B}$-run, $s \in \mathbb{N}$ and assume that for each $i \in V$, i is conditionally neutral on φ in σ at step s. Let $i \in V$ and let us prove that i is conditionally neutral on φ in σ at step $s+1$. Denote $\sigma_s = (j, k)$. If $i \neq k$ then $\Psi^i_{\sigma_{s+1}} = \Psi^i_{\sigma_s}$ by definition, from which we directly get that i is conditionally neutral on φ in σ at step $s+1$. If $i = k$ then $\Psi^i_{\sigma_{s+1}} = \Psi^i_{\sigma_s} \circ_i Bel(\Psi^j_{\sigma_s})$ by definition. Yet j is conditionally neutral on φ in $\mathcal{B}$ in σ at step s, so by Lemma 2, (i) $Bel(\Psi^j_{\sigma_s}) \wedge \varphi \models \bot$ or (ii) $\varphi \models Bel(\Psi^j_{\sigma_s})$; and i is also conditionally neutral on φ in σ at step s, so by Lemma 1, for all worlds ω, ω' such that $\omega, \omega' \models \varphi$, $\omega \simeq^i_{\Psi^i_{\sigma_s}} \omega'$. Then we can see that $\omega \simeq^i_{\Psi^i_{\sigma_{s+1}}} \omega'$, using condition 5 of a gradual assignment in case (i), and using condition 4 of a gradual assignment in case (ii). By Lemma 1 again, we get that i is conditionally neutral on φ in σ at step $s+1$. This shows by induction that for each $\mathcal{B}$-run σ, each step $s \in \mathbb{N}$ and each $i \in V$, i is conditionally neutral on φ in σ at step s, and concludes the proof that $1 \Rightarrow 2$.

We have proved that $1 \Rightarrow 2$, $2 \Rightarrow 3$, and $3 \Rightarrow 1$, which concludes the proof that the three statements are equivalent. □

Lemma 4. *Let $\mathcal{B} \models \mathrm{B}$. If $\varphi \in Out(\mathcal{B})$, then for each agent $i \in V$, i is conditionally neutral on φ in $\mathcal{B}$.*

Proof. Let us prove the contrapositive statement of the proposition. Let φ be formula and assume that there is an agent $i \in V$ that is not conditionally neutral on φ in $\mathcal{B}$. Using Lemma 3, more precisely the fact that ($3 \Rightarrow 1$), we get that for every $\mathcal{B}$-run σ and each step $s \in \mathbb{N}$, there exists $j \in V$ such that j is not conditionally neutral on φ in σ at s; which by Lemma 1, this means that there exists two worlds $\omega, \omega' \models \varphi$ such that $\omega \prec^j_{\Psi^j_{\sigma_s}} \omega'$. By condition 2 of a gradual assignment, $\omega' \not\models Bel(\Psi^j_{\sigma_s})$, so $Bel(\Psi^j_{\sigma_s}) \not\equiv \varphi$; and then, for each $\mathcal{B}$-run σ, $\varphi \not\equiv Out_\sigma(\mathcal{B})$. This shows that $\varphi \notin Out(\mathcal{B})$ and concludes the proof. □

Proposition 6. *If $\varphi \in Out(\mathrm{B})$, then for each $i \in V$, i is observably neutral on φ in* B.

Proof. If $\varphi \in Out(\mathrm{B})$, this means that there exists a BFN $\mathcal{B} \models \mathrm{B}$ such that $\varphi \in Out(\mathcal{B})$. By Lemma 4, each agent $i \in V$ is conditionally neutral on φ in $\mathcal{B}$. By Lemma 3 (more precisely the fact that $1 \Rightarrow 2$) and Lemma 2, each agent $i \in V$ is observably neutral on φ in every $\mathcal{B}$-run σ at step 0, which equivalently means that each agent $i \in V$ is observably neutral on φ in $\mathcal{B}$. By definition of observable neutrality and since $\mathcal{B} \models \mathrm{B}$, each agent $i \in V$ is observably neutral on φ in B. This concludes the proof. □

Lemma 5. *Let $\mathcal{B} \models \mathrm{B}$. If $\varphi \in Out(\mathcal{B})$, then for each $\mathcal{B}$-run σ and step $s \in \mathbb{N}$, $Sup_{\mathcal{B}}(\varphi, \sigma, s) = \{i \in V \mid \varphi \models Bel(\Psi^i_{\sigma_s})\}$.*

Proof. Let $i \in V$ be any agent from $Sup_{\mathcal{B}}(\varphi, \sigma, s)$, i.e., $Bel(\Psi^i_{\sigma_s}) \wedge \varphi \not\models \bot$, and let us prove that $\varphi \models Bel(\Psi^i_{\sigma_s})$. Since $\varphi \in Out(\mathcal{B})$, we know from Lemma 4 that i is conditionally neutral on φ in $\mathcal{B}$, which means that $Bel(\Psi^i \star_{\circ_i} \varphi) \equiv \varphi$. And by Lemma 3 (specifically, the fact that $1 \Rightarrow 2$) we get that $Bel(\Psi^i_{\sigma_s} \star_{\circ_i} \varphi) \equiv \varphi$. Yet since $Bel(\Psi^i_{\sigma_s}) \wedge \varphi \not\models \bot$, from Lemma 2 this means that $\varphi \models Bel(\Psi^i_{\sigma_s})$. This proves that for any $\mathcal{B}$-run σ and step $s \in \mathbb{N}$, $Sup_{\mathcal{B}}(\varphi, \sigma, s) = \{i \in V \mid \varphi \models Bel(\Psi^i_{\sigma_s})\}$. □

Lemma 6. *If $\varphi \in Out(\mathrm{B})$, then $V \setminus V_{\downarrow\varphi} \neq \emptyset$.*

Proof. Assume toward a contradiction that $\varphi \in Out(\mathrm{B})$ and $V \setminus V_{\downarrow\varphi} = \emptyset$. Since $\varphi \in Out(\mathrm{B})$, there exists a BFN $\mathcal{B} \models \mathrm{B}$ such that $\varphi \in Out(\mathcal{B})$. And since $V \setminus V_{\downarrow\varphi} = \emptyset$, we have that $V_{\downarrow\varphi} = V$. By Lemma 5, for each $\mathcal{B}$-run σ, $Sup_{\mathcal{B}}(\varphi, \sigma, 0) = \{i \in V \mid \varphi \models Bel(\Psi^i_{\sigma_0})\}$, which can be written as $V_{\downarrow\varphi} = \{i \in V \mid \varphi \models B^i\}$. This means that for each agent $iinV$, we have that $\varphi \models B^i$, or stated equivalently, $\varphi \models \bigwedge\{B^i \mid i \in V\}$. Since $\varphi \not\models \bot$, $\bigwedge\{B^i \mid i \in V\} \not\models \bot$. This contradicts the assumption that B is inconsistent. This concludes the proof that if $\varphi \in Out(\mathrm{B})$, then $V \setminus V_{\downarrow\varphi} \neq \emptyset$. □

Lemma 7. *Assume that for each agent $i \in V \setminus V_{\downarrow\varphi}$, we have that $BC_\varphi(i) \not\models \bot$. Then for every BFN $\mathcal{B} \models \mathrm{B}$, $\mathcal{B}$-run σ and step $s \in \mathbb{N}$, we have for each agent $i \in V \setminus V_{\downarrow\varphi}$ that $i \in V \setminus Sup_{\mathrm{B}}(\varphi, \sigma, s)$.*

Proof. Assume that for each agent $i \in V \setminus V_{\downarrow\varphi}$, we have that $BC_\varphi(i) \not\models \bot$. This means that each agent $i \in V \setminus V_{\downarrow\varphi}$ can be associated with a world $\omega_i \in \Omega$ such that:

$$\omega_i \models BC_\varphi(i) \tag{1}$$

First, observe that by definition of $V_{\downarrow\varphi}$, for each $i \in V \setminus V_{\downarrow\varphi}$:

$$B^i \wedge \varphi \models \bot \tag{2}$$

And since $i \in C_\varphi(i)$, $BC_\varphi(i) \models B^i$, so from Equations 1 and 2, we get that:

$$\omega_i \models B^i \wedge \neg\varphi \tag{3}$$

Now, let $\mathcal{B} \models \mathrm{B}$ be a BFN and σ be a $\mathcal{B}$-run.

We first intend to prove by induction on the step s that for each $s \in \mathbb{N}$:

$P(s)$: for each $i \in V \setminus V_{\downarrow\varphi}$ and each world $\omega \models \varphi$, $\omega_i \prec^i_{\Psi^i_{\sigma_s}} \omega$

$Q(s)$: for each $i \in V \setminus V_{\downarrow\varphi}$, for each $j \in Reach^-_\varphi(N^-_\varphi(i))$ and each world $\omega \models \varphi$, $\omega_i \preceq^j_{\Psi^j_{\sigma_s}} \omega$

Let us first prove the base case when $s = 0$, i.e., let us prove that $P(0)$ and $Q(0)$ both hold. Let us start with $P(0)$. Assume $i \in V \setminus V_{\downarrow\varphi}$. By Equation 2, $Bel(\Psi^i) \wedge \varphi \models \bot$, and by Equation 3, $\omega_i \models Bel(\Psi^i)$. Then by conditions 1 and 2 of a gradual assignment, we directly get that for each world $\omega \models \varphi$, $\omega_i \prec^i_{\Psi^i} \omega$, or stated equivalently, $\omega_i \prec^i_{\Psi^i_{\sigma_0}} \omega$. This proves that $P(0)$ holds. Let us now prove $Q(0)$. Let $i \in V \setminus V_{\downarrow\varphi}$. Since by Equation 1, $\omega_i \models BC_\varphi(i)$, by definition of $BC_\varphi(i)$ and $C_\varphi(i)$ we get that $\omega_i \models Bel(\Psi^j)$ for each $j \in Reach^-_\varphi(N^-_\varphi(i))$. So, by conditions 1 and 2 of a gradual assignment, we get for each $j \in Reach^-_\varphi(N^-_\varphi(i))$ that $\omega_i \preceq^j_{\Psi^j} \omega$, for each world $\omega \models \varphi$, or stated equivalently, $\omega_i \preceq^j_{\Psi^j_{\sigma_0}} \omega$. This proves that $Q(0)$ holds.

Now, let $s \in \mathbb{N}$, assume that $P(s)$ and $Q(s)$ hold, and let us show that $P(s+1)$ and $Q(s+1)$ hold. Let $\sigma_s = (k, l)$, i.e., $(k, l) \in A$ is the edge triggered at step s in σ. Let us start with $P(s+1)$. Let $i \in V \setminus V_{\downarrow\varphi}$. If $l \neq i$, then $\Psi^i_{\sigma_{s+1}} = \Psi^i_{\sigma_s}$ by definition, and then $P(s+1)$ follows trivially from $P(s)$. So assume that $l = i$, i.e., $\sigma_s = (k, i)$. Then by definition, $\Psi^i_{\sigma_{s+1}} = \Psi^i_{\sigma_s} \circ_i Bel(\Psi^k_{\sigma_s})$. Remark that by $P(s)$, for each world $\omega \models \varphi$, $\omega_i \prec^i_{\Psi^i_{\sigma_s}} \omega$. We now fall into one of the following two cases:

- Case 1: $k \in V \setminus V_{\downarrow\varphi}$. Then by $P(s)$, for each world $\omega \models \varphi$, $\omega_k \prec^k_{\Psi^k_{\sigma_s}} \omega$. Yet $\omega_k \models \neg\varphi$ (cf. Equation 3), so by conditions 1 and 2 of a gradual assignment, $Bel(\Psi^k_{\sigma_s}) \wedge \varphi \models \bot$. Yet we already established that for each world $\omega \models \varphi$, $\omega_i \prec^i_{\Psi^i_{\sigma_s}} \omega$. So, by conditions 5 and 6 of a gradual assignment, we get that for each world $\omega \models \varphi$, $\omega_i \prec^i_{\Psi^i_{\sigma_{s+1}}} \omega$.

- Case 2: $k \in V_{\downarrow\varphi}$. Since $(k, i) \in A$, $k \in N^-_\varphi(i)$, so $k \in Reach^-_\varphi(N^-_\varphi(i))$. Then by $Q(s)$, for each world $\omega \models \varphi$, $\omega_i \preceq^k_{\Psi^k_{\sigma_s}} \omega$. We fall into one of the following two subcases: $\omega_i \models Bel(\Psi^k_{\sigma_s})$ (case 2-a), or $\omega_i \not\models Bel(\Psi^k_{\sigma_s})$ (case 2-b). In case 2-a, since for each world $\omega \models \varphi$, $\omega_i \prec^i_{\Psi^i_{\sigma_s}} \omega$, from conditions 4 and 6 of a gradual assignment we get that for each world $\omega \models \varphi$, $\omega_i \prec^i_{\Psi^i_{\sigma_{s+1}}} \omega$. In case 2-b, since for each world $\omega \models \varphi$, $\omega_i \preceq^k_{\Psi^k_{\sigma_s}} \omega$, conditions 1 and 2 of a gradual assignment tell us that for each world $\omega \models \varphi$, $\omega \not\models Bel(\Psi^k_{\sigma_s})$; and since for each world $\omega \models \varphi$, $\omega_i \prec^i_{\Psi^i_{\sigma_s}} \omega$, from conditions 5 and 6 of a gradual assignment we get that for each world $\omega \models \varphi$, $\omega_i \prec^i_{\Psi^i_{\sigma_{s+1}}} \omega$.

We have proved that $\omega_i \prec^i_{\Psi^i_{\sigma_{s+1}}} \omega$ in every case, which concludes the proof that $P(s+1)$ holds.

Let us now prove $Q(s+1)$. Let $i \in V \setminus V_{\downarrow\varphi}$, and $j \in Reach^-_\varphi(N^-_\varphi(i))$. Recall that $\sigma_s = (k, l)$, i.e., $(k, l) \in A$ is the edge triggered at step s in σ. If $l \neq j$, then $\Psi^j_{\sigma_{s+1}} = \Psi^j_{\sigma_s}$ by definition, and then $Q(s+1)$ follows trivially from $Q(s)$. So assume that $l = j$, i.e., $\sigma_s = (k, j)$. Then by definition, $\Psi^j_{\sigma_{s+1}} = \Psi^j_{\sigma_s} \circ_j Bel(\Psi^k_{\sigma_s})$. Remark that by $Q(s)$, for each world $\omega \models \varphi$, $\omega_i \preceq^j_{\Psi^j_{\sigma_s}} \omega$. We now fall into one of the following two cases:

- Case 1: $k \in V \setminus V_{\downarrow\varphi}$. Then by $P(s)$, for each world $\omega \models \varphi$, $\omega_k \prec^k_{\Psi^k_{\sigma_s}} \omega$. Yet recall that $\omega_k \models \neg\varphi$ (cf. Equation 3), so by conditions 1 and 2 of a gradual assignment, $Bel(\Psi^k_{\sigma_s}) \wedge \varphi \models \bot$. Yet we already established that for each world $\omega \models \varphi$, $\omega_i \preceq^j_{\Psi^j_{\sigma_s}} \omega$. So, by conditions 5 and 6 of a gradual assignment, we get that for each world $\omega \models \varphi$, $\omega_i \preceq^j_{\Psi^j_{\sigma_{s+1}}} \omega$.

- Case 2: $k \in V_{\downarrow\varphi}$. Since $(k, j) \in A$ and $j \in Reach^-_\varphi(N^-_\varphi(i))$, by definition of $Reach^-_\varphi(N^-_\varphi(i))$ we also get that $k \in Reach^-_\varphi(N^-_\varphi(i))$. Then by $Q(s)$, for each world $\omega \models \varphi$, $\omega_i \preceq^k_{\Psi^k_{\sigma_s}} \omega$. We fall into one of the following two subcases: $\omega_i \models Bel(\Psi^k_{\sigma_s})$ (case 2-a), or $\omega_i \not\models Bel(\Psi^k_{\sigma_s})$ (case 2-b). In case 2-a, since for each world $\omega \models \varphi$, $\omega_i \preceq^j_{\Psi^j_{\sigma_s}} \omega$, from conditions 4 and 6 of a gradual assignment we get that for each world $\omega \models \varphi$, $\omega_i \preceq^j_{\Psi^j_{\sigma_{s+1}}} \omega$. In case 2-b, since for each world $\omega \models \varphi$, $\omega_i \preceq^k_{\Psi^k_{\sigma_s}} \omega$, conditions 1 and 2 of a gradual assignment tell us that for each world $\omega \models \varphi$, $\omega \not\models Bel(\Psi^k_{\sigma_s})$; and since for each world $\omega \models \varphi$, $\omega_i \preceq^j_{\Psi^j_{\sigma_s}} \omega$, from conditions 5 and 6 of a gradual assignment we get that for each world $\omega \models \varphi$, $\omega_i \preceq^j_{\Psi^j_{\sigma_{s+1}}} \omega$.

We have proved that $\omega_i \preceq^j_{\Psi^j_{\sigma_{s+1}}} \omega$ in every case, which concludes the proof that $Q(s+1)$ holds.

This concludes the proof by induction on s that for each $s \in \mathbb{N}$, $P(s)$ and $Q(s)$ hold.

We now go back to the proof of our lemma. That is, let $i \in V \setminus V_{\downarrow\varphi}$, we need to prove that for each step $s \in \mathbb{N}$, $i \in V \setminus Sup_{\mathrm{B}}(\varphi, \sigma, s)$. Yet for each $s \in \mathbb{N}$, from $P(s)$ we know that for each world $\omega \models \varphi$, $\omega_i \prec^i_{\Psi^i_{\sigma_s}} \omega$; by conditions 1 and 2 of a gradual assignment, this means that $Bel(\Psi^i_{\sigma_s}) \wedge \varphi \models \bot$, i.e., $i \in V \setminus Sup_{\mathrm{B}}(\varphi, \sigma, s)$. This concludes the proof. □

Proposition 7. *If $\varphi \in Out(\mathrm{B})$, then there exists an agent $i_* \in V$ such that $BC_\varphi(i_*) \models \bot$.*

Proof. Toward a contradiction, assume that $\varphi \in Out(\mathrm{B})$ and that for each $i \in V$, $BC_\varphi(i) \not\models \bot$. In particular, for each $i \in V \setminus V_{\downarrow\varphi}$, $BC_\varphi(i) \not\models \bot$. By Lemma 6, there exists such an agent $i_* \in V \setminus V_{\downarrow\varphi}$, and Lemma 7 tells us that for every BFN $\mathcal{B} \models \mathrm{B}$, $\mathcal{B}$-run σ and step $s \in \mathbb{N}$, we have that $i_* \in V \setminus Sup_{\mathrm{B}}(\varphi, \sigma, s)$, i.e., $Bel(\Psi^{i_*}_{\sigma_s}) \wedge \varphi \models \bot$ by definition of $Sup_{\mathrm{B}}(\varphi, \sigma, s)$, so that $\varphi \notin Out_\sigma(i_*)$, thus $\varphi \notin Out_\sigma(\mathcal{B})$. We got that for every BFN $\mathcal{B} \models \mathrm{B}$, $\varphi \notin Out(\mathcal{B})$, which contradicts the fact that $\varphi \in Out(\mathrm{B})$. This concludes the proof that if $\varphi \in Out(\mathrm{B})$, there exists $i \in V$ such that $BC_\varphi(i) \models \bot$. □

Lemma 8. *Let $\mathcal{B} = \langle (V, A), \vec{\Psi}, \vec{\circ}, \mathcal{S} \rangle$ be a BFN, $i_* \in V$, $p = (i_0, \ldots, i_m)$, $m > 0$, be a path in (V, A) such that $i_* = i_m$, σ be a $\mathcal{B}$-run, $s_* \in \mathbb{N}$, and assume that $\bigwedge\{Bel(\Psi^{i_k}_{\sigma_{s_*}}) \mid k \in \{0, \ldots, m\}\} \not\models \bot$. Then there exist a $\mathcal{B}$-run σ' and a step t such that:*

- *for each $i \in V \setminus \{i_1, \ldots, i_m\}$, $\Psi^i_{\sigma'_t} = \Psi^i_{\sigma_{s_*}}$, and*
- *$Bel(\Psi^{i_*}_{\sigma'_t}) \equiv \bigwedge\{Bel(\Psi^{i_k}_{\sigma_{s_*}}) \mid k \in \{0, \ldots, m\}\}$.*

Proof. Let $\mathcal{B} = \langle (V, A), \vec{\Psi}, \vec{\circ}, \mathcal{S} \rangle$ be a BFN, $i_* \in V$, $p = (i_0, \ldots, i_m)$, $m > 0$, be a path in (V, A) such that $i_* = i_m$, σ be a $\mathcal{B}$-run, $s_* \in \mathbb{N}$, and assume that $\bigwedge\{Bel(\Psi^{i_k}_{\sigma_{s_*}}) \mid k \in \{1, \ldots, m\}\} \not\models \bot$. Let δ be the finite A-sequence $\delta = (e_s)_{s \in \{0, \ldots, m-1\}}$ defined for each $s \in \{0, \ldots, m-1\}$ as $\delta_s = (i_s, i_{s+1})$. So, for instance, if $p = (3, 1, 7, 4)$, then $\delta = ((3, 1), (1, 7), (7, 4))$. By Lemma 13, we know that there exists a $\mathcal{B}$-run σ' with prefix $(\sigma_0, \ldots, \sigma_{s_*-1}, \delta_0, \ldots, \delta_{m-1})$, i.e., for each step $s \in \mathbb{N}$, if $s < s_*$ then $\sigma'_s = \sigma_s$, and if $s_* \leq s \leq s_* + m - 1$, then $\sigma'_s = \delta_{s-s_*}$.

Let us prove by induction that for each step $s \in \{0, \ldots, m\}$, we have that:

$P(s)$: for each $i \in V \setminus \{i_1, \ldots, i_s\}$, $\Psi^i_{\sigma'_{s_*+s}} = \Psi^i_{\sigma_{s_*}}$

$Q(s)$: $\bigwedge\{Bel(\Psi^{i_k}_{\sigma'_{s_*+s}}) \mid k \in \{0, \ldots, m\}\} \not\models \bot$.

$R(s)$: $Bel(\Psi^{i_s}_{\sigma'_{s_*+s}}) \equiv \bigwedge\{Bel(\Psi^{i_k}_{\sigma_{s_*}}) \mid k \in \{0, \ldots, s\}\}$

Let us first prove the base case when $s = 0$, i.e., let us prove that $P(0)$, $Q(0)$ and $R(0)$ both hold. Clearly enough, since σ and σ' have the same prefix up to step $s_* - 1$, we get for each agent $i \in V$ that $\Psi^i_{\sigma'_{s_*}} = \Psi^i_{\sigma_{s_*}}$. This trivially shows that $P(0)$ and $R(0)$ hold. Moreover, the fact that $Q(0)$ holds is also trivial from our initial assumption that $\bigwedge\{Bel(\Psi^{i_k}_{\sigma_{s_*}}) \mid k \in \{0, \dots, m\}\} \not\models \bot$. This concludes the proof for the base case when $s = 0$.

Now, let $s \in \{0, \dots, m-1\}$, assume that $P(s')$, $Q(s')$ and $R(s')$ both hold for all $s' \in \{0, \dots, s\}$, and let us prove that $P(s+1)$, $Q(s+1)$ and $R(s+1)$ hold.

Since $\sigma'_{s_*+s} = \delta_s = (i_s, i_{s+1})$, we have by definition of $\Psi^i_{\sigma'_{s_*+s+1}}$ for each agent $i \in V$ that:

$$\Psi^i_{\sigma'_{s_*+s+1}} = \begin{cases} \Psi^i_{\sigma'_{s_*+s}} \circ_i Bel(\Psi^{i_s}_{\sigma'_{s_*+s}}), & \text{if } i = i_{s+1} \\ \Psi^i_{\sigma'_{s_*+s}}, & \text{otherwise} \end{cases} \tag{4}$$

Let us first prove $P(s+1)$. Let $i \in V \setminus \{i_1, \dots, i_{s+1}\}$. Since $\Psi^i_{\sigma'_{s_*+s+1}} = \Psi^i_{\sigma'_{s_*+s}}$ when $i \in V \setminus \{i_{s+1}\}$ (cf. Equation 4 above), this trivially gives us $\Psi^i_{\sigma'_{s_*+s+1}} = \Psi^i_{\sigma'_{s_*+s}}$ when $i \in V \setminus \{i_1, \dots, i_{s+1}\}$. Yet by $P(s)$, $\Psi^i_{\sigma'_{s_*+s}} = \Psi^i_{\sigma'_{s_*}}$ when $i \in V \setminus \{i_1, \dots, i_s\}$, which holds in particular when $i \in V \setminus \{i_1, \dots, i_{s+1}\}$. We got that for each $i \in V \setminus \{i_1, \dots, i_{s+1}\}$, $\Psi^i_{\sigma'_{s_*+s+1}} = \Psi^i_{\sigma'_{s_*}}$. This proves $P(s+1)$.

Let us now prove $Q(s+1)$. For each agent $i \in V$, based on Equation 4, using $Q(s)$ and Lemma 11, and since $\circ_i$ satisfies (I4), we have that:

$$Bel(\Psi^i_{\sigma'_{s_*+s+1}}) \equiv \begin{cases} Bel(\Psi^i_{\sigma'_{s_*+s}}) \wedge Bel(\Psi^{i_s}_{\sigma'_{s_*+s}}), & \text{if } i = i_{s+1} \\ Bel(\Psi^i_{\sigma'_{s_*+s}}), & \text{otherwise} \end{cases} \tag{5}$$

Clearly, Equation 5 tells us that $\bigwedge\{Bel(\Psi^{i_k}_{\sigma'_{s_*+s+1}}) \mid k \in \{0, \dots, m\}\} \equiv \bigwedge\{Bel(\Psi^{i_k}_{\sigma'_{s_*+s}}) \mid k \in \{0, \dots, m\}\}$. Yet $Q(s)$ tells us that $\bigwedge\{Bel(\Psi^{i_k}_{\sigma'_{s_*+s}}) \mid k \in \{0, \dots, m\}\} \not\models \bot$, thus $\bigwedge\{Bel(\Psi^{i_k}_{\sigma'_{s_*+s+1}}) \mid k \in \{0, \dots, m\}\} \not\models \bot$, which proves $Q(s+1)$.

Lastly, let us prove $R(s+1)$, that is, we need to show that $Bel(\Psi^{i_{s+1}}_{\sigma'_{s_*+s+1}}) \equiv \bigwedge\{Bel(\Psi^{i_k}_{\sigma_{s_*}}) \mid k \in \{0, \dots, s+1\}\}$. From Equation 4 and since $\circ_{i_{s+1}}$ satisfies (I4), we get that $Bel(\Psi^{i_{s+1}}_{\sigma'_{s_*+s+1}}) \equiv Bel(\Psi^{i_{s+1}}_{\sigma'_{s_*+s}}) \wedge Bel(\Psi^{i_s}_{\sigma'_{s_*+s}})$. Yet from $R(s)$, $Bel(\Psi^{i_s}_{\sigma'_{s_*+s}}) \equiv \bigwedge\{Bel(\Psi^{i_k}_{\sigma_{s_*}}) \mid k \in \{0, \dots, s\}\}$. Hence,

$$Bel(\Psi^{i_{s+1}}_{\sigma'_{s_*+s+1}}) \equiv Bel(\Psi^{i_{s+1}}_{\sigma'_{s_*+s}}) \wedge \bigwedge\{Bel(\Psi^{i_k}_{\sigma_{s_*}}) \mid k \in \{0, \dots, s\}\} \tag{6}$$

We consider two cases:

- Case 1: $i_{s+1} \in V \setminus \{i_1, \dots, i_s\}$. In this case, $P(s)$ tells us that $\Psi^{i_{s+1}}_{\sigma'_{s_*+s}} = \Psi^{i_{s+1}}_{\sigma'_{s_*}}$, thus $Bel(\Psi^{i_{s+1}}_{\sigma'_{s_*+s}}) \equiv Bel(\Psi^{i_{s+1}}_{\sigma'_{s_*}})$. Then, Equation 6 can be rewritten as $Bel(\Psi^{i_{s+1}}_{\sigma'_{s_*+s+1}}) \equiv \bigwedge\{Bel(\Psi^{i_k}_{\sigma_{s_*}}) \mid k \in \{0, \dots, s+1\}\}$, which shows that $R(s+1)$ holds.
- Case 2: $i_{s+1} \in \{i_1, \dots, i_s\}$. In this case, let l be the highest index in $\{1, \dots, s\}$ such that $i_{s+1} = i_l$, that is, $i_{s+1} \notin \{i_{l+1}, \dots, i_s\}$ (remark that $l < s$; indeed, $l = s$ would imply that $i_s = i_{s+1}$, which would imply that $(i_s, i_s) \in \sigma'$, i.e., $(i_s, i_s) \in A$, which would contradict the fact that A is irreflexive.) On the one hand, since $l \leq s$, we know that $R(l)$ holds, i.e., $Bel(\Psi^{i_l}_{\sigma'_{s_*+l}}) \equiv \bigwedge\{Bel(\Psi^{i_k}_{\sigma_{s_*}}) \mid k \in \{0, \dots, l\}\}$, yet since $i_l = i_{s+1}$ this can be rewritten as:

$$Bel(\Psi^{i_{s+1}}_{\sigma'_{s_*+l}}) \equiv \bigwedge\{Bel(\Psi^{i_k}_{\sigma_{s_*}}) \mid k \in \{0, \dots, s+1\}\} \tag{7}$$

On the other hand, we know by definition of l that $i_{s+1} \notin \{i_{l+1}, \dots, i_s\}$, and so for each edge $(x, y) \in \{\sigma'_{s_*+l+1}, \dots, \sigma'_{s_*+s}\}$, we have that $y \neq i_{s+1}$. Then by definition, $\Psi^{i_{s+1}}_{\sigma'_{s_*+s}} = \Psi^{i_{s+1}}_{\sigma'_{s_*+(s-1)}} = \dots = \Psi^{i_{s+1}}_{\sigma'_{s_*+l}}$. Using Equation 7, we get that $Bel(\Psi^{i_{s+1}}_{\sigma'_{s_*+s}}) \equiv \bigwedge\{Bel(\Psi^{i_k}_{\sigma_{s_*}}) \mid k \in \{0, \dots, s+1\}\}$. This shows that $R(s+1)$ holds too in this case.

We have now proved that $P(s+1)$, $Q(s+1)$ and $R(s+1)$ both hold, which concludes the proof by induction that for each step $s \in \{0, \dots, m\}$, $P(s)$, $Q(s)$ and $R(s)$ both hold. In particular, $P(m)$ and $R(m)$ hold, and by setting $t = s_* + s$, we get what was to be shown, i.e., that there exist a $\mathcal{B}$-run σ' and a step t such that:

- for each $i \in V \setminus \{i_1, \dots, i_m\}$, $\Psi^i_{\sigma'_t} = \Psi^i_{\sigma_{s_*}}$, and
- $Bel(\Psi^{i_*}_{\sigma'_t}) \equiv \bigwedge\{Bel(\Psi^{i_k}_{\sigma_{s_*}}) \mid k \in \{0, \dots, m\}\}$.

This concludes the proof. □

Lemma 9. *Let $\alpha_1, \dots, \alpha_m$ be m propositional formulas, $m \geq 1$, such that $\varphi \models \bigwedge_{1 \leq i \leq m} \alpha_i$. Let $\circ_L$ be Nayak's lexicographic revision operator on total preorders over worlds. Let Ψ be a total preorder over worlds such that $Bel(\Psi) \wedge \bigwedge_{1 \leq i \leq m} \alpha_i \models \bot$ and $lvl(\Psi, 1) = [\varphi]$. Then $Bel(((\Psi \circ_L \alpha_1) \circ_L \dots) \circ_L \alpha_m) \equiv \varphi$.*

Proof. For $k \in \{1, \dots, m\}$, denote $\Psi_k = ((\Psi \circ_L \alpha_1) \circ_L \dots) \circ_L \alpha_k$ and $\Psi_0 = \Psi$. Let us prove by induction that for each $k \in \{0, \dots, m\}$, we have for all worlds $\omega, \omega' \in \Omega$ that:

$P(k)$: if $\omega, \omega' \models \varphi$, then $\omega \simeq_{\Psi_k} \omega'$, and
$Q(k)$: if $\omega \models \varphi$ and $\omega' \models \neg\varphi \wedge \neg Bel(\Psi)$, then $\omega \prec_{\Psi_k} \omega'$

Let us first prove the base case when $k = 0$, i.e., let us prove that $P(0)$ and $Q(0)$ both hold. $P(0)$ directly follows from the fact that $\Psi_0 = \Psi$ and our intial assumption that $lvl(\Psi, 1) = [\varphi]$. As to $Q(0)$, remark that conditions 1 and 2 of a gradual assignment imply that $lvl(\Psi_0, 0) = [Bel(\Psi)]$; since $lvl(\Psi_0, 1) = [\varphi]$, we get that $\omega \prec_{\Psi_0} \omega'$ for all worlds $\omega \models \varphi$ and $\omega' \models \neg\varphi \wedge \neg Bel(\Psi)$. This concludes the proof for the base case when $k = 0$. Now, let $k \in \{0, \dots, m-1\}$, assume that $P(k)$ and $Q(k)$ both hold. Yet $P(s+1)$ follows from $P(s)$ and the fact that $\varphi \models \alpha_{k+1}$, using condition 4 a gradual assignment. $Q(s+1)$ follows from $Q(s)$ and the fact that $\varphi \models \alpha_{k+1}$, using conditions 4 and 6 of a gradual assignment. This proves by induction that for each $k \in \{0, \dots, m\}$, $P(k)$ and $Q(k)$ hold.

Now, it is easy to see by definition of $\circ_L$ that if ω is a world that is a model of all α_i's and ω' is a world that is not a model of at least one α_i, then ω be comes strictly more plausible than ω' after revision of Ψ by the sequence of all α_i's. Put formally, for all worlds $\omega, \omega' \in \Omega$, if $\omega \models \bigwedge_{1 \leq i \leq m} \alpha_i$ and $\omega' \not\models \bigwedge_{1 \leq i \leq m} \alpha_i$, then $\omega \prec_{\Psi_m} \omega'$. Yet our initial assumptions include that $\varphi \models \bigwedge_{1 \leq i \leq m} \alpha_i$ and $Bel(\Psi) \wedge \bigwedge_{1 \leq i \leq m} \alpha_i \models \bot$, which means that if $\omega \models \varphi$ then $\omega \models \bigwedge_{1 \leq i \leq m} \alpha_i$; and if $\omega' \models Bel(\Psi)$ then $\omega \not\models \bigwedge_{1 \leq i \leq m} \alpha_i$. This gives us that if $\omega \models \varphi$ and $\omega' \models Bel(\Psi)$, then $\omega \prec_{\Psi_m} \omega'$. Together with $Q(m)$, we get that if $\omega \models \varphi$ and $\omega' \models \neg\varphi$, then $\omega \prec_{\Psi_m} \omega'$. And together with $P(m)$ and conditions 1 and 2 of a gradual assignment, we get that $Bel(\Psi_m) \equiv \varphi$, which concludes the proof. □

Lemma 10. *Let $\mathcal{B} \models$ B, σ be a $\mathcal{B}$-run, $s_* \in \mathbb{N}$, and assume that for each agent $i \in V$, i is conditionally neutral on φ in σ at s_*. If there exists an agent $i_* \in V$ such that $Bel(\Psi^{i_*}_{\sigma_{s_*}}) \equiv \varphi$, then $\varphi \in Out(\mathcal{B})$.*

Proof. Let $\mathcal{B}$ be a BFN, σ be a $\mathcal{B}$-run, $s_* \in \mathbb{N}$, assume that for each agent $i \in V$, i is conditionally neutral on φ in σ at step s_*, and let $i_* \in V$ be an agent such that $Bel(\Psi^{i_*}_{\sigma_{s_*}}) \equiv \varphi$. Note that the proof is trivial if $|V| = 1$, so assume $|V| > 1$, and let $p = (i_0, \ldots, i_m)$, $m \geq 1$, $i_0 = i_*$, be a path covering all agents in V, i.e., for each $i \in V$, there is $k \in \{0, \ldots, m\}$ such that $i = i_k$.

Now, for each $k \in \{1, \ldots, m\}$, one can define a finite A-sequence $\delta^{i_{k-1}, i_k} = (e_s)_{s \in \{1, \ldots, x\}}$, where for each $e_s \in \delta^{i_{k-1}, i_k}$, $e_s = (i_{k-1}, i_k)$, $x = \max\{x_i \mid i \in \{1, \ldots, n\}\}$ and x_i is the least integer such that for each epistemic state Ψ, $Bel(\Psi \circ_i^{x_i} \varphi) \models \varphi$. Note that such a finite sequence δ^{i_{k-1}, i_k} always exists for each $k \in \{1, \ldots, m\}$, since for each agent $i \in V$, each change policy $\circ_i$ satisfies (I1*). Then, define δ^p as the concatenation of all sequences δ^{i_{k-1}, i_k} for each $k \in \{1, \ldots, m\}$, i.e., $\delta^p = \delta^{i_0, i_1} \cdot \ldots \cdot \delta^{i_{m-1}, i_m}$.

Now, by Lemma 13, we know that there exists a $\mathcal{B}$-run σ' with prefix the concatenation of $(\sigma_0, \ldots, \sigma_{s_*-1})$ and δ^p. Clearly, by construction of δ^p that for each $k \in \{0, \ldots, m\}$, $Bel(\Psi^{i_k}_{\sigma'_{s_*+|\delta^p|}}) \models Bel(\Psi^{i_*}_{\sigma'_{s_*}})$ (recall that $i_* = i_0$). Yet since σ and σ' are identical up to step s_*, from $Bel(\Psi^{i_*}_{\sigma_{s_*}}) \equiv \varphi$ we also have that $Bel(\Psi^{i_*}_{\sigma'_{s_*}}) \equiv \varphi$. We got that for each $k \in \{0, \ldots, m\}$, $Bel(\Psi^{i_k}_{\sigma'_{s_*+|\delta^p|}}) \models \varphi$; equivalently, for each agent $i \in V$, $Bel(\Psi^{i}_{\sigma'_{s_*+|\delta^p|}}) \models \varphi$.

Yet we assumed that for each agent $i \in V$, i is conditionally neutral on φ in σ at step s, which by Lemma 3 (more precisely the part 3 $\Rightarrow$ 2) implies that for each agent $i \in V$, i is conditionally neutral on φ in σ' at step $s_* + |\delta^p|$. By Lemma 2, each agent $i \in V$ is observably neutral on φ in σ' at step $s_* + |\delta^p|$, which by definition of observable neutrality means that $\varphi \models Bel(\Psi^{i}_{\sigma'_{s_*+|\delta^p|}}) \equiv \varphi$ or $\varphi \wedge Bel(\Psi^{i}_{\sigma'_{s_*+|\delta^p|}}) \models \bot$. Yet we already established that for each agent $i \in V$, $Bel(\Psi^{i}_{\sigma'_{s_*+|\delta^p|}}) \models \varphi$, which means that for each agent $i \in V$, $Bel(\Psi^{i}_{\sigma'_{s_*+|\delta^p|}}) \equiv \varphi$. Then we also get at every step $s \geq s_* + |\delta^p|$ that for each agent $i \in V$, $Bel(\Psi^{i}_{\sigma'_{s_*+|\delta^p|}}) \equiv \varphi$, i.e., $Out_{\sigma'}(i) \equiv \varphi$ which means that $\varphi \in Out(\mathcal{B})$ and concludes the proof. ◻

Proposition 8. *If:*

I. for each agent $i \in V$, i is observably neutral on φ in B*, and*
II. there exists an agent $i_* \in V$ such that $BC_\varphi(i_*) \models \bot$,
then $\varphi \in Out$(B).

Proof. By condition II, let $i_* \in V$ be such that $BC_\varphi(i_*) \models \bot$. Let $\mathcal{B} \models$ B be a BFN $\mathcal{B} = \langle (V, A), \vec{\Psi}, \vec{\circ}, \mathcal{S} \rangle$ such that:

- $\circ_{i_*} = \circ_L$, i.e., the agent i_* is associated with the change policy corresponding to Nayak's lexicographic operator,
- for each agent $i \in V \setminus V_{\downarrow\varphi}$, $lvl(\preceq_{\Psi^i}, 1) = [\varphi]$,

where for each $i \in V \setminus V_{\downarrow\varphi}$, $\preceq_{\Psi^i}$ is the total preorder associated with Ψ^i in the gradual assignment corresponding to $\circ_i$. This BFN $\mathcal{B}$ is constructed such that for each agent $i \in V$, i is conditionally neutral on φ in $\mathcal{B}$ (cf. Lemma 1).

Remark that by condition I, since $Reach^-_\varphi(N^-_\varphi(i_*)) \subseteq V_{\downarrow\varphi}$ by definition, all agents $j \in Reach^-_\varphi(N^-_\varphi(i_*))$ are such that $\varphi \models B^j$; thus, all possible paths $(i_0, \ldots, i_m)$, $m > 0$, of agents in $G_{\downarrow\varphi}$ such that $i_k \in Reach^-_\varphi(N^-_\varphi(i_*))$ for each $k \in \{0, \ldots, m\}$ and $i_m \in N^-_\varphi(i_*)$ are such that $\varphi \models \bigwedge\{B^{i_k} \mid k \in \{0, \ldots, m\}\}$. Then one can apply Lemma 8 successively for all possible such paths, and we get that there exists a $\mathcal{B}$-run σ and a step s_* such that $\Psi^{i_*}_{\sigma_{s_*}} = \Psi^{i_*}$, and for each $j \in N^-_\varphi(i_*)$, $Bel(\Psi^{j}_{\sigma_{s_*}}) \equiv \bigwedge\{Bel(\Psi^{j'}) \mid j' \in Reach^-_\varphi(\{j\})\}$. Stated otherwise, there is a $\mathcal{B}$-run such that at some step, all direct predecessors j of agent i_* have their beliefs corresponding to the conjunction of the beliefs of all ancestors j_* of j in the graph $G_{\downarrow\varphi}$, i.e., in the subgraph of G restricted to the support of φ in B, while the epistemic state of agent i_* has not changed. Then, since $BC_\varphi(i_*) \models \bot$, and since by definition $BC_\varphi(i_*) = \bigwedge\{Bel(\Psi^{j'}) \mid j \in C_\varphi(i_*)\}$ with $C_\varphi(i_*) = \{i_*\} \cup Reach^-_\varphi(N^-_\varphi(i_*))$, we get that:

$$Bel(\Psi^{i_*}_{\sigma_{s_*}}) \wedge \bigwedge\{Bel(\Psi^{j}_{\sigma_{s_*}}) \mid j \in N^-_\varphi(i_*)\} \models \bot \tag{8}$$

Now, denote $N^-_\varphi(i_*) = \{j_1, \ldots, j_r\}$, and let δ be the A-sequence $\delta = ((j_1, i_*), \ldots, (j_r, i_*))$. Let σ' be a $\mathcal{B}$-run with prefix the concatenation of $(\sigma_0, \ldots, \sigma_{s_*-1})$ and δ (such a $\mathcal{B}$-run exists by Lemma 13). Clearly, for each agent $i \in V$, $\Psi^{i}_{\sigma_{s_*}} = \Psi^{i}_{\sigma'_{s_*}}$. Then by Equation 8, by Lemma 9, by definition of $\circ_{i_*} = \circ_L$ and since $\Psi^{i_*}_{\sigma'_{s_*}} = \Psi^{i_*}_{\sigma_{s_*}} = \Psi^{i_*}$, we get that $Bel(\Psi^{i_*}_{\sigma'_{s_*+r}}) \equiv \varphi$.

Then, since for each agent $i \in V$, i is conditionally neutral on φ in $\mathcal{B}$, by Lemma 3 (more precisely the part 1 $\Rightarrow$ 2), for each agent $i \in V$, i is conditionally neutral on φ in σ' at step s_*+r. By Lemma 10 and since $Bel(\Psi^{i_*}_{\sigma'_{s_*+r}}) \equiv \varphi$, we get that $\varphi \in Out(\mathcal{B})$. Since $\mathcal{B} \models$ B, $\varphi \in Out$(B), which concludes the proof. ◻