$\bullet\ f(x) = \begin{cases} 1 & \text{si } x = x_0 \\ 0 & \text{si } x \neq 0 \end{cases}$

$f : [a,b] \to \mathbb{R}$

funciones continuas

Funciones Integrables

$\hookrightarrow f : [a,b] \to \mathbb{R}$ acotada

f integrable $\Leftrightarrow \sup\{L(f,P) : P \in \mathcal{P}_{[a,b]}\} = \inf\{U(f,P) : P \in \mathcal{P}_{[a,b]}\}$

$\hookrightarrow \displaystyle\int_a^b f(x)\,dx.$

Caracterización: $f : [a,b] \to \mathbb{R}$ acotada.

f es integrable en $[a,b] \Longleftrightarrow \forall \varepsilon > 0,\ \exists P_\varepsilon \in \mathcal{P}_{[a,b]}$ tq

$$U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon$$

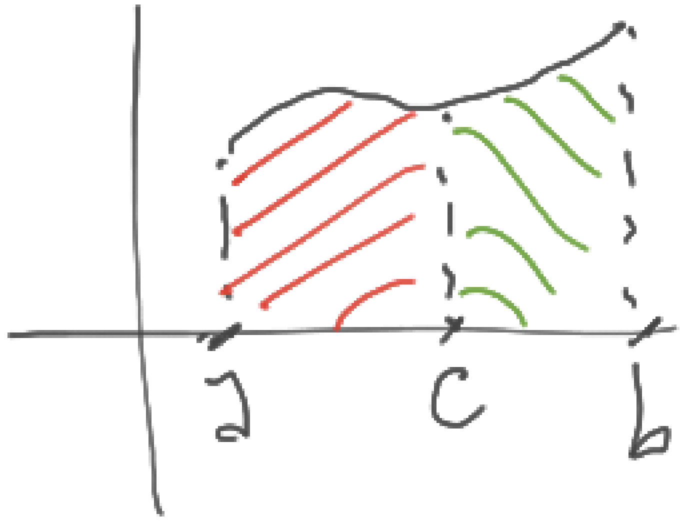
En ese caso, $\displaystyle\int_a^b f(x)\,dx = I$ donde I es el único número tq

$$L(f,P_\varepsilon) \leq I \leq U(f,P_\varepsilon) \quad \forall \varepsilon > 0.$$

T1 $f:[a,b]\to\mathbb{R}$, $c\in(a,b)$

f es integrable en $[a,b] \iff f$ es integrable en $[a,c]$ y $[c,b]$.

Además $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



D/ $\Leftarrow$) Sup. f es int. en $[a,c]$ y $[c,b]$. HIP.

Ents. f es acotada en $[a,c]$ y $[c,b]$,

o sea, $\exists\, m, m', M, M' \,/\, m\le f(x)\le M \;\forall x\in[a,c]$

$m'\le f(x)\le M' \;\forall x\in[c,b]$.

Tomando $\tilde{m}=\min\{m,m'\}$ y $\tilde{M}=\max\{M,M'\}$

resulta $\tilde{m}\le f(x)\le \tilde{M} \;\forall x\in[a,b]$

$\therefore$ f es acotada en $[a,b]$.

Sea $\varepsilon > 0$, debemos probar que existe $P_\varepsilon \in \mathcal{P}_{[a,b]}$ /

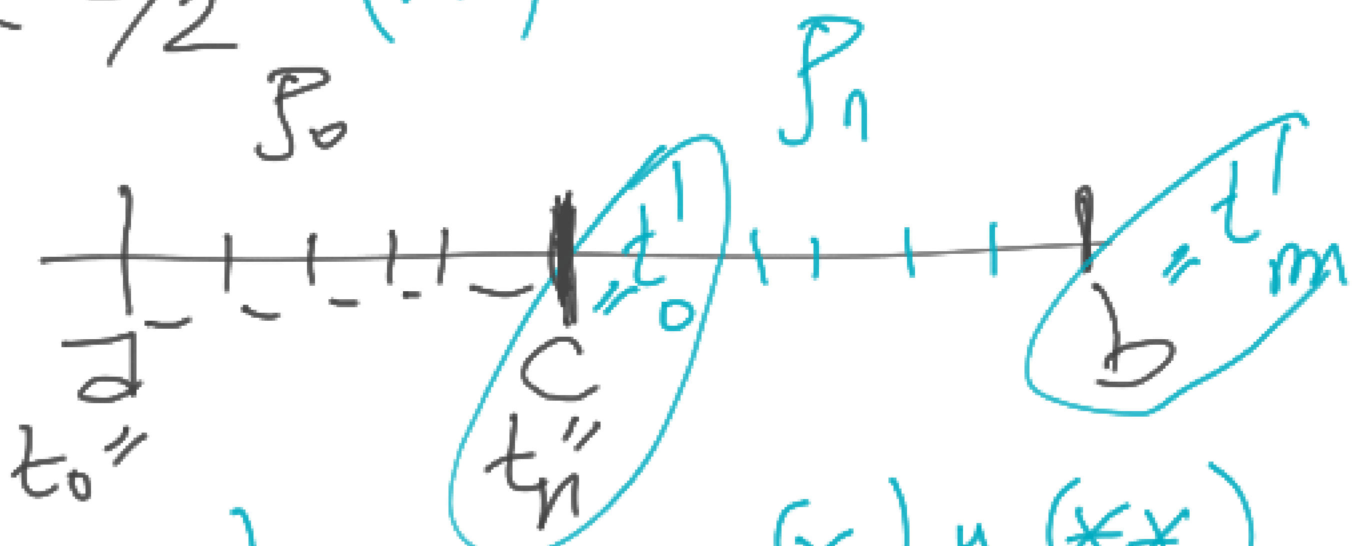
$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$$

Por hip, f es int. en $[a,c]$, luego $\exists\, P_0 \in \mathcal{P}_{[a,c]}$ /

$$\boxed{U(f|_{[a,c]}, P_0) - L(f, P_0)} < \varepsilon/2 \quad (*)$$

Por hip, f es int. en $[c,b]$, luego $\exists\, P_1 \in \mathcal{P}_{[c,b]}$ /

$$U(f, P_1) - L(f, P_1) < \varepsilon/2 \quad (**)$$



Sea $P_\varepsilon = P_0 \cup P_1$. Entonces:

$$L(f, P_\varepsilon) = L(f, P_0) + L(f, P_1)$$

$$\text{y } U(f, P_\varepsilon) = U(f, P_0) + U(f, P_1)$$

$\therefore$ Sumando m.a.m. $(*)$ y $(**)$

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$$

$\therefore$ f es integrable en $[a,b]$

$$\left\{ L(f, P_\varepsilon) = \sum_{i=1}^{n+m} (x_i - x_{i-1}) m_i = \underbrace{\sum_{i=1}^{n} (x_i - x_{i-1}) m_i}_{\displaystyle = \underbrace{\sum_{i=1}^{n} (t_i - t_{i-1}) m_i}_{L(f, P_0)}} + \underbrace{\sum_{i=n+1}^{m+n} (x_i - x_{i-1}) m_i}_{\displaystyle + \underbrace{\sum_{i=1}^{m} (t'_i - t'_{i-1}) m_i}_{L(f, P_1)}} \right.$$

$$P_0 = \{\overset{x_0}{t_0 = a}, \dots, \overset{x_n}{t_n = c}\}$$

$$P_1 = \{t'_0 = c, t'_1, \dots, t'_m = b\}$$

$$P_\varepsilon = P_0 \cup P_1 = \{x_0, \dots, x_{n+m}\} \Big/ \quad x_i = \begin{cases} t_i & i = 0, \dots, n \\ t'_{i-n} & i = n+1, \dots, m+n \end{cases}$$

$$i = n \longrightarrow x_n = t'_{n-n} = t'_0 = c$$

$$i = n+1 \longrightarrow x_{n+1} = t'_{(n+1)-n} = t'_1$$

$$i = n+2 \longrightarrow x_{n+2} = t'_2$$

$$\vdots$$

$$i = n+m \longrightarrow x_{n+m} = t'_m = b$$

Además, por hip.

$$L(f, P_0) \leq \int_a^c f(x)\,dx \leq U(f, P_0)$$

$$+\quad L(f, P_1) \leq \int_c^b f(x)\,dx \leq U(f, P_1)$$

$$\therefore\ \forall \varepsilon > 0,\quad L(f, P_\varepsilon) \leq \underbrace{\int_a^c f(x) + \int_c^b f(x)\,dx}_{I} \leq U(f, P_\varepsilon)$$

$$\therefore\ \int_a^b f(x) = I$$

(T_2) Si f es int. en $[a,b]$, $c \in \mathbb{R}$ $\Big\} \Rightarrow$ cf es int en $[a,b]$ y $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

<u>Idea:</u> $c \geq 0$ $\quad \sup\{f(x) \,/\, t_{i-1} \leq x \leq t_i\} = M_i$

$$\therefore \sup\{cf(x) \,/\, t_{i-1} \leq x \leq t_i\} = cM_i$$

$$\therefore L(cf,P) = \sum_{i=1}^{n} (t_i - t_{i-1})\, c\, m_i = c\sum_{i=1}^{n}(t_i - t_{i-1})\, m_i$$

$$= c\, L(f,P)$$

$$U(cf,P) = c\, U(f,P)$$

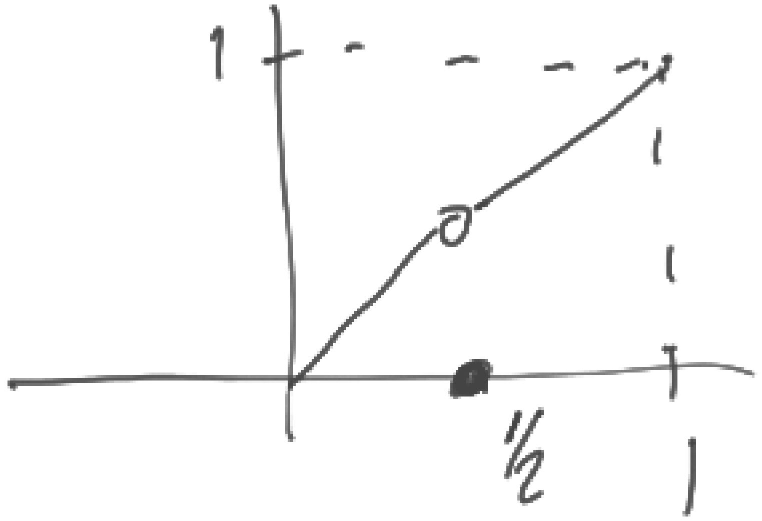
$$\therefore \sup\{L(cf,P) : P \in \mathcal{P}_{[a,b]}\} = c \sup\{L(f,P) : P \in \mathcal{P}_{[a,b]}\}$$

$$\inf\{U(cf,P) : P \in \mathcal{P}_{[a,b]}\} \longrightarrow = c\int_a^b f\,dx$$

(T) f, g integrables en $[a,b] \Rightarrow f+g$ integrable en $[a,b]$

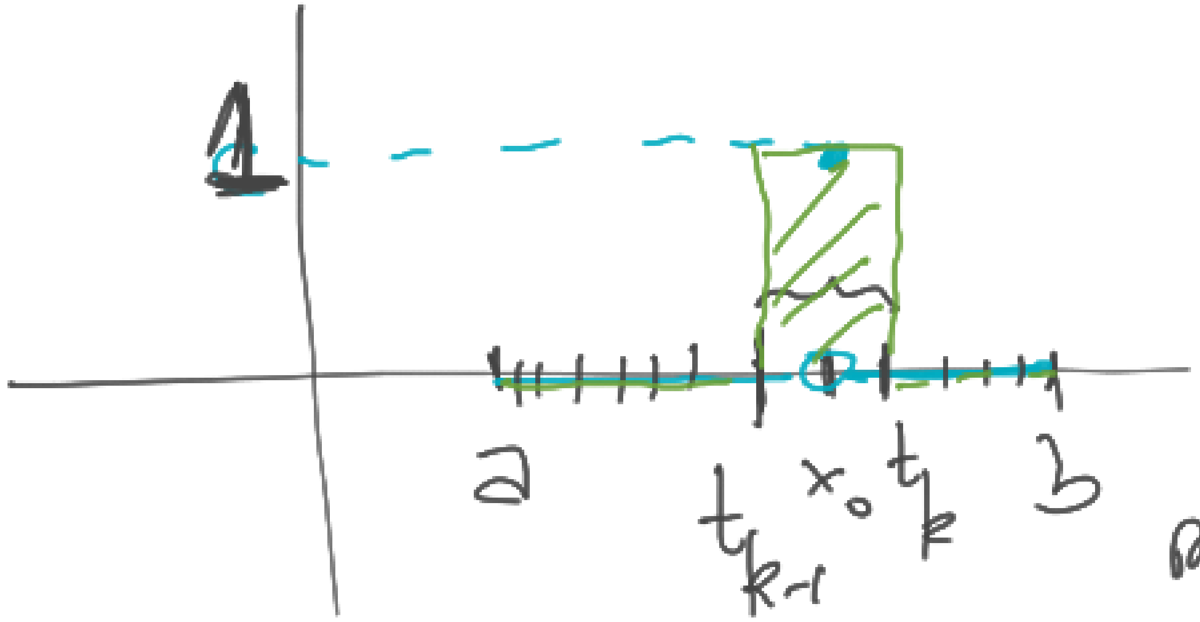
y $\displaystyle\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

Ej: Sea $f : [0,1] \to \mathbb{R}$ / $f(x) = \begin{cases} x & \text{si } x \neq 1/2 \\ 0 & \text{si } x = 1/2 \end{cases}$



$f_{|[0,1/2]} = \begin{cases} x & \text{si } x \neq 1/2 \\ 0 & \text{si } x = 1/2 \end{cases}$

Ej: Sea $f:[a,b]\to\mathbb{R}$ tq $f(x)=\begin{cases}1 & \text{si } x=x_0\\ 0 & \text{si } x\neq x_0\end{cases}$



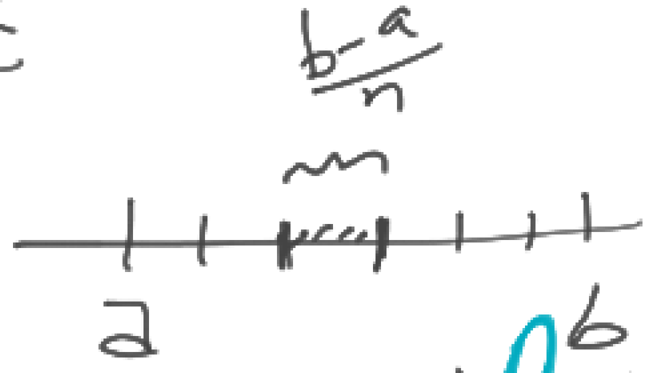
Sea $P=\{t_0=a,\dots,t_n=b\}$ tq $x_0\in[t_{k-1},t_k)$

$m_i=\inf\{f(x): t_{i-1}\le x\le t_i\}=0$

$M_i=\sup\{f(x): t_{i-1}\le x\le t_i\}=\begin{cases}0 & \text{si } i\neq k\\ 1 & \text{si } i=k\end{cases}$

$\therefore L(f,P)=0,\ U(f,P)=(t_k-t_{k-1})$.

Si tomamos $P_n=\{t_0,\dots,t_n\}$ tq $t_i=t_{i-1}+\frac{b-a}{n}=i\,\frac{b-a}{n}$



Ents $L(f,P_n)=0,\ U(f,P_n)=\frac{b-a}{n}$

P.c. $\varepsilon>0$, $\exists n\in\mathbb{N} / \frac{b-a}{n}<\varepsilon$, $U(f,P_n)-L(f,P_n)<\varepsilon$ y $\overset{0}{L(f,P_n)}\le I\le U(f,P_n)$ $\forall n$

$\therefore f$ es int y $\int_a^b f(x)\,dx=0$

Ej: Sea $h(x) = \begin{cases} c & \text{si } x = x_0 \\ 0 & \text{si } x \neq x_0 \end{cases}$ ¿h es integrable?

$h : [a,b] \to \mathbb{R}$

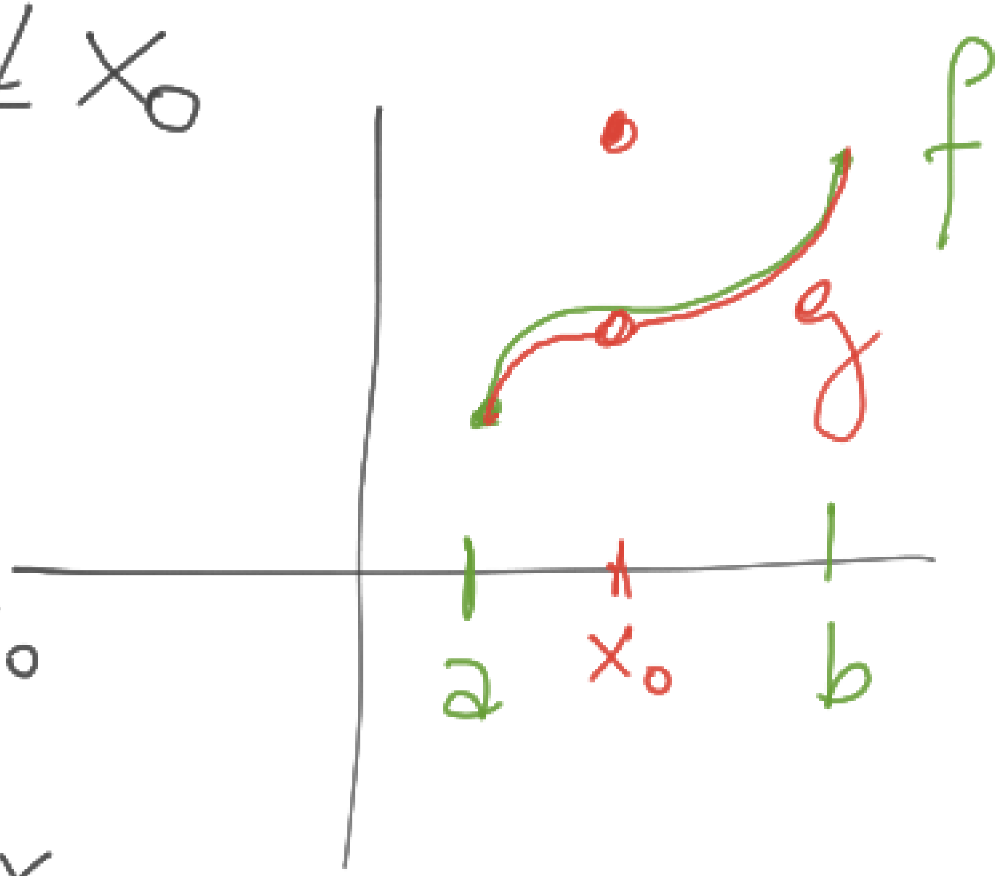
Sí pues $h = cf$

Ej: Sabemos que f es integrable en $[a,b]$.

y $g : [a,b] \to \mathbb{R}$ / $g(x) = f(x)\ \forall x \neq x_0$

Sea $z(x) = g(x) - f(x)$ y sea

$c = g(x_0) - f(x_0)$.



Ents. $z : [a,b] \to \mathbb{R}$ / $z(x) = \begin{cases} c & \text{si } x = x_0 \\ 0 & \text{si } x \neq x_0 \end{cases}$

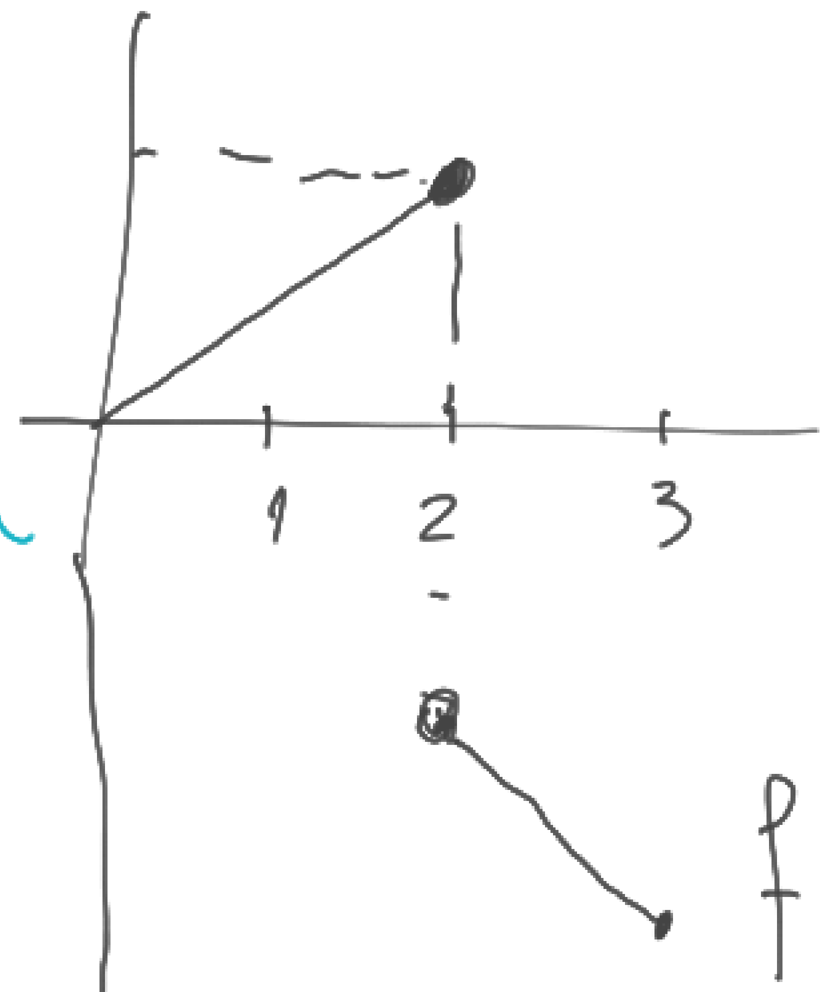
$\left.\begin{array}{l} \therefore\ z \text{ es integrable en } [a,b] \\ f \text{ es integrable en } [a,b] \end{array}\right\} \Rightarrow g = z + f$ es integrable.

$$\int_a^b g(x)\,dx = \underbrace{\int_a^b z\,dx}_{0} + \int_a^b f(x)\,dx$$

Ejemplo: Sea $f(x) = \begin{cases} x & \text{en } [0,2] \\ -2x+2 & \text{en } (2,3] \end{cases}$.

$f : [0,3] \to \mathbb{R}$

$\int_0^3 f = \int_0^2 x\,dx + \underbrace{\int_2^3 -2x+2\,dx}$



¿f integrable?

- $[0,3] = [0,2] \cup [2,3]$

En $[0,2]$ f es integrable por ser continua y $\int_0^2 f(x)\,dx = 2$

En $[2,3]$, $f(x) = \begin{cases} 2 & \text{si } x = 2 \\ -2x+2 & \text{si } x \neq 2 \end{cases}$

Sea $g : [2,3] \to \mathbb{R} \,/\, g(x) = -2x+2$. Ents. $f(x) = g(x)$ $\forall x \neq 2$

$\therefore f$ es int. en $[2,3]$ y $\int_2^3 f(x)\,dx = \int_2^3 g(x)\,dx = \ldots$