8 2019 04 02 Multi-stage games

Definition 42. A multi-stage game consists of the following components:

- A sequence $(0, \dots, K)$ represents the stage of the game;
- there are N players who play repeatedly and chose their actions simultaneously at each stage of the game;
- Each player observes the actions of all N players in all previous stages;
- For each stage $k \in \{0, \dots, K\}$, the action profile of players is given by

$$\underline{\alpha}_k = (\alpha_k^1, \dots, \alpha_k^N)$$

  where α_k^i is the action chosen by player i at stage k;
- The available information prior to stage k, called *history*, is defined as

$$h^k = (\underline{\alpha}_0, \dots, \underline{\alpha}_{k-1});$$

- For every player i at stage k, the feasible set of actions is denoted by $A^i(h^k)$, which depends on the information of history prior to stage k. For example, the resources that have been used in the past can affect the current actions of players. So, at stage k, the action chosen by player i can be viewed as a function of history h^k;
- We would like to include *"no action"* as part of $A^i(h^k)$ for every $i = 1, \dots, N$ and for every $k = 0, \dots, K$. When player i choses "no action", she is inactive at this stage.

Here are some examples for multi-stage games:

1. $K = 0$, Cournot competition.
2. $K = 1$, Stackelberg game. For example $N = 2$, one player is called the leader, and the other called the follower. Then
   - at stage $k = 0$, leader chooses an action $\alpha_0 \in A^{leader}(\emptyset)$. The follower is inactive, namely $A^{follower}(\emptyset) = \{\text{“}no\,action\text{”}\}$.
   - at stage $k = 1$, leader is inactive $A^{leader}(h^1) = \{\text{“}no\,action\text{”}\}$ and the follower chooses an action $\alpha^{follower} \in A^{follower}(h^1)$ where $h^1 = ((\alpha_0, \text{“}no\,action\text{”}))$.
3. $K = 2$, Entry-deterrence game.
   - at stage $k = 0$, incumbent raises money α_0, entrant "no action";
   - at stage $k = 1$, incumbent "no action", entrant choses between "enter" or "not enter";

- at stage $k = 2$, Cournot competition.

Definition 43. A pure strategy for player $i \in \{1, \dots, N\}$ at stage $k \in \{0, \dots, K\}$ is denoted by $\widehat{\alpha}_k^i$.
A pure strategy profile, denoted by $\widehat{\alpha}$, is the collection of pure strategies of all players at all stages except the last one $k = K$, namely

$$\widehat{\alpha} = (\widehat{\alpha}_k^i)_{i=1,\dots,N,\, k=0,\dots,K-1}.$$

It is said to be an admissible pure strategy profile if $\widehat{\alpha}_k^i \in A^i(h^k)$ for all $k = 0, \dots, K-1$ and $i = 1, \dots, N$.

Remark 30. A stage $k = K$, we collect everything and decide the reward or cost for players.

Definition 44. The set of histories up to stage $k \in \{0, \dots, K\}$ is denoted by $H(k)$. So $H(K)$ represents the set of all histories. For each player $i \in \{1, \dots, N\}$, we associate her with a cost function $J^i : H(K) \to \mathbb{R}$.

Very often, the cost function J^i is addictive, i.e.

$$J^i(h^K) = \sum_{k=0}^{K-1} c_k f^i(\alpha_k^i)$$

The coefficients $(c_k)_{k=0,\dots,K-1}$ can be discounted factors, for example, $c_k = c_K \delta^k$ for some $c_K \in \mathbb{R}$ and $\delta \in [0, 1]$.
When $K = \infty$, we need to make sure $\delta < 1$.

Definition 45. A pure strategy profile $\widehat{\alpha}$ is said to be a pure strategy Nash equilibrium if for every $i \in \{1, \dots, N\}$, for every admissible sequence of actions $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_{k-1})$,

$$J^i(\underline{\widehat{\alpha}}_0, \dots, \underline{\widehat{\alpha}}_{K-1}) \leq J^i((\alpha_0, \underline{\widehat{\alpha}}_0^{-i}), \dots, (\alpha_{K-1}, \underline{\widehat{\alpha}}_{K-1}^{-i}))$$

where $(\alpha_k, \underline{\widehat{\alpha}}_k^{-i}) = (\widehat{\alpha}_k^1, \dots, \widehat{\alpha}_k^{i-1}, \alpha_k, \widehat{\alpha}_k^{i+1}, \dots, \widehat{\alpha}_k^N)$ is the action profile at stage k for $k \in \{0, \dots, K-1\}$.

Remark 31. Given a strategy profile $\widehat{\alpha}$ and a player i, a sequence of actions $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_{K-1})$ is said to be admissible if for every $k = 1, \dots, K-1$,

$$\alpha_k \in A^i(h_{\boldsymbol{\alpha}}^k)$$

where $h_{\boldsymbol{\alpha}}^0 = \emptyset$ and $h_{\boldsymbol{\alpha}}^k = ((\alpha_0, \underline{\widehat{\alpha}}_0^{-i}), \dots, (\alpha_{k-1}, \underline{\widehat{\alpha}}_{k-1}^{-i}))$

Definition 46. For every stage $k = 0, \dots, K$, let $G(h^k)$ be a new game depending on the history $h^k = (\underline{\alpha}_0, \dots, \underline{\alpha}_{k-1})$.

- The new strategy profile for N players in game $G(h^k)$ is denoted by $\alpha_{h^k} = (\alpha_l^i)_{i=1,\dots,N,\, l=k,k+1,\dots,K-1}$ (similarly, $\widehat{\alpha}_{h^k}$ for pure strategy profile);

- The new history is denoted by $(h^k, \underline{\alpha}_k, \dots, \underline{\alpha}_{K-1})$;
- The new cost function is a mapping $(\underline{\alpha}_k, \dots, \underline{\alpha}_{K-1}) \mapsto J^i(h^k, \underline{\alpha}_k, \dots, \underline{\alpha}_{K-1})$

A strategy profile $\widehat{\alpha}$ is said to be a *sub-game perfect Nash equilibrium* if for every $k \in \{0, \dots, K\}$, for every $h^k \in H(k)$, the strategy profile $(\widehat{\alpha}_l^i)_{i=1,\dots,N,\, l=k,\dots,K-1}$ is a Nash equilibrium for the sub game $G(h^k)$.

In words, at every stage k, whatever the history is, if players play according to the strategy profile $(\widehat{\alpha}_l^i)_{i=1,\dots,N,\, l=k,\dots,K-1}$ afterwards, then they are in a Nash equilibrium associated to the new game.

Remark 32. We need to clarify the notion of closed loop strategy and open loop strategy. Let α_k^i be the action taken by player i at stage k. For the sake of simplicity, we assume that $A^i(h^k) = \mathbb{R}$ for every $h^k \in H(k)$ for every $k = 0, \dots, K$.

- α_k^i is said to be an open loop strategy if there exists a function $\varphi^i : \{0, \dots, K\} \to \mathbb{R}$ such that $\alpha_k^i = \varphi^i(k)$ for every $k = 0, \dots K$. The function φ^i is decided before the game.
- α_k^i is said to be a closed loop strategy if there exists a function $\varphi_k^i : H(k) \to \mathbb{R}$ such that $\alpha_k^i = \varphi_k^i(h^k)$ for some $h^k \in H(k)$.

It will be rare for open loop strategy profile to have a sub-game perfect Nash equilibrium.

8.1 Prisoner's dilemma

Assume that there are only two players $N = 2$, and the feasible set of actions are $A_1 = A_2 = \{C, D\}$ for every stage k of the game. Here C stands for cooperation and D stands for defection.

Let $T < R < P < S$ where $T, R, P, S \in \mathbb{R}$ and T stands for *temptation*, R stands for *negative rewards* or *cost* , P stands for *punishment*, and S stands for *sucker*. Assume that the cost functions are defined as follow (see also the table 1):

$$J^1(\alpha_1, \alpha_2) = \mathbb{1}_{\alpha_1=C}(R\mathbb{1}_{\alpha_2=C} + T\mathbb{1}_{\alpha_2=D}) + \mathbb{1}_{\alpha_1=D}(S\mathbb{1}_{\alpha_2=C} + P\mathbb{1}_{\alpha_2=D})$$

and

$$J^2(\alpha_1, \alpha_2) = \mathbb{1}_{\alpha_2=C}(R\mathbb{1}_{\alpha_1=C} + T\mathbb{1}_{\alpha_1=D}) + \mathbb{1}_{\alpha_2=D}(S\mathbb{1}_{\alpha_1=C} + P\mathbb{1}_{\alpha_1=D}).$$

The objective of players are to minimize their own cost function. We observe that the choice of relationships $T < R$ and $P < S$ imply that the defection is a preferable strategy for both player 1 and 2.

It can be verified that the strategy (D, D) is a Nash equilibrium.

Lemma 14. The strategy profile $((D, D), \dots, (D, D))$ is a sub-game perfect Nash equilibrium.

player 2\palyer 1	C	D
C	(R,R)	(T,S)
D	(S,T)	(P,P)

Table 1: the cost function table. The tuple $(J^1(\cdot), J^2(\cdot))$ represents the costs of player 1 and player 2 in different situation. The column names represent the strategies chosen by player 1 and the row names are for player 2.

Proof. Step 1: If I am looking for a sub-game perfect Nash equilibrium, the last choice should be (D,D) whatever the history is.

Step 2: One step perturbation principle (for sub-game perfect Nash equilibrium) for the induction part of the proof. □

Remark 33. The result may be different if the number of stages equals to infinity, i.e. $K = \infty$.