The Union-Closed Sets Conjecture

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Introduction

The union-closed conjecture is a well known unsolved problem in combinatorial mathematics. A very nice survey article describes several partial results that have been obtained. Some non-exhaustive examples are, the conjecture is true if:

1. the family contains a singleton-set;
2. the family can be represented as a lower semi-modular lattice.

In this article we will attempt to give a complete proof of the conjecture.

Notations

Let 2^X denotes the power set of X.

Let $[n]$ denotes $\{1, 2, \dots, n\}$.

Let $(F, \cup)$ denotes a union-closed family with $|F| > 1$. That is $\forall A, B \in F, A \cup B \in F$. When the context is clear we can simply write F.

Define the universe of a family F as $U(F) := \bigcup_{X \in F} X$.

Let $G \leq F$ denotes a union-closed sub-family, that is $\forall X, X \in G \implies X \in F$ and G is union-closed.

Define ${}_xF := \{X \in F : x \in X\}$ and ${}_{\bar{x}}F := \{X \in F : x \notin X\}$. This notation is a bit unusual but it allows us to use the bottom left indice for the family's index if our family is part of a sequence.

Conjecture

Let $(F, \cup)$ be a union-closed family.

Then $\exists x \in U(F)$ such that $|{}_xF|/|F| \geq 1/2$.

Plan of a proof

1. For all families F, let $n = |U(F)|$. We show that there exists an isomorphism with $G \leq 2^{[n]}$, $|U(G)| = n$ and if $x_i \in U(F)$ is the i-th most abundant element in F then $i \in [n]$ is the i-th most abundant element in G.

This allows us to concentrate on sub-families $F \leq 2^{[n]}$ such that $|_1F| \geq |_2F| \geq \ldots \geq |_mF|$ and we know that we'll cover all families because they are isomorphic. We qualify a family displaying that property as *regular*.

2. We define $\mathbb{F}(2^{[n]})$ as the collection of all regular sub-families of $2^{[n]}$ and we define $_{[n]}\mathbb{F}(2^{[n]})$ as the collection of all regular sub-families $F \leq 2^{[n]}$ such that $[n] \in F$. We show by induction that if the union-closed conjecture holds for all $_{[n]}\mathbb{F}(2^{[n]})$ then the union-closed conjecture holds for all regular sub-families of $2^{[n]}$. Therefore by the isomorphism result, holds for all union-closed families.

3. We define a chain of regular sub-families $F_0 \geq F_1 \geq \ldots \geq F_{2^n-1}$ such that $|F_i| = 2^n - i$ where $\forall F_i, [n] \in F_i$ and no three consecutive sub-families $F_i \geq F_{i+1} \geq F_{i+2}$ have the same number of sets containing 1, that is either $|_1F_i| \neq |_1F_{i+1}|$ or $|_1F_{i+1}| \neq |_1F_{i+2}|$. We demonstrate that it is always possible to form such a chain. We also demonstrate that the union-closed conjecture holds for all such chains.

4. We define $tree(F)$ as the tree of regular sub-families having a regular family F as root, where G is a child of G' if $G \leq G'$ and $tree(F)$ contains all regular sub-families G such that $|_1F| = |_1G|$. We show that this tree exists and that the union-closed conjecture holds for all families in the tree.

5. We define $chains(2^{[n]})$ as the set of all chains of regular sub-families in $2^{[n]}$ and $\mathbb{T}_n = \{tree(F) : F \text{ is a sub-family in a chain } C, \forall C \in chains(2^{[n]})\}$. We show that $\mathbb{T}_n$ covers $\mathbb{F}(2^{[n]})$. Since the union-closed conjecture holds for all families in $\mathbb{T}_n$ then the union-closed conjecture holds for all families in $\mathbb{F}(2^{[n]})$. Therefore the union-closed conjecture is true for all union-closed families. This concludes the proof.

A Failed Proof Attempt

This proof approach failed because the structure of $\{\mathbb{G}_0, \mathbb{G}_1, \ldots, \mathbb{G}_{2^n-1}\}$ is *too loose*. The proof rely on the idea that we need to remove a set from $_{x_1}G$ and a set from $_{\bar{x}_1}G$ in order to produce a family in $\mathbb{G}_{i+1}$ from a family in $\mathbb{G}_i$. However the structure fails at enforcing this.

Therefore, to make this idea work, we will need a more constraining structure.

In the following attempt, $[n]$ denotes a set of n elements $\{x_1, x_2, \ldots, x_n\}$ rather than $\{1, 2, \ldots, n\}$. We also let $\mathrm{Sub}_{F,x,n} := \{G \leq F : |_xG| = n\}$ denotes the collection of all sub-families G of F where the number of sets in G that contains x is equal to n.

We will proceed by induction. It is easy to verify that it is true for any sub-family $F \leq 2^{[2]}$.

Now suppose the union-closed hypothesis is true for all sub-families $F \leq 2^{[n]}$.

Take a sub-family $G \leq 2^{[n+1]}$. Either $[n+1] \in G$ or $[n+1] \notin G$.

Suppose that $[n+1] \notin G$. Then G is isomorphic to a sub-family $F \leq 2^{[n]}$ and by the induction hypothesis the union-closed hypothesis is true.

Now suppose on the contrary that $[n+1] \in G$.

If $G = 2^{[n+1]}$ then $|_{x_1}G|/|G| = 1/2$ and the union-closed hypothesis is true.

If on the contrary, $G \neq 2^{[n+1]}$, we can consider without loss of generality that $|_{x_1}G| \geq |_{x_i}G|$, for all i such that $1 < i \leq n+1$. Consider the family of sub-families $\{\mathbb{G}_0, \mathbb{G}_1, \ldots, \mathbb{G}_{2^n-1}\}$ where $\mathbb{G}_i := \{G \in \mathrm{Sub}_{2^{[n+1]},x_1,2^n-i} : \forall i$ such that $1 < i \leq n+1, |_{x_1}G| \geq |_{x_i}G|$ and $[n+1] \in G\}$.

For all G in $\mathbb{G}_0$ we can see that $|_{x_1}G| = 2^n$ and $|_{\bar{x}_1}G| \leq 2^n$. Therefore $|_{x_1}G|/|G| \geq 1/2$.

We will make a second induction by supposing the union-closed hypothesis to be true for all $G \in \mathbb{G}_i$ such that $0 \leq i < 2^n - 1$.

We remark that we can obtain any family G' in $\mathbb{G}_{i+1}$ by taking a family G in $\mathbb{G}_i$ and removing a set from G which contains x_1. However, for $|_{x_1}G'| \geq |_{x_i}G'|$ to hold for all i such that $1 < i \leq n+1$, we must also remove a set from G which contains x_2; a set which contains x_3; and so on up to a set which contains x_{n+1}.

Again if $[n+1] \notin G'$, we can conclude by the first induction hypothesis that the union-closed hypothesis is true for G'.

Otherwise we have that $[n+1] \in G'$. Therefore to obtain a set G' in $\mathbb{G}_{i+1}$ from G, we must at least remove a set in $_{x_1}G$ and a set in $_{\bar{x}_1}G$. Thus $|_{x_1}G'|/|G'| \geq (|_{x_1}G| - 1)/(|G| - 2)$ and by the second induction hypothesis $|_{x_1}G'|/|G'| \geq 1/2$.

> "[...] We must at least remove a set in $_{x_1}G$ and a set in $_{\bar{x}_1}G$."
>
> This is where the proof fails. The claim is false as if we pick $\{\{x_1\}, \{x_1, x_2\}\} \in \mathbb{G}_0$ and $\{\{x_1, x_2\}\} \in \mathbb{G}_1$, we can clearly see that the claim doesn't hold.

References

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