The Union-Closed Sets Conjecture

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Introduction

The union-closed conjecture is a well known unsolved problem in combinatorial mathematics. In this article, we will give a short proof by induction.

Notations

Let 2^X denote the power set of X.

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

Let $(F, \cup)$ denote a union-closed family with $|F| > 1$. By union-closed we mean that $\forall A, B \in F, A \cup B \in F$. When the context is clear we can simply write F.

Let $G \leq F$ denote a union-closed family G subset of a union-closed family F. We might informally say that G is a sub-family of F to talk about a union-closed subset of F.

Define $F_x := \{F_i \in F : x \in F_i\}$ and $F_{\bar{x}} := \{F_i \in F : x \notin F_i\}$.

Let $\mathrm{Sub}_{F,x,n} := \{G \leq F : |G_x| = n\}$ denote the set of all sub-families G of F where the number of sets in G that contains x is equal to n.

Theorem

Let $(F, \cup)$ be a union-closed family.

Then $\exists x \in [n]$ such that $|F_x|/|F| \geq 1/2$.

Proof

We will proceed by induction. It is easy to verify that it is true for any sub-family $F \leq 2^{[2]}$.

Now suppose the union-closed hypothesis is true for all sub-families $F \leq 2^{[n]}$.

Take a sub-family $G \leq 2^{[n+1]}$. Either $[n+1] \in G$ or $[n+1] \notin G$.

Suppose that $[n+1] \notin G$. Then $G \leq 2^{[n]}$ and by the induction hypothesis the union-closed hypothesis is true.

Now suppose on the contrary that $[n+1] \in G$.

If $G = 2^{[n+1]}$ then $|G_1|/|G| = 1/2$ and the union-closed hypothesis is true.

If on the contrary, $G \neq 2^{[n+1]}$, we can consider without loss of generality that $|G_1| \geq |G_x|$, for all x such that $1 < x \leq n+1$. Consider the family of sub-families

$\{\mathbb{G}_0, \mathbb{G}_1, \ldots, \mathbb{G}_{2^n-1}\}$ where $\mathbb{G}_i := \{G \in \mathrm{Sub}_{2^{[n+1]},1,2^n-i} : \forall x$ such that $1 < x \leq n+1, |G_1| \geq |G_x|$ and $[n+1] \in G\}$.

For all G in $\mathbb{G}_0$ we can see that $|G_1| = 2^n$ and $|G_{\bar{1}}| \leq 2^n$. Therefore $|G_1|/|G| \geq 1/2$.

We will make a second induction by supposing the union-closed hypothesis to be true for all $\mathbb{G}_i$ such that $0 \leq i < 2^n - 1$.

We remark that we can obtain any family G' in $\mathbb{G}_{i+1}$ by taking a family G in $\mathbb{G}_i$ and removing a set from G which contains 1. However, for $|G'_1| \geq |G'_x|$ to hold for all x such that $1 < x \leq n+1$, we must also remove a set from G which contains 2; a set which contains 3; and so on up to a set which contains $n+1$.

Again if $[n+1] \notin G'$, we can conclude by the first induction hypothesis that the union-closed hypothesis is true for G'.

Otherwise we have that $[n+1] \in G'$. Therefore to obtain a set G' in $\mathbb{G}_{i+1}$ from G, we must at least remove a set in G_1 and a set in $G_{\bar{1}}$. Thus $|G'_1|/|G'| \geq (|G_1| - 1)/(|G| - 2)$ and by the second induction hypothesis $|G'_1|/|G'| \geq 1/2$.

We have covered all the cases and this concludes the proof. $\square$