

The Union-Closed Sets Conjecture

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Introduction

The union-closed conjecture is a well known unsolved problem in combinatorial mathematics. A very nice survey article describes several partial results that have been obtained. Some non-exhaustive examples are, the conjecture is true if:

1. the family contains a singleton-set;
2. the family can be represented as a lower semi-modular lattice.

In this article we will attempt to give a complete proof of the conjecture.

Notations

Let 2^X denotes the power set of X .

Let $[n]$ denotes $\{1, 2, \dots, n\}$.

Let (F, \cup) denotes a union-closed family with $F \neq \{\emptyset\}$. That is $\forall A, B \in F, A \cup B \in F$. When the context is clear we can simply write F .

Define the universe of a family F as $U(F) := \bigcup_{X \in F} X$.

Let $G \leq F$ denotes a union-closed sub-family, that is $\forall X, X \in G \implies X \in F$ and G is union-closed.

Define ${}_x F := \{X \in F : x \in X\}$ and $_{\bar{x}} F := \{X \in F : x \notin X\}$. This notation is a bit unusual but it allows us to use the bottom left indice for the family's index if our family is part of a sequence.

Conjecture

Let (F, \cup) be a union-closed family.

Then $\exists x \in U(F)$ such that $|{}_x F|/|F| \geq 1/2$.

Plan of a proof

1. For all families F , let $n = |U(F)|$. We show that there exists an isomorphism with $G \leq 2^{[n]}$, $|U(G)| = n$ and if $x_i \in U(F)$ is the i -th most abundant element in F then $i \in [n]$ is the i -th most abundant element in G .

This allows us to concentrate on sub-families $F \leq 2^{[n]}$ such that $|_1 F| \geq |_2 F| \geq \dots \geq |_m F|$ and we know that we'll cover all families because they are isomorphic. We qualify a family displaying that property as *regular*.

2. We define $\mathbb{F}(2^{[n]})$ as the collection of all regular sub-families of $2^{[n]}$ and we define $_{[n]}\mathbb{F}(2^{[n]})$ as the collection of all regular sub-families $F \leq 2^{[n]}$ such that $[n] \in F$. We show by induction that if the union-closed conjecture holds for all $_{[n]}\mathbb{F}(2^{[n]})$ then the union-closed conjecture holds for all regular sub-families of $2^{[n]}$. Therefore by the isomorphism result, holds for all union-closed families.
3. We define a chain of regular sub-families $F_0 \geq F_1 \geq \dots \geq F_{2^n-1}$ such that $|F_i| = 2^n - i$ where $\forall F_i, [n] \in F_i$ and no three consecutive sub-families $F_i \geq F_{i+1} \geq F_{i+2}$ have the same number of sets containing 1, that is either $|_1 F_i| \neq |_1 F_{i+1}|$ or $|_1 F_{i+1}| \neq |_1 F_{i+2}|$. We demonstrate that it is always possible to form such a chain. We also demonstrate that the union-closed conjecture holds for all such chains.
4. We define $tree(F)$ as the tree of regular sub-families having a regular family F as root, where G is a child of G' if $G \leq G'$ and $tree(F)$ contains all regular sub-families G such that $|_1 F| = |_1 G|$. We show that this tree exists and that the union-closed conjecture holds for all families in the tree.
5. We define $chains(2^{[n]})$ as the set of all chains of regular sub-families in $2^{[n]}$ and $\mathbb{T}_n = \{tree(F) : F \text{ is a sub-family in a chain } C, \forall C \in chains(2^{[n]})\}$. We show that \mathbb{T}_n covers $\mathbb{F}(2^{[n]})$. Since the union-closed conjecture holds for all families in \mathbb{T}_n then the union-closed conjecture holds for all families in $\mathbb{F}(2^{[n]})$. Therefore the union-closed conjecture is true for all union-closed families. This concludes the proof.

A Failed Proof Attempt

This proof approach failed because the structure of $\{\mathbb{G}_0, \mathbb{G}_1, \dots, \mathbb{G}_{2^n-1}\}$ is *too loose*. The proof rely on the idea that we need to remove a set from $_{x_1} G$ and a set from $_{\bar{x}_1} G$ in order to produce a family in \mathbb{G}_{i+1} from a family in \mathbb{G}_i . However the structure fails at enforcing this.

Therefore, to make this idea work, we need a more constraining structure.

In the following attempt, $[n]$ denotes a set of n elements $\{x_1, x_2, \dots, x_n\}$ rather than $\{1, 2, \dots, n\}$. We also let $Sub_{F,x,n} := \{G \leq F : |_x G| = n\}$ denotes the collection of all sub-families G of F where the number of sets in G that contains x is equal to n .

We will proceed by induction. It is easy to verify that it is true for any sub-family $F \leq 2^{[2]}$.

Now suppose the union-closed hypothesis is true for all sub-families $F \leq 2^{[n]}$.

Take a sub-family $G \leq 2^{[n+1]}$. Either $[n+1] \in G$ or $[n+1] \notin G$.

Suppose that $[n+1] \notin G$. Then G is isomorphic to a sub-family $F \leq 2^{[n]}$ and by the induction hypothesis the union-closed hypothesis is true.

Now suppose on the contrary that $[n+1] \in G$.

If $G = 2^{[n+1]}$ then $|_{x_1} G|/|G| = 1/2$ and the union-closed hypothesis is true.

If on the contrary, $G \neq 2^{[n+1]}$, we can consider without loss of generality that $|_{x_1} G| \geq |_{x_i} G|$, for all i such that $1 < i \leq n+1$. Consider the family of sub-families $\{\mathbb{G}_0, \mathbb{G}_1, \dots, \mathbb{G}_{2^n-1}\}$ where $\mathbb{G}_i := \{G \in \text{Sub}_{2^{[n+1]}, x_1, 2^n-i} : \forall i \text{ such that } 1 < i \leq n+1, |_{x_1} G| \geq |_{x_i} G| \text{ and } [n+1] \in G\}$.

For all G in \mathbb{G}_0 we can see that $|_{x_1} G| = 2^n$ and $|_{\bar{x}_1} G| \leq 2^n$. Therefore $|_{x_1} G|/|G| \geq 1/2$.

We will make a second induction by supposing the union-closed hypothesis to be true for all $G \in \mathbb{G}_i$ such that $0 \leq i < 2^n - 1$.

We remark that we can obtain any family G' in \mathbb{G}_{i+1} by taking a family G in \mathbb{G}_i and removing a set from G which contains x_1 . However, for $|_{x_1} G'| \geq |_{x_i} G'|$ to hold for all i such that $1 < i \leq n+1$, we must also remove a set from G which contains x_2 ; a set which contains x_3 ; and so on up to a set which contains x_{n+1} .

Again if $[n+1] \notin G'$, we can conclude by the first induction hypothesis that the union-closed hypothesis is true for G' .

Otherwise we have that $[n+1] \in G'$. Therefore to obtain a set G' in \mathbb{G}_{i+1} from G , we must at least remove a set in $_{x_1} G$ and a set in $_{\bar{x}_1} G$. Thus $|_{x_1} G'|/|G'| \geq (|_{x_1} G| - 1)/(|G| - 2)$ and by the second induction hypothesis $|_{x_1} G'|/|G'| \geq 1/2$.

“[...] We must at least remove a set in $_{x_1} G$ and a set in $_{\bar{x}_1} G$.”

This is where the proof fails. The claim is false as if we pick $\{\{x_1\}, \{x_1, x_2\}\} \in \mathbb{G}_0$ and $\{\{x_1, x_2\}\} \in \mathbb{G}_1$, we can clearly see that the claim doesn't hold.

References

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