The Union-Closed Sets Conjecture

Notations

Let 2^X denote the power set of X.

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

Let $(F, \cup)$ denote a union-closed family with $|F| > 1$. By union-closed we mean that $\forall A, B \in F, A \cup B \in F$. When the context is clear we can simply write F.

Let $G \leq F$ denote a union-closed family G subset of a union-closed family F. We might informally say that G is a sub-family of F to talk about a union-closed subset of F.

Define $F_x := \{F_i \in F : x \in F_i\}$ and $F_{\bar{x}} := \{F_i \in F : x \notin F_i\}$.

Let $\mathrm{Sub}_{F,x,n} := \{G \leq F : |G_{\in,x}| = n\}$ denote the set of all sub-families G of F where the number of sets in G that contains x is equal to n.

Theorem

Let $(F, \cup)$ be a union-closed family.

Then $\exists x \in [n]$ such that $|F_x|/|F| \geq 1/2$.

Proof

We will proceed by induction. It is easy to verify that it is true for any sub-family $F \leq 2^{[2]}$.

Now suppose the union-closed hypothesis is true for all sub-families $F \leq 2^{[n]}$.

Take a sub-family $G \leq 2^{[n+1]}$. Either

$$[n+1] \in G$$

or

$$[n+1] \notin G.$$

Suppose that $[n+1] \notin G$. Then $G \leq 2^{[n]}$ and by the induction hypothesis the union-closed hypothesis is true.

Now suppose on the contrary that $[n+1] \in G$. If $G = 2^{[n+1]}$ then $|G_1|/|G| = 1/2$.

Consider the family of sub-families $\{\mathbb{G}_0, \mathbb{G}_1, \ldots, \mathbb{G}_{2^n-1}\}$ where $\mathbb{G}_i := \mathrm{Sub}_{G,x,2^n-i}$. For a sub-family H in $\mathbb{G}_i$, we can consider without loss of generality that $|H_1| \geq |H_j|$, for all $1 < j \leq n+1$. Therefore we will consider that $x = 1$ for all $\mathbb{G}_i$.

For all H in $\mathbb{G}_0$ we can see that $|H_1| = 2^n$ and $|H_{\bar{1}}| \leq 2^n$. Therefore $|H_1|/|H| \geq 1/2$.

We will make a second induction by supposing the union-closed hypothesis to be true for all $\mathbb{G}_i$ such that $0 \leq i < n+1$.

We remark that we can obtain a family H' in $\mathbb{G}_{i+1}$ by taking a family H in $\mathbb{G}_i$ and removing a set from H which contains 1. However, for $|H'_1| \geq |H'_j|$ to hold for all $1 < j \leq n+1$, we must also remove a set from H which contains 2; a set which contains 3; and so on up to a set which contains $n+1$.

Since $[n+1] \in G$ we have that $[n+1] \in H'$. Therefore to obtain a set H' in $\mathbb{G}_{i+1}$ from H, we must at least remove a set in H_1 and a set in $H_{\bar{1}}$. Thus $|H'_1|/|H'| \geq (|H_1|-1)/(|H|-2)$ and by the second induction hypothesis $|H'_1|/|H'| \geq 1/2$. This concludes the proof. $\square$