तेजस्वीना वधि तमस्तु

SNPIT & RC

SUBJECT:- ADVANCED ENGINEERING MATHEMATICS

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TOPIC:- LAPLACE TRANSFORM OF PERIODIC FUNCTIONS

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LAPLACE TRANSFORM OF PERIODIC FUNCTIONS

1. PERIODIC SQUARE WAVE
2. PERIODIC TRIANGULAR WAVE
3. PERIODIC SAWTOOTH WAVE
4. STAIRCASE FUNCTION
5. FULL-WAVE RECTIFIER
6. HALF-WAVE RECTIFIER
7. UNIT STEP FUNCTION
8. SHIFTING THEOREM

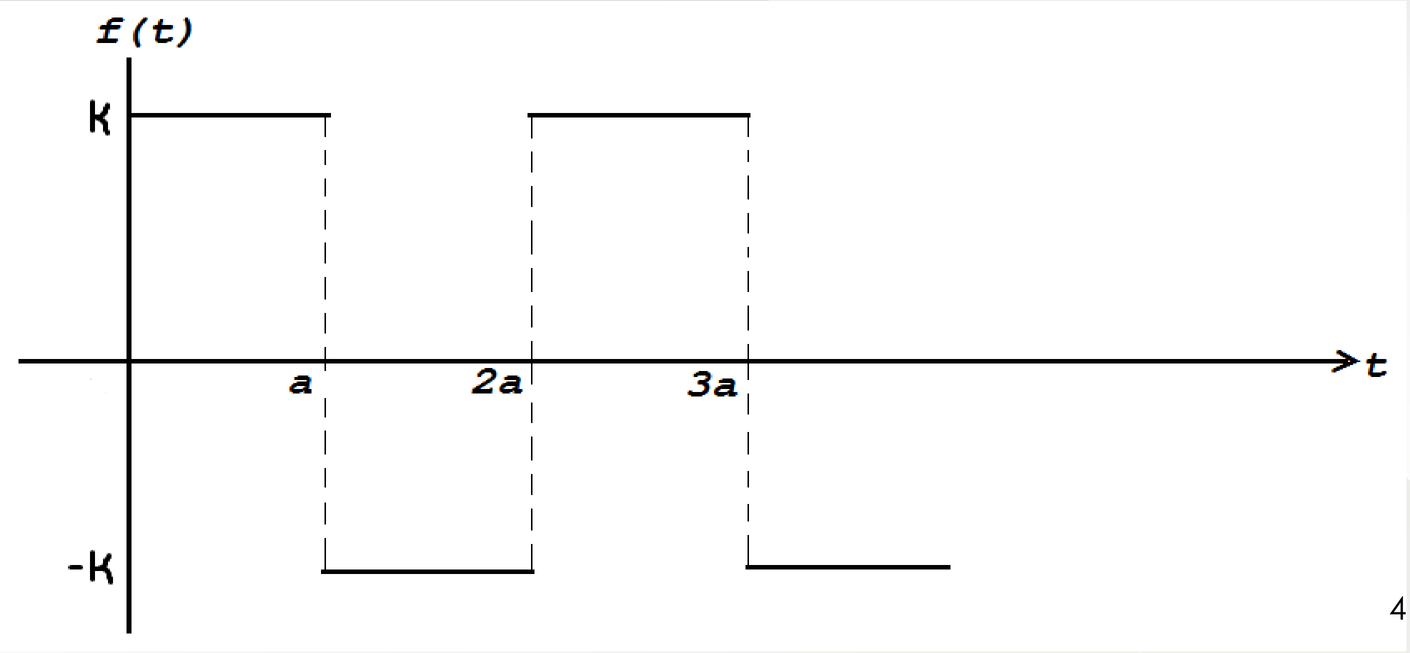
1. PERIODIC SQUARE WAVE

1. Find the Laplace transform of the square wave function of period 2a

   defined as $f(t) = k$ if $0 \leq t < a$

   $= -k$ if $a < t < 2a$

   The graph of square wave is shown in figure



- ANS. :-

since f(t) is a periodic function with period p= 2a

$$L\{f(t)\} = \frac{1}{1-e^{-2as}}\left(\int_0^a ke^{-st}\,dt + \int_0^{2a} -ke^{-st}\,dt\right)$$

$$= \frac{k}{1-e^{-2as}}\left[\left(\frac{e^{-st}}{-s}\right)_0^a - \left(-\frac{e^{-st}}{s}\right)_a^{2a}\right]$$

$$= \frac{k}{1-e^{-2as}}\left[-\frac{e^{-as}}{s} + \frac{1}{s} + \frac{e^{-2as}}{s} - \frac{e^{-as}}{s}\right]$$

$$= \frac{k}{s\left(1-e^{-2as}\right)}\left(1-2e^{-as}+e^{-2as}\right)$$

$$= \frac{k\left(1-e^{-as}\right)^2}{s\left(1+e^{-as}\right)\left(1-e^{-as}\right)}$$

$$= \frac{k}{s}\left(\frac{1-e^{-as}}{1+e^{-as}}\right)$$

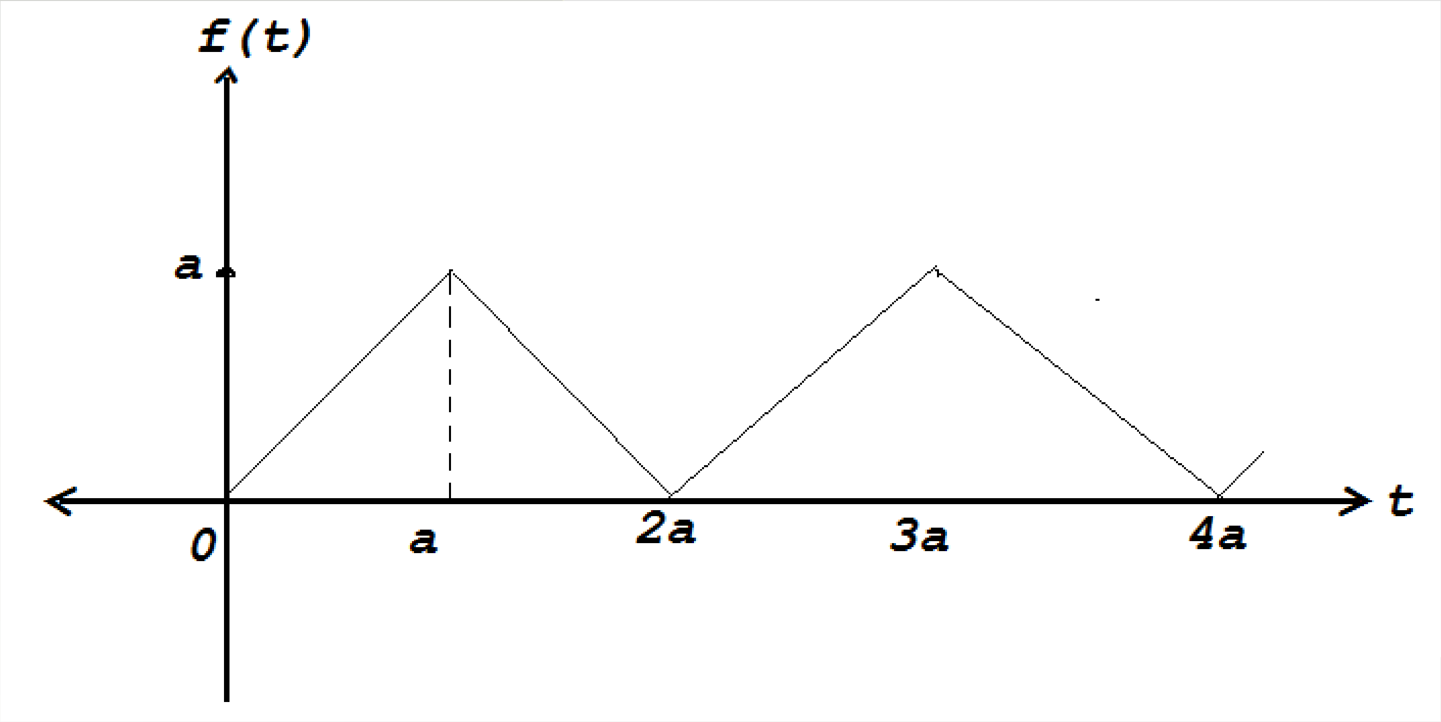
$$= \frac{k}{s}\,\frac{e^{-as/2}}{e^{-as/2}}\left(\frac{e^{as/2}-e^{-as/2}}{e^{as/2}+e^{-as/2}}\right)$$

$$= \frac{k}{s}\tanh\left(\frac{as}{2}\right)$$

$$\therefore L\{f(t)\} = \frac{k}{s}\tanh\left(\frac{as}{2}\right)$$

2. PERIODIC TRIANGULAR WAVE

- Find the Laplace transform of the periodic function shown in figure.



- ANS.:-The function can be represented as

$f(t) = t \qquad 0 < t < a$

$\quad = 2a - t \qquad a < t < 2a$

The function has a period 2a

$$L\{F(t)\} = \frac{1}{1-e^{-2as}}\left(\int_0^{2a} e^{-st} f(t)\,dt\right)$$

$$= \frac{1}{1-e^{-2as}} \left(\int_0^a e^{-st} f(t)dt + \int_a^{2a} e^{-st} f(t)dt \right)$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} t dt + \int_a^{2a} e^{-st} (2a-t)dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\begin{array}{l} \left\{ t\left(-\frac{e^{-st}}{s} \right) - (1)\left(\frac{e^{-st}}{s^2} \right) \right\}_a^{2a} + \\ \left\{ (2a-t)\left(-\frac{e^{-st}}{s} \right) - (-1)\left(\frac{e^{-st}}{s^2} \right) \right\}_a^{2a} \end{array} \right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{-a\,e^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{a\,e^{-as}}{s} + \frac{e^{-2as}}{s^2} - \frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{s^2\,(1-e^{-2as})}(1-2a\,e^{-as}+e^{-2as})$$

$$\therefore L\{f(t)\} = \frac{1}{s^2\,(1-e^{-2as})}\,(1-e^{-as})^2$$

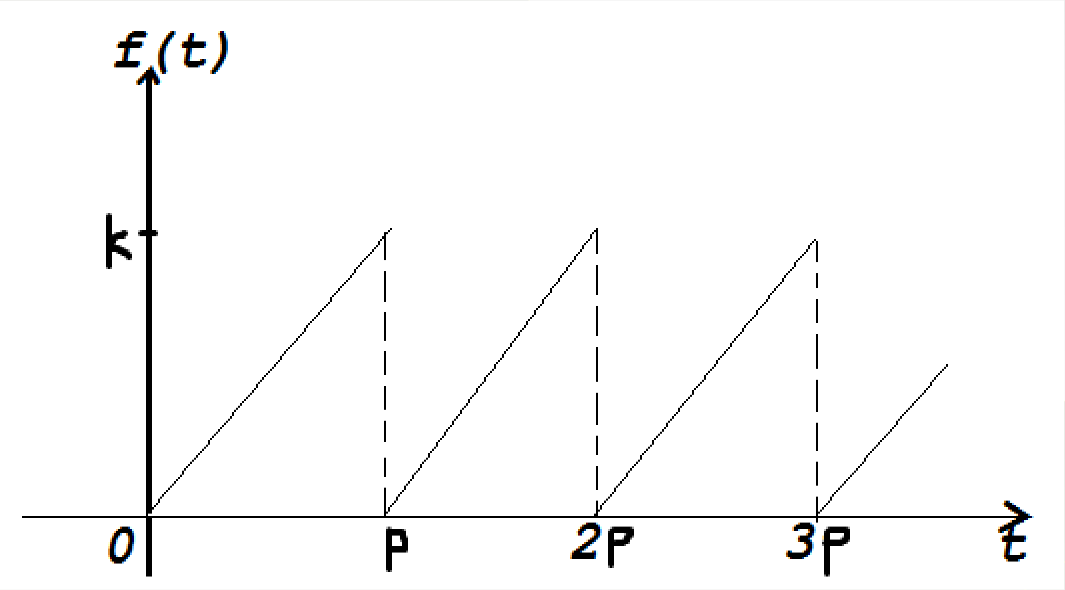
$$\therefore L\{f(t)\} = \frac{1}{s^2}\tan\text{h}\left(\frac{sa}{2}\right)$$

3. PERIODIC SAW TOOTH WAVE

- Find the Laplace transform of the saw tooth wave function given by

  f(t)= $\frac{k}{p}t$ if $0 < t < p$, $f(t+p) = f(t)$

- ANS.:-



Since f(t) is a periodic function with period p.

$$L\{f(t)\} = \frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-ps}}\int_0^p e^{-st} \frac{kt}{p} dt$$

$$= \frac{k}{\mathrm{p}(1-e^{-ps})}\int_0^p e^{-st} t dt$$

$$= \frac{k}{\mathrm{p}(1-e^{-ps})}\left[t\left(\frac{e^{-st}}{-s}\right)-1\left(\frac{e^{-st}}{s^2}\right)\right]_0^p$$

$$= \frac{k}{\mathrm{p}(1-e^{-ps})}\left[\frac{-pe^{-sp}}{s} - \frac{e^{-sp}}{s^2} + \frac{1}{s^2}\right]$$

$$= \frac{k}{(1-e^{-ps})\mathrm{s}^2\ p}\left[\left(1-e^{-sp}\right) - (sp)e^{-sp}\right]$$

$$= \frac{k}{\mathrm{s}^2\ p} - \frac{ke^{-sp}}{\mathrm{s}(1-e^{-sp})}$$

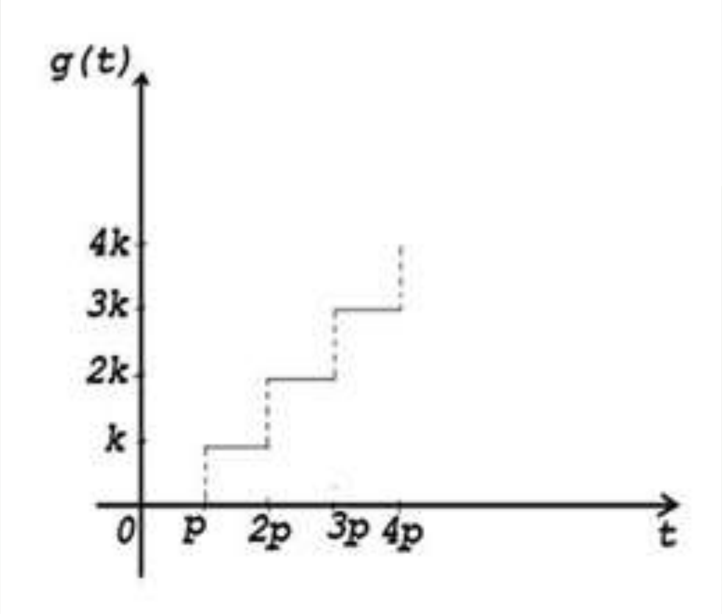
$$\therefore L\{f(t)\} = \frac{k}{\mathrm{s}^2\ p} - \frac{ke^{-sp}}{\mathrm{s}(1-e^{-sp})}$$

4. STAIRCASE FUNCTION

- Find the Laplace transform of the staircase function defined as

$g(t)=kn$ for $np < t < (n+1)p$, where $n=0,1,2,....$

(Note : This is not a periodic function)



- ANS:- If h(t) is a saw tooth wave function defined in example 3 as

$$h(t)= \frac{kt}{p} \text{ for } 0<t<p$$

and $h(t + p) = h(t)$ for all values of t. It is easy to observe from the figure that

$$g(t)=\frac{kt}{p} - h(t) \text{ for } 0 < t < \infty$$

$$\therefore L[\mathrm{g(t)}] = L\left\{\frac{kt}{p}\right\} - L\{h(t)\}$$

$$\therefore L[\mathrm{g(t)}] = \frac{k}{ps^2} - \left[\frac{k}{s^2 p} - \frac{ke^{-sp}}{s\left(1-e^{-sp}\right)}\right]$$

$$\therefore L[\mathrm{g(t)}] = \frac{ke^{-sp}}{s\left(1-e^{-sp}\right)}$$

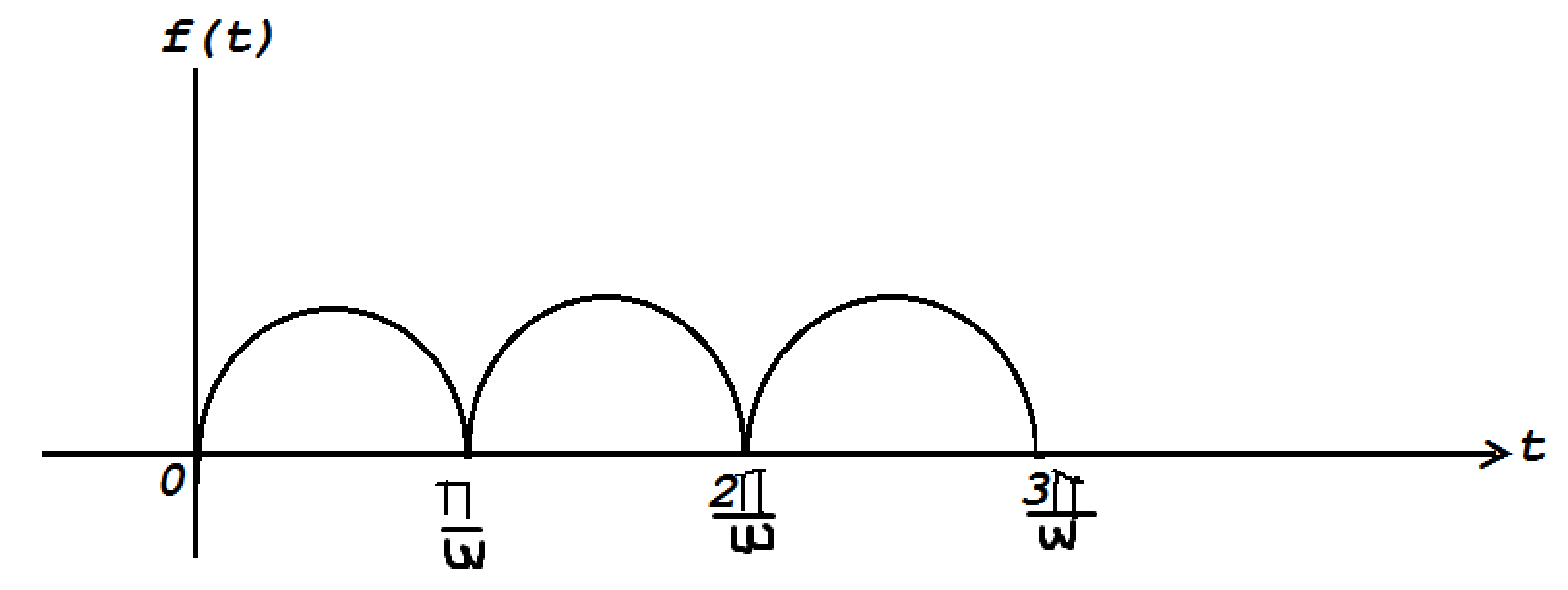
5. FULL-WAVE RECTIFIER

- Find the Laplace transform of the full wave rectification of f(t)=$\left|\sin\omega t\right| \; t \geq 0$

Ans:-The graph of the function f(t)is shown in figure.

Observe that $$\left|\sin\omega\left(t+\frac{\pi}{\omega}\right)\right| = \left|\sin\omega t\right|$$ for any t.

This function is called the full sine-wave rectifier function with period $\frac{\pi}{\omega}$



We may write the definition of f(t) as follows:

f(t)=$\sin \omega t$ for $0 \le t \le \dfrac{\pi}{\omega}$

and $f\left(t+\dfrac{\pi}{\omega}\right)=f(t)$ for all t.

$$\therefore \qquad L[f(t)] = \frac{1}{1-e^{-\frac{s\pi}{\omega}}} \int_0^{\frac{\pi}{\omega}} e^{-st} \sin \omega t dt$$

$$Now \int_0^{\frac{\pi}{\omega}} e^{-st} \sin \omega t dt = \left[\frac{e^{-st}}{s^2+\omega^2} (-s.\sin \omega t - \omega \cos \omega t) \right]_0^{\frac{\pi}{\omega}}$$

$$\int_0^{\frac{\pi}{\omega}} e^{-st} \sin \omega t dt = \frac{1}{s^2+\omega^2} \left[\omega e^{\frac{-s\pi}{\omega}} + \omega \right]$$

$$\int_0^{\frac{\pi}{\omega}} e^{-st} \sin \omega t dt = \frac{\omega}{s^2+\omega^2} \left(1 + e^{\frac{-s\pi}{\omega}} \right)$$

$$\therefore L[f(t)] = \frac{1}{1-e^{\frac{-s\pi}{\omega}}} \cdot \frac{\omega}{s^2+\omega^2}\left(1+e^{\frac{-\pi s}{\omega}}\right)$$

$$\therefore L[f(t)] = \frac{\omega}{s^2+\omega^2} \frac{1+e^{\frac{-\pi s}{\omega}}}{1-e^{\frac{-\pi s}{\omega}}} \cdot \frac{e^{\frac{\pi s}{2\omega}}}{e^{\frac{\pi s}{2\omega}}}$$

$$\therefore L[f(t)] = \frac{\omega}{s^2+\omega^2} \coth\left(\frac{s\pi}{2\omega}\right)$$

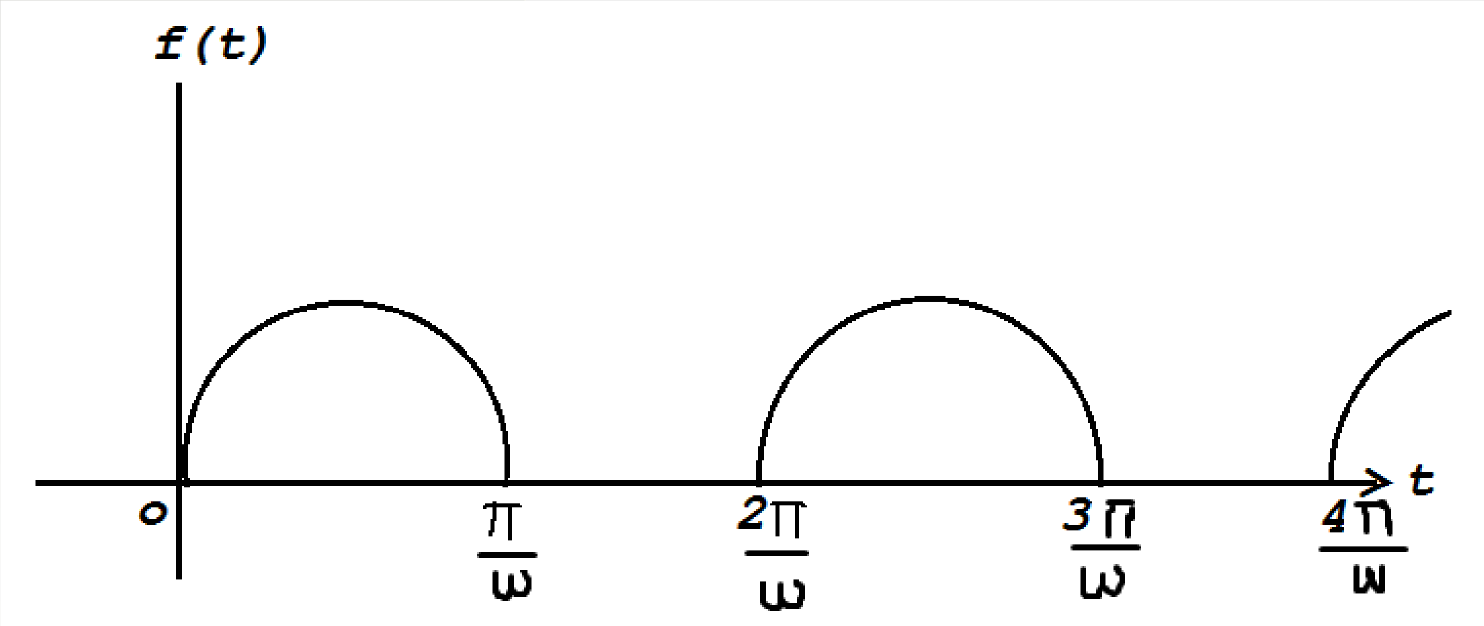
6. HALF-WAVE RECTIFICATION

Find the Laplace transform of half-wave rectification of sinωt defined by

$$f(t)=\begin{cases}\sin wt & if \quad 0 < t < \pi/w \\ 0 & if \quad \pi/w < t < 2\pi/w\end{cases}$$

Where $f\left(t+\dfrac{2n\pi}{w}\right)=f(t)$ For all integer n.

- Ans. The graph of function F(t) is shown in the figure



$$f\left(t+2\pi/w\right)=f(t)\text{ for all }t$$

$$\therefore L\left\{f\left(t\right)\right\}=\frac{1}{1-e^{-\frac{2\pi s}{w}}}\int_{0}^{2\pi/w}e^{-st}f\left(t\right)dt$$

$$=\frac{1}{1-e^{-\frac{2\pi s}{w}}}\int_{0}^{\pi/w}e^{-st}\sin wt\,dt.....(1)$$

$$\text{Now} \int_0^{\pi/w} e^{-st} \sin w\,t\; d\,t = \left[\frac{e^{-st}}{s^2+w^2}\left(-s\ \sin wt - w\cos wt\right) \right]_0^{\pi/w}$$

$$= \frac{1}{s^2+w^2}\left(we^{-\frac{\pi s}{w}} + w \right)$$

$$= \frac{w}{s^2+w^2}\left(1+e^{-\frac{\pi s}{w}} \right)$$

$\therefore$ (1) becomes

$$L\left\{f(t)\right\} = \frac{1}{1-e^{-\frac{2\pi s}{w}}} \quad \frac{w}{s^2+w^2}\left(1+e^{-\frac{\pi s}{w}}\right)$$

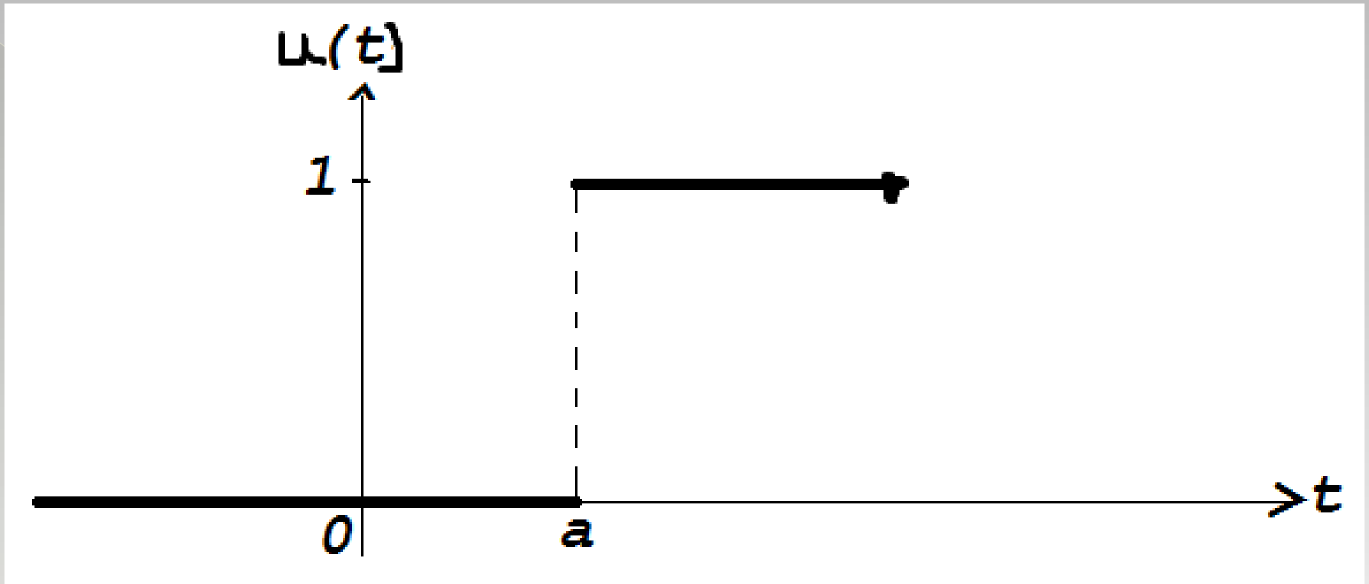
$$= \frac{w}{s^2+w^2} \quad \frac{\left(1+e^{-\frac{\pi s}{w}}\right)}{\left(1+e^{-\frac{\pi s}{w}}\right)\left(1-e^{-\frac{\pi s}{w}}\right)}$$

$$= \frac{w}{\left(s^2+w^2\right)\left(1+e^{-\frac{\pi s}{w}}\right)}$$

7. UNIT STEP FUNCTION (OR HEAVISIDE'S FUNCTION

- The unit step function $u(t - a)$ is defined as

$$u(t - a) = \begin{cases} 0 & \text{if } t < a \quad (a \geq 0) \\ 1 & \text{if } t \geq a \quad \text{figure.} \end{cases}$$



The unit step function is also called the Heaviside function. In particular if a = 0, we have

$$u(t) = 0 \text{ if } t < 0$$
$$= 1 \text{ if } t \geq 0$$

- Laplace transform of unit step function

By definition of Laplace transform,

$$\therefore L\{\mathrm{u}(t-a)\} = \int_0^\infty e^{-st} \, . \, u(t-a) dt$$

$$= \int_0^\infty e^{-st} \, (0) dt + \int_a^\infty e^{-st} \, (1) \, dt$$

$$= \int_0^\infty e^{-st} \;\; dt = \left[\frac{e^{-st}}{-s} \right]_a^\infty = \frac{1}{s} e^{-as}$$

$$\therefore L\{u(t-a)\} = \frac{1}{s}e^{-as} \qquad \therefore L^{-1}\left[\frac{1}{s}e^{-as}\right] = \{u(t-a)\}$$

in particular, if $\text{a} = 0$

$$\therefore L\{u(t)\} = \frac{1}{s} \qquad \therefore L^{-1}\left[\frac{1}{s}\right] = u(t)$$

Second shifting theorem

If $\quad L\{\mathrm{f}(t)\} = \bar{f}(s),\ then$

$$L\{\mathrm{f}(\mathrm{t}-\mathrm{a})\,\mathrm{u}(\mathrm{t}-\mathrm{a})\} = e^{-as}\ \bar{f}(\mathrm{s})$$

Proof : by definition of Laplace transform, we have,

$$L\{f(t-a)\,\mathrm{u}(\mathrm{t}-\mathrm{a})\} = \int_0^\infty e^{-st}\,.\,f(t-a)\,u(t-a)\,dt$$

$$= \int_0^a e^{-st}\ f(\mathrm{u}-a)\,(0)\,\mathrm{dt} + \int_a^\infty e^{-st}\ f(t-a)\,(1)\,dt$$

$$= \int_{a}^{\infty} e^{-st}\ f(\mathrm{u}-a))\,dt, \quad Substituting\ t-a=r$$

$$= \int_{0}^{\infty} e^{-s(a+r)}\ f(\mathrm{r})\,dr$$

$$= e^{-as}\int_{0}^{\infty} e^{-sr}\ f(\mathrm{r})\,dr = e^{-as}\ \bar{f}(s)$$

$$\therefore L\{f(t-a)\,\mathrm{u(t-a)}\} = e^{-as}\ \bar{f}(\mathrm{s}) = e^{-as}\ L[f(t)]$$

$$\therefore L^{-1}[e^{-as}\ \bar{f}(\mathrm{s})] = f(t-a)\,u(t-a)$$

$$Cor.1: L\{f(t)\,u(t-a)\} = e^{-as}\, L[f(t+a)]$$

(alternative form of second shifting theorem)

$$Cor.2: L[u(t-a)] = L\,[1.\,u(t-a)] = e^{-as}\, L(1) = \frac{e^{-as}}{s}$$

$$Cor.3: L[u(t-a)-u(t-b)] = \frac{e^{-as}-e^{-bs}}{s}$$

$$Cor.4: L[f(t)\ \{u(t-a)-u(t-b)] = e^{-as}\, L\{f(t+a)\} - e^{-bs}\, L\{f(t+b)\}$$

REFERENCE BOOK :- DR K.R.KACHOT


THANK
YOU