European Financial Management, Vol. 00, No. 0, 2011, 1–28
doi: 10.1111/j.1468-036X.2011.00635.x

A Pricing Framework for Real Estate Derivatives

Frank J. Fabozzi
EDHEC Business School and EDHEC Risk Institute, 400 Promenade des Anglais, BP 3116 06202 Nice Cedex 3, France
E-mail: frank.fabozzi@edhec.edu

Robert J. Shiller
Yale University, New Haven, CT USA and MacroMarkets LLC
E-mail: robert.shiller@yale.edu

Radu S. Tunaru
University of Kent, Canterbury, Kent Business School, Park Wood Road, CT2 7PE, UK
E-mail: r.tunaru@kent.ac.uk

Abstract

New methods are developed here for pricing the main real estate derivatives — futures and forward contracts, total return swaps, and options. Accounting for the incompleteness of this market, a suitable modelling framework is outlined that can produce exact formulae, assuming that the market price of risk is known. This framework can accommodate econometric properties of real estate indices such as predictability due to autocorrelations. The term structure of the market price of risk is calibrated from futures market prices on the Investment Property Databank index. The evolution of the market price of risk associated with all five futures curves during 2009 is discussed.

Keywords: *derivatives pricing*, *real estate indices*, *incomplete markets*, *market price of risk*, *serial correlation*

JEL classification: *G*13, *G*15, *G*20

We thank John Doukas (the editor of this *Journal*) and the two anonymous referees for their help in improving the content and presentation of this article. Earlier versions of this article were presented at seminars at ICMA Reading and Judge Business School Cambridge as well as at conferences, Campus for Finance 2010 in Vallendar and the conference on Incomplete Markets, 2010 London. We also thank the participants at those events, in particular, Michael Dempster, Markus Harder, Tony Key, and Hashem Pesaran, for helpful comments; special thanks to Stuart Heath at EUREX, London for making the IPD UK futures data available. Correspondence: Frank J. Fabozzi.

1. Introduction

The subprime mortgage crisis which began in the summer of 2007 has emphasised the importance of the real estate market to the global economy. Yet, by comparison with equity, commodity, and debt markets, there has been little innovation with respect to viable derivatives products for the real estate market. The development of risk management procedures for commercial real estate is ahead of its twin sector, residential real estate. However, the sub-prime mortgage crisis has put residential real estate on the top of the list for regulators, government agencies, and investment and commercial banks. Case and Quigley (2007) pointed out that forecasts of home prices impacted the financial markets in a more dramatic way than the direct wealth effects on the economy.

Regarding financial instruments that can be used for hedging real estate portfolios, the turning point for commercial real estate was 2005 when trading in over-the-counter (OTC) total return swaps on the Investment Property Databank (IPD) Index began in the UK. Volume exceeded £3 billion of notional value by the first quarter of 2007, representing more than one-third of the total rate of quarterly trading volume in the cash market for IPD-tracked properties. If the same proportional volume had been traded in the USA for properties tracked by Real Capital Analytics Inc. (RCA), annual derivatives trading volume would have been more than $120 billion. This suggests that the real estate derivatives market has enormous growth opportunities.

Since the inception of the UK property derivatives market in 2005, there has been a steady increase in property derivatives with the volume increasing in general, although post 2009 the activity has slowed down to some extent. Trading activity on IPD derivatives is summarised in Table 1.

In the USA, activity in this sector was slower. Two possible explanations have been proffered as to why in the USA there has been a lack of trading in real estate derivatives. The first is that there is no single commercial property index similar to the IPD Index in the UK. The second explanation is that there appears to be a knowledge gap between real estate traders and derivatives traders. Geltner and Fisher (2007) pointed to a 2006 survey, conducted at the MIT Center for Real Estate, of 37 US real estate investment managers and other likely participants in a derivatives market, who expressed a lack of confidence in how the real estate derivatives should be priced. About 75% of the respondents indicated the lack of reliable models as either an 'important' or a 'very important' barrier. Property derivatives could play an important role for describing timely sentiment indicators for the property market by forming expectations on an annual basis over the tenure of the derivative contract. These new markets should provide enhanced information compared to the real estate investment trust (REIT) market where expectations are biased due to the effects of leveraging and portfolio composition.

While for commercial real estate financial innovation seems to be more advanced, for the home price sector the reality is a bit different. Shiller (2008) has described the establishment recently of risk management vehicles for home prices and emphasises the importance of such instruments in eliminating the inefficiency of the market for single-family homes. Forward contracts, European options, and other structured products are known to have been traded in the OTC market.

From an academic point of view, the literature on the pricing models of real estate derivatives is sparse. The main purpose of this paper is to bridge this gap by offering not just 'another' model, but a framework that (1) captures the econometric properties of real estate indices that have been identified in the literature, (2) leads to exact formulae, when possible, for the main derivatives contingent on the real estate indices (i.e., forwards,

Table 1
Trading volumes

This table reports trading volume activity on derivatives contingent on the IPD All Property Total return UK index, between Q4 of 2005 and Q4 of 2009.

	Q4 2005	Q4 2006	Q4 2007	Q4 2008	Q4 2009
Total outstanding notional (£million)	1,100	6,686	9,023	11,182	9,598
Total notional of trades executed each quarter (£million)	183	864	1,662	979	907
Total outstanding number of trades	80	415	703	1,185	1,317

total return swaps, and options), and (3) allows the market price of risk to be calibrated in one market and applied exogenously for pricing other derivatives contingent on the same underlying property index. This methodology will enable financial players to trade more derivatives on property and contribute to information recovery for future property prices. Various market participants would be able to improve their asset allocation upon further growth in derivatives and policy makers would benefit from being able to infer the market view of future property prices so as to avoid the build-up of property price bubbles.

In this paper, the focus is on real estate derivatives, discussing the structural elements of these contracts and focusing on pricing methods that are theoretically sound and practically feasible. In the next section, the related literature is reviewed and in Section 3 some of the main indices and the contracts available in real estate markets are briefly discussed. Although our focus is on indices and contracts in the USA and UK, the framework can be applied to any other property markets. The main results of this paper are provided in Section 4 and numerical calibration issues are reviewed in Section 5. The final section summarises our conclusions.

2. Literature Review

In a seminal paper dealing with risk management for real estate, Case and Shiller (1989) showed that well-constructed repeat-sales home prices can accurately capture trends in the real estate market. They also made the case for the role of real estate futures markets in improving the efficiency of the home real estate market. The residential property market has been largely untouched by institutional investors, except for a few small marginal market participants, mainly because of the nature of transactions, the contract type, and a lack of knowledge of pricing models. This makes the housing market unique (see Gemmill, 1990). At the granular level, trading in housing property is infrequent, the underlying asset being inhomogeneous and traded in an opaque market where no single market-price exists. The ultimate buyers of residential real estate consider the asset a combination between a financial asset and a consumption good. This market may experience imbalances, with demand changing quickly but supply fixed in the short term, causing price cyclicality (see Chinloy, 1996). The flux of information is asymmetric and agents' price expectations backward looking. Case and Shiller (1989, 1990) found positive serial correlation as well as inertia in house prices and excess returns, concluding that in the USA 'the market for single-family homes is inefficient' (Case and Shiller, 1989).

The use of futures and options for risk management in real estate markets has been discussed by Gemmill (1990), Case *et al*. (1993), Case and Shiller (1996), Shiller and Weiss (1999), Geltner and Fisher (2007), and Bertus *et al*. (2008). Fisher (2005) provided an overview of NCREIF-based swap products. Shiller (2008) discussed derivatives markets in general for home prices.

An early contribution by Titman and Torous (1989) proposed a contingent claim approach for pricing commercial mortgage-backed securities (CMBS) showing that a commercial mortgage is an option contingent on the short-term interest rate and the price of real estate. Shiller (1993) was the first to consider perpetual futures on housing indices. Buttimer *et al*. (1997) analysed a two-state model for pricing derivatives contingent on a real estate index and an interest rate. More specifically, a commercial real estate index total return swap was priced with a bivariate binomial model leading to a positive but negligible swap spread price. Their model was reassessed by Bjork and Clapham (2002) who proved, under their assumptions that the index can be traded, that the price of the swap should be zero within an arbitrage-free framework. Patel and Pereira (2008) extended this line of modelling by taking into consideration counterparty risk and found that real estate index total return swap (TRS) value is no longer zero when counterparty risk is taken into account. Moreover, their results indicate that the spread over a market interest rate such as Libor that the TRS payer must charge is highly dependent on the volatility of index returns and on counterparty default risk, higher volatility of returns and counterparty default risk implying higher spread over market interest rate. Following a similar no-arbitrage approach, Ciurlia and Gheno (2009) focused on taking into account the real estate market sensitivity to the interest rate term structure by employing a two-factor model where the real estate asset value and the spot rate dynamics are jointly modelled. The model is then used to price hypothetical European and American options adopting a bi-dimensional binomial lattice framework.

A different technique has been proposed by van Bragt *et al*. (2009) who developed a risk-neutral valuation procedure for real estate derivatives contingent on autocorrelated indices. Their idea is that the observed property index is a random departure from an unobserved underlying index, representing the fundamental or efficient value of the property market that the observed index tries to reflect. From a theoretical perspective, the pricing of derivatives on indices exhibiting autocorrelation or predictability goes back to Lo and Wang (1995) and Jokivuolle (1998). Van Bragt *et al*. (2009) reconstruct the observed index following an updated method due to Blundell and Ward (1987), consisting of adding multiple lag terms and the accrued value of past observations. This approach also leads to analytical pricing formulas for various real estate derivatives.

One of the most important characteristics of real estate derivatives is the impossibility of short selling for the underlying index. Similar to the case of commodities, the no-arbitrage argument for the forward contract cannot function properly. One major problem with the no-arbitrage approach pricing equivalent to finding a self-financing strategy is that it implies that real property itself or real estate index can be traded at any time, at full market value, in any small unit without any trading cost. This causes problems related to incompleteness of the market, given that we are trying to price exactly those few instruments that are available.

This point of view is shared by Otaka and Kawaguchi (2002) who proposed a market model that consists of three separate markets: (1) the security market, where stocks, bonds, currencies, and derivative securities are traded without friction, (2) the space market given by the rents of buildings, and (3) the property market where the prices of real properties are determined. Because a real estate security cannot be replicated nor hedged perfectly, Otaka and Kawaguchi (2002) consider hedging and pricing of real estate securities under market incompleteness, obtaining a reference price by associating it with a risk-minimising strategy. Baran *et al*. (2008) used the Schwartz and Smith (2000) model for pricing real estate derivatives directly under a martingale measure. Their contribution is an ingenious parameter estimation combining constrained maximum likelihood with Kalman filter.

Geltner and Fisher (2007) proposed a different methodology for pricing real estate forward and swap contracts with an equilibrium argument. They differentiate between a total return index and a capital index and between a transaction-based and appraisal based indices. With their method, the forward price equals the expectation at time zero of the time T value of the index, discounted at the adjusted rate equal to the market equilibrium required ex ante risk premium in the index total return going forward, where the premium is over the riskless rate. This formula can be used for an appraisal index if the lag or momentum effect in the index is taken into consideration when calculating the expectation of the future value of the index and other smoothing effects that may diminish the risk premium associated with the index. Another equilibrium model to price housing index derivatives has been recently developed by Cao and Wei (2010). They extend Lucas' equilibrium model by expanding the model to include a housing market on top of the financial and goods markets. Working with constant relative risk-aversion, mean-reverting dividends, and geometric Brownian motion for the housing index process, closed formulae are obtained for the equilibrium forwards and European vanilla call options. In order to reconcile market prices with equilibrium prices, risk premium measures are introduced and thoroughly analysed.

Another approach proposed by Levin (2009) is based on modelling the house price appreciation rate as the sum of the income inflation rate, a noise factor producing normally distributed, serially uncorrelated, jumps in discrete time, and a dynamical system similar to the spring-mass oscillator with friction and random forces. One of the features accounted for by Levin's model is that the log of home price index will perpetually oscillate around an equilibrium level in the absence of frictions and other noisy terms.

3. Main Characteristics of Real Estate Indices and Contingent Derivatives

3.1 *Indices*

Fisher *et al*. (1994) classified indices into two categories: (1) transactions-based, taking into account the actual transaction-prices over the period, and (2) valuation or appraisal based, derived from valuation-models and continuously updated property-characteristics. In a traditional appraisal-based index, all of the properties in the index population are appraised regularly, and the index returns calculated from a simple aggregation of those appraised values each period. One example is the National Council of Real Estate Investment Fiduciaries (NCREIF) National Index (NPI), on which many US fund managers are benchmarked wholly or partly. Transaction-based indices, however, can be based on the entire population of commercial properties. The IPD family of indices used in the UK and continental Europe on the other hand is determined on an appraisal basis.

In the academic literature, two methodologies have been developed to solve the statistical problems associated with transaction-based indices – the hedonic value procedure and the repeat-sales regression methodology.[1] The hedonic value methodology – first introduced by Adelman and Griliches (1961) and then extended by Rosen (1974) – involves regressing property prices on a function of various characteristics of the properties, such as size, age, location, and quality. The MIT Center for Real Estate, in cooperation with the NCREIF, began publishing in 2006, the first regularly produced hedonic index of commercial property based on the prices of the properties sold from the NCREIF database.

The residential and commercial indices have common statistical characteristics but they also have differences due to the nature of the underlying properties. Residential indices are subject to seasonality which is influenced by the school calendar resulting in more transactions in the summer and early fall compared to the other seasons of the year. This seasonality factor is not observed in commercial properties. Moreover, as an asset class, residential property is considered a combination between a consumption asset and an investment asset. In contrast, commercial property prices are driven mainly by their investment attributes and the resulting economic value is related to both the general and local economic conditions. Here we review the serial correlation properties for a variety of real estate indices in the USA and UK.

Commercial indices. The IPD Annual UK All Property Total Return Index is a commercial real estate index that is compiled from the individual property records of almost 300 annually, quarterly, and monthly valued funds. The total market value of the properties comprising this index is approximately £130 billion. Published once a year, usually on the last working Friday of February, this index can be traced back to December 1980. However, there are historical data going back even farther as illustrated below. The index represents the total combined value of roughly more than 75% of the property assets held by UK institutions, trusts, partnerships, and listed property companies, and just less than half of the total professionally managed UK property investment market. The majority of UK commercial property derivative contracts traded to date have been formally priced off the IPD UK.

The tests for serial correlation for the IPD index shown in Figure 1 indicate an alternation of serial correlation, with negative autocorrelation first appearing at the third lag. This is a long historical series, starting in 1947 for all property in the UK. Similar results are obtained for data starting in 1980.

Residential indices. The repeat-sales methodology – proposed by Bailey *et al*. (1963) – takes into account that individual properties sell more than once. As opposed to appraisal-based indices that may deduct capital expenditures from the appreciation return of the index, the repeat-sales indices do not subtract the effect of capital improvement expenditures. The Standard and Poor's/Case-Shiller Home Price Indexes in the US are the best known repeat-sales indexes and they are calculated for 20 cities. The Office of Federal Housing Oversight also reports repeat-sales house price indexes for

[1] Fisher *et al*. (1994) and Geltner and Pollakowski (2007) discussed various methods for index construction. Alternative techniques for building real estate indices have been described in Mark and Goldberg (1984), Palmquist (1980), and Clapp and Giacotto (1992).

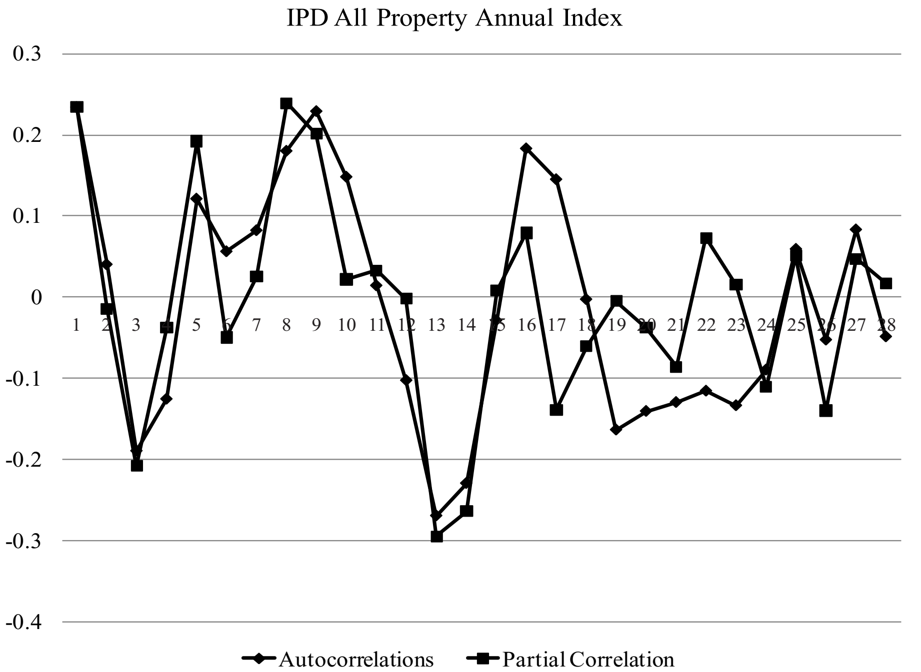

Fig. 1. Autocorrelation and partial autocorrelations for the annual return series of the IPD. All Property UK total return index. Data cover the period between December 1947 to December 2007. The majority of the associated Ljung-Box Q-statistics are significant at either the 90%, 95% or 99% level of confidence

the US.[2] Case and Shiller (1989) note that repeat-sales indices are (1) not subject to the noise caused by a change in the mix of sales, (2) highly autocorrelated, and (3) forecastable with a forecast R-squared at a one-year horizon of about a half.[3] These characteristics explain the results of the autocorrelation tests depicted in Figure 2 for the Composite-10 CSXR Case Shiller home price index, the most representative index for US residential property. There is positive autocorrelation up to two years and negative autocorrelation for longer horizons. One can also observe an oscillatory pattern that is visible in the autocorrelations, most likely due to seasonality effects.

The RPX is a more recent residential index developed by Radar Logic Incorporated (www.radarlogic.com). The RPX index captures owner-occupied housing in 25 US metropolitan statistical areas (MSAs) and it is measured by price per square foot and updated daily. This index can also be used to account for values paid in arms-length residential real estate transactions on a price per square foot basis. There is a global

[2] A historical description of the introduction of this index is given in Shiller (2008). Futures contracts written on this index, with a February quarterly cycle of expiration dates and settled at $250 times the index, were launched for 10 US cities and an aggregate index. The total notional value of futures and options traded since inception reached $612 million on 21 November 2007.

[3] This forecasting characteristic is attributed to the profound illiquidity of the market. Other research, such as Glaeser and Gyourko (2006) and Gyourko *et al*. (2006), provide evidence for the inefficiency of the housing real estate market.

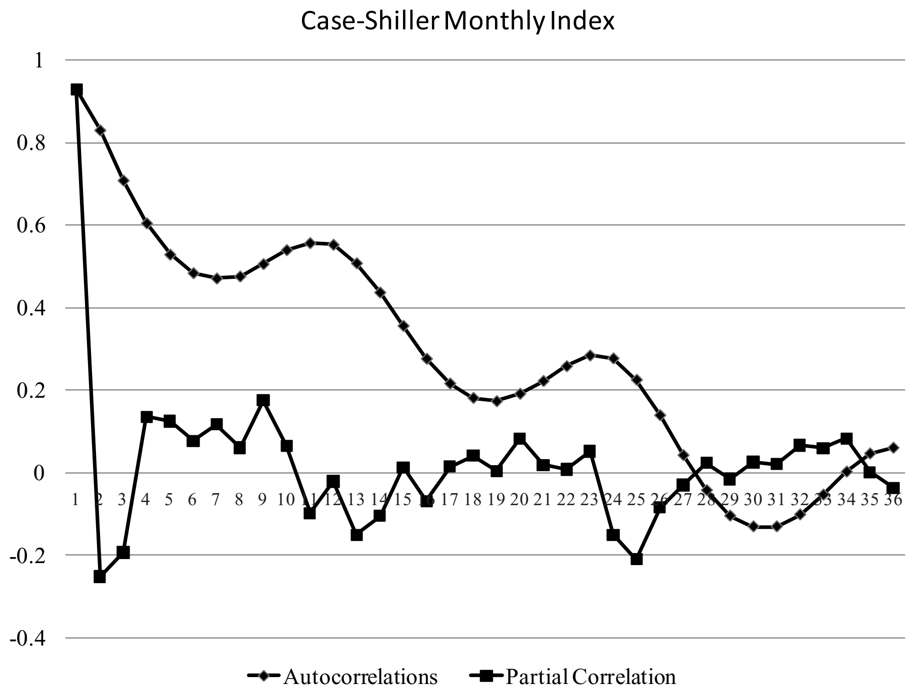

Fig. 2. Autocorrelations and partial correlations for the monthly return series of the Case-Shiller index. Data cover the period January 1987 to November 2008. All Ljung-Box Q-statistics are significant at all levels of confidence: 90%, 95% and 99%

MSA 25 Composite index that reflects the top 25 MSAs, as well as indices in the individual MSA. The RPX index is based on rolling quarterly price fixings. Based on this index, a forward contract and a TRS contract are actively traded.

The serial correlation tests for this index, presented in Figure 3, suggest that there is no autocorrelation for the short or longer horizons. However, the historical series of this index is not very long, starting only in March 2000. More research is needed to understand why this index is different from the majority of other property indices.

The Halifax House Price Index is the UK's main house price series with monthly data covering the whole country going back to January 1983. The index is constructed from one of the largest monthly samples of mortgage data, typically covering around 15,000 house purchases per month. From the data, a 'standardised' house price is derived and property price movements on an apples-for-apples basis (including seasonal adjustments) are calculated over time. The index is seasonally adjusted with the seasonal factors updated monthly. The quarterly series is produced separately. The index is built with the hedonic approach. As with the IPD indices, the econometric tests indicate that there is positive autocorrelation of returns up to about one year and negative autocorrelation afterwards. This can be seen on Figure 4.

Hence, for various real estate indices in the USA and the UK that are constructed with different methodologies, historical data support the conclusion that there is positive autocorrelation short term and negative autocorrelation long term. The empirical evidence presented here also suggests that the degree of autocorrelation is much higher in a house price series compared to a commercial real estate series. This important characteristic means that it is inappropriate to base a model on a simple Brownian

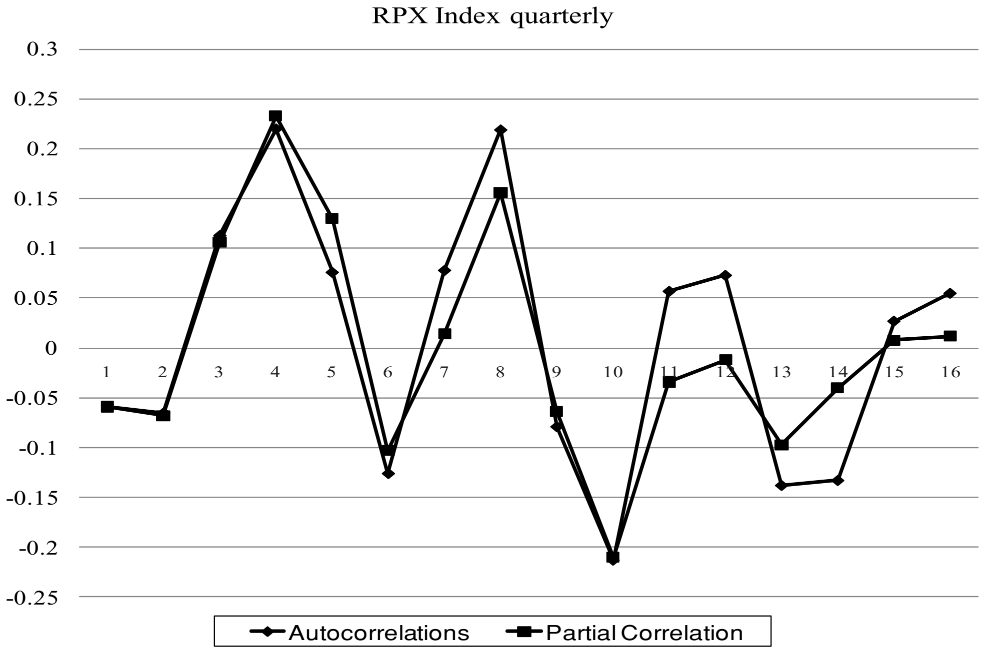

Fig. 3. Autocorrelation and partial correlation of the quarterly returns series for the RPX.CP.28 real estate index. Data cover the period March 2000 to September 2008. All Ljung-Box associated statistics are not significant at the 90%, 95%, or 99% level of confidence

motion or geometric Brownian motion for the index as the underlying for a derivatives contract. In Section 4, a pricing framework that is flexible enough to account for this feature of real estate markets is developed.

3.2 *Contracts*

The typical derivatives now being offered for commercial real estate are futures and forward contracts, total return swaps, options, and structured notes. Structured notes[4] are typically encountered in complex securitisation transactions and differ from contract to contract. Consequently, we will not consider structured notes in this paper.

In May 2006, the Chicago Mercantile Exchange (CME) introduced housing futures and options contracts based on the Case-Shiller indices to enable hedging and speculation

---

[4] It is difficult at this stage of the development of the property derivatives markets to determine hedges for structured finance deals because of the problem of estimation of the appropriate hedge ratio. In addition, the risk of shorting property derivatives on the upside is significant due to the fact that when property markets are having a stronger trend than anticipated, it may lead to losses on the property derivative without necessarily benefiting from an offsetting gain through tighter spreads. This is not only related to the liquidity of these derivatives markets, but also to the nature of the underlying indices and lack of replication with other financial instruments. Although the introduction of options may help alleviate this problem, as of this writing property structured products are hard to hedge. See Fabozzi *et al*. (2010) for a discussion of problems related to real estate assets and securitisation in Europe.

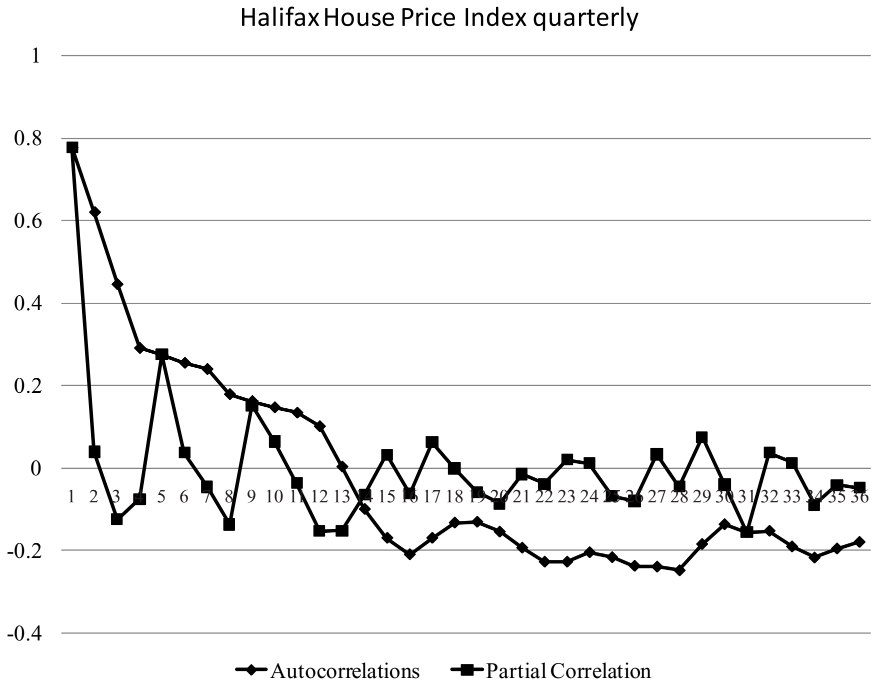

Fig. 4. Autocorrelations and partial autocorrelations for the quarterly return series of Halifax House Price index. Data cover the period 1983 to the first quarter of 2009. All associated Ljung-Box Q-statistics are significant at the 90%, 95% and 99% level of confidence

in US residential real estate. As of 2009, there are futures contracts with maturities extending out 18 months into the future, listed on a quarterly cycle of February, May, August, and November; futures contracts with maturities extending out 19 to 36 months into the future, listed on a bi-annual schedule May and November; and futures contracts with maturities extending out 37 months to 60 months into the future, listed on an annual schedule with November maturity. The futures contracts trade at $250 times the index with a tick of $50, while the options trade on one futures contract with a tick of $10, for a range of strikes at five index point intervals from the previous day close price of the futures on the Case-Shiller Index. There are futures for 10 US cities and also an aggregate index.

Trading on total return swaps contingent on the RPX index started in September 2007. Subsequently, RPX forward contracts began trading in May 2008 and they have become the most liquid contracts based on the RPX index.

Eurex began trading property futures on 9 February 2009. These futures contracts are annual contracts based on the total returns of the IPD index for individual calendar years. The futures contract aims to eliminate counterparty risk, to improve liquidity to the commercial property sector of the real estate property market, and to attract a complete range of potential participants in this asset class. Further futures contracts are going to be launched by Eurex on other IPD property indices, such as UK sector indices (Offices, Retail, Industrial) and other European indices (initially France and Germany) on a demand-led basis.

TRS for commercial real estate were first offered in USA by CSFB in 2005 based on the National Council of Real Estate Investment Fiduciaries Property Index. The NCREIF property index is the underlying for two types of swaps that were traded to date. One is the TRS that allows an investor to synthetically reproduce the economic gains of the index return. The other is an instrument used to swap different NCREIF property sectors. In June 2008, the NCREIF Property Index Total Return Swaps changed from a quarterly to annually payment schedule, all contracts paying at the end of the fourth quarter. In addition, since June 2008, there are two new TRS indices traded, the 4-year and 5-year NCREIF National. In 2005, a TRS engineered by Deutsche Bank and Eurohypo was done on IPD index (UK) while Prudential and British Land did a commercial property swap. The first property swap on the IPD France Offices Annual Index Dec was finalised in 2006 between Merrill Lynch and AXA Real Estate Investment Managers.

In 2003, Goldman Sachs issued the first series of a range of covered warrants based on the Halifax All-Houses All-Buyers seasonally-adjusted index on the London Stock Exchange (LSE). The first option on an IPD index outside the UK was referenced to the German IPD/ DIX Index and it was traded in January 2007 with Goldman Sachs acting as a broker. In August 2007, Morgan Stanley agreed on an exotic swap on Halifax House Price Index with an undisclosed counterparty. This is the UK's first residential property derivative trade that included an embedded exotic option, a 'knock-in put' option allowing the counterparty to gain if the index rises, subject to a maximum payout. The investor's capital is protected unless the index falls below an initially specified value.

MacroShares™ (also called Macros, Macro Securities and Proxy Assets) are new instruments that will allow the management of a much broader array of economic risks than are currently available in the financial market. This new financial instrument is the only listed cash security tracking the housing market. This product, which tracks the S&P/Case-Shiller Composite-(10 City) Home Price Index, permits investors to express either a bullish view on future home prices by purchasing an Up Macro Market (UMM) security or a bearish view on future home prices by purchasing a Down MacroMarket (DMM) security. Moreover, the DMM lets an investor express a bearish view without shorting the market and without a margin account. There is no counterparty risk because the trusts are fully secured by US Treasury securities and cash. The new Home Price Macros (UMM and DMM) are issued with a par value of \$25, and \$50 for a pair consisting of one share of UMM and one share of DMM. For calculations of the index, a leverage factor of 2 is applied. The index (which equals \$25 times the underlying value in the UMM until stop-out) is defined in terms of a reference value equal to the last S&P/Case-Shiller Composite (10-City) Home Price Index when the securities are issued. Each month when a new index value is announced, transfers are made between the UMM trust and the DMM trust so that the UMM trust always has a dollar amount equal to the index, with an obvious limit of \$50. Because of the transfer mechanism and the fact that the index cannot drop more than 100% from the initial value, the contract has a trigger such that when the index hits 50, the trust balances are refunded to UMM so that the DMM receives nothing.

The issuance and redemption of this contract is done as a pair, one UMM share and one DMM share, together for \$50. Thus, the price of the pair stays at \$50 plus accrued interest, and it is only necessary to price the UMM. The DMM price is just \$50 minus the UMM price. The investor in the UMM receives each quarter interest on the balance, and its balance equals the index. At maturity, or at termination trigger, the investor gets the index.

The importance of residential and commercial mortgage portfolios in the financial markets led to the creation of derivatives contingent on these types of assets. Indices of these derivatives (ABX and CMDX) were introduced just before the subprime crisis in order to allow market valuation and improved hedging. Driessen and van Hemert (2011) studied the efficiency of the CMBX market during the subprime crisis, based on a Merton-style structural option pricing model that was calibrated to stock and option prices for the S&P 500 index and several REITs. They could not find strong evidence that CMBX contracts were substantially mispriced before or during the crisis period. A similar conclusion was drawn by Stanton and Wallace (2010) who also argued that the collapse of the CMBS market during the subprime crisis was caused primarily by the rating agencies allowing subordination levels to fall to levels that provided insufficient protection to what were rated as 'safe' tranches. However, investigating the same issue with respect to subprime residential mortgage-backed securities (ABX indices), Stanton and Wallace (2011) found strong evidence that actual prices were too low and inconsistent with mortgage default rates that were observed in the market.

4. Pricing Real Estate Derivatives

4.1 *Pricing framework*

Property derivatives constitute the typical case of an instrument that trades in an incomplete market. The approach proposed in this paper offers a flexible and robust framework for pricing the main property derivatives contracts currently traded worldwide. The theoretical pricing methodology is underpinned by an equivalent martingale probability measure but, since there are many such possible measures, the prices cannot be guaranteed to be unique. The strategy of our approach is to determine the market price of risk using our model and a derivative market such as futures or forwards, which will then allow one to fix the risk-neutral pricing measure for any other derivatives such as options. Therefore, our framework is different from the equilibrium model developed recently by Cao and Wei (2010) in several ways. First, our methodology is underpinned by a continuous-time model specified under the physical measure that accounts for empirical characteristics identified in real estate markets. Based on this modelling outline, pricing formulae for the main property derivatives traded in the financial markets are derived. In other words, the market itself and not the actions of the market participants are modelled. Our calibration of the pricing measure is realised from the futures curves via the market price of risk concept. Cao and Wei (2010) also employ a risk premium concept linking equilibrium model prices to observed market prices. Their risk premium is defined as a factor that equalises the two quantities and is different from the market price of risk used in this research. Within our framework, once the market is completed with information from one property derivatives market, such as futures or forwards, all the other property derivatives contingent on the same index can be priced directly without further intervention. In the equilibrium approach, different risk premia quantities are calculated for different derivatives. There is no contradiction between the two approaches as the risk concepts are very different in nature.

The models envisaged to fall under our framework are mean-reverting continuous-models that exhibit predictability for the drift term. Hence, the general framework under which we work in this paper follows the seminal paper by Harrison *et al.* (1981). We consider a financial market described by a probability space $(\Omega, F, \mathbf{P})$ endowed with a canonical filtration $F = \{F_t\}_{t \geq 0}$ representing the information available to the economic agents. The following objects are intrinsic to this market:

- A stochastic process $\{X_t\}_{t \geq 0}$ for the real estate index.
- A risk-free rate r.
- A liquid bond market for bonds with any maturity. The market price at time t for a zero-coupon bond maturing at T is denoted by $p(t; T)$.
- A money market account given by $dB_t = rB_t dt$. There is the evident relationship between the two rate instruments $p(t, T) = \frac{B_t}{B_T}$ when interest rates are deterministic. Some model pricing derivation is also discussed later on in a wider context of stochastic interest rates.

First and foremost, when building a model for property prices one cannot ignore the empirical evidence in the literature on the characteristics of the time series of prices. What is known so far on property indices—commercial and residential, US or UK, appraisal or transaction based—is that (1) there is a degree of serial correlation that induces a degree of predictability[5] with positive autocorrelation in the short term; (2) there is evidence of negative autocorrelation for long horizons, and; (3) returns of real estate portfolios are mean reverting with risk decreasing over the long horizon.[6] Hence, the real estate market in general is not informationally efficient[7] and returns are not identical and independently distributed (i.i.d). While recognising a high degree of predictability for this asset class, we also agree with Ghysels *et al*. (2007) that economic conditions cannot fully determine the future movements in returns, and, for a commercial real estate index describing the market in aggregate, it is better to model this asset as a financial asset.

A mean-reverting stochastic model for the real estate index on the log scale would be able to account for all these characteristics; modelling on the log scale $Y_t = \log(X_t)$ also ensures that index values always stay positive. The price to pay for switching the scale of modelling is that the long-run mean is not interpretable in the levels of the property index. However, it is more important to have positive model index values than having an easy to interpret model which allows for a negative index. One important question that must be addressed is what is the log price index reverting to? The long-run level towards which the log-series is reverting over long horizons can be conceptualised as anything from a non-constant but deterministic series to a stochastic process. The long-run level gives an indication of the trend at a longer horizon and, in general, may be determined by macroeconomic factors and policy.

The method for determining this trend is external to the pricing framework of real estate derivatives. Therefore, it is assumed that there is a general long-run mean (LRM) trend, satisfying some smoothness or regularity condition. More precisely, if Ψ_t denotes a pre-determined LRM, one can safely assume that this curve is differentiable, at least to the first order, so that $\frac{d\Psi_t}{dt}$ exists. The models proposed here are in the same category with the trended mean-reverting models discussed in Lo and Wang (1995). In the

[5] Ample evidence for real estate markets is described in Mei and Liu (1994).

[6] See MacKinnon and Al Zaman (2009) for a discussion on the effect of return predictability on long-horizon allocations.

[7] One possible explanation advocated by Barkham and Geltner (1995) is that securitised real estate markets lead direct markets.

following the general model under the physical measure[8] is considered

$$dY_t = \left[\frac{d\Psi_t}{dt} - \theta\left(Y_t - \Psi_t\right)\right]dt + \sigma dW_t \tag{1}$$

where θ is the mean-reversion speed parameter, σ is the volatility parameter and $\{W_t\}_{t\geq 0}$ is a Wiener process.

Making the notation $\hat{Y}_t = Y_t - \Psi_t$, the well-known Ornstein-Uhlenbeck (OU) process for the detrended process is obtained

$$d\hat{Y}_t = -\theta\hat{Y}_t dt + \sigma dW_t \tag{2}$$

As Shiller and Weiss (1999) pointed out, the models discussed in Lo and Wang (1995) cannot be applied directly to real estate options because the underlying asset is not costlessly tradable. Nevertheless, Shiller and Weiss (1999) price real estate options using only the assumption that the underlying asset is lognormally distributed and working with the expected rate of return rather than a riskless rate. Here we show how to improve the modelling presented by Lo and Wang (1995) in order to utilise models that can produce serial correlations.

The real estate market is inherently incomplete. We complete the market with the futures contracts. In terms of the dynamics of the underlying real estate index, we change to a pricing measure by modifying the drift with a factor associated with the market price of risk. The pricing of various derivatives can now be done in our framework quite easily since indeed the marginal and conditional distributions are lognormal, as will be demonstrated shortly. The risk-neutral pricing is realised under an equivalent martingale probability measure that changes the model equation only in the drift by a quantity determined by the market price of risk λ and the volatility σ

$$d\hat{Y}_t = [-\theta\hat{Y}_t - \lambda\sigma]dt + \sigma dW_t^Q \tag{3}$$

Here $W_t{}^Q$ is again a Wiener process under an equivalent martingale measure Q. For a given maturity T, the unique value of λ will succeed in pinning down a unique risk-neutral pricing measure Q and therefore ensuring uniqueness of all derivatives, with the same maturity and contingent on the same property underlying index, calculated under this probability measure Q. The term structure of the market price of risk λ is fundamental to the pricing. Our methodology relies on calibrating the term structure of values of λ from market futures prices and subsequently uses the formulae derived here to price other property derivatives such as total return swaps, options, and so on. The risk-neutral model equation is then recovered as

$$dY_t = \left[\frac{d\Psi_t}{dt} - \lambda\sigma - \theta\left(Y_t - \Psi_t\right)\right]dt + \sigma dW_t^Q \tag{4}$$

and after some algebra, the solution to this equation, for any $t < T$, is

$$Y_T = \Psi_T - \frac{\lambda\sigma}{\theta} + \left[Y_t - \Psi_t + \frac{\lambda\sigma}{\theta}\right]e^{\theta(t-T)} + \sigma\int_t^T e^{\theta(u-T)}dW_u^Q \tag{5}$$

[8] Some financial engineering modelling approaches prefer to start directly with the model specification under a risk-neutral measure. Here the model dynamics is given under the physical measure because the parameters estimation, excepting the market price of risk, will be estimated under this probability measure.

implying that

$$
\begin{gathered}
Y_T \mid Y_t \sim N\left(m_{y;t,T}; \sigma^2_{y;t,T}\right) \\
m_{y;t,T} = \Psi_T - \frac{\lambda\sigma}{\theta} + \left(Y_t - \Psi_t + \frac{\lambda\sigma}{\theta}\right) e^{\theta(t-T)}, \quad \sigma^2_{y;t,T} = \frac{\sigma^2}{2\theta}\left[1 - e^{2\theta(t-T)}\right]
\end{gathered} \tag{6}
$$

Therefore, the model underpinning our framework can be interpreted on the index levels $\{X_t\}_{t\geq 0}$ as a lognormal model conditional on the LRM. The log-normality of the real estate index is a helpful feature and consequently derivatives pricing can be done with closed-form solutions. In this manner, great freedom is allowed for selecting an outlook trend for a real estate index prior to pricing a property derivative on that index.

The framework described above can be easily modified to include time-varying volatility when the evolution is deterministic.[9] The most common such specification is a piecewise constant function that takes into consideration the patterns of real estate trade associated with pre-scheduled socio-economic activities such as school calendar, holidays or the end of financial year for taxation purposes. Assuming that the volatility parameter has a functional form $\sigma(t)$, the risk-neutral equation (4) becomes

$$
dY_t = \left[\frac{d\Psi_t}{dt} - \lambda\sigma_t - \theta\left(Y_t - \Psi_t\right)\right] dt + \sigma_t dW_t^Q \tag{7}
$$

The solution to this equation is

$$
Y_T = \Psi_T + \left[Y_t - \Psi_t\right] e^{\theta(t-T)} - \lambda \int_t^T \sigma_u e^{\theta(u-T)} du + \int_t^T \sigma_u e^{\theta(u-T)} dW_u^Q \tag{8}
$$

The normality structure of the set of formulas (6) is preserved with the time-varying volatility

$$
\begin{gathered}
Y_T \mid Y_t \sim N\left(m_{y;t,T}; \sigma^2_{y;t,T}\right) \\
m_{y;t,T} = \Psi_T - \lambda \int_t^T \sigma_u e^{\theta(u-T)} du + \left(Y_t - \Psi_t\right) e^{\theta(t-T)}, \quad \sigma^2_{y;t,T} = \int_t^T \sigma_u^2 e^{2\theta(u-T)} du
\end{gathered} \tag{9}
$$

4.2 *Real estate derivatives pricing*

Here the pricing of futures, forward, total return swap, and European call option contingent on a real estate index is analysed.

---

[9] The volatility can be also considered as a random quantity, for example following the dynamics of a GARCH process or a stochastic volatility type model. However, this type of modelling will add another layer of market incompleteness that will be difficult to resolve given that real estate derivatives are not highly liquid yet as an asset class. Moreover, expanding our methodology to include volatility in a stochastic manner will require most certainly numerical solutions for finding a solution so that the closed-form solution advantage will be lost.

Futures contracts. Denoting with $F(t,T)$ the futures price at time t for maturity T it is known that

$$F(t,T) = E_t^Q[X_T] \tag{10}$$

where Q is the risk-neutral pricing measure. This is a standard formula that looks very simple but the proof does take into account the possibility that the risk-free interest rate r is stochastic. Making use of the log-normality of the underlying process leads to closed-form solutions for the futures price, which is given by the formula

$$F(t,T) = \exp\left\{ m_{y;t,T} + \frac{\sigma^2_{y;t,T}}{2} \right\} \tag{11}$$

with the quantities given explicitly in (6).

Pricing total return swaps. The pricing of total return swaps is slightly more involved. The swap is structured over a given set of tenors $t_0 < t_1 < \cdots < t_N$ that are equidistant in time such that each period is given by $\Delta = t_j - t_{j-1}$ with maturity $t_N - t_0$ (typically five years) and the period taken as quarterly. At the end of each period, one side of the swap has a payment equal to the appreciation (depreciation) of the underlying index, whereas the other side of the swap has a payment equal to a floating rate (i.e., Libor plus a spread) or a fixed rate compounded into the level of the index.

The no-arbitrage price at time $t < t_0$ is calculated under the martingale measure Q associated with the market price of risk λ that has already been identified from the forward (or futures) market. The pricing equation is

$$E_t^Q\left[\sum_{j=1}^{N} e^{-\int_t^{t_j} r_s ds}\left[X_{t_j} - X_{t_{j-1}} - (\Delta \times Libor_{t_{j-1}} + \delta)X_{t_{j-1}}\right]\right] = 0 \tag{12}$$

Following some standard calculations (see Appendix A for details), we obtain the formula

$$\delta = \frac{\sum_{j=1}^{N} e^{-\int_t^{t_j} r_s ds}\left(E_t^Q\left[X_{t_j}\right] - e^{\int_{t_{j-1}}^{t_j} r_s ds} E_t^Q\left[X_{t_{j-1}}\right]\right)}{\sum_{j=1}^{N} e^{-\int_t^{t_j} r_s ds} E_t^Q\left[X_{t_{j-1}}\right]} \tag{13}$$

The expectations in this formula can be calculated exactly by adjusting the results in the formula (6) to derive

$$E_t^Q\left[X_{t_j}\right] = \exp\left(m_{y;t,t_j} + 0.5\sigma^2_{y;t,t_j}\right) \tag{14}$$

It is important to point out that the spread formula in (13) is not necessarily equal to zero. The reason for this, from a theoretical point of view, is that the discounted process $\{e^{-\int_t^T r_s ds} X_T\}_{T \geq t}$ is not necessarily a martingale under Q. Some contributors to the literature[10] decided that this spread should be zero. The over-the-counter total return swaps market on the IPD index in London used a quotation of Libor plus the spread δ. Nevertheless, the zero spread theoretical assessment created considerable resentment among members of the trading community, who were not comfortable with the risk implications of a zero spread over Libor in a TRS contract on an IPD index. Consequently, the quotation mechanism has been changed to a fixed rate versus the total

[10] See Buttimer *et al*. (1997) and Bjork and Clapham (2002) for no-arbitrage models.

return rate. The models suggested in the proposed framework in this paper indicate that it is possible to have a non-zero spread.

Pricing European options. Pricing European options is not difficult since the underlying index is log-normal under the model detailed above. For other structured products or path-dependent exotic options, one can use Monte Carlo simulation taking advantage of the exact solution of the process for the index.

It is straightforward to see that the price at time zero of a European call option with maturity T and exercise price K is given by

$$c = p(0, T)F(0, T)\Phi(d_+) - Kp(0, T)\Phi(d_-) \tag{15}$$

where

$$d_+ = \frac{\ln\left(\frac{F(0,T)}{K}\right) + \frac{\sigma^2_{y;0,T}}{2}}{\sigma_{y;0,T}}, \quad d_- = d_+ - \sigma_{y;0,T}$$

A similar formula can be derived for a European put in a straightforward manner, or simply by applying the put-call parity.

A mean-reverting model with linear trend. For reasons that will become more explicit in Section 5, it is instructive to consider under our pricing framework the case when the LRM is a simple linear deterministic function. The exact specification of the LRM may differ between commercial property and housing, across various indices, and among countries. For example, a housing index is likely to exhibit seasonality effects that will be incorporated in the specification of the associated LRM. Likewise, the analyst may already have a forecasting model for property prices that may include macroeconomics variables, financial markets inputs, and other quantities that he may consider have significant forecasting power. Recalling that these markets inherently have a high degree of predictability, it is easy to see that the specification of the LRM may vary from analyst to analyst. The framework for pricing property derivatives outlined here is not prescriptive in any way. The only requirement is flexibility of specification and some regularity condition requiring that the LRM be a smooth curve. The valuation methodology will be adapted along similar lines described below for any such specification of the LRM.

The linear deterministic LRM employed in this part of the research is more for illustrative purposes. Hence, the following model is considered

$$dY_t = [\beta - \theta\,(Y_t - (\alpha + \beta t))]\,dt + \sigma\,dW_t \tag{16}$$

The process X_t is therefore a generalisation of a geometric mean-reverting OU process since the long-run mean is not a simple constant but an increasing linear trend. The solution to (16) for $s < t$ is

$$Y_t = \alpha + \beta(t - s) + [Y_s - \alpha]\,e^{-\theta(t-s)} + \sigma \int_s^t e^{-\theta(t-u)}dW_u \tag{17}$$

It is not difficult to show that, under the real-world measure, for any $s < t$, $t_1 < t_2$

$$E\,(Y_t \mid Y_s) = \alpha + \beta(t - s) + (Y_s - \alpha)e^{-\theta(t-s)}, \ \mathrm{var}(Y_t \mid Y_s) = \frac{\sigma^2}{2\theta}\left[1 - e^{-2\theta(t-s)}\right] \tag{18}$$

$$\operatorname{cov}\left(Y_{t_2+\tau} - Y_{t_2}, Y_{t_1+\tau} - Y_{t_1}\right) = \begin{cases} \dfrac{\sigma^2}{2\theta} e^{-\theta(t_2+t_1)}[e^{-\theta\tau} - 1][e^{\theta\tau} - 1][e^{2\theta t_1} + e^{-\theta\tau}] & \text{for } t_1 + \tau < t_2 \\ \dfrac{\sigma^2}{2\theta} e^{-\theta(t_2+t_1+\tau)}\left[2e^{\theta(2t_1+\tau)} - 2e^{-\theta\tau} + 2 - e^{2\theta t_1} - e^{2\theta t_2}\right] & \text{for } t_2 < t_1 + \tau \end{cases} \tag{19}$$

The above formulae indicate that the model described by (13) has some desirable properties. The variance of the process is bounded and the autocorrelations of returns over a fixed period of length τ can be positive at short horizons and negative at longer horizons.

5. Numerical and Calibration Issues

In this section, we illustrate the numerical calibration of the mean-reverting model for real estate indices and the motivation for using a mean-reverting stochastic process with a LRM that is non-constant is discussed. First, a useful methodology for the estimation of the parameters driving the model is outlined and then we detail how to recover the market risk values implied by the market futures curves. The LRM specification may vary, but the general modelling outline remains the same. Similarly, analysts may use different methods for estimating the parameters, each method with its pros and cons. Nevertheless, this statistical estimation and trend detection activity takes place outside the realm of property derivatives pricing. The analysis presented below regarding LRM is merely for illustrative purposes.

5.1 *Estimation of parameters*

In order to implement the model, the parameters have to be estimated from historical data. The exception is the market price of risk, which needs to be calibrated from current market prices of one property derivative.

While the model presented above is in continuous time for convenience, for parameter estimation a discrete-time version is required. Thus, suppose that $Y = (\log(X_{t_0}), \log(X_{t_1}), \ldots, \log(X_{t_N}))$ is the vector of data time series. Assuming that $\Psi_t = \Psi_t(\eta)$, the parameters can be estimated with a minimisation procedure, possibly nonlinear, by calculating $\min_\eta ||Y - \Psi||^2$. Here, for simplicity of exposition a linear LRM $\Psi_t = \alpha + \beta t$ is considered, so $\eta = (\alpha, \beta)$. The estimation of the parameter vector η is realised prior to the estimation of the mean-reversion parameter θ and volatility parameter σ. The parameter θ can be determined with the martingale estimation method described in Bibby and Sorensen (1995) and outlined in Appendix B, which is

$$\hat{\theta}^{(N)} = \ln\left(\frac{\sum_{k=1}^{N} \hat{Y}_{k-1}^2}{\sum_{k=1}^{N} \hat{Y}_k \hat{Y}_{k-1}}\right) \tag{20}$$

To estimate the volatility parameter, the classic quadratic variation estimator discussed in Basawa and Rao (1980) can be employed.

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{k=1}^{N} [Y_k - Y_{k-1}]^2 \tag{21}$$

5.2 *The importance of the trend*

The future trend of housing prices plays an important role in assessing the financial stability of individual banks. In the aftermath of the subprime mortgage crisis, the US Federal Reserve Bank and thrift supervisors initiated a Supervisory Capital Assessment Program (SCAP) to determine if the largest US banking organisations have sufficient capital buffers to withstand the impact of an economic environment that is more challenging than is currently anticipated. The capital assessment covers two economic scenarios: a baseline scenario and a more adverse scenario. The baseline assumptions for real GDP growth and the unemployment rate for 2009 and 2010 were assumed to be equal to the average of the projections published by Consensus Forecasts, the Blue Chip survey, and the Survey of Professional Forecasters in February. To account for the possibility that the economy could turn out to be appreciably weaker than expected than in the baseline scenario, the supervisors have also proposed an alternative 'more adverse' scenario. A major component of the economic scenarios is the outlook for housing prices. The projections used for the SCAP scenarios are consistent with the house price path implied by futures prices for the Case-Shiller 10-City Composite index. For the more adverse scenario, house prices are assumed to be about 10% lower at the end of 2010 relative to their level in the baseline scenario.[11] The outlook trend for the Case-Shiller house price index that has been used in the SCAP is shown in Figure 5.

Therefore, making projection of the trend of house prices is going to play a vital role in risk management supervision in the banking sector. Hence, an LRM forecast is likely to be available in all major banks and financial institutions.

5.3 *Calibrating the market price of risk curve*

Here we describe how to apply the pricing methodology proposed above for the futures market to the IPD UK Annual All Property Total Returns index, the representative index for commercial property in the UK. The reason for choosing this index is that there is a futures contract traded on the Eurex exchange since 4 February 2009, so it is possible to complete this market. The futures contract has a nominal value of GBP 50,000 and it is cash settled. There are five successive annual contracts with expiry defined by end of March cycle calendar. Nevertheless, the price is expressed as 100 plus the percentage total return in the year to the end of December. Thus, if X_t is the IPD total returns index at time t, then the final settlement price for the futures contract is given by $100\frac{X_t}{X_{t-1}}$ if one period is one year. For example, if on March 31, 2012 it is known that the values of the IPD index were $X_t = 730.5$ as at December 2011 and $X_{t-1} = 652.5$ as at December

[11] The 'more adverse' scenario was constructed from the historical track record of private forecasters as well as their current assessments of uncertainty. From the historical accuracy of the Blue Chip forecasts made since the late 1970s, the likelihood that the average unemployment rate in 2010 could be at least as high as in the alternative more adverse scenario is roughly 10%. Moreover, the subjective probability assessments provided by participants in the January Consensus Forecasts survey and the February Survey of Professional Forecasters suggests that there is about a 15% chance that real GDP growth could be as least as low, and unemployment at least as high, as assumed in the more adverse scenario Based on the year-to-year variability in housing prices since 1900, and controlling for macroeconomic factors, there is roughly a 10% probability that housing prices will be 10% lower than in the baseline by 2010.

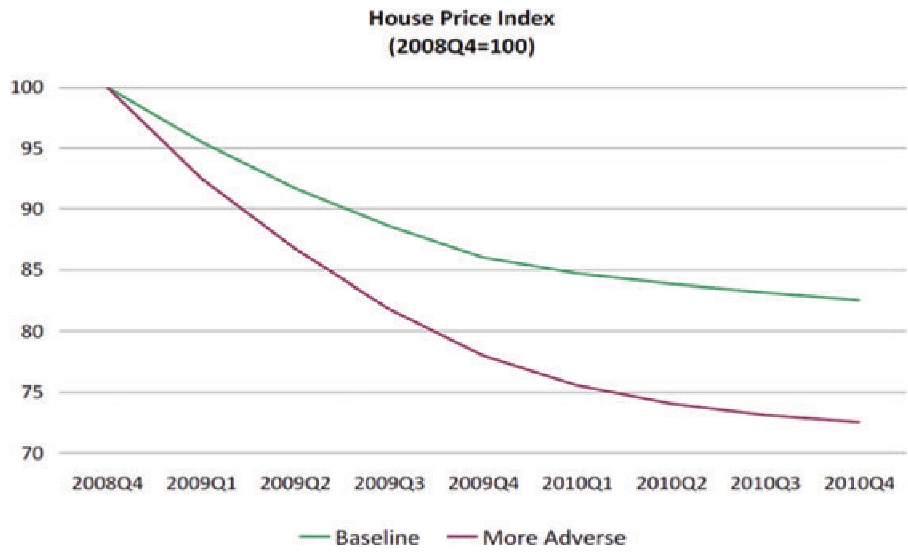

Fig. 5. Hypothetical simulation of the outlook for real estate index from an investment bank perspective. The assumptions for the baseline economic outlook are consistent with the house price path implied by futures prices for the Case-Shiller 10-City Composite index and the average response to a special question on house prices in the latest Blue Chip survey. For the more adverse scenario, house prices are assumed to be about 10% lower at the end of 2010 relative to their level in the baseline scenario

Source: Supervisory Capital Assessment Program conducted by the U.S. Federal bank and thrift supervisors.

2010, then the final settlement price for the one-year futures contract would be taken as 112. It is important to realise that, while the actual payment is done on a March to March cycle, the IPD index is followed on a December to December cycle.

The term structure for the market price of risk is going to be recovered from market futures curves on the IPD index. The data are weekly between 4 February 2009 and 6 January 2010, for all next five yearly maturities. The futures prices are shown in Figure 6. It can be observed that the property derivatives market implied a negative outlook for the commercial property spot market in the UK for the immediate period, with futures prices below 100 on a total return basis. Moreover, the period between February 2009 to the end of March 2009 is characterised by very low futures prices. There are two possible explanations for this. First, in March there is the rollover of the futures contract so the contract with first maturity is likely to be affected the most. However, for the dataset employed, it seems that the second maturity contract, f2, has similar characteristics. The second possible explanation resides in the aftermath of the subprime-liquidity crisis that most likely has induced a negative market sentiment, particularly in relation to property markets. Nevertheless, the last three maturity contracts, f3, f4 and f5, convey a different picture, indicating some optimism for recovery from the beginning of 2011.

There are some stylised facts that are helpful in understanding the dynamics of futures prices. One can observe a drift in prices towards the end of March. This is as expected since the settlement of the futures contracts is done on a yearly basis in March. Moreover, the futures prices seem to be lower in February than in January, particularly for the first two maturities, possibly reflecting market information about the publication of the

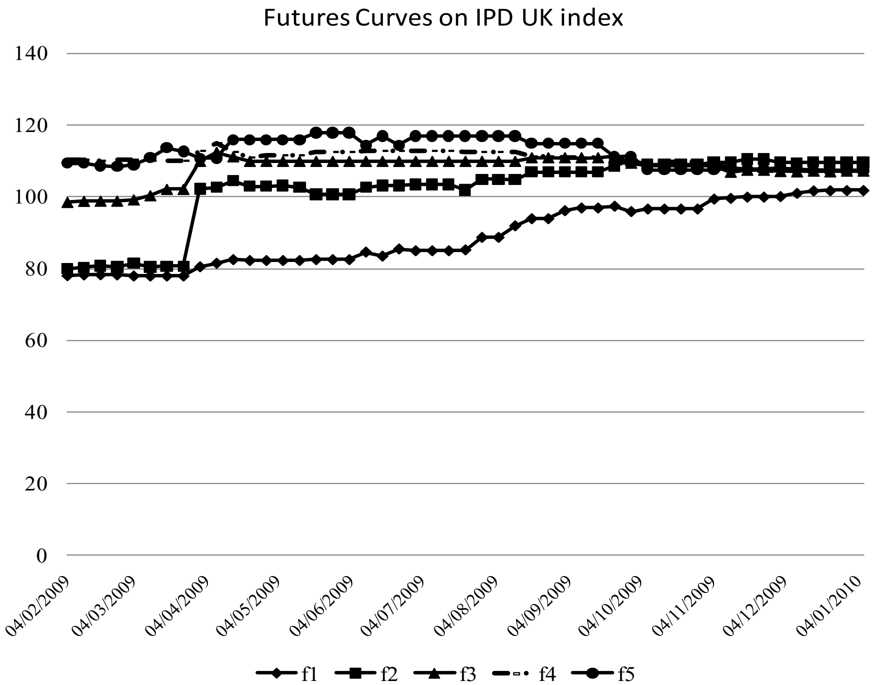

Fig. 6. The weekly futures settlement prices on IPD UK Annual All Property Total Return index for the period 4 February 2009 to 6 January 2010. The contracts f1, f2, f3, f4, f5 are for the next five calendar yearly maturities

Source: Eurex London.

December 2008 value of the IPD index, and also in conjunction with the subprime-liquidity crisis. The market seems to have stabilised somehow after March 2009 and even post optimistic outlooks for the future value of the spot index, but it dropped again from September 2009 albeit not that much as at the beginning of 2009.

The parameters of the mean-reverting process are estimated using the yearly historical time series for the IPD UK index between 1946 and 2009. This series is depicted in Figure 7 on a log-scale and it can be seen that a simple linear trend line fits remarkably well. Nevertheless, this LRM is more of a proxy to a forecasting model embedding a suite of variables reflecting the fundamental value of property.

Hence, for the LRM of IPD index a linear time function $\Psi_t = \alpha + \beta t$ is used. After first estimating the regression parameters α and β, the other parameters required for our model will be estimated using the detrended equation (2). The volatility parameter σ is estimated with formula (21) while the mean-reversion parameter θ is estimated with formula (20). The results are presented in Table 2.

The information on the futures prices on IPD from Eurex, albeit only for the 2009 year, gives a good insight into the perception of the market on the dynamics of commercial property prices in the UK in the future. While the IPD UK index is updated annually, the Eurex futures data have weekly frequency for all five maturities. Some summary statistics of this data are described in Table 3. The volatility encapsulated by the standard deviation of the observed futures prices has a hump term structure with greatest uncertainty for

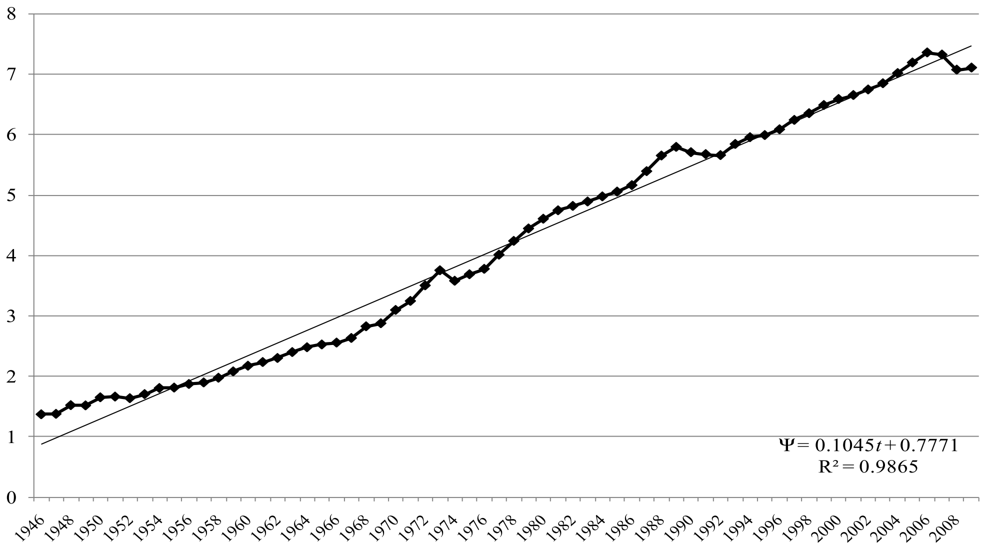

Fig. 7. Fitting a trend to the IPD Total Return Annual UK index, on the log scale. The fitted trend is obtained with OLS fitting. The estimated parameters are $\alpha = 0.7771$ and $\beta = 0.1045$. The goodness of fit measure is $R^2 = 98.65\%$ for annual data for the period 1946 to 2008

Table 2

Model parameter estimates

This table reports the model parameter estimates for the IPD Annual UK All Property index from historical series 1947–2009. The parameters α and β, describing the linear long-run mean, are first estimated with OLS. Then, conditional on these estimates, parameters θ, the speed reversion to the mean, and σ, the volatility, are estimated with the martingale estimation procedure.

α	β	θ	σ
0.7771	0.1045	0.1165	0.131

Table 3

Summary statistics on futures prices

This table reports summary statistics, average and standard deviation, of futures prices on IPD UK Annual Total Return, from weekly data. Source: Eurex.

	Dec 2008	Dec 2009	Dec 2010	Dec 2011	Dec 2012
average	89.29	101.93	107.93	110.71	112.42
std	8.61	9.95	3.81	2.01	3.96

the second year maturity, quite understandable given that these prices were produced right in the aftermath of the subprime crisis.

Now the theoretical futures prices for the IPD index can be determined based on formulae (6) and (11). This procedure is applied for each week in the dataset from Eurex for all five maturities. The theoretical prices cannot be applied without knowing the

Table 4

Calibration of market price of risk

This table reports an example of calibration of term structure of market price of risk from Eurex futures prices on IPD index, at maturity roll change in March 2009. The values of parameter λ are obtained by matching futures market prices with the theoretical futures prices.

Date	Maturity	Dec 2008	Dec 2009	Dec 2010	Dec 2011	Dec 2012
Mar 25, 2009	Eurex Futures price	77.9	80.75	102.25	110	112.75
	Market price of risk λ	1.4693	2.5409	0.7354	0.7062	0.8193
		Dec 2009	Dec 2010	Dec 2011	Dec 2012	Dec 2013
Apr 1, 2009	Eurex Futures price	80.5	102.25	110	112.75	110.75
	Market price of risk λ	2.5862	0.7393	0.7084	0.8213	0.9919

market price of risk parameter λ for each maturity horizon. The market price of risk could be specified exogenously from an equilibrium model such as Geltner and Fisher (2008) for commercial property or Cao and Wei (2010) for housing property, or elicited in-house by expert financial economists. Various methods to gauge the market price of risk may produce different results and overall, there is a degree of model uncertainty how to do this. However, in this paper we shall reverse engineer the values for the market price of risk with different maturities such that the theoretical futures prices from our pricing framework provided by formulae (6) and (11) will match exactly the market futures prices from Eurex. The calculations are easily done with a search algorithm for each of the five maturity contracts. The results in Table 4 describe the change in the term structure of the market price of risk when the roll of maturity of contracts took place in March 2009.

Although the changes in the market price of risk for the last three maturities do not change that much at rollover, for the first two maturities the change is quite significant. It increases for the first maturity contract f1 and decreases for the second maturity contract f2. This behaviour is not surprising since from April 2009 there is another year ahead until settlement of the futures contract f1 while there are two years now for the second maturity contract. Evidently, the entire curve plays a role in the determination of the new term structure of market prices, after a change in maturities. Moreover, the market price of risk for the second maturity prior to the rollover is almost identical to the market price of risk of the first maturity futures *after* the rollover, from 2.54 to 2.58. This transfer seems to be true across maturities, such that the old market price of risk for maturity $i + 1$ becomes the new market price of risk, after rollover, for maturity i.

The recovered curves for the market price of risk for the IPD index relative to the period 4 February 2009 to 6 January 2010, illustrated in Figure 8, reveal some interesting conclusions. The peak of the market premium was in February 2009. This is not surprising considering the repercussions of the subprime-liquidity crisis. Another smaller peak appears towards the end of August 2009, perhaps related to the continuous problems associated with the Royal Bank of Scotland (RBS). From October 2009 the shape of the curves has changed with lower market price of risk for the near future and higher market price of risk for longer term. This is evidence of some optimism for the

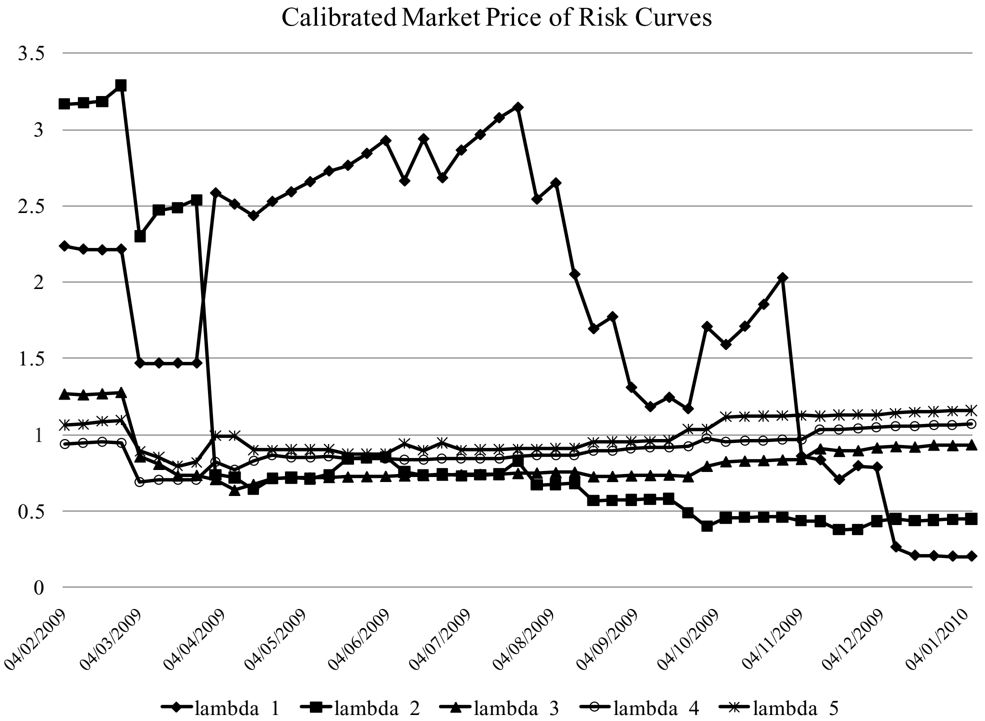

Fig. 8. The evolution of the calibrated values of market price of risk for the five maturities ahead. The implied values for lambda, the market price of risk, are obtained by the minimization of the squared errors between the theoretical model futures value and the market values reported by Eurex

short term and increased uncertainty for the longer term caused by the new taxation regime and regulatory implications for the financial services sector in London.

Some further insight is gained by analysing the curves of the calibrated market price of risk parameter λ for each of the five futures contracts. Everything else being equal, an increase in volatility would imply a decrease in the market price of risk.[12]

6. Conclusions

Traders in real estate markets have been asking for a flexible, non-opaque methodology to enable them to value property derivatives. At the same time, in order to maintain theoretical support, the models need to account for empirical features in these markets and deal with the incomplete character of this asset class. In this paper, a valuation framework has been developed for pricing real estate derivatives such as futures, forward, total return swaps, and European options contracts in closed-form. Moreover, taking advantage of the log-normality of the process representing the index and having

[12] The market price of risk parameter is defined by analogy with the Sharpe ratio as the spread of expected rate of return over the riskless rate divided by the volatility. More technical details when the underlying asset is non-tradable are provided in Bjork (2009).

determined the transition probability density function, it is evident that one can use our methodology to develop Monte Carlo procedures for pricing more exotic products such as contracts with barrier options or structured products. The general set-up takes into account the incompleteness of the real estate market. The family of models suggested for commercial and residential real estate indices have as a core a mean-reverting continuous-time process. Our model allows for positive autocorrelations at short horizon and negative autocorrelations at long horizon in the returns of the underlying index. Even for a non-constant long-run mean trend, pricing formulae for the main property derivatives contracts are derived in closed form, assuming prior calibration or knowledge of the market price of risk.

Here it also has been shown how to calibrate the market price of risk for different maturities from market futures prices. It is important to realise the fundamental role played by futures or forward contracts for the price discovery of other major derivative instruments such as swaps, European call and put options, and so on.

The numerical examples presented in this paper are based on the IPD index for which a thriving market is developing and for which futures data are available from Eurex. It is the first time in the academic literature that these data on property futures contracts traded on Eurex have been analysed.

Appendix A

In this appendix, we detail the calculations leading to the formula for the total return swap spread price. The calculations are done below in full complexity, assuming stochastic interest rates. The pricing equation is

$$E_t^Q\left[\sum_{j=1}^{N} e^{-\int_t^{t_j} r_s ds}\left[X_{t_j} - X_{t_{j-1}} - (\Delta \times Libor_{t_{j-1}} + \delta)X_{t_{j-1}}\right]\right] = 0 \qquad \text{(A.1)}$$

The pricing can be done either under the assumption of a deterministic set of interest rates $\{r_t : 0 < t \leq t_N\}$ or considering the rates as stochastic. The first case is nested in a sense within the second case so we are going to follow the derivation only for the latter, rejoining the discussion on the final formula. Recall that the price at time t of a zero-coupon bond with maturity T is

$$p(t, T) = E_t^Q\left[e^{-\int_t^T r_s ds}\right]. \qquad \text{(A.2)}$$

It is also true that $1 + \Delta \times Libor_{t_{j-1}} = \frac{1}{p(t_{j-1}, t_j)}$ and this relationship will prove useful in our calculations. Equation (21) can be rewritten as

$$\sum_{j=1}^{N} E_t^Q\left[e^{-\int_t^{t_j} r_s ds}\left[X_{t_j} - X_{t_{j-1}} - (\Delta \times Libor_{t_{j-1}} + \delta)X_{t_{j-1}}\right]\right] = 0 \qquad \text{(A.3)}$$

The generic expectation term gives the present no-arbitrage value of the net payment for the period $[t_{j-1}, t_j]$. We shall calculate this term separately and sum up the result at the end.

$$\begin{aligned} &E_t^Q\left[e^{-\int_t^{t_j} r_s ds}\left[X_{t_j} - X_{t_{j-1}} - (\Delta \times Libor_{t_{j-1}} + \delta)X_{t_{j-1}}\right]\right] \\ &= E_t^Q\left[e^{-\int_t^{t_j} r_s ds}\left[X_{t_j} - \frac{1}{p(t_{j-1}, t_j)}X_{t_{j-1}} - \delta X_{t_{j-1}}\right]\right] \end{aligned} \qquad \text{(A.4)}$$

Applying the tower property for conditional expectations implies that

$$E_t^Q\left[e^{-\int_t^{t_j} r_s ds}\frac{1}{p(t_{j-1},t_j)}X_{t_{j-1}}\right] = E_t^Q\left[E_{t_{j-1}}^Q\left[e^{-\int_t^{t_j} r_s ds}\frac{1}{p(t_{j-1},t_j)}X_{t_{j-1}}\right]\right]$$
$$= E_t^Q\left[e^{-\int_t^{t_{j-1}} r_s ds}\frac{1}{p(t_{j-1},t_j)}X_{t_{j-1}}E_{t_{j-1}}^Q\left[e^{-\int_{t_{j-1}}^{t_j} r_s ds}\right]\right]$$
$$= E_t^Q\left[e^{-\int_t^{t_{j-1}} r_s ds}X_{t_{j-1}}\right] \tag{A.5}$$

$$\delta E_t^Q\left[e^{-\int_t^{t_j} r_s ds}X_{t_{j-1}}\right] = \delta E_t^Q\left[E_{t_{j-1}}^Q\left[e^{-\int_t^{t_j} r_s ds}X_{t_{j-1}}\right]\right]$$
$$= \delta E_t^Q\left[e^{-\int_t^{t_{j-1}} r_s ds}X_{t_{j-1}}p(t_{j-1},t_j)\right] \tag{A.6}$$

When interest rates are deterministic, assembling back the three terms leads to

$$e^{-\int_t^{t_j} r_s ds}E_t^Q\left[X_{t_j}\right] - e^{-\int_t^{t_{j-1}} r_s ds}E_t^Q\left[X_{t_{j-1}}\right] - \delta e^{-\int_t^{t_j} r_s ds}E_t^Q\left[X_{t_{j-1}}\right] \tag{A.7}$$

and after some algebra, the TRS spread pricing formula is obtained

$$\delta = \frac{\sum_{j=1}^N e^{-\int_t^{t_j} r_s ds}\left(E_t^Q[X_{t_j}] - e^{\int_{t_{j-1}}^{t_j} r_s ds}E_t^Q[X_{t_{j-1}}]\right)}{\sum_{j=1}^N e^{-\int_t^{t_j} r_s ds}E_t^Q[X_{t_{j-1}}]} \tag{A.8}$$

Appendix B

Here we describe the procedure for estimating the mean-reversion parameter θ with the martingale estimation method described in Bibby and Sorensen (1995). To that end, let $\delta(Y_t;\theta)$ denote the derivative with respect to θ of the drift term, that is

$$\delta(Y_t;\theta) = \Psi_t - Y_t \tag{B.1}$$

The martingale estimation function is

$$M_N(\theta) = \sum_{k=1}^N \frac{\delta(Y_{k-1};\theta)}{\sigma^2}\{Y_k - E(Y_k \mid Y_{k-1})\} \tag{B.2}$$

where the expectation operator is taken here with respect to the real-world measure.

With data for N periods, an efficient estimator $\hat{\theta}^{(N)}$ is obtained by solving the equation

$$M_N(\hat{\theta}^{(N)}) = 0 \tag{B.3}$$

Substituting (21) into (22) and recalling that $\hat{Y}_t = Y_t - \Psi_t$ leads to the nonlinear equation

$$\sum_{k=1}^N \frac{\hat{Y}_{k-1}}{\sigma^2}\{Y_k - E(Y_k \mid Y_{k-1})\} = 0 \tag{B.4}$$

After some algebra, the solution is

$$\hat{\theta}^{(N)} = \ln\left(\frac{\sum_{k=1}^N \hat{Y}_{k-1}^2}{\sum_{k=1}^N \hat{Y}_k\hat{Y}_{k-1}}\right) \tag{B5}$$

References

Adelman, I. and Griliches, Z., 'On an index of quality change', *Journal of the American Statistical Association*, Vol. 56, 1961, pp. 535–48.

Bailey, M., Muth, A., Richard, F. and Nourse, H., 'A regression method for real estate price index construction', *Journal of the American Statistical Association*, Vol. 58, 1963, pp. 922–42.

Baran, L. C., Buttimer, R. J. and Clark, S. P., 'Calibration of a commodity price model with unobserved factors: the case of real estate index futures', *Review of Futures Markets*, Vol. 16, 2008, pp. 455–69.

Barkham, R. and Geltner, D., 'Price discovery in American and British markets' *Real Estate Economics*, Vol. 23, 1995, pp. 21–44.

Basawa, I. V. and Rao, B. L. S. P., *Statistical Inference for Stochastic Processes* (Academic Press, London, 1980).

Bertus, M., Hollans, H. and Swidler, S., 'Hedging house price risk with CME futures contracts: the case of Las Vegas residential real estate', *Journal of Real Estate Finance and Economics*, Vol. 37, 2008, pp. 265–79.

Bibby, M. B. and Sorensen, M., 'Martingale estimating functions for discretely observed diffusion processes', *Bernoulli*, Vol. 1, 1995, 17–39.

Bjork, T., *Arbitrage Theory in Continuous Time*, 3rd edition (Oxford University Press, Oxford, 2009).

Bjork, T. and Clapham, E., 'On the pricing of real estate index linked swaps', *Journal of Housing Economics*, Vol. 11, 2002, pp. 418–32.

Blundell, G. F. and Ward, C. W. R., 'Property portfolio allocation: a multi-factor model', *Journal of Property Research*, Vol. 4, 1987, pp. 145–56.

Van Bragt, D., Francke, M., Kramer, B. and Pelsser, A., 'Risk-neutral valuation of real estate derivatives', *Working Paper* No. 2009–02 (Ortec Finance Research Center, University of Amsterdam 2009).

Buttimer, R. J., Kau, J. B. and Slawson, C.V., 'A model for pricing securities dependent upon a real estate index', *Journal of Housing Economics*, Vol. 6, 1997, pp. 16–30.

Cao, M. and Wei, J., 'Valuation of housing index derivatives', *Journal of Futures Markets*, Vol. 30, 2010, pp. 660–88.

Case, K. E. and Quigley, J. M., 'How housing booms unwind: income effects, wealth effects, and feedbacks through financial markets', *Working Paper* (Wellesley College, 2007).

Case, K. E. and Shiller, R. J., 'The efficiency of the market for single family homes', *American Economic Review*, Vol. 79, 1989, pp. 125–37.

Case, K. E. and Shiller, R. J., 'Forecasting prices and excess returns in the housing market', *AREUEA Journal*, Vol. 18, 1990, pp. 253–73.

Case, K. E. and Shiller, R. J., 'Mortgage default risk and real estate prices: the use of index based futures and options in real estate', *Journal of Housing Research*, Vol. 7, 1996, pp. 243–58.

Case, K. E., Shiller, R. J. and Weiss, A. N., 'Index-based futures and options trading in real estate', *Journal of Portfolio Management*, Vol. 19, 1993, pp. 83–92.

Chinloy, P., 'Real estate cycles: theory and empirical evidence', *Journal of Housing Research*, Vol. 7, 1996, pp. 173–90.

Ciurlia, P. and Gheno, A., 'A model for pricing real estate derivatives with stochastic interest rates', *Mathematical and Computer Modelling*, Vol. 50, 2009, pp. 233–47.

Clapp, J. M. and Giacotto, C., 'Estimating price indices for residential property: a comparison of repeat sales and assessed value methods', *Journal of the American Statistical Association*, Vol. 87, 1992, pp. 300–6.

Driessen, J. and van Hemert, O., 'Pricing of commercial real estate securities during the 2007–2009 financial crisis', 2011 Available at SSRN: http://ssrn.com/abstract=1470249.

Fabozzi, F. J., Shiller, R. J. and Tunaru, R. S., 'Property derivatives for managing European real-estate risk', *European Financial Management*, Vol. 16, 2010, pp. 8–26.

Fisher, J. D., 'New strategies for commercial real estate investment and risk management', *Journal of Portfolio Management*, Vol. 32, 2005, pp. 154–61.

Fisher, J., Geltner, D. and Webb, B. R., 'Value indices of commercial real estate: a comparison of index construction methods', *Journal of Real Estate Finance and Economics*, Vol. 9, 1994, pp.137–64.

Geltner, D. and Pollakowski, H., 'A set of indexes for trading commercial real estate based on the Real Capital Analytics Transaction Prices Database', Report (MIT Center for Real Estate, 2007).

Geltner, D. and Fisher, J., 'Pricing and index considerations in commercial real estate derivatives', *Journal of Portfolio Management*. Special Real Estate Issue, 2007, pp. 99–117.

Geltner, D. and Fisher, J., 'Pricing commercial real estate derivatives', in F. J. Fabozzi, ed., *Handbook of Finance* (John Wiley and Sons, New Jersey, 2008).

Gemmill, G., 'Futures trading and finance in the housing market', *Journal of Property Finance*, Vol. 1, 1990, pp. 196–207.

Ghysels, E., Plazzi, A. and Valkanov, R., 'Valuation in US commercial real estate', *European Financial Management*, Vol. 13, 2007, pp. 472–97.

Glaeser, E. L. and Gyourko, J., 'Housing dynamics', *NBER Working Paper* No. 12787 (December 2006).

Gyourko, J., Mayer, C. and Sinai, T., 'Superstar cities', *NBER Working Paper* No. 12355 (July 2006).

Harrison, M. J. and Pliska, S. R., 'Martingales and stochastic integrals in the theory of continuous trading', *Stochastic Processes and Their Applications*, Vol. 11, 1981, pp. 215–60.

Jokivuolle, E., 'Pricing European options on autocorrelated indices', *Journal of Derivatives*, Vol. 6, 1998, pp. 39–52.

Levin, A., 'Home price derivatives and modeling', *Quantitative Perspectives* (Andrew Davidson and Co., October 2009), pp. 1–36.

Lo, A. W. and Wang, J., 'Implementing option pricing models when asset returns are predictable', *Journal of Finance*, Vol. 50, 1995, pp. 87–129.

Mark, J. H. and Goldberg, M. A., 'Alternative housing price indices: an evaluation', *Journal of the American Real Estate and Urban Economics Association*, Vol. 12, 1984, pp. 30–49.

MacKinnon, G. H. and Al Zaman, A., 'Real estate for the long term: the effect of return predictability on long-horizon allocations', *Real Estate Economics*, Vol. 37, 2009, pp. 117–53.

Mei, J. and Liu, C. H., 'The predictability of real estate returns and market timing', *Journal of Real Estate Finance and Economics*, Vol. 8, 1994, pp. 115–35.

Otaka, M. and Kawaguchi, Y., 'Hedging and pricing of real estate securities under market incompleteness', MTB Investment Technology Institute Co., Ltd., 2002, Tokyo.

Palmquist, R. B., 'Alternative techniques for developing real estate price indexes', *Review of Economics and Statistics*, Vol. 62, 1980, pp. 442–80.

Patel, K. and Pereira, R., 'Pricing property index linked swaps with counterparty default risk', *Journal of Real Estate Finance and Economics*, Vol. 36, 2008, pp. 5–21.

Rosen, S., 'Hedonic prices and implicit markets', *Journal of Political Economy*, Vol. 82, 1974, pp. 33–55.

Schwartz, E. S. and Smith, J. E., 'Short term-variations and long-term dynamics in commodity prices', *Management Science*, Vol. 46, 2000, pp. 893–911.

Shiller, R. J., 'Derivatives markets for home prices', Yale Economics Department Working Paper No. 46 (Cowles Foundation Discussion Paper No. 1648 – 2008).

Shiller, R. J., 'Measuring asset value for cash settlement in derivative markets: hedonic repeated measures indices and perpetual futures', *Journal of Finance*, Vol. 68, 1993, pp. 911–931.

Shiller, R. J. and Weiss, A. N., 'Home equity insurance', *Journal of Real Estate Finance and Economics*, Vol. 19, 1999, pp. 21–47.

Stanton, R. and Wallace, N., 'The bear's lair: indexed credit default swaps and the subprime mortgage crisis', *Review of Financial Studies*, Vol. 24, 2011, pp. 3250–80.

Stanton, R. and Wallace, N. 'CMBS and the role of subordination levels in the crisis of 2007–2009', *NBER Working Paper* No. 16206 (2010).

Titman, S. and Torous, W., 'Valuing commercial mortgages: an empirical investigation of the contingent-claim approach to pricing risky debt', *Journal of Finance*, Vol. 44, 1989, pp. 345–73.