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# The Pricing of Multiclass Commercial Mortgage-Backed Securities

Paul D. Childs, Steven H. Ott, and Timothy J. Riddiough\*

## Abstract

This paper considers the pricing of multiclass commercial mortgage-backed securities. A contingent-claims pricing methodology that overcomes state variable dimensionality problems is developed to examine mortgage pools with many distinct underlying assets and whose loan cash flow values are subject to interest rate uncertainty. Security structure and the correlation structure of collateralizing assets within a pool are found to be important determinants of tranche price and required yield spread. By disentangling default loss risk from default-related call risk, we show it is possible that mezzanine investment classes may require lower yield spreads than higher priority investment classes. Of particular interest is the finding that reduced cash flow volatility obtained through pool diversification may actually decrease the value of the first-loss (junior) tranche. When examining the relationship of pool size and tranche value, we find that five to 10 distinct mortgages are required to realize most of the effects of asset diversification.

## I. Introduction

More than \$100 billion of commercial mortgages have been securitized in the last several years, representing approximately 10 percent of the total \$1 trillion commercial mortgage market (see, e.g., Commercial Real Estate Quarterly, published by Nomura Securities International, Inc., (1996)). By comparison, and suggestive of the growth potential of the commercial mortgage-backed security (CMBS) market, the residential mortgage-backed security (RMBS) market had a similar 10 percent market share in the late 1970s—whereas today one out of every two mortgages is securitized and the total RMBS market capitalization is approximately \$1.5 trillion.<sup>1</sup>

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<sup>1</sup>The residential MBS statistics are from Federal Reserve Board Reports—Mortgage Debt Outstanding.

There are important differences between the investment risks of residential and commercial mortgage-backed securities. Perhaps the biggest difference is that early return of principal with residential-backed pools is due primarily to prepayment, whereas default is typically the major reason for early principal repayment with commercial mortgage pools. Individual mortgages within a RMBS pool are callable at par, but mortgages are insured against default so that credit risk is a secondary concern in security valuation. Interest rate uncertainty and expectations regarding future levels of interest rates are, therefore, major factors in the borrower mortgage termination decision. The interest rate state variable applies across the entire economy—which means, from a modeling perspective, that interest rates alone can be used to determine the call behavior for *all* borrowers in a particular mortgage pool (e.g., Dunn and McConnell (1981), McConnell and Singh (1994)).

In contrast, default is typically the major source of risk to CMBS investors. Unlike residential mortgages, most commercial mortgages originated within the last 15 years contain prepayment lockout or penalty provisions that significantly reduce or eliminate call risk. Default, however, is not readily insurable with the commercial mortgage. Moreover, wide use of the nonrecourse clause limits individual borrower liability to the real estate asset in the event of default. Consequently, and in sharp contrast to the RMBS, CMBS value is contingent on movements in underlying collateral asset value for *each* mortgage in the pool, in addition to being sensitive to changes in the term structure of interest rates.

The purpose of this paper is to employ a contingent-claims methodology to price multiclass commercial mortgage-backed securities that are of varying size and whose mortgages are collateralized with imperfectly correlated assets. As compared to RMBS valuation, the inclusion of property specific state variables for each pooled mortgage adds considerable complexity to the valuation problem. We will also consider a single factor interest rate process as a relevant state variable. Consequently, there are up to  $N+1$  state variables to consider:  $N \geq 1$  property price variables and one interest rate process variable. Furthermore, by allowing for an arbitrary correlation structure among collateral assets, we extend related models of the multiple-asset security that specify either perfect positive correlation or complete independence among collateral assets in the pool (e.g., Boot and Thakor (1993)).

To illustrate security valuation, we consider a stylized collateralized mortgage obligation (CMO) security structure in which return of principal due to scheduled loan amortization or early loan termination is distributed top-down to security-holders. That is, the most senior tranche has strict priority on *all* principal payment cash flows until completely paid down. Once the face value of the senior tranche is zero, all principal flows to the next highest ranking tranche until it too is paid down, and so on. A further distinction between residential and commercial mortgage-backed CMOs is that—due to the inability to insure against default on commercial mortgages—the face amount of the lowest priority tranche is reduced by the shortfall between loan balance and recovery when default occurs. Default risk is, therefore, shifted down through the structure to the lower priority securities. If default rates are high enough, there may be insufficient principal balance in the first-loss securities to cover all losses, and shortfalls will have to be allocated to successively higher tranches.

Methodologically, neither a purely forward nor purely backward approach is entirely satisfactory when attempting to price the CMBS. The advantages of a forward (Monte Carlo) approach are its ability to handle multiple state variables and its ability to distribute cash flows to various payment classes, the priority of which depends on historical allocations. A forward-looking approach is unsatisfactory, however, in its determination of optimal borrower default decision-making, which requires the calculation of the value of waiting to default. Alternatively, contingent-claims models of whole commercial loan valuation like that of Titman and Torous (1989) determine borrower default behavior endogenously and produce default risk premiums that match observed premium levels quite nicely. However, valuation problems with more than three state variables are intractable using their method, since numerical computation time increases exponentially with the number of state variables. Moreover, a purely backward approach is not well suited to allocating path-dependent cash flows to various CMBS tranches.<sup>2</sup>

Instead, we employ a combined backward/forward approach to CMBS valuation that simultaneously reflects endogenous default decision-making by individual borrowers and accurately allocates cash flows to various tranches, while avoiding the dimensionality problems associated with numerous state variables. In the first stage of a two-stage modeling process, we determine the optimal default boundaries for individual borrowers whose mortgages are part of the CMBS pool. When current property value exceeds the boundary value, the borrower continues repaying the loan according to contract terms. Alternatively, hitting the boundary results in default with the property value flowing into the pool as return of principal.<sup>3</sup> The second stage in CMBS valuation is to use a Monte Carlo approach to simulate state variable paths. Utilizing the previously computed default boundary, realized paths determine cash flows that are allocated to the various tranches. Realized tranche values are then calculated over a large number of Monte Carlo trials to determine respective tranche prices. Finally, given the tranche price, a yield-to-maturity is calculated and compared to the yield on a benchmark riskless bond to produce a promised yield spread.

We find that CMBS pricing is especially sensitive to several structural and parameter value inputs. As expected, security design in terms of relative percentage tranche size is important. For parameter values used in this study, senior tranches whose share is 70 percent or less of the total pool are found to be immune from default loss, but not from the effect of early return of principal due to default. Indeed, the senior security is exposed to cash flow timing risk, which results in small but persistently positive yield spreads. Given our CMO payment structure,

<sup>2</sup>Assuming away the dimensionality issue for the moment, one could conceivably handle the allocation problem by first working backward through the pricing grid to determine boundary values, and then moving forward in the same grid to allocate and value cash flows. See Kau et al. (1993) for an application of this type with respect to pricing adjustable rate mortgages, whose cash flows depend on past interest rate state variable realizations.

<sup>3</sup>Our approach has some similarities with that of McConnell and Singh (1994), which determines a critical interest rate at which mortgage prepayment optimally occurs for each refinancing cost class at every point in time throughout the life of the mortgage. In contrast, we must determine a critical property price at which default occurs at each point in time throughout the life of the mortgages *and for each feasible spot rate of interest* that may occur at a given time  $t$ . Thus, McConnell and Singh determines critical interest rates over the mortgage term, whereas we determine a three-dimensional surface of critical property prices as a function of interest rates and time.

mezzanine level security classes are better protected against this cash flow timing risk in addition to being well protected against default loss risk. This generates the surprising result that, in some scenarios, the senior securityholders may actually require higher yield spreads than certain lower priority security classes.

Lower priority tranches are found to be quite sensitive to variation in property value and, therefore, to default risk. The correlation structure among collateral assets in the pool, in particular, is found to be an important determinant of tranche value. We find that better pool diversification (i.e., lower overall pooled cash flow volatility) may actually *lower* tranche value to low priority classes that are expected to be out-of-the-money with respect to full return of principal. This occurs because the payoff function to low-rated security classes is convex over the relevant range of likely asset value outcomes, thus suggesting that higher cash flow volatility may actually increase security value. We also address the question of how large the securitized pool must be to realize most of the effects of asset value diversification. In a pool in which loans have similar contractual features and size characteristics, and where the underlying collateral asset values are at least weakly positively correlated, we find that only five to 10 mortgages are needed for the mezzanine and low-rated securities to largely realize diversification effects.

The remainder of the paper is organized as follows. The next section develops the pricing model and numerical solution methodology. Allocation of cash flows among the prioritized payment classes also is discussed. Section III details the pricing results, where we focus attention primarily on the relative effects of security design and asset diversification. In the final section, we compare observed and modeled yield spreads to summarize our findings.

## II. Model Specification and Pricing Methodology

### A. Mortgages and Underlying State Variables

Consider a pool that is composed of  $N \geq 1$  commercial mortgages. Each mortgage is secured by a single income-producing property. Property value,  $P$ , is determined by the following system of log-normal stochastic differential equations,

$$(1) \quad dP = \mu(P)dt + S(P)dZ_P,$$

where  $\mu(P) = [(\alpha_i - \delta_i)P_i]$  is an  $N$ -dimensional vector with  $\alpha_i$  denoting the total expected return to property  $i$  and  $\delta_i$  the rate of dividend payout to the particular property;  $S(P)$  is an  $N \times N$  matrix with elements such that  $\text{cov}[dP_i, dP_j] = \rho_{ij}\sigma_i\sigma_jP_iP_jdt$ ,  $\rho_{ij}$  being the instantaneous correlation coefficient for changes in  $P_i$  and  $P_j$ , and where  $\sigma_i$  denotes the standard deviation of property returns to property  $i$ ; and  $Z_P$  is an  $N$ -dimensional standardized Wiener process. The covariance matrix of changes in state variables,  $SS'$ , is restricted to be positive semi-definite.

Commercial mortgage values also depend on interest rates, which are assumed to move stochastically. We model the entire term structure of interest rates using

the single-factor, square root model of Cox, Ingersoll, and Ross (1985b).<sup>4</sup> The specific form of the process is as follows,

$$(2) \quad dr = \kappa(\nu - r)dt + \sigma_r \sqrt{r} dz_r,$$

where  $r$  is the current spot rate of interest,  $\kappa$  is the speed of reversion parameter,  $\nu$  is the long-run value towards which the spot rate is expected to revert,  $\sigma_r$  is an interest rate volatility parameter, and  $z_r$  is a standardized Wiener process. Changes in interest rates may also be correlated with changes in property prices. Denote this instantaneous correlation coefficient as  $\rho_{r,P_i}$ . For simplicity, we let the market price of interest rate risk equal zero.

Basic mortgage contractual provisions are as follows. Assume that each mortgage is nonrecourse and that there is no cross-collateralization among individual mortgages in the pool with other mortgages either within or outside the pool.<sup>5</sup> Consistent with Titman and Torous (1989), it is also assumed that prepayment will not occur due to lock-out provisions specified in the mortgage contract. Default is, therefore, the only early termination risk to consider with respect to security valuation. The mortgages are fixed rate and fixed payment, each with a continuous rate of mortgage payment. The rate of mortgage payment in part depends on the loan size relative to the property value, the contract rate of interest, the loan term, and the period of time over which the loan amortizes.<sup>6</sup> For simplicity, we will assume that all mortgage contracts and initial property values in the pool are identical.

## B. Determination of Borrower Default Behavior

As a first step in CMBS pricing, individual borrower default decision criteria must be established. Because each mortgage is independently collateralized by a unique asset, default decisions are made noncooperatively among borrowers in the pool. Thus, each borrower will minimize each mortgage value by defaulting if property value falls to or below mortgage value (inclusive of the value of the default option). In this subsection, we present mortgage value dynamics and the boundary conditions under which default occurs.

Consider mortgage  $i$ ,  $i = 1, \dots, N$ , whose value,  $M^i$ , is contingent on collateral property price,  $P_i$ , the risk-free spot interest rate,  $r$ , and current time  $t$ , where time left until loan maturity is  $\tau = T - t$ . Standard arguments that incorporate the state

<sup>4</sup>McConnell and Singh (1994) has examined the sensitivity of residential MBS price to alternative single factor interest rate model specifications and found that pricing differences are small, as long as the current term structure is adequately duplicated. In our application, in which the mortgage termination decision is even less sensitive to changes in interest rates (as compared to prepayment decisions with residential loans), particular model specification matters even less. Hence, we choose a process that is likely to be familiar to most readers.

<sup>5</sup>Cross-collateralization is sometimes known to occur in CMBS. This type of contract could be handled using our pricing methodology by recognizing that multiple mortgages, once cross-collateralized, can be treated as one "bundled" mortgage with its own unique set of risk characteristics. See Childs et al. (1996) for further background.

<sup>6</sup>The rate of mortgage payment also depends on state variable parameters and default boundary conditions as they impact mortgage value, factors that we discuss shortly.

variables shown in equations (1) and (2) result in the following mortgage pricing equation,

$$(3) \quad \begin{aligned} & 1/2\sigma_i^2 P_i^2 M_{P_i P_i}^i + \sigma_i P_i \sigma_r \sqrt{r} \rho_{r, P_i} M_{r, P_i}^i + 1/2\sigma_r^2 r M_{rr}^i \\ & + (r - \delta_i) P_i M_{P_i}^i + \kappa(\nu - r) M_r^i + M_i + m_i = r M^i, \end{aligned}$$

where subscripts on  $M^i$  denote partial derivatives,  $m_i$  is the continuous rate of payment for mortgage  $i$ , and each property is assumed to be a traded security.<sup>7</sup> At the beginning of the loan term ( $t = 0$ ), we assume that the mortgage coupon rate is set such that the mortgage value,  $M^i(P_i, r, 0)$ , equals the initial loan balance,  $F_i(0)$ . In other words, a stated coupon rate of interest,  $c_i$ , is chosen such that the continuous rate of mortgage payment,  $m_i$ , results in debt that is valued at par at loan origination.

Default boundary conditions are now specified as follows.<sup>8</sup> At loan maturity (i.e.,  $t = T$ ), mortgage value is  $\min\{F_i(T), P_i\}$ , where  $F_i(T) = M^i(P_i, r, T) \geq 0$  is the balloon payment required at loan maturity. Thus, the maturity default condition is

$$(4) \quad \begin{aligned} & \text{Final Payment: } P_i > F_i(T), \\ & \text{Default: } P_i \leq P_i^*(r, T) = F_i(T), \end{aligned}$$

where  $P_i^*(r, T)$  is the critical property value at or below which default optimally occurs. Prior to loan maturity, the default-free boundary condition that determines whether continued payment or default occurs is

$$(5) \quad \begin{aligned} & \text{Continued Payment: } P_i > M^i(P_i, r, t), \\ & \text{Default: } P_i \leq P_i^*(r, t) = M^i(P_i^*, r, t), \end{aligned}$$

where  $P_i^*(r, t)$  is determined endogenously. Note that, in lieu of satisfaction of the stated loan terms, debtholders receive the property value as return of principal when the default condition holds.

Thus, for any feasible pair  $(r, t)$ , there exists an  $N$ -dimensional vector of critical property values,  $P^*(r, t) = [P_i^*(r, t)]$ . This vector of critical property prices, which is found using the traditional backward pricing equation approach, can now be used to determine the return of individual mortgage principal and interest as it flows into the total mortgage pool.

<sup>7</sup>“Standard arguments” include assumptions requiring that the assets are continuously tradable at no cost, and that they can be sold short. These assumptions obviously do not perfectly describe commercial real estate asset and debt markets. However, if markets are dynamically complete, an alternative (and perhaps more realistic) set of assumptions could be used to determine a (slightly modified) pricing equation. See, for example, Grenadier ((1995), p. 304) for a discussion of similar issues in the context of asset leasing markets.

<sup>8</sup>See Titman and Torous (1989) for a complete set of boundary conditions needed to fully specify the pricing problem. Like Titman and Torous, we ignore the costs of borrower and lender default in the pricing of commercial real estate debt. Although these respective costs may be significant, their effects are offsetting so that the net impact on debt value may be trivial. In lieu of any empirical evidence to the contrary, we choose to keep the analysis as simple as possible. Including these costs, if desired, would be straightforward.

### C. Realized Mortgage Cash Flows

The accounting of individual mortgage and aggregated pool cash flows is discussed in this subsection. Since repayment decisions are made independently on each mortgage within the pool, pooled cash flows are simply the sum of individual mortgage cash flows. In classifying and prioritizing these cash flows, a continuous time approach is presented to be consistent with the theoretical determination of mortgage price and default boundary discussed above. It is important to note the distinction between continuous rates of scheduled mortgage payment and the discrete payments that follow from either a balloon payment at the end of the loan term or a property value recovery due to default. Much of the following discussion is structured based on this dichotomy between continuous flows and discrete payments.

Let  $A(t)$  be the set of indexed numbers associated with mortgages that are still active at time  $t$ .<sup>9</sup> Let  $F_n(t)$  be the remaining loan balance of mortgage  $n$ , where  $n \in A(t)$ .<sup>10</sup> At this time, the total *rate* of continuous cash flow to the pool is the sum of the mortgage payment rates coming from active individual mortgages. That is,

$$(6a) \quad \text{TF}^{\text{PL}}(t) = \sum_{n \in A(t)} m_n.$$

The total rate of cash flow can be decomposed into a rate of interest cash flow (IF) and, if the loan is amortizing, a rate of principal cash flow (PF). These flow rates at time  $t$  can be expressed as

$$(6b) \quad \text{IF}^{\text{PL}}(t) = \sum_{n \in A(t)} c_n F_n(t),$$

$$(6c) \quad \text{PF}^{\text{PL}}(t) = \text{TF}^{\text{PL}}(t) - \text{IF}^{\text{PL}}(t).$$

Now consider the receipt of discrete cash payments. Each mortgage has at most one discrete cash payment that occurs during the life of the loan. If the loan repays as scheduled and is not fully amortizing, a balloon payment of the remaining loan balance is made at time  $T$ . If, on the other hand, a particular loan defaults at some time  $t_n^d \leq T$ , the property is recovered with value  $P_n^*(r, t_n^d)$  and an accounting loss of  $L_n(t_n^d) = F_n(t_n^d) - P_n^*(r, t_n^d)$  is realized.<sup>11</sup> For any other time  $t \neq t_n^d$ ,  $L_n(t) = 0$ . Thus, for a particular active mortgage  $n$  at some time  $t$ ,  $0 < t \leq T$ , the discrete payment to the mortgage is

$$(7) \quad \text{DP}_n(t) = \begin{cases} F_n(t), & \text{if } t = T \text{ and if mortgage } n \in A(T) \\ P_n^*(r, t_n^d), & \text{if } t = t_n^d \leq T \\ 0, & \text{if } t \neq t_n^d \text{ and if } t \neq T. \end{cases}$$

<sup>9</sup>For example, if  $N = 6$ , then  $A(t) = \{1, 3, 4, 6\}$  would indicate that mortgages indexed as 2 and 5 have previously defaulted.

<sup>10</sup>This loan balance is given by  $F_n(t) = (m_n/c_n)[1 - e^{-c_n(\hat{T}-t)}]$ , where  $c_n$  is the endogenously determined mortgage rate of interest and  $\hat{T} \geq T$  is the term over which the loan is amortized.

<sup>11</sup>Because the analysis is done in continuous time, no interest will have accumulated from the last payment date to the default date. If the analysis were to occur in discrete time, the loss would also be adjusted for the interest owed since the last payment. An accounting of interest owed in discrete time—an amount that is added to the realized loss—is important since this interest must be made available to tranche investors who are promised their scheduled interest payment over the relevant time increment.



Summing the individual discrete payments and accompanying losses occurring at time  $t$  results in pooled discrete payments and losses, which can be stated as

$$(8) \quad DP^{\text{PL}}(t) = \sum_{n \in A(t)} DP_n(t),$$

$$(9) \quad L^{\text{PL}}(t) = \sum_{n \in A(t)} L_n(t).$$

#### D. Allocation of Pooled Cash Flows to Each Security

We will consider a CMBS with three principal-based investment classes (tranches)—which we label as senior, mezzanine, and junior—and a residual interest only (IO) class.<sup>12</sup> Cash flow accounting of principal and interest to the various tranches is somewhat involved. In general, the highest priority tranche receives interest on the remaining tranche face value and *all* pooled principal until retired (where principal includes scheduled amortization from nondefaulted loans and unscheduled property recoveries from loan defaults). Lower priority tranches receive interest only until each sequentially becomes the highest ranking remaining tranche. The residual IO tranche—whose cash flow is affected by historical principal allocations—is not part of the principal allocation scheme since its stated face value is zero.

Commercial mortgage-backed security investors, unlike their residential MBS counterparts, must account for shortfalls in return of principal due to borrower default. Although principal is allocated top-down, default loss—the difference between mortgage loan balance and property value at the time of default—is allocated bottom-up. Default losses, therefore, reduce the face value of the lowest-rated principal-based tranche, which, in turn, reduces scheduled interest payments going forward. If default losses cumulatively exceed the face value of the lowest rated tranche, losses are then allocated to the next highest ranking tranche until fully allocated.<sup>13</sup>

We will now formalize the accounting of prioritized cash flows. The face value of the pooled mortgage debt is  $F^{\text{PL}}(t) = \sum_{i=1}^N F_i(t)$ ,  $0 \leq t \leq T$ . At  $t = 0$ , this amount is allocated among the three principal-based investment classes that we consider—senior, mezzanine, and junior—such that  $F_S(t) + F_M(t) + F_J(t) \leq F^{\text{PL}}(t)$ . This relationship will continue to hold for all  $t > 0$  due to allocations of principal and default losses.<sup>14</sup> Let  $c_S$ ,  $c_M$ , and  $c_J$  be the coupon rates for the principal-based tranches. These coupon rates must satisfy  $\max\{c_S, c_M, c_J\} \leq \min\{c_1, \dots, c_N\}$  to guarantee that sufficient pooled interest cash flow is available for distribution as interest to the principal-based tranches.

<sup>12</sup>Of course, other structures are possible. For example, a finer tranche structure is certainly feasible and is quite often employed. If the pool is over-collateralized, a residual principal tranche will also be included.

<sup>13</sup>We have considered the possibility that there could be excess principal once the pool is retired. This would occur if mortgage values are such that default occurs when property values in sum exceed the mortgage balances. We have performed simulations that verify that state realizations of this type do not occur given the interest rate process assumed in equation (2).

<sup>14</sup>This inequality is strict when the pool is “overcollateralized,” in the sense that issued principal claims are less than the aggregate pool balance. We do not consider a residual principal tranche, so the face value relationship above holds with equality in our subsequent analysis.

The rate of continuous cash flow to the pool as of time  $t$  can now be sequentially allocated to the various tranches from top to bottom. This rate of continuous income flow is composed of principal and interest, of which we consider principal first,

$$(10a) \quad PF_S(t) = \begin{cases} PF^{\text{PL}}(t), & \text{if } F_S(t) > 0. \\ 0, & \text{otherwise} \end{cases}$$

$$(10b) \quad PF_M(t) = \begin{cases} PF^{\text{PL}}(t), & \text{if } F_S(t) = 0 \text{ and } F_M(t) > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$(10c) \quad PF_J(t) = \begin{cases} PF^{\text{PL}}(t), & \text{if } F_S(t) = F_M(t) = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Notice that the entire rate of principal cash flow is to the senior tranche until it is retired. Once paid off, priority of principal allocation shifts to the next most senior tranche, and so on.

The total rate of continuous cash flow to any principal-based tranche is simply the sum of the flow rate of payments from interest and principal. Thus,

$$(11a) \quad TF_Q(t) = c_Q F_Q(t) + PF_Q(t), \quad \text{for } Q \in \{S, M, J\}.$$

The rate of cash flow to the residual interest-only tranche also must be accounted for. This rate of cash flow is

$$(11b) \quad TF_{\text{IO}} = IF^{\text{PL}} - c_S F_S(t) - c_M F_M(t) - c_J F_J(t).$$

The IO cash flow rate will generally decline over time as the total pool balance declines through scheduled amortization and return of principal through defaults.

Next consider the discrete cash payments. Discrete principal payments are allocated to the most senior tranche first. Any discrete principal payment amounts in excess of the remaining face value of the more senior tranche goes toward paying down the next most senior tranche. Formally, this prioritization is expressed as

$$(12a) \quad DP_S(t) = \min \{F_S(t), DP^{\text{PL}}(t)\},$$

$$(12b) \quad DP_M(t) = \min \{F_M(t), DP^{\text{PL}}(t) - DP_S(t)\},$$

$$(12c) \quad DP_J(t) = DP^{\text{PL}}(t) - DP_S(t) - DP_M(t),$$

$$(12d) \quad DP_{\text{IO}}(t) = 0.$$

As previously discussed, losses associated with loan default must be allocated to the most junior tranche first. If the default loss allocation exceeds the remaining face value, the tranche is simply eliminated and any remaining loss is shifted to the next higher ranking tranche. The formal prioritization of losses occurs as follows,

$$(13a) \quad L_J(t) = \min \{F_J(t), L^{\text{PL}}(t)\},$$

$$(13b) \quad L_M(t) = \min \{F_M(t), L^{\text{PL}}(t) - L_J(t)\},$$

$$(13c) \quad L_S(t) = L^{\text{PL}}(t) - L_J(t) - L_M(t).$$

Finally, the face values of the respective tranches can be adjusted once the above allocations are known. The face value of a tranche at time  $t$  is the original face

value less the cumulative continuously paid principal to date, less the cumulative discrete payments and losses to date. Formally,

$$(14) \quad F_Q(t) = F_Q(0) - \int_0^t PF_Q(s)ds - \sum_{s \in [0,t]} [DP_Q(s) + L_Q(s)], \quad Q \in \{S, M, J\},$$

where  $s \in [0, t]$  indicates the points in time at which discrete payments occur.<sup>15</sup>

## E. Tranche Valuation and Numerical Solution Technique

Respective CMBS tranches can now be valued. Based on the CMO cash flow allocation structure defined above, expectations regarding tranche cash flow at each point in time over the debt term can be formed. These expected cash flows reflect boundary conditions previously discussed in terms of the scheduled rate of cash flow principal and interest and the discrete return of principal through a balloon payment or through mortgage default. By Lemma 4 of Cox, Ingersoll, and Ross (1985a), the value of a particular tranche or residual piece at time  $t$  can be expressed as

$$(15) \quad V_Q(P, r, t) = E_{P,r,t} \left[ \int_t^T TF_Q(s) e^{-\int_t^s r(u)du} ds + \sum_{s \in [t,T]} DP_Q(s) e^{-\int_t^s r(u)du} \right],$$

where  $Q \in \{S, M, J, IO\}$  indexes the payment class under consideration, and the expectation is taken with respect to the risk-adjusted state variable processes.

To implement the valuation of the mortgage-backed securities, we extend the two-stage technique of McConnell and Singh (1994) to handle  $N+1$  state variables. In the first stage, individual mortgages are valued using dynamic programming. The second stage utilizes Monte Carlo simulation to value the securities, given the mortgage rates and default boundaries obtained from the first stage. This technique is now explained in more detail.

Since borrower default decisions are made on a loan-by-loan basis independently of the other borrowers, individual mortgage value and the respective default boundary depend only on interest rates and the collateral property price. Thus, the first stage is made up of  $N$  dynamic programming problems, each with two (potentially correlated) stochastic variables. Notice that computation time is, at worst, linear in the number of properties. Each dynamic program uses the method of Hull and White (1990) and iterates over the mortgage rate until the loan is priced at par.<sup>16</sup> For each time and interest rate pair, cubic interpolation is used to find the default boundary  $P_i^*(r, t)$ .

<sup>15</sup>The discrete payments and losses are nonzero at most  $N$  times over the life of the pool, so the sum of cumulative discrete payments and losses over  $[0, t]$  is well defined.

<sup>16</sup>McConnell and Singh (1994) use the implicit finite difference method in their dynamic programming stage. In our problem, an extra stochastic variable (property price) is required, which significantly complicates the problem. The multidimensional technique found in Section V of Hull and White (1990) is well suited for our problem due to its relative ease of implementation and the fact that the method is stable and converges to the "true" contingent-claim value. Additional technical background regarding our valuation technique is shown in the Appendix.

The second stage uses the  $N$  respective mortgage rates (which are all the same in our simulations) and  $V \times W$  element matrices ( $V = T/\Delta t$ ,  $W = (r^{\max} - r^{\min})/\Delta r$ ) of critical property prices to perform a Monte Carlo simulation in which expected cash flow allocations and subsequent tranche values for the CMBS are determined.<sup>17</sup> For each trial, a path of  $N$  property price and one interest rate states—all of which may be correlated with one another—is created by sampling the risk-adjusted versions of the distributions specified in equations (1) and (2). Default occurs if realized property value falls to the default boundary; otherwise payments occur as scheduled.<sup>18</sup> Cash flows are allocated and tranche balances are calculated as discussed in the previous subsection. To determine current payment class values, cash flows are discounted to the present based on the path of spot interest rates. Thus, for each completed trial (i.e., state variable sample path realizations and cash flow allocations through time  $T$ ), a realized investment value is obtained for each of three standard investment classes and the residual IO tranche. A total of 10,000 Monte Carlo trials are generated to ensure sufficient convergence to the “true” tranche values (as expressed in equation (15)).<sup>19</sup>

For ease of comparison, rather than state security class values,  $V_Q$ , given a coupon rate of payment,  $c_Q$ , for each tranche, we will state the required *yield spread* above that associated with a benchmark riskless bond. Determining the promised yield-to-maturity to a particular security requires making an internal rate of return calculation based on the tranche value obtained in equation (15), with future cash flows to each tranche being generated under the assumption that they are received as promised (i.e., no defaults occur). Calculating the benchmark yield for each security class requires using the simulation to estimate expected cash flow at each payment date over the security’s life, where the “riskless” bond price is calculated by assuming the expected cash flows are certain cash flows. The benchmark yield-to-maturity is calculated using the “riskless” bond price and the expected cash flows.<sup>20</sup> To test for sensitivity of payment class value, yield spreads are generated for a variety of CMBS security structures, mortgage contract specifications, and state variable parameter values.

<sup>17</sup>  $r^{\min}$  is generally positive rather than zero since, given the mean-reverting process of Cox, Ingersoll, and Ross (1985b), the Hull and White (1990) method that we implement during the first stage has only nondownward branching when  $r$  becomes small enough. The quantity  $r^{\max}$  is also finite, for analogous reasons.

<sup>18</sup> As part of the first stage, interpolation is done across property prices to establish the boundary of the continuation region for each interest rate in the grid. A similar interpolation is performed for interest rates that do not fall exactly on the interest rate node value.

<sup>19</sup> To ensure the accuracy of the numerical solutions at 10,000 trials, simulations with up to 50,000 were performed—which yielded no qualitative differences in the results. Solutions to the CMBS tranche values are obtained using a discretized version of the two-step procedure described previously. In doing so, we consider 48 time steps per year. Because all of the individual mortgages are priced at par, initial pool value equals the face value of the mortgages comprising the pool. We have verified that the value of the individual tranches add up to the total pool value. Indeed, in all parameter value cases considered in this paper, the sum of all tranche values obtained through Monte Carlo analysis are within 0.10 percent of the total pool value determined in the first stage of the numerical analysis.

<sup>20</sup> Note that the benchmark riskless yield is computed based on the *expected* effects of default on the timing of the cash flows. This expected cash flow approach provides a superior measure of yield spread as compared to simply calculating a benchmark yield based on anticipation of *promised* cash flows. For example, in an upward sloping term structure environment, use of a benchmark based on promised as opposed to expected cash flows will generally downwardly bias the measure of yield spread.

### III. Pricing Results

#### A. Mortgage Contract, Parameter Value, and Security Design Specification

Valuation solutions are obtained by making the following model input assumptions. We examine a pool of  $N = 6$  commercial mortgages, where all mortgage contract specifications are identical to one another.<sup>21</sup> Each of these mortgages is originated at a 75-percent loan-to-value ratio. The loan term-to-maturity is seven years, with payments amortized over a 25-year term. Properties securing these mortgages are assumed to have a dividend payout rate of 8.5 percent. Two alternative property volatility structures are considered:  $\sigma_i = 0.15$  and  $\sigma_i = 0.20$ ,  $i = 1, \dots, N$ . We also examine three relatively straightforward correlation structures among properties:  $\rho_{ij} = 0$ ,  $\rho_{ij} = 0.5$ , and  $\rho_{ij} = 1$ , for  $i \neq j$ ,  $i, j = 1, \dots, N$ . Two interest rate/property price correlation cases are also considered:  $\rho_{r,P_i} = 0$  and  $\rho_{r,P_i} = 0.2$ ,  $i = 1, \dots, N$ , respectively. Lastly, required yield spreads are calculated under two different term structure of interest rate environments: one that we refer to as *steep* and the other that we refer to as *flat*. Fixing  $\kappa = 0.25$ ,  $\nu = 0.09$ ,  $\sigma_r = 0.075$  (see equation (2)), a lower initial spot rate of interest,  $r_0 = 0.06$ , will provide a steeper term structure while a higher initial spot rate of interest,  $r_0 = 0.09$ , generates a flatter term structure. All of the loan contract/parameter combinations considered are consistent with values observed in the marketplace.<sup>22</sup>

For each of these loan contract/parameter value combinations, a required yield spread is calculated for three principal-based investment classes and an interest-only investment class. Security structure, expressed as a percent of the pool allocated to each principal-based investment class, is  $F_S(0):F_M(0):F_I(0) = 70:10:20$ ,  $70:20:10$ , and  $70:25:5$ . A 70-percent share for the senior tranche is commonplace, so we hold this number constant. We vary the relative share between the mezzanine and junior tranches to examine price sensitivity as the junior piece becomes increasingly risky (i.e., as its share percentage declines). Coupon rates paid to each of the respective tranches depend on the shape of the term structure of interest rates, and are chosen such that the senior tranche approximately prices at par.

<sup>21</sup>Typical pool sizes range from one to several hundred mortgages. We choose  $N = 6$  to balance two opposing considerations. On one hand, because fundamental valuation behavior can be reflected with only a small number of pooled mortgages (e.g., pool diversification effects are largely realized with five to 10 mortgages), we can keep  $N$  relatively small. On the other hand, we want to consider a pool that is larger than  $N = 3$  to demonstrate the model's ability to efficiently handle many state variables.

<sup>22</sup>Loan-to-value ratios in the 75-percent range with seven-year terms-to-maturity are commonplace in the commercial mortgage marketplace, as are partially amortizing loans. This contract structure is also consistent with studies of whole loan commercial mortgage pricing (e.g., Titman and Torous (1989), Childs et al. (1996)). A dividend payout rate of 8.5 percent is consistent with recent commercial mortgage capitalization rates as reported by the American Council of Life Insurers (see Holland et al. (1995)). Property volatilities in the 15- to 20-percent range are in line with recent experience (again, see Holland et al. (1995)). Hartzell et al. (1986) examines the correlation structure of alternative property types (e.g., office, retail, industrial, etc.) and finds correlations that are generally between 0 and 0.50. We consider perfect positive correlation among property types as an extreme case with which to compare these other structures. The correlation between changes in short-term interest rates and commercial real estate prices has been found to be near zero (e.g., Goetzmann and Ibbotson (1990)). As a result, we consider correlations of 0 and 0.20. Alternative term structures are in line with recent experience and are considered because default risk premiums are shown to differ purely based on the shape of the yield curve (again, see Titman and Torous (1989)).

The coupon rates are  $c_S = 0.075$ ,  $c_M = 0.080$ , and  $c_J = 0.080$  in the steep term structure case, and are  $c_S = 0.090$ ,  $c_M = 0.095$ , and  $c_J = 0.095$  in the flat term structure case.<sup>23</sup>

## B. Numerical Results

We first discuss senior tranche yield spreads. Table 1 shows yield spreads are small, ranging from eight to 20 basis points. At a 70-percent pool share, this payment class is, for all practical purposes, immune from default loss risk, which results in a relative lack of yield spread variability over property volatility and property correlation structure. The positive spreads are due to the stochastic nature of interest rates. As seen from the default boundary condition specified in equation (5), higher probabilities of borrower default correspond to low interest rate realizations. Given the greater propensity of borrower default for low interest rate outcomes, the senior securityholder experiences a return of principal that resembles interest rate call risk. This effect is factored into the security's yield. Therefore, although the senior tranche may be immune from default loss, it is not immune to the effects of the early return of principal that follow from loan default.

This call risk effect is also responsible for the slight decrease in yield spread as pool diversification decreases (i.e., as  $\rho_{ij} \rightarrow 1$ ) and the slight increase in yield spread as interest rate/property value correlation increases (i.e., as  $\rho_{r,p_i} \rightarrow 0.2$ ). Since there is a substantial probability that the *entire* pool pays off as promised when  $\rho_{ij} = 1$ , the loans that comprise the pool are slightly less vulnerable to economy-wide interest rate effects on default risk. On the other hand, a well diversified pool results in a potentially wide dispersion of property values and, therefore, greater potential sensitivity to the impact of interest rate on borrower default and ultimately security cash flows. In the case of interest rate/property value correlation, positive correlation means that lower property prices are more likely to be accompanied with higher debt values. This results in higher probabilities of loan default at low realized interest rates and, therefore, higher required security yields to compensate for this increased "call" risk.

Required yield spreads for the mezzanine tranche are shown in Table 2. First notice that the spread increases as the mezzanine tranche share increases. Because almost all of the default risk in the pool is shifted to the lower 30 percent of the pool (encompassing both the mezzanine and junior tranches), an increasing mezzanine share reduces the default "buffer" provided by the first-loss piece to result in a higher risk security. Also, observe the rather sharp differences in required yield spread as a function of property value correlation structure. For certain parameter value combinations, these spreads are seen to vary by more than 200 basis points strictly as a function of collateral asset correlation structure. Because the mezzanine tranche is buffered against default loss risk, its return-of-principal payoff function is typically concave over the relevant region. Increased property

<sup>23</sup> As discussed earlier, the tranche coupon rates are restricted to be less than the equilibrium rate paid on any of the mortgages in the pool. The minimum endogenously-determined risky whole mortgage rate in the steep term structure environment is 0.0819, and in the flat term structure environment is 0.0958. Both of these rates result from parameter values  $\sigma_i = 0.15$  and  $\rho_{r,p_i} = 0$ .

TABLE 1  
Required Spreads above Benchmark Yield as a Function of Term Structure Environment,  
Asset Price-Interest Rate Correlation,  $\rho_{r,P_i}$ , Asset Price Volatility,  $\sigma_{ij}$ , and Asset Price  
Correlation,  $\rho_{ij}$ : Senior Tranche

| All Security Structures   |             |                      |                      |                     |                      |
|---------------------------|-------------|----------------------|----------------------|---------------------|----------------------|
|                           |             | Steep Term Structure |                      | Flat Term Structure |                      |
| $\sigma_{ij}$             | $\rho_{ij}$ | $\rho_{r,P_i} = 0$   | $\rho_{r,P_i} = 0.2$ | $\rho_{r,P_i} = 0$  | $\rho_{r,P_i} = 0.2$ |
| 0.15                      | 0           | 0.0010               | 0.0015               | 0.0011              | 0.0017               |
|                           | 0.5         | 0.0009               | 0.0015               | 0.0010              | 0.0016               |
|                           | 1           | 0.0008               | 0.0013               | 0.0008              | 0.0013               |
| 0.20                      | 0           | 0.0012               | 0.0019               | 0.0012              | 0.0020               |
|                           | 0.5         | 0.0011               | 0.0019               | 0.0011              | 0.0019               |
|                           | 1           | 0.0010               | 0.0017               | 0.0009              | 0.0017               |
| Range for Benchmark Yield |             | 0.0726–0.0731        |                      | 0.0892              |                      |

Required yield spreads above the benchmark yield are reported for a variety of state variable parameter values. The benchmark yield is determined assuming that the expected cash flows to the security match those of an otherwise identical riskless bond, and varies as a function of the term structure of interest rates and other parameter values used in a given simulation. Parameter values apply to every property in the pool (of which there are  $N = 6$ ). A security structure in which the senior tranche is allocated 70 percent of the pool is assumed. All mortgages in the pool are seven years in term, partially amortizing over a 25-year term, and have 75-percent initial loan-to-value ratios. The payout rate on each property,  $\delta_{ji}$ , is assumed to equal 0.085. Interest rate parameter values are as follows: Steep term structure:  $\kappa = 0.25, \nu = 0.09, \sigma_r = 0.075, r_0 = 0.06$ ; flat term structure:  $\kappa = 0.25, \nu = 0.09, \sigma_r = 0.075, r_0 = 0.09$ . Time increments are  $\Delta t = 1/48$ . The coupon rates paid to the senior, mezzanine, and junior tranches are 0.075, 0.08, and 0.08, respectively, for the steep term structure; and, 0.09, 0.095, and 0.095, respectively, for the flat term structure.

diversification—resulting in decreased pooled cash flow volatility—is, therefore, beneficial in terms of increasing the likelihood of full return of principal.

Interestingly, when the junior security has a 20-percent pool share and when the pool is well diversified in the sense that  $\rho_{ij} = 0, i \neq j$ , yield spreads are actually *lower* than spreads associated with the senior security. This is due to two related structural effects. First, in this case, the mezzanine security is fully protected against default losses. Second, due to the senior tranche having priority on return of principal, the mezzanine security is also insulated from cash flow timing uncertainty that follows from default-induced return of principal. The result is a security that is (essentially) fully protected against both default loss risk and default-related call risk.

Other parameter value effects on mezzanine tranche pricing are as follows. Primarily due to the impact of default loss risk, required yield spreads are higher when property volatility increases from  $\sigma_i = 0.15$  to  $\sigma_i = 0.20$ . A similar outcome results when  $\rho_{r,P_i}$  increases from 0 to 0.2. There are also significant differences depending on the shape of the yield curve. To see this, recall that both term structures have the same long-term rate parameter of  $\nu = 0.09$ . The initial spot rate is much lower in the steeper term structure environment; hence, expected discount rates over the life of the seven-year loan are higher in the flatter term structure. Higher expected discount rates decrease the required default risk premium, ceteris

TABLE 2  
Required Spreads above Benchmark Yield as a Function of Term Structure Environment,  
Asset Price-Interest Rate Correlation,  $\rho_{r,P_i}$ , Security Structure, Asset Price Volatility,  $\sigma_{ij}$ ,  
and Asset Price Correlation,  $\rho_{ij}$ : Mezzanine Tranche

|                           |             | Security Structure<br>(Tranche Allocations)              |          |         | Security Structure<br>(Tranche Allocations)             |          |         |
|---------------------------|-------------|--|----------|---------|---|----------|---------|
| $\sigma_{ij}$             | $\rho_{ij}$ | 70:10:20   | 70:20:10 | 70:25:5 | 70:10:20  | 70:20:10 | 70:25:5 |
|                           |             | Panel A.<br>Steep Term Structure<br>$\rho_{r,P_i} = 0$   |          |         | Panel B.<br>Flat Term Structure<br>$\rho_{r,P_i} = 0$   |          |         |
| 0.15                      | 0           | 0.0000   | 0.0011   | 0.0081  | 0.0000  | 0.0001   | 0.0036  |
|                           | 0.5         | 0.0003   | 0.0059   | 0.0133  | 0.0002  | 0.0024   | 0.0076  |
|                           | 1           | 0.0037   | 0.0166   | 0.0221  | 0.0017  | 0.0098   | 0.0152  |
| 0.20                      | 0           | 0.0002   | 0.0111   | 0.0281  | 0.0000  | 0.0050   | 0.0185  |
|                           | 0.5         | 0.0055   | 0.0224   | 0.0352  | 0.0017  | 0.0136   | 0.0249  |
|                           | 1           | 0.0229   | 0.0424   | 0.0481  | 0.0108  | 0.0305   | 0.0369  |
| Range for Benchmark Yield |             | 0.0733–0.0740  |          |         | 0.0891–0.0892   |          |         |
|                           |             | Panel C.<br>Steep Term Structure<br>$\rho_{r,P_i} = 0.2$ |          |         | Panel D.<br>Flat Term Structure<br>$\rho_{r,P_i} = 0.2$ |          |         |
| 0.15                      | 0           | 0.0000   | 0.0017   | 0.0101  | 0.0000  | 0.0002   | 0.0047  |
|                           | 0.5         | 0.0004   | 0.0070   | 0.0151  | 0.0004  | 0.0029   | 0.0089  |
|                           | 1           | 0.0041   | 0.0181   | 0.0240  | 0.0022  | 0.0106   | 0.0167  |
| 0.20                      | 0           | 0.0002   | 0.0133   | 0.0307  | 0.0000  | 0.0064   | 0.0207  |
|                           | 0.5         | 0.0060   | 0.0244   | 0.0376  | 0.0019  | 0.0152   | 0.0272  |
|                           | 1           | 0.0233   | 0.0443   | 0.0505  | 0.0111  | 0.0320   | 0.0389  |
| Range for Benchmark Yield |             | 0.0732–0.0740  |          |         | 0.0891–0.0892   |          |         |

Required yield spreads above the benchmark yield are reported for a variety of state variable parameter values and security structures. The benchmark yield is determined assuming that the expected cash flows to the security match those of an otherwise identical riskless bond, and varies as a function of the term structure of interest rates and other parameter values used in a given simulation. Parameter values apply to every property in the pool (of which there are  $N = 6$ ). For each of these parameter value constellations, three security structures are examined: 70:10:20, 70:20:10, and 70:25:5 percentage splits between the senior, mezzanine, and junior tranches, respectively. All mortgages in the pool are seven years in term, partially amortizing over a 25-year term, and have 75-percent initial loan-to-value ratios. The payout rate on each property,  $\delta_i$ , is assumed to equal 0.085. Interest rate parameter values are as follows: Steep term structure:  $\kappa = 0.25$ ,  $\nu = 0.09$ ,  $\sigma_r = 0.075$ ,  $r_0 = 0.06$ ; flat term structure:  $\kappa = 0.25$ ,  $\nu = 0.09$ ,  $\sigma_r = 0.075$ ,  $r_0 = 0.09$ . Time increments are  $\Delta t = 1/48$ . The coupon rates paid to the senior, mezzanine, and junior tranches are 0.075, 0.08, and 0.08, respectively, for the steep term structure; and, 0.09, 0.095, and 0.095, respectively, for the flat term structure.

paribus, since the present value impact of default loss is lower when discount rates are higher.

Now consider the junior (or first-loss) tranche results shown in Table 3. This tranche is quite sensitive to security structure, where decreasing the pool share results in a higher likelihood that it will be out-of-the-money in terms of receiving substantial repayment of principal. Also note that the magnitude of these spreads



is typically rather high, with required yields-to-maturity that can easily exceed returns expected to the underlying collateral asset.

TABLE 3  
Required Spreads above Benchmark Yield as a Function of Term Structure Environment, Asset Price-Interest Rate Correlation,  $\rho_{r,P_i}$ , Security Structure, Asset Price Volatility,  $\sigma_{ij}$ , and Asset Price Correlation,  $\rho_{ij}$ : Junior Tranche

| $\sigma_{ij}$             | $\rho_{ij}$ | Security Structure<br>(Tranche Allocations)              |          |         | Security Structure<br>(Tranche Allocations)             |          |         |
|---------------------------|-------------|--|----------|---------|---|----------|---------|
|                           |             | 70:10:20   | 70:20:10 | 70:25:5 | 70:10:20  | 70:20:10 | 70:25:5 |
|                           |             | Panel A.<br>Steep Term Structure<br>$\rho_{r,P_i} = 0$   |          |         | Panel B.<br>Flat Term Structure<br>$\rho_{r,P_i} = 0$   |          |         |
| 0.15                      | 0           | 0.0398   | 0.0892   | 0.1603  | 0.0283  | 0.0622   | 0.1225  |
|                           | 0.5         | 0.0401   | 0.0746   | 0.1086  | 0.0286  | 0.0566   | 0.0882  |
|                           | 1           | 0.0390   | 0.0476   | 0.0492  | 0.0290  | 0.0405   | 0.0419  |
| 0.20                      | 0           | 0.0846   | 0.1856   | 0.2846  | 0.0651  | 0.1479   | 0.2426  |
|                           | 0.5         | 0.0815   | 0.1351   | 0.1796  | 0.0647  | 0.1147   | 0.1578  |
|                           | 1           | 0.0695   | 0.0745   | 0.0759  | 0.0595  | 0.0663   | 0.0673  |
| Range for Benchmark Yield |             | 0.0716–0.0738  |          |         | 0.0891–0.0893   |          |         |
|                           |             | Panel C.<br>Steep Term Structure<br>$\rho_{r,P_i} = 0.2$ |          |         | Panel D.<br>Flat Term Structure<br>$\rho_{r,P_i} = 0.2$ |          |         |
|                           |             | 70:10:20   | 70:20:10 | 70:25:5 | 70:10:20  | 70:20:10 | 70:25:5 |
|                           |             |  |          |         |   |          |         |
| 0.15                      | 0           | 0.0434   | 0.0965   | 0.1638  | 0.0310  | 0.0687   | 0.1296  |
|                           | 0.5         | 0.0436   | 0.0800   | 0.1135  | 0.0314  | 0.0622   | 0.0943  |
|                           | 1           | 0.0424   | 0.0518   | 0.0535  | 0.0317  | 0.0448   | 0.0463  |
| 0.20                      | 0           | 0.0890   | 0.1889   | 0.2811  | 0.0690  | 0.1542   | 0.2430  |
|                           | 0.5         | 0.0856   | 0.1395   | 0.1826  | 0.0688  | 0.1201   | 0.1624  |
|                           | 1           | 0.0737   | 0.0789   | 0.0803  | 0.0633  | 0.0710   | 0.0718  |
| Range for Benchmark Yield |             | 0.0718–0.0738  |          |         | 0.0891–0.0893   |          |         |

Required yield spreads above the benchmark yield are reported for a variety of state variable parameter values and security structures. The benchmark yield is determined assuming that the expected cash flows to the security match those of an otherwise identical riskless bond, and varies as a function of the term structure of interest rates and other parameter values used in a given simulation. Parameter values apply to every property in the pool (of which there are  $N = 6$ ). For each of these parameter value constellations, three security structures are examined: 70:10:20, 70:20:10, and 70:25:5 percentage splits between the senior, mezzanine and junior tranches, respectively. All mortgages in the pool are seven years in term, partially amortizing over a 25-year term, and have 75-percent initial loan-to-value ratios. The payout rate on each property,  $\delta_i$ , is assumed to equal 0.085. Interest rate parameter values are as follows: Steep term structure:  $\kappa = 0.25, \nu = 0.09, \sigma_r = 0.075, r_0 = 0.06$ ; flat term structure:  $\kappa = 0.25, \nu = 0.09, \sigma_r = 0.075, r_0 = 0.09$ . Time increments are  $\Delta t = 1/48$ . The coupon rates paid to the senior, mezzanine, and junior tranches are 0.075, 0.08, and 0.08, respectively, for the steep term structure; and, 0.09, 0.095, and 0.095, respectively, for the flat term structure.

Perhaps the most notable aspect of our results concerns the effect of property value correlation on junior tranche yield. As the pool becomes better diversified, required yields generally *increase* substantially.<sup>24</sup> The intuition for this result

<sup>24</sup>This volatility effect with subordinated debt was first recognized by Black and Cox (1976). The difference between our finding and theirs is that *correlation* structure between multiple assets is re-

follows from a recognition that the return-of-principal payoff function for the junior tranche has characteristics similar to an out-of-the-money call option. Expected losses do not change as a function of correlation structure, but the variance of these losses does change. The cost of increased pool cash flow volatility is slight, due to payoff function convexity and the fact that the junior piece is expected to be out-of-the-money with respect to full return of principal. However, the benefit to higher pool cash flow volatility is that there is an increased probability of low losses—which results in a higher expected payoff and, therefore, decreases the required yield to this security. Not surprisingly, the correlation structure effect is strongest when the first-loss piece has the low 5-percent share.

To see the out-of-the-money option analogy more precisely, in Figure 1, we show the loss density function on a \$100,  $N = 20$ , CMBS pool for alternative property value correlation structures. Due to the central limit theorem, the zero correlation case turns out to be approximately normally distributed with a loss density mode of about \$8. As correlations between collateralizing assets increase, the loss density function shows greater variance—and eventually becomes bimodal as asset correlations approach one. Indeed, given a pool composed of perfectly correlated collateral assets (or equivalently, a single-asset CMBS), default risk is literally all or nothing—where the probability of zero losses is quite high. As a result, first-loss tranche value is higher—and, therefore, required spreads are lower—when cash flow volatility as determined by property value correlation structure is higher.

Lastly, yields-to-maturity for the IO residual tranche are shown in Table 4. Except for property volatility,  $\sigma_i$ —which has a powerful effect on the expected timing of loan termination—yield spreads are relatively insensitive to alternative mortgage/CMBS pool design or economic environment. For example, given tranche coupon rates used in this study, allocation of cash flow within the various principal-based tranches has little effect on IO price. Similarly, yield spread is fairly insensitive to differences in property value correlation, although the property value correlation/call risk effect described with senior security pricing is responsible for having a slight impact on pricing.

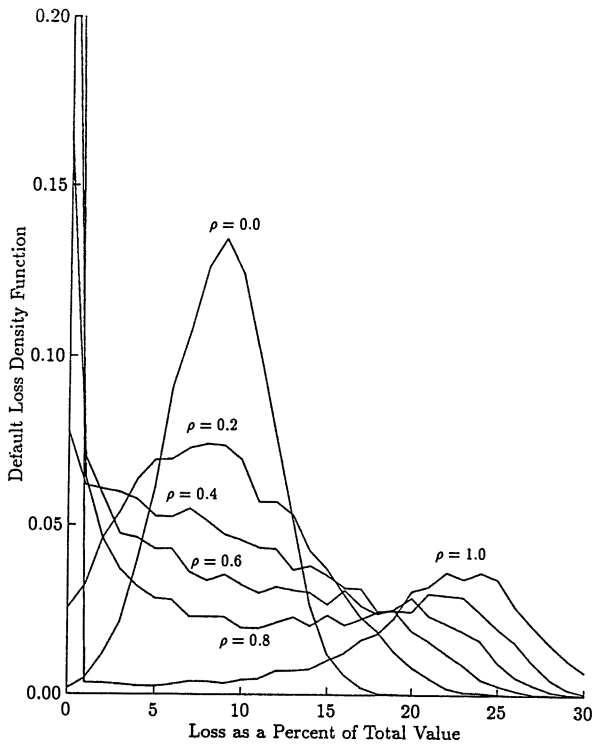
Summing up, these results suggest that the correlation structure of properties underlying the pooled mortgages plays a significant and rather unique role in CMBS pricing, especially when compared to more well-known asset-backed securities. Because of the prioritization of losses on a bottom-up basis, better pool diversification can actually reduce first-loss tranche value due to principal payoff function convexity and the out-of-the-money nature of this security. On the other hand, the investors in higher priority pieces will generally find that better pool diversification increases value since lower cash flow volatility reduces the likelihood that losses will ever be allocated to their tranche.

Given these rather substantial diversification effects, a natural follow-up question is: how many individual mortgages in a CMBS pool are required to realize most of the effects from asset diversification? We answer this question with Figure 2, given several different security design and correlation structure combinations.

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sponsible for our pricing outcomes, whereas *volatility* of a single asset drives the pricing results in the Black and Cox model.

FIGURE 1  
CMBS Pool Default Loss Density as a Function of Asset Correlation Structure



This figure shows the probability density functions for CMBS default losses given a total pool composed of  $N = 20$  mortgages and with total pool value of \$100. Alternative correlation structures in increments of 0.2 are shown. Note that as correlations increase, pool cash flow volatility increases, but expected loss to the pool remains constant. Other parameter values used to generate the graph are: steep term structure,  $\sigma_i = 0.20$  and  $\rho_i, \rho_j = 0$  for all  $i$  collateralizing properties. All other model inputs are as described in the text.

Because security value rather than yield provides a more standardized measure of the effects of diversification, we show tranche market value as a percent of face value on the vertical axis and the number of properties in the pool ( $N$ ) on the horizontal axis. These graphs vividly demonstrate the opposing effects of pool diversification on mezzanine and junior tranche value. Overall, it appears that, at least for the chosen model parameter values, pool sizes of five to 10 mortgages are required to realize a majority of the total potential effect of diversification.<sup>25</sup>

<sup>25</sup>These findings, in part, depend on the fact that all mortgages are the same size in our analysis, the fact that parameter values are identical for each loan, and the fact that the pool is valued at par at time 0. Having one mortgage that is much larger than the other mortgages in the pool, for example, would require a relatively larger number of smaller-value loans to realize significant diversification effects.

TABLE 4

Required Spreads above Benchmark Yield as a Function of Term Structure Environment, Asset Price-Interest Rate Correlation,  $\rho_{r,P_i}$ , Asset Price Volatility,  $\sigma_{ij}$ , and Asset Price Correlation,  $\rho_{ij}$ : Interest-Only Tranche

|                           |             | All Security Structures |                      |                     |                      |
|---------------------------|-------------|-------------------------|----------------------|---------------------|----------------------|
|                           |             | Steep Term Structure    |                      | Flat Term Structure |                      |
| $\sigma_{ij}$             | $\rho_{ij}$ | $\rho_{r,P_i} = 0$      | $\rho_{r,P_i} = 0.2$ | $\rho_{r,P_i} = 0$  | $\rho_{r,P_i} = 0.2$ |
| 0.15                      | 0           | 0.0349                  | 0.0386               | 0.0346              | 0.0385               |
|                           | 0.5         | 0.0348                  | 0.0383               | 0.0342              | 0.0380               |
|                           | 1           | 0.0312                  | 0.0348               | 0.0289              | 0.0331               |
| 0.20                      | 0           | 0.0556                  | 0.0598               | 0.0538              | 0.0588               |
|                           | 0.5         | 0.0560                  | 0.0602               | 0.0540              | 0.0589               |
|                           | 1           | 0.0548                  | 0.0591               | 0.0519              | 0.0567               |
| Range for Benchmark Yield |             | 0.0705–0.0706           |                      | 0.0895              |                      |

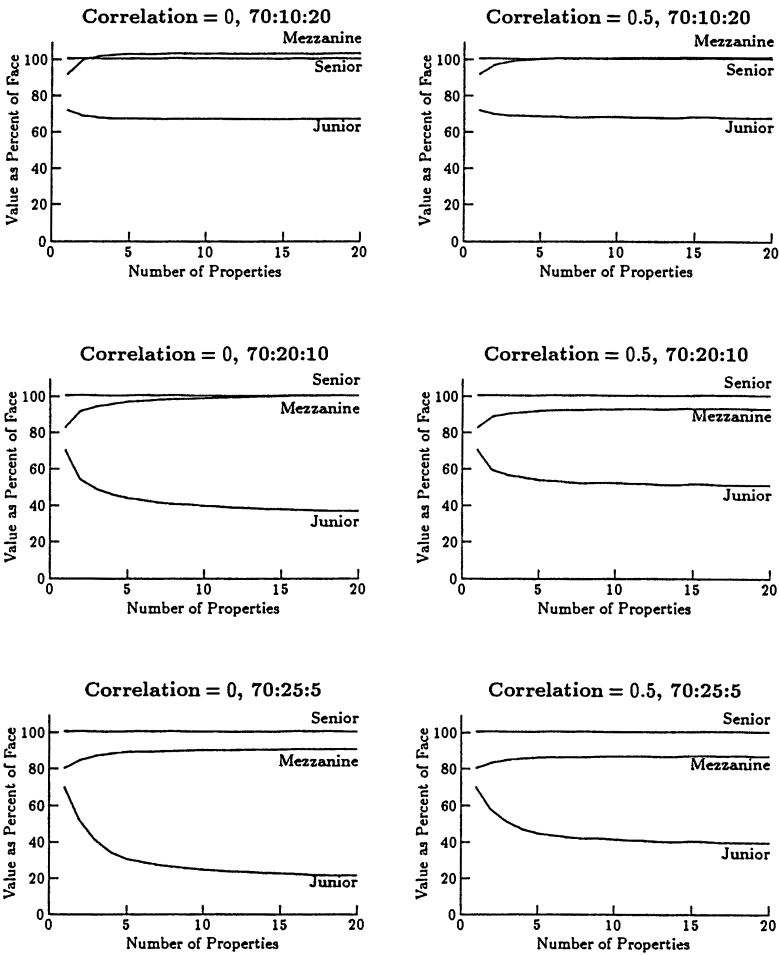
Required yield spreads above the benchmark yield are reported for a variety of state variable parameter values. The benchmark yield is determined assuming that the expected cash flows to the security match those of an otherwise identical riskless bond, and varies as a function of the term structure of interest rates and other parameter values used in a given simulation. Parameter values apply to every property in the pool (of which there are  $N = 6$ ). The interest-only tranche is allocated residual interest after the principal-based tranches receive their coupon interest. As a result, yield spread is independent of percent allocation between the mezzanine and junior tranches when their stated coupon rates are equal (which is the case in these simulations). All mortgages in the pool are seven years in term, partially amortizing over a 25-year term, and have 75-percent initial loan-to-value ratios. The payout rate on each property,  $\delta_i$ , is assumed to equal 0.085. Interest rate parameter values are as follows: Steep term structure:  $\kappa = 0.25, \nu = 0.09, \sigma_r = 0.075, r_0 = 0.06$ ; Flat term structure:  $\kappa = 0.25, \nu = 0.09, \sigma_r = 0.075, r_0 = 0.09$ . Time increments are  $\Delta t = 1/48$ . The coupon rates paid to the senior, mezzanine, and junior tranches are 0.075, 0.08, and 0.08, respectively, for the steep term structure; and, 0.09, 0.095, and 0.095, respectively, for the flat term structure.

IV. Summary and Discussion

This paper considers the valuation of a new and increasingly important class of structured debt: the commercial mortgage-backed security. These securities present a unique set of valuation issues that differ sharply from those associated with most other asset-backed securities—most notably, the residential MBS. An appropriate determination of default risk is critical to accurate CMBS pricing, which means that property value underlying each mortgage in the pool must be considered as a state variable. To value such a security, we employ a combined backward/forward simulation approach. This allows us to exploit the independence of borrower default decisions and, consequently, to overcome dimensional-ity problems associated with a large number of state variables.

Using data adopted from Duff & Phelps (1994) and Corcoran (1995), Table 5 shows that our model provides prices and yields that are generally in line with yields observed in the marketplace, although with some apparent differences. For example, given the parameter values employed in this study, our analysis indicates that senior investment classes that are allocated 70 percent or less of the loan pool are essentially free from default loss risk, which suggests pricing comparable to

FIGURE 2  
The Effect of Pool Size and Asset Correlation Structure on Tranche Value



Six graphs are shown for different combinations of security structure and correlation structure among properties. The number of properties in the pool is shown on the horizontal axis and tranche value as a percent of face value is shown on the vertical axis. Other parameters used to obtain these graphs are: steep term structure,  $\sigma_i = 0.20$  and  $\rho_{r_i, P_i} = 0$  for all  $i$  collateralizing properties. All other model inputs are as described in the text.

a AAA-rated corporate bond. Currently observed yield spreads of 70–100 basis points on senior CMBS investment classes suggest that there may be other non-default risks to consider or that pricing may be conservative given investor and rating agency unfamiliarity with this new debt security. To the extent that the latter effect is relevant, we should observe a gradual narrowing of spreads over comparable duration Treasury bond yields.

In the course of our analysis, we consider varying structures for the more junior payment classes, and find ranges of prices that vary substantially over a

TABLE 5  
Comparison of Observed and Modeled Yield Spreads

| Investment Class | Pool Share    | Observed<br>Yield Spreads | Modeled<br>Yield Spreads |
|------------------|---------------|---------------------------|--------------------------|
| Senior           | 70 percent    | 70–100 basis points       | 10–20 basis points       |
| Mezzanine        | 10–25 percent | 100–350 basis points      | 0–400 basis points       |
| Junior           | 5–20 percent  | 450–1500 basis points     | 300–1800 basis points    |

For equivalent pool share percentages, this table compares observed market yield spreads with those predicted by the model for various investment classes. Observed yield spread ranges are from a Duff & Phelps (1994) report on CMBS, as adopted by Corcoran (1995). Yield spreads are interpolated, since the investment class percentage cutoffs from the stylized security employed by Corcoran do not exactly match the pool shares used in this paper. Yield spreads are based on an assumption of a multiasset pool of newly originated loans. Model yield spread ranges are adopted from Tables 1, 2, and 3 in this paper under the assumption that property value correlations are equal to 0.5.

plausible range of model input parameter values and that generally coincide with reported yields by the dealers of these securities. Although most simulations produce expected effects, the impact of property diversification on security value contradicts the conventional wisdom with respect to the first-loss payment class. Indeed, investment banks and rating agencies have recently touted the general *positive* effects of increased pool diversification on CMBS value. Our results suggest that better diversification may actually decrease first-loss tranche value, a finding that, in our opinion, merits close examination by those who are intimately involved in this new and developing market.

## Appendix

This appendix briefly outlines the valuation technique. The first step of the valuation is to use dynamic programming for each mortgage to determine mortgage rates and default boundaries. Then, using the mortgage rates and default boundaries as inputs to a second stage, the next step is a Monte Carlo simulation that values the tranches of the CMBS.

Each dynamic program has two stochastic state variables: interest rate and the property price of an individual property. To improve efficiency, we transform the state variables to constant variance variables before building the lattice. First, consider the dynamics of the short-term interest rate in a risk-neutral world. We assume that the interest rate,  $r$ , is a CIR square root process,

$$dr = \kappa(\nu - r)dt + \sigma_r\sqrt{r}dz_r.$$

The transformed variable  $X_1 \equiv \sqrt{r}$  has constant variance. A straightforward application of Ito's Lemma provides the dynamics for  $X_1$ ,

$$dX_1 = \left( \frac{\alpha_1}{X_1} - \alpha_2 X_1 \right) dt + \frac{\sigma_r}{2} dz_r,$$

where,

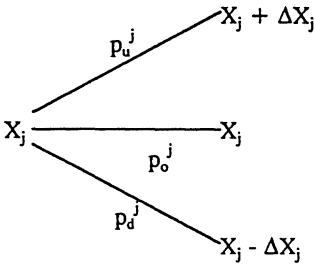
$$\alpha_1 = \frac{4\kappa\nu - \sigma_r^2}{8} \quad \text{and} \quad \alpha_2 = \frac{\kappa}{2}.$$

Each of the property prices  $P_i$ ,  $i = 1, \dots, N$ , follows geometric Brownian motion. The usual transform  $X_{2i} \equiv \ln P_i$  produces a constant variance variable. In a risk-neutral world, the dynamics of  $X_{2i}$  are

$$dX_{2i} = \left( r - \delta_i - \frac{1}{2}\sigma_i^2 \right) dt + \sigma_i dz_{2i}.$$

The interest rate and property price may be correlated, so  $E[dz_1 dz_{2i}] = \rho_i dt$ .

We use the trinomial method of Hull and White (1990) to discretize the valuation problem. The normal branching of each individual process is<sup>26</sup>



for  $j = 1, \dots, N + 1$ . The grid spacing is determined by

$$\Delta X_1 = \frac{\sigma_r}{2} \sqrt{3 \Delta t}, \quad \Delta X_{2i} = \sigma_i \sqrt{3 \Delta t}.$$

The marginal probabilities associated with each possible state outcome are given in Hull and White (1990).

The tree for *both* interest rate and property price is produced using the same grid spacing as defined above. Nine branches emanate from each node corresponding to the nine pair-wise combinations of up, over, and down from the individual processes. The joint probabilities are the product of the marginal probabilities adjusted to account for correlation between the processes, as outlined in Hull and White (1994). To recover actual state variable values at any node, simply invert the transforms to find  $r = X_1^2$  and  $P_i = e^{X_{2i}}$ .

This fully defined tree allows mortgage valuation for a given mortgage coupon rate. We iterate over the mortgage coupon payment to find the mortgage interest rate that prices the mortgage at its face value. Additionally, for each discrete time period, we approximate the continuous default boundary by interpolating near the property price/interest rate nodes where default occurs.

Next, given the mortgage rates and default boundaries from the first step, a Monte Carlo simulation is used to value the tranches of the CMBS. Multiple property price/interest rate paths are simulated based on the risk-neutralized stochastic processes for the state variables that reflect any correlation between individual property prices and/or the interest rate. As part of this process, we must take  $(N + 1)$  standard normal uncorrelated deviates and transform them to have the appropriate correlation. Let  $\Sigma$  be the  $(N + 1) \times (N + 1)$  variance-covariance

<sup>26</sup>For the interest rate process, alternative branching occurs at the top node and bottom nodes of the grid. See Hull and White (1990) for a complete description.

matrix for the  $N$  property prices and the interest rate. Let  $\mathbf{T} = [\tau_{jk}]$  be a lower triangular decomposition of  $\Sigma$  such that  $\Sigma = \mathbf{T}\mathbf{T}'$ .<sup>27</sup> Let  $\mathbf{Z}$  be an  $(N + 1)$  vector of independent standard normal variates. We utilize the  $\mathbf{T}$  matrix to transform the independent draws  $\mathbf{Z}$  to a vector of draws  $\mathbf{Y}$  that has the appropriate correlation and standard deviation,

$$Y_j = \sum_{k=1}^j \tau_{jk} Z_k, \quad \text{for } j = 1, \dots, (N + 1).$$

Given the interest rate, property prices, mortgage rate, and default boundary at any time period, the cash flows from the mortgage coupons and any default proceeds can be allocated to the individual tranches of the CMBS. Cash flows are discounted at the spot rate of interest to determine the value of each tranche at time zero.

## References

- Black, F., and J. C. Cox. "Valuing Corporate Securities: Some Effects of Bond Indenture Provisions." *Journal of Finance*, 31 (May 1976), 351–367.
- Boot, A. W. A., and A. V. Thakor. "Security Design." *Journal of Finance*, 48 (Sept. 1993), 1349–1378.
- Childs, P. D.; S. H. Ott; and T. J. Riddiough. "The Value of Recourse and Cross-Default Clauses in Commercial Mortgage Contracting." *Journal of Banking and Finance*, 20 (April 1996), 511–536.
- Corcoran, P. J. "Debt and Equity in the New Real Estate Markets." *Real Estate Finance*, (Fall 1995), 73–82.
- Cox, J. C.; J. E. Ingersoll; and S. A. Ross. "An Intertemporal General Equilibrium Model of Asset Prices." *Econometrica*, 53 (March 1985a), 363–384.
- . "A Theory of the Term Structure of Interest Rates." *Econometrica*, 53 (March 1985b), 385–407.
- Duff & Phelps Rating Company. "The Rating of Commercial Mortgage-Backed Securities." (Oct. 1994).
- Dunn, K. B., and J. J. McConnell. "Valuation of GNMA Mortgage-Backed Securities." *Journal of Finance*, 36 (June 1981), 599–616.
- Grenadier, S. R. "Valuing Lease Contracts: A Real-Options Approach." *Journal of Financial Economics*, 38 (July 1995), 297–331.
- Goetzmann, W. N., and R. G. Ibbotson. "The Performance of Real Estate as an Asset Class." *Journal of Applied Corporate Finance*, 13 (Fall 1990), 65–76.
- Hartzell, D.; J. Hekman; and M. Miles. "Diversification Categories in Investment Real Estate." *AREUEA Journal*, 14 (Summer 1986), 230–259.
- Holland, S. A.; S. H. Ott; and T. J. Riddiough. "Uncertainty and the Rate of Commercial Real Estate Development." Unpubl. Manuscript, The Economics Institute (1995).
- Hull, J., and A. White. "Valuing Derivative Securities Using the Explicit Difference Method." *Journal of Financial and Quantitative Analysis*, 25 (March 1990), 87–100.
- . "Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models." *Journal of Derivatives*, 2 (Winter 1994), 37–48.
- Kau, J. B.; D. C. Keenan; W. S. Muller; and J. F. Epperson. "Option Theory and Floating-Rate Securities with a Comparison of Adjustable and Fixed-Rate Mortgages." *Journal of Business*, 66 (Oct. 1993), 595–618.
- McConnell, J. J., and M. Singh. "Rational Prepayments and the Valuation of Collateralized Mortgage Obligations." *Journal of Finance*, 49 (July 1994), 891–921.
- Nomura Securities International, Inc.—Global Real Estate Research. "Commercial Real Estate Quarterly," (Jan. 1996).
- Titman, S., and W. N. Torous. "Valuing Commercial Mortgages: An Empirical Investigation of the Contingent-Claims Approach to Pricing Risky Debt." *Journal of Finance*, 44 (June 1989), 345–373.
- Tong, Y. L. *The Multivariate Normal Distribution*. New York, NY: Springer-Verlag (1990).

<sup>27</sup> See Proposition 8.1.1 in Tong (1990) for a formula for the  $\mathbf{T}$  matrix.