Physica A 479 (2017) 309–323

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Pricing real estate index options under stochastic interest rates

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HIGHLIGHTS

- We price real estate index options under stochastic interest rates.
- The non-tradable and mean-reverting of the real estate index are considered.
- We propose a modified finite difference method that adopts the non-uniform grids.
- Constant interest rate models lead to the mispricing of options.

ARTICLE INFO

Article history:
Received 30 August 2016
Received in revised form 1 January 2017
Available online 8 March 2017

Keywords:
Real estate derivatives
Option pricing
Stochastic interest rate
Finite difference method

ABSTRACT

Real estate derivatives as new financial instruments are not merely risk management tools but also provide a novel way to gain exposure to real estate assets without buying or selling the physical assets. Although real estate derivatives market has exhibited a rapid development in recent years, the valuation challenge of real estate derivatives remains a great obstacle for further development in this market. In this paper, we derive a partial differential equation contingent on a real estate index in a stochastic interest rate environment and propose a modified finite difference method that adopts the non-uniform grids to solve this problem. Numerical results confirm the efficiency of the method and indicate that constant interest rate models lead to the mispricing of options and the effects of stochastic interest rates on option prices depend on whether the term structure of interest rates is rising or falling. Finally, we have investigated and compared the different effects of stochastic interest rates on European and American option prices.

1. Introduction

Real estate is the largest class of asset in the world and plays a vital role in world economy. However, compared with the diverse innovative derivatives in the financial market, limited attempts have been made to create a liquid real estate derivatives market for investors to hedge the real estate price risk. The subprime mortgage crisis that began in the United States and spread all over the world has emphasized the risk embedded in the real estate market.

Real estate derivatives contingent on real estate indices came into existence in the 1990s. However, this market did not take off until early this century. Since its inception in the UK in 2005, the UK real estate derivatives market has developed and made a huge effect on the growth of global real estate derivatives market in notional amounts and trading volumes. In recent years, the total over-the-counter return swap trading, which is contingent on the IPD annual UK index, exhibited great development. On February 9, 2009, Eurex began to trade real estate futures based on the IPD annual UK index. The United

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http://dx.doi.org/10.1016/j.physa.2017.03.009
0378-4371/

States, which remains the heart of the world's financial system, has emphasized the importance of real estate derivatives over the last few years. The NCREIF property index (NPI) is the underlying index that enables an instrument to track price movements in the commercial real estate market in the US. The Case–Shiller indices futures and options contracts were then introduced in the Chicago Mercantile Exchange in 2006. Real estate derivatives as new risk management instruments indeed attract attention from investors in real estate market. Fisher [1] and Clayton [2] described the development of the real estate derivatives market in the US and Fabozzi et al. [3] reviewed derivative instruments that can manage commercial real estate price risk and interest rate risk.

In addition to its role to hedge real estate price risk, real estate derivatives are also gateways for investors to gain exposure to the price movements of real estate assets without buying or selling physical assets. The traditional way to gain access to real estate investments is either to buy the physical assets directly or to enter this market through a fund or a real estate investment trust (REITs). However, both approaches have their own limitations when gaining access to real estate. On the one hand, direct investments are risky and time-consuming, and they require considerable due diligence, tax, and transaction costs. On the other hand, indirect investments, such as REITs, have low correlation with the price movements of real estate and behave as a fixed income security. Real estate derivatives can overcome these shortcomings. They provide a new way to easily gain access to real estate investment where changes in real estate price are fully reflected.

Although the real estate derivatives market has displayed a rapid development in the last 10 years, the trading volume and liquidity are not comparable to financial derivatives partly because real estate derivatives are not widely accepted by investors as new investment instruments. The more significant obstacle for the development of this market may be due to the lack of reliable pricing models. Geltner and Fisher [4] conducted a survey at the MIT Center for Real Estate. The results revealed that 37 real estate investment managers and other likely participants, who are the potential investors in real estate market, showed lack of confidence in pricing real estate derivatives. Accordingly, 75% of the participants considered the lack of reliable pricing models as either an important or a very important barrier.

Early research on the pricing of real estate derivatives were based on the no-arbitrage approach, which is no different from the stock option pricing problem under the Black–Scholes model framework developed by Black and Scholes [5]. Buttimer et al. [6] employed a bivariate binominal pricing model for the total return swap contingent on a real estate index and a stochastic interest rate. They found a non-zero but negligible swap spread price. Björk and Clapham [7] revisited the model proposed by Buttimer et al. [6] and proved that the theoretical price of the total return swap is equal to zero when using the no-arbitrage approach. The previous non-zero numerical result is attributed to the specific numerical approximations. Ciurlia and Gheno [8] assumed that the real estate index movement satisfies a geometric Brownian motion, and used a bi-dimensional binominal lattice to price real estate index options when considering the term structure of interest rates.

In fact, one of the most notable characteristics of real estate derivatives is the non-tradability of the underlying index, which is different from commodity derivatives and stock derivatives. This characteristic strongly violates the Black–Scholes assumptions and hinders the traditional no-arbitrage approach. With regard to the incompleteness of the real estate market, Case and Shiller [9] discussed index-based futures and options as risk management instruments. Fabozzi et al. [10] argued that the traditional no-arbitrage approach is not applicable to price real estate derivatives because the underlying index cannot be traded. In response to this pricing problem, Geltner and Fisher [4] compared the characteristics of appraisal-based indices and transaction-based indices. They then presented an equilibrium model for pricing real estate forwards and total return swap contracts in consideration of the real estate index characteristics. Cao and Wei [11] proposed another equilibrium valuation framework in an incomplete market. Assuming the presence of a mean-reverting aggregate dividend process, a constant relative risk aversion utility function, and a geometric Brownian motion for real estate index movement, the closed-form of forward and European options were obtained. A new method for pricing real estate derivatives was then developed by Fabozzi et al. [12]. Fabozzi et al. [12] considered the econometric properties of real estate indices and the incompleteness of real estate market by using the real estate futures market to complete this market. Assuming the market price of the real estate index risk is known, the closed-form of futures, the European options, and the total return swap contingent on a real estate index were obtained.

According to an informal survey of 70 commercial real estate investors in the PRMIA webinar on February 27, 2013 [3], investors need real estate derivatives with long term to maturity. The survey indicated that 20% of the respondents expected to trade IPD annual UK index futures annual maturities up to 10 years. About 27% expected to trade annual maturities up to 20 years, whereas 53% expected to trade semi-annual maturities up to 20 years. In this case, the volatility of interest rate plays an important role in the valuation of real estate derivatives when the maturity date is long.

Hence, in this paper, we consider the real estate index option pricing problem under stochastic interest rates and investigate how constant interest rate models lead to the mispricing of European and American options. With regard to this question, the partial differential equation (hereafter PDE) contingent on a real estate index and a stochastic interest rate is obtained and a modified finite difference method is adopted to solve this two-factor PDE. We find that the effects of stochastic interest rates on European option prices depend on whether the term structure of interest rates is rising or falling. Constant interest rate models overprice option prices when the term structure of interest rates is rising, but these models underprice option prices when the term structure is falling. In addition, the correlation between the real estate index and the interest rate movement has opposite effects on European call and put prices. By decomposing the American option into the corresponding European option and early exercise premium, we find that stochastic interest rates have similar effects on American put prices and European put prices, however, the results still present some different features.

The rest of the paper is organized as follows. Section 2 describes the model formulation of real estate derivatives under stochastic interest rates. Section 3 introduces the finite difference method used in this paper. Section 4 presents the numerical analysis and seeks insight into how stochastic interest rates affect option pricing. Section 5 concludes the paper and gives some suggestions for further research. A detailed discussion of bilinear interpolation is presented in the Appendix.

2. Model formulation

Fabozzi et al. [12] analyzed the commercial and residential real estate indices in the US and UK. They found that real estate indices exhibit a positive autocorrelation in the short term and a negative autocorrelation in the long term. Consequently, a mean reverting stochastic model is proposed to measure the real estate index X_t movement as follows

$$dY_t = \left[\frac{d\psi_t}{dt} - \theta(Y_t - \psi_t)\right]dt + \sigma_1 dW_t^1, \tag{1}$$

where Y_t is the log scale of X_t, $Y_t = \log(X_t)$, ψ_t is the long run mean trend of the real estate index in log scale which can be calibrated through the historical data, θ is the mean-reversion speed parameter, σ_1 is the volatility parameter, and W_t^1 is a Wiener process. The stochastic model shows that the real estate market is not efficient, whereas the drift term exhibits predictability. In a seminal paper, Case and Shiller [13] adopted the repeat sales method to study the efficiency of home prices in US, in which they found the inertia in housing prices.

The stochastic interest rate model considered in this paper is the CIR model [14] as follows

$$dr_t = a(b - r_t)dt + \sigma_2\sqrt{r_t}dW_t^2, \tag{2}$$

where r_t is the instantaneous interest rate, b is the long-term reversion target, a determines the speed of adjustment, σ_2 is the interest rate standard deviation at time t, and W_t^2 is another Wiener process correlated with W_t^1. We assume a constant correlation between W_t^1 and W_t^2, $dW_t^1 dW_t^2 = \rho dt$. The advantages of this model are that the interest rate can never become zero, and the analytic solution for the discount bond is derived.

The method proposed in this paper considers the non-tradability of the real estate index and the key point is to know the market price of real estate index risk. Real estate market is essentially an incomplete market. However this market can be completed with information from any real estate derivative, such as real estate futures. Once we have known how investors value the real estate index risk, then all the other real estate derivatives contingent on the same index can be priced directly under the no-arbitrage framework. For example, in this paper the pricing problem of the real estate index options under stochastic interest rates can be replicated by the real estate futures contingent on the same index and a bond. Or, options can be priced in term of the other options since the real estate index options for different maturity dates share the same market price of risk.

Let $f(t, X_t, r_t)$ be the price of real estate index derivative dependent on a real estate index and a stochastic interest rate. By applying the multivariate version of Ito's lemma, the stochastic process of real estate index derivative is given by

$$\begin{aligned}\mu(Y_t, r_t, t) \;=\; & \frac{1}{f}\left\{\frac{\partial f}{\partial t} + \left[\frac{d\psi_t}{dt} - \theta(Y_t - \psi_t)\right]\frac{\partial f}{\partial Y_t} + a(b - r_t)\frac{\partial f}{\partial r_t} + \frac{1}{2}\sigma_1^2\frac{\partial^2 f}{\partial Y_t^2} + \frac{1}{2}\sigma_2^2 r_t\frac{\partial^2 f}{\partial r_t^2}\right.\\ & \left. + \rho\sigma_1\sigma_2\sqrt{r_t}\frac{\partial^2 f}{\partial Y_t \partial r_t}\right\},\end{aligned} \tag{3}$$

$$\sigma_Y(Y_t, r_t, t) = \frac{\sigma_1}{f}\frac{\partial f}{\partial Y_t}, \qquad \sigma_r(Y_t, r_t, t) = \frac{\sigma_2\sqrt{r_t}}{f}\frac{\partial f}{\partial r_t}. \tag{4}$$

The above model has two stochastic factors. Given that the underlying real estate index and an interest rate are non-tradable assets, we need to construct a riskless hedging portfolio that contains Δ_1, Δ_2, Δ_3 units of derivatives with three different maturities T_1, T_2, T_3. By letting $\prod$ be the value of the portfolio, the rate of return of this portfolio over time dt is denoted as

$$\begin{aligned} d\prod =\; & [\Delta_1\mu(T_1)f_1 + \Delta_2\mu(T_2)f_2 + \Delta_3\mu(T_3)f_3]dt + [\Delta_1\sigma_Y(T_1)f_1 + \Delta_2\sigma_Y(T_2)f_2 + \Delta_3\sigma_Y(T_3)f_3]dW_t^1\\ & + [\Delta_1\sigma_r(T_1)f_1 + \Delta_2\sigma_r(T_2)f_2 + \Delta_3\sigma_r(T_3)f_3]dW_t^2. \end{aligned} \tag{5}$$

Suppose Δ_1, Δ_2, Δ_3 let the stochastic terms to be zero. These values lead to the three equations below

$$\begin{aligned}&\Delta_1\sigma_Y(T_1)f_1 + \Delta_2\sigma_Y(T_2)f_2 + \Delta_3\sigma_Y(T_3)f_3 = 0\\ &\Delta_1\sigma_r(T_1)f_1 + \Delta_2\sigma_r(T_2)f_2 + \Delta_3\sigma_r(T_3)f_3 = 0\\ &[\Delta_1\mu(T_1)f_1 + \Delta_2\mu(T_2)f_2 + \Delta_3\mu(T_3)f_3]dt = r_t(\Delta_1 f_1 + \Delta_2 f_2 + \Delta_3 f_3)dt.\end{aligned}$$

Namely,

$$\begin{bmatrix}\sigma_Y(T_1)f_1 & \sigma_Y(T_2)f_2 & \sigma_Y(T_3)f_3\\ \sigma_r(T_1)f_1 & \sigma_r(T_2)f_2 & \sigma_r(T_3)f_3\\ (\mu(T_1) - r_t)f_1 & (\mu(T_2) - r_t)f_2 & (\mu(T_3) - r_t)f_3\end{bmatrix}\begin{bmatrix}\Delta_1\\ \Delta_2\\ \Delta_3\end{bmatrix} = 0. \tag{6}$$

Given the no-arbitrary principle, the above equation possesses nontrivial solutions, where the last row of the equation is the linear combination of the first two rows. This equation leads to the following relationship

$$\mu(Y_t, r_t, t) - r_t = \lambda(Y_t, t)\sigma_Y(Y_t, r_t, t) + q(r_t, t)\sigma_r(Y_t, r_t, t), \tag{7}$$

where $\lambda(Y_t, t)$ and $q(r_t, t)$ are the market price of the real estate index and interest rate risk. In the following section, we simply write it as λ and q, respectively. Substituting Eqs. (3) and (4) into Eq. (7), we can obtain the following PDE for derivatives dependent on a real estate index and a stochastic interest rate

$$\begin{aligned} &-\frac{\partial f}{\partial t} - \left[\frac{d\psi_t}{dt} - \lambda\sigma_1 - \theta(Y_t - \psi_t)\right]\frac{\partial f}{\partial Y_t} - [a(b - r_t) - q\sigma_2\sqrt{r_t}]\frac{\partial f}{\partial r_t} - \frac{1}{2}\sigma_1^2\frac{\partial^2 f}{\partial Y_t^2} \\ &-\frac{1}{2}\sigma_2^2 r_t\frac{\partial^2 f}{\partial r_t^2} - \rho\sigma_1\sigma_2\sqrt{r_t}\frac{\partial^2 f}{\partial Y_t\partial r_t} + r_t f = 0. \end{aligned} \tag{8}$$

This paper focuses on the option pricing problem. Hence, we must solve the PDE above, subject to appropriate terminal and boundary conditions. The specific terminal and boundary conditions for European and American options show below.

2.1. European option pricing

For ease of understanding, we can translate Eq. (8) into the following PDE about X_t

$$\begin{aligned} &-\frac{\partial f}{\partial t} - \left[\frac{d\psi_t}{dt} - \theta(\log(X_t) - \psi_t) + \frac{1}{2}\sigma_1^2 - \lambda\sigma_1\right]X_t\frac{\partial f}{\partial X_t} - [a(b - r_t) - q\sigma_2\sqrt{r_t}]\frac{\partial f}{\partial r_t} \\ &-\frac{1}{2}X_t^2\sigma_1^2\frac{\partial^2 f}{\partial X_t^2} - \frac{1}{2}\sigma_2^2 r_t\frac{\partial^2 f}{\partial r_t^2} - \rho X_t\sqrt{r_t}\sigma_1\sigma_2\frac{\partial^2 f}{\partial X_t\partial r_t} + r_t f = 0. \end{aligned} \tag{9}$$

For notional convenience, we define differential operator L to enable Eq. (9) to be expressed as $Lf = 0$. The terminal conditions for European call and put options are the payoffs

$$f_c(T, X_t, r_t) = g_c(X_t, r_t) = (X_T - K)^+, \tag{10}$$

$$f_p(T, X_t, r_t) = g_p(X_t, r_t) = (K - X_T)^+, \tag{11}$$

where T is the maturity time and K is the strike price.

To solve Eq. (9), the following boundary conditions are needed. The boundary conditions for call option are

$$f_c(t, 0, r_t) = 0, \tag{12}$$

$$\lim_{x\to\infty} f_c(t, X_t, r_t) = (X_t - K)e^{-P(t,T)(T-t)}, \tag{13}$$

$$\lim_{r\to\infty} \frac{\partial f_c}{\partial r_t} = 0, \tag{14}$$

$$\frac{\partial f}{\partial t} + \left[\frac{d\psi_t}{dt} - \theta(\log(X_t) - \psi_t) + \frac{1}{2}\sigma_1^2 - \lambda\sigma_1\right]X_t\frac{\partial f}{\partial X_t} + ab\frac{\partial f}{\partial r_t} + \frac{1}{2}X_t^2\sigma_1^2\frac{\partial^2 f}{\partial X_t^2} = 0 \quad \text{at } r_t = 0. \tag{15}$$

The boundary conditions for put option are

$$f_p(t, 0, r_t) = Ke^{-P(t,T)(T-t)}, \tag{16}$$

$$\lim_{x\to\infty} f_p(t, X_t, r_t) = 0, \tag{17}$$

$$\lim_{r\to\infty} \frac{\partial f_p}{\partial r_t} = 0, \tag{18}$$

$$\frac{\partial f}{\partial t} + \left[\frac{d\psi_t}{dt} - \theta(\log(X_t) - \psi_t) + \frac{1}{2}\sigma_1^2 - \lambda\sigma_1\right]X_t\frac{\partial f}{\partial X_t} + ab\frac{\partial f}{\partial r_t} + \frac{1}{2}X_t^2\sigma_1^2\frac{\partial^2 f}{\partial X_t^2} = 0 \quad \text{at } r_t = 0. \tag{19}$$

The boundary conditions (12), (13), (16) and (17) become obvious for call and put options when the real estate index tends to be zero or extremely large, while boundary conditions (15) and (19) are obtained by $r_t = 0$. In Eqs. (13) and (16), $P(t, T)$ is zero-coupon interest rate obtained from the closed-form of the CIR model. Eqs. (14) and (18) show that when the interest rate is extremely large, a marginal increase in the interest rate has little effects on call and put option prices.

2.2. American option pricing

In the absence of cash flows of the real estate index, only American put option pricing is considered in this paper. American valuation problem is a free-boundary problem [15–18]. Thus, a popular approach to solve this free-boundary PDE is to reformulate it as a linear complementarity problem (LCP). Owing to the early exercise possibility of the American put option, the exercise constraint is $f(t, X_t, r_t) \geq (K - X_t)^+$. In the region where the exercise constraint is inactive, $f(t, X_t, r_t)$ satisfies $Lf = 0$. By combining the relations above, the LCP of American put option becomes

$$\begin{cases} Lf(t, X_t, r_t) \geq 0, & f(t, X_t, r_t) \geq g_p(X_t, r_t) \\ Lf(t, X_t, r_t)(f(t, X_t, r_t) - g_p(X_t, r_t)) = 0, & \end{cases}, \tag{20}$$

To solve Eq. (20) with terminal condition (11) and boundary conditions (17), (18), and (19), we impose another condition $f(t, 0, r_t) = K$ when the real estate index tends to be zero.

3. A modified finite difference method

A number of numerical methods can be used to price options which have no analytic formulas. Finite difference method is one of the most popular methods to solve this problem. For instance, Gong et al. [19] used the finite difference method to solve the convertible bonds based on a multi-stage compound-option model. Dibeh et al. [20] adopted the finite difference method to price European options in a post-crash environment and Ankudinova et al. [21] used this method in several transaction cost models.

In this paper, we apply a modified finite difference method for the discretization of Eq. (9) based on the Euler forward time stepping method [22–26] and a non-uniform grid method [27] in real estate index axis. The implicit finite difference method is conventionally considered superior to the explicit finite difference method for option pricing. This is mainly because the explicit method is confronted with the severe stability constraint, which causes the conditional method to become stable only when the time step is sufficiently small. However, the transformation matrix in Eq. (9) which is dependent on time t at each iteration increases the difficulty of implementing the implicit method or modified implicit method, such as ADI method [28]. Hence, the Euler forward method is used in this paper to solve Eq. (9) numerically.

For the sake of numerical implementation, the real estate index axis, interest rate axis, and time axis in this section are limited to $[0, X_{\max}] \times [0, r_{\max}] \times [T_0, T]$. A finite difference approximation is constructed on the grid

$$(X_t, r_t, t) \in [0 = X_0, \ldots, X_m = X_{\max}] \times [0 = r_0, \ldots, r_n = r_{\max}] \times [T_0 = t_0, \ldots, t_l = T]. \tag{21}$$

The real estate index axis, interest rate axis, and time axis are discretized into m, n and l pieces. For $i = 0, \ldots, m$, $j = 0, \ldots, n$, $k = 0, \ldots, l$, the option price at grid (i, j, k) is $f_{i,j}^k = f(X_i, r_j, t_k)$.

We apply a non-uniform grid method in the X_t direction to improve the accuracy of the finite difference. Compared with uniform grids method, non-uniform grids cause more grids to lie in the neighborhood of the strike price K, $K \in [X_{left}, X_{right}] \in [0, X_{\max}]$. The application of such non-uniform grids is attributed to the payoff of the option, which is not discontinuous at strike price K. Thus, it is natural to have more grids near the strike price because this is the domain of interest in applications. The type of non-uniform grids employed in this paper refers to Haentjens [27]. By letting $m > 0$ and $d > 0$, the equidistant points $\xi_{\min} = \xi_0 < \xi_1 < \cdots < \xi_m = \xi_{\max}$ are defined by

$$\xi_{\min} = \sinh^{-1}\left(\frac{X_{left}}{d}\right), \tag{22}$$

$$\xi_{\text{int}} = \frac{X_{right} - X_{left}}{d}, \tag{23}$$

$$\xi_{\max} = \xi_{\text{int}} + \sinh^{-1}\left(\frac{X_{\max} - X_{right}}{d}\right). \tag{24}$$

Grids $0 = X_0, \ldots, X_m = X_{\max}$ then transform into

$$X_i = \varphi(\xi_i)(0 \leq i \leq m), \tag{25}$$

where

$$\varphi(\xi) = \begin{cases} X_{left} + d\sinh(\xi) & \xi_{\min} \leq \xi < 0 \\ X_{left} + d\xi & 0 \leq \xi \leq \xi_{\text{int}} \\ X_{right} + d\sinh(\xi - \xi_{\text{int}}) & \xi_{\text{int}} < \xi \leq \xi_{\max}. \end{cases} \tag{26}$$

This transformation makes the grid uniform inside the interval $[X_{left}, X_{right}]$, whereas grids are non-uniform outside. d as a scaling parameter, controls the density of grids that lie inside the interval. In addition, this transformation may result in the little chance that the real estate index and the interest rate that determined beforehand is a grid point, thus a bilinear interpolation is applied to solve this problem in the Appendix.

Then a finite difference discretization, which applies a non-uniform grid in X_t direction, a uniform grid in r_t direction based on central differences and a uniform grid in t direction, are obtained as follows

$$\frac{\partial f}{\partial t} = \frac{f_{i,j}^k - f_{i,j}^{k-1}}{\Delta t},$$

$$\frac{\partial f}{\partial X} = \alpha_{-1} f^k_{i-1,j} + \alpha_0 f^k_{i,j} + \alpha_1 f^k_{i+1,j},$$
$$\frac{\partial f}{\partial r} = \frac{f^k_{i,j+1} - f^k_{i,j-1}}{2\Delta r},$$
$$\frac{\partial^2 f}{\partial X^2} = \beta_{-1} f^k_{i-1,j} + \beta_0 f^k_{i,j} + \beta_1 f^k_{i+1,j},$$
$$\frac{\partial^2 f}{\partial r^2} = \frac{f^k_{i,j+1} + f^k_{i,j-1} - 2f^k_{i,j}}{\Delta r^2},$$
$$\frac{\partial f}{\partial X \partial r} = \frac{-\alpha_{-1}}{2\Delta r} f^k_{i-1,j-1} + \frac{\alpha_{-1}}{2\Delta r} f^k_{i-1,j+1} + \frac{-\alpha_0}{2\Delta r} f^k_{i,j-1} + \frac{\alpha_0}{2\Delta r} f^k_{i,j+1} + \frac{-\alpha_1}{2\Delta r} f^k_{i+1,j-1} + \frac{\alpha_1}{2\Delta r} f^k_{i+1,j+1},$$

with

$$\Delta X_i = X_i - X_{i-1}, \qquad \Delta r = \frac{r_{\max}}{n}, \qquad \Delta t = \frac{T - T_0}{l},$$
$$\alpha_{-1} = \frac{-\Delta X_{i+1}}{\Delta X_i(\Delta X_i + \Delta X_{i+1})}, \qquad \alpha_0 = \frac{\Delta X_{i+1} - \Delta X_i}{\Delta X_i \Delta X_{i+1}}, \qquad \alpha_1 = \frac{\Delta X_i}{\Delta X_{i+1}(\Delta X_i + \Delta X_{i+1})},$$
$$\beta_{-1} = \frac{2}{\Delta X_i(\Delta X_i + \Delta X_{i+1})}, \qquad \beta_0 = \frac{-2}{\Delta X_i \Delta X_{i+1}}, \qquad \beta_1 = \frac{2}{\Delta X_{i+1}(\Delta X_i + \Delta X_{i+1})}.$$

Eq. (9) can be transformed into

$$f^{k-1}_{i,j} = D^1_{i,j} f^k_{i-1,j-1} + D^2_j f^k_{i,j-1} + D^3_{i,j} f^k_{i+1,j-1} + D^4_{i,j} f^k_{i-1,j} + D^5_{i,j} f^k_{i,j} + D^6_{i,j} f^k_{i+1,j} + D^7_{i,j} f^k_{i-1,j+1} + D^8_j f^k_{i,j+1} + D^9_{i,j} f^k_{i+1,j+1}, \tag{27}$$

where

$$D^1_{i,j} = \frac{\rho\sigma_1\sigma_2 X_i \Delta X_{i+1} \Delta t \sqrt{j}}{2\sqrt{\Delta r}\,\Delta X_i(\Delta X_i + \Delta X_{i+1})},$$
$$D^2_{i,j} = \frac{\sigma_2^2 \Delta t j}{2\Delta r} - \frac{\rho\sigma_1\sigma_2 X_i(\Delta X_{i+1} - \Delta X_i)\Delta t\sqrt{j}}{2\sqrt{\Delta r}\,\Delta X_i \Delta X_{i+1}} - \frac{a(b - \Delta r j) - q\sigma_2\sqrt{j}}{2\sqrt{\Delta r}},$$
$$D^3_{i,j} = -\frac{\rho\sigma_1\sigma_2 X_i \Delta X_i \Delta t\sqrt{j}}{2\sqrt{\Delta r}\,\Delta X_{i+1}(\Delta X_i + \Delta X_{i+1})},$$
$$D^4_{i,j} = \frac{\sigma_1^2 X_i^2 \Delta t}{\Delta X_i(\Delta X_i + \Delta X_{i+1})} - \frac{[\beta - \theta(\log(X_i) - \alpha - \beta\Delta t k) - \lambda\sigma_1 + 0.5\sigma_1^2] X_i \Delta X_{i+1} \Delta t}{\Delta X_i(\Delta X_i + \Delta X_{i+1})},$$
$$D^5_{i,j} = 1 + \frac{[\beta - \theta(\log(X_i) - \alpha - \beta\Delta t k) - \lambda\sigma_1 + 0.5\sigma_1^2] X_i(\Delta X_{i+1} - \Delta X_i)\Delta t}{\Delta X_i \Delta X_{i+1}} - \Delta r \Delta t j - \frac{\sigma_2^2 \Delta t j}{\Delta r} - \frac{\sigma_1^2 X_i^2 \Delta t}{\Delta X_i \Delta X_{i+1}},$$
$$D^6_{i,j} = \frac{\sigma_1^2 X_i^2 \Delta t}{\Delta X_i(\Delta X_i + \Delta X_{i+1})} + \frac{[\beta - \theta(\log(X_i) - \alpha - \beta\Delta t k) - \lambda\sigma_1 + 0.5\sigma_1^2] X_i \Delta X_i \Delta t}{\Delta X_{i+1}(\Delta X_i + \Delta X_{i+1})},$$
$$D^7_{i,j} = -\frac{\rho\sigma_1\sigma_2 X_i \Delta X_{i+1} \Delta t\sqrt{j}}{2\sqrt{\Delta r}\,\Delta X_i(\Delta X_i + \Delta X_{i+1})},$$
$$D^8_{i,j} = \frac{\sigma_2^2 \Delta t j}{2\Delta r} + \frac{\rho\sigma_1\sigma_2 X_i(\Delta X_{i+1} - \Delta X_i)\Delta t\sqrt{j}}{2\sqrt{\Delta r}\,\Delta X_i \Delta X_{i+1}} + \frac{a(b - \Delta r j) - q\sigma_2\sqrt{j}}{2\sqrt{\Delta r}},$$
$$D^9_{i,j} = \frac{\rho\sigma_1\sigma_2 X_i \Delta X_i \Delta t\sqrt{j}}{2\sqrt{\Delta r}\,\Delta X_{i+1}(\Delta X_i + \Delta X_{i+1})}.$$

The boundary condition at $r_t = 0$ can be discretized into

$$f^{k-1}_{i,j} = E^1_{i,j} f^k_{i-1,j} + E^2_{i,j} f^k_{i,j} + E^3_{i,j} f^k_{i+1,j} + E^4 f^k_{i,j+1}, \tag{28}$$

where

$$E^1_{i,j} = D^4_{i,j}, \qquad E^3_{i,j} = D^6_{i,j}, \qquad E^4 = \frac{ab}{\Delta r},$$
$$E^2_{i,j} = 1 + \frac{[\beta - \theta(\log(X_i) - \alpha - \beta\Delta t k) - \lambda\sigma_1 + 0.5\sigma_1^2] X_i(\Delta X_{i+1} - \Delta X_i)\Delta t}{\Delta X_i \Delta X_{i+1}} - \frac{\sigma_1^2 X_i^2 \Delta t}{\Delta X_i \Delta X_{i+1}} - \frac{ab}{\Delta r}.$$

By specifically dealing with the terminal and boundary conditions, the implementation of the explicit method becomes straightforward. Using the relationship above, we can work step-by-step back down the grid at time t.

Table 1
IPD annual UK index parameters.

α	β	θ	σ_1	λ
0.7771	0.1045	0.1165	0.131	0.7

Table 2
Stochastic interest rate model parameters.

a	b	σ_2	q
0.3	0.05	0.1	0

4. Numerical results

Since real estate derivatives based on the IPD annual UK index are actively traded in the market, we take this index as an example to price real estate index options under the pricing framework in this paper. The parameters regarding the IPD annual UK index movement refer to the empirical analysis in Fabozzi et al. [12] and the specific parameters are provided in Table 1, where the long run mean of IPD UK index is $\psi_t = \alpha + \beta t$, θ controls the mean-reversion speed, σ_1 is the volatility parameter. The market price of risk λ can be obtained from the real estate futures traded in Eurex which reflects investors' attitude to the real estate market as the time change and we simply assume that it is constant in this paper. Moreover, the strike price is assumed to be 1500 and $t_0 = 65$ in the numerical analysis below.

The stochastic interest rate model parameters refer to Medvedev and Scaillet [29] and the specific parameters are as in Table 2 where a is the mean-reversion speed parameter, b is long-term average, σ_2 is the interest rate volatility, q is the market price of interest rate risk and assumed to be $q = 0$.

4.1. European option pricing under constant interest rates

Analytic solutions of real estate index options under constant interest rates can be obtained in Fabozzi et al. [12]. However, we still highlight the importance of this part because it would be insightful to understand the characteristics of real estate index options compared with other options, such as stock options. The results provide as a benchmark in studying the European option pricing under stochastic interest rates and the American option pricing problem. The outcomes can also test the validity and accuracy of the modified finite difference method in this paper. Tables 3 and 4 give the numerical results of call and put options under constant interest rates when time to maturity is 1 and 2 years, respectively. The parameters in the finite difference method are $K = 1500$, $X_{left} = 1200$, $X_{right} = 1800$, $X_{\max} = 6000$, $r_{\max} = 0.2$, $d = 40$. Variable d controls the density of the grids in the domain near the strike price.

From Tables 3 and 4, we find that the option prices obtained by the finite difference method converge to the analytic solutions, wherein the absolute errors are in the range of 10^{-3} to 10^{-5} and the absolute error is largest at the strike price K due to discontinuous of payoff of the option. Compared with stock options, real estate index options under constant interest rates exhibit some different characteristics. For the stock options, each increase of the interest rate will increase the call option price, but will decrease the put option price. However, European call and put options contingent on the real estate index will exhibit lower prices with each interest rate increases. Moreover, numerical results reveal that as the market price of the real estate index risk increases, the European call prices decrease while the European put prices increase. This is because the market price of risk is the excess return that the market wants as compensation for risk taking and a high market price of the real estate index risk means investors maintain a pessimistic outlook on the future housing price, thus investors pay less for the call options while pay more for the put options, and vice versa.

4.2. European option pricing under stochastic interest rates

Due to the long term to maturity of real estate derivatives, we price real estate index options under stochastic interest rates in this section. The market price of real estate index risk is assumed to be 0.7 below and the stochastic interest rate model parameters refer to Table 2.

Tables 5 and 6 indicate that when the real estate index and interest rates are uncorrelated, constant interest rate models yield higher option prices compared with stochastic interest rate models for both European call and put options when the term structure of interest rates rises, because such models do not explain the likelihood of interest rate rising. By contrast, when the term structure of interest rates is falling, constant interest rate models yield lower option prices since stochastic interest rate models do explain such likelihood. These price differences are due to the mean-reverting nature of stochastic interest rates. The case of the initial interest rate being equal to its long-term mean is slightly different. We find that stochastic interest rate models yield higher prices, because interest rate volatility increases the option prices. These results can also be seen in Figs. 1 and 2 directly. Figs. 1 and 2 reveal the European call and put prices at different real estate index and initial interest rate levels under both constant and stochastic interest rates when the maturity date is 2 years.

Table 3
Numerical results and true values for call option under constant interest rates.

Time to maturity	$T = 1$					$T = 2$				
Real estate index	1400	1450	1500	1550	1600	1400	1450	1500	1550	1600
Market price of real estate risk: $\lambda = 0.4$										
$r_{t0} = 0.03$	96.7111	128.3356	163.8559	202.5376	243.6435	224.1075	261.4820	300.3389	340.3408	381.1911
	96.7119	**128.3364**	**163.8578**	**202.5381**	**243.6437**	**224.1160**	**261.4873**	**300.3473**	**340.3493**	**381.1971**
$r_{t0} = 0.05$	94.7961	125.7944	160.6113	198.5271	238.8190	215.3226	251.2287	288.5665	327.0022	366.2487
	94.7969	**125.7952**	**160.6132**	**198.5276**	**238.8193**	**215.3282**	**251.2343**	**288.5705**	**327.0041**	**366.2501**
$r_{t0} = 0.07$	92.9190	123.3035	157.4310	194.5960	234.0901	206.8796	241.3788	277.2501	314.1768	351.8805
	92.9198	**123.3043**	**157.4328**	**194.5965**	**234.0903**	**206.8851**	**241.3832**	**277.2555**	**314.1820**	**351.8893**
Market price of real estate risk: $\lambda = 0.7$										
$r_{t0} = 0.03$	66.1456	91.3737	120.8636	154.0862	190.3970	143.7653	173.1608	204.6480	237.9003	272.6010
	66.1456	**91.3738**	**120.8652**	**154.0862**	**190.3969**	**143.7652**	**173.1602**	**204.6472**	**237.8972**	**272.5951**
$r_{t0} = 0.05$	64.8359	89.5645	118.4703	151.0351	186.6269	138.1282	166.3711	196.6237	228.5722	261.9122
	64.8358	**89.5645**	**118.4719**	**151.0351**	**186.6268**	**138.1280**	**166.3704**	**196.6228**	**228.5691**	**261.9065**
$r_{t0} = 0.07$	63.5520	87.7909	116.1245	148.0444	182.9315	132.7122	159.8476	188.9140	219.6098	251.6425
	63.5520	**87.7910**	**116.1260**	**148.0444**	**182.9314**	**132.7120**	**159.8470**	**188.9131**	**219.6068**	**251.6370**
Market price of real estate risk: $\lambda = 1$										
$r_{t0} = 0.03$	42.9390	62.0009	85.3162	112.6458	143.5496	84.2458	105.3585	128.8454	154.4993	182.0790
	42.9386	**62.0005**	**85.3175**	**112.6455**	**143.5493**	**84.2451**	**105.3573**	**128.8440**	**154.4951**	**182.0708**
$r_{t0} = 0.05$	42.0888	60.7732	83.6268	110.4153	140.7071	80.9425	101.2273	123.7933	148.4414	174.9396
	42.0884	**60.7729**	**83.6281**	**110.4150**	**140.7068**	**80.9418**	**101.2262**	**123.7920**	**148.4372**	**174.9317**
$r_{t0} = 0.07$	41.2553	59.5698	81.9709	108.2289	137.9210	77.7687	97.2582	118.9393	142.6209	168.0801
	41.2549	**59.5695**	**81.9722**	**108.2286**	**137.9206**	**77.7680**	**97.2570**	**118.9380**	**142.6169**	**168.0726**

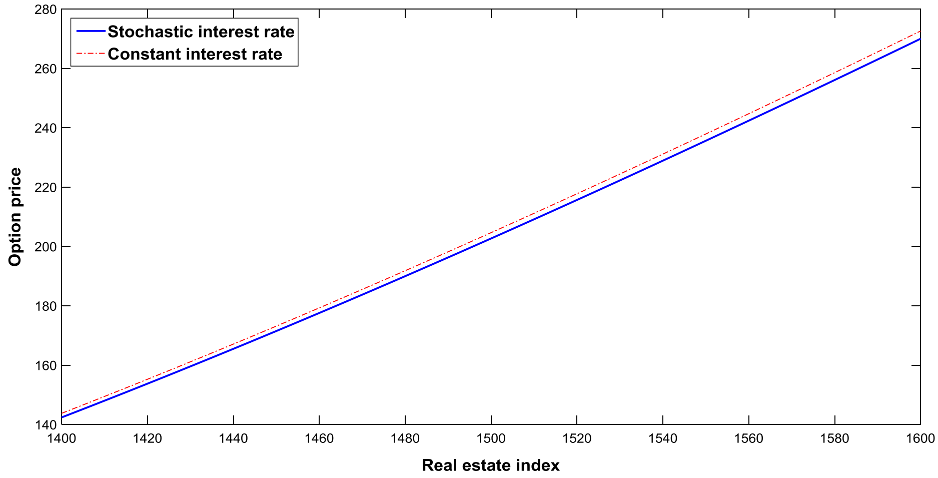
Notes: This table compares the numerical results of call option based on the numerical method in this paper with analytic solutions in [12] (in bold) under different market price of real estate index risks and initial interest rates. The parameters of the real estate index are in Table 1. The strike price is assumed to be 1500, the real estate index ranges from 1400 to 1600 and T is time to maturity, supposing 1 year or 2 years. r_{t0} denotes the initial interest rate.

Table 4
Numerical results and true values for put option under constant interest rates.

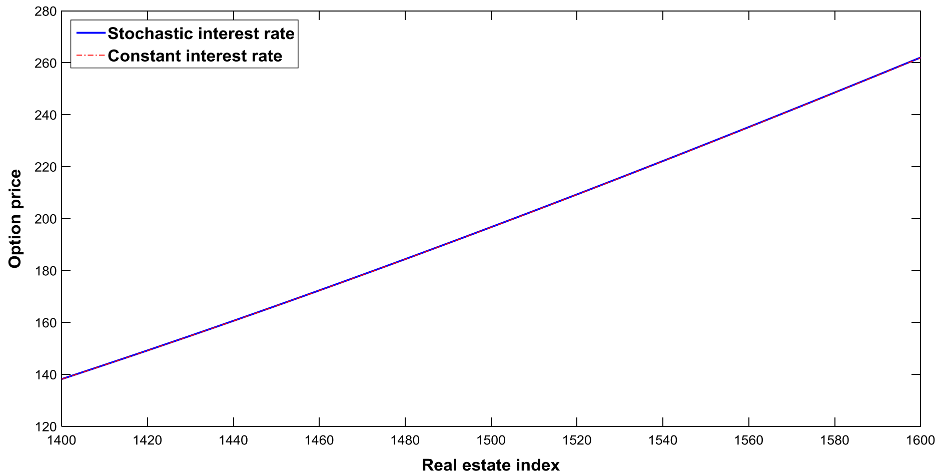
Time to maturity	$T = 1$					$T = 2$				
Real estate index	1400	1450	1500	1550	1600	1400	1450	1500	1550	1600
Market price of real estate risk: $\lambda = 0.4$										
$r_{t0} = 0.03$	53.1592	37.2207	25.3579	16.8302	10.8937	31.4282	23.5492	17.4815	12.8673	9.3964
	53.1605	**37.2221**	**25.3605**	**16.8312**	**10.8945**	**31.4302**	**23.5510**	**17.4836**	**12.8685**	**9.3973**
$r_{t0} = 0.05$	52.1065	36.4837	24.8558	16.4969	10.6780	30.1958	22.6258	16.7961	12.3628	9.0279
	52.1079	**36.4850**	**24.8583**	**16.4979**	**10.6788**	**30.1978**	**22.6275**	**16.7980**	**12.3639**	**9.0289**
$r_{t0} = 0.07$	51.0747	35.7613	24.3636	16.1703	10.4666	29.0118	21.7386	16.1375	11.8780	8.6739
	51.0761	**35.7626**	**24.3661**	**16.1712**	**10.4673**	**29.0137**	**21.7403**	**16.1394**	**11.8791**	**8.6748**
Market price of real estate risk: $\lambda = 0.7$										
$r_{t0} = 0.03$	77.1916	56.5888	40.4212	28.1535	19.1352	59.7803	46.9888	36.5899	28.2450	21.6251
	77.1920	**56.5894**	**40.4233**	**28.1540**	**19.1356**	**59.7815**	**46.9900**	**36.5918**	**28.2459**	**21.6258**
$r_{t0} = 0.05$	75.6630	55.4683	39.6208	27.5961	18.7563	57.4363	45.1464	35.1552	27.1375	20.7771
	75.6635	**55.4688**	**39.6229**	**27.5966**	**18.7567**	**57.4374**	**45.1475**	**35.1570**	**27.1383**	**20.7779**
$r_{t0} = 0.07$	74.1648	54.3699	38.8363	27.0496	18.3849	55.1841	43.3761	33.7768	26.0734	19.9625
	74.1653	**54.3705**	**38.8383**	**27.0501**	**18.3853**	**55.1853**	**43.3772**	**33.7785**	**26.0742**	**19.9632**
Market price of real estate risk: $\lambda = 1$										
$r_{t0} = 0.03$	106.5945	81.4945	60.8151	44.3111	31.5365	101.6025	83.3847	67.8215	54.6942	43.7499
	106.5945	**81.4946**	**60.8169**	**44.3113**	**31.5366**	**101.6030**	**83.3852**	**67.8229**	**54.6946**	**43.7502**
$r_{t0} = 0.05$	104.4838	79.8809	59.6109	43.4337	30.9121	97.6186	80.1152	65.1622	52.5496	42.0344
	104.4838	**79.8809**	**59.6127**	**43.4338**	**30.9122**	**97.6191**	**80.1156**	**65.1635**	**52.5500**	**42.0348**
$r_{t0} = 0.07$	102.4149	78.2991	58.4306	42.5737	30.3000	93.7909	76.9738	62.6071	50.4891	40.3862
	102.4149	**78.2992**	**58.4323**	**42.5738**	**30.3001**	**93.7914**	**76.9743**	**62.6085**	**50.4895**	**40.3866**

Notes: This table compares the numerical results of put option based on the numerical method in this paper with analytic solutions in [12] (in bold) under different market price of real estate index risks and initial interest rates. The parameters of the real estate index are in Table 1. The strike price is assumed to be 1500, the real estate index ranges from 1400 to 1600 and T is time to maturity, supposing 1 year or 2 years. r_{t0} denotes the initial interest rate.

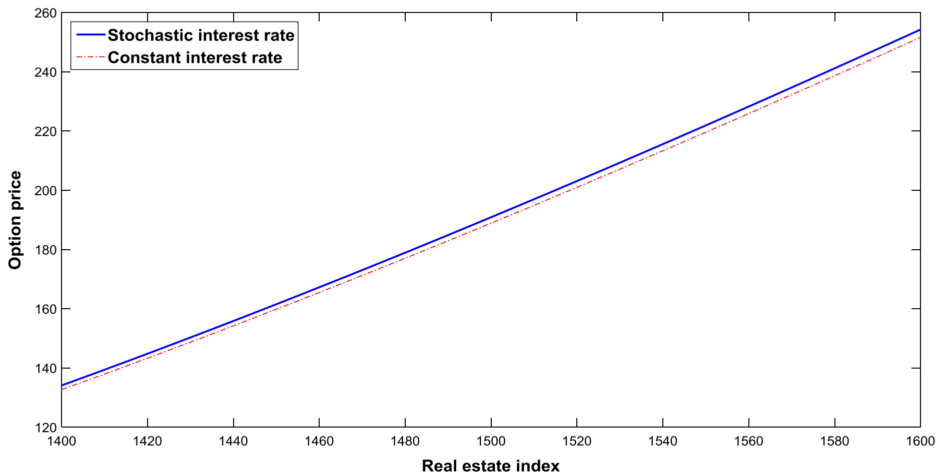
It is no surprise that European call and put options exhibit similar features under stochastic interest rates, because the effects of interest rate variations on European call and put prices are equal in sign under constant interest rate models. In the sight of time to maturity, we find that stochastic interest rate models have more significant effects on option prices under longer maturity, which further emphasizes the importance of considering the stochastic interest rates on option pricing due to the longer maturity date of real estate derivatives compared with stock derivatives or commodity derivatives.



(a) $r_{t0} = 0.03$.

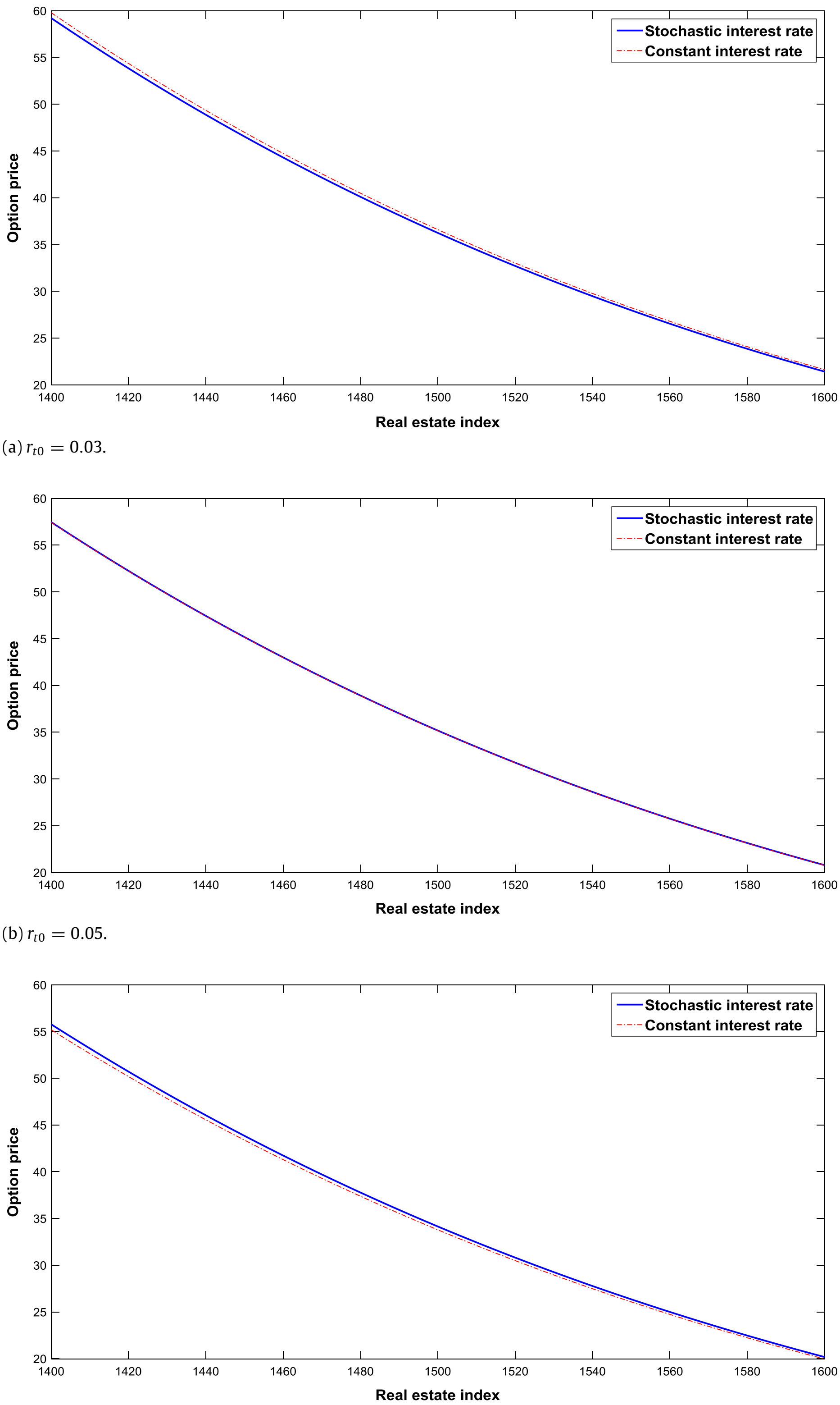


(b) $r_{t0} = 0.05$.



(c) $r_{t0} = 0.07$.

Fig. 1. European call prices at different initial interest rates when $T = 2$ ($\rho = 0$).



(a) $r_{t0} = 0.03$.

(b) $r_{t0} = 0.05$.

(c) $r_{t0} = 0.07$.

Fig. 2. European put prices at different initial interest rates when $T = 2$ ($\rho = 0$).

Table 5
European call option under stochastic interest rates.

Time to maturity	$T=1$					$T=2$				
Real estate index	1400	1450	1500	1550	1600	1400	1450	1500	1550	1600
Constant interest rate model										
$r_{t0}=0.03$	66.1456	91.3738	120.8652	154.0862	190.3969	143.7652	173.1602	204.6472	237.8972	272.5951
$r_{t0}=0.05$	64.8358	89.5645	118.4719	151.0351	186.6268	138.1280	166.3704	196.6228	228.5691	261.9065
$r_{t0}=0.07$	63.5520	87.7910	116.1260	148.0444	182.9314	132.7120	159.8470	188.9131	219.6068	251.6370
Stochastic interest rate model										
When $\rho=-0.3$										
$r_{t0}=0.03$	66.1762	91.3845	120.8469	154.0241	187.8405	143.3567	172.5866	203.8837	236.9185	269.1017
$r_{t0}=0.05$	65.0970	89.8867	118.8573	151.4782	184.7256	139.3449	167.7384	198.1366	230.2197	261.4731
$r_{t0}=0.07$	64.0266	88.4025	116.8873	148.9594	181.6458	135.4127	162.9898	192.5113	223.6666	254.0142
When $\rho=0$										
$r_{t0}=0.03$	65.9695	91.1300	120.5429	153.6750	187.4542	142.3873	171.5004	202.6857	235.6169	267.7109
$r_{t0}=0.05$	64.8410	89.5712	118.4809	151.0463	184.2476	138.1888	166.4434	196.7091	228.6693	259.8168
$r_{t0}=0.07$	63.7319	88.0390	116.4542	148.4625	181.0959	134.1144	161.5359	190.9092	221.9271	252.1559
When $\rho=0.3$										
$r_{t0}=0.03$	65.7565	90.8653	120.2310	153.3185	187.0594	141.3767	170.3719	201.4479	234.2772	266.2829
$r_{t0}=0.05$	64.5790	89.2457	118.0969	150.6071	183.7613	136.9929	165.1077	195.2433	227.0821	258.1248
$r_{t0}=0.07$	63.4314	87.6660	116.0138	147.9586	180.5380	132.7786	160.0435	189.2709	220.1527	250.2642

Notes: This table reports numerical results of call option under stochastic interest rates with different initial interest rates and ρ values. The parameters of the real estate index and stochastic interest rate model are in Tables 1 and 2. The strike price is assumed to be 1500, the real estate index ranges from 1400 to 1600 and T is time to maturity, supposing 1 year or 2 years. r_{t0} denotes the initial interest rate.

Table 6
European put option under stochastic interest rates.

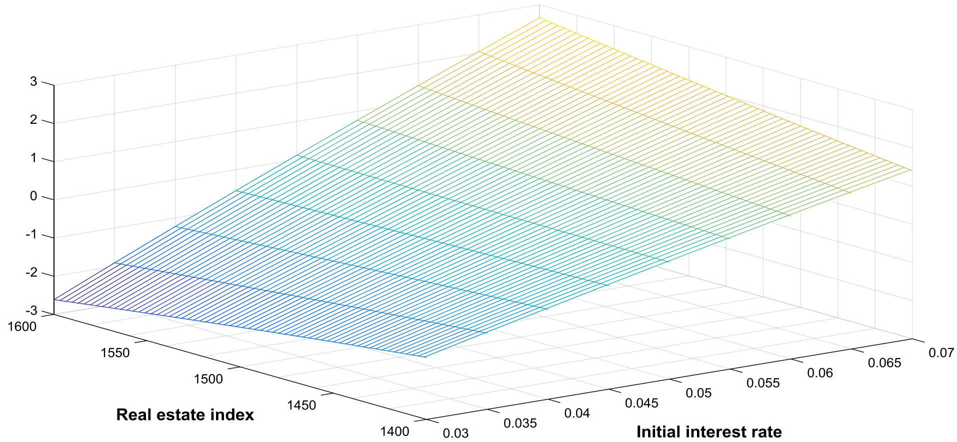
Time to maturity	$T=1$					$T=2$				
Real estate index	1400	1450	1500	1550	1600	1400	1450	1500	1550	1600
Constant interest rate model										
$r_{t0}=0.03$	77.1920	56.5894	40.4233	28.1540	19.1356	59.7815	46.9900	36.5918	28.2459	21.6258
$r_{t0}=0.05$	75.6635	55.4688	39.6229	27.5966	18.7567	57.4374	45.1475	35.1570	27.1383	20.7779
$r_{t0}=0.07$	74.1653	54.3705	38.8383	27.0501	18.3853	55.1853	43.3772	33.7785	26.0742	19.9632
Stochastic interest rate model										
When $\rho=-0.3$										
$r_{t0}=0.03$	76.7742	56.2609	40.1749	27.9703	19.4994	58.6997	46.1058	35.8781	27.6748	21.5490
$r_{t0}=0.05$	75.4081	55.2549	39.4529	27.4650	19.1454	56.8615	44.6553	34.7442	26.7962	20.8620
$r_{t0}=0.07$	74.0752	54.2742	38.7497	26.9734	18.8014	55.0975	43.2647	33.6581	25.9553	20.2049
When $\rho=0$										
$r_{t0}=0.03$	76.9858	56.4381	40.3157	28.0792	19.5833	59.2076	46.5388	36.2402	27.9746	21.7973
$r_{t0}=0.05$	75.6689	55.4726	39.6261	27.5989	19.2483	57.4617	45.1665	35.1716	27.1497	21.1545
$r_{t0}=0.07$	74.3745	54.5237	38.9482	27.1268	18.9191	55.7675	43.8348	34.1346	26.3492	20.5308
When $\rho=0.3$										
$r_{t0}=0.03$	77.1936	56.6074	40.4513	28.1834	19.6617	59.6886	46.9437	36.5780	28.2516	22.0251
$r_{t0}=0.05$	75.9260	55.6827	39.7941	27.7281	19.3458	58.0361	45.6506	35.5755	27.4813	21.4274
$r_{t0}=0.07$	74.6702	54.7658	39.1418	27.2757	19.0315	56.4130	44.3793	34.5891	26.7226	20.8381

Notes: This table reports numerical results of put option under stochastic interest rates with different initial interest rates and ρ values. The parameters of the real estate index and stochastic interest rate model are in Tables 1 and 2. The strike price is assumed to be 1500, the real estate index ranges from 1400 to 1600 and T is time to maturity, supposing 1 year or 2 years. r_{t0} denotes the initial interest rate.

Fig. 3(a) and (b) show the effects of stochastic interest rates on option prices for both call and put options. It is obvious that constant interest rate models overprice option prices when the term structure of interest rates is rising, but underprice option prices when the term structure is falling. Specifically, stochastic interest rate models have larger influences on the in-the-money options compared with out-of-the-money options.

We further analyze the effects of different correlation coefficients ρ between the real estate index and the interest rate movement on option prices. Financial data indicate that $\rho < 0$ and many scholars consider that the low interest rate is an important booster in magnifying the housing bubble in the US and urges the subprime mortgage crisis.

Tables 5 and 6 give the numerical results when the real estate index and interest rates are correlated. As the correlation coefficient increases, call prices decrease while put prices increase. The positive correlation between the underlying index and spot interest rate implies that the statement of a call option with a larger payoff (high underlying index) is discounted more heavily because of a higher interest rate and that the statement of a put option with a larger payoff (low underlying index) is discounted more slightly due to a lower interest rate. On the contrary, the negative correlation between the underlying index and an interest rate has an opposite effect on call and put option prices. As a result, the correlation coefficient parameter has an opposite effect on call and put options. These numerical results have different features from



(a) European call.

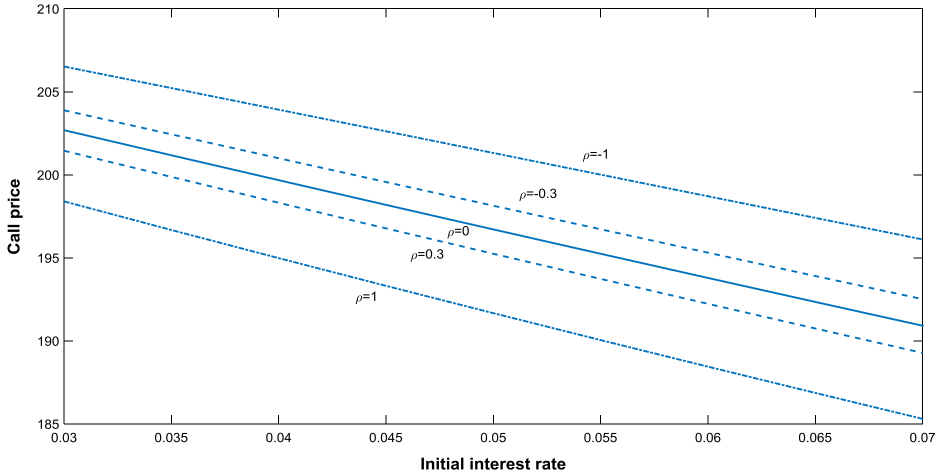
(b) European put.

Fig. 3. Price difference (stochastic interest rate–constantinterest rate, $T = 2$) at different real estate index and initial interest rate levels.

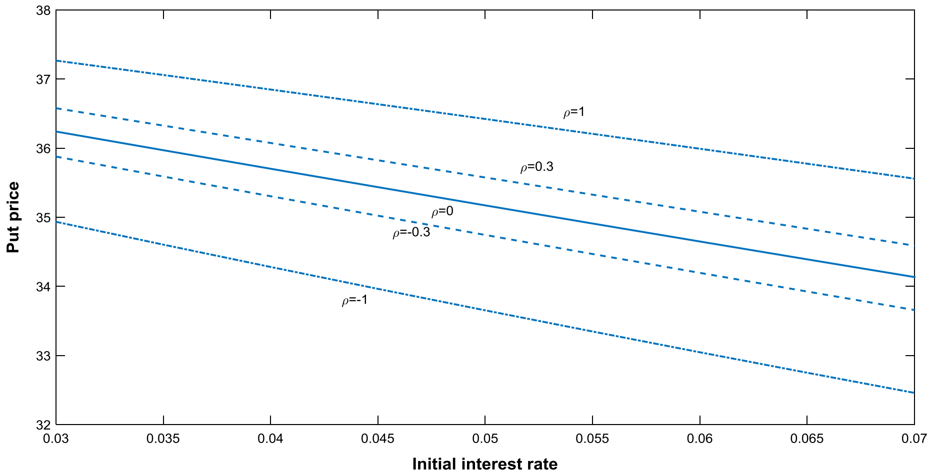
the results in Ciurlia et al. [8]. This is due to the different specifications of real estate index evolutions. The underlying index evolution in their paper is a geometric Brownian motion, whereas the real estate index evolution in this paper is a mean-reverting stochastic process. Fig. 4(a) and (b) describe the sensitivity of option prices to the change in correlation coefficients and show more directly that the correlation between the real estate index and stochastic interest rates has an opposite effect on call and put option prices, which highlights the importance of including a correlation coefficient into the model.

4.3. American option pricing

Finally, we analyze the effects of stochastic interest rates on American puts. Fig. 5 shows that stochastic interest rates have similar influences on American puts and European puts. Constant interest rate models overprice option prices when the term structure of interest rates is rising, but underprice option prices when the term structure is falling. However, compared with Fig. 3(b), we can find two interesting differences. First, the magnitude of the stochastic interest rates has a more significant effect on European puts than on American puts. Second, stochastic interest rates have larger effect on in-the-money European options, whereas Fig. 5 does not manifest this phenomenon in American options. The effects of stochastic interest rates on American puts can be decomposed into the effect on the early exercise premium and corresponding European puts, $\Delta P^A = \Delta EEP + \Delta P^E$. Thus, two extreme cases exist when the real estate index is sufficiently large or tends to be zero. The first case indicates that the early exercise premium is negligible, $\Delta EEP = 0$. The effects of stochastic interest rates on American puts become almost equal to the effect on European puts, $\Delta P^A \approx \Delta P^E$. The second case indicates that American options must be exercised due to the deep in-the-money option, that is, $\Delta P^A \approx 0$. Hence, the effects of stochastic interest rates on American put options is between these two extreme cases when the real estate index is near the strike price. Consequently, the effects of stochastic interest rates on American put options is smaller in magnitude than on European put options and as the real estate index decreases, early exercise premium increases which causes an inconsistent effect on in-the-money American and European options.



(a) Call price functions ($X = 1500$, $K = 1500$, $T = 2$).



(b) Put price functions ($X = 1500$, $K = 1500$, $T = 2$).

Fig. 4. European option prices at different correlation coefficients.

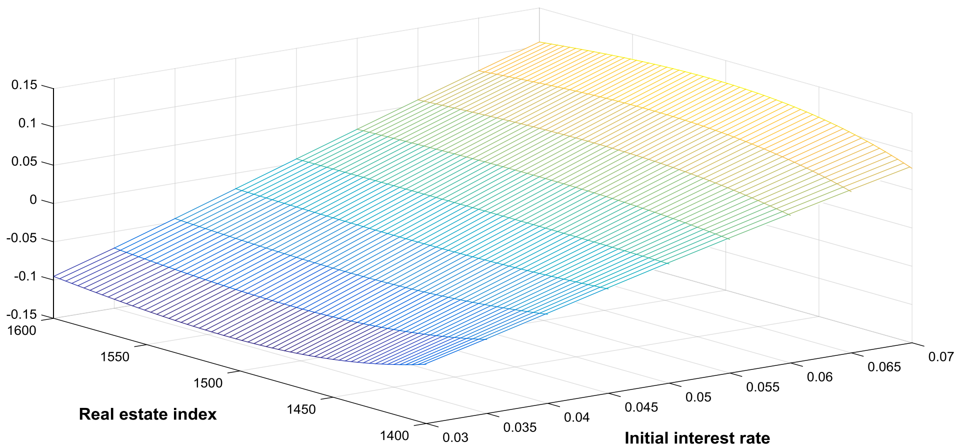


Fig. 5. American put price difference (stochastic interest rate–constant interest rate, $T = 2$) at different real estate index and initial interest rate levels.

Fig. 6 shows the effects of different correlation coefficient parameters on American put prices when $X = 1500$ and $K = 1500$. A comparison between Figs. 4 and 6(b) reveal that different correlation coefficients have slighter effects on

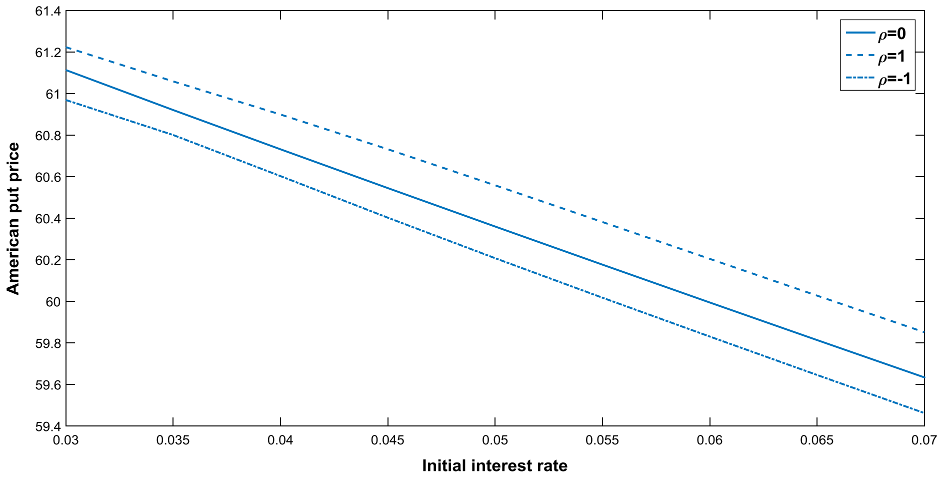


Fig. 6. American put prices at different correlation coefficients ($T = 2$).

American prices. A positive correlation coefficient parameter increases the European put price but decreases the early exercise premium, while a negative coefficient decreases the European put price but increases the early exercise premium. Consequently, we obtain the numerical results as shown in Fig. 6. Based on the analysis above, we find that the early exercise possibility of American options makes it less sensitive to different model specifications compared with European options.

5. Conclusion

Interest in real estate derivatives has surged over the last few years because they provide a new way to hedge real estate price risk. Moreover, real estate derivatives allow investors, especially asset managers and portfolio managers, to improve portfolio diversification by gaining exposure to real estate assets more quickly and effectively. However, the absence of reliable valuation models restricts the further development of this market in terms of trading volumes and liquidity. In this paper, we propose a PDE approach that considers the non-tradable characteristics of the real estate index and price the real estate index options under stochastic interest rates. Numerical results reveal some insights into how stochastic interest rates affect European and American option prices. Future research can extend the pricing model presented in this paper to other exotic options, and structured products, subject to appropriate terminal and boundary conditions. Empirical studies can also test the essential standpoints in this paper.

Acknowledgments

This work is supported by National Natural Science Foundation of China (NSFC No. 71231005) and National Natural Science Foundation of China (NSFC No. 71671076). The authors also express their appreciation to the main editor Eugene Stanley and the anonymous reviewer for their comments and suggestions.

Appendix. Bilinear interpolation

If the real estate index and spot interest rate are determined in advance, it would be unlikely that (X_t, r_t) is just a grid point obtained by the finite difference method. Thus, bilinear interpolation is applied to solve this problem. Suppose that (X_t, r_t) lies in a certain square $X_l \le X_t \le X_r$ and $r_d \le r_t \le r_u$. Then, bilinear interpolation becomes essential in applying the linear interpolation twice in the X_t and r_t directions. The direct formula is as follows

$$\begin{aligned} f(X_t, r_t, t) \approx {} & \frac{f(X_r, r_u, t)}{(X_r - X_l)(r_u - r_d)}(X_r - X_t)(r_u - r_t) + \frac{f(X_l, r_u, t)}{(X_r - X_l)(r_u - r_d)}(X_t - X_l)(r_u - r_t) \\ & + \frac{f(X_r, r_d, t)}{(X_r - X_l)(r_u - r_d)}(X_r - X_t)(r_t - r_d) + \frac{f(X_l, r_d, t)}{(X_r - X_l)(r_u - r_d)}(X_t - X_l)(r_t - r_d). \end{aligned} \tag{A.1}$$

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