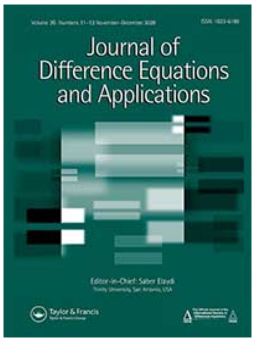

Journal of Difference Equations and Applications

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/gdea20

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To cite this article: Mbakisi Dube & Kailash C. Patidar (2020) A robust nonstandard finite difference scheme for pricing real estate index options, Journal of Difference Equations and Applications, 26:11-12, 1471-1493, DOI: 10.1080/10236198.2020.1852226

To link to this article: https://doi.org/10.1080/10236198.2020.1852226

Published online: 26 Nov 2020.

JOURNAL OF DIFFERENCE EQUATIONS AND APPLICATIONS
2020, VOL. 26, NOS. 11–12, 1471–1493
https://doi.org/10.1080/10236198.2020.1852226

A robust nonstandard finite difference scheme for pricing real estate index options

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ABSTRACT
Real estate assets can be used to store capital, generate income through rentals and can act as collateral for debt instruments. Common risk management mechanisms for real estate investments use portfolio diversification techniques, but these techniques require large amounts of capital to work effectively. Real estate index derivatives offer an alternative mechanism for managing risks associated with real estate investments. They also increase market liquidity by providing a path for individuals who do not own real estate assets to participate in the real estate market. We consider the problem of pricing real estate index derivative contracts. The market is incomplete, so it is completed by futures derivatives on the same real estate index. A dynamic hedging strategy is employed leading to a parabolic partial differential equation with coefficients which are dependent on time and the real estate index. We then construct a nonstandard finite difference method to price European and American real estate index options. The scheme utilizes complete cubic spline interpolants of the option prices at time-dependent backtrack points on the spatial grid. Bounds for the global error are theoretically established. Numerical experiments are carried out to illustrate the accuracy of the scheme.

ARTICLE HISTORY
Received 9 June 2020
Accepted 12 November 2020

KEYWORDS
Computational finance; option pricing; real estate index options; spline interpolation; finite difference methods

2020 MATHEMATICS SUBJECT CLASSIFICATIONS
35Q91; 35K20; 65M06; 65M12

1. Introduction

The global real estate market is huge, with estimates showing that 30–40% of capital value in developed European countries is derived from real estate [12]. In addition, the European real estate investment peaked at 230 billion euros in 2007 further illustrating the sheer enormity of the amount of funds committed to this asset class [27]. Considering the size of the investment contained in real estate assets, it is therefore important to understand how risk is managed in this asset class particularly through the use of derivatives.

Derivative contracts have prices at which they are bought and sold in the market. The counter-parties of a derivative contract seek a fair price in order for it to be tradeable. Pricing models for derivative contracts often use frameworks based on the seminal study by Black and Scholes [4]. In the area of real estate derivative pricing several studies have

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also been done, see, for example, [3,5–7,9,10,13,15,16,30]. These studies have had varying degrees of success in deriving fair prices for real estate derivatives.

Geltner and Fisher [14] priced real estate index-based forward and swap contracts by employing equilibrium rules which incorporate the characteristics of the index. These rules present arbitrage as a way of enforcing equilibrium such that the price stemming from the future-spot parity applies. Cao and Wei [6] priced contingent claims based on the housing index. These are traded on the Chicago Mercantile Exchange. Since the indices are non-tradeable, they used an equilibrium framework. Using a constant relative risk aversion utility function and a mean-reverting model they demonstrated that the values of housing index-based contingent claims rely only on parameters describing the underlying index.

Another approach is presented by Van Bragt *et al.* [30]. Considering a discrete-time setting for the index, they used a risk-neutral valuation technique to price derivatives. They viewed the index as a non-tradeable asset and the real estate market as incomplete. They also assumed that a sufficiently liquid market for futures and swaps on real estate exists. This formed the basis for their pricing methodology allowing them to replicate more complex contingent claims. In the process, they captured the serial autocorrelation of a real estate index by using an auto regressive model for the observed index. They then developed the risk-neutral model and priced different derivatives through closed-form formulas. They also considered the case when the interest rate is stochastic.

Ciurlia and Gheno [9] utilized a two-factor model where the spot rate and the real estate value are jointly modelled. They argued that this model approach is necessitated by the dependence of real estate assets on the term structure of interest rate. They relied on the risk-neutral valuation approach as used in the derivation of the classic Black–Scholes model. They used a bi-dimensional binomial lattice framework to obtain prices for European and American options.

Buttimer *et al.* [5] presented a two-state model for determining the price of derivative contracts which have a real estate index along with interest rate as underlying variables. They used a bivariate binomial tree to determine derivative prices. In the model, the property index inherently incorporated the volatility in the property market. The framework was applied to pricing commercial real estate index linked swaps. Following Buttimer *et al.* [5] and Bjork and Clapham [3] also considered commercial real estate index linked swaps. They presented an arbitrage argument in a continuous-time model that resulted in a price of zero for these derivatives. They provided economic and mathematical arguments that support their stand point. They concluded that the nonzero value obtained by Buttimer *et al.* [5] is due to approximation errors which were made in their theoretical and numerical frameworks. They mentioned that the result they obtained is applicable in a general sense.

In [13], Fabozzi *et al.* proposed a framework that relied on a mean-reverting stochastic model for the property index. Essentially the market incompleteness was catered for through the use of the market price of risk that was calibrated in one market then utilized in pricing other derivative securities contingent on the same underlying index, in another market. This study relied on the usage of a real estate index as the underlying. They also presented the exact prices for forwards, options and total return swaps.

In line with the approach used in Fabozzi *et al.* [13] and Gong and Zou [16] utilized a binomial model to price real estate derivatives using a transformation technique that relied on constant volatility. They suggested that to improve the likelihood of convergence of the discrete-time approximation to the continuous-time model, for the European and American option prices, a Richardson extrapolation can be used.

In their study, Dai and Gong [10] derived a time-dependent linear parabolic differential equation of the Black–Scholes type through a delta hedging approach. The underlying was taken to be a real estate pricing index on which has an associated futures contract. This futures contract was used to complete the market. The interest rate was assumed to be stochastic and the model was analysed numerically through a modified finite difference method. They discovered that a model that employs a constant interest rate may result in spurious prices in the case of European and American options.

The rest of the paper is structured as follows. Section 2 presents the mathematical model. In Section 3, a nonstandard finite difference scheme is developed. A theoretical error analysis for the full method is given in Section 4. Section 5 presents results from numerical experiments to illustrate the accuracy of our numerical scheme. Conclusions and the scope for further research are discussed in Section 6.

2. Mathematical model

This section presents a mathematical model that is based on a framework developed by Fabozzi *et al.* [13]. The formulation of the model equations follows from [11,18,19].

Consider a financial market $(\Omega, \mathcal{F}, \mathbf{P})$ equipped with a filtration $\mathcal{F}_{t\geq 0}$. The following is a list of mathematical objects linked to this market (see [10,11,13,16,18,19] for further details):

- A stochastic process S_t for the real estate index. The empirical findings of most property indices show that they have serial correlation leading to some predictability with positive and negative autocorrelation in the short-term and long-term, respectively. The returns of real estate portfolios are mean reverting with decreasing risk in the long run.
- A risk-free bond rate r.
- The market price of a zero-coupon bond $p(t,T)$ at time t with a maturity T.
- A money account $\mathrm{d}B_t = rB_t\,\mathrm{d}t$. Note that when the interest rate is deterministic

$$p(t,T)B_T = B_t.$$

To capture the properties of the real estate index we employ a mean reverting stochastic model with $Y_t = \text{In}\, S_t$ for $S_t \in (0, +\infty)$ given as

$$\mathrm{d}Y_t = \mu(Y_t, t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \tag{1}$$

where

$$\mu(Y_t, t) = \left[\frac{\mathrm{d}\Psi_t}{\mathrm{d}t} - \theta(Y_t - \Psi_t)\right].$$

We have the following terms; θ is the mean-reversion speed parameter, σ is the volatility, $W_{t\geq 0}$ is a Wiener process and Ψ_t denotes the long-run mean(LRM) which is continuous and differentiable to the first order. In this study, we let $\Psi_t = \alpha + \beta t$ [10,11,13,16,18,19]. The LRM is calibrated using historical data of the Investment Property Databank (IPD) UK Annual Property Total Returns Index, which is an index for commercial real estate in the UK.

Here, we present a dynamic hedging trading strategy. Let $v(Y_t, t)$ be the price of the real estate index-based option with maturity T and strike price K, that is a function of the real estate index $Y_t \in (-\infty, +\infty)$ and time $t \in [0, T]$. Applying Itô's Lemma on $v(Y_t, t)$, we obtain

$$\mathrm{d}v = \left(\frac{\partial v}{\partial t} + \mu\frac{\partial v}{\partial Y_t} + \frac{1}{2}\sigma^2\frac{\partial^2 v}{\partial Y_t^2}\right)\mathrm{d}t + \sigma\frac{\partial v}{\partial Y_t}\,\mathrm{d}W_t. \tag{2}$$

Constructing a risk-free portfolio Π that contains Δ_1 units of derivative v_1 with maturity T_1 and $-\Delta_2$ units of derivative v_2 with maturity T_2 such that

$$\Pi = \Delta_1 v_1 - \Delta_2 v_2. \tag{3}$$

Now the change in the portfolio after time $\mathrm{d}t$ is

$$\begin{aligned}\mathrm{d}\Pi = {} & \Delta_1\left[\left(\frac{\partial v_1}{\partial t} + \mu\frac{\partial v_1}{\partial Y_t} + \frac{1}{2}\sigma^2\frac{\partial^2 v_1}{\partial Y_t^2}\right)\mathrm{d}t + \sigma_1\frac{\partial v_1}{\partial Y_t}\,\mathrm{d}W_t\right] \\ & - \Delta_2\left[\left(\frac{\partial v_2}{\partial t} + \mu\frac{\partial v_2}{\partial Y_t} + \frac{1}{2}\sigma^2\frac{\partial^2 v_2}{\partial Y_t^2}\right)\mathrm{d}t + \sigma_2\frac{\partial v_2}{\partial Y_t}\,\mathrm{d}W_t\right].\end{aligned} \tag{4}$$

In order to eliminate risk in this portfolio, we hedge the random factors of $\mathrm{d}W_t$ by choosing

$$\Delta_1 = \left(\sigma_1\frac{\partial v_1}{\partial Y_t}\right)^{-1} \tag{5}$$

and

$$\Delta_2 = \left(\sigma_1\frac{\partial v_2}{\partial Y_t}\right)^{-1}. \tag{6}$$

The no-arbitrage principle states that $\mathrm{d}\Pi = r\Pi\,\mathrm{d}t$. Using this along with Equation (3), we obtain

$$\begin{aligned}& \Delta_1\left(\frac{\partial v_1}{\partial t} + \mu\frac{\partial v_1}{\partial Y_t} + \frac{1}{2}\sigma^2\frac{\partial^2 v_1}{\partial Y_t^2}\right)\mathrm{d}t \\ & \quad - \Delta_2\left(\frac{\partial v_2}{\partial t} + \mu\frac{\partial v_2}{\partial Y_t} + \frac{1}{2}\sigma^2\frac{\partial^2 v_2}{\partial Y_t^2}\right)\mathrm{d}t = r(\Delta_1 v_1 - \Delta_2 v_2)\,\mathrm{d}t.\end{aligned} \tag{7}$$

Rearranging and collecting like terms together we obtain a constant $\lambda(Y_t, t)$ market price of risk described by

$$\lambda = \left(\sigma\frac{\partial v}{\partial Y_t}\right)^{-1}\left(\frac{\partial v}{\partial t} + \mu\frac{\partial v}{\partial Y_t} + \frac{1}{2}\sigma^2\frac{\partial^2 v}{\partial Y_t^2} - rv\right). \tag{8}$$

This results in the following parabolic partial differential equation with non-constant coefficients given as

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 v}{\partial Y_t^2} + \left[\frac{\mathrm{d}\Psi_t}{\mathrm{d}t} - \theta(Y_t - \Psi_t) - \lambda\sigma\right]\frac{\partial v}{\partial Y_t} - rv = 0. \tag{9}$$

If we consider a European call option with maturity T and strike price K on the real estate index S_t, Equation (9) will have the terminal condition

$$v(Y_T, T) = \max\left(\exp(Y_T) - K, 0\right), \tag{10}$$

with the associated boundary conditions

$$\lim_{Y_t \to -\infty} v(Y_t, t) = 0 \tag{11}$$

and

$$\lim_{Y_t \to +\infty} v(Y_t, t) = (\exp(Y_t) - K)) \exp(-r(T - t). \tag{12}$$

For a European put option the terminal condition for Equation (9) is

$$v(Y_T, T) = \max(K - \exp(Y_T), 0) \tag{13}$$

with the associated boundary conditions

$$\lim_{Y_t \to -\infty} v(Y_t, t) = K \exp(-r(T - t)) \tag{14}$$

and

$$\lim_{Y_t \to +\infty} v(Y_t, t) = 0. \tag{15}$$

Fabozzi *et al.* [13] obtained an analytical solution for the European option pricing problem. The American call option price $v(Y_t, t)$ satisfying Equation (9) over the domain $-\infty \leq Y_t < B(t)$ with $v(Y_t, t) = \max(\exp(Y_T - K, 0)$ for $B(t) \leq Y_t$ has the terminal condition

$$v(Y_T, T) = \max(\exp(Y_T) - K, 0) \tag{16}$$

with the associated boundary conditions

$$\lim_{Y_t \to -\infty} v(Y_t, t) = 0 \tag{17}$$

and

$$\lim_{Y_t \to +\infty} v(Y_t, t) = \exp(Y_t) - K. \tag{18}$$

The American put option satisfies Equation (9) over the domain $B(t) \leq Y_t$ with $v(Y_t, t) = \max(K - \exp(Y_t), 0)$ for $-\infty \leq Y_t < B(t)$ has the terminal condition

$$v(Y_T, T) = \max(K - \exp(Y_t), 0) \tag{19}$$

accompanied by boundary conditions

$$\lim_{Y_t \to -\infty} v(Y_t, t) = K \tag{20}$$

and

$$\lim_{Y_t \to +\infty} v(Y_t, t) = 0. \tag{21}$$

The term $B(t)$ in the models for the American option represents the exercise boundary that is unknown and is implicitly defined in the models [19]. The partial differential equations along with their associated terminal-boundary values as described for the American options do not have exact closed-form solutions. In the following sections we present numerical schemes that can be used to approximate the solutions of these models.

3. Exact time-stepping scheme

Before we proceed further in this section, we would like to mention that the exact time-stepping scheme used in this paper should not be confused with other types of exact schemes, such as the Lagrangian method. This scheme should be understood in the context of the definition by Mickens [23], whereby such schemes are regarded as the important steps towards the construction of nonstandard finite difference methods. Lubuma and Patidar [21] have also used such an exact time-stepping scheme in solving a singularly perturbed advection-reaction equation in the context of nonstandard finite difference methods. Arenas *et al.* [2] have used a similar approach in solving a nonlinear Black-Scholes equation.

In this section, we construct an exact time-stepping scheme to numerically approximate the solution of Equation (9). We first represent Equation (9) as

$$\frac{\partial v}{\partial t} + \left[\beta - \theta(\ln(S_t) - (\alpha + \beta t)) - \lambda\sigma + \frac{1}{2}\sigma^2\right] S_t \frac{\partial v}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 v}{\partial S_t^2} - rv = 0. \tag{22}$$

Suppose $S_t = (L - G)s + G$, such that $S_t \in [G, L] \subset [0, +\infty)$ and $\tau = T - t$. This implies that $s \in [0, 1]$ and $\tau \in [0, T]$. Letting $v(s, t) = u(s, \tau)$ Equation (22) becomes

$$\frac{\partial u}{\partial \tau} + n(s,\tau)\frac{\partial u}{\partial s} + m(s,\tau)\frac{\partial^2 u}{\partial s^2} + ru = 0, \tag{23}$$

where

$$n(s,\tau) = -\left(\beta - \theta\ln((L-G)s + G) + \theta\alpha + \theta\beta(T-\tau) - \lambda\sigma + \frac{1}{2}\sigma^2\right)\frac{(L-G)s+G}{L-G} \tag{24}$$

and

$$m(s,\tau) = -\frac{1}{2}\sigma^2\left(\frac{(L-G)s+G}{L-G}\right)^2. \tag{25}$$

The initial and boundary conditions become

$$u(S_0, 0) = \max(K - S_0, 0),$$
$$u(L, \tau) = K\exp(-r\tau)),$$

and

$$u(G, \tau) = 0.$$

3.1. Approximation of the advection-reaction part

In this subsection, we present an exact time-stepping scheme for the advection-reaction part of Equation (23). Consider the advection–reaction equation

$$\frac{\partial u}{\partial \tau} + \tilde{n}(s,\tau)\frac{\partial u}{\partial s} = -ru. \tag{26}$$

Let $u(s,0) = f(s)$ be the initial condition for Equation (26) where $f(s)$ is a given function. The following definition will be used in the rest of the paper.

Definition 3.1: Consider a numerical scheme given by

$$U^m(s) = F(V^{m-1}(s), \Delta\tau, m), \tag{27}$$

where $\Delta\tau$ is step size in time and $U^m(s)$ is the numerical solution at time $m\Delta\tau$. Assuming that it has the solution

$$U^m(s) = G(U^0(s), \Delta\tau, m) = G(f(s), \Delta\tau, m), \tag{28}$$

then the numerical scheme is said to be an exact time-stepping scheme if $U^m(s) = u(s, m\Delta\tau)$ for any arbitrary $\Delta\tau$ and at every s [8,21,23].

We develop an exact time-stepping scheme for the advection-reaction part of Equation (23) given by Equation 26. The exact time-stepping scheme is a type of nonstandard finite difference (NSFD) scheme. These NSFD schemes provide numerically stable algorithms for approximating solutions of differential equations. Detailed discussions on them can also be found in [21–26,28,29]. Introducing the term

$$\tilde{n}(s,\tau) = (a\tau + b\ln((L-G)s + G) + c)\frac{(L-G)s+G}{L-G}, \tag{29}$$

where a, b and c are real constants. Through the use of the method of characteristics the general solution of (28) is

$$u(s,\tau) = f(d)\,\mathrm{e}^{-r\tau}, \tag{30}$$

where $d = s(0)$ is the initial condition of the ordinary differential equation

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = \tilde{n}(s,\tau). \tag{31}$$

Letting $u_1 = \ln((L-G)s + G)$, we have

$$\mathrm{d}u_1 = \frac{(L-G)\,\mathrm{d}s}{(L-G)s+G}. \tag{32}$$

This means that

$$\frac{\mathrm{d}u_1}{\mathrm{d}\tau} = a\tau + bu_1 + c. \tag{33}$$

Introducing the variable $u_2 = a\tau + bu_1 + c$, we obtain

$$\frac{\mathrm{d}u_2}{\mathrm{d}\tau} = a + bu_2. \tag{34}$$

Solving Equation (34) and substituting back the variables we obtain the solution

$$\ln((L-G)s(\tau)+G) = \frac{\mathrm{e}^{b\tau+bK_1}}{b^2} - \frac{a}{b^2} - \frac{a\tau}{b} - \frac{c}{b}. \tag{35}$$

Using the initial condition $s(0) = d$ the constant of integration K_1 becomes

$$K_1 = \frac{1}{b}\ln(b^2\ln((L-G)d+G)+a+cb). \tag{36}$$

Finally,

$$\ln((L-G)s(\tau)+G) = \left(\ln((L-G)d+G) + \frac{a}{b^2} + \frac{c}{b}\right)\mathrm{e}^{b\tau} - \frac{a}{b^2} - \frac{a\tau}{b} - \frac{c}{b}. \tag{37}$$

Making d the subject we obtain

$$d = \frac{\exp\left(\mathrm{e}^{-b\tau}(\ln((L-G)s+G) + \frac{a}{b^2} + \frac{a\tau}{b} + \frac{c}{b}) - \frac{a}{b^2} - \frac{c}{b}\right) - G}{L-G}. \tag{38}$$

Substituting d into Equation (30), we obtain the general solution

$$u(s,\tau) = f\left(\frac{\exp(\mathrm{e}^{-b\tau}(\ln((L-G)s+G) + \frac{a}{b^2} + \frac{a\tau}{b} + \frac{c}{b}) - \frac{a}{b^2} - \frac{c}{b}) - G}{L-G}\right)\mathrm{e}^{-r\tau}. \tag{39}$$

The following theorem provides a description of the exact time-stepping scheme.

Theorem 3.1: *The exact time-stepping scheme for Equation* (26) *is given by*

$$\frac{U^{m+1}(s^{(m+1)}) - U^m(\bar{s}^m)}{\delta(\Delta\tau)} = -rU^m(\bar{s}^m), \tag{40}$$

where

$$\delta(\Delta\tau) = \frac{1-\mathrm{e}^{-r\Delta\tau}}{r} \tag{41}$$

and

$$\bar{s}^{(m)} = \frac{1}{L_1}\left((L_1 s^{(m+1)} + G)\mathrm{e}^{-(\frac{a}{b^2} + \frac{am\Delta\tau}{b} + \frac{c}{b})\mathrm{e}^{b\Delta\tau} + \frac{a}{b^2} + \frac{a((m+1)\Delta\tau)}{b} + \frac{c}{b}}\right)^{\mathrm{e}^{-b\Delta\tau}} - \frac{G}{L_1}. \tag{42}$$

with $L_1 = L - G$. The backtrack point $\bar{s}^{(m)}$ is associated with $s^{(m+1)}$ on the spatial grid at time $(m+1)\Delta\tau$ for

$$\Delta\tau = \frac{T}{M}, \tag{43}$$

with $m = 0, 1, \ldots, M$.

Proof: Consider the relationship

$$u\left(\frac{1}{L_1}\left((L_1 s + G)e^{-\left(\frac{a}{b^2}+\frac{a\tau}{b}+\frac{c}{b}\right)e^{b\Delta\tau}+\frac{a}{b^2}+\frac{a(\tau+\Delta\tau)}{b}+\frac{c}{b}}\right)^{e^{-b\Delta\tau}} - \frac{G}{L_1}, \tau\right)$$
$$= f\left(\frac{\exp\left(e^{-b(\tau+\Delta\tau)}\left(\ln(L_1 s + G) + \frac{a}{b^2} + \frac{a(\tau+\Delta\tau)}{b} + \frac{c}{b}\right) - \frac{a}{b^2} - \frac{c}{b}\right) - G}{L_1}\right) e^{-r\tau}. \tag{44}$$

From Equation (39), we obtain

$$u(s, \tau + \Delta\tau)$$
$$= f\left(\frac{\exp\left(e^{-b(\tau+\Delta\tau)}\left(\ln(L_1 s + G) + \frac{a}{b^2} + \frac{a(\tau+\Delta\tau)}{b} + \frac{c}{b}\right) - \frac{a}{b^2} - \frac{c}{b}\right) - G}{L_1}\right) e^{-r(\tau+\Delta\tau)}. \tag{45}$$

This leads to

$$u(s, \tau + \Delta\tau)$$
$$= u\left(\frac{1}{L_1}\left((L_1 s + G)e^{-\left(\frac{a}{b^2}+\frac{a\tau}{b}+\frac{c}{b}\right)e^{b\Delta\tau}+\frac{a}{b^2}+\frac{a(\tau+\Delta\tau)}{b}+\frac{c}{b}}\right)^{e^{-b\Delta\tau}} - \frac{G}{L_1}, \tau\right) e^{-r\Delta\tau}. \tag{46}$$

If we let

$$\bar{s}^{(m)} = \frac{1}{L_1}\left((L_1 s^{(m+1)} + G)e^{-\left(\frac{a}{b^2}+\frac{am\Delta\tau}{b}+\frac{c}{b}\right)e^{b\Delta\tau}+\frac{a}{b^2}+\frac{a((m+1)\Delta\tau)}{b}+\frac{c}{b}}\right)^{e^{-b\Delta\tau}} - \frac{G}{L_1}, \tag{47}$$

where $\bar{s}^m$ is referred to as the backtrack point associated with the point $s^{(m+1)}$ at time $(m + 1)\Delta\tau$. Finally, we obtain the exact time-stepping scheme

$$\frac{U^{m+1}(s^{(m+1)}) - U^m(\bar{s}^m)}{\delta(\Delta\tau)} = -rU^m(\bar{s}^m), \tag{48}$$

where

$$\delta(\Delta\tau) = \frac{1 - e^{-r\Delta\tau}}{r}. \tag{49}$$

■

The left-hand side of the numerical scheme (40) can be viewed as a backward nonstandard finite difference approximation of the characteristic derivative

$$\frac{Du}{D\tau} = \frac{\partial u}{\partial \tau} + \tilde{n}(s, \tau)\frac{\partial u}{\partial s}. \tag{50}$$

In the next subsection, we obtain a NSFD scheme for Equation (23).

3.2. Nonstandard finite difference (NSFD) scheme for the full partial differential equation

In order to determine the NSFD scheme for Equation (23), we combine the exact time-stepping scheme given by Equation (40) with a finite difference approximation of the diffusion term on a time independent spatial grid $\gamma = \{s_0, s_1, \dots, s_N\}$ where $s_0 = 0 < s_1 < \dots < 1 = s_N$.

In order to improve the accuracy of the approximation of the diffusion term, we employ a nonuniform mesh with relatively many points in a given interval $[s_l, s_r] \subset [0, 1]$ containing the strike price K [17]. This helps mitigate the detrimental impact on the numerical scheme of discontinuous derivatives associated with the initial value function at the strike price K. In any case, the region of interest in applications is the region near the strike price. Defining the mesh points of γ using the transformation

$$s_i = \varphi(\xi_i), \tag{51}$$

where $\xi_{\min} = \xi_0 < \xi_1 < \dots < \xi_N = \xi_{\max}$ are equally spaced points with

$$\xi_{\min} = \sinh^{-1}\left(\frac{s_0 - s_l}{d_1}\right), \tag{52}$$

$$\xi_g = \frac{s_r - s_l}{d_1}, \tag{53}$$

$$\xi_{\max} = \xi_g + \sinh^{-1}\left(\frac{s_N - s_r}{d_1}\right), \tag{54}$$

and

$$\varphi(\xi) = \begin{cases} s_l + d_1 \sinh(\xi) & \text{if } \xi_{\min} \le \xi < 0, \\ s_l + d_1 \xi & \text{if } 0 \le \xi \le \xi_g, \\ s_r + d_1 \sinh(\xi - \xi_g) & \text{if } \xi_g < \xi \le \xi_{\max}. \end{cases}$$

The mesh γ described above is uniform inside $[s_l, s_r]$ and nonuniform outside. The strike price $K \in [s_l, s_r]$. The term $d_1 > 0$ regulates the concentration of points $s_i \in [s_l, s_r]$. As d_1 decreases the number of spatial points in $[s_l, s_r]$ increases.

Let $\Delta\xi = \xi_1 - \xi_0$. It can be seen that the mesh is smooth so there exist real constants $C_0, C_1, C_2 > 0$ such that the mesh widths $\Delta s_i = s_i - s_{i-1}$, uniformly satisfy $C_0 \Delta\xi \le \Delta s_i \le C_1 \Delta\xi$ and $|\Delta s_{i+1} - \Delta s_i| \le C_2 (\Delta\xi)^2$ in $[i, N]$. The transformation described above is obtained from [17]. The diffusion term on this mesh is approximated by

$$\frac{\partial^2 u(s_i, \tau_m)}{\partial s^2} \approx \rho_{-1} U_{i-1}^m + \rho_0 U_i^m + \rho_1 U_{i+1}^m. \tag{55}$$

Here,

$$\rho_{-1} = \frac{2}{\Delta s_i (\Delta s_i + \Delta s_{i+1})}, \tag{56}$$

$$\rho_0 = \frac{-2}{\Delta s_i \Delta s_{i+1}}, \tag{57}$$

and

$$\rho_1 = \frac{2}{\Delta s_{i+1}(\Delta s_i + \Delta s_{i+1})}. \tag{58}$$

This finite difference has a second order truncation error provided that the mesh is smooth and $u(s,\tau)$ is continuously differentiable, [17].

The scheme for Equation (23) is therefore given by

$$\frac{U_i^{m+1} - U^m(\bar{s}_i^m)}{\delta(\Delta\tau)} + m(s_i,(m+1)\Delta\tau)(\rho_{-1}U_{i-1}^m + \rho_0 U_i^m + \rho_1 U_{i+1}^m) + rU^m(\bar{s}_i^m) = 0, \tag{59}$$

where $U_i^0 = u(s_i,0) = \max(K - s_i, 0)$ for $i = 0,1,\dots,N$, $U_0^m = K\exp(-r(m\Delta\tau))$ and $U_N^m = 0$ for $m = 0,1,\dots,M$. Here, U_i^{m+1} denotes the approximate solution at the point s_i and $U^m(\bar{s}_i^m)$ is the approximate solution at the point $\bar{s}_i^m$.

The value of $U^m(\bar{s}_i^m)$ will be approximated by a cubic spline interpolation $I_U(\bar{s}_i^m)$ of the values U_i^m in such a way that the ith spatial grid point corresponds to the ith backtrack point.

4. Theoretical error analysis

In this section, we present global error estimates for the full scheme. Consider the nonuniform mesh $\gamma = \{s_0, s_1, \dots, s_N\}$ where $\Delta s_i = s_i - s_{i-1} > 0$ such that $s_0 = 0 < s_1 < \cdots < s_{l_g-1} < s_{l_g} < s_{l_g+1} < \cdots < s_{r_g-1} < s_{r_g} < s_{r_g+1} < \cdots < s_N = 1$ with $s_l = s_{l_g}$ and $s_r = s_{r_g}$. From definition the mesh is uniform inside $[s_{l_g}, s_{r_g}]$. Since $\Delta s_i = s_i - s_{i-1}$, we obtain $s_{r_g} - s_{l_g} = z\Delta s_{r_g}$ where the integer $z \geq 0$. It can be seen that z is inversely proportional to d_1. Hence for a constant $K_z > 0$ we have $z = K_z d_1^{-1}$. Using $s_{l_g} = s_{r_g} - K_z d_1^{-1}\Delta s_{r_g}$ and $s_{r_g} = K_z d_1^{-1}\Delta s_{r_g} + s_{l_g}$ we obtain

$$\begin{aligned}\xi_{\min} &= \sinh^{-1}\left(\frac{0 - (s_{r_g} - K_z d_1^{-1}\Delta s_{r_g})}{d_1}\right)\\ &= \sinh^{-1}\left(\frac{-s_{r_g}}{d_1} + \frac{K_z\Delta s_{r_g}}{d_1^2}\right).\end{aligned} \tag{60}$$

We also obtain

$$\xi_{\max} = \frac{K_z\Delta s_{r_g}}{d_1^2} + \sinh^{-1}\left(\frac{1 - s_{l_g}}{d_1} - \frac{K_z\Delta s_{r_g}}{d_1^2}\right). \tag{61}$$

It can be seen that the mesh is smooth so there exist real constants $C_0, C_1 > 0$ such that the mesh widths $\Delta s_i = s_i - s_{i-1}$ uniformly satisfy $C_0\Delta\xi \leq \Delta s_i \leq C_1\Delta\xi$, [17]. Consider the uniform mesh grids defined by

$$\Delta\xi = \frac{\xi_{\max} - \xi_{\min}}{N}. \tag{62}$$

Substituting Equations (60) and (61) into Equation (62), we obtain

$$\Delta\xi = \frac{1}{N}\left(\frac{K_z\Delta s_{r_g}}{d_1^2} + \sinh^{-1}\left(\frac{1-s_{l_g}}{d_1} - \frac{K_z\Delta s_{r_g}}{d_1^2}\right) - \sinh^{-1}\left(\frac{-s_{r_g}}{d_1} + \frac{K_z\Delta s_{r_g}}{d_1^2}\right)\right).$$

Introducing a constant $K_y \geq 1$, we obtain

$$\Delta\xi \leq \frac{1}{N}\left(\frac{K_y\Delta s_{r_g}}{d_1^2} + \sinh^{-1}\left(\frac{K_y\Delta s_{r_g}}{d_1^2}\right) - \sinh^{-1}\left(\frac{-K_y\Delta s_{r_g}}{d_1^2}\right)\right).$$

Let

$$x = \frac{K_y\Delta s_{r_g}}{d_1^2}.$$

Introducing a constant $K_x > 1$, we use Taylor's series expansion of $\sinh^{-1}$ with the assumption

$$\left|\frac{K_y\Delta s_{r_g}}{d_1^2}\right| < 1 \tag{63}$$

to obtain

$$\begin{aligned}\Delta s_i &\leq C_1\Delta\xi \\ &\leq \frac{C_1}{N}\left(x + x - \frac{x^3}{6} + \frac{3x^5}{40} - \frac{15x^7}{336} + \cdots\right. \\ &\quad \left. - \left((-x) - \frac{(-x)^3}{6} + \frac{3(-x)^5}{40} - \frac{15(-x)^7}{336} + \cdots\right)\right) \\ &\leq \frac{C_1}{N}K_x x. \end{aligned} \tag{64}$$

Introducing a constant $K_w > 0$, we obtain

$$\Delta s_i \leq K_w\Delta s_{r_g}. \tag{65}$$

Note that

$$\Delta s_{r_g} = \frac{s_{r_g} - s_{l_g}}{z}.$$

It can be seen that Δs_{r_g} is implicitly affected by the values of s_{r_g}, s_{l_g}, d_1 and N.

Now let

$$j(i) = \{j : |\bar{s}_i^m - s_j| = \min_k|\bar{s}_i^m - s_k|\}. \tag{66}$$

For constants $K_u, K_v > 0$, we obtain

$$\begin{aligned}|\bar{s}_i^m - s_{j(i)}| &\leq \min\left(\frac{\Delta s_i}{2}, K_v\Delta\tau\right) \\ &\leq K_u \min\left(\frac{\Delta s_{r_g}}{2}, \Delta\tau\right). \end{aligned} \tag{67}$$

For the backtrack point ([21])

$$\bar{s}_i^m = \frac{1}{L_1}\left((L_1 s_i + G)\mathrm{e}^{-\left(\frac{a}{b^2}+\frac{am\Delta\tau}{b}+\frac{c}{b}\right)\mathrm{e}^{b\Delta\tau}+\frac{a}{b^2}+\frac{a((m+1)\Delta\tau)}{b}+\frac{c}{b}}\right)^{\mathrm{e}^{-b\Delta\tau}} - \frac{G}{L_1},$$

the term $K_v \Delta\tau$ arises when

$$\left|1 - \frac{1}{L_1}\left((L_1 + G)\mathrm{e}^{-\left(\frac{a}{b^2}+\frac{am\Delta\tau}{b}+\frac{c}{b}\right)\mathrm{e}^{b\Delta\tau}+\frac{a}{b^2}+\frac{a((m+1)\Delta\tau)}{b}+\frac{c}{b}}\right)^{\mathrm{e}^{-b\Delta\tau}} + \frac{G}{L_1}\right| < \frac{\Delta s_i}{2}.$$

Consider the expression $I_U^{(2)}(s) = \sum_0^N m_i l_i(s)$, where l_i is the piecewise linear Lagrange basis function associated with the node s_i and m_i is an unknown coefficient called a moment of I_U [1]. In this section, we will use the following theorem:

Theorem 4.1: *Suppose that, almost everywhere, $f \in C^4([0,1])$ and $\gamma = \{s_0, s_1, \dots, s_N\}$ is a nonuniform grid on $[0,1]$ with $\Delta s_i = \Delta s_i - \Delta s_{i-1}$. The vector of moments m_i of the complete cubic spline I_f interpolating f on γ is denoted by $\mathbf{m} \in \mathbb{R}^{N+1}$. The vector of true second-derivatives $f^{(2)}(s_i)$ at the nodes of γ is denoted by $\mathbf{f} \in \mathbb{R}^{N+1}$. Then for a constant $K_t > 0$, we have*

$$\begin{aligned}\|\boldsymbol{m} - \boldsymbol{f}\|_\infty = \max_{0\le i\le N}\left|m_i - f^{(2)}(s_i)\right| &\le \frac{3}{4}\|f^{(4)}\|_\infty\left((\Delta s_i)^2 + (\Delta s_{i+1})^2\right)\\ &\le \frac{3K_t}{2}\|f^{(4)}\|_\infty(\Delta s_{r_g})^2,\end{aligned} \tag{68}$$

where $\|\,\|_\infty$ denotes the norm

$$\|f\|_\infty = \sup_{[0,1]}|f(s)|.$$

Proof: A proof for this theorem can be found on p. 75 in [1]. ■

The following theorem gives bounds for the interpolation errors.

Theorem 4.2: *Suppose that, almost everywhere, $f \in C^4([0,1])$ and let $\gamma = \{s_0, s_1, \dots, s_N\}$ be a grid on $[0,1]$ with $\Delta s_i = s_i - s_{i-1}$. If I_f is the complete spline interpolating f on γ, then there exists a constant $K_q > 0$ such that*

$$|f(\bar{s}_i^m) - I_f(\bar{s}_i^m)| \le K_q\|f^{(4)}\|_\infty(\min((\Delta s_{r_g})^4, (\Delta s_{r_g})^2(\Delta\tau)^2) + (\Delta s_{r_g})^4).$$

Proof: We use the approach described by Allen and Isaacson [1] to establish a bound for

$$|f^{(3)}(\bar{s}_i^m) - I_f^{(3)}(\bar{s}_i^m)|.$$

Then we proceed using that result to establish another bound for

$$|f^{(2)}(\bar{s}_i^m) - I_f^{(2)}(\bar{s}_i^m)|$$

and so on.

Using the term defined in (66) and the definition of a complete cubic spline I_f interpolating f on γ we obtain

$$|f^{(3)}(\bar{s}_i^m) - I_f^{(3)}(\bar{s}_i^m)| = \left| \frac{m_{j(i)+1} - m_{j(i)}}{\Delta s_{j(i)+1}} - f^{(3)}(\bar{s}_i^m) \right| \\ \leq A + B, \tag{69}$$

where

$$A = \left| \frac{m_{j(i)+1} - f^{(2)}(s_{j(i)+1})}{\Delta s_{j(i)+1}} - \frac{m_{j(i)} - f^{(2)}(s_{j(i)})}{\Delta s_{j(i)+1}} \right|$$

and

$$B = \left| \frac{f^{(2)}(s_{j(i)+1}) - f^{(2)}(\bar{s}_i^m)}{\Delta s_{j(i)+1}} - \frac{f^{(2)}(s_{j(i)}) - f^{(2)}(\bar{s}_i^m)}{\Delta s_{j(i)+1}} - f^{(3)}(\bar{s}_i^m) \right|.$$

Introducing some constant $K_s \geq 1$, we use (68) to obtain

$$A \leq \left| \frac{\|f^{(4)}\|_\infty \frac{3K_t}{2} (\Delta s_{r_g})^2}{\Delta s_{j(i)+1}} \right| + \left| \frac{\|f^{(4)}\|_\infty \frac{3K_t}{2} (\Delta s_{r_g})^2}{\Delta s_{j(i)+1}} \right| \\ = \left| \frac{\|f^{(4)}\|_\infty \frac{3K_t}{2} (\Delta s_{r_g})^2}{K_s \Delta s_{r_g}} \right| + \left| \frac{\|f^{(4)}\|_\infty \frac{3K_t}{2} (\Delta s_{r_g})^2}{K_s \Delta s_{r_g}} \right|. \tag{70}$$

To obtain a bound for B we use expression (67) and Taylor's series expansions about the point $\bar{s}_i^m$. Finding the points $\eta_1, \eta_2 \in (s_{j(i)+1}, s_{j(i)})$ and defining

$$C = \frac{1}{\Delta s_{j(i)+1}} (s_{j(i)+1} - \bar{s}_i^m) f^{(3)}(\bar{s}_i^m) + \frac{1}{2\Delta s_{j(i)+1}} (s_{j(i)+1} - \bar{s}_i^m)^2 f^{(4)}(\eta_1)$$

and

$$D = -\frac{1}{\Delta s_{j(i)+1}} (s_{j(i)} - \bar{s}_i^m) f^{(3)}(\bar{s}_i^m) - \frac{1}{2\Delta s_{j(i)+1}} (s_{j(i)} - \bar{s}_i^m)^2 f^{(4)}(\eta_2) - f^{(3)}(\bar{s}_i^m).$$

Introducing a constant $K_r > 0$, we obtain

$$B = |C + D| \\ \leq \left| \frac{1}{2\Delta s_{j(i)+1}} (s_{j(i)+1} - \bar{s}_i^m)^2 f^{(4)}(\eta_1) \right| + \left| \frac{1}{2\Delta s_{j(i)+1}} (s_{j(i)} - \bar{s}_i^m)^2 f^{(4)}(\eta_2) \right| \\ \leq \left| \frac{1}{2K_s \Delta s_{r_g}} \left(K_r \Delta s_{r_g} \right)^2 f^{(4)}(\eta_1) \right| + \left| \frac{1}{2K_s \Delta s_{r_g}} \left(K_u \min\left(\frac{\Delta s_{r_g}}{2}, \Delta\tau \right) \right)^2 f^{(4)}(\eta_2) \right|. \tag{71}$$

Combining these inequalities and using a constant $K_q > 0$, we have

$$|f^{(3)}(\bar{s}_i^m) - I_f^{(3)}(\bar{s}_i^m)| \leq K_q \frac{\|f^{(4)}\|_\infty}{\Delta s_{r_g}} (\min((\Delta s_{r_g})^2, (\Delta\tau)^2) + (\Delta s_{r_g})^2). \tag{72}$$

We can integrate to get the remaining cases $f^{(2)} - I_f^{(2)}$ and so on. So using the Fundamental Theorem of Calculus and the triangle inequality we complete the proof and obtain

$$|f(\bar{s}_i^m) - I_f(\bar{s}_i^m)| \leq K_q \|f^{(4)}\|_\infty (\min((\Delta s_{r_g})^4, (\Delta s_{r_g})^2(\Delta\tau)^2) + (\Delta s_{r_g})^4). \tag{73}$$

■

The following theorem gives the global error bounds for the NSFD scheme on a smooth nonuniform spatial grid. Note that the initial value function is not smooth at the strike price. This lack of smoothness is handled through the grid transformation we presented in the previous section. Other studies use some approximation technique near the point where the initial value function is not smooth, see, e.g. [23].

Theorem 4.3: *Let, almost everywhere, $u(s,\tau) \in C^{(4)}([0,1]) \times C([0,T])$ be the solution of Equation (23). Also let the approximate solution U_i^{m+1} be defined by Equation (59) on a smooth nonuniform grid γ where $I_{U_i^m}(\bar{s}_i^m)$ is the complete cubic spline interpolant of $U^m(\bar{s}_i^m)$. Then there exists a constant $M_d > 0$ such that*

$$\|u(\cdot,\tau^m) - U^{m+1}(\cdot)\|_\infty \leq M_d M_b \left(\frac{(\Delta s_{r_g})^2}{\Delta\tau} + \Delta\tau \right)$$

for

$$M_b = \max\left(\max_{0\leq k\leq m+1} \left\| \frac{\partial^4 u^k}{\partial s^4} \right\|_\infty, \sup_{[0,1]\times[0,T]} \frac{\partial^2 u^*}{\partial^2 \mathcal{T}}, \sup_{[0,1]\times[0,T]} \frac{\partial u^*}{\partial \mathcal{T}} \right).$$

Proof: Let e_i^{m+1} be the truncation error at time $(m+1)\Delta\tau$ for the NSFD scheme on a fixed spatial grid with a nonuniform grid size $\Delta s_i = s_i - s_{i-1}$. If we let $\tau^m = (m)\Delta\tau$, we define the truncation error as

$$\begin{aligned} \mathrm{e}_i^{m+1} = &\left[\frac{u_i^{m+1} - u^m(\bar{s}_i^m)}{\delta(\Delta\tau)} - \left(\frac{\partial u(s_i,\tau^{m+1})}{\partial\tau} + n(s_i,\tau^{m+1}) \frac{\partial u(s_i,\tau^{m+1})}{\partial s} \right) \right] \\ &+ \left[m(s_i,\tau^{m+1})(\rho_1 U_{i+1}^{m+1} + \rho_0 U_i^{m+1} + \rho_{-1} U_{i-1}^{m+1}) - m(s_i,\tau^{m+1}) \frac{\partial^2 u(s_i,\tau^{m+1})}{\partial s^2} \right] \\ &+ [ru^m(\bar{s}_i^m) - ru(s_i,\tau^{m+1})]. \end{aligned} \tag{74}$$

Using Taylor's series expansions and the Mean Value Theorem, we obtain

$$|\mathrm{e}_i^{m+1}| = \mathcal{O}\left(\frac{\partial^2 u^*}{\partial \mathcal{T}^2} \Delta\tau \right) + \mathcal{O}\left(\left\| \frac{\partial^3 u^{m+1}}{\partial s^3} \right\|_\infty |(\Delta s_{i+1} - \Delta s_i)| \right) + \mathcal{O}\left(\frac{\partial u^*}{\partial \mathcal{T}} \Delta\tau \right), \tag{75}$$

where $\frac{\partial u^*}{\partial \mathcal{T}}$ and $\frac{\partial^2 u^*}{\partial \mathcal{T}^2}$ are, respectively, the first and second tangential derivatives along the characteristic line between (s_i, τ^{m+1}) and $(\bar{s}_i^m, \tau^m)$.

We define the difference between the exact and approximate solution as $\mathcal{E} = u - U$. Substituting $\mathcal{E}$ into Equation (59), we obtain

$$\frac{\mathcal{E}_i^{m+1} - \mathcal{E}^m(\bar{s}_i^m)}{\delta(\Delta\tau)} + m(s_i, \tau^{m+1})(\rho_1 \mathcal{E}_{i+1}^{m+1} + \rho_0 \mathcal{E}_i^{m+1} + \rho_{-1}\mathcal{E}_{i-1}^{m+1}) + r\mathcal{E}^m(\bar{s}_i^m) = \mathrm{e}_i^{m+1},$$

$\mathcal{E}_i^0 = 0$, $\mathcal{E}_0^m = 0$ and $\mathcal{E}_N^m = 0$. Rearranging terms we obtain

$$\mathcal{E}_i^{m+1} = (1 - r\delta(\Delta\tau))\mathcal{E}^m(\bar{s}_i^m) + M + \delta(\Delta\tau)\mathrm{e}_i^{m+1},$$

where

$$\begin{aligned} M &= -\frac{m(s_i, \tau^{m+1})\delta(\Delta\tau)}{\Delta s_i \Delta s_{i+1}(\Delta s_i + \Delta s_{i+1})}(2\Delta s_i(\mathcal{E}_{i+1}^{m+1} - \mathcal{E}_i^{m+1}) + 2\Delta s_{i+1}(\mathcal{E}_{i-1}^{m+1} - \mathcal{E}_i^{m+1})) \\ &= \frac{m(s_i, \tau^{m+1})\delta(\Delta\tau)}{\Delta s_i \Delta s_{i+1}(\Delta s_i + \Delta s_{i+1})}(2\Delta s_i(\mathcal{E}_i^{m+1} - \mathcal{E}_{i+1}^{m+1}) + 2\Delta s_{i+1}(\mathcal{E}_i^{m+1} - \mathcal{E}_{i-1}^{m+1})). \end{aligned} \tag{76}$$

Suppose that

$$\mathcal{E}_{i_1}^{m+1} = u_{i_1}^{m+1} - U_{i_1}^{m+1} = \min_{1 \le i \le N-1}(u_i^{m+1} - U_i^{m+1}), \tag{77a}$$

$$\mathcal{E}_{i_2}^{m+1} = u_{i_2}^{m+1} - U_{i_2}^{m+1} = \max_{1 \le i \le N-1}(u_i^{m+1} - U_i^{m+1}). \tag{77b}$$

Choosing the indices i_1 and i_2 according to the definition given by Equations (77a) and (77b) M is nonpositive for $i = i_1$ and is nonnegative for $i = i_2$, [23] implying that

$$\mathcal{E}_{i_1}^{m+1} \ge (1 - r\delta(\Delta\tau))\mathcal{E}^m(\bar{s}_{i_1}^m) + \delta(\Delta\tau)\,\mathrm{e}_{i_1}^{m+1}, \tag{78a}$$

$$\mathcal{E}_{i_2}^{m+1} \le (1 - r\delta(\Delta\tau))\mathcal{E}^m(\bar{s}_{i_2}^m) + \delta(\Delta\tau)\,\mathrm{e}_{i_2}^{m+1}. \tag{78b}$$

As a result for any $j \in \{1, \ldots, N-1\}$, we have

$$\begin{aligned} |\mathcal{E}_j^{m+1}| &\le \max_{1 \le i \le N-1} |(1 - r\delta(\Delta\tau))\mathcal{E}^m(\bar{s}_i^m) + \delta(\Delta\tau)\,\mathrm{e}_i^{m+1}| \\ &\le \max_{1 \le i \le N-1} |(1 - r\delta(\Delta\tau))\mathcal{E}^m(\bar{s}_i^m)| + \max_{1 \le i \le N-1} |\delta(\Delta\tau)\,\mathrm{e}_i^{m+1}|. \end{aligned} \tag{79}$$

The inequalities in (78) imply that

$$\max_{1 \le i \le N-1} |\mathcal{E}_i^{m+1}| \le \max_{1 \le i \le N-1} |(1 - r\delta(\Delta\tau))\mathcal{E}^m(\bar{s}_i^m)| + \max_{1 \le i \le N-1} |\delta(\Delta\tau)\,\mathrm{e}_i^{m+1}|. \tag{80}$$

Since we are using a complete cubic spline interpolation we have

$$\max_{1 \le i \le N-1} |\mathcal{E}^m(\bar{s}_i^m)| \le \max_{1 \le i \le N-1} |\mathcal{E}_i^m| + \max_{1 \le i \le N-1} |u(\bar{s}_i^m) - I_u(\bar{s}_i^m)|.$$

Combining this into (80) and using Theorem 4.2, we obtain

$$\max_{1 \le i \le N-1} |\mathcal{E}_i^{m+1}| \le (1 - r\delta(\Delta\tau)) \left(\max_{1 \le i \le N-1} |\mathcal{E}_i^m| \right)$$

$$+ K_q(1 - r\delta(\Delta\tau)) \left\| \frac{\partial^4 u}{\partial s^4} \right\|_\infty \left(\min((\Delta s_{r_g})^4, (\Delta s_{r_g})^2(\Delta\tau)^2) + (\Delta s_{r_g})^4 \right)$$
$$+ \delta(\Delta\tau) \max_{1\le i\le N-1} |\mathrm{e}_i^{m+1}|. \tag{81}$$

Using Taylor's series, it is easy to show that

$$\delta(\Delta\tau) = \mathcal{O}(\Delta\tau)$$

and

$$1 - r\delta(\Delta\tau) \approx 1 - r\Delta\tau \le 1 + E\Delta\tau$$

for some positive constant E. Since

$$\Delta\tau(\min((\Delta s_{r_g})^4, (\Delta s_{r_g})^2(\Delta\tau)^2) + (\Delta s_{r_g})^4) \le (\min((\Delta s_{r_g})^4, (\Delta s_{r_g})^2(\Delta\tau)^2) + (\Delta s_{r_g})^4)$$

for a constant $K_p > 0$ we obtain

$$\max_{1\le i\le N-1} |\mathcal{E}_i^{m+1}|$$
$$\le (1 + E\Delta\tau)) \max_{1\le i\le N-1} |\mathcal{E}_i^m|$$
$$+ K_p \left(\left\| \frac{\partial^4 u}{\partial s^4} \right\|_\infty \left(\min\left(\frac{(\Delta s)^4}{\Delta\tau}, (\Delta s)^2\Delta\tau \right) + \frac{(\Delta s)^4}{\Delta\tau} \right) + \max_{1\le i\le N-1} |\mathrm{e}_i^{m+1}| \right) \Delta\tau. \tag{82}$$

At time zero the complete cubic spline interpolation function of the approximate solution U_i^0 is equal to that of the exact solution u_i^0. Therefore, the error associated with the backtrack point has the following bound:

$$\max_{1\le i\le N-1} |\mathcal{E}^0(\bar{s}_i^m)| \le K_q \left\| \frac{\partial^4 u^0}{\partial s^4} \right\| (\min((\Delta s)^4, (\Delta s)^2(\Delta\tau)^2) + (\Delta s)^4).$$

Introducing a constant $K_o > 0$, we obtain

$$\left\| \frac{\partial^3 u^{m+1}}{\partial s^3} \right\|_\infty |(\Delta s_{i+1} - \Delta s_i)| \le C_2 \left\| \frac{\partial^3 u^{m+1}}{\partial s^3} \right\|_\infty (\Delta\xi)^2$$
$$\le K_o \left\| \frac{\partial^3 u^{m+1}}{\partial s^3} \right\|_\infty (\Delta s_{r_g})^2. \tag{83}$$

Iterating inequality (82) and using constants $K_n, K_m > 0$, we obtain

$$\max_{1\le i\le N-1} |\mathcal{E}_i^{m+1}| \le \sum_{k=0}^{m} (1 + E\Delta\tau))^k K_p \Delta\tau \max_{0\le k\le m} \left\| \frac{\partial^4 u^k}{\partial s^4} \right\|_\infty \min\left(\frac{(\Delta s_{r_g})^4}{\Delta\tau}, (\Delta s_{r_g})^2\Delta\tau \right)$$
$$+ \sum_{k=0}^{m} (1 + E\Delta\tau))^k K_p \Delta\tau \max_{0\le k\le m} \left\| \frac{\partial^4 u^k}{\partial s^4} \right\|_\infty \left(\frac{(\Delta s_{r_g})^4}{\Delta\tau} \right)$$

$$
\begin{aligned}
&+\sum_{k=0}^{m}(1+E\Delta\tau))^k K_p \Delta\tau \max_{1\le i\le N-1}|\mathrm{e}_i^{m+1}| \\
&\le \sum_{k=0}^{m}\mathrm{e}^{Ek\Delta\tau}K_p\Delta\tau M_1 \min\left(\frac{(\Delta s_{r_g})^4}{\Delta\tau},(\Delta s_{r_g})^2\Delta\tau\right) \\
&+\sum_{k=0}^{m}\mathrm{e}^{Ek\Delta\tau}K_p\Delta\tau\left(\frac{M_1(\Delta s_{r_g})^4}{\Delta\tau}+\max_{1\le i\le N-1}|\mathrm{e}_i^{m+1}|\right) \\
&\le (m+1)\mathrm{e}^{ET}K_p\Delta\tau M_1\left(\min\left(\frac{(\Delta s_{r_g})^4}{\Delta\tau},(\Delta s_{r_g})^2\Delta\tau\right)+\frac{(\Delta s_{r_g})^4}{\Delta\tau}\right) \\
&+(m+1)\mathrm{e}^{ET}K_p\Delta\tau \max_{1\le i\le N-1}|\mathrm{e}_i^{m+1}| \\
&\le T\mathrm{e}^{ET}K_n M_1\left(\min\left(\frac{(\Delta s_{r_g})^4}{\Delta\tau},(\Delta s_{r_g})^2\Delta\tau\right)+\frac{(\Delta s_{r_g})^4}{\Delta\tau}\right) \\
&+T\mathrm{e}^{ET}K_n M_2((\Delta s_{r_g})^2+\Delta\tau) \\
&\le T\mathrm{e}^{ET}K_n\left(M_1\left(\min\left(\frac{(\Delta s_{r_g})^2}{\Delta\tau},\Delta\tau\right)+\frac{(\Delta s_{r_g})^2}{\Delta\tau}\right)\right. \\
&\left.+M_2((\Delta s_{r_g})^2+\Delta\tau)\right) \\
&\le T\mathrm{e}^{ET}K_m\left(M_1\left(\frac{(\Delta s_{r_g})^2}{\Delta\tau}\right)+M_2((\Delta s_{r_g})^2+\Delta\tau)\right),
\end{aligned} \tag{84}
$$

where T is the final time instance with the terms

$$M_1=\max_{0\le k\le m}\left\|\frac{\partial^4 u^k}{\partial s^4}\right\|_\infty$$

and

$$M_2=\max\left(\left\|\frac{\partial^4 u^{m+1}}{\partial s^4}\right\|_\infty,\sup_{[0,1]\times[0,T]}\frac{\partial^2 u^*}{\partial^2\mathcal{T}},\sup_{[0,1]\times[0,T]}\frac{\partial u^*}{\partial\mathcal{T}}\right).$$

Introducing a constant $M_d>0$, we obtain

$$\max_{1\le i\le N-1}|\mathcal{E}_i^{m+1}|\le M_d M_b\left(\frac{(\Delta s_{r_g})^2}{\Delta\tau}+\Delta\tau\right)$$

for

$$M_b=\max\left(\max_{0\le k\le m+1}\left\|\frac{\partial^4 u^k}{\partial s^4}\right\|_\infty,\sup_{[0,1]\times[0,T]}\frac{\partial^2 u^*}{\partial^2\mathcal{T}},\sup_{[0,1]\times[0,T]}\frac{\partial u^*}{\partial\mathcal{T}}\right),$$

completing the proof. ■

When the mesh is uniform, the mesh size is greater than or equal to the mesh size Δs_{r_g} of $[s_l,s_r]$ and Theorem 4.3 follows directly.

Table 1. Results obtained by standard and nonstandard finite difference methods applied to a European style real estate index put option for $T = 1$.

n	FD M_a	FD M_s	NSFD1 M_a	NSFD1 M_s	NSFD2 M_a	NSFD2 M_s
40	1.2378	4.3910-01	2.3040-01	9.1600-02	1.3000-02	6.5000-03
80	1.3990-01	3.7800-02	3.8200-02	1.0200-02	4.1904-04	1.2474-04
160	2.3000-02	4.7000-03	4.8000-03	9.6887-04	6.7105-07	1.3251-07
320	1.5000-02	2.4000-03	1.3000-03	2.0328-04	5.3380-12	7.9627-13

Table 2. Results obtained by standard and nonstandard finite difference methods applied to a European style real estate index put option for $T = 0.75$.

n	FD M_a	FD M_s	NSFD1 M_a	NSFD1 M_s	NSFD2 M_a	NSFD2 M_s
40	7.0280-01	2.4900-01	3.6790-01	1.3700-01	4.8200-02	1.7100-02
80	4.9400-02	1.3300-02	1.2200-02	3.3000-03	2.9406-04	7.8739-05
160	3.9000-03	7.8841-04	8.1306-04	1.6180-04	6.2429-08	1.7320-08
320	1.0000-03	1.5831-04	1.2510-04	1.8850-05	3.4855-13	5.4476-14

5. Numerical results and discussion

In this section, we present some numerical simulation results for pricing European and American style real estate index options. We present the maximum absolute error (M_a) and the mean square norm (M_s) for an implicit finite difference scheme (FD), the NSFD scheme with uniform spatial grid sizes (NSFD1) and the NFSD with nonuniform spatial grid sizes (NSFD2). In line with [10,11,15,16,18,19], we let

$$M_a = \max_{1 \le i \le n} |U(s_i, \tau_0) - u(s_i, \tau_0)| \tag{85}$$

and

$$M_s = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (U(s_i, \tau_0) - u(s_i, \tau_0))^2}. \tag{86}$$

The formulae for the analytic solution $u(s, \tau)$ for the European call option price is given by Fabozzi *et al.* [13]. The analytical solution for the European put option is found through the put-call parity. Tables 1 and 2 display the results from the numerical experiments when using $M = 2000$, $S_{\tau_0} = 1700$, $\tau_0 = 0.01$, $L = 2000$, $s_l = 0.45$, $s_r = 0.9$, $d_1 = 0.002$, $r = 0.05$ and $K = 1600$.

The parameters of the mean-reverting process are estimated using the yearly historical time series for the IPD UK index from the year 1946 to the year 2009. These are estimated as $\alpha = 0.7771$, $\beta = 0.1045$, $\theta = 0.1165$ and $\sigma = 0.131$, [13].

The market price of risk λ is obtained when the theoretical futures prices for different maturities calculated from the model by Fabozzi *et al.* [13] are equated with the market futures prices from Eurex, for the period December 2009 to December 2013. We set $\lambda = 0.7$, see [10,11,15,16,18,19].

Tables 1 and 2 show that as the number of spatial grid points increases so does the accuracy of the NSFD2. This method is generally more accurate than the FD or the NSFD1 scheme. The NSFD schemes are generally more accurate than the standard finite difference scheme.

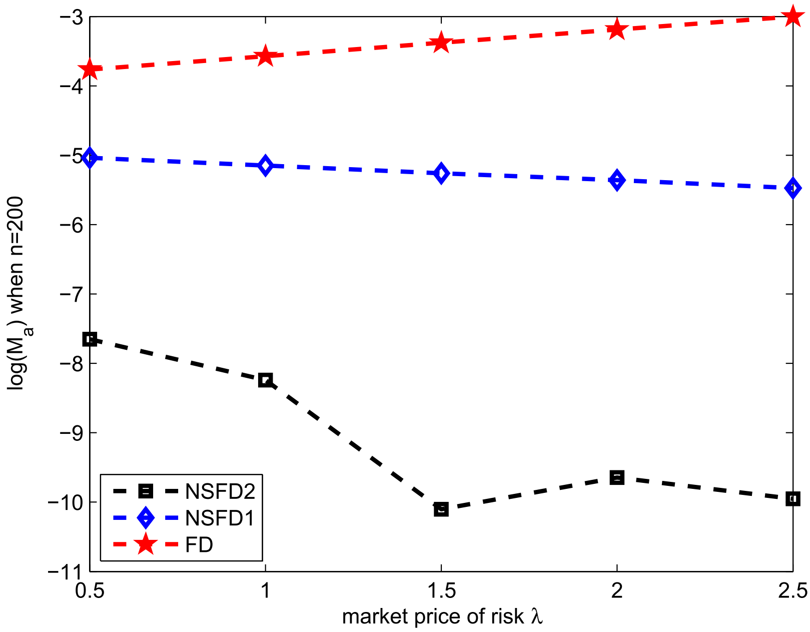

Figure 1. Values of $\log(M_a)$ for a European put option for varying λ for the NSFD1, NSFD2 and FD schemes using $n = 200$.

The market price of risk changes according to the futures contract used in estimating it. This is consistent with the empirical evidence provided by Fabozzi *et al.* [13] where estimates show that $\lambda \in (0, 3)$. This captures changes in the perception of market risk.

Figure 1 displays how the market price of risk varies with the logarithm of M_a. The log errors associated with NSFD2 are in general smaller than those exhibited by NSFD1 and FD. The FD method diverges as the number of grid points increases. This may be due to a violation of stability conditions. These results shown in Figure 1 imply wider applicability of our NSFD scheme for different market prices of risk.

The difference between a European and American option is on when the option can be exercised. Whereas a European option can be exercised at the expiration time an American option can be exercised at any time during the lifetime of the option. To approximate the American option prices on a time grid, we assume that each exercise date coincides with time steps on the grid. To capture this aspect for a put option we include the optimal exercise condition in our finite difference schemes at each time step for all stock price levels, that is if

$$U_i^{m+1} < (K - S_i),$$

then

$$U_i^{m+1} = K - S_i.$$

This approximation becomes more accurate for smaller values of $\Delta\tau$ [20].

The exact solution of the American option pricing problem is not available. So in order to determine the error on the maximum norm we use the double mesh principle. We define

Table 3. Results obtained by standard and nonstandard finite difference methods applied to an American style real estate index put option.

n	FD M_{a1}	NSFD1 M_{a1}	NSFD2 M_{a1}
20	50.0006	2.5000-02	7.5000-03
40	24.9982	1.2500-02	3.8000-03
80	12.4439	6.3000-03	1.9000-03
160	10.8316	3.1000-03	9.3750-04

the error as the double mesh difference M_{a1} given by

$$M_{a1}^{n} = \max_{1\le i\le n} |U(s_{i^{(n)}}, \tau_0) - U(s_{i^{(2n)}}, \tau_0)|. \tag{87}$$

The terms $i^{(n)}$ and $i^{(2n)}$ represent that the numerical solutions are obtained when n and $2n$ grid points are respectively used on the spatial grid. We use $\tau = 5 \times 10^{-4}$, $T = 1$, $d_1 = 2 \times 10^{-16}$, $s_l = 0.5$, $s_r = 0.8$ and $S_{\tau_0} = 800$, with the other parameter values remaining the same as in the European option pricing case. These errors are presented in Table 3.

Table 3 shows that as the number of spatial grid points increase the NSFD2 scheme has a greater increase in accuracy as compared to the NSFD1 and FD schemes. This means nonuniform grid sizes with a higher concentration near the strike price improve the accuracy of the NSFD scheme when pricing American options. The NSFD schemes generally outperform the FD scheme.

6. Concluding remarks and scope for further research

This study considers the problem of pricing European and American real estate index options within the framework proposed by Fabozzi *et al.* [13]. A hedging strategy is utilized that uses two derivatives that are dependent on the real estate index. As a result, a terminal-boundary value variable coefficient parabolic partial differential equation for the option price is obtained. This partial differential equation of option prices can be viewed as consisting of two sub equations which are the advection reaction equation and the diffusion equation. We develop an appropriate nonstandard finite difference scheme [8,21,23]. This scheme is then combined with a complete cubic spline interpolation approximation and a finite difference scheme for the diffusion. A theoretical analysis shows that the global error is bounded. Numerical simulations prove the accuracy of the proposed scheme in pricing European and American put options. The scheme that uses nonuniform grid sizes is shown to be more accurate in the neighbourhood of the strike price than the scheme that uses fixed grid sizes. The scheme is also shown to be applicable for different market conditions. Future research can extend our study by considering the effects of transaction costs, stochastic volatility and jumps in the real estate index model. Our NSFD scheme can also be used to approximate solutions of the partial differential equation-based models for the formation of biofilm-forming microbes and for ground water solute pollution problems.

Acknowledgments

We thank the anonyms referees for their fruitful comments and suggestions which helped in improving the presentation of the work in this manuscript. K.C.P.'s research was also supported by the South African National Research Foundation, Grant No: CPRR160413162033.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

KCP's research was also supported by the South African National Research Foundation [grant number CPRR160413162033].

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