Risk-Neutral Valuation of Real Estate Derivatives

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This article proposes a novel and intuitive risk-neutral valuation model for real estate derivatives. The authors first model the underlying efficient market price of real estate and then construct the observed index value with an adaptation of the price update rule by Blundell and Ward [1987]. The resulting index behavior can easily be analyzed and closed-form pricing solutions are derived for forwards, swaps, and European put and call options. They demonstrate the application of the model by valuing a put option on a house price index. Autocorrelation in the index returns appears to have a large impact on the option value. They also study the effect of an over- or undervalued real estate market. The observed effects are significant and as expected.

Recently, interest in real estate derivatives has surged. This interest has been fueled by, among other things, the introduction of real estate futures on the Chicago Mercantile Exchange (CME) in 2006. These futures give investors the opportunity to directly manage house price risk. Currently, trading is possible using 20 regional indexes and two composite indexes. See research by Bertus et al. [2008] and Shiller [2008] for more information. Geltner and Fisher [2007] and Fabozzi et al. [2009, 2011] also provide a good overview of other real estate derivatives markets, such as swap trading on the U.K. Investment Property Database index (IPD) or the U.S. NCREIF Property Index (NPI).

Currently, the most mature property derivatives market is the U.K. IPD derivatives market. At the end of 2008, some GBP 19.3 billion of swaps referenced IPD indexes. In the beginning of 2009, trading in IPD derivatives decreased significantly, however, mostly because fewer deals between banks were executed with Lehman Brothers exiting the market and several other banks cutting back on new business activities. The U.S. CME futures market does not yet have much liquidity, with only occasional trades. Property derivatives markets in France and Germany are also still very small.

In this article, we develop a novel and intuitive risk-neutral valuation model for real estate derivatives. Our main goal is to value derivatives that are coupled to private real estate indexes with a significant degree of autocorrelation. It is well known from the real estate literature (see, e.g., Geltner et al. [2003] for an overview) that autocorrelation can occur in appraisal-based indexes because appraisers slowly update past prices with new market information. Transaction-based indexes can also exhibit a positive autocorrelation because private real estate markets are less informationally efficient than public securities markets. As a result, the price discovery and information aggregation functions of the private real estate market are less effective. This can cause noisy prices and inertia in asset values (and returns); see Shiller [2008, p. 4] and the references given in that paper.

A significantly positive autocorrelation implies a (partial) predictability of future returns and opportunities for arbitrageurs. It is not possible, however, to trade the assets that constitute a private real estate index in a liquid market and at low costs. In practice, the index is thus not a tradable asset and arbitrage possibilities are very limited. This can also cause significant problems for suppliers of real estate derivatives because they cannot easily trade the underlying assets and (delta) hedge their positions. Nevertheless, derivatives markets for forward and swap contracts are emerging in recent years.

Geltner and Fisher [2007] noted, however, that a 2006 survey of (potential) market participants identified a lack of confidence in how real estate derivatives should be priced. They also noted that this concern is understandable, because the underlying asset cannot be traded in a frictionless market. This makes it impossible to use classic pricing formulas for derivatives (such as the relationship between spot and forward prices), because these formulas only apply under strict no-arbitrage assumptions. Geltner and Fisher [2007] argued, however, that the valuation of real estate derivatives is still possible using equilibrium pricing rules, provided that the dynamic behavior of the underlying real estate index is properly taken into account. In this article, we take the next step by proposing a quantitative risk-neutral valuation model that can be used for actual pricing purposes.

A small body of related research exists in the equity option literature. Lo and Wang [1995] studied the effect of predictability of asset returns in a continuous-time model. They propose an adjustment of the Black and Scholes [1973] pricing formula for stock options to account for the effect of predictability. Jokivuolle [1998] developed a discrete-time model to derive an analytical pricing formula for options on a stock index that exhibits positive correlation due to infrequent trading of the underlying stocks. He assumed that the unobservable true liquidation value of the index follows a random walk process. The observed (autocorrelated) index is then modeled as the weighted average of current and past returns. More recently, Liao and Chen [2006] derived a closed-form formula for a European option on an asset with returns following a first-order moving average process. Fabozzi et al. [2011] used mean-reverting continuous-time models, which exhibit predictability for the drift term, for deriving closed-form solutions of the main property derivatives traded in the financial markets.

The real estate literature also contains a few pioneering papers on risk-neutral valuation. Early examples of risk-neutral valuation techniques are given by Kau et al. [1990], Buetow and Albert [1998] and Buttimer and Kau [1997]. The Buttimer and Kau paper is especially important because it describes how a risk-neutral valuation model can be used to value derivatives that are related to commercial real estate indexes. Throughout their paper, these authors assume that the real estate index follows a random walk process with drift. By construction, such a process leads to uncorrelated index returns. They noted, however, that their model can also be used in case of autocorrelated indexes, provided that a proper transformation can be found to switch (back and forth) between the autocorrelated index (which is observed) and the uncorrelated variable (which is explicitly modeled).

A different approach is followed by Shiller and Weiss [1999] in their paper on home equity insurance. They first fitted the observed real estate returns with a simple autoregressive (AR) model with one lag. Using this model, the conditional returns and volatilities can be determined analytically. The assumption was then made that options on the house price index can be valued using an adaptation of the familiar Black and Scholes [1973] equation. This adaptation consists of replacing the expected risk-free return with the expected real-world return and the implied volatility with the estimated value from the AR model. One aspect of this approach is adopted by us, namely modeling the real estate returns with an AR model. We view the second, heuristic, step (in which the real-world return is directly used as an input for a risk-neutral valuation formula) as problematic, however.

Our approach circumvents this problem by following the approach of Jokivuolle [1998]. We thus explicitly model the (underlying) "efficient market" value of the real estate index and then construct the observed index. We argue that an adaptation of the price update rule proposed by Blundell and Ward [1987] serves this purpose well. The first modification of this update rule is straightforward and consists of adding multiple lag terms. This leads to an AR model that can be estimated using standard econometric techniques.

A second modification is more fundamental from a valuation perspective and consists of using the accrued value of past observations. Empirical results confirm that the volatility and autocorrelation of a (transaction-based) Dutch house price index can be replicated quite well (on an annual basis) with our model. As a second example, we consider monthly data for the U.S. 10-city S&P/Case-Shiller house price index. Our analysis shows that modeling seasonality and stochastic volatility is important for such monthly data.

The remainder of this article is organized as follows. First, we introduce our theoretical framework and analyze the properties of the real-world and risk-neutral process for real estate indexes with autocorrelation. We also explain how the real estate model can be coupled to a stochastic interest rate model. The subsequent section contains closed-form pricing formulas for forwards, swaps, and European options. We then estimate the real estate model using historical information for house price indexes in the Netherlands and the United States and go on to discuss the valuation of a European put option on a house price index using Monte Carlo simulation. We also assess the quality of the derived closed-form option pricing formula.

THEORETICAL FRAMEWORK

Notational Conventions

Our risk-neutral model consists of a discrete-time model for the observed real estate index in combination with continuous-time models for the efficient market process of real estate and for interest rates. The current point in time is denoted as $t = 0$. Time is measured with respect to the period between two price updates of the real estate index. Unless stated otherwise (and without loss of generality), we assume that the time step between two price updates is equal to one year. Hence, $t = 1$ corresponds to one year ahead, $t = 2$ to two years ahead, and so on. In the continuous-time models, the non-integer points in time are also sampled. To avoid confusion, we therefore denote the continuous-time variable with τ in the remainder of this article.

Real-World Process

Price update model. We model the real-world process of a real estate index with an adaptation of the price update rule proposed by Blundell and Ward [1987].[1] They suggest that the new price is a weighted sum of the current market price and the last period's price. More precisely, they propose the following price update rule:[2]

$$a(t) = Ky(t) + (1 - K)a(t - 1) \tag{1}$$

where $a(t)$ is the current price, $a(t - 1)$ is the previous price, $y(t)$ is the "true" market price, and K is a constant, $0 \leq K \leq 1$. The parameter K is commonly referred to as the "confidence" parameter. If K is close to 1, the market price $y(t)$ is weighted heavily; if K is small, the emphasis is more on the previous price $a(t - 1)$. The simple price update rule in Equation (1) is frequently used to model appraisal smoothing in real estate indexes. See Geltner et al. [2003] for an extensive overview of research in this area. We show in this article that this price update rule can also be used to describe the dynamic behavior of autocorrelated transaction-based indexes.

The model in Equation (1) is equivalent to an exponentially weighted moving average (EWMA) model, see Hull [2009, pp. 479–480]. By substituting the expression for $a(t - 1)$ in $a(t)$, the expression for $a(t - 2)$ in $a(t - 1)$, and so on, we find that

$$a(t) = K\sum_{i=1}^{m}(1 - K)^{i-1} y(t - i + 1) + (1 - K)^m a(t - m) \tag{2}$$

where $1 \leq m \leq t$. This equation shows that the current value $a(t)$ partly consists of a basket of previous y-terms, where the weight of these terms decreases at an exponential speed (as controlled by the K parameter). By setting m equal to t we also see that the weight of the index value at time 0, $a(0)$ is equal to $(1 - K)^t$ at time t.

It is important to note that the price update rule in Equation (1) does *not* account properly for the time value of money because the previous value $a(t - 1)$ is not accrued. From a valuation perspective, this leads to a systematic underperformance of the real estate index. To correct for this effect, we adapt the price update rule in Equation (1) and accrue the past index value with the expected (annual) return π:

$$a(t) = Ky(t) + (1 - K)(1 + \pi)a(t - 1) \tag{3}$$

Using accrued prices is common practice when appraisers set new prices for real estate objects. In this case, the reference price level is often formed by previous transactions for similar objects with a correction for the price increase (or decrease) of the real estate market up to the current point in time. The expected return is not modeled in detail at this point to keep the analysis as simple as possible. In practice, the expected return may depend (positively) on the amount of risk associated with the real estate investment. In the risk-neutral model the expected return is coupled directly to the level of the interest rate, as we explain in detail later in this section.

Substitution of $a(t-1)$ in $a(t)$, $a(t-2)$ in $a(t-1)$, and so on, again yields the EWMA form of Equation (3):

$$a(t) = K\sum_{i=1}^{m}(1-K)^{i-1}y^*(t-i+1) + (1-K)^m a^*(t-m) \tag{4}$$

where $1 \le m \le t$ and

$$\begin{aligned} a^*(t-m) &\equiv a(t-m)(1+\pi)^m, \\ y^*(t-i+1) &\equiv y(t-i+1)(1+\pi)^{i-1} \end{aligned} \tag{5}$$

Equation (4) is thus equivalent to Equation (2) if we accrue past values.

We can easily determine the evolution of annual returns based on Equation (3):

$$r^a(t) = K\frac{y(t-1)}{a(t-1)}r^y(t) + (1-K)\frac{a(t-2)}{a(t-1)}(1+\pi)r^a(t-1) \tag{6}$$

where $r^a(t) \equiv a(t)/a(t-1) - 1$ is the index return and $r^y(t) \equiv y(t)/y(t-1) - 1$ is the unobserved return, both using annual compounding. A much simpler expression is derived when the index series are expressed in logarithms; see Geltner et al. [2003]. In this case, continuously compounded returns can be expressed as log differences:

$$r_c^a(t) = K^* r_c^y(t) + (1-K^*)r_c^a(t-1) \tag{7}$$

and thus

$$r_c^y(t) = \frac{1}{K^*}r_c^a(t) - \frac{1-K^*}{K^*}r_c^a(t-1) \tag{8}$$

where $r_c^a(t)$ is the index return and $r_c^y(t)$ is the unobserved market return, both using continuous compounding. The parameter K^* has a similar interpretation as the confidence parameter K. This parameter determines what fraction of the index return is explained by the unobserved market return (the remaining fraction is explained by the past index return). Note that the effect of accrual disappears when we take log differences (that is, when we assume that past values accrue with the same return π).

The efficient market process. We now assume that the underlying market returns follow a random walk process with drift:

$$r_c^y(t) = \pi + \varepsilon(t) \tag{9}$$

where $\varepsilon(t)$ is a normally distributed, serially uncorrelated noise term with zero mean and variance σ_ε^2. Note that the drift parameter π is assumed to be constant here to keep the analysis as simple as possible. In successive periods of appreciation and depreciation of the price levels, however, this assumption is not always valid. A more appropriate specification would then be to allow π to change over time. An example of such a model is a local linear trend model, which is used by, for example, Francke [2010].

The confidence parameter K^* can be calculated from the first-order autoregressive (AR) process that we obtain by substituting Equation (9) in Equation (7):

$$r_c^a(t) = K^*\pi + (1-K^*)r_c^a(t-1) + K^*\varepsilon(t) \tag{10}$$

K^* is thus equal to 1 minus the first-order autocorrelation of the index returns. In practice, we can also neglect the difference between K in Equation (6) and K^* in Equation (7) because (on average) $y(t-1) \approx a(t-1)$ and $\frac{a(t-2)}{a(t-1)} \approx \frac{1}{1+\pi}$. Under these simplifying assumptions, the functional form of Equations (6) and (7) becomes the same. Annually and continuously compounded returns also have almost the same first-order autocorrelation.[3] It thus follows that $K \approx K^*$.

A word of caution is appropriate at this point. The assumption that the underlying market returns follow a random walk with drift is probably too strong for the private real estate market because these markets are less informationally efficient than public securities markets; see also Geltner et al. [2003]. We should therefore be careful not to directly equate the underlying random walk process with the true market process. A better interpretation would be to state that the observed index returns can be modeled using an underlying efficient market process in combination with the price update rule in Equation (1). In the remainder of this article, we therefore refer to $y(t)$ as the efficient market price at time t. We will show later in this section that assuming an underlying efficient market process makes it possible to easily analyze the properties of the constructed real estate index with autocorrelation. This facilitates the derivation of several pricing formulas in the next section.

Price update rules with multiple lags. The price update rule in Equation (1) can easily be extended with multiple lag terms. For the general case of p lags (with $p \geq 1$), we have

$$a(t) = Ky(t) + \sum_{i=1}^{p} \omega_i a(t-i), \text{ where } K + \sum_{i=1}^{p} \omega_i = 1 \quad (11)$$

The generalization of Equation (10) then becomes:

$$r_c^a(t) = K^*\pi + \sum_{i=1}^{p} \omega_i^* r_c^a(t-i) + K^*\varepsilon(t), \quad \text{where } K^* + \sum_{i=1}^{p} \omega_i^* = 1 \quad (12)$$

Equation (12) is an AR model of order p. To estimate the expected return π, the weights ω_i^* and the variance σ_ε^2 of an AR(p) model different approaches can be followed; see Lütkepohl [2006] or Steehouwer [2005, pp. 43–129] for detailed overviews. Note that the restriction that the sum of the weights should be equal to 1 does not complicate the estimation of the model since both K^* and π are free parameters.

A simple approach is to estimate the model parameters with an ordinary least squares (OLS) regression method. This method basically minimizes the one-step-ahead prediction errors. An alternative approach is to choose the weights in such a way that the autocovariance function of the AR process is exactly equal to the autocovariance function of the observed real estate index. This correspondence can be achieved by using the Yule–Walker equations (see Steehouwer [2005, p. 46]).

To decide which model is most appropriate several order selection criteria have been proposed in the literature (see, e.g., Steehouwer [2005, pp. 82–84]). These selection criteria typically choose the model order in such a way that the prediction error is minimized while putting a penalty on the number of parameters estimated. The estimation of the real estate model is discussed in detail later in the article.[4]

Seasonality. Seasonality in real estate returns can become important when modeling quarterly or monthly returns. Let us assume that we have already modeled the seasonally adjusted index $a(t)$ using Equation (1) or Equation (11). We can then add a seasonal component $g(t)$ to obtain the index value with seasonality, $\tilde{a}(t)$:

$$ln(\tilde{a}(t)) = ln(a(t)) + g(t) \quad (13)$$

or, equivalently:

$$\tilde{a}(t) = a(t)\exp(g(t)) \quad (14)$$

Different approaches can be used to estimate the $g(t)$ function. A natural assumption is to assume that seasonality does not have a net effect on an annual basis. For simplicity, one can also assume that the seasonal pattern is constant over time. Given these assumptions, one could then use so-called dummy variables in the OLS regression. These dummy variables are equal to 1 for the respective periods. For example, a January dummy is equal to 1 for all January (log) returns and zero for all other months; a February dummy is equal to 1 for all February returns and zero otherwise, and so on. The $g(t)$ function is then easily constructed using the estimated weights of the dummy variables. Another (even simpler) method consists of detrending the log index and then fitting a (shifted) sine function with a period of one year to the data. The first approach (i.e., a regression on monthly dummies) is used later in the article, in the subsection "Example 2: U.S. House Price Index."

Risk-Neutral Process

In a risk-neutral world, all individuals are indifferent to risk and expect to earn on all securities a return equal to the (instantaneous) risk-free rate. Assuming that the world is risk neutral greatly facilitates the valuation of options: the option payoffs can simply be discounted along the path of the short rate for each scenario. It is also important to note that risk-neutral valuation gives the correct price of an option in *all* worlds (also the risk-averse real world), not just in the risk-neutral world.

Process for interest rates. We model the evolution of the short interest rate in this article with the familiar one-factor Hull–White (HW) model (see Hull [2009, pp. 688–689]). Within the large family of interest rate models, the HW model is a typical example of a no-arbitrage model. Such a model produces interest-rate scenarios which are consistent with the current term structure. This no-arbitrage feature is extremely important for option pricing applications, because a small error in the underlying bond prices can cause large errors in the price of interest-rate options, see Hull [2009, p. 686].

Technically speaking, the one-factor HW model assumes that the risk-neutral process for the nominal short rate, r_N, is as follows:

$$dr_N(\tau) = \kappa\left(\frac{\theta(\tau)}{\kappa} - r_N(\tau)\right)d\tau + \sigma_1 dZ_1(\tau) \tag{15}$$

We denote time in this equation with the symbol τ to indicate that we now use a continuous-time model. This model assumes that the short interest rate fluctuates around the mean-reversion level $\theta(\tau)/\kappa$. The parameter κ controls the amount of mean reversion. The θ function is deterministic and chosen in such a way that the model satisfies the no-arbitrage constraint. The one-factor HW model is in fact an extension of the Vasicek [1997] model in the sense that the mean-reversion level is time-dependent instead of constant. σ_1 controls the volatility of the Wiener process dZ_1.

Price update model. We now derive the risk-neutral process for real estate indexes with autocorrelation. The risk-neutral process for the evolution of the index value can be derived analogously to Equation (3):

$$\begin{aligned} &a(t) = Ky(t) + (1-K)\pi(t)a(t-1), \\ &\text{where } \pi(t) \equiv \exp\left(\int_{t-1}^{t} r_N(\tau)d\tau\right)\exp(-q) \end{aligned} \tag{16}$$

The expected return π is thus a time-dependent function in a risk-neutral world and depends on the level of the (short) interest rate and the direct return. More precisely, the term $\exp\left(\int_{t-1}^{t} r_N(\tau)d\tau\right)$ is the risk-free return on a bank account between time $t-1$ and t. The term $\exp(-q)$ is a correction for the direct return q associated with real estate investments. By setting q equal to zero a total return index is modeled.

Substitution of $a(t-1)$ in $a(t)$, $a(t-2)$ in $a(t-1)$, and so on, again yields the following EWMA form of Equation (16):

$$a(t) = K\sum_{i=1}^{m}(1-K)^{i-1}y^*(t-i+1) + (1-K)^m a^*(t-m) \tag{17}$$

where $1 \leq m \leq t$ and

$$\begin{aligned} a^*(t-m) &\equiv a(t-m)\exp\left(\int_{t-m}^{t} r_N(\tau)d\tau\right)\exp(-qm), \\ y^*(t-i+1) &\equiv y(t-i+1)\exp\left(\int_{t-i+1}^{t} r_N(\tau)d\tau\right)\exp(-q(i-1)) \end{aligned} \tag{18}$$

Equation (16) can be extended for the general price update model with p lag terms:

$$a(t) = Ky(t) + \sum_{i=1}^{p}\omega_i a^*(t-i) \quad \text{where} \quad K + \sum_{i=1}^{p}\omega_i = 1 \tag{19}$$

The EWMA form of Equation (19) can also be derived. Let us assume that $T > t \geq 0$. We can now substitute $a^*(T-1)$ in $a(T)$, $a^*(T-2)$ in $a^*(T-1)$, and so on, until an expression is obtained with only the terms $y^*(t+1), \ldots, y^*(T)$ and $a^*(t), \ldots, a^*(t-p+1)$:

$$a(T) = \sum_{i=1}^{T-t} c_i y^*(t+i) + \sum_{i=1}^{p} d_i a^*(t-i+1) \tag{20}$$

where $c_{T-t} = K$. Explicit expressions for the c_i and d_i coefficients of this equation can be determined using a software package that is able to perform symbolic algebra calculations.[5]

The efficient market process. We also need to specify the risk-neutral process for the underlying efficient market price. Analogously to Equation (9), we use a random walk process with drift (geometric Brownian motion):

$$dy(\tau) = (r_N(\tau) - q)y(\tau)d\tau + \sigma_2 y(\tau)dZ_2 \tag{21}$$

where the volatility σ_2 is constant and dZ_2 follows a Wiener process. By means of Ito's lemma, it can be shown that ln $y(\tau)$ is governed by the following process (see Hull [2009, pp. 270–271]):

$$d\ln y(\tau) = \left(r_N(\tau) - q - \sigma_2^2/2\right)d\tau + \sigma_2 dZ_2 \qquad (22)$$

For numerical reasons, Equation (22) is commonly used in practice instead of Equation (21). Note that these equations are equivalent to the Black and Scholes [1973] price process for a dividend paying stock in case of stochastic interest rates.

Model extensions: Real interest rates, inflation, stochastic volatility. It is also possible to model real interest rates and inflation in a consistent way. Brigo and Mercurio [2006, pp. 646–647], for example, developed a consistent risk-neutral model for nominal and real interest rates as well as the CPI index. To keep the analysis as simple as possible, we do not discuss such an extended model in this article. Including inflation may be very important for practical applications, however, because real estate cash flows (like rental income or maintenance costs) are often inflation-linked.

Another extension consists of modeling stochastic volatility. This is especially important when considering high-frequency data, like monthly or quarterly returns. An example is given later for monthly U.S. house price data. A quite general stochastic volatility model is the constant elasticity of variance (CEV) model:

$$\begin{cases} dy(\tau) = (r_N(\tau) - q)y(\tau)d\tau + \sqrt{V(\tau)}\,y(\tau)dZ_2, \\ dV(\tau) = \lambda(\gamma - V(\tau))dt + \sigma_3 V(\tau)^{\beta} dZ_3 \end{cases}$$

This model is a natural extension of the geometric Brownian motion in Equation (21) and has for example been studied by Jones [2003]. The λ parameter controls the speed of mean reversion of the variance $V(\tau)$. The γ parameter denotes the mean reversion level, and σ_3 controls the volatility of the variance process. The initial variance should, of course, also be specified as a boundary condition. The elasticity parameter β must satisfy $0.5 \le \beta \le 1.0$ to retain the uniqueness of option prices. Both limiting cases are in fact well-known stochastic volatility models. For $\beta = 1/2$, we have the model of Heston [1993] and for $\beta = 1$ we have the continuous-time GARCH model as in Nelson [1990]. Maximum likelihood estimation of the CEV model parameters, based on option prices, is discussed in detail in an excellent paper by Aït-Sahalia and Kimmel [2007].

Martingale Properties

The efficient market process. If there are no arbitrage opportunities, the expected price of a traded security has to increase in the same way as a bank account in a risk-neutral world, see Hull [2009, p. 630]. To verify this no-arbitrage restriction, we consider the realization of the efficient market price $y(t)$ and the nominal bank account $B_N(t) \equiv B(0)\exp\left(\int_0^t r_N(\tau)d\tau\right)$ up to time t and determine the expected value of the ratio $y(T)/B_N(T)$ for $T > t \ge 0$. Let us first consider the situation where all direct returns are reinvested in the index (i.e., we have a total return index). This situation can also be modeled by setting the direct return q equal to zero. We then have that (see Appendix A.1 for the proof):

$$E_Q\left[\frac{y(T)}{B_N(T)} \mid F_t\right] = \frac{y(t)}{B_N(t)} \qquad (23)$$

where $E_Q[y(T)/B_N(\mathrm{T}) \mid F_t]$ means that the expected value of $y(T))/B_N(T)$ in a risk-neutral world and conditional on the filtration up to time t is considered. The expected value of $y(T)/B_N(T)$ is thus constant for $T > t \ge 0$. That is, this ratio is a zero-drift (martingale) process. A total return index thus satisfies the martingale requirement for traded securities if its dynamics is governed by Equation (22).

Note that a price index with $q > 0$ is not a tradable asset, comparable to the situation for an index of dividend-paying stocks. Consequently, the martingale property is not satisfied by a price index if $q > 0$. In this case,

$$E_Q\left[\frac{y(T)}{B_N(T)} \mid F_t\right] = \frac{y(t)}{B_N(t)}\exp\left(-q(T-t)\right) \qquad (24)$$

see again Appendix A.1. A price index is thus not a tradable asset if direct returns are paid out.

The real estate index process. We now consider the realization of the real estate index $a(t)$ up to time t and determine the expected value of the ratio $a(T)/B_N(T)$ for $T > t \ge 0$. In Appendix A, we also prove that

$$E_Q\left[\frac{a(T)}{B_N(T)} \mid F_t\right] = \frac{\exp\left(-q(T-t)\right)}{B_N(t)} \times \left[y(t)(1-\alpha_{K,T}(t)) + a(t)\alpha_{K,T}(t)\right] \qquad (25)$$

where $\alpha_{K,T}(t) \equiv (1-K)^{T-t}$. $a(T)/B_N(T)$ is thus a martingale if $a(t) = \gamma(t)$ and $q = 0$. Because $a(t) = \gamma(t)$ holds in general only when $K = 1$; Equation (16) does *not* represent (the risk-neutral process of) a tradable asset when $K < 1$. Arbitrage opportunities would thus exist in case of a complete market when trading an autocorrelated real estate index. The reverse argument also holds: the index value may well be different from the efficient market price, but active trading in the index is not possible in this case; otherwise arbitrageurs would quickly force the index value toward the efficient market price.

Another important observation is that the future development of a total return real estate index with autocorrelation (i.e., $q = 0$ and $K < 1$) is unbiased if the index is in equilibrium at time t (i.e., when $a(t) = \gamma(t)$). By "unbiased," we here mean that $E_Q[a(T)/B_N(T) = a(t)/B_N(t)$ for $T > t$. Stated otherwise, if the real estate index starts from an equilibrium situation, the expected return is in line with the return on a risk-free bank account. As a consequence, the pricing formulas for linear instruments (forwards and swaps) all collapse to the classic no-arbitrage formulas if a total return real estate index is in equilibrium at the valuation date. This will be proved more formally in the next section.

A generalization of Equation (25) also exists for the price update model with more than one lag term, as specified in Equation (19). Using the same procedure as in Appendix A.2, the counterpart of Equation (25) follows:

$$E_Q\left[\frac{a(T)}{B_N(T)} \mid F_t\right] = \frac{\exp\left(-q(T-t)\right)}{B_N(t)}\left[\gamma(t)\sum_{i=1}^{T-t} c_i + \sum_{i=1}^{p} d_i \hat{a}(t-i+1)\right],$$

where

$$\hat{a}(t-i+1) \equiv a(t-i+1)\exp\left(\int_{t-i+1}^{t} r_N(\tau)d\tau\right)\exp\left(-q(i-1)\right) \tag{26}$$

Incorporating seasonality is also straightforward (see Equation (14)):

$$E_Q\left[\frac{\tilde{a}(T)}{B_N(T)} \mid F_t\right] = \exp(g(T))E_Q\left[\frac{a(T)}{B_N(T)} \mid F_t\right] \tag{27}$$

where $\tilde{a}(t)$ is the index value with seasonality.

Theoretical Framework: Conclusion

We have developed a simple and intuitive risk-neutral model for autocorrelated real estate indexes. This model can be coupled to existing risk-neutral models for interest rates, inflation, stochastic volatility etc. By studying the martingale properties of the real estate index we find that the no-arbitrage restriction is only satisfied under very specific conditions (i.e., for total return indexes without autocorrelation). In general, arbitrage possibilities thus exist. These cannot be exploited easily, however, since the underlying index cannot be traded actively. We will use the derived results in the next section to derive pricing formulas for various real estate derivatives.

PRICING FORMULAS

In this section, we derive pricing formulas for derivatives that are linked to autocorrelated real estate indexes. To keep the analysis as transparent as possible, we first present results for the simple price update model with one lag term and then for a model with multiple lags.

Forwards

We can easily determine the price of a forward contract on a real estate index. Let us assume that the forward contract expires at time $T > t$ and that the agreed-upon delivery price is $F_T(t)$. For the owner of the forward contract, the payoff at time T is then equal to the difference between $F_T(t)$ and the index $a(T)$. If we denote the price of this contract at time t as $f(t)$, we have that

$$f(t) = B_N(t)E_Q\left[\frac{F_T(t)-a(T)}{B_N(T)}\middle| F_t\right] \tag{28}$$

Using Equation (25) we then arrive at the following result:

$$f(t) = D(t,T)F_T(t) - \exp\left(-q(T-t)\right)[\gamma(t)(1-\alpha_{K,T}(t)) + a(t)\alpha_{K,T}(t)] \tag{29}$$

where $D(t,T)$ denotes the price at time t of a zero-coupon bond that matures at time $T > t$. The scaling factor $\alpha_{K,T}(t)$ is equal to $(1 - K)^{T-t}$. The price of this forward contract is equal to zero if

$$F_T(t) = \frac{\exp(-q(T-t))}{D(t,T)}[y(t)(1-\alpha_{K,T}(t)) + a(t)\alpha_{K,T}(t)] \quad \text{if } f(t) = 0 \tag{30}$$

When the index value $a(t)$ is also equal to the efficient market price $y(t)$, we obtain the classic relationship between the (spot) value of the index and the forward price:

$$F_T(t) = \frac{\exp(-q(T-t))}{D(t,T)}a(t) \quad \text{if } y(t) = a(t) \text{ and } f(t) = 0 \tag{31}$$

This analysis is easily extended to more general price update models with seasonality and multiple lag terms. By substituting Equation (27) in Equation (28), we arrive at:

$$\tilde{F}_T(t) = \frac{\exp(g(T))\exp(-q(T-t))}{D(t,T)} \times \left[y(t)\sum_{i=1}^{T-t} c_i + \sum_{i=1}^{p} d_i \hat{a}(t-i+1) \right] \tag{32}$$

with $\tilde{F}_T$ the forward price including the seasonal component.

Geltner and Fisher [2007] noted that the forward market can signal that the real estate market is over- or undervalued. Equation (30) makes this price discovery function of the forward market explicit: using an estimate for the confidence parameter K, together with the actual forward price $F(T)$ and the index value $a(t)$, the underlying efficient market price $y(t)$ can be derived. The accuracy of the extracted efficient market price is of course strongly depending on the degree of liquidity and density in the forward market. A reliable price reporting system is also crucial. Geltner and Fisher [2007] mentioned that the U.K. IPD swap market appears to be performing the price discovery function well because IPD swap prices have fallen dramatically in 2006, even as the IPD index itself has continued to climb. The IPD swap market has thus correctly signaled overvaluation in the U.K. property market.

If there is a liquid forward market, it also becomes possible to (delta) hedge movements of the underlying efficient market price. For example, if we calculate the sensitivity $\partial F_T(t)/\partial y(t)$ using Equation (30), we find that

$$\partial F_T(t) / \partial y(t) = \frac{\exp(-q(T-t))}{D(t,T)}[1 - \alpha_{K,T}(t) + K\alpha_{K,T}(t)] \tag{33}$$

where we have used Equation (16) to determine $\partial a(t)/\partial y(t)$. Changes of the efficient market price $y(t)$ are thus directly reflected in changes of the forward price $F_T(t)$. This is an important result because we implicitly assumed (see previous discussion) that continuous trading in the underlying efficient market index is possible. This, obviously, cannot be achieved by trading in the primary real estate market (due to a limited liquidity and high trading costs). Using forward contracts, it however becomes possible to replicate the efficient market process in good approximation, provided this secondary market is sufficiently liquid. This also provides a method to replicate the cash flows of more complicated real estate derivatives (like options) using delta hedging. A liquid forward market would thus serve as the foundation of the risk-neutral valuation method developed in this article.

We should also note that in case of stochastic interest rates forward and futures prices are not equal. This is caused by the daily settlement procedure for futures contracts. Assume, for instance, that the real estate index is strongly positively correlated with interest rates. When the real estate index increases, the gain of a long futures contract is invested with a high probability at an above-average interest rate. The opposite holds when the real estate index drops and the resulting loss probably needs to be financed at a below-average interest rate. It thus follows that in case of a positive correlation between the real estate index and interest rates a long futures contract will be more attractive than a long forward contract. Other factors may also cause significant differences between forward and futures contracts (like taxes and transaction costs).

Swaps

We assume that the swap contract starts at time $T_0 \geq t$ and ends at time $T_n > T_0$. To fix the notation: the owner of a receiver swap receives the price return of the real estate index in each period and pays the floating rate. The floating payments are based on the index values at the beginning of each period. The floating rate can, for example, be the LIBOR spot rate. We also assume (without loss of generality) that the cash flows are swapped annually. Results for a total return index can be obtained by setting q equal to zero in the following equations.

We first determine the value of the swaplet that is active during the time interval $[T_{k-1}, T_k]$, where $1 \leq k \leq n$. If we denote the price of this swaplet as $\prod_k(t)$, we can use the following result by Björk and Clapham [2002]:[6]

$$\Pi_k(t) = LB_N(t)E_Q\left[\frac{a(T_k)}{B_N(T_k)} - \frac{a(T_{k-1})}{B_N(T_{k-1})}\Big|F_t\right] \quad (34)$$

where L is a scaling parameter which can be used to set the notional amount of the swap to the right amount.[7] The total value of the swap, $\prod(t)$, is thus equal to

$$\Pi(t) = \sum_{k=1}^{n}\Pi_k(t) = LB_N(t)E_Q\left[\frac{a(T_n)}{B_N(T_n)} - \frac{a(T_0)}{B_N(T_0)}\Big|F_t\right] \quad (35)$$

Substituting Equation (25) and rearranging terms, we find that

$$\begin{aligned}\Pi(t) = L\exp\left(-q(T_N - t)\right)&\left[y(t)(1-\alpha_{K,T_N}(t))\right.\\ &\left.+ a(t)\alpha_{K,T_N}(t)\right] - L\exp\left(-q(T_0 - t)\right)\\ &\times\left[y(t)(1-\alpha_{K,T_0}(t)) + a(t)\alpha_{K,T_0}(t)\right] \quad (36)\end{aligned}$$

When $a(t) = y(t)$ and $q = 0$, the value of the swap contract is equal to zero. The same holds if $K = 1$ and $q = 0$. This is in line with the result obtained by Björk and Clapham [2002], who prove that the value of a total return real estate swap is exactly equal to zero if the real estate index follows a random walk process with drift.[8]

Results for the general model with multiple lags and seasonality can easily be derived by substituting Equation (27) in Equation (35). If the swap market is sufficiently liquid it also becomes possible to (delta) hedge movements of the underlying efficient market price. For example, if we calculate the sensitivity $\partial\prod_T(t)/\partial y(t)$ using Equations (36) and (16), we find that

$$\begin{aligned}\partial\Pi(t)/\partial y(t) = L\exp&\left(-q(T_N - t)\right)\\ &\left[1-\alpha_{K,T_N}(t) + K\alpha_{K,T_N}(t)\right]\\ &-L\exp\left(-q(T_0 - t)\right)\\ &\times\left[1-\alpha_{K,T_0}(t) + K\alpha_{K,T_0}(t)\right] \quad (37)\end{aligned}$$

A liquid swap market can thus also be used to delta hedge more complicated derivatives. If there is no access to either a liquid forward or swap market the risk-neutral valuation approach cannot be applied. One should then resort to methods developed for pricing in incomplete markets. A good example of this approach is given in the paper by Syz and Vanini [2011]. They studied the effect of market frictions (like transaction costs, transaction time, and short sale constraints) to explain why property returns are swapped against a rate that can deviate significantly from LIBOR.

European Options

We value the option at time t. We consider a European option that expires at time $T > t \geq 0$ and cannot be exercised before that date. If $K = 1$, an exact pricing formula exists. This formula is a modification of the familiar Black [1976] equation. The crucial modification is an adjustment of the implied volatility parameter to account for the effect of stochastic interest rates. This adjusted volatility, denoted as $\sigma_2^*(t,T)$, can be calculated as follows, see Brigo and Mercurio [2006, p. 888]:

$$\begin{aligned}\sigma_2^{*\,2}(t,T) = V(t,T) + \sigma_2^2(T-t)&\\ + 2\rho\frac{\sigma_1\sigma_2}{\kappa}&\left(T - t - \frac{1}{\kappa}(1 - exp(-\kappa(T-t))\right)\end{aligned} \quad (38)$$

with

$$\begin{aligned}V(t,T) = \frac{\sigma_1^2}{\kappa^2}&\left(T - t + \frac{2}{\kappa}exp(-\kappa(T-t))\right.\\ &\left.-\frac{1}{2\kappa}exp(-2\kappa(T-t)) - \frac{3}{2\kappa}\right) \quad (39)\end{aligned}$$

and ρ the correlation between the Wiener processes for the short interest rate (see Equation (15)) and the efficient market process (see Equation (22)).

If $K < 1$, a simple, approximating pricing formula can be derived. Equation (17) shows that the index value at time T is equal to a weighted sum of $T - t$ lognormal distributions. We thus have an Asian basket option. To value this option, we first calculate the first moment M_1 and the second moment M_2 of the exact probability distribution at time T, see Hull [2009, pp. 578–579]:

$$\begin{aligned} M_1 &= M_{1,0} + \sum_{i=1}^{T-t} M_{1,i}, \\ M_{1,0} &= \exp\left((r_{N,T}(t) - q)(T-t)\right) a(t)(1-K)^{(T-t)}, \\ M_{1,i} &= \exp\left((r_{N,T}(t) - q)(T-t)\right) \gamma(t) K (1-K)^{i-1} \end{aligned} \tag{40}$$

and

$$\begin{aligned} M_2 = &\sum_{i=1}^{T-t} M_{1,i}^2 \exp\left({\sigma_2^*}^2(t, t+i)\right) \\ &+ \sum_{i<j} M_{1,i} M_{1,j} \exp\left({\sigma_2^*}^2(t, t+i)\right) \end{aligned} \tag{41}$$

Using these moments, we can fit the exact distribution with an approximating lognormal distribution. This approach was first proposed by Levy [1992]. The forward price $F_T(t)$ and the implied volatility σ can then be approximated using the following equation (see also Hull [2009, p. 565]):

$$F_T(t) = M_1; \quad \sigma = \sqrt{\frac{1}{T-t} \ln\left(\frac{M_2}{M_1}\right)} \tag{42}$$

Closed-form pricing formulas for European put and call options are then given by the familiar Black [1976] price for an option on a forward contract:

$$\begin{aligned} p(t) &= \exp\left(-r_{N,T}(t)(T-t)\right)\left[XN(-d_2) - F_T(t)N(-d_1)\right], \\ c(t) &= \exp\left(-r_{N,T}(t)(T-t)\right)\left[F_T(t)N(d_1) - XN(d_2)\right] \end{aligned} \tag{43}$$

where

$$d_1 = \frac{\ln(F_T(t)/X) + \sigma^2 (T-t)/2}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t} \tag{44}$$

$p(t)$ here denotes the price of a put option, $c(t)$ the price of a call option, and X the strike price.

Equations (41)–(44) are also valid for models with multiple lag terms and seasonality. Equation (40) should be generalized, however, in this case. The proper form is

$$\begin{aligned} M_1 &= M_{1,0} + \sum_{i=1}^{T-t} M_{1,i}, \\ M_{1,0} &= \exp(g(T)) \exp\left((r_{N,T}(t) - q)(T-t)\right) \sum_{i=1}^{p} d_i \hat{a}(t-i+1), \\ M_{1,i} &= \exp(g(T)) \exp\left((r_{N,T}(t) - q)(T-t)\right) \gamma(t) c_i \end{aligned} \tag{45}$$

The accuracy of this option pricing formula is tested later in this article. It is important to note that alternative pricing techniques for Asian options are available in the literature. Lord [2006] provided an overview of the current state of the art in this field. A detailed discussion of these advanced methods is outside the scope of this article. We certainly recommend them, however, when option pricing with a very high accuracy is required. We also note that an accurate valuation of American options is possible using the least squares Monte Carlo method by Longstaff and Schwartz [2001].

Pricing Formulas: Conclusions

Our real estate valuation model can be analyzed theoretically and closed-form pricing formulas have been derived. The formulas for forwards and swaps are exact. These formulas collapse to the classic no-arbitrage results if the real estate index follows an efficient market process. If this is not the case, due to autocorrelation in the index returns, deviations from the no-arbitrage price occur if the index level is not equal to the efficient market price.

Given actual market prices for forwards or swaps, the derived pricing formulas can thus be used to estimate the difference between the current index level and the efficient market price. This facilitates the price discovery process: information about market over- or undervaluation can be extracted from the derivatives markets for forwards or swaps. This, in turn, can also make the primary real estate market more efficient because the price update process becomes more effective. We also demonstrated that forward and swap contracts can be used to replicate the underlying efficient market process. Given a liquid forward or swap market the developed risk-neutral valuation model is thus well founded, and more complicated derivatives can be hedged and priced.

For European real estate options, a simple, approximate closed-form solution is derived. Because these options are essentially Asian (basket) options, existing algorithms for Asian stock options can be applied to further improve the accuracy. Accurate valuation of American real estate options is also possible using the Monte Carlo method proposed by Longstaff and Schwartz [2001].

The accuracy of the proposed valuation model is of course strongly dependent on the ability of the valuation model to properly capture the dynamic behavior of the real estate index. Model selection and estimation issues are therefore discussed in detail in the next section.

ESTIMATION OF THE MODEL

Our real estate model could be calibrated (at least partly) using market data if a liquid real estate option market would exist. The standard approach for equity option models is, for example, to fit the model parameters as well as possible to prices of equity options with different maturities and strike levels. The market-implied volatilities (and sometimes also the market-implied correlations) are thus the key inputs for the model estimation. This approach is currently not feasible for real estate, however, due to a lack of trading in real estate options. We thus have to resort to an estimation of the model using historical index data.

We therefore base our analysis on historical data for the Dutch and U.S. residential markets. We first provide an overview of the data in the next subsection. Calibration results for the Dutch and U.S. interest rate models are then presented. We subsequently present calibration results for the Dutch and U.S. real estate models.

We mainly focus on annual historical data for the Dutch real estate market and monthly historical data for the U.S. real estate model. Note that it would of course be possible to do everything at the monthly interval. This would, however, limit the accessibility of this article, because the analysis at the monthly interval requires much more advanced model estimation techniques and does not directly lead to analytical prices for real estate derivatives (due to seasonality and stochastic volatility, which become important at the monthly interval).

Description of the Data and Model Assumptions

In general, calibration results highly depend on the underlying data.[9] Therefore, it is important to investigate the data before calibrating the model. As a first example, we consider a transaction-based index of Dutch house prices. We use monthly returns for the period December 1973–March 2011.[10] The index for the December 1973–December 1994 period is based on data from the Dutch Association of Realtors and Property Consultants (NVM), see van Bussel and Mahieu [1996]. The index for the January 1995–March 2011 period is from Statistics Netherlands (CBS). For more information about the latter index, see De Vries et al. [2009].

As a second example, we consider the S&P/Case-Shiller Home Price Indices (HPI).[11] These indexes measure the residential housing market in metropolitan regions across the United States. All indexes are constructed using the repeat sales pricing technique.

EXHIBIT 1
Summary Statistics

		Yearly		Monthly	
	Period	1973–2010	1987–2010	12/1973–3/2011	2/1987–3/2010
NL.	Average	0.0536	0.0563	0.0535	0.0556
	St. dev.	0.0809	0.0483	0.1178	0.0720
	Skewness	0.6677	0.4112	1.0660	0.2738
	Excess Kurt.	1.5437	0.5992	4.3780	0.2321
	Acf(1)	0.7316	0.6809	0.3814	0.3276
	Acf(2)	0.3161	0.3398	0.2864	0.4679
	Acf(3)	0.0455	0.2579	0.2977	0.4375
	Acf(6)			0.3265	0.4883
	Acf(12)			0.2550	0.4346
	Acf(24)			0.1276	0.2983
U.S.	Average		0.0379		0.0365
	St. dev.		0.0874		0.1088
	Skewness		–0.7899		–0.7671
	Excess Kurt.		0.7526		1.0274
	Acf(1)		0.6915		0.9372
	Acf(2)		0.3329		0.8200
	Acf(3)		–0.0519		0.6777
	Acf(6)				0.4127
	Acf(12)				0.6305
	Acf(24)				0.3752

This methodology collects data on single-family home re-sales and captures re-sold sale prices to form sale pairs. The S&P/Case-Shiller HPI are calculated monthly and published with a two-month lag. The index point for each reporting month is based on sales pairs found for that month and the preceding two months. This index family consists of 20 regional indexes and 2 composite indexes as aggregates of the regions. We here consider the 10-city composite index, which tracks the house price in the original 10 S&P/Case-Shiller indexes, because more historical data are available (compared with the 20-city composite index). We use the monthly index levels for the February 1987–March 2011 period. Exhibit 2 contains an overview of the statistics of the Dutch house prices and the S&P/Case-Shiller HPI.

In Exhibit 2, Panel A, we show the development of the transaction-based index of Dutch house prices and the corresponding annual log returns. We observe that house prices increased until 1977, when they experienced a sharp fall. From the mid-1980s until the early 2000s, house prices exhibited a sharp increase. The most recent period, the late 2000s, which coincides with the global financial crisis, witnessed a decrease of house prices.

Exhibit 2, Panel B, shows the development of the S&P/Case-Shiller HPI and the corresponding monthly log returns. Notice the sharp increase (more than 250%) of the house price index in the period 1997 through 2006, followed by the sharp decline in prices in the last years. Looking more closely, we can also see that the log returns exhibit an oscillatory pattern with a period of approximately one year. This is due to a seasonal effect. In addition, the volatility of the log returns appears to be non-stationary over time. For example, the volatility in the quiet upward trending market (1995–2005) was much lower than during the recent market crash.

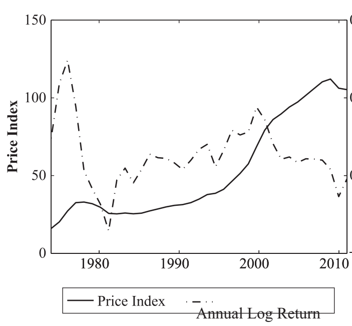
The total expected return on owner-occupied housing is the expected house price appreciation plus a convenience yield, see De Jong et al. [2007]. A convenience yield represents the (non-monetary) benefits from the housing services. When we assume that the convenience yield is a constant fraction of the house value, we can model this aspect by setting the direct return q equal to the convenience yield. De Jong et al. [2007] referred to the convenience yield as an imputed rent and give an estimate of 0.67% a year for the U.S. housing market. The same percentage is used in this article for the Dutch housing market. Note that this estimation of the convenience yield is lower than a typical rental rate. This is due to related expenses for house owners, such as depreciation, maintenance and repairs, property taxes, insurance, and mortgage interest payments.

It is also important to specify the ratio of the initial index price level $a(0)$ and the efficient market price $y(0)$. If $a(0) > y(0)$, the house market is overvalued; if $a(0) < y(0)$, the house market is undervalued. The question of whether or not the Dutch housing market is overvalued has been investigated by Francke et al. [2009] using different models. Unfortunately, all models estimate the overvaluation of the Dutch market differently, ranging between approximately 0% and 12% overvaluation. Because the precise amount of overvaluation thus cannot be determined very accurately, we first take a neutral stance in the next section by assuming that the house market is in equilibrium. The effect of over- or undervaluation is then studied in a sensitivity analysis.

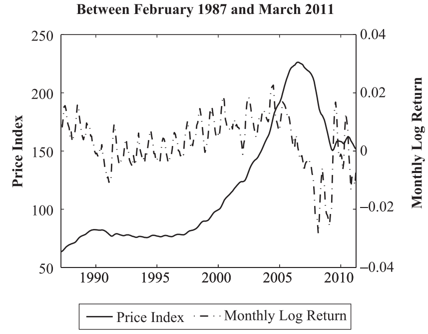
EXHIBIT 2

Overview of Dutch and U.S. House Price Index Development

Panel A: The CBS/NVM Index of Dutch House Prices Between December 1973 and March 2011



Panel B: Development of the 10 City Composite S&P/Case-Shiller Index Between February 1987 and March 2011



Calibration of the Interest Rate Model

The valuation date is March 31, 2011. For the euro area, we use the euro (zero-coupon) swap curve as published by Bloomberg as the reference nominal interest rate curve. We estimate the stochastic interest rate model, a continuous-time, one-factor Hull–White model, using market prices of forward-at-the-money options on euro swap contracts (data also from Bloomberg). The two parameters of the one-factor Hull–White model (the mean-reversion parameter κ and the volatility parameter σ_1) are estimated using a large set of swaptions, with option and swap maturities ranging from 1 to 15 years. Swaption prices are typically quoted in terms of implied (Black) volatilities. Exhibit 3 gives a graphical overview of these volatilities.

We use the Levenberg–Marquardt least-squares algorithm to find the optimal model parameters. The Hull–White parameters with the best fit are a mean reversion κ of ≈ 0.0341 and a volatility σ_1 of ≈ 0.0097. A comparison between the model and market prices is shown in Exhibit 3, Panel B. In this exhibit, we show the difference between the model and market-implied volatility for the entire set of swaptions. The average absolute error is equal to 0.68 percentage point; the maximum absolute error is 2.16 percentage points (for a two-year option on a one-year swap). We also used a more elaborate two-factor Hull–White model. This does not improve the results significantly, however, so we continue with the one-factor model in the remainder of this article. The correlations between the Wiener processes for the short interest rate (see Equation (15)) and the efficient market process (see Equation (22)) are estimated using historical data. More precisely, we use the correlation between historical changes in the short interest rate and the derived efficient market returns. This correlation is equal to 0.16.

The U.S. interest rate model is calibrated in the same way as the Euro interest rate model, i.e., the same data range for swaptions is used and also the same optimization procedure. The Hull–White parameters with the best fit are a mean reversion equal to ≈ 0.0625 and a volatility parameter equal to ≈ 0.0146. The correlation parameter is also estimated using historical data and is equal to 0.45.

Example 1: Dutch House Price Index

Using annual historical data, we compare the quality of price update models with up to three lags. The parameters of Equation (12) are estimated with ordinary least squares (OLS) regression using annual log returns for the 1977–2010 period. By applying the augmented Dickey–Fuller test, the null hypothesis of a unit root is rejected at the 5% level for these returns, that is, the process is stationary. The main estimation results are summarized in Exhibit 4.

Because we assume in our valuation model that the efficient market returns follow a random-walk process with drift, it is important to check whether the residuals indeed have a serial correlation close to zero. In this case, the Durbin–Watson test statistic should be close to 2. Exhibit 4 shows that this is indeed the case for the models with two or three lags.

There exist several order selection criteria to select the best AR model. These criteria typically choose the

EXHIBIT 3
Calibration Results of the Hull–White Model

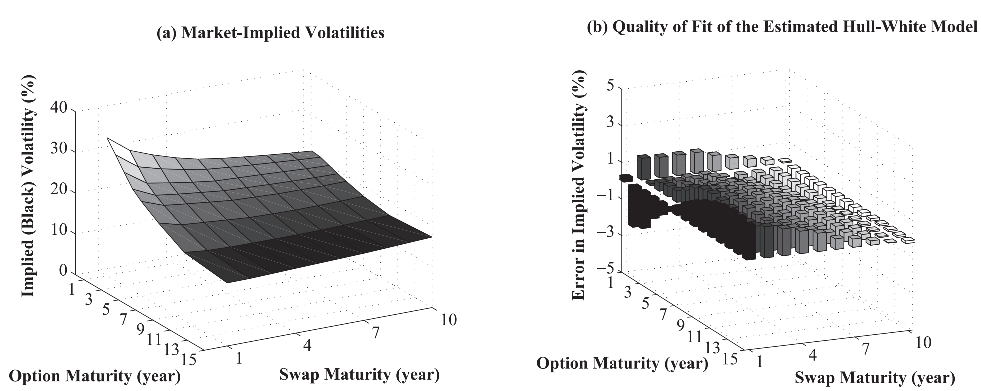


EXHIBIT 4

Characteristics of the Estimated Price Update Models

	1 Lag	2 Lags	3 Lags
K^*	0.394	0.458	0.530
w_1^*	0.606	0.878	0.902
w_2^*	N.A.	–0.336	–0.387
w_3^*	N.A.	N.A.	–0.045
π	0.025	0.035	0.028
$K^*\pi$	0.010	0.016	0.015
σ_ϵ	0.102	0.093	0.082
$K^*\sigma_\epsilon$	0.040	0.043	0.043
Durbin-Watson	1.380	1.870	1.870

model order in such a way that the prediction error is minimized while putting a penalty on the number of parameters estimated. The prediction error is here measured using the maximum likelihood residual variance. This variance is not corrected for the number of parameters estimated. The number of parameters is equal to the order of the AR model plus an additional parameter for the strength of the random walk process. We use the final prediction error criterion (FPE), the Akaike information criterion (AIC), the Schwarz criterion (SC), and the Hannan and Quinn criterion (HQ).[12] The order for which the value of the criterion is minimized is seen as the model that is closest to the true model and is therefore the "optimal" order. Each of the criteria assumes that the models are estimated including a constant term. Exhibit 5 shows that the model with two lags is unanimously selected.

When we select a model with two lags, one additional historical observation (1976) can be used (compared with the model with three lags). We therefore reran the calibration for the model with two lags, including this additional observation. This results in the following model parameters: $K^* = 0.404$, $w_1^* = 1.028$, $w_2^* = -0.431$, $\pi = 0.041$, $K^*\pi = 0.017$, $\sigma_\varepsilon = 0.111$, $K^*\sigma_\varepsilon = 0.045$. This model is used in the remainder of this article.

EXHIBIT 5

Selecting the Optimal Order of the Estimated Price Update Models

	1 Lag	2 Lags	3 Lags	Selected order
1. *FPE*	0.038	0.034	0.036	2
2. *AIC*	–3.276	–3.386	–3.330	2
3. *SC*	–3.186	–3.252	–3.151	2
4. *HQ*	–3.245	–3.340	–3.269	2

Example 2: U.S. House Price Index

We calibrate a monthly price update model of the U.S. S&P/Case-Shiller Home Price Index in this section. The price update model that we consider is thus different than the annual price update model that we used in the previous section. For higher frequency data, aspects like stochastic volatility and seasonality become more important and these effects are therefore explicitly modeled here.

We model seasonality and stochastic volatility as follows. First, we estimate an autoregressive (AR) model with seasonal dummies using OLS (see the earlier description). We allow for at most 14 lags—that is, a lookback period of at most 14 months.[13] We estimate the optimal model using the automatic model selection option in PCGive.[14] We then remove all AR coefficients that are not significant and extend this model with a GARCH(1,1) stochastic volatility model (see Bollerslev [1986]). This model can be written in the following form:

$$\sigma(t)^2 = \alpha_0 + \alpha_1\, r_c^a(t-1)^2 + \beta_1\, \sigma(t-1)^2 \tag{46}$$

where $\sigma(t)$ is the volatility of the monthly log return $r_c^a(t)$. The AR coefficients, the coefficients of the seasonal dummies and the GARCH(1,1) coefficients (α_0, α_1, and β_1) are then determined by maximum likelihood estimation in PCGive. In order to generate risk-neutral scenarios using the GARCH(1,1) model, the

EXHIBIT 6

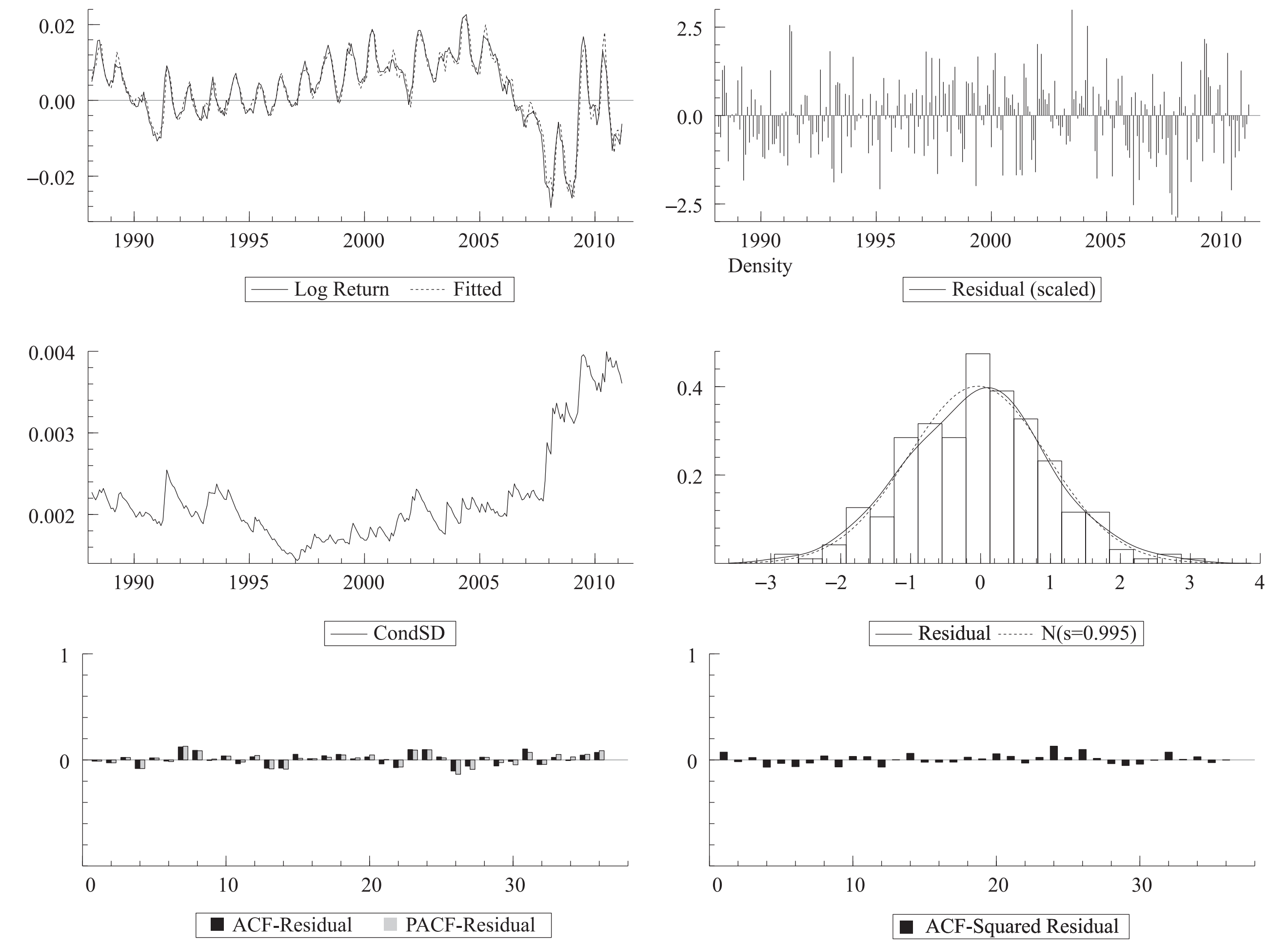
Characteristics of the Estimated Case-Shiller Model, Including Seasonality and Stochastic Volatility

Model with 14 Lags	Coefficients	Standard Error	*t*-Stat	*P*-Value
$r_c^a(t-1)$	1.145950	0.03714	33.1	0.000
$r_c^a(t-3)$	–0.367669	0.07172	–5.76	0.000
$r_c^a(t-4)$	0.127003	0.05567	2.55	0.011
$r_c^a(t-11)$	0.182534	0.03585	5.15	0.000
$r_c^a(t-13)$	–0.130242	0.03579	–3.53	0.000
$r_c^a(March)$	0.00142174	0.00046	2.34	0.020
$r_c^a(April)$	0.00167801	0.00046	3.10	0.002
$r_c^a(July)$	–0.00143169	0.00048	–2.80	0.005
α_0	3.599e-8	9.621e-8	0.450	0.653
α_1	0.061430	0.02898	2.07	0.040
β_1	0.937042	0.04248	24.1	0.000

Notes: The AR model consists of 14 lags, of which only the 5 significant coefficients are included. Three seasonal dummies (for the months February, March, and June) are also included. The α_0, α_1 and β_1 parameters are the coefficients of the GARCH(1,1) stochastic volatility model.

E X H I B I T 7

Estimation Results for the 10-City Composite S&P/Case-Shiller Index



corresponding parameters α_0, α_1, and β_1 are projected to the continuous counterpart. For more information, refer to Hull [2009, p. 482].

The characteristics of the estimated model are summarized in Exhibit 6. The parameters of Equation (12) are estimated with OLS regression using monthly log returns for the March 1988–March 2011 period.

The residuals of the regression have a serial correlation close to zero: the Portmanteau test statistic is equal to 36.1 (with p-value 0.2418). The residuals are also (in good approximation) normally distributed: the normality test statistic is equal to 0.63 (with a P-value of 0.72). The results are shown in more detail in Exhibit 7.

The top left-hand side graph shows how well the model fits the historical data period. The top right graph displays the (scaled) residuals. The evolution of conditional volatility is displayed in the middle left graph. Notice the increasing volatility in the most recent period. The histogram in the middle right graph shows the histogram of the residuals. Notice that these are (approximately) normally distributed. The bottom graphs display the autocorrelation functions (ACF) of the residuals. These autocorrelations should be close to zero if the model fits the data well. This is indeed the case. The applied GARCH(1,1) model thus successfully describes the stochastic volatility component that is present in the data.

MODEL APPLICATION: DERIVATIVE PRICING

The developed valuation model for real estate derivatives is explored further in this section. As an example, we consider a house owner who buys a (hypothetical) at-the-money put option with a maturity of 10 years on his/her house. This option can only be exercised at the maturity date; that is, it is a European option. The underlying index is the Dutch house price index that we introduced in the previous section. We use the price update model with two lags. The direct return q (i.e., the convenience yield) is set equal to 0.67%.

It is important to note that a property derivative market currently does not exist in the Netherlands (a liquid derivatives market also does not exist in the United States at this moment). The example given in this section is thus only provided to illustrate the developed valuation framework. We should also note that our valuation model assumes that continuous trading in the underlying efficient market price is possible. We explained before that it is possible to replicate continuous trading in the underlying index using forward or swap contracts. Once a liquid forward or swap market has emerged in the Netherlands, the applied valuation framework can thus be applied to value (arbitrary complex) property derivatives. In practice, it is of course also important to distinguish between global and more local real estate risk. Derivatives trading typically focuses on the main (metropolitan) areas, which can make it more difficult to hedge house price risk in local areas using derivatives.

As an interesting side line: in the Netherlands, a real-life option on the house price exists. Under certain (specific) conditions the so-called Waarborgfonds Eigen Woningen (WEW) pays out to the bank (the lender) when the owner of the mortgage is not able to fulfill his payments. This guarantee is backed up by the Dutch government if the buffer of the WEW turns out to be insufficient in the future.

Option Pricing Results

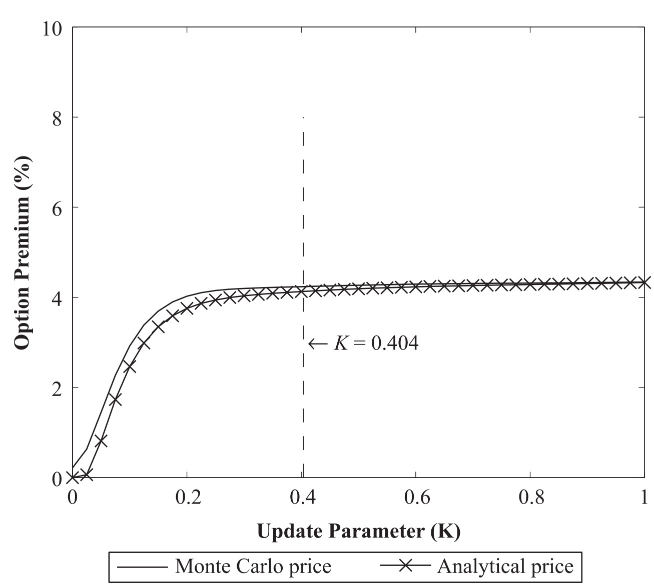
We value the 10-year at-the-money put option using Monte Carlo simulation. This is done by generating 1,000,000 risk-neutral scenarios and discounting the option payoffs back to time zero along the path of the short interest rate.[15] To determine the effect of autocorrelation in the index returns on the option value, we also generated results with alternative model parameters. For these alternative models, the confidence parameter K is varied between zero and 1. The other weights (w_1 and w_2) are proportionally scaled up or down in order to keep the sum of all weights equal to one. Note that we assume in this section that the current real estate index level $a(0)$ is equal to the current efficient market level $y(0)$; see Equation (19). The impact of overvaluation (when $a(0) > y(0)$) or undervaluation (when $y(0) < a(0)$) will be studied in the next section.

The results are shown in Exhibit 8. Recall that returns are highly correlated if $K = 0$. If $K = 1$, returns are almost completely uncorrelated. Also keep in mind that the estimated value of K is equal to 0.404, as indicated in the exhibit. The option premiums are expressed as percentages of the notional amount. Exhibit 8 clearly shows that the option premiums decrease when the autocorrelation of the returns increases (i.e., K decreases). This is due to the smaller (cumulative) volatility of the autocorrelated real estate returns.

The quality of the approximate analytical pricing formula that we derived previously is also investigated in Exhibit 8. This analytical pricing formula is very accurate for high values of the confidence parameter K. In this case, the terminal probability distribution of the index value is determined to a large extent by only a few lognormal distributions. The terminal distribution function can be fitted well with a single lognormal distribution in this case, so the approximation error is small. When K decreases, the terminal probability distribution

Exhibit 8

Price of a 10-Year at-the-Money Put Option on a House Price Index as a Function of the Confidence Parameter *K*



of the index value becomes the sum of a series of different lognormal functions. As a result, the fit with one lognormal function deteriorates. However, the quality of the approximation remains quite satisfactory. For alternative approximation methods, the interested reader is referred to the overview in Lord [2006].

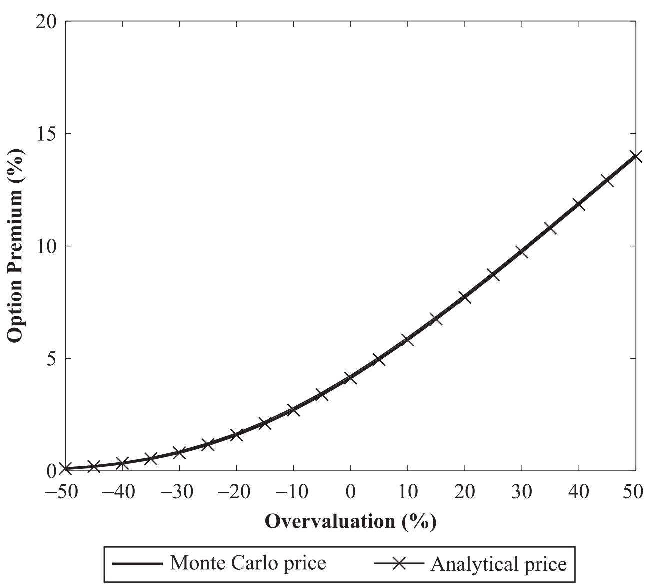
Effect of Over- or Undervaluation

We can model overvaluation (undervaluation) of the real estate market by setting the current efficient market level $y(0)$ lower (higher) than the current index level $a(0)$; see Equation (19). The results are shown in Exhibit 9.

In Exhibit 9, the confidence parameter is set equal to the default value (0.404). Overvaluation is measured as $(a(0) - y(0)) / y(0)$. The option premiums increase, as expected, when the initial index level is higher than the efficient market price (and vice versa). Exhibit 9 also demonstrates that the agreement between the Monte Carlo price and the analytical price is very good in cases of under- or overvaluation. Information about the degree of under- or overvaluation of the real estate market may be obtained by using information in the forward or swap markets (see Geltner and Fisher [2007] and previous discussion in this article) or by using information in the public real estate market (see Geltner et al. [2003]).

EXHIBIT 9

Effect of Over- or Undervaluation on the Option Price



CONCLUSIONS

We proposed a new and intuitive risk-neutral valuation model for real estate derivatives that are linked to autocorrelated indexes. Following Jokivuolle [1998], we first modeled the (unobserved) underlying market fluctuations using a simple random walk process with drift. We then reconstructed the observed index using an adaptation of the price update rule by Blundell and Ward [1987].

The first modification of the update rule by Blundell and Ward [1987] is straightforward and consists of adding multiple lag terms. This leads to an autoregressive (AR) model that can be estimated using standard econometric techniques. A second modification is more fundamental from a valuation perspective and consists of using the accrued value of past observations. We showed, using real (annual and monthly) data, that this model is able to reproduce the dynamic behavior of a transaction-based house price index with autocorrelation. For high-frequency data (like monthly house prices), aspects like seasonality and stochastic volatility can become important. It is possible to model such phenomena as well within the developed framework.

The resulting model has also been analyzed analytically and closed-form pricing solutions have been derived for forwards, swaps, and European put and call options. The developed model can be applied once a liquid forward or swap market has been established. In this case, it becomes possible to (approximately) replicate the underlying efficient market process. The risk-neutral assumption of continuous trading in the underlying asset is then satisfied, and arbitrarily complex derivatives can be hedged and priced. Given actual market prices for forwards or swaps, the derived pricing formulas can also be used to estimate the difference between the current index level and the efficient market price. This facilitates the price discovery process: information about market over- or undervaluation can be extracted from the derivatives markets for forwards or swaps. This, in turn, can also make the primary real estate market more efficient because the price update process is facilitated.

As an example, we valued a (hypothetical) European put option on a house price index. We first generated benchmark (Monte Carlo) results and then tested our (approximate) closed-form pricing formula. This example highlights the strong effect of autocorrelation in the underlying index on the option price. As is well known from the real estate literature, a high degree of autocorrelation reduces the (annual) volatility of the real estate index returns, compared with the (annual) volatility of the underlying "true" market price. This causes lower option prices, because the time value of the option decreases in this case. Using the proposed model, the effect of over- or undervaluation of the real estate market is also studied. The observed effects are significant and as expected.

Our technique is quite general and can be applied for different purposes. First, it can be used to price existing derivatives in real estate markets (see the examples in Buttimer and Kau [1997], Bertus et al. [2008], and Geltner and Fisher [2007]). Our technique can also be used for the valuation of so-called hybrid forms of sales (see Kramer [2008]). In this case, a housing corporation sells a house with a discount to the tenant. In addition, there is a profit and loss sharing mechanism when the house is sold in the future. By determining the present value of the future profits and losses, the corporation can determine whether the initial discount (given to the home buyer) is reasonable. This information can also be used when the corporation reports on a pure market-value basis and includes the present value of future profits and losses on the balance sheet.

APPENDIX

A. PROOFS

A.1 Proof of Equation (24)

Using Equation (22) we find that

$$E_Q\left[\frac{y(T)}{B_N(T)}\,|\,F_t\right] = E_Q\left[\frac{y(t)\exp\left(\int_t^T r_N(\tau)d\tau - \sigma_2^2(T-t)/2 - q(T-t) + \sigma_2 Z_2(T-t)\right)}{B_N(t)\exp\left(\int_t^T r_N(\tau)\,d\tau\right)}\,|\,F_t\right] \tag{A-1}$$

The right-hand side of this expression can be simplified to

$$\frac{y(t)}{B_N(t)} = \exp(-q(T-t))\exp(-\sigma_2^2(T-t)/2)E_Q[\exp(\sigma_2 Z_2(T-t))\,|\,F_t] \tag{A-2}$$

Since we also have that

$$E_Q[\exp(\sigma_2 Z_2(T-t))\,|\,F_t] = \exp\,(\sigma_2^2(T-t)/2) \tag{A-3}$$

we arrive at Equation (24).

A.2 Proof of Equation (25)

The proper starting point for the analysis is Equation (17), since this equation enables us to write a(T) as a basket of previous (accrued) efficient market prices and the (accrued) index value at time t. This becomes clear when we set t equal to T and m equal to $T - t$ in Equation (17):

$$a(T) = K\sum_{i=1}^{T-t}(1-K)^{i-1}y^*(T-i+1) + (1-K)^{T-t}a^*(t) \tag{A-4}$$

Let us first determine whether the accrued prices y*($T - i + 1$), at time T and conditional on the filtration up to time t, are martingales for $1 \le i \le T - t$ if $q = 0$. This is indeed the case, since

$$\begin{aligned} E_Q\left[\frac{y^*(T-i+1)}{B_N(T)} \mid F_t\right] &= E_Q\left[\frac{y(T-i+1)\exp\left(\int_{T-i+1}^{T} r_N(\tau)\,d\tau\right)}{B_N(T-i+1)\exp\left(\int_{T-i+1}^{T} r_N(\tau)\,d\tau\right)}\exp(-q(i-1)) \mid F_t\right] \\ &= E_Q\left[\frac{y(T-i+1)}{B_N(T-i+1)}\exp(-q(i-1)) \mid F_t\right] \end{aligned} \tag{A-5}$$

Using Equation (24) we also find that

$$E_Q\left[\frac{y^*(T-i+1)}{B_N(T)} \mid F_t\right] = \frac{y(t)}{B_N(t)}\exp(-q(T-t)) \tag{A-6}$$

The following result then easily follows:

$$E_Q\left[\frac{a(T)}{B_N(T)} \mid F_t\right] = \frac{\exp(-q(T-t))}{B_N(t)}\left(y(t)K\sum_{i=1}^{T-t}(1-K)^{i-1} + a(t)(1-K)^{T-t}\right) \tag{A-7}$$

Since $K\sum_{i=1}^{T-t}(1-K)^{i-1} = 1-(1-K)^{T-t}$, Equation (25) is obtained.

ENDNOTES

Research for this article took place when David van Bragt was affiliated with Ortec Finance.

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[1]A much earlier application of this update rule can be found in Brown [1959].

[2]Equation (1) assumes that the price update rule is constant over time. Generalizations with time-varying parameters can be found in Brown and Matysiak [2002].

[3]We can, at any given point in time, convert a continuously compounded return $r_c^a(t)$ into an annually compounded return $r^a(t)$ with the relation $r^a(t) = \exp(r_c^a(t)) - 1$. Let us now consider a given time series for $r_c^a(t)$. We can convert these returns into annually compounded returns (using this relation). The temporal correlation between the two time series is almost the same because changes in $r^a(t)$ and $r_c^a(t)$ are in first-order approximation the same.

[4]A natural extension of the price update model is to model the constant K by a time-dependent function, so that the constant K can change with market circumstances. This extension, however, is not investigated in this article.

[5]The following Mathematica program for example writes $a(10)$ in the form of Equation (20) for a price update model with two lags:

```
a[t_] := Ky[t]+ w1 a[t − 1] + (1 − K − w1)a[t − 2] (*price update equation with two lags*)
a[0] = a0 (*index value at time 0*)
a[−1] = am1 (*index value at time −1*)
```

Simplify [a[10]] (*evaluate a(10) and simplify the expression*).

[6]We do not assume (as is the case in Björk and Clapham [2002]) that direct returns are reinvested. This situation is, however, easily obtained by setting q equal to zero in the derived equations.

[7]The notional is thus not a fixed amount but is adjusted periodically by the appreciation and depreciation of the index; see the description in Buttimer and Kau [1997, p. 22].

[8]Patel and Pereira [1996] have extended this result by considering the effect of counterparty default risk. They find that the swap price is no longer equal to zero in this case because a compensation for the additional risk is required. Syz and Vanini [2011] studied the effect of market frictions (like transaction costs, transaction time, and short sale constraints) on the real estate swap market.

[9]We note that the choice of historical data (for example, the region, type, and so on) depends on the application at hand.

[10]The annual returns are calculated using the monthly returns in a certain year.

[11]More information about the Case-Shiller indexes, including historical data and information about the index construction, can be found on http://www.homeprice.standardandpoors.com.

[12]The constant *c* that is used in the HQ criterion is set equal to 1.1.

[13]Results do not improve significantly when we use more than 14 lags.

[14]See Doornik and Hendry [2007].

[15]We use antithetic sampling to reduce the standard error of the Monte Carlo estimate.

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