Dynamical System
- Orbit of non linear planar system -

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1 Hyperbolic Equilibria

1.1 Equilibria

In this part, we consider this planar non linear system:

$$\begin{cases} \dot{x} = x - 2xy \\ \dot{y} = \dfrac{x^2}{2} - y \end{cases}$$

First of all, we write this equation to find the equilibrium point:

$$\begin{cases} 0 = x - 2xy \\ 0 = \dfrac{x^2}{2} - y \end{cases} \tag{1}$$

With the second equation in (1) we have:

$$y = \frac{x^2}{2}$$

and by replacing in the first equation in (1) we obtain:

$$x - 2xy = 0 \qquad \Longleftrightarrow x - 2\frac{x^3}{2} = 0 \qquad \Longleftrightarrow x - x^3 = 0 \qquad \Longleftrightarrow x = x^3$$

so by this last equation we have x: it can be x=1, x=-1, x=0.

We infer that:

- when x=1, with second equation in (1), we find y=$\frac{1}{2}$
- when x=-1, with the second equation in (1), we find y=$\frac{1}{2}$
- when x=0, with the second equation in (1),we find y=0

To conclude, we obtain this three equilibrium point: (1,$\frac{1}{2}$); (0,0); (-1,$\frac{1}{2}$).

1.2 Matrix of the linearized system

After finding each equilibrium point, we have to find the matrix of the linearized system around this points, the eigenvalues and describe the type and the stability of these equilibrium point.

First, we define the function f(x,y) as:

$$\text{f(x,y)=}\begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} x - 2xy \\ \frac{x^2}{2} - y \end{pmatrix}$$

We also define the Jacobian of this matrix as:

$$\text{Jf(x,y)=}\begin{pmatrix} \frac{df_1(x,y)}{dx} & \frac{df_1(x,y)}{dy} \\ \frac{df_2(x,y)}{dx} & \frac{df_2(x,y)}{dy} \end{pmatrix} = \begin{pmatrix} 1-2y & -2x \\ x & -1 \end{pmatrix}$$

Equilibrium point (0,0)

We begin with the equilibrium point (0,0):

$$\text{Jf(0,0)=}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This matrix is only composed with the value on its diagonal. We deduce that the eigenvalues are λ_1=1 and λ_2=-1.
We can deduce that the equilibrium point (0,0) is a **saddle** by the definition 1 (λ_1 <0< λ_2) and by the theorem 2, that this point is **unstable** (because any saddle is unstable). It is also **hyperbolic** by the definition 3 ($\text{Re}(\lambda_1) \neq 0$; $\text{Re}(\lambda_2) \neq 0$).

Equilibrium point (1,$\frac{1}{2}$)

Let's continue with the equilibrium point (1,$\frac{1}{2}$):

$$\text{Jf}(1,\frac{1}{2}) = \begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix} \text{ and let A } = \begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix}$$

- det(A)= 2
- tr(A)= -1
- λ^2 - tr(A)λ+det(A)= 0

With this three equations above we can write this:

$$\lambda^2 + \lambda + 2 = 0 \iff r^2 + r + 2 = 0$$

$$\Delta = b^2 - 4 * a * c = 1 - 8 = -7$$

$\Delta < 0$ so the solutions are complex and $r_{1,2} = \frac{-b \pm i * \sqrt{-\Delta}}{2 * a} = \frac{-1 \pm i * \sqrt{7}}{2}$

- $r_1 = \frac{-1+i\sqrt{7}}{2}$
- $r_2 = \frac{-1-i\sqrt{7}}{2}$

We infer that the equilibrium point is **focus** by definition 1 ($\lambda_{1,2} = \alpha \pm \text{i}\beta$). It is also **hyperbolic** because Re(r_1)$\neq$0 and (Re(r_2)$\neq$0 (by the definition 3). Moreover, the trace is negative so we deduce that the point is **stable**.

Equilibrium point (-1,$\frac{1}{2}$)

Let's finish with the equilibrium point (-1,$\frac{1}{2}$):

$$\text{Jf}(-1,\frac{1}{2}) = \begin{pmatrix} 0 & 2 \\ -1 & -1 \end{pmatrix} \text{ and let A } = \begin{pmatrix} 0 & 2 \\ -1 & -1 \end{pmatrix}$$

- det(A)= 2
- tr(A)= -1
- λ^2 - tr(A)λ+det(A)= 0

With this three equations above we can write this:

$$\lambda^2 + \lambda + 2 = 0 \iff r^2 + r + 2 = 0$$

$$\Delta = b^2 - 4 * a * c = 1 - 8 = -7$$

$\Delta < 0$ so the solutions are complex and $r_{1,2} = \frac{-b \pm i\sqrt{-\Delta}}{2*a} = \frac{-1 \pm i\sqrt{7}}{2}$

- $r_1 = \frac{-1+i\sqrt{7}}{2}$
- $r_2 = \frac{-1-i\sqrt{7}}{2}$

We infer that the equilibrium point is **focus** by definition 1 ($\lambda_{1,2} = \alpha \pm i\beta$). It is also **hyperbolic** because $Re(r_1) \neq 0$ and $(Re(r_2) \neq 0$ (by the definition 3). Moreover, the trace is negative, we deduce that the point is **stable**.

1.3 Phase Portrait

In this part, we will draw the phase portrait. In the mean time, we have:

$$\begin{cases} \dot{x} = x - 2xy \\ \dot{y} = \dfrac{x^2}{2} - y \end{cases}$$

This system has 3 equilibrias: (0,0); $(1,\frac{1}{2})$; $(-1,\frac{1}{2})$.

- $y=0 \Rightarrow \dot{x}=x$

x	$-\infty$		0		$+\infty$
x		$-$	0	$+$	

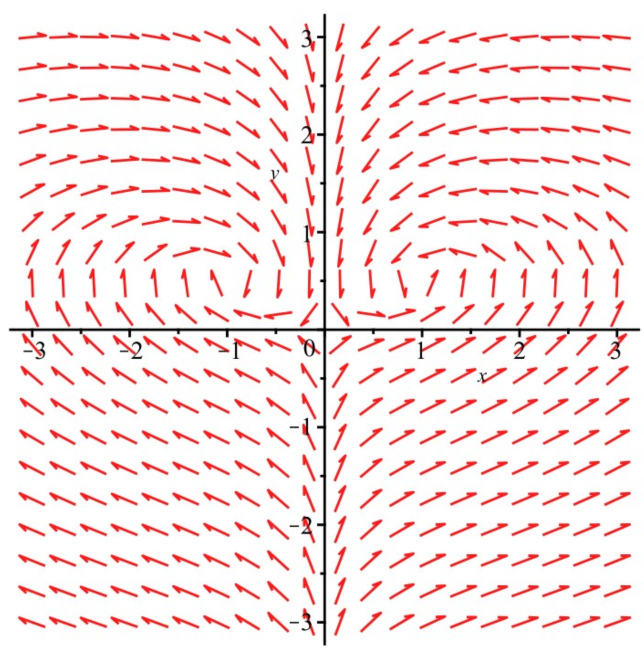
We obtain this phase portrait for y=0.

- x=0 ⇒ $\dot{y}$=-y

y	$-\infty$		0		$+\infty$
$-y$		$+$	0	$-$	

We obtain this phase portrait for x=0.

Then, we draw the final phase portrait. It looks like:



To put in a nutshell, we have found 3 equilibrium points with different types and stability. Moreover, we have drawn phases portraits to see where the flow goes, and finish by drawing the final one (see above).

2 Non hyperbolic equilibria (Lotka-Volterra)

2.1 Equilibria

In this second part, we consider this system:

$$\begin{cases} \dot{x} = x - xy \\ \dot{y} = -0.3y + 0.3xy \end{cases}$$

First, we write this equation to find the equilibrium point:

$$\begin{cases} 0 = x - xy \\ 0 = -0.3y + 0.3xy \end{cases} \tag{2}$$

With the first equation in (2) we have:

$$yx = x \Rightarrow y = 1$$

and by replacing in the second equation in (2) we obtain:

$$-0.3 + 0.3x = 0 \Rightarrow x = 1$$

We find the first equilibrium point (1,1).
Moreover, when x=0 in the first equation in (2), we see that in the second equation in (2), y=0.
And then we find the other equilibrium point (0,0).

Finally, we obtain this two equilibrium points: (0,0); (1,1).

We put f(x,y) as:

$$\text{f(x,y)=}\begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} x - xy \\ -0.3y + 0.3xy \end{pmatrix}$$

We also define the Jacobian of this matrix as:

$$\text{Jf(x,y)=}\begin{pmatrix} \frac{df_1(x,y)}{dx} & \frac{df_1(x,y)}{dy} \\ \frac{df_2(x,y)}{dx} & \frac{df_2(x,y)}{dy} \end{pmatrix} = \begin{pmatrix} 1-y & -x \\ 0.3y & 0.3(x-1) \end{pmatrix}$$

Equilibrium point (1,1)

Let's check this equilibrium point (1,1):

$$\text{Jf(1,1)=}\begin{pmatrix} 0 & -1 \\ 0.3 & 0 \end{pmatrix} \text{ and let A } = \begin{pmatrix} 0 & -1 \\ 0.3 & 0 \end{pmatrix}$$

- det(A)= 0.3
- tr(A)= 0
- λ^2 - tr(A)λ+det(A)= 0

With this three equations above we can write this:

$$\lambda^2 + 0.3 = 0 \iff r^2 + 0.3 = 0$$

$$\Delta = b^2 - 4 * a * c = -4 * 0.3 = -1.2$$

$\Delta < 0$ so the solutions are complex and $r_{1,2} = \frac{-b \pm i\sqrt{-\Delta}}{2*a} = \frac{\pm i\sqrt{1.2}}{2}$

- $r_1 = \frac{i\sqrt{1.2}}{2}$
- $r_2 = \frac{-i\sqrt{1.2}}{2}$

We infer that the equilibrium point (1,1) is a **center** by definition 1 ($\lambda_{1,2} = \pm$ iβ) so it is **stable**.
Moreover, it **is not hyperbolic** because Re(r_1)=Re(r_2)=0 (by the definition 3).

2.2 Integral of Lotka-Volterra system

We have the system below:

$$\begin{cases} dx = x - xy \\ dy = -0.3y + 0.3xy \end{cases} \tag{3}$$

Let implement the $\frac{firstequation}{secondequation}$ of the third system.

We obtain: $$\frac{dx}{dy} = \frac{x(1-y)}{0.3y(x-1)} \tag{4}$$

$$\iff \frac{x-1}{x}dx = \frac{1-y}{0.3y}dy \iff \int \frac{x-1}{x}dx = \int \frac{1-y}{0.3y}dy$$
$$\iff x - ln|x| = \frac{1}{0.3}(ln|y| - y) + C \ (C \in \mathbb{R})$$

Finally, we have: H(x,y)= x + $\frac{y}{0.3}$ - ln$|x|$ - $\frac{1}{0.3}$ ln$|y|$
This is a first integral in $\cup$ =(0,∞)(0,∞).

2.3 Phase Portrait

In this part, we will draw the phase portrait. In the mean time, we have:

$$\begin{cases} \dot{x} = x - xy \\ \dot{y} = -0.3y + 0.3xy \end{cases}$$

This system has 2 equilibrias: (0,0); (1,1)

- x=0 $\Rightarrow \dot{y} = -0.3y$

y	$-\infty$		0		$+\infty$
$-0.3y$		$+$	0	$-$	

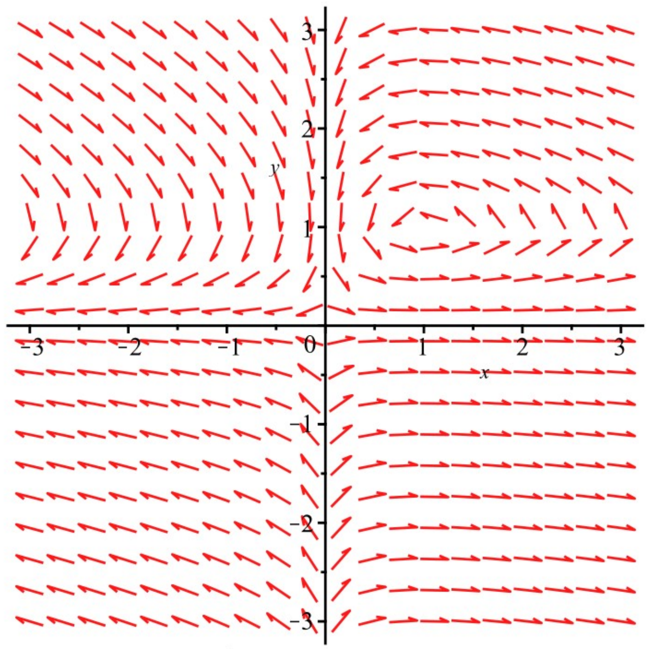
We obtain this phase portrait for x=0.

- y=0 $\Rightarrow \dot{x} = x$

x	$-\infty$		0		$+\infty$
x		$-$	0	$+$	

We obtain this phase portrait for x=0.

Then, we draw the final phase portrait. It looks like:



To sum up this second hand, we proove that the equilibrium point (1,1) is not hyperbolic and have defined its type and stability. Then, we have found the integral of Lotka-Volterra system and draw the phase portrait of this system.

NB: In order to not overload the axis, I prefered drawing separately the phase portrait of the axis x and the axis y, but I did not copy them on the final phase portrait.