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Decoupled approach to multidisciplinary design optimization under uncertainty

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Abstract We propose solution methods for multidisciplinary design optimization (MDO) under uncertainty. This is a class of stochastic optimization problems that engineers are often faced with in a realistic design process of complex systems. Our approach integrates solution methods for reliability-based design optimization (RBDO) with solution methods for deterministic MDO problems. The integration is enabled by the use of a deterministic equivalent formulation and the first order Taylor's approximation in these RBDO methods. We discuss three specific combinations: the RBDO methods with the multidisciplinary feasibility method, the all-at-once method, and the individual disciplinary feasibility method. Numerical examples are provided to demonstrate the procedure.

Keywords Multidisciplinary systems · Reliability · Design optimization · Stochastic optimization · Chance-constrained programming

1 Introduction

Multidisciplinary design optimization (MDO) is a problem of finding a set of system design parameters, subject to constraints, such that a prescribed performance measure is optimized. The term "multidisciplinary" refers to the fact that an analysis of the underlying system involves more than one engineering discipline that could be coupled. We consider two or more disciplines coupled if, for any pair of disciplines, an analysis of one requires the output from the other. An example of MDO problems is the design of an aircraft's wing, in which the goal is to find a wing profile that

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minimizes the drag while keeping the stress within the prescribed limit. The engineering disciplines involved in this example are aerodynamics and structural mechanics. Here, the two disciplines are coupled because to perform the structural analysis, the wing's pressure distribution, an output from aerodynamic analysis, is needed as an input; and in turn, to perform the aerodynamic analysis, the wing's deformation, an output from structural analysis, is needed. An important characteristic of the design optimization problem for complex engineering systems, including MDO, is that the system analysis normally involves computationally intensive evaluations of implicit ("black-box") functions.

During the past decade, several solution techniques for deterministic MDO problems have been developed. However, these methods can be inadequate in real-world applications since they do not explicitly take uncertainty into account. In the context of MDO, Batill et al. (2000) qualitatively describe the effect of uncertainty in the design process. Realistic engineering systems are inevitably affected by uncertainty that can arise from many sources, e.g., material properties, manufacturing tolerances, and loading conditions that are not typically known with certainty. In a physics-based model, uncertainty may arise from the inaccuracy of the mathematical model or the inadequate understanding of the underlying physical phenomenon. As a result, there is much interest in developing solution methods for MDO problems under uncertainty.

The MDO solution methods proposed by Koch et al. (2002), Mahadevan and Gantt (2000), and Oakley et al. (1998) are based on the traditional techniques from RBDO that have been developed for structural design since the 1970's. These traditional RBDO techniques estimate the system reliability (via reliability index) within the optimization algorithm and are often referred to as nested-loop (or double-loop) methods. Empirical evidence suggests that nested-loop methods are less computationally efficient than recent RBDO methods in which the reliability analysis and optimization are either directly integrated or performed sequentially. The reliability analysis and optimization are decoupled in both of these approaches. We include the term decoupled in the paper's title to indicate this feature of our approach. Detailed discussions of the integrated and sequential approaches are given in (Chiralaksanakul and Mahadevan 2005) and references cited therein. We utilize particular types of integrated and sequential methods in our solution procedures for probabilistic MDO problems. In particular, we use in the proposed procedure an integrated approach known as the single-loop method developed by Chen et al. (1997) and Liang et al. (2004) and a sequential approach known as the sequential optimization and reliability analysis (SORA) method developed by Du and Chen (2004) in the context of single-disciplinary RBDO. In (Koch et al. 2002) and (Oakley et al. 1998), response surfaces are utilized (within the nested-loop methods) in order to improve computational efficiency in solving probabilistic MDO problems.

Our approach to probabilistic MDO problems is based on the existing methods for deterministic MDO problems and the recent and more efficient RBDO methods. In particular, we present a framework to integrate either the integrated or sequential methods with the existing methods for deterministic MDO problems. The key idea that underlies these RBDO approaches is the deterministic equivalent formulation of an RBDO problem via the quantile function. We exploit the deterministic equivalent formulation to construct a deterministic approximation of the probabilistic

MDO problem and then solve the resulting approximation problem with the solution methods for deterministic MDO problems. Two of the MDO challenges suggested by Batill et al. (2000) that we address are (i) to explain MDO concepts in terms of basic mathematical principles and (ii) to develop a rationale approach for decision-making in probabilistic MDO problems. The proposed framework also provides an alternative in handling the so-called compatibility requirement when the uncertainty is explicitly considered in MDO problems.

The solution methods of deterministic MDO problems that we integrate with the RBDO method are the multidisciplinary feasibility (MDF) method, the all-at-once (AAO) method, and the individual disciplinary feasibility (IDF) method. All three methods are based on nonlinear programming algorithms. Their difference lies in how the system analysis is handled. As we will describe in Sect. 3, the MDF method performs the system analysis for a given design "outside" the optimization algorithm while the AAO method performs it "inside." The IDF method decouples the system analysis (by disciplines) and performs each disciplinary analysis outside the optimization algorithm. The AAO method is also known as the simultaneous analysis and design method in Haftka (1985), and the sequential compatibility constraint method in Braun et al. (1995). Braun et al. (1995) report that the AAO method is computationally more efficient than the MDF method for deterministic launch-vehicle design optimization. Further computational studies of these methods for deterministic problems are conducted by Alexandrov and Kodiyalam (1998) and Hulme and Bloebaum (2000). Detailed descriptions of the MDF, AAO, and IDF methods can be found in (Cramer et al. 1994; Hulme and Bloebaum 2000).

Another class of solution methods for deterministic MDO problems is suitable for design optimization when one attempts to decompose the MDO problem by engineering discipline in order to achieve disciplinary autonomy, i.e., each group of designing engineers can fully control the design parameters in their own discipline. Examples of decomposition-based methods are collaborative optimization (CO), developed by Braun and Kroo (1996), and bi-level integrated system synthesis (BLISS), developed by Sobieszczanski-Sobieski et al. (1998) and Sobieszczanski-Sobieski and Kodiyalam (2001).

The rest of the paper is organized as follows. Section 2 presents a statement of the probabilistic MDO problem. Solution methods for probabilistic MDO problems are detailed in Sect. 3. Then, we illustrate the proposed methods with three numerical examples in Sect. 4 and conclude in Sect. 5.

2 Problem statement

We begin by considering the following deterministic MDO problem

$$\min_{x \in X} \quad c(x), \tag{1a}$$

$$\text{s.t.} \quad g(x, u(x), v(x)) \geq 0, \tag{1b}$$

where $u(x)$ and $v(x)$ are intermediate variables (also called behavior or state variables) defined by

$$h_1(x, u, v) = 0, \tag{2}$$

$$h_2(x, u, v) = 0. \tag{3}$$

System design variables, $x_1, \ldots, x_n$, are the components of $x \in \mathbb{R}^n$ and the cost function to be minimized is denoted by $c(x)$. Following the convention in the RBDO literature, we use the scalar-valued limit-state function, g, to indicate system failure when it is strictly less than zero. For notational simplicity, we represent other deterministic constraints on x that do not involve the intermediate variables by $X \subseteq \mathbb{R}^n$. For example, to enforce a geometric constraint $f(x) = 0$ and the lower and upper bound, x_l and x_u, we can define set X as $\{x \in \mathbb{R}^n : f(x) = 0, x_l \leq x \leq x_u\}$. We emphasize that u and v depend on x through (2) and (3) and we show this dependence explicitly in (1b). Equations (2) and (3) govern the physics of the underlying system and are coupled. In the earlier mentioned example of an aircraft's wing design optimization, h_1 and h_2 may represent the structural and aerodynamic analysis while $u(x)$ and $v(x)$ may represent the wing deformation and pressure distribution for design x. If $u(x)$ and $v(x)$ satisfy (2) and (3) simultaneously, then they are said to be compatible (for design x).

In practical engineering applications, h_1 and h_2 are implicit functions, i.e., they involve numerical analysis (or simulation) performed by computer. In our example of aircraft wing design, the structural analysis may be carried out by a finite element analysis while the aerodynamic analysis by a computational fluid dynamics analysis. As a result, the evaluation of g and/or its gradient needed by the optimization algorithm also involves those numerical procedures (or simulations). In the MDO literature, solving (2) and (3) for compatible $u(x)$ and $v(x)$ is referred to as multidisciplinary analysis. Disciplinary analysis refers to solving either (2) or (3), e.g., finding u such that (2) is satisfied for a given x and v.

Now, consider the following probabilistic variation of (1):

$$\min_{x \in X} \quad c(x), \tag{4a}$$

$$\text{s.t.} \quad P(g(x, \tilde{\xi}, u(x, \tilde{\xi}), v(x, \tilde{\xi})) \leq 0) \leq \alpha, \tag{4b}$$

where $u(x, \xi)$ and $v(x, \xi)$ are defined by

$$h_1(x, \xi, u, v) = 0, \tag{5}$$

$$h_2(x, \xi, u, v) = 0. \tag{6}$$

In (4), $\tilde{\xi}$ is a vector of random variables, $\tilde{\xi}_1, \ldots, \tilde{\xi}_m$, that model the uncertainty in the system. We use tilde ($\tilde{}$) to distinguish the random variable from its realization, e.g., ξ is a realization of random variable $\tilde{\xi}$. We assume that the joint probability distribution of $\tilde{\xi}_1, \ldots, \tilde{\xi}_m$, is known. Constraint (4b) specifies that x is feasible if the probability of system failure is less than a prespecified level α. Strictly speaking, functions g, h_1, h_2, u, and v in (4), (5), and (6) are not the same as those defined in (1), (2), and (3). However, we do not introduce new symbols in order to simplify the notation, and their meanings are usually clear from the context. In (4b), we show explicit dependence of u and v on x and $\tilde{\xi}$. We remark that (4–6) may not be feasible even if its deterministic counterpart (1–3) is feasible.

We will use (4–6), which contains only one probabilistic constraint and two system equations, in our development in order to clearly convey the underlying idea of the proposed method, and to avoid tedious algebraic manipulations that do not

provide additional insights. Extensions to MDO problems with two or more probabilistic constraints and with either coupled or uncoupled multidisciplinary system equations are straightforward, and we indicate them as appropriate. In addition, we limit ourselves to problems with a deterministic cost function since the main focus of the paper is on the probabilistic constraint aspect of the MDO problem. To apply the proposed methods (in Sect. 3) to MDO problems with a stochastic cost function, e.g., $c(x, \tilde{\xi}, u(x, \tilde{\xi}), v(x, \tilde{\xi}))$, we can approximate it with $c(x, E\tilde{\xi}, u(x, E\tilde{\xi}), v(x, E\tilde{\xi}))$ as normally done in the RBDO and MDO literature. With appropriate convexity of c in ξ, this approximation provides bounds on the optimal value function (e.g., via Jensen's inequality). Symbol E denotes the mathematical expectation operator.

3 RBDO-based methods for probabilistic MDO problems

It is important to observe that if we can solve (5) and (6) explicitly for u and v as expressions of x and ξ, and substitute these expressions into (4b), then we can express (4) as

$$\min_{x \in X} \quad c(x), \tag{7a}$$

$$\text{s.t.} \quad P(g(x, \tilde{\xi}) \leq 0) \leq \alpha. \tag{7b}$$

Formulation (7) is the type of mathematical programming model widely used in RBDO; therefore, the MDO problem defined by (4–6) can be viewed as a class of RBDO problems in which the limit-state function contains intermediate variables that must be evaluated via multidisciplinary system equations.

For completeness and ease of understanding, we briefly discuss the idea behind the single-loop and SORA methods in the context of single discipline design optimization before applying these methods to probabilistic MDO problems. Details of these RBDO methods are given in (Chiralaksanakul and Mahadevan 2005; Du and Chen 2004; Liang et al. 2004).

Both single-loop and SORA methods are based on a deterministic formulation via the quantile function of random variable $g(x, \tilde{\xi})$. Tu et al. (1999) refers to this type of RBDO methods as performance measure approach. The α-level quantile, $a(x, \alpha)$, of $g(x, \tilde{\xi})$ is given by

$$a(x, \alpha) = \min\{a' \in \mathbb{R} : P(g(x, \tilde{\xi}) \leq a') \geq \alpha\}, \tag{8}$$

where $\alpha \in [0, 1]$. If the cumulative distribution function (CDF) of $g(x, \tilde{\xi})$ is continuous and strictly increasing for each $x \in X$, then it can be shown that (7) is equivalent to

$$\min_{x \in X} \quad c(x), \tag{9a}$$

$$\text{s.t.} \quad a(x, \alpha) \geq 0, \tag{9b}$$

i.e., x^* solves (7) if and only if it solves (9). Inverse first order reliability method (FORM) approximates $a(x, \alpha)$ by first transforming $\tilde{\xi}_1, \ldots, \tilde{\xi}_m$, to uncorrelated standard normal random variables, $\tilde{\eta}_1, \ldots, \tilde{\eta}_m$, via a transformation $T : \mathbb{R}^m \to \mathbb{R}^m$, i.e.,

$\eta = T(\xi)$ and then applying a first-order Taylor's expansion of g. To distinguish between the original and transformed functions, we use a hat ($\hat{}$) to denote the transformed function; for example, the transformed limit-state function is denoted $\hat{g}(x, \eta) \equiv g(x, T^{-1}(\eta))$. Note that the transformation T is in general inexact and can be implicit. RBDO methods that are based on FORM or its inverse may suffer from the transformation error if the original random variables are not normal. The recent work of Du and Sudjianto (2004) in the area of reliability analysis propose a computationally viable approach in reducing the transformation error due to the use of FORM or its inverse.

The first-order Taylor's approximation (with respect to η) of $\hat{g}$ at $\eta = \eta_0$ is

$$\hat{g}(x, \eta) \approx \hat{g}(x, \eta_0) + \nabla^T \hat{g}(x, \eta_0)(\eta - \eta_0), \tag{10}$$

where the superscript T denotes the transpose of a vector. Substituting (10) into (8), we obtain a quantile approximation $a_1(x, \alpha) = \hat{g}(x, \eta_0) - \nabla^T \hat{g}(x, \eta_0)\eta_0 + \Phi^{-1}(\alpha)\|\nabla \hat{g}(x, \eta_0)\|$. When η_0 is chosen to be an optimal solution of the inverse FORM, i.e.,

$$\min_{\eta} \quad \hat{g}(x, \eta), \tag{11a}$$

$$\text{s.t.} \quad \|\eta\| = -\Phi^{-1}(\alpha) \quad (\text{for } \alpha \in [0, 0.5]), \tag{11b}$$

the KKT optimality conditions of (11) yields $\nabla^T \hat{g}(x, \eta^*)\eta^* - \Phi^{-1}(\alpha)\|\nabla \hat{g}(x, \eta^*)\| = 0$ where η^* is an optimal solution of (11). As a result, we have that $a_1(x, \alpha) = \hat{g}(x, \eta^*)$. In words, the first-order quantile approximation of $\hat{g}$ corresponds to the optimal value of the inverse FORM optimization problem. The value of $-\Phi^{-1}(\alpha)$ is typically determined through a target reliability index β_t. In practice, β_t is non-negative; thus, the practical range of α should be between 0 and 0.5 (as shown in (11b)). Inverse FORM has been recently exploited in many applications; see (Du et al. 2004; Tu et al. 1999; Youn and Choi 2004) and the references cited therein.

Using $a_1(x, \alpha) = \hat{g}(x, \eta^*)$, we can state a deterministic approximation for (7) as

$$\min_{x \in X} \quad c(x), \tag{12a}$$

$$\text{s.t.} \quad \hat{g}(x, \eta^*) \geq 0. \tag{12b}$$

The SORA and single-loop methods are based on (12). They differ in how η^* is approximated and updated. The SORA method approximates η^* via inverse FORM as indicated in the statement in Fig. 1. Du and Chen (2004) originally develop the SORA method through the notion of "shifting distance." The algorithmic statement in Fig. 1 is an alternative interpretation for the SORA method that does not involve the notion of shifting distance. It gives us insights to the SORA algorithm so that we can apply it to solve probabilistic MDO problems. We note that the SORA statement in Fig. 1 is mathematically equivalent to that of Du and Chen (2004).

The single-loop method of Chen et al. (1997) and Liang et al. (2004) does not solve (11) directly to approximate η^* as the SORA method does (in Step 1) but it

Step 0 **Initialization**

Let x^0 be a given initial design, and ϵ be a stopping tolerance. Set $k = 0$.

Step 1 **Computing the quantile approximation of design x^k**

Solve $\min_\eta\{\hat{g}(x^k, \eta) : \| \eta \| = -\Phi^{-1}(\alpha)\}$. Denote its solution by η^k.

Step 2 **Stopping rules**

If $\hat{g}(x^k, \eta^k) \geq 0$, and $|c(x^k) - c(x^{k-1})| \leq \epsilon$, then STOP.

Design x^k is a candidate for an optimal design.

(For $k = 0$, set $c(x^{k-1}) = \infty$ to avoid premature stopping.)

Step 3 **Finding design x^{k+1}**

Solve $\min_{x \in X}\{c(x) : \hat{g}(x, \eta^k) \geq 0\}$. Denote its solution by x^{k+1}.

Set $k = k + 1$ and goto Step 1.

Fig. 1 A statement of the SORA method

instead computes a k-iteration approximation of η^* by

$$\eta^k = \frac{-\nabla \hat{g}(x^k, \eta^{k-1})}{\|\nabla \hat{g}(x^k, \eta^{k-1})\|}\beta_t, \tag{13}$$

where design x^k is specified by the optimization algorithm at iteration k and η^{k-1} is an estimate of η^* from previous iteration $k - 1$. Note that (13) is motivated by the KKT conditions of the optimization problem associated with inverse FORM. While solving (11) to approximate η^* may require several calculations of the gradient of $\hat{g}$ (with respect to η), the updating formula (13) requires only one calculation of the gradient. We are not aware of convergence proof for both the SORA and single-loop methods.

For problems with multiple probabilistic constraints, the quantile approximation (in Step 1) must be computed for each probabilistic constraint, thereby yielding different η's. The deterministic approximation (in Step 3) then uses corresponding η to express $\hat{g}(x, \eta) \geq 0$. Examples 1 and 3 in Sect. 4 illustrates the method for problems with two probabilistic constraints.

One difficulty in solving deterministic MDO problems is how to effectively handle the multidisciplinary system equations that enforce the compatibility of intermediate variables between disciplines. This in fact leads to the development of different MDO methods such as the MDF, AAO, and IDF methods. In the stochastic setting, the difficulty is even compounded because we need to deal with multidisciplinary system equations that can contain random variables. The approach that we propose in this paper addresses this issue for the stochastic case by employing the single and sequential RBDO methods (discussed above) to construct a quantile-based deterministic approximation of probabilistic MDO problems. The quantile-base deterministic approximation specifies precisely the realization of the random variables for which the multidisciplinary system equations must be evaluated and how its outputs are used

Step 0 **Initialization**

Let x^0 be a given initial design, and ϵ be a stopping tolerance. Set $k = 0$.

Step 1 **Computing the quantile approximation of design x^k**

Solve $\min_\eta\{\hat{g}(x^k, \eta, \hat{u}(x^k, \eta), \hat{v}(x^k, \eta)) : \| \eta \| = -\Phi^{-1}(\alpha)\}$, where $\hat{u}(x^k, \eta)$ and $\hat{v}(x^k, \eta)$ are computed from (15) and (16). Denote its solution by η^k.

Step 2 **Stopping rules**

If $\hat{g}(x^k, \eta^k, \hat{u}(x^k, \eta^k), \hat{v}(x^k, \eta^k)) \geq 0$, and $|c(x^k) - c(x^{k-1})| \leq \epsilon$, then STOP. Design x^k is a candidate for an optimal design.

(For $k = 0$, set $c(x^{k-1}) = \infty$ to avoid premature stopping.)

Step 3 **Finding design x^{k+1}**

Solve $\min_{x \in X}\{c(x) : \hat{g}(x, \eta^k, \hat{u}(x, \eta^k), \hat{v}(x, \eta^k)) \geq 0\}$, where $\hat{u}(x, \eta^k)$ and $\hat{v}(x, \eta^k)$ are given by (15) and (16). Denote its solution by x^{k+1}.

Set $k = k + 1$ and goto Step 1.

Fig. 2 A statement of the SORA method for probabilistic MDO problems

in the optimization algorithm. Using $T(\xi) = \eta$, we can transform (4) to

$$\min_{x \in X} \quad c(x), \tag{14a}$$

$$\text{s.t.} \quad P(\hat{g}(x, \tilde{\eta}, \hat{u}(x, \tilde{\eta}), \hat{v}(x, \tilde{\eta})) \leq 0) \leq \alpha, \tag{14b}$$

where $\hat{u}(x, \eta)$ and $\hat{v}(x, \eta)$ are defined by

$$\hat{h}_1(x, \eta, \hat{u}, \hat{v}) = 0, \tag{15}$$

$$\hat{h}_2(x, \eta, \hat{u}, \hat{v}) = 0. \tag{16}$$

A statement of the SORA method for (14) is shown in Fig. 2. Observe that the optimization problem in Step 3 is a deterministic approximation under a fixed η^k. As a result, we can employ either the MDF, AAO, or IDF method to execute Step 3. Updating η^k in Step 1 can be either performed via the SORA method or the formula (13). If the SORA method is chosen, then either the MDF, AAO, or IDF method can be then again used to solve the optimization subproblem. If (13) is chosen, then we do not need to use an MDO method since there is no optimization in Step 1. Therefore, we will discuss in the next three sections the proposed methods based on the SORA algorithm (as shown in Fig. 2). In this way, the algorithmic differences of Step 1 and 3 under three MDO methods can be clearly illustrated while it is also convenient to discuss how the algorithm can be modified in order to use (13) to update η^k. To show the difference between the MDF, AAO, and IDF methods, we include diagrams of these methods in Figs. 3, 6, and 9, respectively.

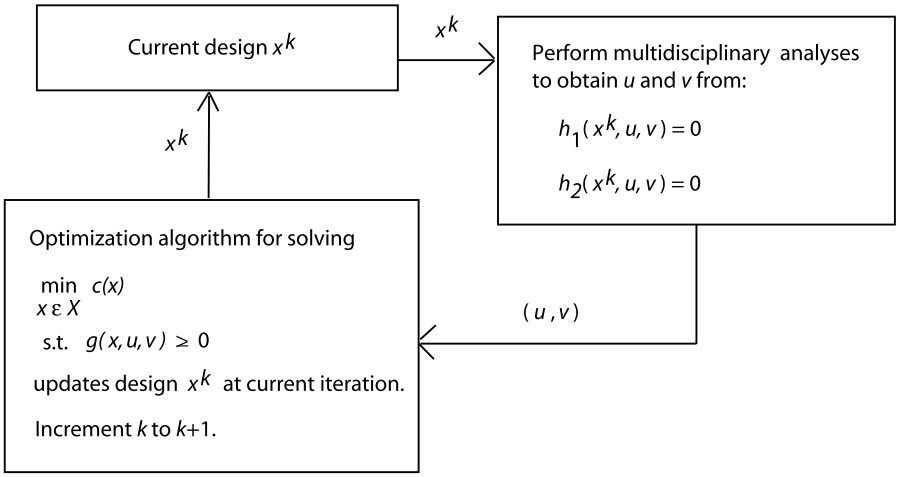

Fig. 3 Diagram of the MDF method

3.1 The RBDO/MDF method

The MDF method treats $\hat{g}$ in (14b) as a function of x and η. As a result, a calculation of $\hat{g}$ and/or its gradient for a given x and η, needed by the optimization algorithm, entails a calculation for compatible $\hat{u}$ and $\hat{v}$ via a multidisciplinary analysis, which is performed separately from the optimization algorithm. In particular, at each iteration of the optimization algorithm to solve the subproblem in Step 1 of Fig. 2, a multidisciplinary analysis is performed to obtain $\hat{u}(x^k, \eta)$ and $\hat{v}(x^k, \eta)$. Then, $\hat{u}(x^k, \eta)$ and $\hat{v}(x^k, \eta)$ are used in the calculation of $\hat{g}(x^k, \eta, \hat{u}(x^k, \eta), \hat{v}(x^k, \eta))$ and/or its gradient. Similarly, a multidisciplinary analysis is performed in Step 3 of Fig. 2 to obtain $\hat{u}(x, \eta^k)$ and $\hat{v}(x, \eta^k)$. These values are then used in the calculation of $\hat{g}(x, \eta^k, \hat{u}(x, \eta^k), \hat{v}(x, \eta^k))$ and/or its gradient. Note that x is fixed at x^k in Step 1 while η is fixed at η^k in Step 3. We do not need to specify an initial value of $\hat{u}$ and $\hat{v}$ since they are not optimization variables in the MDF method. In each multidisciplinary analysis, an initial value of $\hat{u}$ and $\hat{v}$ can be arbitrarily chosen.

The efficiency of the MDF method depends on the optimization algorithm chosen and the number of disciplinary analyses performed throughout the course of that algorithm. As discussed above, a multidisciplinary analysis needs to be performed for a fixed x and η whenever the optimization algorithm calls for the value of $\hat{g}$ and/or its gradient. Each multidisciplinary analysis may require a certain number of disciplinary analyses, each of which can be a time-consuming numerical procedure (or simulation). The number of disciplinary analyses may also increase when the system equations are coupled. In an extreme case, if the optimization algorithm is inefficient (or not suitable for the problem) and requires a large number of function and gradient evaluations and each multidisciplinary analysis also requires a large number of disciplinary analyses, then a lot of computational effort will be expected.

Step 1′ **Computing the quantile approximation of design** x^k

Solve $\min_{\eta,\hat{u},\hat{v}}\{\hat{g}(x^k,\eta,\hat{u},\hat{v}) : \| \eta \| = -\Phi^{-1}(\alpha), \hat{h}_1(x^k,\eta,\hat{u},\hat{v}) = 0,$
$\hat{h}_2(x^k,\eta,\hat{u},\hat{v}) = 0\}$. Denote its solution by $(\eta^k,\hat{u}^k,\hat{v}^k)$.

Fig. 4 A statement of the RBDO/AAO method to execute Step 1 of Fig. 2

Step 3′ **Finding design** x^{k+1}

Solve $\min_{x,\hat{u},\hat{v}}\{c(x) : \hat{g}(x,\eta^k,\hat{u},\hat{v}) \geq 0, \hat{h}_1(x,\eta^k,\hat{u},\hat{v}) = 0,$
$\hat{h}_2(x,\eta^k,\hat{u},\hat{v}) = 0, x \in X\}$. Denote its solution by $(x^{k+1},\hat{u}^{k+1},\hat{v}^{k+1})$.
Set $k = k+1$ and goto Step 1′.

Fig. 5 A statement of the RBDO/AAO method to execute Step 1 of Fig. 2

3.2 The RBDO/AAO method

The AAO method incorporates the system equations into the optimization problem as constraints, and treats the intermediate variables as optimization variables. To do so, we replace Step 1 and 3 in Fig. 2 with Step 1′ and 3′ in Figs. 4 and 5.

If the optimization algorithm used in Step 1′ and 3′ attempts to maintain feasibility at each iteration, i.e., to compute compatible $\hat{u}$ and $\hat{v}$, then the AAO method is the same as the MDF method. When the feasibility is not maintained, $\hat{u}$ and $\hat{v}$ need not be compatible and the computational efficiency can be improved as reported by Braun et al. (1995) in the context of solving a deterministic MDO problem and by Du and Chen (2003) in the context of conducting a reliability analysis of a multidisciplinary system. Our computational results in Sect. 4 also shows that the RBDO/AAO and RBDO/IDF method are more efficient than the RBDO/MDF method in solving probabilistic MDO problems. In addition, to calculate $\hat{h}_1$ or $\hat{h}_2$ or their gradients, $\hat{h}_1$ or $\hat{h}_2$ needs to be evaluated at the value of x, $\hat{u}$, and $\hat{v}$ that are specified by the optimization algorithm; therefore, $\hat{h}_1$ and $\hat{h}_2$ need not be zero during the optimization algorithm if the feasibility is not maintained. As we will see, this is a major distinction between the AAO and IDF method. These disciplinary analyses can also be performed simultaneously and may be suitable for a parallel or distributed-computing implementation.

In Step 1′, the initial value of $\hat{u}$ and $\hat{v}$ in the first major iteration can be arbitrarily chosen. In the numerical examples in Sect. 4, we set it to the value of $\hat{u}$ and $\hat{v}$ obtained from solving (14) with the random variable replaced by its mean, which is referred to as the mean-value problem. The constraint of the mean-value problem is $\hat{g}(x, E\tilde{\eta}, \hat{u}(x, E\tilde{\eta}), \hat{v}(x, E\tilde{\eta})) \geq 0$. The major iteration refers to the iteration counter k in Fig. 4 (so that it is distinguished from the iteration of the algorithm used to solve the subproblem in Step 1). In subsequent major iterations, $\hat{u}$ and $\hat{v}$ can be initialized to the optimal value obtained previously. This is the default initialization for almost all non-linear programming solvers. The initialization of $\hat{u}$ and $\hat{v}$ in Step 3′ is performed in the same fashion as in Step 1′.

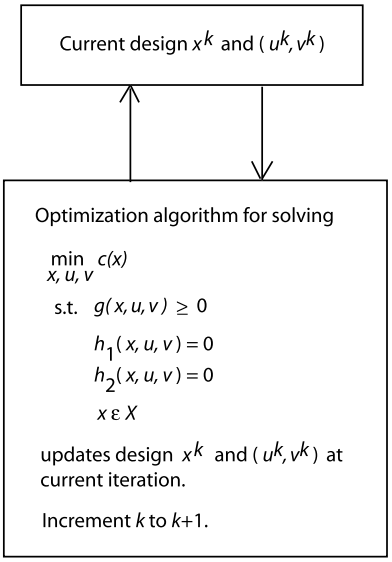

Fig. 6 Diagram of the AAO method

Step $1''$ **Computing the quantile approximation of design** x^k

Solve $\min_{\eta,\hat{u},\hat{v}}\{\hat{g}(x^k, \eta, \hat{u}, \hat{v}) : \| \eta \| = -\Phi^{-1}(\alpha), \hat{u} - \hat{u}_s = 0, \hat{v} - \hat{v}_s = 0\}$ where $\hat{u}_s$ and $\hat{v}_s$ are defined by $\hat{h}_1(x^k, \eta, \hat{u}_s, \hat{v}) = 0$ and $\hat{h}_2(x^k, \eta, \hat{u}, \hat{v}_s) = 0$. Denote its solution by $(\eta^k, \hat{u}^k, \hat{v}^k)$.

Fig. 7 A statement of the IDF method to execute Step 1 of Fig. 2

There are two potential disadvantages of the AAO method. First, the evaluation of $\hat{h}_1$ and $\hat{h}_2$ for incompatible $\hat{u}$ and $\hat{v}$ may be difficult or impossible since the computer codes involved are typically designed for only compatible $\hat{u}$ and $\hat{v}$. Second, when the number of system equations grow large, the number of functions and/or its gradient evaluations for the corresponding constraints increases and can become a bottleneck.

3.3 The RBDO/IDF method

The IDF method attempts to mitigate the disadvantages (described earlier) of the AAO method by creating surrogate variables of $\hat{u}$ and $\hat{v}$, denoted $\hat{u}_s$ and $\hat{v}_s$, in order to maintain compatibility within each discipline. To do so, we replace Step 1 and 3 in Fig. 2 with Step $1''$ and $3''$ in Figs. 7 and 8. The value of $\hat{u}$ and $\hat{v}$ in the optimization subproblem in Step $1''$ and $3''$ of the RBDO/IDF method can be initialized in the same way as the RBDO/AAO method described in Sect. 3.2.

Note that the surrogate variables are implicit functions defined via $\hat{h}_1 = 0$ and $\hat{h}_2 = 0$. If the optimization algorithm used in Step $1''$ and $3''$ attempts to maintain feasibility at each iteration, then the IDF method is the same as AAO and MDF method. When the feasibility is not maintained, the IDF method is different from the AAO

Step 3″ **Finding design** x^{k+1}

Solve $\min_{x,\hat{u},\hat{v}}\{c(x):\ \hat{g}(x,\eta^k,\hat{u},\hat{v}) \geq 0, \hat{u}-\hat{u}_s=0, \hat{v}-\hat{v}_s=0, x\in X\}$ where $\hat{u}_s$ and $\hat{v}_s$ are defined by $\hat{h}_1(x,\eta^k,\hat{u}_s,\hat{v})=0$ and $\hat{h}_2(x,\eta^k,\hat{u},\hat{v}_s)=0$. Denote its solution by $(x^{k+1},\hat{u}^{k+1},\hat{v}^{k+1})$. Set $k=k+1$ and goto Step 1″.

Fig. 8 A statement of the IDF method to execute Step 3 of Fig. 2

method in that it requires a disciplinary analysis to compute $\hat{u}_s$ or $\hat{v}_s$ or their gradients. For example, to evaluate the constraint $\hat{u}-\hat{u}_s=0$ in Step 3″, we need to solve $\hat{h}_1(x,\eta^k,\hat{u}_s,\hat{v})=0$ for $\hat{u}_s$ at the value of x and $\hat{v}$ specified by the optimization algorithm. Recall that in the AAO method no disciplinary analysis is performed and $\hat{h}_1$ may not be zero at the value of x, $\hat{u}$, and $\hat{v}$ specified by the optimization algorithm. If (15) and (16) can be manipulated so that they can be expressed as

$$\hat{u}=\hat{h}_1'(x,\eta,\hat{u},\hat{v}), \tag{17a}$$

$$\hat{v}=\hat{h}_2'(x,\eta,\hat{u},\hat{v}) \tag{17b}$$

then the IDF method is (computationally) identical to the AAO method.

The IDF method differs from the MDF method when the feasibility is not maintained because the surrogate variables allow the compatibility within each discipline to be satisfied while the interdisciplinary compatibility may or may not. To illustrate, let us consider Step 3″. Here, the compatibility within each discipline is satisfied by solving for $\hat{u}_s$ and $\hat{v}_s$ such that $\hat{h}_1(x,\eta^k,\hat{u}_s,\hat{v})=0$ and $\hat{h}_2(x,\eta^k,\hat{u},\hat{v}_s)=0$ for the value of $x,\hat{u}$ and $\hat{v}$ specified by the optimization algorithm. Since $\hat{u}-\hat{u}_s$ and $\hat{v}-\hat{v}_s$ may not be zero in the IDF method, the interdisciplinary compatibility may not necessarily satisfied. In the MDF method, a multidisciplinary analysis is, however, performed and it ensures that $\hat{u}-\hat{u}_s=0$ and $\hat{v}-\hat{v}_s=0$ as well as $\hat{h}_1(x,\eta^k,\hat{u}_s,\hat{v})=0$ and $\hat{h}_2(x,\eta^k,\hat{u},\hat{v}_s)=0$, i.e., both types of compatibility must be satisfied. Note that in the MDF method, it is equivalent to express both types of compatibility as $\hat{h}_1(x,\eta^k,\hat{u},\hat{v})=0$ and $\hat{h}_2(x,\eta^k,\hat{u},\hat{v})=0$ because $\hat{u}=\hat{u}_s$ and $\hat{v}=\hat{v}_s$ (and we did so in Sect. 3.1).

In both AAO and IDF method, the value of $\hat{u}$ and $\hat{v}$ obtained in Figs. 5 and 8, respectively, are the intermediate variables corresponding to the incumbent design x^{k+1} and realization η^k. They are computed in the process of finding an optimal design. The actual value of the intermediate variables can be determined (via the multidisciplinary system equations) only after the optimal design is implemented and the realization of the random variables is observed.

4 Numerical examples

We illustrate the proposed methods with three numerical examples. The first problem is a probabilistic MDO problem constructed from a chance-constrained linear

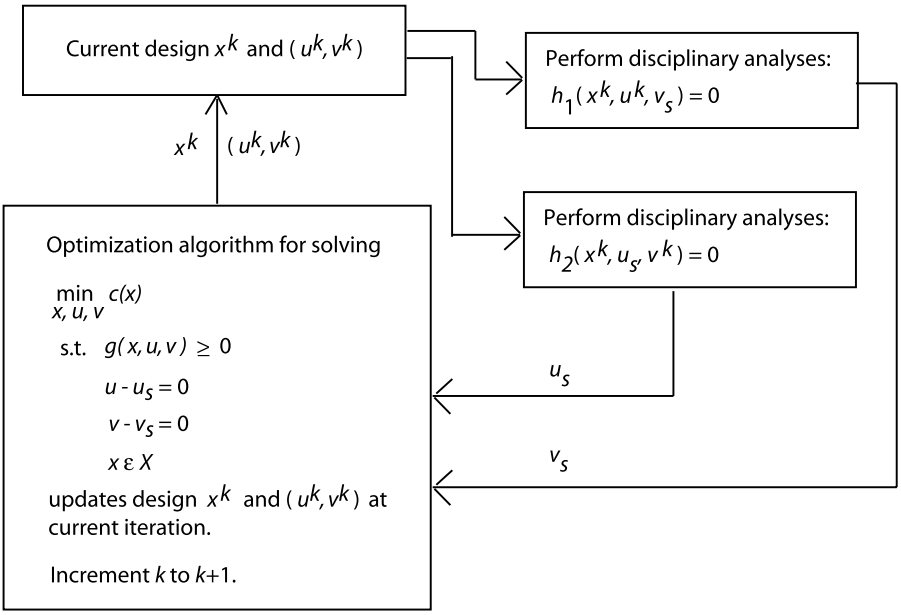

Fig. 9 Diagram of the IDF method

program. While it is simple enough that we can analytically perform each step of the proposed methods, it yet illustrates all features of these methods in dealing with probabilistic MDO problems, i.e., how the quantile-based deterministic approximation is constructed, how the compatibility requirement is handled in the stochastic setting, and how the method works when there are multiple probabilistic constraints. The fact that the constraints are linear only affect the choice of the algorithm used to solve the optimization subproblem, i.e., linear versus nonlinear programming algorithms. The second problem is a probabilistic MDO problem generated from CASCADE system (Hulme and Bloebaum 2000). This problem is non-linear and also exhibits all the feature of MDO problems. The third problem is a probabilistic variant of the electronic packaging problem. Its deterministic version is a part of the MDO test suite maintained by NASA Langley Research Center (LaRC) (Padula et al. 1996). Additional applications and computational experiments are given by Smith and Mahadevan (2005) and Smith (2004), respectively.

4.1 Example 1: maximum distance problem

Consider a deterministic linear program:

$$\max_{x \geq 0} \quad x_2, \tag{18a}$$

$$\text{s.t.} \quad x_1 + x_2 \leq 1, \tag{18b}$$

$$x_1 - x_2 \geq 0. \tag{18c}$$

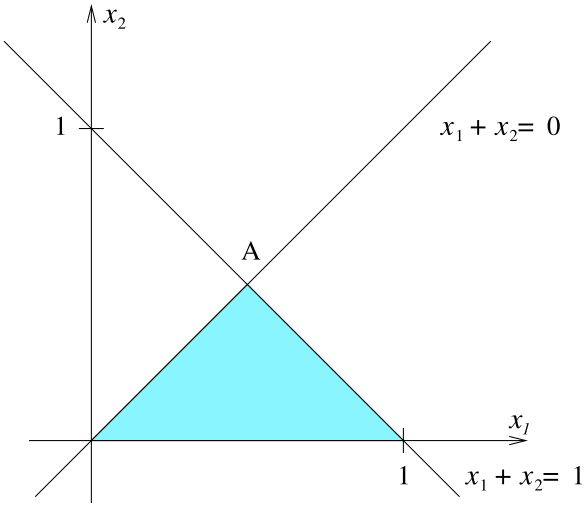

Fig. 10 A geometric representation of the deterministic linear program in Example 1

It is easy to verify that (18) is equivalent to finding a point inside the shaded triangle in Fig. 10 such that its distance (along x_2-axis) from the base of the triangle on the x_1-axis is maximum. The unique optimal solution of (18) is $(0.500, 0.500)$, denoted by point A in Fig. 10.

We construct a probabilistic variant of (18) by replacing the right-hand-side of (18b) with $\tilde{c}_1$, the coefficient of x_1 in (18c) with $(3 - \tilde{c}_2)/2$, and enforcing these constraints probabilistically. Random variables $\tilde{c}_1$ and $\tilde{c}_2$ are assumed to be independent and normally distributed with mean $\mu_1 = \mu_2 = 1.00$ and standard deviation $\sigma_1 = \sigma_2 = 0.100$. Furthermore, we let $c = (c_1, c_2)$ and define intermediate variables u and v by

$$h_1(x, c, u, v) \equiv c_2 x_1 + 2x_2 - u + v = 0, \tag{19}$$

$$h_2(x, c, u, v) \equiv 3x_1 - u - v = 0, \tag{20}$$

so that the probabilistic variant of (18) can be expressed as

$$\max_{x \geq 0} \quad x_2, \tag{21a}$$

$$\text{s.t.} \quad P\left(\tilde{c}_1 - u(x, \tilde{c}) + \frac{1}{2}(\tilde{c}_2 + 1)x_1 \leq 0\right) \leq \alpha_1, \tag{21b}$$

$$P(v(x, \tilde{c}) \leq 0) \leq \alpha_2, \tag{21c}$$

where α_1 and α_2 are real numbers between 0 and 1. We set $\alpha_1 = \alpha_2 = 1.30 \times 10^{-3}$ (corresponding to the target reliability index of 3.00). Beginning with (21), we can obtain (18) by eliminating u and v via (19) and (20), and fixing c_1 and c_2 at their means. Thus, (18) is the mean-value problem associated with (21). We can solve (21) analytically for its unique optimal solution: $(0.378, 0.322)$.

Since $\tilde{c}_1$ and $\tilde{c}_2$ are independent and normally distributed, transformation $T(\cdot)$ is given by $\tilde{c}_i = \mu_i + \sigma_i \tilde{\eta}_i$, $i = 1, 2$, where $\tilde{\eta}_i$, $i = 1, 2$, are independent standard normal

Table 1 Number of CONOPT iterations needed by the proposed RBDO-based methods for Example 1

	RBDO methods used to update η^k	
	SORA	Single-loop
RBDO/MDF	87	5
RBDO/AAO	85	4
RBDO/IDF	85	4

random variables. Let $\eta = (\eta_1, \eta_2)$. The transformed probabilistic MDO problem is

$$\max_{x \geq 0} \quad x_2, \tag{22a}$$

$$\text{s.t.} \quad P\left(\mu_1 + \sigma_1\tilde{\eta}_1 - \hat{u}(x, \tilde{\eta}) + \frac{1}{2}(\mu_2 + \sigma_2\tilde{\eta}_2 + 1)x_1 \leq 0\right) \leq \alpha_1, \tag{22b}$$

$$P(\hat{v}(x, \tilde{\eta}) \leq 0) \leq \alpha_2, \tag{22c}$$

where

$$\hat{h}_1(x, c, \hat{u}, \hat{v}) = \mu_2 + \sigma_2\eta_2 x_1 + 2x_2 - \hat{u} + \hat{v} = 0, \tag{23}$$

$$\hat{h}_2(x, c, \hat{u}, \hat{v}) = 3x_1 - \hat{u} - \hat{v} = 0. \tag{24}$$

We implement the RBDO/MDF, RBDO/AAO, and RBDO/IDF methods for Example 1 using optimization modeling language GAMS and nonlinear programming solver CONOPT. In Example 1, the stopping tolerance ϵ is set to 1.00×10^{-3} and $(0.500, 0.500)$ is used as an initial design. All three methods yield the analytical solution of $(0.378, 0.322)$. Computational effort for the three methods is proportional to the number of CONOPT iterations shown in Table 1.

We compute the gradient analytically in Example 1. Since the system equations $\hat{h}_1$ and $\hat{h}_2$ of Example 1 are linear and they can be rewritten the same form as (17), the computational results of the IDF method for this example is identical to that of the AAO method.

4.2 Example 2: CASCADE test problem

Example 2 is derived from a deterministic MDO problem generated from CASCADE system (Hulme and Bloebaum 2000). This test problem has 2 constraints and 4 design variables. There are 2 coupled disciplines, each of which contains 2 behavior variables and 2 nonlinear equations. The problem generated by CASCADE is given as subroutines in C programming language. We implement MDF, AAO, and IDF methods in MATLAB to solve this deterministic version and obtain an optimal design for the deterministic problem to be $(3814.5, 120.50, 29.300, 5000.0)$ with objective value of -221.56. In order to obtain a stochastic version, we make two coefficients of each constraint to be normal random variables with coefficient of variation ranging between 10% and 20%. The RBDO/MDF, RBDO/AAO, and RBDO/IDF methods are implemented in MATLAB to solve the stochastic version. We use target reliability index of 3 (corresponding to $\alpha = 1.3000 \times 10^{-3}$) and the

Table 2 Number of disciplinary analyses performed by each proposed RBDO-based method for Example 2

	RBDO methods used to update η^k	
	SORA	Single-loop
RBDO/MDF	5015	1940
RBDO/AAO	117	81
RBDO/IDF	117	81

stopping tolerance ϵ of 1.0000×10^{-5}. All three methods yield comparable solution of $(3529.00, 159.00, 11.500, 4934.8)$ with corresponding optimal objective value of -206.65. Computational effort for each method is reported in Table 2. The gradient for updating η^k in the single-loop method is approximated by forward finite difference.

The system equations of Example 2 (generated by CASCADE) is of the form given by (17); thus, the RBDO/AAO and RBDO/IDF method turn out to be the same as we discussed in Sect. 3.3. To verify the obtained solution, we employ Monte Carlo sampling to estimate the probability of constraint violation to be 1.40×10^{-4} and 1.50×10^{-4} for each constraint (with 95% half width of 1.58×10^{-6} and 2.34×10^{-6}). These values exceed the value specified by the target reliability index by 1.00×10^{-4} and 2.00×10^{-4}, respectively. This error can be attributed to the first order Taylor approximation employed in the single-loop and SORA methods.

4.3 Example 3: electronic packaging problem

A formulation of the deterministic electronic packaging problem can be stated as

$$\max_{x_L \le x \le x_U} \quad u_1(x), \tag{25a}$$

$$\text{s.t.} \quad u_{11}(x) \le T_1, \tag{25b}$$

$$u_{12}(x) \le T_2, \tag{25c}$$

$$u_4(x) = u_5(x). \tag{25d}$$

There are 8 design variables, $x_1, \ldots, x_8$, and 12 intermediate variables, $u_1, \ldots, u_{12}$. The goal is to find x such that the power density, u_1, is maximized. Constraints (25b) and (25c) require that the temperature at each resistor, u_{11} and u_{12}, do not exceed T_1 and T_2, respectively. Constraint (25d) ensures that the electrical current passing through each resistor, u_4 and u_5, is equal. The lower and upper bound on x are x_L and x_U, respectively. The system equations defining $u_1, \ldots, u_{12}$, are

$$u_1 = \frac{u_{10}}{x_1 x_2 x_3}, \quad u_5 = \frac{u_2 u_8}{u_2 + u_3}, \quad u_9 = \frac{u_2 u_3}{u_2 + u_3},$$

$$u_2 = x_5(1 + x_6(u_{11} - T_0)), \quad u_6 = u_4^2 u_2, \quad u_{10} = u_8^2 u_9,$$

$$u_3 = x_7(1 + x_8(u_{12} - T_0)), \quad u_7 = u_5^2 u_3, \quad u_{11} = h_{11}(x_1, x_2, x_3, x_4, u_6, u_7),$$

$$u_4 = \frac{u_3 u_8}{u_2 + u_3}, \quad u_8 = \frac{V}{u_9}, \quad u_{12} = h_{12}(x_1, x_2, x_3, x_4, u_6, u_7),$$

where T_0 is the temperature at which the nominal resistance is measured, V is input voltage, and h_{11} and h_{12} are implicit functions through which the temperature at each resistor is calculated.

Two disciplines involved in the electronic packaging problem are electrical and heat transfer, which are coupled through u_2, u_3, u_{11}, and u_{12}. A FORTRAN subroutine, called HEATSNK (available at mdob.larc.nasa.gov), has been developed to perform multidisciplinary analyses for a fixed x, i.e., HEATSNK takes x as an input and outputs $u_1(x), \ldots, u_{12}(x)$.

We implement the MDF method in MATLAB to solve (25) with the parameters given in (Alexandrov and Kodiyalam 1998):

$$
\begin{aligned}
&T_0 = 20.000, \qquad T_1 = T_2 = 85.000, \qquad V = 10.000,\\
&x_L = (0.050000, 0.050000, 0.010000, 0.0050000,\\
&\qquad 10.000, 0.0040000, 10.000, 0.0040000),\\
&x_U = (0.15000, 0.15000, 0.10000, 0.050000, 1000.0,\\
&\qquad 0.0090000, 1000.0, 0.0090000)
\end{aligned}
$$

The gradient is approximated by finite differencing procedures (via MATLAB optimization algorithms). Using (0.15000, 0.15000, 0.10000, 0.050000, 100.00, 0.0060000, 100.00, 0.0060000) as a starting point, we obtain (0.050000, 0.050000, 0.010000, 0.050000, 10.000, 0.0040000, 10.000, 0.0040000) as an optimal design. Under this design, the power density is $u_1 = 6391.6 \times 10^2$ and the temperature at each resistor is $u_{11} = 82.370$ and $u_{12} = 82.910$. Thus, constraints (25b) and (25c) are not active. The objective function value for this design is the same as that reported by Alexandrov and Kodiyalam (1998). In order to have active constraints, we reduce T_1 and T_2 to 75.000. With these new values of T_1 and T_2, we obtain the following solution to (25):

$$
\begin{aligned}
&\hat{x} = (0.050000, 0.050000, 0.010000, 0.050000,\\
&\qquad 12.306, 0.0040000, 12.285, 0.0040000)\\
&u_1(\hat{x}) = 5336.2 \times 10^2, \qquad u_{11}(\hat{x}) = 74.550, \qquad u_{12}(\hat{x}) = 75.000.
\end{aligned}
$$

We attempt to implement in MATLAB the AAO method for (25) by incorporating all system equations as constraints; however, we cannot obtain the same solution as that from the MDF method. We believe that these numerical problems are caused by incorporating too many system equations as equality constraints. To reduce the number of equality constraints for the AAO method, we reformulate (25) as follows:

$$
\begin{aligned}
\max_{x,u'} \quad & \frac{V^2(u'_2 + u'_3)}{x_1 x_2 x_3 u'_2 u'_3}, && \text{(26a)}\\
\text{s.t.} \quad & u'_{11} \le T_1, \qquad u'_{12} \le T_2, && \text{(26b)}\\
& u'_2 = u'_3, && \text{(26c)}\\
& u'_2 = x_5(1 + x_6(u'_{11} - T_0)), && \text{(26d)}\\
& u'_3 = x_7(1 + x_8(u'_{12} - T_0)), && \text{(26e)}
\end{aligned}
$$

$$u'_{11} = h'_{11}(x_1, x_2, x_3, x_4, u'_2, u'_3), \tag{26f}$$

$$u'_{12} = h'_{12}(x_1, x_2, x_3, x_4, u'_2, u'_3), \tag{26g}$$

$$x_L \le x \le x_U, \tag{26h}$$

where $u' = (u'_2, u'_3, u'_{11}, u'_{12})$. Constraints (26d–26g) are derived from system equations (by eliminating all intermediate variables except u_2, u_3, u_{11}, and u_{12}). We implement the AAO method in MATLAB to solve (26) and obtain the same solution as that of the MDF method. The computational effort to obtain $\hat{x}$ is summarized in Table 3. The computational work of carrying out one analysis in the MDF method is to evaluate function h_{11} and h_{12} while that in the AAO method is to evaluate function h'_{11} and h'_{12}. We note that the IDF method for (26) is computationally identical to the AAO method since u'_2, u'_3, u'_{11}, and u'_{12} can be separated out in (26d–26g) as discussed in Sect. 3.3.

To turn the electronic packaging problem into a probabilistic MDO problem, we assume that there are (additive) normally-distributed random errors $\tilde{\xi}_1$ and $\tilde{\xi}_2$ associated with x_5 and x_7, respectively; hence, $u_2 = (x_5 + \xi_1)(1 + x_6(u_{11} - T_0))$ and $u_3 = (x_7 + \xi_2)(1 + x_8(u_{12} - T_0))$.

Let $\mu = (\mu_1, \mu_2)$ and $\sigma = (\sigma_1^2, \sigma_2^2)$ be the mean and variance of $\tilde{\xi} = (\tilde{\xi}_1, \tilde{\xi}_2)$. The transformed u_2 and u_3 can then be expressed as

$$\hat{u}_2 = (x_5 + \mu_1 + \sigma_1\eta_1)(1 + x_6(\hat{u}_{11} - T_0)),$$

$$\hat{u}_3 = (x_7 + \mu_2 + \sigma_2\eta_2)(1 + x_8(\hat{u}_{12} - T_0)),$$

where $\tilde{\eta} = (\tilde{\eta}_1, \tilde{\eta}_2)$ are independent standard normal random variable. Note that the intermediate variables become random since they depend on $\tilde{\eta}$. We consider the following probabilistic variant of (25):

$$\max_{x_L \le x \le x_U} \quad \hat{u}_1(x, E\tilde{\eta}), \tag{27a}$$

$$\text{s.t.} \quad P(T_1 - \hat{u}_{11}(x, \tilde{\eta}) \le 0) \le \alpha_1, \tag{27b}$$

$$P(T_2 - \hat{u}_{12}(x, \tilde{\eta}) \le 0) \le \alpha_2, \tag{27c}$$

$$\hat{u}_4(x, E\tilde{\eta}) = \hat{u}_5(x, E\tilde{\eta}). \tag{27d}$$

Using an approximation of the objective function as discussed in Sect. 2, we minimize $\hat{u}_1(x, E\tilde{\eta})$ instead of $E\hat{u}_1(x, \tilde{\eta})$ in (27). In addition, we enforce the equality of the electrical current via (27d). An alternative to (27d) is to enforce the equality constraints as $P(\|\hat{u}_4(x, \tilde{\eta}) - \hat{u}_5(x, \tilde{\eta})\| \ge \epsilon) \le \alpha_3$ where any type of norm can be used for $\|\cdot\|$.

For Example 3, the stopping tolerance of 1.00000×10^{-5} is used throughout our implementation. With $\mu = (0.00000, 0.00000), \sigma = (0.30000, 0.90000), \alpha_1 = \alpha_2 =$

Table 3 Number of disciplinary analyses required for solving deterministic electronic packaging problem

	MDF	AAO	IDF
No. of analyses	711	88	88

1.3000×10^{-3}, and the same value of parameters as that used to obtain $\hat{x}$, we implement the RBDO/MDF method to solve (27) and obtain the following solution:

$$x^* = (0.050000, 0.050000, 0.010000, 0.050000,$$
$$12.358, 0.0090000, 14.845, 0.0040000),$$
$$\text{with } \hat{u}_1(x^*, E\tilde{\eta}) = 4508.9 \times 10.$$

We also implement the RBDO/AAO method based on the formulation (26) and obtain the same solution as the RBDO/MDF method. Again, the RBDO/IDF method (based on (26)) yields the same computational result as that of the RBDO/AAO method. The computational effort of each method is summarized in Table 4. In the RBDO/AAO method that uses the single-loop scheme to update η^k, we could not obtain convergence. In fact, the final solution is infeasible. We believe that this is due to two reasons. First, the single-loop method provides an inaccurate estimate of the quantile. In contrast, the SORA method gives uses an exact quantile of the design obtained in the previous iteration. We note that similar behaviors of divergence have been observed in RBDO context as reported by Youn et al. (2004). Second, some or all of the compatibility requirements are relaxed in the IDF and AAO methods; thus, the accuracy of the quantile estimate becomes worse.

To validate the solution obtained from the RBDO-based method, we simulate the performance of design x^* and the mean-value design $\hat{x}$. The quantities of interest (related to the objective and constraint functions of (27)) are estimated by the standard sample mean and listed in the first column of Table 5. We randomly sample from the distribution of $\tilde{\eta}$ in our estimation procedure since we only want to verify these solutions and do not concern with efficiency. The estimations for $x = x^*$ and $x = \hat{x}$ in Table 5 are based on the same set of 50,000 observations of $\tilde{\eta}$; thus, the difference in the sample mean estimator under $\hat{x}$ and x^* is purely due to the quality of the solution. The 95% half-width of the confidence interval associated with the sample mean estimator is listed under "95% HW" in Table 5. We point that the probability that the temperature of the resistor will exceed the temperature limit is 0.30000 and 0.50000 under design $\hat{x}$ while this probability is within the prescribed limit under design x^*.

We remark that accurate gradient estimation plays an important role in the numerical optimization procedure that involves multidisciplinary analysis (especially when the underlying function is implicit). In particular, we have experienced numerical difficulties (e.g., stalling or non-convergence) when we first implement the proposed method for Example 3 in GAMS using CONOPT as a nonlinear programming solver. In the GAMS/CONOPT implementation, we need to implement our own finite difference procedure. When we instead implement the proposed methods in MATLAB,

Table 4 Number of disciplinary analyses performed in each RBDO-based method for Example 3

	RBDO methods used to update η^k	
	SORA	Single-loop
RBDO/MDF	6804	8004
RBDO/AAO	1550	Not converge
RBDO/IDF	1550	Not converge

Table 5 Estimation results by Monte Carlo random sampling to compare design x^* obtained from the proposed method with design $\hat{x}$ obtained from solving the mean-value problem. The same set of 50,000 samples are used for estimating these quantities under x^* and $\hat{x}$

Quantity estimated	Design $x = x^*$		Design $x = \hat{x}$	
	Sample mean	95% HW	Sample mean	95% HW
$P(\hat{u}_{11}(x,\tilde{\eta}) \geq 75.000)$	0.00000	0.00000	0.30000	0.18000×10^{-4}
$P(\hat{u}_{12}(x,\tilde{\eta}) \geq 75.000)$	0.13600×10^{-2}	0.14446×10^{-5}	0.50036	0.19600×10^{-4}
$P(\lvert\hat{u}_4(x,\tilde{\eta}) - \hat{u}_5(x,\tilde{\eta})\rvert \geq 0.030000)$	0.36952	0.18921×10^{-4}	0.51730	0.19588×10^{-4}
$E\hat{u}_1(x,\tilde{\eta})$	4516.9×10	0.51489	5349.3×10	0.72315
$E\hat{u}_{11}(x,\tilde{\eta})$	68.438	0.25240×10^{-4}	74.580	0.32168×10^{-4}
$E\hat{u}_{12}(x,\tilde{\eta})$	68.896	0.68224×10^{-4}	75.165	0.96645×10^{-4}
$E\lvert\hat{u}_4(x,\tilde{\eta}) - \hat{u}_5(x,\tilde{\eta})\rvert$	0.26802×10^{-1}	0.80638×10^{-6}	0.37263×10^{-1}	0.11307×10^{-5}

which provides a more sophisticated finite difference procedure, these numerical difficulties disappear. However, we still have a numerical difficulty in MATLAB due to imposing too many equality constraints in our model for the AAO method. We believe that this issue is generally encountered for other nonlinear optimization problems with a large number of equality constraints as well.

5 Conclusion

We have presented a mathematical framework that integrates the RBDO methods with existing deterministic MDO methods to solve MDO problems under uncertainty. The proposed framework exploits the deterministic equivalent formulation based on the quantile function. Specifically, we have described in details the RBDO/MDF, RBDO/AAO, and RBDO/IDF method. In each of these methods, two different ways to form the quantile approximation have been shown, i.e., via either the single-loop or SORA method.

To demonstrate the proposed methods, we have applied them to three numerical examples. The first example is derived from a chance-constrained linear program. The solution obtained from the RBDO/MDF, RBDO/AAO, and RBDO/IDF methods matches with the analytical solution. The second example is derived from CASCADE system (Hulme and Bloebaum 2000). The solutions obtained from all three methods are approximately the same for the second example. The third example is a probabilistic variant of the electronic packaging problem. The RBDO/IDF method for these three examples is computationally identical to the RBDO/AAO method due to the form of the system equations.

Computational efficiency of the RBDO methods and the MDF, AAO, and IDF method have been studied in the RBDO and MDO literature in other contexts than the probabilistic MDO. Our computational results are consistent with those reported in the literature. In particular, the RBDO/AAO and RBDO/IDF methods require less computational effort than the RBDO/MDF method in all three examples. In addition, the deterministic MDO methods integrated with the single-loop method, when converge, can be more efficient than those with the SORA method as observed in Examples 1 and 2. The AAO and IDF methods can cause divergence when integrated

with the single-loop method as observed in Example 3. This divergence is due to an inaccurate estimate of the quantile of the single-loop method and the relaxation of compatibility requirements in the IDF and AAO methods.

The challenges that other probabilistic MDO problems pose are (i) how to properly handle probabilistic equality constraints, (ii) variable scaling in multidisciplinary analysis and nonlinear optimization, (iii) convergence of multidisciplinary analysis when design variables vary over a wide range, and (iv) accuracy of the finite difference approximation of the gradient of implicit functions involving multidisciplinary equations.

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