UNIVERSIDAD TÉCNICA
FEDERICO SANTA MARÍA

Departamento de Matemática

Existence of solutions of some fractional problems

Tesis presentado por

Nicolás González Muñoz

como requisito parcial para optar al grado de
Magíster en Ciencias, Mención Matemática

Director de Tesis:

Salomón Alarcón

Valparaíso

2021

UNIVERSIDAD TÉCNICA
FEDERICO SANTA MARÍA

Departamento de Matemática

Existence of solutions of some fractional problems

Tesis[1] presentada por

Nicolás González Muñoz

como un requisito parcial para optar al grado de
Magíster en Ciencias, Mención Matemática

Director de Tesis:

Salomón Alarcón

Examinadores:

Leonelo Iturriaga

Erwin Topp

Valparaíso
2021

[1]Material de referencia. Su uso no involucra ninguna responsabilidad por parte del autor o la institución.

Antecedentes

Trabajo y Autor

Título de la Tesis: Existence of solutions of some fractional problems.

Autor: Nicolás González Muñoz.

Observación: Trabajo presentado como requisito parcial para optar al grado de Magíster en Ciencias, Mención Matemática de la Universidad Técnica Federico Santa María

Comisión Evaluadora

Director de Tesis: Salomón Alarcón.

Doctor en Ciencias de la Ingeniería con Mención en Modelación Matemática. Universidad de Chile. Santiago, Chile. 2006.
Profesor del Departamento de Matemática. *Universidad Técnica Federico Santa María.*

Salomón Alarcón

Examinador: Leonelo Iturriaga.

Doctor en Ciencias con Mención en Matemática, *Universidad de Chile.* Santiago, Chile. 2006.
Profesor del Departamento de Matemática. *Universidad Técnica Federico Santa María.*
.

Leonelo Iturriaga

Examinador: Erwin Topp.

Doctor en Ciencias de la Ingeniería Mención Modelamiento Matemático, *Universidad de Chile (en cotutela con Université François Rabelais de Tours)*, Francia, 2014.
Profesor Asistente Departamento de Matemática y Ciencia de la Computación. *Universidad de Santiago de Chile.*

Erwin Topp

Valparaíso, 2021.

Resumen

En este trabajo se estudia el problema

$$\begin{cases} -\mathcal{D}v(t) = \lambda c(t) f(t, v(t), \ldots, D^n v(t)), & 0 < t < 1 \\ D^k v(0) = 0, & 0 \le k \le m \quad \text{si } m \ge 0 \\ \mathcal{A}_i(D^{n_{i,j}} v(c_{i,j}))_{j=m+1}^{[\alpha]-1} = 0, & \forall\, i \in \{m+1, \ldots, [\alpha]-1\} \end{cases}$$

donde $\mathcal{D}$ es o bien, el operador D^α o ${}^C D^\alpha$ con $\alpha > 1$, o el operador $D^\beta\,(q(t)D^\alpha)$ con $\alpha > 1$ y $\beta > 0$; $\lambda > 0$ es un parámetro, $c \in L^1(0,1)$ es no negativa, $f : [0,1] \times \mathbb{R}^{n+1} \to \mathbb{R}$ es una función tal que $f(t,\cdot)$ es localmente lipschitz para c.t.p. $t \in [0,1]$ y $f(\cdot, x) \in L^\infty(0,1)$ para todo $x \in \mathbb{R}^{n+1}$; $m \in \mathbb{Z}_{\ge -1}$ y $n \in \mathbb{N}_0$ son tales que $n \ge 0$ y $n, m < [\alpha] - 1$, $\{\mathcal{A}_i\}_{i=m+1}^{[\alpha]-1} \subset C(\mathbb{R}^{[\alpha]-m-1}; \mathbb{R})$ son operadores lineales, $\{n_{i,j}\}_{i,j=m+1}^{[\alpha]-1} \subset \mathbb{R}$ y $\{c_{i,j}\}_{i,j=m+1}^{[\alpha]-1} \subset [0,1]$. Se mostrará que bajo ciertas suposiciones sobre la función f y la condiciones de borde del problema, este admite al menos una solución positiva en $C[0,1]$ usando un método de punto fijo, y que bajo otras supocisiones sobre f el problema admite una infinidad de soluciones positivas, y que estas satisfacen que $v_\lambda \to 0$ cuando $\lambda \to \infty$ usando métodos de punto fijo combinados con un nuevo enfoque para abordar el problema.

Abstract

In this work will be stutied the problem

$$\begin{cases} -\mathcal{D}v(t) = \lambda c(t) f(t, v(t), \dots, D^n v(t)), & 0 < t < 1 \\ D^k v(0) = 0, & 0 \le k \le m \quad \text{if } m \ge 0 \\ \mathcal{A}_i(D^{n_{i,j}} v(c_{i,j}))_{j=m+1}^{[\alpha]-1} = 0, & \forall\, i \in \{m+1, \dots, [\alpha]-1\} \end{cases}$$

where $\mathcal{D}$ is, either, the operator D^α or ${}^C D^\alpha$ with $\alpha > 1$, or the operator $D^\beta\,(q(t) D^\alpha)$ with $\alpha > 1$ and $\beta > 0$; $\lambda > 0$ is a parameter, $c \in L^1(0,1)$ is nonnegative, $f : [0,1] \times \mathbb{R}^{n+1} \to \mathbb{R}$ is such that $f(t, \cdot)$ is locally lipschitz for a.e. $t \in [0,1]$ and $f(\cdot, x) \in L^\infty(0,1)$ for all $x \in \mathbb{R}^{n+1}$; $m \in \mathbb{Z}_{\ge -1}$ and $n \in \mathbb{N}_0$ are such that $n \ge 0$ and $n, m < [\alpha]-1$, $\{\mathcal{A}_i\}_{i=m+1}^{[\alpha]-1} \subset C(\mathbb{R}^{[\alpha]-m-1}; \mathbb{R})$ are linear operators, $\{n_{i,j}\}_{i,j=m+1}^{[\alpha]-1} \subset \mathbb{R}$ and $\{c_{i,j}\}_{i,j=m+1}^{[\alpha]-1} \subset [0,1]$. It will be shown that, under certain assumptions about f and the boundary conditions of the problem, it admits at least one positive solution on $C[0,1]$ using a fixed point method, and that under some other assumptions about f the problem admits an infinity of positive solutions, and that they satisfy that $v_\lambda \to 0$ when $\lambda \to \infty$ using fixed point methods combined with a new approach to address the problem.

Agradecimientos

Este trabajo es aquel que me permite finalizar mis estudios universitarios, y me abre un mundo de posibilidades. En este sentido, deseo agradecer a todas las personas que me han apoyado y guiado, y me han ayudado a hacer esto posible; aquellos que me criaron hasta ser la persona que soy hoy, quienes me han apoyado en los momentos más duros, aquellos de los cuales he aprendido, y aquellos que me han ayudado a financiarme.

En primer lugar, deseo agradecer a aquellos que ya no están en este mundo, mis abuelos, Oscar y Graciela, quienes entregaron su sabiduría no solo a mí, sino que a toda mi familia; mis padres, Humberto y Rosa, quienes me criaron y se preocupan por mi hasta el día de hoy; mi hermana, Claudia, quien me ha ayudado en todo lo que ha podido desde que nací; mi tío, Carlos, quien ha apoyado a mi familia en los momentos difíciles; y, en general, a toda mi familia, quienes me han animado a alcanzar mis metas.

Quiero agradecer a mi profesor y tutor Salomón Alarcón, quien me guió y apoyó en esta última etapa. A la comisión que participa en mi proceso de evaluación, Leonelo Iturriaga y Erwin Topp, por su disposición a reunirnos y revisar mi trabajo.

A mis profesores de la universidad Patricio Guzmán, Pedro Gajardo, Pablo Aguirre, Rodrigo Lecaros, Alberto Mercado, Alfredo Calderón, Eduardo Valenzuela, Fabián Contreras, Alexander Quass, Alejandro Allendes, Humberto Villalobos y Felipe Osorio, de quienes me siento orgulloso de haber aprendido.

A mis compañeros, Martín Hernández, Simón Masnú, Fabián Ramírez y Rogelio Arancibia, con quienes he compartido risas y nos hemos apoyado en nuestros estudios.

Al congreso COMCA 2021, quienes me permitieron participar y compartir mi trabajo con otros investigadores del mayor nivel .

Finalmente, quiero agradecer a Beca de Estudios de Magíster Científico UTFSM 2021, junto con el proyecto FONDECYT 1211766, quienes hicieron posible que pudiera financiar mis estudios, y a dedicarme completamente a mi trabajo.

Prefacio

Este documento fue escrito durante el año 2021 y corresponde al trabajo de tesis hecho por el autor para calificar al grado de Magíster en Ciencias, Mención Matemática. En este trabajo se analizan tópicos sobre ecuaciones diferenciales fraccionarias, en el área de ecuaciones diferenciales no lineales. Producto de este trabajo se contribuye en aquellas áreas con varios resultados en el contexto de ecuaciones diferenciales fraccionarias unidimensionales y se relacionan estos resultados con problemas de mayor dimensión.

Resumen de los contenidos del texto

En este documento se estudia la existencia y multiplicidad de soluciones positivas en algunas ecuaciones diferenciales fraccionarias unidimensionales que involucra a los operadores fraccionarios de Riemann-Liouville y de Caputo, así como el comportamiento asintótico de dichas soluciones, para luego relacionar dichos resultados a problemas de mayor dimensión.

En el capítulo 1 se definen los principales objetos de análisis, así como las motivaciones físicas y matemáticas de las definiciones antes mencionadas. Se presentan los problemas a estudiar, y se describen brevemente los métodos para resolver los problemas propuestos.

En el capítulo 2 se presentan algunos resultados preliminares que son ampliamente conocidos en el área, y que serán útiles en el desarrollo de este trabajo.

En el capítulo 3 se construyen funciones de Green generales para los operadores fraccionarios estudiados en este documento, y se estudian las condiciones bajo las cuales el problema está bien puesto.

En el capítulo 4 se prueban algunos lemas útiles relacionados con las funciones de Green encontradas en el capítulo anterior, los cuales son utilizados para probar la existencia de soluciones del problema. Luego, dicha demostración es extendida para abarcar el estudio de multiplicidad de soluciones y comportamiento asintótico de las mismas.

En el capítulo 5 se conecta el análisis hecho en capítulos anteriores a problemas de mayor dimensión, y los resultados anteriores son utilizados en el estudio sobre existencia de soluciones radiales de este problema.

Finalmente, en el capítulo 6 se sintetizan las principales conclusiones del trabajo realizado, señalando específicamente el potencial impacto de este trabajo, así como algunas conjeturas sobre posibles extensiones y generalizaciones del mismo, constuyendo las bases para posibles trabajos futuros.

Propósito del texto

Este documento tiene un propósito expositivo, dado que su principal objetivo es mostrar los conocimientos adquiridos por el autor. Su intención es contrastar los resultados obtenidos en su análisis, usanso métodos comunes para todo lo problemas planteados, revelando su naturaleza en cuanto a cambios en su estructura analítica.

El propósito del capítulo 1 es presentar y motivar el estudio de los objetos matemáticos que aborda este trabajo. Se desarrolla una descripción del contexto matemático y físico de los problemas, entregando detalles técnicos e históricos sobre la evolución y el estado de arte del área.

En el capítulo 2 se describen algunos resultados preliminares, siendo la mayor parte de ellos conocidos en el área. Adicionalmente, se presentan algunas herramientas que son de utilidad en la construcción de funciones de Green.

Los capítulos 3 y 4 son dependientes entre sí, dado que en el capítulo 3 se desarrollan algunas herramientas usadas en el capítulo 4 para resolver los problemas propuestos en el mismo. Debido a que la estructura de los problemas unidimensionales estudiados es similar, en el capítulo 3 se construyen funciones de Green para cada problema, mientras que en el capítulo 4 se resuelven dichos problemas. En este sentido, es posible contrastar la estructura de estos problemas más fácilmente.

El capítulo 5 también es dependiente del capítulo 4, dado que los resultados desarrollados en este último son utilizados para resolver los problemas en el primero.

El capítulo 6 es presentado como un reporte, debido a que busca sintetizar el trabajo realizado a lo largo del documento.

Finalmente, se provee al lector de un apéndice, de forma que la lectura de este documento sea lo más fluida posible. Este contiene algunos resultados técnicos que tienen un rol secundario en este trabajo, o que son conocidos en el área. Sin embargo, este apéndice puede ser de utilidad para obtener un entendimiento completo de este documento, así como de los resultados desarrollados en este.

Notación

$\mathbb{N}$: Conjunto de todos los números naturales.

$\mathbb{Z}$: Conjunto de todos los números enteros.

$\mathbb{R}$: Conjunto de todos los números reales.

$\mathbb{C}$: Conjunto de todos los números complejos.

$A_{\prec b}$: Conjunto de todos los elementos a del conjunto parcialmente ordenado $(A, \prec)$ tales que $a \prec b$.

$\overline{\Omega}$: Clausura del conjunto $\overline{\Omega}$ sobre un espacio topológico dado.

$C(\Omega)$: Espacio de todas las funciones continuas sobre Ω.

$C(\overline{\Omega})$: Espacio de funciones uniformemente continuas sobre $\overline{\Omega}$.

$C^n(\Omega)$: Espacio de funciones $n-$veces continuamente diferenciables sobre Ω.

$C^n(\overline{\Omega})$: Espacio de todas las funciones $f \in C^n(\Omega)$ tales que $D^m f \in C(\overline{\Omega})$ para todo $0 \leq m \leq n$.

$C_c^\infty(\Omega) = \mathcal{D}(\Omega)$: Espacio de funciones test sobre Ω.

$\mathcal{D}'(\Omega)$: Espacio de distribuciones. Dual topológico de $\mathcal{D}(\Omega)$.

$\mathcal{S}$: Espacio de Shwartz. Espacio de todas las funciones de clase $C^\infty(\mathbb{R}^n)$ de decaimiento rápido.

$\mathcal{S}'$: Espacio de distribuciones temperadas, dual topológico del espacio de Shwartz.

Tabla de Contenidos

Chapter 1

Introduction

In this chapter the historical background and the general context of this work is presented. First, the history of fractional calculus is briefly commented, mentioning some notorious fractional operators, and presenting the classical definition of the Riemann-Liouville fractional integral and derivative operators, as well as the Caputo operator and some generalized fractional operators.

After that, the main problems to study on this work are presented, as well as some similar works researched before by other people, and some hypothesis are set that will be used on the following chapters. Also, the state of art on this field is mentioned.

Finally, some known tools are presented, that will be useful for the purposes of this work. The methods used to solve the problems are presented, together with the objectives and expected results.

Onwards in this text, on the set $\mathbb{C}$ will be considered the partial order $\leq$ defined as $x \leq y$ if and only if $Re(x) \leq Re(y)$. Analogously, the relation $<$ is defined on $\mathbb{C}$ as $x < y$ if and only if $Re(x) < Re(y)$.

1.1 What is fractional calculus?

The concepts of integral and derivative of integer order are well known. The derivative $\frac{\partial^n y}{\partial x^n}$ describes the changes on the variable y with respect of the variable x. But then, one question arise: Can the concept of derivative be extended so that n can be a fraction?. In fact, in 1695 L'Hôpital sent a letter to Leibniz asking what the derivative of $\frac{\partial^n y}{dx^n}$ was when $n = 1/2$. Then, the problem was considered by several mathematicians over time.

So, fractional calculus was developed as a generalization of classical calculus, and today there are many applications of this generalization, like physics, chemistry, biology, economy, and even probability theory.

1.2 Some known fractional operators

Over time, several fractional operators were defined, with some related to each other. There some of the most used are mentioned.

1.2.1 Riemann-Liouville Fractional Derivative

The object that motivated fractional calculus, and one of the text's objects of study. In 1811 Adrien-Marie Legendre introduced the Gamma function, which was rewrote by Euler in its modern form, given by

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx, \quad Re(z) > 0$$

and was also proven that

$$\Gamma(n) = (n-1)!$$

for any positive integer n. That is, the Gamma function generalize the factorial operator to complex numbers with positive real part.

Then, in 1823, Augustin Louis Cauchy proved his formula for repeated integration, that is, given a continuous function f over the real line, if the nth repeated integral of f is defined as as

$$f^{(-n)}(x) = \int_a^x \int_a^{\sigma_1} \cdots \int_a^{\sigma_{n-1}} f(\sigma_n)d\sigma_n \cdots d\sigma_2 d\sigma_1$$

then

$$f^{(-n)}(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}f(t)dt$$

All of the above combined give us the following definition.

Definition 1.2.1. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function, where $a < b$, $a \in \mathbb{R}\cup\{-\infty\}$ and $b \in \mathbb{R}\cup\{\infty\}$. Then the Riemann-Liouville fractional integral of order α is defined as*

$$_aJ_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad \alpha \in \mathbb{C}, \quad Re(\alpha) > 0$$

and $J^\alpha f(x) = f(x)$ if $\alpha = 0$.

If $a = 0$, the previous notation is simplified as

$$_aJ_x^\alpha f(x) = J^\alpha f(x)$$

So, with that now is possible to integrate a continuous function "a fractional number of times". Furthermore, with the fact that, for any continuous function f we have that

$$Df^{(-1)}(x) = f(x)$$

then it's possible to define the following.

Definition 1.2.2. *Let $f : [a, b] \to \mathbb{R}$ be a function such that $f \in C[a, b]$, where $a < b$, $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$. Then the Riemann-Liouville fractional derivative of order α is defined as*

$$ {}_aD_x^\alpha f(x) = D^{[\alpha]}J^{[\alpha]-\alpha}f(x), \quad \alpha \in \mathbb{C}, \quad Re(\alpha) > 0 $$

and if $\alpha = 0$, $D^\alpha f(x) = f(x)$.

If $a = 0$, the previous is simply denote

$$ {}_0D_x^\alpha f(x) = D^\alpha f(x) $$

and if $Re(\alpha) < 0$, the following notation could be used,

$$ {}_aD_x^\alpha f(x) = {}_aJ_x^{-\alpha} f(x) $$

So, in fact, the Riemann-Liouville fractional derivative answers the question that motivated fractional calculus. Moreover, by construction, if α is a positive integer, then the Riemann-Liouville fractional derivative coincides with the classical derivative.

1.2.2 Caputo Fractional Derivative

The Caputo fractional derivative is defined in a very similar way as the Riemann-Liouville fractional derivative.

Definition 1.2.3. *Let $f : [a, b] \to \mathbb{R}$ be a function such that $f \in AC^{[\alpha]}[a, b]$, where $a < b$, $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$. Then the Caputo fractional derivative of order α is defined as*

$$ {}_a^CD_x^\alpha f(x) = J^{[\alpha]-\alpha}D^{[\alpha]}f(x), \quad \alpha \in \mathbb{C}, \quad Re(\alpha) > 0 $$

and if $\alpha = 0$, then ${}^CD^\alpha f(x) = f(x)$.

If $a = 0$, we denote

$$ {}_0^CD_x^\alpha f(x) = {}^CD^\alpha f(x) $$

So, the idea behind the construction of the Caputo and Riemann-Liouville fractional derivatives are essentially the same, but inverting the order of the derivative and integral. Furthermore, under certain conditions, both operators coincide.

1.2.3 Riesz Potential

The Riemann-Liouville fractional integral can be extended to spaces of higher dimension in several ways. However, Riesz potentially is probably the most used.

Definition 1.2.4. *Let $0 < \alpha < n$ and let $f \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \frac{n}{\alpha}$. The Riesz potential $I_\alpha f$ is defined as*

$$ (I_\alpha f)(x) = \frac{1}{c_\alpha} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy $$

1.2.4 Fractional Laplacian

Maybe the most used fractional operator for problems on higher dimensions, it's defined in several ways, with the most of them being essentially the same. One of them is presented here.

Definition 1.2.5. *Let $s \in (0,1)$ and $f : \mathbb{R}^n \to \mathbb{R}$ a function. The fractional laplacian of order s, $(-\Delta)^s$, is defined as*

$$c \int_{\mathbb{R}} \frac{f(x) - f(y)}{|x-y|^{n+2s}} dy$$

where $c > 0$ is a fixed constant.

In the definition above, constant c can be chosen in several ways, depending on the context. In some way, fractional laplacian is an extension of the laplacian operator, since $\lim_{s \to 1^-} (-\Delta)^s f = \Delta f$ for f in a suitable functional space. Furthermore, this operator is sometimes used to define some fractional spaces, like fractional sobolev spaces $W^{r,p}$.

Also, under some conditions on the dimension n, the fractional laplacian is the inverse of the Riesz potential over a suitable space of generalized functions.

1.2.5 More General Fractional Operators

Note that, in most of the fractional operators shown previously, they can be written as a convolution with a fixed kernel function. In this sense, all those operators can be generalized by choosing an arbitrary, but suitable, kernel function.

For example, Riemann-Liouville fractional integral can be written as

$$_aJ_x^\alpha f(x) = f * p_\alpha(x)$$

where $*$ is the convolution operator, and $p_\alpha : [a, \infty) \to \mathbb{R}_{\geq 0}$ is the polynomial

$$p_\alpha(x) = \frac{1}{\Gamma(\alpha)} (x-a)^{\alpha - 1}$$

So, we can generalize this operator by setting

$$_a\overline{J}_x f(x) = f * g$$

for some suitable function g, depending on the domain of the operator.

1.3 Motivation on Physics

The classical derivatives represent a local property of a function: their change over time on a given point x, with respect with a certain variable. However, in physics there are several phenomenons that can't be modeled in that way. Take for example the representation of the viscosity of a pseudo-plastic fluid, that is, a non Newtonian fluid whose viscosity decrease under shear strain. Since the deformation propagates along all the fluid, the viscosity on one point depends on shear strain applied to all the fluid. Then, it can't be represented with a local operator.

However, note that all of the fractional operators defined previously are non local, that is, the value obtained on one point depends, at the same time, either, on the point's previous states and/or the states of all the points over a certain region around the said point. Because of that, fractional operators have a heavy physical significance, and are commonly used to model phenomenons with "memory", like fluid viscosity, or the position of a particle.

In particular, the Riemann-Liouville fractional derivative models a phenomenon where perturbations propagate over a region of the space, generally having less influence in the farthest points.

1.4 Problems to analyze and state of art

It was mentioned before that the focus of this work will be in the study of problems related to the Riemann-Liouville and Caputo fractional derivatives. Specifically, three problems will be studied. In all the three cases, the objective is to study the well posedness of the problem, the existence of non negative solutions and its asymptotic behaviour when $\lambda \to \infty$.

1.4.1 Unidimensional problem

In particular, in this text is studied the problem of general boundary conditions

$$\begin{cases} -\mathcal{D}v(t) = \lambda c(t) f(t, v(t), \dots, D^n v(t)), & 0 < t < 1 \\ D^k v(0) = 0, & 0 \le k \le m \quad \text{if } m \ge 0 \\ \mathcal{A}_i(D^{n_{i,j}} v(c_{i,j}))_{j=m+1}^{[\alpha]-1} = 0, & \forall\, i \in \{m+1, \dots, [\alpha]-1\} \end{cases} \tag{1.4.1}$$

where $\mathcal{D}$ is, either, the operator D^α or ${}^C D^\alpha$ with $\alpha > 1$, or the operator $D^\beta\left(q(t) D^\alpha\right)$ with $\alpha > 1$ and $\beta > 0$. In general, we consider $m \in \mathbb{Z}_{\ge -1}$ and $n \in \mathbb{N}_0$ that are such that $n \ge 0$ and $n, m < [\alpha] - 1$, $\{\mathcal{A}_i\}_{i=m+1}^{[\alpha]-1} \subset C(\mathbb{R}^{[\alpha]-m-1}; \mathbb{R})$ are linear operators, $\{n_{i,j}\}_{i,j=m+1}^{[\alpha]-1} \subset \mathbb{R}$ and $\{c_{i,j}\}_{i,j=m+1}^{[\alpha]-1} \subset [0,1]$.
Here $(D^{n_{i,j}} v(c_{i,j}))_{j=m+1}^{[\alpha]-1} = \left(D^{n_{i,m+1}} v(c_{i,m+1}), D^{n_{i,m+2}} v(c_{i,m+2}), \dots, D^{n_{i,[\alpha]-1}} v(c_{i,[\alpha]-1})\right)$

On this context, problem (1.4.1) have been widely studied in both, the Riemann-Liouville case and the Caputo case. When α is a positive integer, in which case the Riemann-Liouville and the Caputo operators coincide with the usual derivative, problem (1.4.1) has captured the attention of many researchers, since that in this case the problem can be studied by means of classical ODE theory. In particular, from the Peano and the Picard theorems [38], its direct that if $c(t) f(t, \mathbf{u}(t))$ is continuous, then problem (1.4.1) has a solution, and if it's locally lipschitz, then the solution is locally unique. Also there are some works on the multiplicity of solutions when α is an integer (see, for example, [15]), but at the knowledge of the author, there is little research on this topic on the fractional case (see [25, 27]).

Despite the latter, there have been not much research related to problem (1.4.1) on the fractional case at this type of generality. One of the first interesting approaches to this problem in the fractional case is the one by Bai and Lü [13], which proved the existence and multiplicity of positive solutions for the problem

$$\left\{ \begin{array}{rcl} D^\alpha u(t) + f(t,u(t)) & = & 0 \quad 0<t<1 \\ u(0)=u(1) & = & 0, \end{array} \right. \tag{1.4.2}$$

for the standard Riemann-Liouville operator, where $1 < \alpha \leq 2$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous, satisfying some growth conditions, using a fixed-point method over a cone due to Krasnoselskii. Unfortunately, there the case $f(t,x) = x^p$ doesn't satisfy this hypothesis for non $p > 1$. Then, Jiang and Yuan [18] studied the same problem, and obtained the same conclusion, but with weaker assumptions on the function f. Moreover, in [18] the case $f(t,x) = x^p$ can be included, but not the case $f(t,x) = e^x - 1$.

After that, Chai and Hu [26] studied the more general problem

$$\left\{ \begin{array}{rcl} D^\alpha u(t) + f(t,u(t)) & = & 0 \quad 0<t<1 \\ u(0)=u'(0)=\cdots=u^{(n-2)} & = & 0, \quad D^{\alpha-1}u(\eta) = kD^{\alpha-1}u(1), \end{array} \right. \tag{1.4.3}$$

where $k > 1$, $\eta \in (0,1)$, $n-1 < \alpha \leq n$, $n \geq 3$ and $f : [0,1]\times[0,\infty) \to [0,\infty)$ is continuous, and satisfy some other conditions (that don't allow us to include functions of the type $f(t,x) = x^p$). There, under a similar approach as before, they showed the existence of positive solutions of problem (1.4.3).

By other side, for $2 < \alpha < 3$, one of the first approaches on the Caputo case is the one by Karakostas [23], who studied conditions of non-existence of solutions for problems

$$\left\{ \begin{array}{rcl} D^\alpha x(t) + \lambda a(t) f(x(t)) & = & 0 \quad 0<t<1 \\ x(0)=x'(0)=x'(1) & = & 0, \end{array} \right. \tag{1.4.4}$$

where $a : (0,1) \to [0,\infty)$ and $f : [0,\infty) \to [0,\infty)$ are continuous and $\int_0^1 a(s)ds > 0$, and

$$\left\{ \begin{array}{rcl} {}^C D^\alpha x(t) + f(t,x(t)) & = & 0 \quad 0<t<1 \\ x(0)=x'(0) & = & 0, \quad x(1) = \lambda \int_0^1 x(s)ds \end{array} \right. \tag{1.4.5}$$

where $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous and $0 < \lambda < 2$. In the problem (1.4.4), Karakostas found that if $\sup_{|u|>0} \frac{f(u)}{|u|} < C_1$ where $C_1 < \infty$, with C_1 depending on the Green function associated to problem (1.4.4), then problem (1.4.4) has no solution. In a similar way, if $\sup_{0<t<1} \sup_{|u|>0} \frac{f(t,u)}{|u|} < C_2$, where $C_2 < \infty$, with C_2 depending on the Green function associated to problem (1.4.5), he proved that problem (1.4.5), has no solution.

After that, still for $2 < \alpha < 3$, Sun and Wang [25] studied the problem

$$\left\{ \begin{array}{rcl} {}^C D^\alpha x(t) + f(t,x(t)) & = & 0 \quad 0<t<1 \\ x(0)=x'(0) & = & 0, \quad x(1) = \lambda \int_0^1 x(s)ds \end{array} \right. \tag{1.4.6}$$

where $f : [0,1] \times [0,\infty) \times \mathbb{R} \to [0,\infty)$ is continuous and $0 < \lambda < 2$, and found that, under some hypothesis over f, problem (1.4.6) admits at least three positive solutions. However, they can't consider nonlinearities of the type x^p.

There are also some works were the authors study problems similar to problem (1.4.1), but on a different context, e.g, with other types of nonlinearities (see [21, 27]), with different types of boundary conditions (see [17, 24, 33, 36]), or including a system of equations (see [30, 31, 34]), or with other fractional operator (see [22, 32, 35]). A matter of interest here is that this text study the problem in a greater grade of generality and under new boundary conditions. Furthermore, problems (1.4.2)-(1.4.6) are included in our study of problem (1.4.1).

It's worth to emphatize that, in general, the hypothesis considered in this work on the nonlinearity are different from the ones on [13], [26], [23] and [25], so this reaserch serves as a complement to theirs.

Motivated by the works mentioned before, this text focus on the well-posedness of the problem, existence, multiplicity and asymptotic behaviour over the parameter λ of solutions of problem (1.4.1). Furthermore, some bounds for the solutions found are obtained. In particular, the boundary conditions of problem (1.4.1) generalize the ones on problems (1.4.2)-(1.4.6), since both cases are included, as mentioned before, so each result obtaind by the research on this work can be applied to those problems. Besides, a more general dependence of the function f on the solution u and its derivatives is considered, making explicit its dependence on derivatives of higher order. Finallly, are also taken in consideration other type of nonlinearities different than those in (1.4.2)-(1.4.6).

On this research will be fundamental to study the Green function associated to the operator $\mathcal{D}$, and the boundary conditions of problem (1.4.1), since some of its properties are inherited by the fixed point operator. Furthermore, since the Green function is the fundamental solution of equation (1.4.1), its study will be useful on the research of problem (1.4.1) when other nonlinearities are considered, or even in other problems related to this operator. The Green function has been object of many researches, see for instance [13, 18, 26], but never with a generality about the operator neither novelty on the boundary conditions as in this work.

1.4.2 Higher Dimensionality fractional problem

Finally, using the results developed for problem (1.4.1), it's linked to the fractional problem of the form

$$\sum_{i=0}^{[\alpha]-1} g_i(x)\nabla^\alpha v(x), \quad x \in \Omega \subset \mathbb{R}^n$$

for suitable boundary conditions, where

$$\nabla^\alpha v(x) = \left(\frac{\partial^{[\alpha]}}{\partial x_1^{[\alpha]}} J_r^{[\alpha]-\alpha} v(x), \ldots, \frac{\partial^{[\alpha]}}{\partial x_n^{[\alpha]}} J_r^{[\alpha]-\alpha} v(x) \right)$$

and

$$J_r^\alpha v(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (|x| - t)^{\alpha-1} v\left(\frac{tx}{|x|}\right) dt$$

Here, the operator J_r^α will be called the "Riemann-Liouville fractional radial integral" operator.

In this case, a radial, non negative solution will be found on some bounded domain $\Omega \subset \mathbb{R}^n$, where $n \geq 2$. In particular, the domain Ω will consider as an annular domain.

1.5 Method applied

In this work, some techniques are applied that are usually used on PDE problems.

First, a Green function is built for the proposed problems. To do that, the first step is to extend the fractional operators involved in the problems to a suitable distributional space, that contains, at least, the space is presumed the solution lives, and the Dirac's delta distribution. Then, the operator will be studied further, determining its kernel and right-inverse, if it exists. Then to build the Green function, the right-inverse of the operator is applied to the Dirac's delta distribution, and the elements on the base on the operator's kernel are added to the result.

Then, an explicit expression is found for the Green function associated to the given problem conditions. In the process, some conditions will be found under which the proposed problem is well posed, and some useful properties of the Green function will be proven.

Once the Green function is expressed in the most explicit way, the problem is transformed to an equivalent fixed point problem related to the Green function, proposing a suitable operator, concluding the existence of solutions using some known fixed point theorems.

Finally, the proof is refined to study the asymptotic behaviour and multiplicity of the solutions found, refining some norm estimates derived from the existence results.

The main goal then is to prove the existence of non negative solutions $v \in C^n[0,1]$ for problem (1.4.1) in each case, as well as to prove that the said solution satisfies that $v \to 0$ as $\lambda \to \infty$, then link problem (1.4.1) to the fractional problem presented on section 1.4.2.

Chapter 2

Preliminaries

In this chapter, some useful tools are presented, that are very used on the topics of ODE and PDE, like the Laplace transform and some fixed point theorems, also recalling the concept of convolution.

After that,the Riemann-Liouville fractional integral and derivative operators are extended to a space of distributions using the concepts convolution operator and distributional derivatives to do that. The same process is used to extend the Caputo fractional derivative.

Finally, some properties of those operators are presented, together with a variation of a concatenated fractional derivative operator.

2.1 Convolution

Recalling the concept of convolution is very important for this work, since this is essential to extend some concepts to the distribution space. Initially, the convolution operator is defined as follows.

Definition 2.1.1. *Let $N \geq 1$ and let $\Omega \subset \mathbb{R}^N$ be an open set. Let $f, g : \Omega \to \mathbb{R}$ be a continuous, integrable functions. The convolution between f and g, $f * g$, is defined as as*

$$(f * g)(x) = \int_{\mathbb{R}^N} \overline{f}(y)\overline{g}(x-y)dy = \int_{\mathbb{R}^N} \overline{g}(y)\overline{f}(x-y)dy$$

where $\overline{f}$ and $\overline{g}$ are the extensions by 0 over $\mathbb{R}^N$ of f and g, respectively. In this case, $f * g$ is also a continuous function, by algebra of continuous functions.

In particular, if $f, g : [a, b] \to \mathbb{R}$, where $a, b \in \mathbb{R}$, then the convolution between f and g is defined as

$$(f * g)(x) = \int_a^x f(y)g(x-y)dy = \int_a^x g(y)f(x-y)dy$$

From the previous definition, is possible to define the convolution between a tempered distribution and a function in $\mathcal{S}$.

Definition 2.1.2. *Let $f \in \mathcal{S}'$ and $g \in \mathcal{S}$. The convolution between f and g is defined as*

$$f * g(x) = \langle f, \tau_{-x} g \rangle_{\mathcal{S}' \times \mathcal{S}}$$

where τ_x is the translation distribution, that is, $\tau_{-x} g = g(x - \cdot)$.

Finally, the convolution between tempered distributions is set as follows.

Definition 2.1.3. *Let $f, g \in \mathcal{S}'$, with at least one of them with compact support. The convolution between f and g is defined as the tempered distribution $f * g$ that satisfies the identity*

$$(f * g) * \phi = f * (g * \phi), \quad \forall \phi \in \mathcal{S}$$

2.2 Laplace transform

Laplace transform is known to be useful to solve a wide range of ODE. This operator used to be defined over a subset of the space of continuous functions, which grow at most exponentially at infinity. The definition given for functions defined over an interval $[0, \infty)$ is usually as follows.

Definition 2.2.1. *Let $f : [a, \infty) \to \mathbb{R}$ be a continuous function which grows at most exponentially at infinity. The Laplace transform of f, denoted as $\mathcal{L}f$ or F, is defined as*

$$F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty f(t) e^{-st} dt$$

Despite that definition, the Laplace transform can be extended to a subspace of the temperated distributions $\mathcal{S}'$, where $\mathcal{S}$ is the Shwartz space of all function $\phi \in C^\infty(\mathbb{R}^N; \mathbb{R})$ decaying faster at infinity than every polynomial over $\mathbb{R}$. To do that, we still need to define some more concepts.

Definition 2.2.2. *A tempered distribution $f \in \mathcal{S}'$ is said to be of finite order if there exist a continuous function $h : (-\infty, \infty) \to \mathbb{R}$ and $n \in \mathbb{N}_0$, such that*

$$f = D^n h$$

in the sense of distributions. If n is the lowest non negative integer such that the previous is satisfied, it's said that f is of order n.

With that, the Laplace transform can be extended as follows.

Definition 2.2.3. *Let $f \in \mathcal{S}'$ be of order $n \in \mathbb{N}_0$, and let $h : (-\infty, \infty) \to \mathbb{R}$ be a continuous function such that*

$$f = D^n h$$

if $h(t) = 0$ for all $t < 0$, and if there exists $\sigma \in \mathbb{R}$ such that $\mathcal{L}\{h(t)\}$ is absolutely convergent for all $Re(s) > \sigma$, then the Laplace transform of f is defined as

$$\mathcal{L}\{f\} = F = s^n \mathcal{L}\{h\}$$

The space of tempered distributions that satisfy this conditions is denoted as $\mathcal{F}$.

From the previous, and from the known properties of the Laplace transform, one can get the following.

Lemma 2.2.1. *Let be $f, g \in \mathcal{F}$ and $\lambda \in \mathbb{R}$, then:*

1. $\mathcal{L}\{f + \lambda g\} = \mathcal{L}\{f\} + \lambda \mathcal{L}\{g\}$, $\forall\, \lambda \in \mathbb{R}$,
2. $\mathcal{L}$ *is invertible over* $\mathcal{F}$.
3. $\mathcal{L}\{D^k f\} = s^k \mathcal{L}\{f\}$, $\forall\, k \in \mathbb{N}_0$,
4. $\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$,
5. $\mathcal{L}\{(-t)^k f\} = D^k \mathcal{L}\{f\}$,
6. $\mathcal{L}\{\tau_b f\} = e^{-bs}\mathcal{L}\{f\}$, *where* τ_b *is the translation distribution.*
7. $\mathcal{L}\{\delta\} = 1$, *where* δ *is the Dirac's delta distribution.*

2.3 Extending some fractional operators

The Riemann-Liouville fractional integral and derivative operators were defined on subsection 1.2.1 over a subset of the space of continuous functions $C(a, b)$, where onwards will be assumed that $a = 0$ and $b = 1$. It was also mentioned that R-L fractional integral can be seen as a convolution between functions, where

$$J^\alpha f = f * p_\alpha$$

$p_\alpha(x) = x^{\alpha-1}\mu(x)$, and $\mu(t)$ is the Heavieside distribution, defined as

$$\mu(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 1 \end{cases} \tag{2.3.1}$$

Since p_α is a tempered distribution (with compact support, if it's extended by 0 over $\mathbb{R} \setminus (0, 1)$), is possible to convolute it with any tempered distribution f, so that one can define the following.

Definition 2.3.1. *Let $f \in \mathcal{S}'$. The Riemann-Liouville fractional integral of f is defined as*

$$J^\alpha f = f * p_\alpha$$

Using distributional derivatives, the concept of fractional derivative can also be extended as follows.

Definition 2.3.2. *Let $f \in \mathcal{S}'$. The Riemann-Liouville fractional derivative of f is defined as*

$$D^\alpha f = D^{[\alpha]} J^\alpha f$$

where, in this case, D^n denotes the distributional derivative of order n.

Definition 2.3.3. *Let $f \in \mathcal{S}'$ be a tempered distribution. The Caputo fractional derivative of f is defined as*

$$^{C}D^{\alpha} f = J^{[\alpha]-\alpha} D^{[\alpha]} f$$

where the operator $D^{[\alpha]}$ is the distributional derivative of order $[\alpha]$.

Note that, with this extensions, one can apply those operators to, both, continuous functions and the Dirac's delta distribution, which is what was pursued. Moreover, for more general fractional operators, is possible to give the following definition:

Definition 2.3.4. *Let $\mathcal{L}$ be a linear operator with the property that there exists $g \in C(0,1) \cap L^1(0,1)$ and $n \in \mathbb{N}_0$ such that, either,*

$$\mathcal{L}f = D^n(f * g), \quad \forall f \in AC^n[0,1]$$

or

$$\mathcal{L}f = (f * D^n g), \quad \forall f \in C^n[0,1]$$

where D^n is the derivative of order n. The extension $\mathcal{L} : \mathcal{S}' \to \mathcal{S}'$ is defined as

$$\mathcal{L}f = D^n(f * g)$$

or

$$\mathcal{L}f = (D^n f * g)$$

respectively, where D^n is now taken in the distributional sense.

Since in the definition above, the function g is a tempered distribution (with compact support when in particular assumed to be 0 over $\mathbb{R} \setminus (0,1)$), the extension presented is well defined.

2.4 Some properties of the R-L fractional operators

Now that all the necessary extensions and definitions were given, there are some properties of the R-L fractional operators that are useful to know. First some properties related to the Laplace transform are shown.

Lemma 2.4.1. *Let $\alpha \in \mathbb{C}$ be such that $Re(\alpha) > 0$, and let $f \in \mathcal{F}$ be a tempered distribution such that $supp(f) \subset [a, \infty)$. Then*

$$\mathcal{L}\{J^{\alpha} f\} = s^{-\alpha} \mathcal{L}\{f\}$$

Proof. One have

$$\begin{aligned}\mathcal{L}\{J^\alpha f\} &= \frac{1}{\Gamma(\alpha)}\mathcal{L}\{f * p_\alpha \mu\}\\ &= \frac{1}{\Gamma(\alpha)}\mathcal{L}\{p_\alpha \mu\}\mathcal{L}\{f\}\\ &= s^{-\alpha}\mathcal{L}\{f\}\end{aligned}$$

□

From lemma 2.4.1 one can get the following property related to a semi-group property of the fractional integral.

Lemma 2.4.2. *Let be $\alpha, \beta \in \mathbb{C}$ such that $Re(\alpha), Re(\beta) > 0$, and $f \in \mathcal{F}$ be a tempered distribution. Then*

$$J^\alpha J^\beta f = J^\beta J^\alpha f = J^{\alpha+\beta} f$$

Proof. From lemma 2.4.1

$$\begin{aligned}\mathcal{L}\{J^\alpha J^\beta f\} &= s^{-\alpha}\mathcal{L}\{J^\beta f\}\\ &= s^{-\alpha}s^{-\beta}\mathcal{L}\{f\}\\ &= s^{-(\alpha+\beta)}\mathcal{L}\{f\}\end{aligned}$$

Then

$$J^\alpha J^\beta f = \mathcal{L}^{-1}\{s^{-(\alpha+\beta)}\mathcal{L}\{f\}\} = J^{\alpha+\beta} f$$

The rest of the demonstration follows exchanging α and β on the previous calculus. □

Lemma 2.4.2 is used to show the next lemma.

Lemma 2.4.3. *For all $\alpha \in \mathbb{C}$ such that $Re(\alpha) > 0$ and all $f \in \mathcal{F}$*

$$D^\alpha J^\alpha f = f$$

Proof. By the definition of fractional derivative, in combination with lemma 2.4.2,

$$\begin{aligned}D^\alpha J^\alpha f &= D^{[\alpha]} J^{[\alpha]-\alpha} J^\alpha f\\ &= D^{[\alpha]} J^{[\alpha]} f\end{aligned}$$

where $[\cdot]$ is the ceil function, and $D^{[\alpha]}$ is the distributional derivative of order $[\alpha]$. Then,

$$\mathcal{L}\{D^{[\alpha]} J^{[\alpha]} f\} = s^{[\alpha]}\mathcal{L}\{J^{[\alpha]} f\} = s^{[\alpha]}s^{-[\alpha]}\mathcal{L}\{f\} = \mathcal{L}\{f\}$$

Applying the inverse Laplace transform, one gets the result. □

Like usual derivatives, in general the identity $J^\alpha D^\alpha f = f$ isn't true. However, it won't affect the results obtained in what follows. Also, it is worth mentioning that, in general, the operators D^α and D^β don't commute, even if $\alpha = \beta$.

Also, we have the following lemma.

Lemma 2.4.4. *Let $\alpha \in \mathbb{C}$ be such that $Re(\alpha)$. Then the operator J^α is injective over $\mathcal{F}$.*

Proof. Since J^α is a linear operator, it suffices to prove that

$$J^\alpha f = 0 \quad \Leftrightarrow \quad f = 0$$

Let $f \in \mathcal{F}$. Then

$$\mathcal{L}\{J^\alpha f\} = s^{-\alpha}\mathcal{L}\{f\}$$

since $\mathcal{L}$ is biyective over $\mathcal{F}$, then

$$J^\alpha f = 0 \quad \Leftrightarrow \quad \mathcal{L}\{J^\alpha f\} = 0 \quad \Leftrightarrow \quad \mathcal{L}\{f\} = 0$$

so that $f = 0$ on $\mathcal{F}$, concluding that J^α is inyective over $\mathcal{F}$. □

The following lemma will be useful to build Green functions.

Lemma 2.4.5. *For each $\alpha \in \mathbb{C}$ such that $Re(\alpha) > 0$,*

$$Ker(D^\alpha) = Span\langle\{t^{k+\alpha-[\alpha]}\}_{k=0}^{[\alpha]-1}\rangle$$

Proof. Remember that

$$D^\alpha = D^{[\alpha]}J^{[\alpha]-\alpha}$$

where $[\alpha] \in \mathbb{N}$, so that $D^{[\alpha]}$ is the weak derivative of order $[\alpha]$. On the other hand,

$$Ker(D^{[\alpha]}) = Span\langle\{t^k\}_{k=0}^{[\alpha]-1}\rangle$$

Then,

$$f \in Ker(D^\alpha) \quad \Leftrightarrow \quad J^{[\alpha]-\alpha}f \in Ker(D^{[\alpha]})$$

which is equivalent to the existence of $\{a_k\}_{k=0}^{[\alpha]-1} \subset \mathbb{R}$ such that

$$s^{-([\alpha]-\alpha)}\mathcal{L}\{f\} = \mathcal{L}\{J^{[\alpha]-\alpha}f\} = \sum_{k=0}^{[\alpha]-1} a_k\mathcal{L}\{t^k\} = \sum_{k=0}^{[\alpha]-1} a_k\frac{k!}{s^{k+1}}$$

that is,

$$\mathcal{L}\{f\} = \sum_{k=0}^{[\alpha]-1} a_k \mathcal{L}\{t^k\} = \sum_{k=0}^{[\alpha]-1} a_k \frac{k!}{s^{[\alpha]-\alpha+k+1}}$$

Applying inverse Laplace transform follows that

$$f(t) = \sum_{k=0}^{[\alpha]-1} a_k \frac{t^{k+\alpha-[\alpha]}}{k!\,\Gamma(k+\alpha-[\alpha]+1)}$$

obtaining the result.

□

Finally, the following formulas will be useful.

Lemma 2.4.6. *1. Given $\alpha \geq 0$ and $q > -1$, then*

$$I^\alpha t^q = \frac{\Gamma(q+1)}{\Gamma(1+q+\alpha)} t^{q+\alpha}, \quad t > 0$$

2. Given $q > -1$ and $\alpha \geq 0$ such that $q - \alpha > -1$, then

$$D^\alpha t^q = \frac{\Gamma(q+1)}{\Gamma(q-\alpha+1)} t^{q-\alpha}, \quad t > 0$$

With all the above, almost all the ingredients needed for the construction of Green functions have been mentioned. However, one more thing is necessary to prove the existence results in this work.

2.5 Fixed point theorems

Fixed point methods are very common on the topic of non linear PDEs. In particular, this section will focus in two known results: Banach fixed point theorem and Krasnosel'skii theorem. First, the Banach fixed point theorem is enunciated.

Theorem 2.5.1. *(Banach fixed point) Let X be a complete metric space and let $T : X \to X$ a continuous operator. If there exists $L < 1$ such that*

$$d_X(Tu, Tv) \leq L d_X(u, v), \quad \forall\, u, v \in X$$

Then T have a unique fixed point.

Next, Krasnosel'skii theorem will be enunciated, but first, some more background is needed.

Definition 2.5.1. *Let X be a Banach space. A subset $K \subset X$ is said to be a cone if it satisfies all the following conditions.*

(C1) *K is not empty and $K \neq \{0\}$.*

(C2) *K is closed.*

(C3) *K is convex.*

(C4) *For all $\lambda \geq 0$, $\lambda K \subset K$.*

(C5) *$K \cap (-K) = \{0\}$.*

Definition 2.5.2. *Given a cone $K \subset X$ on a Banach space, the partial order $\leq$ with respect of K is defined as*

$$x \leq y \quad \text{if and only if} \quad y - x \in K$$

It's also written as

$$x \nleq y \quad \text{if and only if} \quad x \leq y \quad \text{is false}$$

Definition 2.5.3. *Let K be an ordered cone. Let $0 < r < R$ be given. The following set*

$$K(r,R) := \{x \in K : r \leq \|x\| \leq R\}$$

will be called a conic shell, which interior border will be given by $K_r := \{x \in K : \|x\| = r\}$, and its exterior border by $K_R := \{x \in K : \|x\| = R\}$.

Definition 2.5.4. *Let X, Y be Banach spaces, $\Omega \subset X$ and a map $F : \Omega \to Y$. It's said that F is completely continuous if it's continuous and maps bounded subsets of Ω to precompact subsets.*

Theorem 2.5.2. *(Krasnosel'skii) Let X be a Banach space and $K \subset X$ a cone on X. Let $T : K \to K$ be a completely continuous operator. For $0 < r < R$,*

1. *(Compressive form) T have a fixed point on the conic shell $K(r,R)$ if*

$$\begin{aligned} \|Tx\| &\geq \|x\| \quad \forall\, x \in K_r \\ \text{and} \quad \|Tx\| &\leq \|x\| \quad \forall\, x \in K_R \end{aligned}$$

2. *(Expansive form) T a fixed point on the conic shell $K(r,R)$ if*

$$\begin{aligned} \|Tx\| &\leq \|x\| \quad \forall\, x \in K_r \\ \text{and} \quad \|Tx\| &\geq \|x\| \quad \forall\, x \in K_R \end{aligned}$$

Since this last result is very known on degree theory, the proof of this theorem is omitted.

Chapter 3

Construction of Green functions

Now that all the necessary ingredients were gathered, the focus in this chapter is to construct Green functions for problem (1.4.1).

First, the Green function associated to the problem is built in each case in the general case, i.e, without considering boundary conditions, so one can get some degrees of freedom when determining it explicitly.

After that, some sufficient conditions will be set for the problem to be well posed, so that the Green function is, at least, uniformly continuous.

Finally, there is obtained an explicit expression for the Green function for each case.

Despite the main object of study of this document is the problem (1.4.1), in this chapter we take into consideration the non homogeneous problem

$$\begin{cases} -\mathcal{D}v(t) = \lambda c(t) f(t, v(t), \dots, D^n v(t)), & 0 < t < 1 \\ D^k v(0) = 0, & 0 \leq k \leq m \quad \text{if } m \geq 0 \\ \mathcal{A}_i(D^{n_{i,j}} v(c_{i,j}))_{j=m+1}^{[\alpha]-1} = d_i, & \forall\, i \in \{m+1, \dots, [\alpha]-1\} \end{cases} \tag{3.0.1}$$

where $\{d_i\}_{i=m+1}^{[\alpha]-1} \subset \mathbb{R}$ and the other parameters are the same as in problem (1.4.1).

Onwards, the notations $\frac{d^n}{dt^n}(G(t,s)) = D^n G(t,s)$, $\frac{d^{[\alpha]}}{dt^{[\alpha]}}{}_0 \int_0^t (t-\tau)^{[\alpha]-\alpha-1} G(\tau, s) d\tau = D^\alpha G(t,s)$ and $\int_0^t (t-\tau)^{[\alpha]-\alpha-1} D^{[\alpha]} G(\tau, s) d\tau = {}^C D^\alpha$ will be used.

It's worth to mention that all the results that appear in this section are original, and are the result of the work of the author, unless the opossite is indicated.

3.1 Building Green Functions

3.1.1 Riemann-Liouville case

For the purpose of this work, is necessary to look for the Green function of problem (1.4.1) when $\mathcal{D} = D^\alpha$, that is, to find a distribution $G(t,s)$ such that for all $t \in [0,1]$,

$$-D^\alpha G(t,s) = \delta(t-s) \quad 0 < s < 1$$

so that

$$-D^\alpha G(t,s) f(s) = f(t), \quad \forall f \in C[0,1]$$

with δ the Dirac distribution and $\alpha \in \mathbb{C}$ such that $Re(\alpha) > 0$. From lemma 2.4.3 one have that if $f \in \mathcal{F}$, which is the case of the distribution δ, then

$$D^\alpha J^\alpha f = f$$

Meanwhile, 2.4.5 leads to

$$Ker(D^\alpha) = Span\langle \{t^{k+\alpha-[\alpha]}\}_{k=0}^{m-1} \rangle$$

Collecting all this ideas, then $G(t,s)$ has the form

$$G(t,s) = -J^\alpha \delta(t-s) + \sum_{k=0}^{[\alpha]-1} a_k(s) t^{k+\alpha-[\alpha]}$$

where

$$\begin{aligned} -J^\alpha \delta(t-s) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \delta(\tau - s) d\tau \\ &= \begin{cases} 0 & 0 \le t \le s \le 1 \\ -\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} & 0 \le s \le t \le 1 \end{cases} \end{aligned}$$

With that, the Green function should be of the form

$$G(t,s) = \begin{cases} \sum_{k=0}^{[\alpha]-1} a_k(s) t^{k+\alpha-[\alpha]} & 0 \le t \le s \le 1 \\ \sum_{k=0}^{[\alpha]-1} a_k(s) t^{k+\alpha-[\alpha]} - \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} & 0 \le s \le t \le 1 \end{cases}$$

where functions $a_k(s)$ are such that G satisfy some border conditions (eventually fractional). The above can be written in the simpler form

$$G(t,s) = \sum_{k=0}^{[\alpha]-1} a_k(s) t^{k+\alpha-[\alpha]} - \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} \mu(t-s) \tag{3.1.1}$$

where μ is the Heaviside function defined in (2.3.1). Thus, one gets the following lemma.

Lemma 3.1.1. *The function*

$$G(t,s) = \sum_{k=0}^{[\alpha]-1} a_k(s) t^{k+\alpha-[\alpha]} - \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}\mu(t-s)$$

defined on (3.1.1) *is the (general) Green function for the operator* $-D^\alpha$*.*

Proof. By construction of G,

$$-D^\alpha G(t,s) = D^\alpha \left[\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}\mu(t-s)\right] = \frac{1}{\Gamma(\alpha)} D^{[\alpha]} J^{[\alpha]-\alpha}((t-s)^{\alpha-1}\mu(t-s))$$

Then,

$$\begin{aligned}
\mathcal{L}\{D^{[\alpha]}J^{[\alpha]-\alpha}((t-s)^{\alpha-1}\mu(t-s))\}(r) &= r^{[\alpha]}\mathcal{L}\{J^{[\alpha]-\alpha}((t-s)^{\alpha-1}\mu(t-s))\}(r) \\
&= r^{\alpha-[\alpha]} r^{[\alpha]} e^{-sr}\frac{\Gamma(\alpha)}{r^\alpha} \\
&= \Gamma(\alpha)e^{-rs}
\end{aligned}$$

Thus,

$$\mathcal{L}(D^\alpha G(t,s))(r) = \frac{1}{\Gamma(\alpha)}\Gamma(\alpha)e^{-rs} = e^{-rs}$$

so that

$$G(t,s) = \mathcal{L}^{-1}\{e^{-rs}\} = \delta(t-s)$$

□

Remark 3.1.1. *Note that functions* $a_k(s)$ *in the Green function remain free. That is because of the dependence on border conditions. For example, if the following problem is considered*

$$\begin{cases} D^\alpha u(t) = 0 & 0 < t < 1 \\ u(0) = u'(0) = u''(0) = u''(1) = 0 & \end{cases}$$

then for each $s \in [0,1]$*, the Green function for this equation satisfies the conditions*

$$G(0,s) = G'(0,s) = G''(0,s) = G''(1,s) = 0$$

From which the Green function can be written explicitly as

$$G(t,s) = (1-s)^{\alpha-3}t^{\alpha-1} - (t-s)^{\alpha-1}\mu(t-s)$$

3.1.2 Concatenated case

Let $q : (0,1) \to \mathbb{R}$ be a positive, continuous function. Some other properties of the function q will be specified later.

Like in the Riemann-Liouville case, is necessary to find a distribution $G(t,s)$ such that for all $t \in [0,1]$,

$$-D^{\beta}\left(qD^{\alpha}G(t,s)\right) = \delta(t-s) \quad 0 < s < 1$$

so that

$$-D^{\beta}\left(qD^{\alpha}G(t,s)\right)f(s) = f(t), \quad \forall\, f \in C[0,1]$$

In this case some extra considerations are needed, since the operator $\mathcal{D} = -D^{\beta}\left(qD^{\alpha}\cdot\right)$ is more complex than the operator in the Riemann-Liouville case. First, it's essential to find the kernel of $\mathcal{D}$, which leads to the next lemma.

Lemma 3.1.2. *Let $\alpha, \beta \in \mathbb{C}$ be such that $Re(\alpha), Re(\beta) > 0$, and let $q \in C(0,1)$ be such that $q(t) > 0$ for all $t \in (0,1)$. Then*

$$Ker(\mathcal{D}) = Span\left\langle \left\{t^{k+\alpha-[\alpha]}\right\}_{k=0}^{[\alpha]-1}, \left\{J^{\alpha}\left(\frac{t^{k+\beta-[\beta]}}{q(t)}\right)\right\}_{k=0}^{[\beta]-1}\right\rangle$$

Proof. From lemma 2.4.5,

$$Ker(D^{\alpha}) = Span\langle\{t^{k+\alpha-[\alpha]}\}_{k=0}^{[\alpha]-1}\rangle$$
$$Ker(D^{\beta}) = Span\langle\{t^{k+\beta-[\beta]}\}_{k=0}^{[\beta]-1}\rangle$$

So, is necessary to look for a function $f(t)$ such that for some $0 \le k \le [\beta]-1$ the following is verified.

$$q(t)D^{\alpha}f(t) = t^{k+\beta-[\beta]}$$

defining $f(t) := J^{\alpha}g(t)$, the above is equivalent to

$$D^{\alpha}J^{\alpha}g(t) = g(t) = \frac{t^{k+\beta-[\beta]}}{q(t)}$$

and since any element in $Ker(D^{\alpha})$ can be written as

$$\sum_{k=0}^{[\alpha]-1} a_k t^{k+\alpha-[\alpha]}$$

all the above leads to conclude that

$$f(t) = J^{\alpha}\left(\frac{t^{k+\beta-[\beta]}}{q(t)}\right) + \sum_{k=0}^{[\alpha]-1} a_k t^{k+\alpha-[\alpha]}$$

Thus,

$$Ker(\mathcal{A}) = Span\left\langle \left\{t^{k+\alpha-[\alpha]}\right\}_{k=0}^{[\alpha]-1}, \left\{J^{\alpha}\left(\frac{t^{k+\beta-[\beta]}}{q(t)}\right)\right\}_{k=0}^{[\beta]-1}\right\rangle$$

□

Lemma 3.1.2 combined with the fact that

$$\mathcal{D}J^{\alpha}\left(\frac{1}{q}J^{\beta}\right)f = f, \quad \forall f \in \mathcal{S}'$$

allows to construct the said Green function.

First, one has that

$$\begin{aligned} -J^{\beta}(\delta(t-s)) &= -\frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\delta(\tau-s)d\tau \\ &= \begin{cases} 0 & 0 \le t \le s \le 1 \\ -\frac{1}{\Gamma(\beta)}(t-s)^{\beta-1} & 0 \le s \le t \le 1 \end{cases} \end{aligned}$$

From that,

$$-J^{\alpha}\left(\frac{1}{q(t)}J^{\beta}(\delta(t-s))\right) = \begin{cases} 0 & 0 \le t \le s \le 1 \\ -\frac{1}{\Gamma(\beta)\Gamma(\alpha)}\int_s^t (t-\tau)^{\alpha-1}\frac{(\tau-s)^{\beta-1}}{q(\tau)}d\tau & 0 \le s \le t \le 1 \end{cases}$$

So, defining

$$Q(t,s) = \int_0^t (t-\tau)^{\alpha-1}\frac{(\tau-s)^{\beta-1}}{q(\tau)}\mu(\tau-s)d\tau$$

then the Green function has the form

$$G(t,s) = \sum_{k=0}^{[\alpha]-1} a_k(s)t^{k+\alpha-[\alpha]} + \sum_{k=0}^{[\beta]-1} b_k(s)J^{\alpha}\left(\frac{t^{k+\beta-[\beta]}}{q(t)}\right) - \frac{1}{\Gamma(\beta)\Gamma(\alpha)}Q(t,s) \tag{3.1.2}$$

So, the previous function is the candidate to be the Green function. All the calculus made in this section, leads to the following result.

Lemma 3.1.3. *The function defined on (3.1.2), given by*

$$G(t,s) = \sum_{k=0}^{[\alpha]-1} a_k(s)t^{k+\alpha-[\alpha]} + \sum_{k=0}^{[\beta]-1} b_k(s)J^{\alpha}\left(\frac{t^{k+\beta-[\beta]}}{q(t)}\right) - \frac{1}{\Gamma(\beta)\Gamma(\alpha)}Q(t,s)$$

is the (general) Green function associated to the operator $\mathcal{D}$.

Proof. The proof follows by the previous construction. □

Remark 3.1.2. *In this case, the Green function can only be explicitly determined if we know the function $q(t)$.*

In particular, the function $J^\alpha\left(\frac{t^{\beta-[\beta]}}{q(t)}\right)$ has the following properties.

Lemma 3.1.4. *Let $\alpha,\beta,\gamma\in\mathbb{C}$ be such that $Re(\beta)>0$, $Re(\alpha)>1$ and $Re(\gamma)<Re(\alpha)$.*

1. *If $0<Re(\alpha-\gamma)<1$ and*
$$h(t)=\frac{t^{\beta-[\beta]}}{q(t)}\in L^p(0,1)$$
for some $p\geq 1$, with $p'=\frac{p}{p-1}$ verifying that
$$p'<\frac{1}{1+Re(\gamma-\alpha)}$$
then $D^\gamma J^\alpha(h)(t)\in C[0,1]$.
2. *If $1\leq Re(\alpha-\gamma)$, and there exists $1\leq p\leq\infty$ such that*
$$h(t)=\frac{t^{\beta-[\beta]}}{q(t)}\in L^p(0,1)$$
with $p'=\frac{p}{p-1}$, then $D^\gamma J^\alpha(h)(t)\in C[0,1]$.

In any case, $J^\alpha(h)\in C^m[0,1]$, with $m=\left[\alpha-\frac{1}{p}\right]-1$.

Proof. Note that, since $0<Re(\gamma)<Re(\alpha)$, then
$$D^\gamma J^\alpha(h)(t)=J^{\alpha-\gamma}(h)(t)=\int_0^t(t-\tau)^{\alpha-\gamma-1}\frac{\tau^{\beta-[\beta]}}{q(\tau)}d\tau$$

So,

1. If $0<Re(\alpha-\gamma)<1$, then $t^{\alpha-\gamma-1}\in L^r(0,1)$ for all $1\leq r<\frac{1}{1+\gamma-\alpha}$. In particular, $t^{\alpha-\gamma-1}\in L^{p'}(0,1)$. Since $h\in L^p(0,1)$, then $(t-\cdot)^{\alpha-\gamma-1}\frac{\cdot^{\beta-[\beta]}}{q(\cdot)}\in L^1(0,t)$ for all $t>0$, and then $D^\gamma J^\alpha(h)\in C[0,1]$.
2. If $Re(\alpha-\gamma)\geq 1$, then $t^{\alpha-\gamma-1}\in L^r(0,1)$ for all $1\leq r\leq\infty$. In particular, $t^{\alpha-\gamma-1}\in L^{p'}(0,1)$. Since $h\in L^p(0,1)$, then $(t-\cdot)^{\alpha-\gamma-1}\frac{\cdot^{\beta-[\beta]}}{q(\cdot)}\in L^1(0,1)$, and then $D^\gamma J^\alpha(h)\in C[0,1]$.

From 2, taking $\gamma\in\mathbb{N}$ such that $Re(\gamma)\leq Re(\alpha)-1$, then $J^\alpha(h)\in C^{m_1}[0,1]$, where $m_1=\max\{n\in\mathbb{N}:n\leq Re(\alpha)-1\}$. From 1, since
$$p'<\frac{1}{1+Re(\gamma-\alpha)}\qquad\Leftrightarrow\qquad p'(1+Re(\gamma-\alpha))<1$$

$$\begin{aligned}
&\Leftrightarrow \qquad 1 + Re(\gamma - \alpha) < \frac{1}{p'} \\
&\Leftrightarrow \qquad Re(\gamma) < Re(\alpha) + \frac{1}{p'} - 1 = Re(\alpha) - \frac{1}{p}
\end{aligned}$$

so is obtained that

$$Re(\gamma) < Re(\alpha) - \frac{1}{p}$$

Then, $J^\alpha(h) \in C^m[0,1]$, where $m = \left[Re(\alpha) - \frac{1}{p}\right] - 1$. □

Remark 3.1.3. *In particular, the continuity of* $J^\alpha\left(\frac{t^{\beta-[\beta]}}{q(t)}\right)$ *implies that* $D^n J^\alpha\left(\frac{t^{\beta-[\beta]}}{q(t)}\right)|_{t=0} = 0$ *for all* $n \leq m$*. Indeed, since* $t^{\alpha-n-1} \in L^p(0,1)$ *and* $\frac{t^{\beta-[\beta]}}{q(t)} \in L^{p'}(0,1)$*, then*

$$\begin{aligned}
D^n J^\alpha\left(\frac{t^{\beta-[\beta]}}{q(t)}\right) &= \int_0^t (t-\tau)^{\alpha-n-1}\frac{\tau^{\beta-[\beta]}}{q(\tau)}d\tau \\
&\leq \left\|(t-\cdot)^{\alpha-n-1}\right\|_{L^p(0,t)}\left\|\frac{\tau^{\beta-[\beta]}}{q(\tau)}\right\|_{L^{p'}(0,t)} \\
&\leq \left\|(t-\cdot)^{\alpha-n-1}\right\|_{L^p(0,t)}\left\|\frac{\tau^{\beta-[\beta]}}{q(\tau)}\right\|_{L^{p'}(0,1)} \\
&= \left(\frac{t^{(\alpha-n-1)p+1}}{(\alpha-n-1)p+1}\right)^{\frac{1}{p}} M \\
&= t^{\alpha-n-\frac{1}{p'}}\overline{M}
\end{aligned}$$

so that

$$0 \leq \lim_{t\to 0^+} D^n J^\alpha\left(\frac{t^{\beta-[\beta]}}{q(t)}\right) \leq \lim_{t\to 0^+} t^{\alpha-n-\frac{1}{p'}}\overline{M} = 0$$

On the other hand, the function Q has the following properties.

Lemma 3.1.5. *Let* $\alpha, \beta \in \mathbb{C}$ *be such that* $Re(\alpha) > 1$, $Re(\beta) > 0$*. If there exists* $1 \leq p \leq \infty$ *such that*

$$\sup_{s\in(0,1)}\left\|\frac{(\cdot - s)^{\beta-1}}{q(\cdot)}\right\|_{L^p(s,1)} < \infty$$

then for all $\gamma \in \mathbb{C}$ *such that* $0 < Re(\gamma) < Re(\alpha) - \frac{1}{p}$ *is verified that* $D^\gamma Q(s,s) = 0$*. Furthermore,* $Q \in C([0,1]\times[0,1])$ *and* $Q(\cdot,s) \in C^m[0,1]$ *for all* $s \in [0,1]$*, where* $m = [Re(\alpha) - \frac{1}{p}] - 1 \geq 0$*.*

Proof. Recall that

$$Q(t,s) = J^{\alpha}\left(\frac{(t-s)^{\beta-1}\mu(t-s)}{q(t)}\right)$$

so, given that $Re(\gamma) < Re(\alpha) - \frac{1}{p} \leq Re(\alpha)$, then

$$D^{\gamma}Q(t,s) = J^{\alpha-\gamma}\left(\frac{(t-s)^{\beta-1}\mu(t-s)}{q(t)}\right)$$

Since $Re(\alpha-\gamma)-1 > 1-\frac{1}{p} = -\frac{1}{p'}$, then $t^{\alpha-\gamma-1} \in L^{p'}(0,1)$, and since $\sup_{s\in(0,1)} \left\|\frac{(\cdot-s)^{\beta-1}}{q(\cdot)}\right\|_{L^p(s,1)} < \infty$ for all $t \in [0,1]$, then $(t-\cdot)^{\alpha-\gamma-1}\frac{(\cdot-s)^{\beta-1}}{q(\cdot)} \in L^1(s,t)$ for all $s \in [0,1]$ and $t \in (s,1]$. Thus,

$$\begin{aligned}
0 &\leq \lim_{t\to s^+} D^{\gamma}Q(t,s)\\
&= \lim_{t\to s^+}\int_s^t (t-\tau)^{\alpha-\gamma-1}\frac{(\tau-s)^{\beta-1}}{q(\tau)}d\tau\\
&\leq \lim_{t\to s^+}\|(t-\cdot)^{\alpha-\gamma-1}\|_{L^{p'}(s,t)}\left\|\frac{(\cdot-s)^{\beta-1}}{q(\cdot)}\right\|_{L^p(s,t)}\\
&= \lim_{t\to s^+}\frac{(t-s)^{(\alpha-\gamma-1)+\frac{1}{p'}}}{((\alpha-\gamma-1)p'+1)^{\frac{1}{p'}}}\left\|\frac{(\cdot-s)^{\beta-1}}{q(\cdot)}\right\|_{L^p(s,t)}\\
&\leq M\lim_{t\to s^+}\frac{(t-s)^{(\alpha-\gamma-1)+\frac{1}{p'}}}{((\alpha-\gamma-1)p'+1)^{\frac{1}{p'}}}\\
&= 0
\end{aligned}$$

On the other hand, as $\mu(t-s) = 0$ for all $t < s$, it follows that $\lim_{t\to s^-} D^{\gamma}Q(t,s) = 0$, concluding that $D^{\gamma}Q(s,s) = 0$. Moreover, from the previous calculus, and since $Re(\alpha) > 1$, then $(t-\cdot)^{\alpha-1}\frac{(\cdot-s)^{\beta-1}}{q(\cdot)} \in L^1(s,t)$ for all $0 \leq s < t \leq 1$, and given that $Q(t,s) = 0$ for all $t \leq s$, then $Q \in C([0,1]\times[0,1])$.

Finally, since $(t-\cdot)^{\alpha-\gamma-1}\frac{(\cdot-s)^{\beta-1}}{q(\cdot)} \in L^1(s,t)$ for all $0 \leq s < t \leq 1$ and all $\gamma \in \mathbb{C}$ such that $Re(\gamma) < Re(\alpha) - \frac{1}{p}$, taking $\gamma = k \in \mathbb{N}$ for each $1 \leq k < Re(\alpha) - \frac{1}{p}$, and since $D^kQ(t,s) =$ for all $t \leq s$, then $D^kQ \in C([0,1]\times[0,1])$, concluding that $D^kQ(\cdot,s) \in C[0,1]$ for all $s \in [0,1]$ and for all $0 \leq k < Re(\alpha) - \frac{1}{p}$. □

Also, the following lemma allows to affirm that the obtained Green function can't be simplified in the form

$$\sum_{k=0}^{[\alpha]-1} a_k(s)t^{k+\alpha-[\alpha]} + \sum_{k=n_0}^{n_1} b_k(s)J^{\alpha}\left(\frac{t^{k+\beta-[\beta]}}{q(t)}\right) - \frac{1}{\Gamma(\beta)\Gamma(\alpha)}Q(t,s)$$

where $n_0 > 0$ or $n_1 < [\beta]-1$.

Lemma 3.1.6. *Let $\alpha, \gamma \in \mathbb{C}$ be such that $Re(\alpha) > 0$ and $Re(\gamma) > -1$. Let $q \in C(0,1)$ be a positive function such that*

$$J^\alpha \left(\frac{t^\gamma}{q(t)} \right) \in C[0,1]$$

Then for all $c \in \mathbb{R}$ and for all $n \in \mathbb{N}_0$ such that $n \leq [\alpha] - 1$,

$$J^\alpha \left(\frac{t^\gamma}{q(t)} \right) \neq ct^{n+\alpha-[\alpha]}$$

Proof. Suppose by contradiction that there exists some $c \in \mathbb{R}$ and some $n \in \mathbb{N}_0$ such that $n \leq [\alpha] - 1$ and

$$J^\alpha \left(\frac{t^\gamma}{q(t)} \right) = ct^{n+\alpha-[\alpha]}$$

If $c = 0$, then by the injectivity of J^α the function $q \in C(0,1)$ is such that $\frac{t^\gamma}{q(t)} \equiv 0$, which is not possible.

If $c \neq 0$, then applying Laplace transform follows that

$$s^{-\alpha} \mathcal{L} \left\{ \frac{t^\gamma}{q(t)} \right\} (s) = c \frac{\Gamma(n + \alpha - [\alpha] + 1)}{s^{n+\alpha-[\alpha]+1}}$$

which implies that

$$\mathcal{L} \left\{ \frac{t^\gamma}{q(t)} \right\} (s) = c \frac{\Gamma(n + \alpha - [\alpha] + 1)}{s^{n-[\alpha]+1}}$$

so that

$$\frac{t^\gamma}{q(t)} = c \frac{\Gamma(n + \alpha - [\alpha] + 1)}{\Gamma(n - [\alpha] + 1)} \mathcal{L}^{-1} \left\{ \frac{\Gamma(n - [\alpha] + 1)}{s^{n-[\alpha]+1}} \right\}$$

which only makes sense if $n - [\alpha] > -1$, so that $n > [\alpha] - 1$, which concludes the proof. □

3.1.3 Caputo case

It is known that Riemann-Liouville and Caputo fractional derivatives are related through the following lemma.

Lemma 3.1.7. *Let $\alpha > 0$ and let $f \in C^{[\alpha]}[0,b]$. Then*

$$D^\alpha f(t) = {}^C D^\alpha f(t) + \sum_{k=0}^{[\alpha]-1} \frac{t^{k-\alpha}}{\Gamma(k - \alpha + 1)} f^{(k)}(0)$$

However, it require some differentiation hypothesis that are not satisifed by the Green function of the Riemann-Liouville case. Despite that, it's still possible to build a general Green function for the operator ${}^C D^\alpha$ following the same steps as in the construction of the Green function for the Riemann-Liouville case.

Now, the construction of a Green function for the operator ${}^C D^\alpha$ is as follows.

First, is already known that the set $\{t^k\}_{k=0}^{[\alpha]-1}$ is a base for the kernel of $D^{[\alpha]}$, so it is also a base for the kernel of ${}^C D^\alpha$. Now note that, for all $f \in \mathcal{S}' \cap \mathcal{F}$,

$$\begin{aligned}\mathcal{L}\{{}^C D^\alpha f\}(\tau) &= \mathcal{L}\{J^{[\alpha]-\alpha} D^{[\alpha]} f\}(\tau) \\ &= \tau^{\alpha-[\alpha]} \mathcal{L}\{D^{[\alpha]} f\}(\tau) \\ &= \tau^{\alpha-[\alpha]} \tau^{[\alpha]} \mathcal{L}\{f\}(\tau) \\ &= \tau^\alpha \mathcal{L}\{f\}(\tau)\end{aligned}$$

so, if $f(t,s) \in \mathcal{S}' \cap \mathcal{F}$ is a distribution such that ${}^C D^\alpha f(t,s) = \delta(t-s)$, then

$$\tau^\alpha \mathcal{L}\{f(\cdot,s)\}(\tau) = \mathcal{L}\{\delta(\cdot - s)\}(\tau) = e^{-s\tau}$$

so that

$$\mathcal{L}\{f(\cdot,s)\}(\tau) = \tau^{-\alpha} e^{-s\tau} = \frac{1}{\Gamma(\alpha)} \left(\frac{\Gamma(\alpha)}{\tau^\alpha}\right) e^{-s\tau}$$

applying inverse Laplace transform, it's obtain that

$$f(t,s) = \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} \mu(t-s)$$

Then, adding the elements on the kernel of ${}^C D^\alpha$, the function $G(t,s)$ defined as

$$G_C(t,s) = \sum_{k=0}^{[\alpha]-1} a_k(s) t^k - \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} \mu(t-s) \tag{3.1.3}$$

is the general Green function for the operator ${}^C D^\alpha$.

Note that the Green functions for the operator D^α and ${}^C D^\alpha$ are essentially the same, but with different kernel functions. Moreover, the kernel functions of ${}^C D^\alpha$ are smoother than the ones of D^α, so one can expect that G_C is smoother than G. It will be seen in the following subsection.

3.2 Making explicit the Green Function

Now consider the boundary conditions of problem (3.0.1), so the Green function can be express in a more explicit form in each case. In this section, some conditions will be found on constants $m \in \mathbb{Z}_{\geq -1}$, $\{n_{i,j}\}_{i,j=m+1}^{[\alpha]-1} \subset \mathbb{R}$, $\{c_{i,j}\}_{i,j=m+1}^{[\alpha]-1} \subset [0,1]$ and $\{d_i\}_{i=m+1}^{[\alpha]-1}$ (see definition of problems (1.4.1) and (3.0.1)), so the Green function satisfies some desirable properties.

3.2.1 Riemann-Liouville Case

Note that if $[\alpha] \leq 1$, then the function G defined on (3.1.1) that solves the problem

$$\begin{cases} -D^\alpha G(t,s) = \delta(t-s), & (t,s) \in (0,1)\times(0,1) \\ D^k G(t,s) = 0 & 0 \leq k \leq m \\ \mathcal{A}_i\left(D^{n_{i,j}} G(c_{i,j},s)\right)_{j=m+1}^{[\alpha]-1} = d_i, & m+1 \leq i \leq [\alpha]-1 \end{cases}$$

is not in $C([0,1]\times[0,1])$, so it will be assumed that $[\alpha] > 1$.

Also note that, since

$$G(t,s) = \sum_{k=0}^{[\alpha]-1} a_k(s) t^{k+\alpha-[\alpha]} - \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}\mu(t-s)$$

and $[\alpha] \geq Re(\alpha)$, then $Re(\alpha) \neq [\alpha]$ and $G \in C([0,1]\times[0,1])$ implies that $a_0 \equiv 0$. So, if $a_k \in C[0,1]$, then $a_0 \equiv 0$ if and only if G satisfies the condition $G(0,s) = 0$, for all $s \in [0,1]$.

Moreover, note that if $\gamma \in (\alpha - [\alpha], 1 + \alpha - [\alpha])$, then lemma 2.4.6 leads to

$$D^\gamma G(t,s) = \sum_{k=0}^{[\alpha]-1} \frac{\Gamma(k+1+\alpha-[\alpha])}{\Gamma(k+1+\alpha-[\alpha]-\gamma)} a_k(s) t^{k+\alpha-[\alpha]-\gamma} - \frac{1}{\Gamma(\alpha-\gamma)}(t-s)^{\alpha-1-\gamma}\mu(t-s)$$

where $Re(\alpha)-1-Re(\gamma) > Re(\alpha)-1+[\alpha]-Re(\alpha)-1 \geq 0$, $k+Re(\alpha)-[\alpha]-Re(\gamma) \geq 0$ for all $k \geq 1$, and $Re(\alpha)-[\alpha]-Re(\gamma) < 0$. So $a_0 \equiv 0$ if and only if $D^\gamma G(0,s) = 0$ for all $s \in [0,1]$, where $Re(\gamma) \in (Re(\alpha)-[\alpha], 1+Re(\alpha)-[\alpha])$.

In general, applying this idea iteratively, if $a_k \equiv 0$ for all $0 \leq k \leq m < [\alpha]-1$, and $a_k \in C[0,1]$ for all $m < k \leq [\alpha]-1$, then $a_{m+1} \equiv 0$ if and only if $D^\gamma G(0,s) = 0$ for all $s \in [0,1]$, where $Re(\gamma) \in (m+1+Re(\alpha)-[\alpha], m+2+Re(\alpha)-[\alpha])$.

So, if $a_k \equiv 0$ for all $0 \leq k \leq m < [\alpha]-1$, and $a_k \in C[0,1]$ for all $m < k \leq [\alpha]-1$, then $G \in C([0,1]\times[0,1])$, with $G(\cdot,s) \in C^{m-1}[0,1]$ for all $s \in [0,1]$. All of that can be summarized the following lemma.

Lemma 3.2.1. *Let $\alpha \in \mathbb{C}$ be such that $[\alpha] > 1$ and $m \in \mathbb{N}_0$ such that $0 \leq m < [\alpha]-1$, and let $\{\gamma_k\}_{k=0}^m$ be such that $Re(\gamma_k) \in (k+Re(\alpha)-[\alpha], k+1+Re(\alpha)-[\alpha])$ for all $0 \leq k \leq m$. Assume that the Green function G defined on (3.1.1) satisfies that $D^{\gamma_k} G(0,s) = 0$ for all $s \in [0,1]$ and all $0 \leq k \leq m$.*

Let $\{\mathcal{A}_i\}_{i=m+1}^{[\alpha]-1} \subset C(\mathbb{R}^{[\alpha]-m-1};\mathbb{R}) \setminus \{0\}$ be a set of (possibly nonlinear) functions. Let $\{n_{i,k}\}_{i,k=m+1}^{[\alpha]-1} \subset \mathbb{R}$, $\{c_{i,k}\}_{i,k=m+1}^{[\alpha]-1} \subset [0,1]$ and $\{d_i\}_{i=m+1}^{[\alpha]-1} \subset \mathbb{R}$. If for all $s \in [0,1]$ the system

$$\mathcal{A}_i(D^{n_{i,k}} G(c_{i,k},s))_{k=m+1}^{[\alpha]-1} = d_i, \quad i = m+1,\ldots,[\alpha]-1$$

admits a unique vector of solutions $(a_k(s))_{k=m+1}^{[\alpha]-1} \subset C[0,1]$, then $G \in C([0,1]\times[0,1])$, with $G(\cdot,s) \in C^m[0,1]$ for all $s \in [0,1]$, and G is given by

$$G(t,s) = \sum_{k=m+1}^{[\alpha]-1} a_k(s) t^{k+\alpha-[\alpha]} - \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}\mu(t-s)$$

Remark 3.2.1. *In the previous lemma, if $\alpha \in \mathbb{N}$, it's enough to assume that $(a_k(s))_{k=m+1}^{[\alpha]-1} \subset C[0,1]$ to obtain that $G \in C([0,1] \times [0,1])$, with $G(\cdot, s) \in C^{[\alpha]-1}[0,1]$ for all $s \in [0,1]$.*

Now a particular case of the previous lemma is studied. Let $\{\gamma_i\}_{i=0}^{m}$ be as in lemma 3.2.1 and assume that $D^{\gamma_i} G(0,s) = 0$ for all $0 \leq i \leq m < [\alpha] - 1$, and let $\{n_{i,j}\}_{i,j=m+1}^{[\alpha]-1} \subset \mathbb{R}$, $\{c_{i,j}\}_{i,j=m+1}^{[\alpha]-1} \subset (0,1]$ and $\{d_i\}_{i=m+1}^{[\alpha]-1} \subset \mathbb{R}$. Let $\{\mathcal{A}_i\}_{i=m+1}^{[\alpha]-1} \subset \mathcal{L}(\mathbb{R}^{[\alpha]-m-1}; \mathbb{R})$ be linear operators, that is, there exists $\nu_i = (\nu_{i,j})_{j=m+1}^{[\alpha]-1} \in \mathbb{R}^{[\alpha]-m-1}$ such that $\mathcal{A}_i x = \langle \nu_i, x \rangle$ for all $x \in \mathbb{R}^{[\alpha]-m-1}$. Suppose that G satisfy the conditions

$$\mathcal{A}_i (D^{n_{i,j}} G(c_{i,j}, s))_{j=m+1}^{[\alpha]-1} = d_i, \quad i = m+1, \ldots, [\alpha] - 1 \tag{3.2.1}$$

Note that from lemma 2.4.6,

$$D^{n_{i,j}} G_1(t,s) = \sum_{k=0}^{[\alpha]-1} a_k(s) \frac{\Gamma(k+1+\alpha-[\alpha])}{\Gamma(1+k+\alpha-[\alpha]-n_{i,j})} t^{k+\alpha-[\alpha]-n_{i,j}} - \frac{1}{\Gamma(\alpha - n_{i,j})} (t-s)^{\alpha-1-n_{i,j}} \mu(t-s)$$

where the previous calculus are valid only if $Re(n_{i,j}) < Re(\alpha)$ and if $Re(n_{i,j}) < k + Re(\alpha) - [\alpha] + 1$ for all $i, j, k = 0, \ldots, [\alpha] - 1$ such that $a_k \not\equiv 0$ a.e. In particular, since $a_k \equiv 0$ for all $0 \leq k \leq m$, the previous conditions are equivalent to the condition $Re(n_{i,j}) < Re(\alpha) + m - [\alpha] + 1$

So, system (3.2.1) can be written as a linear problem on $x(s) = (a_k(s))_{k=m+1}^{[\alpha]-1}$, so that system (3.2.1) is equivalent to problem.

$$Ax(s) = b(s)$$

where $A = (\nu_{i,j})_{i,j=m+1}^{[\alpha]-1} \in \mathcal{M}_{([\alpha]-m-1)\times([\alpha]-m-1)}$ and $b(s) \in C([0,1]; \mathbb{R}^{[\alpha]-m-1})$ is given by

$$b(s) = \left(d_i + \sum_{j=m+1}^{[\alpha]-1} \frac{\nu_{i,j}}{\Gamma(\alpha - n_{i,j})} (c_{i,j} - s)^{\alpha - n_{i,j} - 1} \mu(c_{i,j} - s) \right)_{i=m+1}^{[\alpha]-1}$$

In this way, the system (3.2.1) is l.i., or equivalently, the problem is well posed if and only if the constant matrix A is invertible, so that

$$x(s) = A^{-1} b(s)$$

Note that, from the latter, all functions $a_k(s)$ have the form

$$a_k(s) = \sum_{i=m+1}^{[\alpha]-1} \gamma_{i,k} \left(d_i + \sum_{j=m+1}^{[\alpha]-1} \frac{\nu_{i,j}}{\Gamma(\alpha - n_{i,j})} (c_{i,j} - s)^{\alpha - n_{i,j} - 1} \mu(c_{i,j} - s) \right)$$

where $A^{-1} = (\gamma_{i,k})_{i,k=m+1}^{[\alpha]-1}$. Then, $a_k \in C[0,1]$ for all $k \in \{m+1, \ldots, [\alpha]-1\}$, and

$$G(t,s) = \sum_{k=m+1}^{[\alpha]-1} \left(t^{k+\alpha-[\alpha]} \sum_{i=m+1}^{[\alpha]-1} \gamma_{i,k} \left(d_i + \sum_{j=m+1}^{[\alpha]-1} \frac{\nu_{i,j}}{\Gamma(\alpha - n_{i,j})} (c_{i,j} - s)^{\alpha - n_{i,j} - 1} \mu(c_{i,j} - s) \right) \right) - \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} \mu(t-s)$$

So, $G \in C([0,1] \times [0,1])$, with $G(\cdot, s) \in C^m[0,1]$ for all $s \in [0,1]$.

This the previous can be summarized in the next lemma.

Lemma 3.2.2. *Let $\alpha \in \mathbb{C}$ be such that $Re(\alpha) > 1$, and suppose that $D^{\gamma_k} G(0,s) = 0$ for all $0 \le k \le m < [\alpha] - 1$, where $Re(\gamma_k) \in (k + Re(\alpha) - [\alpha], k + 1 + Re(\alpha) - [\alpha])$. Let $\{n_{i,j}\}_{i,j=m+1}^{[\alpha]-1} \subset (-\infty, Re(\alpha) + m - [\alpha] + 1)$, $\{c_{i,j}\}_{i,j=m+1}^{[\alpha]-1} \subset (0,1]$ and $\{d_i\}_{i=m+1}^{[\alpha]-1} \subset \mathbb{R}$. Let $\{\mathcal{A}_i\}_{i=m+1}^{[\alpha]-1} \subset \mathcal{L}(\mathbb{R}^{[\alpha]-m-1}; \mathbb{R})$ be linear operators. Suppose that G satisfy the conditions*

$$\mathcal{A}_i (D^{n_{i,j}} G(c_{i,j}, s))_{j=m+1}^{[\alpha]-1} = d_i, \quad i = m+1, \ldots, [\alpha] - 1$$

and rewrite it in its equivalent form

$$Ax(s) = b(s)$$

If the constant matrix A is non singular, then problem (4.1.2) is well posed in the sense that its Green function is continuous over $[0,1]$. Furthermore, $G(\cdot, s) \in C^m[0,1]$ for all $s \in [0,1]$. Moreover, in this case G is given by

$$G(t,s) = \sum_{k=m+1}^{[\alpha]-1} \left(t^{k+\alpha-[\alpha]} \sum_{i=m+1}^{[\alpha]-1} \gamma_{i,k} \left(d_i + \sum_{j=m+1}^{[\alpha]-1} \frac{\nu_{i,j}}{\Gamma(\alpha - n_{i,j})} (c_{i,j} - s)^{\alpha - n_{i,j} - 1} \mu(c_{i,j} - s) \right) \right) - \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} \mu(t-s)$$

Remark 3.2.2. *Note that $\mu(c_{i,j} - s) = 0$ for all $s < c_{i,j}$ and all $m+1 \le i,j \le [\alpha] - 1$. In particular, if $c_{i,j} = 0$, then $\mu(c_{i,j} - s) = 0$ for all $s \in [0,1]$. So, since*

$$a_k(s) = \sum_{i=m+1}^{[\alpha]-1} \gamma_{i,k} \left(d_i + \sum_{j=m+1}^{[\alpha]-1} \frac{\nu_{i,j}}{\Gamma(\alpha - n_{i,j})} (c_{i,j} - s)^{\alpha - n_{i,j} - 1} \mu(c_{i,j} - s) \right)$$

then a_k' is undefined at most on the points $c_{i,j}$ such that $c_{i,j} > 0$. If one defines

$$c = \min \{ c_{i,j} > 0 \, : \, m+1 \le i,j \le [\alpha] - 1 \}$$

then $a_k \in C^1[0,c)$ for all $m+1 \le k \le [\alpha] - 1$. In particular, a_k is locally lipschitz on 0 for all k.

In the appendix A.2 can be found another property of G which will be used later.

3.2.2 Concatenated Case

Let $\alpha, \beta \in \mathbb{C}$ be such that $Re(\alpha) > 1$ and $Re(\beta) > 0$. Remember that in this case the Green function is given by

$$G(t,s) = \sum_{k=0}^{[\alpha]-1} a_k(s) t^{k+\alpha-[\alpha]} + \sum_{k=0}^{[\beta]-1} b_k(s) J^{\alpha}\left(\frac{t^{k+\beta-[\beta]}}{q(t)}\right) - \frac{1}{\Gamma(\beta)\Gamma(\alpha)} Q(t,s)$$

where

$$Q(t,s) = \int_0^t (t-\tau)^{\alpha-1} \frac{(\tau-s)^{\beta-1}}{q(\tau)} \mu(\tau-s) d\tau$$

Here is assumed that $q \in C(0,1)$ is a positive function such that there exists $1 \le p \le \infty$ such that

$$H(t) = \frac{t^{\beta-[\beta]}}{q(t)} \in L^p(0,1)$$

and

$$\sup_{s\in(0,1)} \left\| \frac{(\cdot - s)^{\beta-1}}{q(\cdot)} \right\|_{L^p(s,1)} < \infty$$

So lemmas 3.1.4 and 3.1.5 can be applied to obtain that $J^{\alpha}\left(\frac{t^{\beta-[\beta]}}{q(t)}\right) \in C^n[0,1]$, $Q \in C([0,1]\times[0,1])$ and $Q(\cdot,s) \in C^n[0,1]$ for all $s \in [0,1]$, where $n = \left[\alpha - \frac{1}{p}\right] - 1 \ge 0$. Moreover, since $J^{\alpha}\left(\frac{t^{\beta-[\beta]}}{q(t)}\right) \in C^n[0,1]$, then

$$H_k = J^{\alpha}\left(\frac{t^{k+\beta-[\beta]}}{q(t)}\right) \in C^n[0,1], \quad \forall k \ge 0$$

so that, if $\alpha \neq [\alpha]$ then $G \in C([0,1]\times[0,1])$ if and only if $a_k \in C[0,1]$ for all $k = 0,\dots,[\alpha]-1$, $b_k \in C[0,1]$ for all $k = 0,\dots,[\beta]-1$ and $a_0 \equiv 0$. In particular, since $H_k(0) = 0$ for all $k = 0,\dots,[\beta]-1$ and $Re(\alpha)-[\alpha] \in (-1,0]$, then $a_0 \equiv 0$ if and only if $G(0,s) = 0$ for all $s \in [0,1]$.

Moreover, take $Re(\gamma) \in (Re(\alpha)-[\alpha], 1+Re(\alpha)-[\alpha])$, and note that

$$\begin{aligned} &D^{\gamma}G(t,s) \\ &= \sum_{k=0}^{[\alpha]-1} \frac{\Gamma(k+1+\alpha-[\alpha])}{\Gamma(k+1+\alpha-[\alpha]-\gamma)} a_k(s) t^{k+\alpha-[\alpha]-\gamma} + \sum_{k=0}^{[\beta]-1} b_k(s) J^{\alpha-\gamma}\left(\frac{t^{k+\beta-[\beta]}}{q(t)}\right) - \frac{1}{\Gamma(\beta)\Gamma(\alpha)} D^{\gamma}Q(t,s) \end{aligned}$$

where the previous calculus have sense in general if $Re(\gamma) < Re(\alpha)$ and $Re(\gamma) < k + Re(\alpha) - [\alpha] + 1$ for all $k \ge 0$ such that $a_k \not\equiv 0$. Also, by lemmas 3.1.4 and 3.1.5,

$$J^{\alpha-\gamma}\left(\frac{t^{k+\beta-[\beta]}}{q(t)}\right)\Bigg|_{t=0} = 0, \quad \forall k \ge 0$$

and

$$D^{\gamma}Q(0,s)=0, \quad \forall\, s\in[0,1]$$

By all of that, one has that $a_0\equiv 0$ if and only if $D^{\gamma}G(0,s)=0$ for all $s\in[0,1]$, where $Re(\gamma)\in(Re(\alpha)-[\alpha],1+Re(\alpha)-[\alpha])$.

Furthermore, applying this idea iteratively, one has that if $m\in\mathbb{N}_0$ is such that $m\leq[\alpha]-2$ and $m\leq n$, and if $a_k\equiv 0$ for all $0\leq k<m$ and $a_k\in C[0,1]$ for all $m\leq k\leq n$, then $a_m\equiv 0$ if and only if $D^{\gamma}G(0,s)=0$ for all $s\in[0,1]$, where $Re(\gamma)\in(m+Re(\alpha)-[\alpha],\min\{m+1+Re(\alpha)-[\alpha],n\})$.

Combining all the above, if $D^{\gamma_i}G(0,s)=0$ for all $s\in[0,1]$ and for all $0\leq i\leq m$, where $Re(\gamma_i)\in(i+Re(\alpha)-[\alpha],\min\{i+1+Re(\alpha)-[\alpha],[\alpha-\frac{1}{p}]-1\})$, and that $a_k\in C[0,1]$ for all $m<k\leq[\alpha]-1$, then $G\in C([0,1]\times[0,1])$ and $G(\cdot,s)\in C^m[0,1]$ for all $s\in[0,1]$.

Now consider $\{\mathcal{A}_i\}_{i=m+1}^{[\alpha]+[\beta]-1}\subset C(\mathbb{R}^{[\alpha]+[\beta]-m-1};\mathbb{R})$ be a set of (possibly nonlinear) functions, and assume that the Green function defined on (3.1.2) satisfies the conditions

$$\mathcal{A}_i(D^{\gamma_{i,k}}G(c_{i,k},s))_{k=m+1}^{[\alpha]+[\beta]-1}, \quad i=m+1,\dots,[\alpha]+[\beta]-1 \tag{3.2.2}$$

then G is uniquely defined if and only if system (3.2.2) admits a unique vector of solutions $(a_k)_{k=m+1}^{[\alpha]+[\beta]-1}\subset C[0,1]$. Furthermore, in this case, $G\in C([0,1]\times[0,1])$ and $G(\cdot,s)\in C^n[0,1]$ for all $s\in[0,1]$. This is summarized in the following lemma.

Lemma 3.2.3. *Let $\alpha,\beta\in\mathbb{C}$ be such that $Re(\alpha)>1$ and $Re(\beta)>0$, and let G be the Green function defined on (3.1.2). Define $n=\left[\alpha-\frac{1}{p}\right]-1$, where $1\leq p\leq\infty$, and let $m\in\mathbb{N}_0$ be such that $m\leq n$ and $m\leq[\alpha]-2$. Let $\{n_{i,j}\}_{i,j=m+1}^{[\alpha]+[\beta]-1}\subset(-\infty,m+Re(\alpha)-[\alpha]+1)$, $\{c_{i,j}\}_{i,j=m+1}^{[\alpha]+[\beta]-1}\subset[0,1]$ and $\{d_i\}_{i=m+1}^{[\alpha]+[\beta]-1}\subset\mathbb{R}$, and assume that $q\in C(0,1)$ is a positive function such that*

$$H(t)=\frac{t^{\beta-[\beta]}}{q(t)}\in L^p(0,1)$$

and

$$\sup_{s\in(0,1)}\left\|\frac{(\cdot-s)^{\beta-1}}{q(\cdot)}\right\|_{L^p(s,1)}<\infty$$

and

$$D^{n_{i,j}}J^{\alpha}\left(\frac{t^{k+\beta-\beta}}{q(t)}\right)\Bigg|_{t=c_{i,j}}$$

exists for all $k=0,\dots,[\beta]-1$, and

$$D^{n_{i,j}}Q(c_{i,j},s)$$

exists for all $s \in [0,1]$.

Let $\{\mathcal{A}_i\}_{i=m+1}^{[\alpha]+[\beta]-1} \subset C(\mathbb{R}^{[\alpha]+[\beta]-m-1};\mathbb{R})$ *be a set of (possibly nonlinear) functions, and suppose that* G *satisfies* $D^{\gamma_i}G(0,s) = 0$ *for all* $s \in [0,1]$ *and all* $0 \leq i \leq m \leq n$, *where* $Re(\gamma_i) \in (i + Re(\alpha) - [\alpha], \min\{i+1+Re(\alpha)-[\alpha], [\alpha - \frac{1}{p}] - 1\})$, *and*

$$\mathcal{A}_i(D^{n_{i,k}}G(c_{i,k},s))_{k=m+1}^{[\alpha]+[\beta]-1} = d_i, \quad i = m+1,\dots,[\alpha]+[\beta]-1$$

if the previous system admits a unique vector of solutions $((a_k)_{k=m+1}^{[\alpha]-1}, (b_k)_{k=0}^{[\beta]-1}) \subset C[0,1]$, *then*

$$G(t,s) = \sum_{k=m+1}^{[\alpha]-1} a_k(s)t^{k+\alpha-[\alpha]} + \sum_{k=0}^{[\beta]-1} b_k(s)J^\alpha\left(\frac{t^{k+\beta-[\beta]}}{q(t)}\right) - \frac{1}{\Gamma(\beta)\Gamma(\alpha)}Q(t,s)$$

$G \in C([0,1]\times[0,1])$ *and* $G(\cdot,s) \in C^m[0,1]$ *for all* $s \in [0,1]$.

Now the focus will be in a particular case of the previous lemma. Let n, m, $\{n_{i,j}\}_{i,j=m+1}^{[\alpha]+[\beta]-1}$, $\{c_{i,j}\}_{i,j=m+1}^{[\alpha]+[\beta]-1}$ and $\{d_i\}_{i=m+1}^{[\alpha]+[\beta]-1}$ as in the previous lemma. Also, let $\{\mathcal{A}_i\}_{i=m+1}^{[\alpha]+[\beta]-1} \subset C(\mathbb{R}^{[\alpha]+[\beta]-m-1};\mathbb{R})$ be linear functions, that is, there exists $\{\nu_{i,j}\}_{j=m+1}^{[\alpha]+[\beta]-1} \subset \mathbb{R}$ such that

$$\mathcal{A}_i x = \sum_{j=m+1}^{[\alpha]+[\beta]-1} \nu_{i,j}x_{j-m}, \quad \forall\, x = (x_1,\dots,x_{[\alpha]+[\beta]-m-1}) \in \mathbb{R}^{[\alpha]+[\beta]-m-1}$$

It will be assumed that G satisfies $D^{\gamma_i}G(0,s) = 0$ for all $s \in [0,1]$ and $0 \leq i \leq m$, where $Re(\gamma_i) \in [i, \min\{i+1+Re(\alpha)-[\alpha], [\alpha-\frac{1}{p}]-1\})$, and

$$\mathcal{A}_i(D^{n_{i,j}}G(c_{i,j},s))_{j=m+1}^{[\alpha]+[\beta]-1} = d_i, \quad i = m+1,\dots,[\alpha]+[\beta]-1 \tag{3.2.3}$$

It will also be assumed that $q \in C(0,1)$ is a positive function such that

$$H(t) = \frac{t^{\beta-[\beta]}}{q(t)} \in L^p(0,1)$$

and

$$\sup_{s\in(0,1)} \left\| \frac{(\cdot - s)^{\beta-1}}{q(\cdot)} \right\|_{L^p(s,1)} < \infty$$

and

$$D^{n_{i,j}}J^\alpha\left(\frac{t^{k+\beta-[\beta]}}{q(t)}\right)\bigg|_{t=c_{i,j}}$$

exists for all $k = 0,\dots,[\beta]-1$, and

$$D^{n_{i,j}}Q(c_{i,j},s)$$

exists for all $s \in [0,1]$.

Note that

$$
\begin{aligned}
D^{n_{i,j}}G(c_{i,j},s) &= \sum_{k=m+1}^{[\alpha]-1} \frac{\Gamma(k+1+\alpha-[\alpha])}{\Gamma(k+1+\alpha-[\alpha]-n_{i,j})} a_k(s) c_{i,j}^{k+\alpha-[\alpha]-n_{i,j}} \\
&+ \sum_{k=0}^{[\beta]-1} b_k(s) J^{\alpha-n_{i,j}} \left(\frac{t^{k+\beta-[\beta]}}{q(t)} \right)\Bigg|_{t=c_{i,j}} - \frac{1}{\Gamma(\beta)\Gamma(\alpha)} D^{n_{i,j}} Q(c_{i,j},s) \\
&= \sum_{k=m+1}^{[\alpha]-1} a_{i,j,k} a_k(s) + \sum_{k=0}^{[\beta]-1} b_{i,j,k} b_k(s) - \frac{1}{\Gamma(\beta)\Gamma(\alpha)} D^{n_{i,j}} Q(c_{i,j},s)
\end{aligned}
$$

where

$$
\begin{aligned}
a_{i,j,k} &= \frac{\Gamma(k+1+\alpha-[\alpha])}{\Gamma(k+1+\alpha-[\alpha]-n_{i,j})} c_{i,j}^{k+\alpha-[\alpha]-n_{i,j}} \\
b_{i,j,k} &= J^{\alpha-n_{i,j}} \left(\frac{t^{k+\beta-[\beta]}}{q(t)} \right)\Bigg|_{t=c_{i,j}}
\end{aligned}
$$

so that system (3.2.4) is linear with respect of $x(s) = ((a_k(s))_{k=m+1}^{[\alpha]-1}, (b_k(s))_{k=0}^{[\beta]-1})$. So, system (3.2.4) can be written as

$$
Ax(s) = b(s)
$$

for some constant matrix A, where $b(s)$ is given by

$$
b(s) = \left(d_i + \sum_{j=m+1}^{[\alpha]+[\beta]-1} \frac{\nu_{i,j}}{\Gamma(\beta)\Gamma(\alpha)} D^{n_{i,j}} Q(c_{i,j},s) \right)_{i=m+1}^{[\alpha]+[\beta]-1}
$$

So, G is uniquely defined if and only if A is non singular, and if that is the case,

$$
\begin{aligned}
a_k(s) &= \sum_{i=m+1}^{[\alpha]+[\beta]-1} \gamma_{i,k} \left(d_i + \sum_{j=m+1}^{[\alpha]+[\beta]-1} \frac{\nu_{i,j}}{\Gamma(\beta)\Gamma(\alpha)} D^{n_{i,j}} Q(c_{i,j},s) \right) \\
b_k(s) &= \sum_{i=m+1}^{[\alpha]+[\beta]-1} \delta_{i,k} \left(d_i + \sum_{j=m+1}^{[\alpha]+[\beta]-1} \frac{\nu_{i,j}}{\Gamma(\beta)\Gamma(\alpha)} D^{n_{i,j}} Q(c_{i,j},s) \right)
\end{aligned}
$$

where $\gamma_{i,k}, \delta_{i,k} \in \mathbb{C}$, for each i,k, are constants. Then

$$
G(t,s) = \sum_{k=m+1}^{[\alpha]-1} t^{k+\alpha-[\alpha]} \left(\sum_{i=m+1}^{[\alpha]+[\beta]-1} \gamma_{i,k} \left(d_i + \sum_{j=m+1}^{[\alpha]+[\beta]-1} \frac{\nu_{i,j}}{\Gamma(\beta)\Gamma(\alpha)} D^{n_{i,j}} Q(c_{i,j},s) \right) \right)
$$

$$+\sum_{k=0}^{[\beta]-1} J^{\alpha}\left(\frac{t^{k+\beta-[\beta]}}{q(t)}\right)\left(\sum_{i=m+1}^{[\alpha]+[\beta]-1}\delta_{i,k}\left(d_i+\sum_{j=m+1}^{[\alpha]+[\beta]-1}\frac{\nu_{i,j}}{\Gamma(\beta)\Gamma(\alpha)}D^{n_{i,j}}Q(c_{i,j},s)\right)\right)$$
$$-\frac{1}{\Gamma(\beta)\Gamma(\alpha)}Q(t,s)$$

So that $G \in C([0,1]\times[0,1])$ and $G(\cdot,s)\in C^m[0,1]$ for all $s\in[0,1]$.

All the above is summarized in the following lemma.

Lemma 3.2.4. *Let $\alpha,\beta\in\mathbb{C}$ be such that $Re(\alpha)>1$ and $Re(\beta)>0$, and let $\{n_{i,j}\}_{i,j=m+1}^{[\alpha]+[\beta]-1}\subset(-\infty,m+Re(\alpha)-[\alpha]+1)$, $\{c_{i,j}\}_{i,j=m+1}^{[\alpha]+[\beta]-1}\subset[0,1]$, and $\{d_i\}_{i=m+1}^{[\alpha]+[\beta]-1}\subset\mathbb{R}$, where $m\in\mathbb{N}_0$ is such that $m\le n=\left[\alpha-\frac{1}{p}\right]-1$. Also, let $\{\mathcal{A}_i\}_{i=m+1}^{[\alpha]+[\beta]-1}\subset C(\mathbb{R}^{[\alpha]+[\beta]-m-1};\mathbb{R})$ be linear functions, that is, there exists $\{\nu_{i,j}\}_{j=m+1}^{[\alpha]+[\beta]-1}\subset\mathbb{R}$ such that*

$$\mathcal{A}_i x=\sum_{j=m+1}^{[\alpha]-[\beta]-1}\nu_{i,j}x_{j-m},\quad \forall\, x=(x_1,\dots,x_{[\alpha]+[\beta]-m-1})\in\mathbb{R}^{[\alpha]+[\beta]-m-1}$$

Also, let $\{\gamma_i\}_{i=0}^m$ be such that $Re(\gamma_i)\in(i+Re(\alpha)-[\alpha],\min\{i+1+Re(\alpha)-[\alpha],Re(\alpha)-\frac{1}{p}\})$ for all $0\le i\le m$.

Suppose that G satisfies $D^{\gamma_i}G(0,s)=0$ for all $s\in[0,1]$ and $0\le m\le n$, and

$$\mathcal{A}_i(D^{n_{i,j}}G(c_{i,j},s))_{j=m+1}^{[\alpha]+[\beta]-1}=d_i,\quad i=m+1,\dots,[\alpha]+[\beta]-1 \tag{3.2.4}$$

and rewrite the system as

$$Ax(s)=b(s)$$

where A is a constant matrix and $b(s)=((a_k)_{k=m+1}^{[\alpha]-1},(b_k)_{k=0}^{[\beta]-1})$.

Also assume that $q\in C(0,1)$ is a positive function such that

$$H(t)=\frac{t^{Re(\alpha+\beta)-[\beta]-1}}{q(t)}\in L^p(0,1)$$

and

$$\sup_{s\in(0,1)}\left\|\frac{(\cdot-s)^{\beta-1}}{q(\cdot)}\right\|_{L^p(s,1)}<\infty$$

for some $1\le p\le\infty$, and

$$D^{n_{i,j}}J^{\alpha}\left(\frac{t^{k+\beta-[\beta]}}{q(t)}\right)\Bigg|_{t=c_{i,j}}$$

exists for all $k=0,\dots,[\beta]-1$, and

$$D^{n_{i,j}}Q(c_{i,j},s)$$

exists for all $s \in [0,1]$.

If the matrix A *is non singular, then the Green function defined on the Caputo case is uniquely defined, with*

$$G(t,s) = \sum_{k=m+1}^{[\alpha]-1} t^{k+\alpha-[\alpha]} \left(\sum_{i=m+1}^{[\alpha]+[\beta]-1} \gamma_{i,k} \left(d_i + \sum_{j=m+1}^{[\alpha]+[\beta]-1} \frac{\nu_{i,j}}{\Gamma(\beta)\Gamma(\alpha)} D^{n_{i,j}} Q(c_{i,j},s) \right) \right)$$
$$+ \sum_{k=0}^{[\beta]-1} J^{\alpha} \left(\frac{t^{k+\beta-[\beta]}}{q(t)} \right) \left(\sum_{i=m+1}^{[\alpha]+[\beta]-1} \delta_{i,k} \left(d_i + \sum_{j=m+1}^{[\alpha]+[\beta]-1} \frac{\nu_{i,j}}{\Gamma(\beta)\Gamma(\alpha)} D^{n_{i,j}} Q(c_{i,j},s) \right) \right)$$
$$- \frac{1}{\Gamma(\beta)\Gamma(\alpha)} Q(t,s)$$

Moreover, $G \in C([0,1]\times[0,1])$ *and* $G(\cdot,s) \in C^m[0,1]$ *for all* $s \in [0,1]$.

In the appendix A.2 can be found another property of G which will be used later.

3.2.3 Caputo Case

First note that the Green function G defined on (3.1.3), given by

$$G(t,s) = \sum_{k=0}^{[\alpha]-1} a_k(s) t^k - \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} \mu(t-s)$$

satisfies that $G \in C([0,1],[0,1])$ if and only if $Re(\alpha) > 1$ and $a_k \in C[0,1]$ for all $k \in \{0,\dots,[\alpha]-1\}$, and in that case, $G(\cdot,s) \in C^{[\alpha]-2}[0,1]$ for all $s \in [0,1]$. Furthermore, note that

$$D^{[\alpha]-1} G(t,s) = \Gamma([\alpha]) a_{[\alpha]-1}(s) - \frac{1}{\Gamma(\alpha-[\alpha]+1)} (t-s)^{\alpha-[\alpha]} \mu(t-s)$$

so that $D^{[\alpha]-1}G(t,s)$ is continuous if and only if $t < s$. In this case,

$$D^{[\alpha]-1} G(t,s) = \Gamma([\alpha]) a_{[\alpha]-1}(s), \quad \forall\, s \in [0,1]$$

so that $a_{[\alpha]-1} \equiv 0$ if and only if $D^{[\alpha]-1}G(t,s) = 0$ for all $s \in [0,1]$ and $t < s$. However, $D^{[\alpha]-1}G(t,s)$ never is continuous on $t = s$.

Also, note that

$$G(0,s) = a_0(s)$$

so that $a_0 \equiv 0$ if and only if $G(0,s) = 0$ for all $s \in [0,1]$. Moreover, if $\gamma \in [0,\beta_0)$, where $\beta_0 = 1 + \alpha - [\alpha]$ if $Re(\alpha) \in (1,2]$ and $\beta_0 = 1$ if $Re(\alpha) > 2$, then $Re(\alpha-\gamma) - 1 > 0$ and

$$D^{\gamma} G(t,s) = \sum_{k=0}^{[\alpha]-1} \frac{\Gamma(k+1)}{\Gamma(k+1-\gamma)} a_k(s) t^{k-\gamma} - \frac{1}{\Gamma(\alpha-\gamma)} (t-s)^{\alpha-\gamma-1} \mu(t-s)$$

where the previous calculus is valid if $Re(\gamma) < Re(\alpha)$ and $Re(\gamma) < k+1$ for all $k \geq 0$ such that $a_k \not\equiv 0$.

So, $D^\gamma G(t,s) \in C[0,1]$ if and only if $a_0 \equiv 0$, which is equivalent to the condition $D^\gamma G(0,s) = 0$ for all $s \in [0,1]$.

Moreover, for all $n \in \mathbb{N}$ with $n \leq [\alpha] - 2$ one has that, if $\gamma \in [n, \beta_n)$, where $\beta_n = n + 1 + \alpha - [\alpha]$ if $\alpha \in (n, n+1]$ and $\beta_n = n+1$ if $\alpha > n+1$, then $\alpha - \gamma - 1 > 0$ and

$$D^\gamma G(t,s) = \sum_{k=n}^{[\alpha]-1} \frac{\Gamma(k+1)}{\Gamma(k+1-\gamma)} a_k(s) t^{k-\gamma} - \frac{1}{\Gamma(\alpha-\gamma)}(t-s)^{\alpha-\gamma-1}\mu(t-s)$$

where $D^\gamma G \in C([0,1]\times[0,1])$ $a_k \equiv 0$ for all $0 \leq k \leq n$ and $a_k \in C[0,1]$ for all $n < k \leq [\alpha]-1$. Applying this inductively, one gets that if $a_k \equiv 0$ for all $0 \leq k \leq m \leq [\alpha]-3$, then $a_{m+1} \equiv 0$ if and only if $D^\gamma G(0,s) = 0$ for all $s \in [0,1]$ and some $\gamma \in [m+1, \beta_{m+1})$, where $\beta_{m+1} = m + 2 + \alpha - [\alpha]$ if $\alpha \in (m+1, m+2]$ and $\beta_{m+1} = m+2$ if $\alpha > m+2$.

In the same way, if $\gamma \geq 0$, then

$${}^C D^\gamma G(t,s) = \sum_{k=[\gamma]}^{[\alpha]-1} \frac{k!}{[\gamma]!} a_k(s) J^{[\gamma]-\gamma}(t^{k-[\gamma]}) - \frac{1}{\Gamma(\alpha-\gamma)} J^{[\gamma]-\gamma}\left((t-s)^{\alpha-[\gamma]-1}\mu(t-s)\right)$$

but in this case, if $\gamma \neq [\gamma]$, then ${}^C D^\gamma G(0,s) = 0$ for all $s \in [0,1]$. In the case $\gamma = [\gamma]$, it follows that ${}^C D^{[\gamma]} G(0,s) = D^{[\gamma]} G(0,s)$, so one falls in the case previously analysed. All that motivates the following lemma.

Lemma 3.2.5. *Let $\alpha \in \mathbb{C}$ be such that $Re(\alpha) > 1$. Let $m \in \mathbb{Z}$ such that $m \geq -1$. Let $\{\gamma_i\}_{i=0}^m$, with $m \leq [\alpha]-2$, be such that $\gamma_i \in [i, \beta_i)$, where $\beta_i = i+1+\alpha-[\alpha]$ if $\alpha \in (i, i+1]$ and $\beta_i = i+1$ if $\alpha > i+1$, for all $0 \leq i \leq m$.*

Let's assume that $D^{\gamma_i} G(0,s) = 0$ for all $i = 0, \dots, m$. Define

$$I := \{n \in \mathbb{N} \,:\, n > m\}$$

Let $\{n_{i,j}\}_{i,j\in I} \subset (-\infty, \min\{Re(\alpha), m+1\})$, $\{c_{i,j}\}_{i,j\in I} \subset [0,1]$ and $\{d_i\}_{i\in I} \subset \mathbb{R}$. Let $\{\mathcal{A}_i\}_{i\in I} \subset C(\mathbb{R}^{[\alpha]-m-1};\mathbb{R})$ be a set of (possibly nonlinear) functions. If the system of equations

$$\mathcal{A}_i(D^{n_{i,j}} G(c_{i,j}, s))_{j\in I} = d_i, \quad \forall\, i \in I$$

admits a unique vector of solutions $(a_k)_{k\in I} \subset C[0,1]$, then

$$G(t,s) = \sum_{k\in I} a_k(s) t^k - \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}\mu(t-s)$$

$G \in C([0,1]\times[0,1])$ and $G(\cdot, s) \in C^{[\alpha]-2}$ for all $s \in [0,1]$.

Now, a particular case of the previous lemma will be studied. Suppose that α, $\{\gamma_i\}_{i=0}^m$, I, $\{n_{i,j}\}_{i,j\in I} \subset (-\infty, \min\{Re(\alpha), m+1\})$, $\{c_{i,j}\}_{i,j\in I}$ and $\{d_i\}_{i\in I}$ are as in the previous lemma. Let $\{\mathcal{A}_i\}_{i\in I} \subset C(\mathbb{R}^{[\alpha]-m-1};\mathbb{R})$ be linear functions, so that there exists $\nu_i = (\nu_{i,j})_{j\in I} \in \mathbb{R}^{[\alpha]-m-1}$ such that

$$\mathcal{A}_i x = \langle \nu, x\rangle, \quad \forall\, x \in \mathbb{R}^{[\alpha]-m-1}$$

Suppose that the Green function G defined on (3.1.3) satisfy the conditions ${}^C D^{\gamma_i} G(0,s)$ for all $s \in [0,1]$ and $i = 0, \dots, m$, and

$$\mathcal{A}_i (D^{n_{i,j}} G(c_{i,j}, s))_{j\in I} = d_i, \quad \forall i \in I \tag{3.2.5}$$

From the calculus from before follows that $D^{n_{i,j}} G(c_{i,j}, s)$ is well defined for all $i, j \in I$, since $G(\cdot, s) \in C^{[\alpha]-2}$. Moreover,

$$D^{n_{i,j}} G(c_{i,j}, s) = \sum_{\substack{k\in I\\ k\geq n}} a_k(s) c_{i,j}^{k-n_{i,j}} \frac{\Gamma(k+1)}{(k-n_{i,j})!} - \frac{1}{\Gamma(\alpha - n_{i,j})} (c_{i,j}-s)^{\alpha - n_{i,j}-1} \mu(c_{i,j}-s)$$

so that system (3.2.5) is linear on $(a_k(s))_{k\in I}$. Then, one can rewrite system (3.2.5) as

$$Ax(s) = b(s)$$

where A is a constant matrix, $x(s) = (a_k)_{k\in I}$ and $b(s)$ is given by

$$b(s) = \left(d_i + \sum_{j\in I} \frac{\nu_{i,j}}{\Gamma(\alpha - n_{i,j})} (c_{i,j}-s)^{\alpha-n_{i,j}-1} \mu(c_{i,j}-s) \right)_{i\in I}$$

From that, follows that the functions $a_k(s)$ have the form

$$a_k(s) = \sum_{i\in I} \delta_{i,k} \left(d_i + \sum_{j\in I} \frac{\nu_{i,j}}{\Gamma(\alpha - n_{i,j})} (c_{i,j}-s)^{\alpha-n_{i,j}-1} \mu(c_{i,j}-s) \right)$$

and therefore,

$$\begin{aligned} G(t,s) = &\sum_{k\in I} \delta_{i,k} \left(d_i + \sum_{j\in I} \frac{\nu_{i,j}}{\Gamma(\alpha - n_{i,j})} (c_{i,j}-s)^{\alpha-n_{i,j}-1} \mu(c_{i,j}-s) \right) t^k \\ &- \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} \mu(t-s) \end{aligned}$$

where $\delta_{i,k} \in \mathbb{C}$ are constants for all i, k. All that is summarized in the following lemma.

Lemma 3.2.6. *Let $\alpha \in \mathbb{C}$ be such that $Re(\alpha) > 1$. Let $m \in \mathbb{Z}$ such that $m \geq -1$. Let $\{\gamma_i\}_{i=0}^m$, with $-1 \leq m \leq [\alpha]-2$ be such that $\gamma_i \in [i, \beta_i)$, where $\beta_i = i+1+\alpha-[\alpha]$ if $\alpha \in (i, i+1]$ and $\beta_i = i+1$ if $\alpha > i+1$, for all $0 \leq i \leq m_0$.*

Let's assume that $D^{\gamma_i} G(0,s) = 0$ for all $i = 0, \dots, m$. Define

$$I := \{n \in \mathbb{N} \,:\, 0 \leq n \leq m\}$$

Let $\{n_{i,j}\}_{i,j\in I} \subset (-\infty, \min\{Re(\alpha), m+1\})$*,* $\{c_{i,j}\}_{i,j\in I} \subset [0,1]$ *and* $\{d_i\}_{i\in I} \subset \mathbb{R}$*. Let* $\{\mathcal{A}_i\}_{i\in I} \subset C(\mathbb{R}^{[\alpha]-m-1};\mathbb{R})$ *be a set of linear functions. Assume that* G *also satisfies*

$$\mathcal{A}_i(D^{n_{i,j}}G(c_{i,j},s))_{j\in I} = d_i, \quad \forall\, i \in I$$

and rewrite the system above as

$$Ax(s) = b(s)$$

with $x(s) = (a_k)_{k\in I}$*. If* A *is non singular, then* G *is uniquely defined, with*

$$G(t,s) = \sum_{k\in I} \delta_{i,k} \left(d_i + \sum_{j\in I} \frac{\nu_{i,j}}{\Gamma(\alpha - n_{i,j})}(c_{i,j}-s)^{\alpha-n_{i,j}-1}\mu(c_{i,j}-s) \right) t^k - \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}\mu(t-s)$$

for some constants $\delta_{i,k}, \nu_{i,k} \in \mathbb{C}$*,* $i,k \in I$*. Moreover,* $G \in C([0,1]\times[0,1])$ *and* $G(\cdot,s) \in C^{[\alpha]-2}$ *for all* $s \in [0,1]$*.*

In the appendix A.2 can be found another property of G which will be used later.

Chapter 4

Main Results

In this chapter, since now the Green function of problem (4.1.2) is known in each case, the focus in this section is to study their existence of solutions, and its asymptotic behaviour.

First, a fixed point operator will be defined for this problems from their Green functions, and then, its continuity and compactness will be studied. Specifically, it will be proven that those operators are completely continuous under certain conditions.

Then, the existence of solutions will be studied in two cases, using both, Banach and Krasnosel'skii fixed point theorems.

Finally, both problems will be researched further, analysing the multiplicity of solutions and asymptotic behaviour of its solutions, when $\lambda \to \infty$.

Onward, wit will be assumed that all the hypothesis of lemmas 3.2.2, 3.2.4 or 3.2.6, according to the problem studied, are satisfied, so that the Green function G of this problem is uniquely defined, with $G \in C([0,1] \times [0,1])$ and $G(\cdot, s) \in C^m[0,1]$ for all $s \in [0,1]$.

Also, sometimes will be considered the following conditions on f

C1) $f(\cdot, x) \in L^\infty(0,1)$ for all $x \in \mathbb{R}^{n+1}$,

C2) $f(t, \cdot)$ is locally lipschitz for a.e. $t \in [0,1]$,

H1) $f(t, \cdot)$ is lipschitz for a.e. $t \in [0,1]$,

H2) $f(t,x) \geq 0$ for a.e. $(t,x) \in [0,1] \times [0,\infty) \times \mathbb{R}^n$,

H3)

$$\liminf_{\|x\|_\infty \to \infty} \frac{f(t,x)}{\|x\|_\infty} = \infty, \quad \text{a.e. } t \in [0,1]$$

and

H4) There exists constants $p > 1$ and $c_1 \geq 0$ such that

$$0 \leq \limsup_{\|x\|_\infty \to 0^+} \frac{f(t,x)}{\|x\|_\infty^p} \leq c_1, \quad \text{a.e. } \ t \in [0,1]$$

and the conditions on the Green function.

G1) if $G(t,s) > 0$ for all $(t,s) \in (0,1) \times (0,1)$,

G2) if for each $0 \leq k \leq n$ there exists a measurable function $h_k : (0,1) \to (0,1)$ such that $\max_{t\in[0,1]} |D^k G(t,s)| = D^k G(h_k(s), s)$ for all $s \in (0,1)$, and there exist $0 < a < b \leq 1$ and $\sigma = \sigma(a,b,\alpha)$ such that for all $0 \leq k \leq n$ $D^k G(t,s) \geq \sigma D^k G(h_k(s), s)$ for all $(t,s) \in [a,b] \times (0,1)$.

Finally, if $u \in C^k[0,1]$ for some $k \in \mathbb{N}_0$, it will be denoted $\mathbf{u}^k(t) = (u(t), \ldots, D^k u(t))$.

4.1 Existence of solutions

In this section will be considered the problem

$$\begin{cases} -\mathcal{D}v(t) = \lambda c(t) f(t, v(t), \ldots, D^n v(t)), & 0 < t < 1 \\ D^k v(0) = 0, & 0 \leq k \leq m \quad \text{if } m \geq 0 \\ \mathcal{A}_i(D^{n_{i,j}} v(c_{i,j}))_{j=m+1}^{[\alpha]-1} = 0, & \forall\, i \in \{m+1, \ldots, [\alpha]-1\} \end{cases} \tag{4.1.1}$$

where $\mathcal{D}$ is, either, the operator D^α or ${}^C D^\alpha$ with $\alpha \in \mathbb{R}$, $\alpha > 1$, or the operator $D^\beta\left(q(t) D^\alpha\right)$ with $\alpha, \beta \in \mathbb{R}$, $\alpha > 1$ and $\beta > 0$. In general, it's considered $m, n \in \mathbb{N}_0$ that are such that $0 \leq n \leq m < [\alpha] - 1$, $\{\mathcal{A}_i\}_{i=m+1}^{[\alpha]-1} \subset C(\mathbb{R}^{[\alpha]-m-1}; \mathbb{R})$ are linear operators, $\{n_{i,j}\}_{i,j=m+1}^{[\alpha]-1} \subset \mathbb{R}$ and $\{c_{i,j}\}_{i,j=m+1}^{[\alpha]-1} \subset [0,1]$.

It's also assumed that $\lambda > 0$ is a parameter, $c \in L^1(0,1)$ is not the null function, and $f : [0,1] \times \mathbb{R}^{n+1} \to \mathbb{R}$ satisfies conditions **C1** and **C2**.

Consider the operator $T : C^n[0,1] \to C^n[0,1]$ given by

$$Tv(t) = \int_0^1 G(t,s) \lambda c(s) f(s, \mathbf{v}^n(s)) ds$$

where G is the Green function for problem (4.1.2). Since, by lemmas A.2.1, A.2.2 and A.2.3, the family $\{D^k G(\cdot, s)\}_{s\in[0,1]}$ is uniformly equicontinuous, then given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|t_1 - t_2| < \delta \quad \Rightarrow \quad |G(t_1, s) - G(t_2, s)| < \epsilon, \quad \forall s \in [0,1]$$

By the other side, if $v \in C^n[0,1]$, then there exists $K > 0$ such that $\|D^k v\|_\infty \leq K$ for all $0 \leq k \leq n$. Also, since $f(\cdot, x) \in L^\infty(0,1)$ for all $x \in \mathbb{R}^{n+1}$ and $f(t, \cdot) \in C(\mathbb{R}^{n+1})$ for a.e. $t \in [0,1]$, in particular $f \in L^\infty((0,1) \times \Omega)$ for all $\Omega \subset \mathbb{R}^{n+1}$ bounded. Then, there exists

$M > 0$ such that $|f(t, \mathbf{v}^n(t))| \leq M$ for a.e. $t \in [0,1]$, and then, given $t_1, t_2 \in [0,1]$ and $0 \leq k \leq m$,

$$\begin{aligned}|D^kTv(t_1) - D^kTv(t_2)| &\leq \int_0^1 |D^kG(t_1,s) - D^kG(t_2,s)||\lambda|c(s)f(s,\mathbf{v}^n(s))|ds \\ &\leq M\int_0^1 |D^kG(t_1,s) - D^kG(t_2,s)||\lambda|c(s)|ds\end{aligned}$$

since c is not the null function, then $\|c\|_1 > 0$, so taking $\delta > 0$ such that

$$|G(t_1,s) - G(t_2,s)| < \frac{\epsilon}{M\|c\|_1}, \quad \forall\, |t_1 - t_2| < \delta, \forall\, s \in [0,1]$$

then, for $|t_1 - t_2| < \delta$ one gets that

$$M\int_0^1 |D^kG(t_1,s) - D^kG(t_2,s)||\lambda|c(s)|ds \leq M\int_0^1 \frac{\epsilon}{M\|c\|_1}|c(s)|ds = \epsilon$$

so that $D^kTv \in C[0,1]$ for all $0 \leq k \leq m$, which implies that, for all $v \in C^n[0,1]$, $Tv \in C^m[0,1]$, so T is well defined.

On the other hand, by construction of G,

$$\begin{aligned}\mathcal{D}Tv(t) &= \lambda\int_0^1 \mathcal{D}G(t,s)\lambda c(s)f(s,\mathbf{v}^n(s))ds \\ &= \lambda\int_0^1 \delta(t-s)\lambda c(s)f(s,\mathbf{v}^n(s))ds \\ &= \lambda c(t)f(t,\mathbf{v}^n(t))\end{aligned}$$

Also, since $\mathcal{A}_i(D^{n_{i,j}}G(c_{i,j},s))_{j=m+1}^{[\alpha]-1} = 0$ and $\mathcal{A}_i$ is linear for all $i = m+1,\ldots,[\alpha]-1$, follows that

$$\mathcal{A}_i(D^{n_{i,j}}Tv(c_{i,j}))_{j=m+1}^{[\alpha]-1} = 0 \cdot \lambda\int_0^1 c(s)f(s,\mathbf{v}^n(s)) = 0$$

Finally, since $D^{\gamma_k}G(0,s) = 0$ for all $0 \leq k \leq m$, then

$$D^{\gamma_k}Tv(0) = 0, \quad \forall\, 0 \leq k \leq m$$

then, Tv is solution of the problem

$$\begin{cases} \mathcal{D}Tv(t) = c(t)f(t,\mathbf{v}^n(t)), & 0 < t < 1 \\ D^{\gamma_k}Tv(0) = 0, & 0 \leq k \leq m \\ \mathcal{A}_i(D^{n_{i,j}}Tv(c_{i,j}))_{j=m+1}^{[\alpha]-1} = 0, & \forall\, i \in \{m+1,\ldots,[\alpha]-1\} \end{cases} \tag{4.1.2}$$

for all $v \in C^n[0,1]$. In particular, $v \in C^n[0,1]$ is a solution of problem (4.1.1) if and only if v is a fixed point of T, and if it is the case, then $v = Tv \in C^m[0,1]$. Now will be proven an important property of T.

Lemma 4.1.1. *Let $G \in C([0,1] \times [0,1])$ be the Green function of problem (4.1.2). If $f : [0,1] \times \mathbb{R}^{n+1} \to \mathbb{R}$ satisfies conditions **C1** and **C2**, and $c \in L^1(0,1)$, then the operator $T : C^n[0,1] \to C^n[0,1]$ defined by*

$$Tv(t) = \int_0^1 G(t,s)\lambda c(s) f(s, \mathbf{v}^n(s))ds$$

is completely continuous.

Proof. First is asserted that T is continuous over $X = C^n[0,1]$. To see this note that, as f is locally lipschitz over its second variable, in particular f is lipschitz on its second variable over $\overline{\Omega}$ for all $\Omega \subset \mathbb{R}^{n+1}$ bounded. Additionally, given $u, v \in C^n[0,1]$, in particular they are evenly bounded, that is, there exists $M > 0$ such that $\|u\|_{C^n[0,1]} \leq M$ and $\|v\|_{C^n[0,1]} \leq M$. Since $f(t, \cdot)$ is lipschitz continuous over $[-M, M]^{n+1}$ for almost every $t \in [0,1]$, there exists $L = L(M) > 0$ such that

$$\begin{aligned} |f(t, \mathbf{u}^n(t)) - f(t, \mathbf{v}^n(t))| &\leq L \sup_{0 \leq k \leq n} |D^k u(t) - D^k v(t)|, \quad \text{a.e. } t \in [0,1] \\ &\leq L\|u - v\|_{C^n[0,1]}, \quad \text{a.e. } t \in [0,1] \end{aligned}$$

with what, for a.e. $t \in [0,1]$ and for all $0 \leq k \leq n$, since $G \in C([0,1] \times [0,1])$ and $G(\cdot, s) \in C^m[0,1]$ for all $s \in [0,1]$ we get that

$$\begin{aligned} |D^k Tu(t) - D^k Tv(t)| &= \left| \int_0^1 D^k G(t,s)\lambda c(s)(f(s, \mathbf{u}^n(s)) - f(s, \mathbf{v}^n(s)))ds \right| \\ &\leq \int_0^1 |D^k G(t,s)| \lambda |c(s)| |f(s, \mathbf{u}^n(s)) - f(s, \mathbf{v}^n(s))| ds \\ &\leq L \int_0^1 \max_{0 \leq t, r \leq 1} |D^k G(t,r)| \lambda |c(s)| \|u - v\|_{C^n[0,1]} ds \\ &\leq \lambda L \|u - v\|_{C^n[0,1]} \|D^k G\|_{L^\infty((0,1)^2)} \int_0^1 |c(s)| ds \\ &= \lambda L_k \|u - v\|_{C^n[0,1]} \end{aligned}$$

As the previous bound don't depend on t, defining $\hat{L} = \max_{0 \leq k \leq n} L_k$ and if we take the maximum over $t \in [0,1]$ follows that

$$\|Tu - Tv\|_{C^n[0,1]} \leq \lambda \hat{L} \|u - v\|_{C^n[0,1]} \tag{4.1.3}$$

Now note that, as $X = C^n[0,1]$ is a Banach space with the norm $\sup_{0 \leq k \leq n} \|D^k \cdot\|_\infty$, then all Cauchy sequences over X converge on X. With that on mind, consider a sequence $\{u_k\}_{k \in \mathbb{N}} \subset X$ such that $u_k \to u$ on X. Then the sequence $\{u_k\}_{k \in \mathbb{N}}$ is of Cauchy, and therefore, for all $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$\|u_j - u_k\|_{C^n[0,1]} < \epsilon, \quad \forall j, k \geq k_0$$

So, inequality (4.1.3) leads to

$$\|Tu_j - Tu_k\|_{C^n[0,1]} \leq \lambda \hat{L} \|u_j - u_k\|_{C^n[0,1]} < \hat{L}\epsilon$$

so that $\{Tu_k\}_{k\in\mathbb{N}}$ is also a Cauchy's sequence over X, and thus converge on X. On the other hand, one has that

$$\|Tu_k - Tu\|_{C^n[0,1]} \leq \lambda \hat{L} \|u_k - u\|_{C^n[0,1]} \to 0 \quad \text{when } k \to \infty$$

Thus, $Tu_k \to Tu$ when $k \to \infty$, concluding that T is continuous over $C^n[0,1]$.

Now it will be proven that T maps bounded sets on precompact sets. Let $B \subset X$ be a bounded set, then there exists $K > 0$ such that $\|v\|_{C^n[0,1]} \leq K$ for all $v \in B$. On the other hand, as $f(t,\cdot)$ is continuous over $\mathbb{R}^{n+1}$ for a.e. $t \in [0,1]$, in particular $f \in L^\infty([0,1] \times [-K,K]^{n+1})$, whence there exists $M > 0$ such that for all $v \in B$ it's verified that $\|f(t,\mathbf{v}^n(t))\|_\infty \leq M$ for a.e. $t \in [0,1]$. So, given $t \in [0,1]$ and $0 \leq k \leq n$,

$$\begin{aligned}
|D^k Tv(t)| &= \left| \int_0^1 D^k G(t,s) \lambda c(s) f(s,\mathbf{v}^n(s)) ds \right| \\
&\leq \lambda \int_0^1 \|D^k G\|_{L^\infty((0,1)^2)} |c(s)||f(s,\mathbf{v}^n(s))| ds \\
&\leq \lambda M \|D^k G\|_{L^\infty((0,1)^2)} \int_0^1 |c(s)| ds \\
&= \lambda M_k < \infty
\end{aligned}$$

Since the previous bound is uniform for each $0 \leq k \leq n$, taking maximum over $t \in [0,1]$ is obtained that

$$\|D^k Tv\|_\infty \leq \lambda M_k, \quad \forall\, v \in B$$

and therefore

$$\|Tv\|_{C^n[0,1]} \leq \lambda \sup_{0\leq k\leq n} M_k = \lambda \hat{M}, \quad \forall\, v \in B \tag{4.1.4}$$

From the latter it's obtained that $T(B)$ is bounded on X. Now consider $0 \leq t_1 \leq t_2 \leq 1$, $0 \leq k \leq n$ and $v \in B$, then

$$\begin{aligned}
&|D^k T(v(t_1)) - D^k T(v(t_2))| \\
&= \left| \int_0^1 D^k G(t_1,s) \lambda c(s) f(s,\mathbf{v}^n(s)) ds - \int_0^1 D^k G(t_2,s) \lambda c(s) f(s,\mathbf{v}^n(s)) ds \right| \\
&= \left| \int_0^1 \left(D^k G(t_1,s) - D^k G(t_2,s) \right) \lambda c(s) f(s,\mathbf{v}^n(s)) ds \right| \\
&\leq \int_0^1 \left| D^k G(t_1,s) - D^k G(t_2,s) \right| \lambda |c(s) f(s,\mathbf{v}^n(s))| ds \\
&\leq \lambda \|D^k G(t_1,\cdot) - D^k G(t_2,\cdot)\|_\infty M \int_0^1 |c(s)| ds \\
&= \bar{M} \lambda \|D^k G(t_1,\cdot) - D^k G(t_2,\cdot)\|_\infty
\end{aligned}$$

Since, by lemma A.2.1, for each $0 \leq k \leq m$, $\{D^k G(\cdot, s)\}_{s\in[0,1]}$ is a family of uniformly continuous function on $C[0,1]$, it follows that for each $0 \leq k \leq n$ the family $\{D^k Tv\}_{v\in B}$ is uniformly equicontinuous over $C[0,1]$, and in consequence, $\{Tv\}_{v\in B}$ is uniformly continuous over $C^n[0,1]$.

Given that $[0,1]$ is compact and $T(B) \subset C^n[0,1]$ is bounded and uniformly equicontinuous, from the Arzelá-Ascoli theorem, applied to each $0 \leq k \leq n$, one gets that $T(B)$ is precompact on $C^n[0,1]$.

As T is continuous and maps bounded sets in precompacts, then T is completely continuous over $C^n[0,1]$. □

Remark 4.1.1. *If $G \notin C([0,1]\times[0,1])$ but $G \in C^m([0,1]; L^p(0,1))$ for some $1 \leq p \leq \infty$, and for all $0 \leq k \leq n$ the set $\{D^k G(\cdot, s)\}_{s\in(0,1)}$ satisfies that for all $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\|G(t_1, s) - G(t_2, s)\|_{L^p(0,1)} < \epsilon, \quad \forall\, |t_1 - t_2| < \delta, \quad \forall\, s \in (0,1)$$

if $c \in L^{p'}(0,1)$, then the previous lemma still holds.

Main Theorem 4.1.1. *Suppose that the Green function of problem (4.1.1) satisfies conditions **G1** and **G2**, and that f satisfies conditions **C1** and **C2**. Then*

1. *if f satisfies condition **H1**, then for all $\lambda > 0$ small enough, problem (4.1.1) has a unique solution $v \in C^m[0,1]$. If $f(t,0) = 0$, then the solution is the trivial $v \equiv 0$.*
2. *If f satisfies conditions **H2**, **H3** and **H4**, and if $c \not\equiv 0$ is nonnegative, then for all $\lambda > 0$, problem (4.1.1) has at least one positive solution $v \in C^m[0,1]$.*

Proof. First the item 2 of the theorem will be proven, so it's assumed that $f : [0,1] \times \mathbb{R}^{n+1} \to \mathbb{R}$ is locally lipschitz on its second variable such that $f(\cdot, x) \in L^\infty(0,1)$ for all $x \in \mathbb{R}^{n+1}$, is non negative over $[0,1]\times[0,\infty)\times\mathbb{R}^n$ and that satisfies **H3** and **H4**, and that $c \in L^1(0,1)$ is non negative and different from the null function. Consider the operator $T : C^n[0,1] \to C^n[0,1]$ defined as

$$Tv(t) = \int_0^1 G(t,s)\lambda c(s) f(s, \mathbf{v}^n(s))ds$$

From lemma 4.1.1 it's known that T is completely continuous over X. It's also known from the calculus made at the start of this section that $v \in C^m[0,1]$ is a solution of (4.1.1) if and only if is a fixed point of T. Next define the cone

$$\mathcal{C} = \{v \in X \,:\, v \geq 0,\ \min_{a\leq t\leq b} \|\mathbf{v}^n(t)\|_\infty \geq \sigma \|v\|_{C^n[0,1]}\}$$

where a, b and σ are the constants from property **G2**. In particular, as $\mathcal{C}$ is a closed subset of $C^n[0,1]$, then T is completely continuous over $\mathcal{C}$.

Now it's asserted that T maps $\mathcal{C}$ into itself. In fact, if $v \in C^n[0,1]$ is non negative, since G, c and f are non negative functions a.e, it follows that Tv is non negative. On the other hand, for every $0 \leq k \leq n$,

$$\begin{aligned}
\min_{a\leq t\leq b} D^k Tv(t) &= \min_{a\leq t\leq b} \int_0^1 D^k G(t,s)\lambda c(s) f(s, \mathbf{v}^n(s))ds \\
&\geq \min_{a\leq t\leq b} \int_0^1 \sigma D^k G(h_k(s),s)\lambda c(s) f(s, \mathbf{v}^n(s))ds \\
&= \sigma \int_0^1 D^k G(h_k(s),s)\lambda c(s) f(s, \mathbf{v}^n(s))ds \\
&\geq \sigma \max_{0\leq t\leq 1} \int_0^1 D^k G(t,s)\lambda c(s) f(s, \mathbf{v}^n(s))ds \\
&= \sigma \|D^k Tv\|_\infty
\end{aligned}$$

and then,

$$\sigma \|Tv\|_{C^n[0,1]} \leq \max_{0\leq k\leq n} \min_{a\leq t\leq b} D^k Tv(t) \leq \min_{a\leq t\leq b} \max_{0\leq k\leq n} D^k Tv(t) = \min_{a\leq t\leq b} \|\mathbf{Tv}^n(t)\|_\infty$$

so that $Tv \in \mathcal{C}$

Now, given that f satisfies **H4**, for all $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < \|x\|_\infty \leq \delta$, then

$$\left| \frac{f(t,x)}{\|x\|_\infty^p} - \frac{c_1}{2} \right| \leq \frac{c_1}{2} + \epsilon, \quad \text{a.e. } t \in [0,1]$$

So that, in particular

$$\frac{f(t,x)}{\|x\|_\infty^p} \leq c_1 + \epsilon$$

Taking $\epsilon \leq \kappa$ for some $\kappa > 0$, follows that

$$f(t,x) \leq (c_1 + \kappa)\|x\|_\infty^p = c_2 \|x\|_\infty^p, \quad \forall\, 0 < \|x\|_\infty \leq \delta$$

Therefore, if $v \in \mathcal{C}$ is such that $0 < \|v\|_{C^n[0,1]} \leq \delta$, and $0 \leq k \leq n$, then

$$\begin{aligned}
\|D^k Tv\|_\infty &= \max_{0\leq t\leq 1} \left| \int_0^1 D^k G(t,s)\lambda c(s) f(s, \mathbf{v}^n(s))ds \right| \\
&\leq \lambda \max_{0\leq t\leq 1} \int_0^1 |D^k G(t,s)||c(s)||f(s, \mathbf{v}^n(s))|ds \\
&\leq \lambda \max_{0\leq t\leq 1} \int_0^1 c_2 |D^k G(h_k(s),s)||c(s)|\|\mathbf{v}^n(s)\|_\infty^p ds \\
&\leq c_2 \lambda \|v\|_{C^n[0,1]}^p \int_0^1 |D^k G(h_k(s),s)||c(s)|ds \\
&= \lambda \hat{c}_k \|v\|_{C^n[0,1]}^p
\end{aligned}$$

and thus

$$\|Tv\|_{C^n[0,1]} \leq \lambda \|v\|^p_{C^n[0,1]} \max_{0\leq k\leq m} \hat{c}_k = \hat{c}\lambda \|v\|^p_{C^n[0,1]}$$

Furthermore, if v also satisfies

$$\lambda \hat{c} \|v\|^{p-1}_{C^n[0,1]} \leq 1$$

Then

$$\|Tv\|_{C^n[0,1]} \leq \lambda \hat{c} \|v\|^p_{C^n[0,1]} \leq \|v\|_{C^n[0,1]}$$

By other side, given that f satisfies **H3**, then for all $N > 0$ there exists $C > 0$ large enough such that if $\|x\|_\infty \geq C$, then $f(t,x) \geq N\|x\|_\infty$ for a.e. $t \in [0,1]$. Now note that, since $\max_{0\leq t\leq 1} |D^k G(t,s)| = D^k G(h_k(s), s)$ for almost all $s \in [0,1]$, then $D^k G(h_k(s), s) \geq 0$ for almost all $s \in [0,1]$. Since $\sigma > 0$, and $D^k G(t,s) \geq \sigma D^k G(h_k(s), s)$ for all $(t,s) \in [a,b] \times [0,1]$, then $D^k G(t,s) \geq 0$ for all $t \in [a,b]$, and almost all $s \in [0,1]$. Therefore, for each $t \in [a,b]$ and $0 \leq k \leq n$ one has that

$$\begin{aligned}
\|D^k Tv\|_\infty &= \max_{0\leq r\leq 1} \left| \int_0^1 D^k G(r,s)\lambda c(s) f(s, \mathbf{v}^n(s)) ds \right| \\
&\geq \int_0^1 D^k G(t,s)\lambda c(s) f(s, \mathbf{v}^n(s)) ds \\
&\geq \int_a^b D^k G(t,s)\lambda c(s) f(s, \mathbf{v}^n(s)) ds \\
&\geq \int_a^b D^k G(t,s)\lambda c(s) N \|\mathbf{v}^n(s)\|_\infty ds \\
&\geq \lambda N \left(\min_{a\leq s\leq b} \|\mathbf{v}^n(s)\|_\infty \right) \int_a^b D^k G(t,s) c(s) ds \\
&\geq \lambda N \sigma \|v\|_{C^n[0,1]} \int_a^b \sigma D^k G(h_k(s), s) c(s) ds \\
&= \lambda N \sigma^2 \|v\|_{C^n[0,1]} \int_a^b D^k G(h_k(s), s) c(s) ds
\end{aligned}$$

so that

$$\|Tv\|_{C^n[0,1]} \geq \lambda N \sigma^2 \|v\|_{C^n[0,1]} \max_{0\leq k\leq n} \int_a^b D^k G(h_k(s), s) c(s) ds$$

Then, considering N large enough, so that

$$\lambda \sigma^2 N \max_{0\leq k\leq n} \int_a^b D^k G(h_k(s), s) c(s) ds \geq 1$$

and $v \in \mathcal{C}$ such that $\min_{a\le t\le b}\|\mathbf{v}^n(t)\|_\infty \ge C$, in particular, such that $\|v\|_{C^n[0,1]} \ge \frac{C}{\sigma}$, follows that

$$\|Tv\|_{C^n[0,1]} \ge \lambda\sigma^2 N\|v\|_{C^n[0,1]} \max_{0\le k\le n}\int_a^b D^k G(h_k(s),s)c(s)ds \ge \|v\|_{C^n[0,1]}$$

So, taking either δ small enough, or C large enough if necessary, T satisfies the hypothesis from Krasnosel'skii's theorem in its expansive form for

$$r \le \min\left\{\delta, \frac{1}{(\hat{c}\lambda)^{\frac{1}{p-1}}}\right\}$$

and

$$R \ge \frac{C}{\sigma}, \quad \text{such that } \lambda\sigma^2 N_C \max_{0\le k\le n}\int_a^b D^k G(h_k(s),s)c(s)ds > 1$$

concluding that T has at least one fixed point v on $\mathcal{C}$, which satisfies

$$r \le \|v\|_\infty \le R$$

that is, problem (4.1.1) has at least one non negative solution different from the null function. Finally, since $Tv \in C^m[0,1]$ and v is a fixed point of T, then $v \in C^m[0,1]$.

Moreover, assume by contradiction that $v(t_0) = 0$ for some $t_0 \in (0,1)$. Since $v \in C[0,1]$ is a fix point of T, then

$$0 = v(t_0) = \int_0^1 G(t_0,s)\lambda c(s)f(s,\mathbf{v}^n(s))ds$$

since $G(t_0,s) > 0$ and $c(s)f(s,\mathbf{v}^m(s)) \ge 0$ for a.e. $s \in (0,1)$, the previous identity implies that $c(s)f(s,\mathbf{v}^m(s)) = 0$ for a.e. $s \in (0,1)$, and therefore

$$v(t) = \int_0^1 G(t,s)\lambda c(s)f(s,\mathbf{v}^n(s))ds = 0, \quad \forall\, t \in [0,1]$$

so v is the trivial solution, which is a contradiction, concluding that $v(t) > 0$ for all $t \in (0,1)$.

Now, item 1 will be proven. In this case, $f : [0,1]\times\mathbb{R}^{n+1}$ is lipschitz continuous on its second variable and is such that $f(\cdot,x) \in L^\infty(0,1)$ for all $x \in \mathbb{R}^{n+1}$, and $c \in L^1(0,1)$. In particular, there exists $L > 0$ such that, for a.e. $s \in [0,1]$,

$$|f(s,\mathbf{u}^n(s)) - f(s,\mathbf{v}^n(s))| \le L\|\mathbf{u}^n(s) - \mathbf{v}^n(s)\|_\infty, \quad \forall\, u,v \in C^n[0,1]$$

So that, for all $0 \le k \le n$,

$$|D^k Tu(t) - D^k Tv(t)| \le \left|\int_0^1 |D^k G(t,s)|\lambda c(s)(f(s,\mathbf{u}^n(s)) - f(s,\mathbf{v}^n(s)))ds\right|$$

$$\leq \int_0^1 \lambda |D^k G(h_k(s),s)||c(s)|L\|\mathbf{u}^n(s)-\mathbf{v}^n(s)\|_\infty ds$$
$$\leq \|u-v\|_{C^n[0,1]}\lambda L \int_0^1 |D^k G(h_k(s),s)||c(s)|ds$$

Then, taking maximum over $t \in [0,1]$, follows that

$$\|D^k Tu - D^k Tv\|_\infty \leq \|u-v\|_{C^n[0,1]}\lambda L \int_0^1 |D^k(h_k(s),s)||c(s)|ds$$

so that

$$\|Tu-Tv\|_{C^n[0,1]} \leq \|u-v\|_{C^n[0,1]}\lambda L \max_{0\leq k\leq n}\int_0^1 |D^k G(h_k(s),s)||c(s)|ds$$

In particular, taking

$$\lambda < \frac{1}{L\max_{0\leq k\leq n}\int_0^1 |D^k G(h_k(s),s)||c(s)|ds} \tag{4.1.5}$$

it's obtained that

$$\|Tu-Tv\|_{C^n[0,1]} \leq \bar{L}\|u-v\|_{C^n[0,1]}, \quad \forall\, u,v \in C[0,1]$$

where

$$\bar{L} = \lambda L \max_{0\leq k\leq n}\int_0^1 |D^k G(h_k(s),s)||c(s)|ds < 1$$

So, given that T is continuous and $X = C^n[0.1]$ is a complete metric space, from the Banach fixed point theorem follows that T has a unique fixed point $v \in C^n[0,1]$, that is, problem (4.1.1) has a unique solution for all $\lambda > 0$ that satisfies (4.1.5). Moreover, since $Tv \in C^m[0,1]$ and v is a fixed point of T, then $v \in C^m[0,1]$.

□

Remark 4.1.2. 1. *The previous result is valid for all the three cases studied, since the proof only uses the properties of G.*

2. *From the uniqueness result on the first part of theorem 4.1.1 we have that, if $f(t,0) = 0$ for a.e. $t \in [0,1]$, then problem 4.1.1 has no non trivial solutions. Moreover, since condition **H4** implies that $f(t,0) = 0$ for a.e. $t \in [0,1]$, then 0 is always a solution of problem (4.1.1) on the second part of theorem 4.1.1.*

One can go a bit further than the previous theorem. Indeed, if f only is locally lipschitz on its second variable, and $c \in L^1(0,1)$, then one can apply Banach fixed point theorem over bounded, closed subsets of $C^n[0,1]$ to locally extend the previous result of existence. In fact, define

$$B_R = \{v \in C^n[0,1] \,:\, \|v\|_{C^n[0,1]} \leq R\}$$

Then given $R > 0$, $0 \le k \le n$ and $u, v \in B_R$,

$$\begin{aligned}\|D^kTu - D^kTv\|_\infty &\le \max_{0\le t\le 1}\int_0^1 |D^kG(t,s)|\lambda|c(s)||f(s,\mathbf{u}^n(s)) - f(s,\mathbf{v}^n(s))|ds \\ &\le \int_0^1 \lambda|D^k(h_k(s),s)||c(s)|L(R)\|\mathbf{u}^n(s) - \mathbf{v}^n(s)\|_\infty ds \\ &\le \|u-v\|_{C^n[0,1]}\lambda L(R)\int_0^1 |D^kG(h_k(s),s)||c(s)|ds\end{aligned}$$

Thus, if

$$\lambda < \frac{1}{L(R)\max_{0\le k\le n}\int_0^1 |D^kG(h_k(s),s)||c(s)|ds} \tag{4.1.6}$$

Then the Banach fixed point theorem can be applied to obtain that for all $\lambda > 0$ satisfying (4.1.6) there exists a unique solution $v \in C^m[0,1]$ of (4.1.1) satisfying $\|v\|_{C^n[0,1]} \le R$, where now v not necessarily is non negative, which leads to the following corollary.

Corollary 4.1.1. *Suppose that there exists a measurable function $h : [0,1] \to [0,1]$ such that $|G(t,s)| \le |G(h(s),s)|$ for all $(t,s) \in [0,1]\times[0,1]$. Also suppose that $f : [0,1]\times\mathbb{R}^{n+1} \to \mathbb{R}$ is a locally lipschitz function on its second variable such that $f(\cdot,x) \in L^\infty(0,1)$ for all $x \in \mathbb{R}^{n+1}$, and that $c \in L^1(0,1)$. Let $L(R)$ be the lipschitz constant of f over $[0,1]\times[-R,R]^{n+1}$. Then for all $\lambda > 0$ such that*

$$\lambda < \frac{1}{L(R)\max_{0\le k\le n}\int_0^1 |D^kG(h_k(s),s)||c(s)|ds}$$

there exists a unique solution $v \in C^m[0,1]$ of problem (4.1.1) such that $\|v\|_{C^n[0,1]} \le R$.

Furthermore, thanks to the fixed point method one can obtain the following result.

Theorem 4.1.1. *If G is, either, positive or negative over $(0,1)\times(0,1)$, and if $c(t)f(t,x)$ is, either, nonnegative or nonpositive over $(0,1)\times\mathbb{R}^{m+1}$, then problem (4.1.1) doesn't admit a solution $u \in C(0,1)$ changing sign.*

Proof. If $u \in C(0,1)$ is a solution of problem (4.1.1), then $u(t) = Tu(t)$ for all $t \in (0,1)$. If u changes sign, then there exists $t_0 \in (0,1)$ such that $u(t_0) = 0$, so that

$$Tu(t_0) = \int_0^1 G(t_0,s)\lambda c(s)f(s,\mathbf{u}^m(s))ds = 0$$

since, either, $G(t_0,s) > 0$ or $G(t_0,s) < 0$ for all $s \in (0,1)$, then the previous equality implies that $c(t)f(t,\mathbf{u}^n(t)) = 0$ for a.e $t \in (0,1)$, so in particular u is a solution of $-D^\alpha u = 0$, and from theorem 4.1.1 is obtained that u is the trivial solution, which is a contradiction. ∎

4.2 Asymptotic Behaviour and Multiplicity of Solutions

4.2.1 Multiplicity of solutions

Note that in the proof of theorem 4.1.1 one can apply Krasnosel'skii theorem for all $r, R > 0$ that satisfies

$$r \leq \min\left\{\delta, \frac{1}{(\hat{c}\lambda)^{\frac{1}{p-1}}}\right\} \tag{4.2.1}$$

and

$$R \geq \frac{C}{\sigma}, \quad \text{such that } \lambda\sigma^2 N_C \max_{0\leq k\leq n} \int_a^b D^k G(h_k(s), s)c(s)ds \geq 1 \tag{4.2.2}$$

since

$$\|Tv\|_{C^n[0,1]} \geq \|v\|_{C^n[0,1]}, \quad \forall\, v \in \mathcal{C} \text{ such that } \|v\|_{C^n[0,1]} \geq R \tag{4.2.3}$$

$$\|Tv\|_{C^n[0,1]} \leq \|v\|_{C^n[0,1]}, \quad \forall\, v \in \mathcal{C} \text{ such that } \|v\|_{C^n[0,1]} \leq r \tag{4.2.4}$$

where, at the same time,

$$\hat{c} = \hat{c}(\kappa) = (c_1 + \kappa) \max_{0\leq k\leq n} \int_0^1 |D^k(h_k(s), s)||c(s)|ds, \tag{4.2.5}$$

for $\kappa > 0$ fixed, then $\delta = \delta(\kappa) > 0$ is chosen so that

$$f(t, x) \leq (c_1 + \kappa)\|t\|_\infty^p, \quad \forall\, 0 < \|x\|_\infty \leq \delta \tag{4.2.6}$$

and for $N > 0$, then $C = C(N) > 0$ is chosen so that

$$f(t, x) \geq N\|x\|_\infty, \quad \forall\, \|x\|_\infty \geq C \tag{4.2.7}$$

Based on the above, define the functions $\overline{\delta}, \underline{C}, \underline{N} : (0, \infty) \to (0, \infty)$ as

$$\overline{\delta}(\kappa) = \min\left\{\kappa, \sup\left\{\delta > 0 \,:\, \left|\frac{f(t,x)}{\|x\|_\infty^p} - c_1\right| \leq \kappa, \quad \forall\, 0 < \|x\|_\infty \leq \delta\right\}\right\}$$

$$\underline{C}(N) = \inf\left\{C > 0 \,:\, \frac{f(t,x)}{\|x\|_\infty} \geq N, \quad \forall\, \|x\|_\infty \geq C\right\}$$

and

$$\underline{N}(\lambda) = \frac{1}{\lambda\sigma^2 \max_{0\leq k\leq n} \int_a^b D^k G(h_k(s), s)c(s)ds}$$

At the same time, define the functions $\overline{r} : (0, \infty) \times (0, \infty) \to (0, \infty)$ and $\underline{R} : (0, \infty) \to (0, \infty)$ as

$$\overline{r}(\kappa, \lambda) = \min\left\{\overline{\delta}(\kappa), \frac{1}{(\hat{c}(\kappa)\lambda)^{\frac{1}{p-1}}}\right\}$$

$$\underline{R}(\lambda) = \underline{C}(\underline{N}(\lambda))\sigma^{-1}$$

Onwards, it's assumed that $\kappa, \lambda > 0$.

It's asserted that $\overline{\delta}(\kappa)$ satisfies condition (4.2.6) for all $\kappa > 0$, and that $\underline{C}(N)$ satisfies condition (4.2.7) for all $N > 0$. Indeed, suppose by contradiction that there exists $\kappa > 0$ such that $\overline{\delta}(\kappa)$ doesn't satisfy (4.2.6), then there exists $\delta > 0$ such that

$$\operatorname*{ess\,sup}_{t\in[0,1]} f(t,x) \leq (c_1+\kappa)\|x\|_\infty^p \quad \forall\, 0 < \|x\|_\infty < \delta$$

and $f(t,\bar{x}) > (c_1+\kappa)\delta^p$ for some $\|\bar{x}\|_\infty = \delta$ and t on a set of positive measure. Then, since $f(t,\cdot)$ is continuous for a.e. $t \in [0,1]$, there exists $\mu > 0$ such that $f(t,x) > (c_1+\kappa)\delta^p$ for all $x \in B(\bar{x},\mu)$, which is a contradiction, since $B(\bar{x},\mu)$ contains elements with norm lesser than δ. A similar argument leads to the conclusion that $\underline{C}(N)$ satisfies condition (4.2.7) for all $N > 0$.

Now given that $f(t,\cdot) \in C(\mathbb{R}^{n+1};\mathbb{R})$ for a.e. $t \in [0,1]$ one has that $\frac{f(t,\cdot)}{\|\cdot\|_\infty^p}, \frac{f(t,\cdot)}{\|\cdot\|_\infty} \in C(\mathbb{R}^{n+1} \setminus \{0\})$ for a.e. $t \in [0,1]$. With the latter, by the definition of functions $\overline{\delta}, \underline{C}$ and $\underline{N}(\lambda)$, those verify

$$\underline{N}(\lambda)\lambda\sigma^2 \max_{0\leq k\leq n} \int_a^b D^k G(h_k(s),s)c(s)ds = 1 \tag{4.2.8}$$

$$f(t,x) \leq (c_1+\kappa)\|x\|_\infty^p, \quad \forall\, 0 < \|x\|_\infty \leq \overline{\delta}(\kappa), \text{ a.e. } t \in [0,1]$$

and

$$f(t,x) \geq \underline{N}(\lambda)\|x\|_\infty, \quad \forall\, \|x\|_\infty \geq \overline{C}(\underline{N}(\lambda)), \text{ a.e. } t \in [0,1]$$

so that for each $r \leq \overline{r}(\kappa,\lambda)$ and for all $R \geq \underline{R}(\lambda)$ is verified that

$$\begin{aligned} \|Tv\|_{C^n[0,1]} &\geq \|v\|_{C^n[0,1]}, \quad \forall\, v \in \mathcal{C} \text{ such that } \|v\|_{C^n[0,1]} \geq R \\ \|Tv\|_{C^n[0,1]} &\leq \|v\|_{C^n[0,1]}, \quad \forall\, v \in \mathcal{C} \text{ such that } \|v\|_{C^n[0,1]} \leq r \end{aligned}$$

and the solutions found for the problem (4.1.1) verify, in all the three cases,

$$\min\{r,R\} \leq \|v\|_\infty \leq \max\{r,R\}$$

Note that, since

$$\begin{aligned} \|Tv\|_{C^n[0,1]} &\geq \|v\|_{C^n[0,1]}, \quad \forall\, v \in \mathcal{C} \text{ such that } \|v\|_{C^n[0,1]} \geq \underline{R}(\lambda) \\ \|Tv\|_{C^n[0,1]} &\leq \|v\|_{C^n[0,1]}, \quad \forall\, v \in \mathcal{C} \text{ tal que } \|v\|_{C^n[0,1]} \leq \overline{r}(\kappa,\lambda) \end{aligned}$$

one has that if there exist $\lambda, \kappa > 0$ such that $\underline{R}(\lambda) < \overline{r}(\kappa,\lambda)$, then for all $r, R > 0$ such that $\underline{R}(\lambda) \leq R$, $r \leq \overline{r}(\kappa,\lambda)$ and $r < R$, is possible to apply the Krasnosel'skii theorem, since in this case $\|Tv\|_{C^n[0,1]} = \|v\|_{C^n[0,1]}$ for all $\underline{R}(\lambda) \leq \|v\|_{C^n[0,1]} \leq \overline{r}(\kappa, \lambda)$, to get a solution $v_{r,R} \in C^m[0,1]$ such that

$$r \leq \|v_{r,R}\|_{C^n[0,1]} \leq R$$

Furthermore, since r and R are arbitrary, one can take $t \to R$ to obtain a solution $v_R \in C^m[0,1]$ such that

$$\|v_R\|_{C^n[0,1]} = R$$

so, in fact, there is a family of solutions $\{v_r\}_{r \in [\underline{R}(\lambda), \overline{r}(\kappa,\lambda)]}$ satisfying $\|v_r\|_{C^n[0,1]} = r$ for each $r \in [\underline{R}(\lambda), \overline{r}(\kappa, \lambda)]$.

Then, the purpose of this section is to find such λ and κ. To do that, first note that $\underline{R}(\lambda) < \overline{r}(\kappa, \lambda)$ if and only if

$$\sigma^{-1} \inf \left\{ C > 0 \, : \, \frac{f(t,x)}{\|x\|_\infty} \geq \frac{1}{\lambda \sigma^2 M}, \quad \forall \|x\|_\infty \geq C \right\} < \min \left\{ \overline{\delta}(\kappa), \frac{1}{(\hat{c}(\kappa)\lambda)^{\frac{1}{p-1}}} \right\} \tag{4.2.9}$$

where

$$M = \max_{0 \leq k \leq n} \int_a^b D^k G(h_k(s), s) c(s) ds$$

Note that, if there exists $\epsilon > 0$ such that

$$\frac{f(t,x)}{\|x\|_\infty} \geq \frac{1}{\lambda \sigma^2 M}, \quad \forall \, \|x\|_\infty \geq \sigma \min \left\{ \overline{\delta}(\kappa), \frac{1}{(\hat{c}(\kappa)\lambda)^{\frac{1}{p-1}}} \right\} - \epsilon \tag{4.2.10}$$

then condition (4.2.9) is satisfied. Now let's define

$$\lambda_1(\kappa) = \frac{1}{\hat{c}(\kappa) A^{p-1} \overline{\delta}(\kappa)^{p-1}}$$

where $A > 0$ is a constant to fix later. Then, for all $\lambda \leq \lambda_1(\kappa)$

$$\frac{1}{(\hat{c}(\kappa)\lambda)^{\frac{1}{p-1}}} \geq \frac{1}{(\hat{c}(\kappa)\lambda_1(\kappa))^{\frac{1}{p-1}}} \geq A \overline{\delta}(\kappa)$$

so that

$$\min \left\{ \overline{\delta}(\kappa), \frac{1}{(\hat{c}(\kappa)\lambda)^{\frac{1}{p-1}}} \right\} \geq \overline{\delta}(\kappa) \min\{1, A\}, \quad \forall \, \lambda \leq \lambda_1(\kappa)$$

By other side, given $\epsilon = \epsilon(\kappa) > 0$ such that $\epsilon(\kappa) < \sigma \overline{\delta}(\kappa)$, define

$$\lambda_2(\kappa) = \inf \left\{ \lambda > 0 \, : \, \frac{f(t,x)}{\|x\|_\infty} \geq \frac{B}{\lambda \sigma^2 M}, \quad \forall \, \|x\|_\infty \geq \sigma \overline{\delta}(\kappa) - \epsilon(\kappa) \right\}$$

where $B > 0$ is a constant to fix later. Then, by definition, for all $\lambda \geq \lambda_2(\kappa)$

$$\frac{f(t,x)}{t} \geq \frac{B}{\lambda \sigma^2 M}, \quad \forall \|x\|_\infty \geq \sigma \overline{\delta}(\kappa) - \epsilon(\kappa) > 0$$

Now, note that

$$0 \leq \lim_{\kappa \to 0^+} \overline{\delta}(\kappa) \leq \lim_{\kappa \to 0^+} \kappa = 0$$

so that $\overline{\delta}(0^+) = 0$. On the other side, since

$$\lim_{\kappa \to \infty} \sup \left\{ \delta > 0 \,:\, \left| \frac{f(t,x)}{\|x\|_\infty^p} - c_1 \right| \leq \kappa, \quad \forall\, 0 < \|x\|_\infty \leq \delta \right\} = \infty$$

then

$$\lim_{\kappa \to \infty} \overline{\delta}(\kappa) = \lim_{\kappa \to \infty} \min \left\{ \kappa, \sup \left\{ \delta > 0 \,:\, \left| \frac{f(t,x)}{\|x\|_\infty^p} - c_1 \right| \leq \kappa, \quad \forall\, 0 < \|x\|_\infty \leq \delta \right\} \right\} = \infty$$

so, since $\hat{c}$ is continuous over $\kappa > 0$ and $\hat{c}(0^+) = c_1 K > 0$, and $p > 1$, then

$$\lim_{\kappa \to 0^+} \lambda_1(\kappa) = \lim_{\kappa \to 0^+} \frac{1}{\hat{c}(\kappa) A^{p-1} \overline{\delta}(\kappa)^{p-1}} = \infty$$
$$\lim_{\kappa \to \infty} \lambda_1(\kappa) = 0$$

Now, note that, since

$$\lim_{\|x\|_\infty \to 0^+} \frac{f(t,x)}{\|x\|_\infty^p} = c_1$$

then

$$\lim_{\|x\|_\infty \to 0^+} \frac{f(t,x)}{\|x\|_\infty} = \lim_{\|x\|_\infty \to 0^+} \frac{f(t,x)}{\|x\|_\infty^p} \|x\|_\infty^{p-1} = 0$$

and since $\overline{\delta}(\kappa) \to 0^+$ when $\kappa \to 0^+$, and since $0 < \epsilon(\kappa) < \sigma \overline{\delta}(\kappa)$, so does $\epsilon(\kappa)$, then

$$\lim_{\kappa \to 0^+} \lambda_2(\kappa) = \lim_{\kappa \to 0^+} \inf \left\{ \lambda > 0 \,:\, \frac{f(t,x)}{\|x\|_\infty} \geq \frac{B}{\lambda \sigma^2 M}, \quad \forall \|x\|_\infty \geq \sigma \overline{\delta}(\kappa) - \epsilon(\kappa) \right\} = \infty$$

In a similar way, since

$$\lim_{\|x\|_\infty \to \infty} \frac{f(t,x)}{\|x\|_\infty} = \infty$$

and $\overline{\delta}(\kappa) \to \infty$ when $\kappa \to \infty$, then choosing $\epsilon(\kappa)$ such that there exists a constant $0 < D < 1$ such that

$$\epsilon(\kappa) \leq \sigma D \overline{\delta}(\kappa) \tag{4.2.11}$$

then

$$\lim_{\kappa\to\infty} \sigma\overline{\delta}(\kappa) - \epsilon(\kappa) \geq \lim_{\kappa\to\infty} \sigma(1-D)\overline{\delta}(\kappa) = \infty$$

which leads to

$$\lim_{\kappa\to\infty} \lambda_2(\kappa) = \lim_{\kappa\to\infty} \inf\left\{\lambda > 0 \,:\, \frac{f(t,x)}{\|x\|_\infty} \geq \frac{B}{\lambda\sigma^2 M}, \quad \forall\, \|x\|_\infty \geq \sigma\overline{\delta}(\kappa) - \epsilon(\kappa)\right\} = 0$$

Now, note that

$$\begin{aligned}
\frac{\lambda_2(\kappa)}{\lambda_1(\kappa)} &= \hat{c}(\kappa)A^{p-1}\overline{\delta}(\kappa)^{p-1} \inf\left\{\lambda > 0 \,:\, \frac{f(t,x)}{\|x\|_\infty} \geq \frac{B}{\lambda\sigma^2 M}, \quad \forall\, \|x\|_\infty \geq \sigma\overline{\delta}(\kappa) - \epsilon(\kappa)\right\} \\
&= \inf\left\{\lambda\hat{c}(\kappa)A^{p-1}\overline{\delta}(\kappa)^{p-1} > 0 \,:\, \frac{f(t,x)}{\|x\|_\infty} \geq \frac{B}{\lambda\sigma^2 M}, \quad \forall\, \|x\|_\infty \geq \sigma\overline{\delta}(\kappa) - \epsilon(\kappa)\right\} \\
&= \inf\left\{\lambda > 0 \,:\, \frac{f(t,x)}{\|x\|_\infty} \geq \frac{\hat{c}(\kappa)A^{p-1}\overline{\delta}(\kappa)^{p-1}B}{\lambda\sigma^2 M}, \quad \forall\, \|x\|_\infty \geq \sigma\overline{\delta}(\kappa) - \epsilon(\kappa)\right\}
\end{aligned}$$

Since

$$\overline{\delta}(\kappa) \leq \sup\left\{\delta > 0 \,:\, \left|\frac{f(t,x)}{\|x\|_\infty^p} - c_1\right| \leq \kappa, \quad \forall 0 < \|x\|_\infty \leq \delta\right\}$$

then, in particular,

$$\frac{f(t,x)}{\|x\|_\infty^p} \leq c_1 + \kappa, \quad \forall \sigma\overline{\delta}(\kappa) - \epsilon(\kappa) \leq \|x\|_\infty \leq \overline{\delta}(\kappa)$$

and then

$$\frac{f(t,x)}{\|x\|_\infty} \leq (c_1+\kappa)\|x\|_\infty^{p-1} \leq (c_1+\kappa)\overline{\delta}(\kappa)^{p-1}, \quad \forall \sigma\overline{\delta}(\kappa) - \epsilon(\kappa) \leq \|x\|_\infty \leq \overline{\delta}(\kappa)$$

so that inequality

$$\frac{f(t,x)}{\|x\|_\infty} \geq \frac{\hat{c}(\kappa)A^{p-1}\overline{\delta}(\kappa)^{p-1}B}{\lambda\sigma^2 M}, \quad \forall\, \|x\|_\infty \geq \sigma\overline{\delta}(\kappa) - \epsilon(\kappa)$$

implies that

$$\frac{\hat{c}(\kappa)\overline{\delta}(\kappa)^{p-1}A^{p-1}B}{\lambda\sigma^2 M} \leq (c_1+\kappa)\overline{\delta}(\kappa)^{p-1}$$

so that

$$\lambda \geq \frac{\hat{c}(\kappa)A^{p-1}B}{(c_1+\kappa)\sigma^2 M}$$

Now recall that

$$\hat{c}(\kappa) = (c_1 + \kappa)K$$

and thus

$$\lambda \geq \frac{KA^{p-1}B}{\sigma^2 M}$$

Therefore, taking $A, B > 0$ such that

$$0 < A^{p-1}B < \frac{\sigma^2 M}{2K} \tag{4.2.12}$$

then

$$\lambda \geq \frac{1}{2}$$

which implies that

$$\frac{\lambda_2(\kappa)}{\lambda_1(\kappa)} \leq \frac{1}{2}, \quad \forall \kappa > 0 \tag{4.2.13}$$

or equivalently,

$$\lambda_2(\kappa) \leq \frac{\lambda_1(\kappa)}{2} < \lambda_1(\kappa), \quad \forall \kappa > 0$$

Therefore, since for all $\lambda \geq \lambda_2(\kappa)$,

$$\frac{f(t)}{t} \geq \frac{B}{\lambda\sigma^2 M}, \quad \forall \|x\|_\infty \geq \sigma\bar{\delta}(\kappa) - \epsilon(\kappa) > 0$$

Then, for all $\lambda_2(\kappa) \leq \lambda$,

$$\inf\left\{C > 0 : \frac{f(t)}{t} \geq \frac{B}{\lambda\sigma^2 M}, \quad \forall \|x\|_\infty \geq C\right\} \leq \sigma\bar{\delta}(\kappa) - \epsilon(\kappa)$$

thus, taking $B = 1$ and

$$A < \left(\frac{\sigma^2 M}{2K}\right)^{\frac{1}{p-1}} < \left(\frac{1}{2}\right)^{\frac{1}{p-1}}$$

so that condition (4.2.12) is satisfied, and taking

$$\sigma\bar{\delta}(\kappa) > \epsilon(\kappa) = \frac{\sigma}{2}(2 - A)\bar{\delta}(\kappa) > \sigma(1 - A)\bar{\delta}(\kappa)$$

then the condition (4.2.11) is satisfied, and inequality (4.2.13) holds. Moreover, one has that

$$\sigma\bar{\delta}(\kappa) - \epsilon(\kappa) < \sigma\bar{\delta}(\kappa) - \sigma(1 - A)\bar{\delta}(\kappa)$$

$$
\begin{aligned}
&= \sigma A\overline{\delta}(\kappa) \\
&= \sigma\overline{\delta}(\kappa)\min\{1, A\}
\end{aligned}
$$

and, since for all $\lambda \leq \lambda_1(\kappa)$,

$$
\min\left\{\overline{\delta}(\kappa), \frac{1}{(\hat{c}(\kappa)\lambda)^{\frac{1}{p-1}}}\right\} \geq \overline{\delta}(\kappa)\min\{1, A\}, \quad \forall\,\lambda \leq \lambda_1(\kappa)
$$

then $\lambda_2(\kappa) < \lambda_1(\kappa)$ for all $\kappa > 0$. Moreover, since $\lambda_1(0,\infty) = (0,\infty)$, then there exists $\kappa > 0$ such that $\lambda_1(\kappa) = \frac{4}{3}\lambda$, and since $\lambda_2(\kappa) \leq \frac{1}{2}\lambda_1(\kappa) = \frac{2}{3}\lambda$, then for all $\lambda > 0$ there exists $\kappa > 0$ such that $\lambda_2(\kappa) < \lambda < \lambda_1(\kappa)$, in which case follows that

$$
\begin{aligned}
\inf\left\{C > 0 : \frac{f(t)}{t} \geq \frac{B}{\lambda\sigma^2 M}, \quad \forall\,\|x\|_\infty \geq C\right\} &< \sigma\overline{\delta}(\kappa)\min\{1, A\} \\
&< \overline{\delta}(\kappa)\min\{1, A\} \\
&\leq \min\left\{\overline{\delta}(\kappa), \frac{1}{(\hat{c}(\kappa)\lambda)^{\frac{1}{p-1}}}\right\}
\end{aligned}
$$

so (4.2.9) holds. Now recall that

$$
\begin{aligned}
\lim_{\kappa\to 0^+}\lambda_1(\kappa) &= \lim_{\kappa\to 0^+}\lambda_2(\kappa) = \infty \\
\lim_{\kappa\to\infty}\lambda_1(\kappa) &= \lim_{\kappa\to\infty}\lambda_2(\kappa) = 0
\end{aligned}
$$

then for all $\lambda > 0$ there exists $\kappa_\lambda > 0$ such that (4.2.9) is satisfied. with that,

$$
\underline{R}(\lambda) < \overline{r}(\kappa_\lambda, \lambda), \quad \forall\,\lambda > 0
$$

summarizing all that, follows the following result.

Main Theorem 4.2.1. *for all $\lambda > 0$, there exists $0 < r_\lambda < R_\lambda$ such that problem (4.1.1) admits a family of solutions $\{v_r\}_{r\in[r_\lambda,R_\lambda]} \subset C^m[0,1]$ satisfying that $\|v_r\|_{C^n[0,1]} = r$. Moreover, those solutions also satisfy that*

$$
\lim_{\lambda\to\infty}\|v_{r_\lambda}\|_{C^n[0,1]} = 0
$$

4.2.2 Asymptotic Behaviour

Now note that

$$
\lim_{\lambda\to\infty}\underline{N}(\lambda) = \lim_{\lambda\to\infty}\frac{1}{\lambda\sigma^2\max_{0\leq k\leq n}\int_a^b D^k G(h_k(s), s)c(s)ds} = 0
$$

On the other hand, given that f is non negative over $[0,1]\times[0,\infty)\times\mathbb{R}^n$, in particular for a.e. $t \in [0,1]$

$$
\frac{f(t,x)}{\|x\|_\infty} \geq 0, \forall\, x \in [0,\infty)\times\mathbb{R}^n
$$

so that

$$\lim_{N\to 0^+} \underline{C}(N) = \lim_{N\to 0^+} \inf \left\{ C > 0 \,:\, \frac{f(t,x)}{\|x\|_\infty} \geq N, \quad \forall \, \|x\|_\infty \geq C \right\} = 0$$

and therefore

$$\lim_{\lambda\to\infty} \underline{R}(\lambda) = \lim_{\lambda\to\infty} \underline{C}(\underline{N}(\lambda))\sigma^{-1} = 0$$

Now recall that on the proof of theorem 4.2.1, one has that $\lambda_2(\kappa) < \lambda < \lambda_1(\kappa)$, and since λ_1 is non increasing, then $\lambda \to \infty$ implies that $\lambda_1 \to \infty$, which also implies that $\kappa \to 0^+$. Since $\kappa \to 0^+$ implies that $\lambda_2(\kappa) \to \infty$, one concludes that the conditions $\kappa \to 0^+$ and $\lambda \to \infty$ are equivalent.

By other way, by definition,

$$\overline{\delta}(\kappa) \leq \kappa, \quad \forall \, \kappa > 0$$

so that

$$\overline{r}(\kappa, \lambda) \leq \overline{\delta}(\kappa) \leq \kappa$$

In this way,

$$0 \leq \lim_{\lambda\to\infty} \overline{r}(\kappa,\lambda) = \lim_{\kappa\to 0^+} \overline{r}(\kappa,\lambda) \leq \lim_{\kappa\to 0^+} \kappa = 0, \quad \forall \, \kappa > 0$$

So, follows that

$$0 = \lim_{\lambda\to\infty} \min\{\overline{r}(\kappa,\lambda), \underline{R}(\lambda)\} \leq \lim_{\lambda\to\infty} \|v\|_{C^n[0,1]} \leq \lim_{\lambda\to\infty} \max\{\overline{r}(\kappa,\lambda), \underline{R}(\lambda)\} = 0$$

concluding that the solutions of the problem (4.1.1) satisfy

$$\lim_{\lambda\to\infty} \|v\|_{C^n[0,1]} = 0$$

concluding that $v \to 0$ in $C^n[0,1]$ when $\lambda \to \infty$.

4.3 Examples of applications

On this section, some examples are given of applications of theorem 4.1.1.

First, an example on the Riemann-Liouville case is shown.

Example 4.3.1. Let $\alpha > 2$ and $n \in \mathbb{N}$ be such that $n \leq [\alpha] - 2$, and let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous, locally lipschitz on its second variable, non negative function over $[0,1] \times [0,\infty)$ and such that it verifies **H3** and **H4**. Let $c \in C(0,1) \cap L^1(0,1)$ be a non negative function. Consider the problem

$$\begin{cases} -D^\alpha u(t) = \lambda c(t) f(t,u(t)) & 0 < t < 1 \\ D^k u(0) = 0, & \forall \, k \in \{0, \dots, [\alpha]-2\} \\ D^n u(1) = 0 \end{cases} \tag{4.3.1}$$

where $n \in \mathbb{N}$ is such that $n \leq [\alpha] - 2$. Then for all $\lambda > 0$ the problem (4.3.1) has at least one positive solution on $C^{[\alpha]-2}[0,1]$.

Proof. Given that $D^k u(0) = 0$ for all $0 \leq k \leq [\alpha] - 2$, it follows that the Green function of problem (4.3.1) has the form

$$G(t,s) = t^{\alpha-1}\gamma(1-s)^{\alpha-1-n}\prod_{j=0}^{n-1}(\alpha-1-j) - \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}\mu(t-s)$$

By the other side, as

$$\begin{aligned}&D^n G(1,s)\\ &= t^{\alpha-1-n}\gamma(1-s)^{\alpha-1-n}\left(\prod_{j=0}^{n-1}(\alpha-1-j)\right)^2 - \frac{\prod_{j=0}^{n-1}(\alpha-1-j)}{\Gamma(\alpha)}(t-s)^{\alpha-1-n}\mu(t-s)\end{aligned}$$

given that $D^n v(1) = 0$, it follows that

$$D^n G(1,s) = \left(\prod_{j=0}^{n-1}(\alpha-1-j)\right)(1-s)^{\alpha-1-n}\left[\gamma\left(\prod_{j=0}^{n-1}(\alpha-1-j)\right) - \frac{1}{\Gamma(\alpha)}\right]$$

so that

$$\gamma = \frac{1}{\Gamma(\alpha)\left(\prod_{j=0}^{n-1}(\alpha-1-j)\right)}$$

and therefore

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\left[t^{\alpha-1}(1-s)^{\alpha-n-1} - (t-s)^{\alpha-1}\mu(t-s)\right]$$

Next, it's proven that G satisfies conditions **G1** and **G2** to then apply theorem (4.1.1).

1. It's clear that $G(t,s) > 0$ for all $0 < t \leq s < 1$. On the other hand, if $0 < s \leq t < 1$, then

$$\begin{aligned}G(t,s) &= \frac{1}{\Gamma(\alpha)}\left[t^{\alpha-1}(1-s)^{\alpha-n-1} - (t-s)^{\alpha-1}\right]\\ &= \frac{1}{\Gamma(\alpha)}\left[t^{n}(t-ts)^{\alpha-n-1} - (t-s)^{\alpha-1}\right]\\ &> \frac{1}{\Gamma(\alpha)}\left[t^{n}(t-s)^{\alpha-n-1} - (t-s)^{\alpha-1}\right]\\ &= \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-n-1}\left[t^n - (t-s)^n\right]\\ &\geq 0\end{aligned}$$

   concluding that G satisfies property **G1**.

2. Note that $G(t,s)$ is differentiable for $t \in (0,1)$, where

$$G'(t,s) = \frac{\alpha-1}{\Gamma(\alpha)}\left[t^{\alpha-2}(1-s)^{\alpha-n-1} - (t-s)^{\alpha-2}\mu(t-s)\right]$$

Also note that $G'(t,s) > 0$ for all $0 < t \leq s < 1$. By the other side, given that $\alpha > n+1 \geq 2$ and $\alpha > 2$, follows that if $0 < s \leq t < 1$, then

$$\begin{aligned}
G'(t,s) &= \frac{\alpha-1}{\Gamma(\alpha)}\left[t^{\alpha-2}(1-s)^{\alpha-n-1} - (t-s)^{\alpha-2}\right] \\
&= \frac{\alpha-1}{\Gamma(\alpha)}\left[t^{n-1}(t-ts)^{\alpha-n-1} - (t-s)^{\alpha-2}\right] \\
&> \frac{\alpha-1}{\Gamma(\alpha)}\left[t^{n-1}(t-s)^{\alpha-n-1} - (t-s)^{\alpha-2}\right] \\
&= \frac{\alpha-1}{\Gamma(\alpha)}(t-s)^{\alpha-n-1}\left[t^{n-1} - (t-s)^{n-1}\right] \\
&\geq 0
\end{aligned}$$

concluding that for all $s \in (0,1)$, G is strictly increasing over $t \in (0,1)$, so that

$$\max_{0\leq t\leq 1} G(t,s) = G(1,s), \quad \forall\, s \in (0,1)$$

Furthermore,

$$\begin{aligned}
&G(t,0) = \frac{1}{\Gamma(\alpha)}\left[t^{\alpha-1} - t^{\alpha-1}\right] = 0, \quad \forall\, t \in [0,1] \\
&G(t,1) = 0, \quad \forall t \in [0,1]
\end{aligned}$$

so that

$$\max_{0\leq t\leq 1} G(t,s) = G(1,s), \quad \forall\, s \in [0,1]$$

3. From the latter, given $a \in (0,1)$, if $a \leq t \leq s \leq 1$, then

$$\begin{aligned}
\frac{G(t,s)}{G(1,s)} &\geq \frac{G(a,s)}{G(1,s)} \\
&= \frac{a^{\alpha-1}(1-s)^{\alpha-n-1}}{(1-s)^{\alpha-n-1}[1-(1-s)^n]} \\
&= \frac{a^{\alpha-1}}{1-(1-s)^n} \\
&\geq a^{\alpha-1}
\end{aligned}$$

On the other hand, if $0 \leq s \leq t \leq 1$, with $a \leq t$, then

$$\frac{G(t,s)}{G(1,s)} = \frac{t^{\alpha-1}(1-s)^{\alpha-n-1} - (t-s)^{\alpha-1}}{(1-s)^{\alpha-n-1}[1-(1-s)^n]}$$

Given that $G(1,s) > 0$ for all $s \in (0,1)$ and, in the same way, $G(t,s) > 0$ for all $(t,s) \in [a,1] \times (0,1)$, follows that

$$\frac{G(t,s)}{G(1,s)} > 0, \quad \forall\, (t,s) \in [a,1] \times (0,1)$$

Moreover, note that, given that $t \geq s$ and G is increasing in t, if $s \geq a$ it's obtained that

$$\frac{G(t,s)}{G(1,s)} \geq \frac{G(s,s)}{G(1,s)} = \frac{s^{\alpha-1}(1-s)^{\alpha-n-1}}{(1-s)^{\alpha-n-1}[1-(1-s)^n]} = \frac{s^{\alpha-1}}{1-(1-s)^n}$$

In particular, taking $s \to 1^-$, follows that

$$\lim_{s\to 1^-} \frac{G(t,s)}{G(1,s)} \geq \lim_{s\to 1^-} \frac{s^{\alpha-1}}{1-(1-s)^n} = 1$$

Finally,

$$\begin{aligned}
&\lim_{s\to 0^+} \frac{G(t,s)}{G(1,s)} \\
&= \lim_{s\to 0^+} \frac{t^{\alpha-1}(1-s)^{\alpha-n-1} - (t-s)^{\alpha-1}}{(1-s)^{\alpha-n-1}[1-(1-s)^n]} \\
&= \lim_{s\to 0^+} \frac{(\alpha-1)(t-s)^{\alpha-2} - (\alpha-n-1)t^{\alpha-1}(1-s)^{\alpha-n-2}}{n(1-s)^{\alpha-n-1}(1-s)^{n-1} - (\alpha-n-1)(1-s)^{\alpha-n-2}[1-(1-s)^n]} \\
&= \frac{t^{\alpha-2}[(\alpha-1)-(\alpha-n-1)t]}{n} \\
&\geq a^{\alpha-1}
\end{aligned}$$

From all the above, follows that there exists $\sigma = \sigma(a,\alpha) > 0$ such that

$$G(t,s) \geq \sigma G(1,s), \quad \forall\,(t,s) \in [a,1]\times[0,1]$$

so G verifies **G2**.

In that way, from theorem 4.1.1 one gets the existence of positive solution. □

Therefore, proving the existence of solutions for a given fractional problem can be reduced to prove properties of the Green function associated with it.

Next, another example is given, this time for the concatenated case.

Corollary 4.3.1. *Let $\beta > 1$ and $n \in \mathbb{N}$ be such that $n \leq [\beta]-1$. Then for all $\lambda > 0$ the problem*

$$\begin{cases} D^\beta Dv(t) = \lambda c(t) f(v(t)), & 0 < t < 1 \\ D^k v(0) = 0, & \forall\, k \in \{0,\dots,[\beta]-2\} \\ D^n v(1) = 0 & \end{cases} \tag{4.3.2}$$

has at least one positive solution $v \in C[0,1]$.

Proof. Note that, in this case,

$$Q(t,s) = \int_0 t(t-\tau)^{\beta-1}\mu(\tau-s)d\tau = \frac{1}{\beta}(t-s)^\beta$$

meanwhile

$$J^1(t^{k+\beta-[\beta]}) = \frac{\Gamma(k+\beta-[\beta]+1)}{\Gamma(k+\beta-[\beta]+2)} t^{k+\beta-[\beta]+1}, \quad \forall\, k \in \mathbb{N}_0$$

Thus, the Green function of the problem (4.3.2) has the form

$$G(t,s) = \sum_{k=0}^{[\beta]-1} \frac{\Gamma(k+\beta-[\beta]+1)}{\Gamma(k+\beta-[\beta]+2)} t^{k+\beta-[\beta]+1} \left[\delta_k \frac{(1-s)^{\beta-n}}{\beta\Gamma(\beta)} \prod_{j=0}^{n-1}(\beta-j)\right] - \frac{1}{\beta\Gamma(\beta)}(t-s)^\beta \mu(t-s)$$

Moreover, from conditions $D^k v(0) = 0$, $k \in \{0,\ldots,[\beta]-2\}$ then

$$G(t,s) = \frac{\Gamma(\beta)}{\Gamma(\beta+1)} t^\beta \left[\delta \frac{(1-s)^{\beta-n}}{\beta\Gamma(\beta)} \prod_{j=0}^{n-1}(\beta-j)\right] - \frac{1}{\beta\Gamma(\beta)}(t-s)^\beta \mu(t-s)$$

Finally, from condition $D^k v(1) = 0$, $k \in \{1,\ldots,[\beta]-1\}$ one has that

$$G(t,s) = \frac{1}{\beta\Gamma(\beta)} \left[t^\beta(1-s)^{\beta-n} - (t-s)^\beta \mu(t-s)\right]$$

Now, it's asserted that G satisfies **G1** and **G2**.

1. First note that $G(t,s) > 0$ for all $0 < t \leq s < 1$. By other way, if $0 < s \leq t < 1$, then

$$\begin{aligned} G(t,s) &= \frac{1}{\beta\Gamma(\beta)} \left[t^\beta(1-s)^{\beta-n} - (t-s)^\beta\right] \\ &= \frac{1}{\beta\Gamma(\beta)} \left[t^n(t-ts)^{\beta-n} - (t-s)^\beta \mu(t-s)\right] \\ &> \frac{1}{\beta\Gamma(\beta)} \left[t^n(t-s)^{\beta-n} - (t-s)^\beta \mu(t-s)\right] \\ &= \frac{1}{\beta\Gamma(\beta)}(t-s)^{\beta-n} \left[t^n - (t-s)^n \mu(t-s)\right] \\ &\geq 0 \end{aligned}$$

   therefore, $G(t,s) > 0$ for all $(t,s) \in (0,1) \times (0,1)$.

2. Note that G is differentiable for all $t \in (0,1)$, with

$$G'(t,s) = \frac{1}{\Gamma(\beta)} \left[t^{\beta-1}(1-s)^{\beta-n} - (t-s)^{\beta-1}\mu(t-s)\right]$$

   It's immediate that $G'(t,s) > 0$ for all $0 < t \leq s < 1$. On the other hand, if $0 < s \leq t < 1$, given that $\beta > n \geq 1$, then

$$G'(t,s) = \frac{1}{\Gamma(\beta)} \left[t^{\beta-1}(1-s)^{\beta-n} - (t-s)^{\beta-1}\right]$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\beta)} \left[t^{n-1}(t-ts)^{\beta-n} - (t-s)^{\beta-1}\right] \\
&> \frac{1}{\Gamma(\beta)} \left[t^{n-1}(t-s)^{\beta-n} - (t-s)^{\beta-1}\right] \\
&= \frac{1}{\Gamma(\beta)}(t-s)^{\beta-n} \left[t^{n-1} - (t-s)^{n-1}\right] \\
&\geq 0
\end{aligned}
$$

so that for each $s \in (0,1)$ one gets that

$$\max_{0 \leq t \leq 1} G(t,s) = G(1,s)$$

Furthermore,

$$
\begin{aligned}
&G(t,0) = \frac{1}{\beta\Gamma(\beta)} \left[t^{\beta-1} - t^{\beta-1}\right] = 0, \quad \forall\, t \in [0,1] \\
&G(t,1) = 0, \quad \forall\, t \in [0,1]
\end{aligned}
$$

so that

$$\max_{0 \leq t \leq 1} G(t,s) = G(1,s), \quad \forall\, s \in [0,1]$$

3. Let $a \in (0,1)$. For $a \leq t \leq s \leq 1$ one has that

$$
\begin{aligned}
\frac{G(t,s)}{G(1,s)} &\geq \frac{G(a,s)}{G(1,s)} \\
&= \frac{a^\beta(1-s)^{\beta-n}}{(1-s)^{\beta-n}[1-(1-s)^n]} \\
&= \frac{a^\beta}{1-(1-s)^n} \\
&\geq a^\beta > 0
\end{aligned}
$$

By other way, if $0 \leq s \leq t \leq 1$, with $a \leq t$, given that $G(t,s) > 0$ for all $(t,s) \in [a,1] \times (0,1)$, follows that

$$\frac{G(t,s)}{G(1,s)} > 0, \quad \forall\, (t,s) \in [a,1] \times (0,1)$$

Now note that, given that G is increasing for $t \in [0,1]$, for $t \geq s$ one gets that

$$\frac{G(t,s)}{G(1,s)} \geq \frac{G(s,s)}{G(1,s)} = \frac{s^\beta(1-s)^{\beta-n}}{(1-s)^{\beta-n}[1-(1-s)^n]} = \frac{s^\beta}{1-(1-s)^n}$$

So,

$$\lim_{s \to 1^-} \frac{G(t,s)}{G(1,s)} \geq \lim_{s \to 1^-} \frac{s^\beta}{1-(1-s)^n} = 1$$

Finally, for each $a \leq t \leq 1$

$$\begin{aligned}
\lim_{s\to 0^+} \frac{G(t,s)}{G(1,s)} &= \lim_{s\to 0^+} \frac{t^\beta (1-s)^{\beta-n} - (t-s)^\beta}{(1-s)^{\beta-n}[1-(1-s)^n]} \\
&= \lim_{s\to 0^+} \frac{\beta(t-s)^{\beta-1} - (\beta-n)t^\beta(1-s)^{\beta-n-1}}{n(1-s)^{\beta-n}(1-s)^{n-1} - (\beta-n)(1-s)^{\beta-n-1}[1-(1-s)^n]} \\
&= \frac{t^{\beta-1}(\beta-(\beta-n)t)}{n} \\
&\geq a^{\beta-1}
\end{aligned}$$

concluding that there exists $\sigma = \sigma(a,\alpha) > 0$ such that

$$G(t,s) \geq \sigma G(1,s), \quad \forall\, (t,s) \in [a,1]\times[0,1]$$

From all the above, from theorem 4.1.1 follows that the problem (4.3.2) has at least one positive solution $v \in C[0,1]$.

□

Finally, an example for the Caputo case is given.

Example 4.3.2. Let $\alpha > 2$ and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, locally lipschitz, non negative function over $[0,\infty)$ and such that it verifies **H3** and **H4**. Let $c \in C(0,1) \cap L^1(0,1)$ be a non negative function. Consider the problem

$$\begin{cases} -{}^C D^\alpha u(t) = \lambda c(t) f(u(t)) & 0 < t < 1 \\ D^k u(0) = 0, & 0 \leq k \leq [\alpha]-1,\ k \neq 1 \\ u'(1) = 0 \end{cases} \tag{4.3.3}$$

Then for all $\lambda > 0$ the problem (4.3.3) has at least one positive solution.

Proof. Given that $D^k u(0) = 0$ for all $k > 1$, it follows that the Green function of problem (4.3.1) has the form

$$G(t,s) = a_0(s) + a_1(s)t - \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}\mu(t-s)$$

and since $G(0,s) = 0$, follows that $a_0 \equiv 0$, so that

$$G(t,s) = a_1(s)t - \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}\mu(t-s)$$

Also, given that $DG(1,s) = 0$, it follows that

$$a_1(s) = \frac{\alpha-1}{\Gamma(\alpha)}(1-s)^{\alpha-2}$$

so that

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\left[(\alpha-1)(1-s)^{\alpha-2}t - (t-s)^{\alpha-1}\mu(t-s)\right]$$

Next, the properties of G are verified to then apply theorem (4.1.1).

1. It's clear that $G(t,s) > 0$ for all $0 < t \leq s < 1$. On the other hand, if $0 < s \leq t < 1$, then

$$\begin{aligned}
G(t,s) &= \frac{1}{\Gamma(\alpha)}\left[(\alpha-1)(1-s)^{\alpha-2}t - (t-s)^{\alpha-1}\right] \\
&= \frac{1}{\Gamma(\alpha)}\left[(\alpha-1)(1-s)^{\alpha-2}t - (t-s)^{\alpha-2}(t-s)\right] \\
&> \frac{1}{\Gamma(\alpha)}\left[(\alpha-1)(1-s)^{\alpha-2}t - (t-s)^{\alpha-2}t\right] \\
&> \frac{1}{\Gamma(\alpha)}\left[(\alpha-1)(1-s)^{\alpha-2}t - (1-s)^{\alpha-2}t\right] \\
&= \frac{(1-s)^{\alpha-2}t}{\Gamma(\alpha)}\left[\alpha-1-1\right] \\
&= \frac{\alpha-2}{\Gamma(\alpha)}(1-s)^{\alpha-2}t \\
&> 0
\end{aligned}$$

concluding property 1.

2. Note that G is differentiable for $t \in (0,1)$, where, taking $s \in (0,1)$,

$$\begin{aligned}
G'(t,s) &= \frac{1}{\Gamma(\alpha)}\left[(\alpha-1)(1-s)^{\alpha-2} - (\alpha-1)(t-s)^{\alpha-2}\mu(t-s)\right] \\
&> \frac{1}{\Gamma(\alpha)}\left[(\alpha-1)(1-s)^{\alpha-2} - (\alpha-1)(1-s)^{\alpha-2}\right] \\
&= 0
\end{aligned}$$

so $G'(t,s) > 0$ for all $(t,s) \in (0,1) \times (0,1)$. Now, note that

$$G'(t,0) = \frac{\alpha-1}{\Gamma(\alpha)}\left[1 - t^{\alpha-2}\right] > 0, \quad \forall\, 0 < t < 1$$

and

$$G(t,1) = 0, \quad \forall\, 0 \leq t \leq 1$$

so that

$$\max_{0 \leq t \leq 1} G(t,s) = G(1,s), \quad \forall\, s \in [0,1]$$

3. From the latter, given $a \in (0,1)$ one has that, if $a \leq t \leq s \leq 1$, then

$$\begin{aligned}
\frac{G(t,s)}{G(1,s)} &\geq \frac{G(a,s)}{G(1,s)} \\
&= \frac{(\alpha-1)(1-s)^{\alpha-2}a - (a-s)^{\alpha-1}\mu(a-s)}{(\alpha-1)(1-s)^{\alpha-2} - (1-s)^{\alpha-1}}
\end{aligned}$$

Since $G \in C([0,1]\times[0,1])$ and $G(t,s) > 0$ for all $0 < t, s < 1$, then

$$\frac{G(t,s)}{G(1,s)} > 0, \quad \forall (t,s) \in (0,1)\times(0,1)$$

and since $0 < a \le t$ and $G(t,s) \ge G(a,s)$ for all $(t,s) \in [a,1]\times[0,1]$, it's enough to analyze the cases $s \in \{0,1\}$. First, note that

$$\begin{aligned}\lim_{s\to 0^+}\frac{G(a,s)}{G(1,s)} &= \lim_{s\to 0^+}\frac{(\alpha-1)(1-s)^{\alpha-2}a-(a-s)^{\alpha-1}}{(\alpha-1)(1-s)^{\alpha-2}-(1-s)^{\alpha-1}}\\ &= \frac{(\alpha-1)a-a^{\alpha-1}}{\alpha-2}\\ &> 0\end{aligned}$$

and that

$$\begin{aligned}\lim_{s\to 1^-}\frac{G(a,s)}{G(1,s)} &= \lim_{s\to 1^-}\frac{(\alpha-1)(1-s)^{\alpha-2}a}{(\alpha-1)(1-s)^{\alpha-2}-(1-s)^{\alpha-1}}\\ &= \lim_{s\to 1^-}\frac{(\alpha-1)a}{\alpha-1-(1-s)}\\ &= \frac{(\alpha-1)a}{\alpha-2}\\ &> 0\end{aligned}$$

then, for all $a \in (0,1)$ there exists $\sigma = \sigma(\alpha,1) > 0$ such that

$$G(t,s) \ge \sigma G(1,s), \quad \forall (t,s) \in [a,1]\times[0,1]$$

In that way, from theorem 4.1.1 follows that the existence of a positive solution, for this example. □

Remark 4.3.1. *In this work it's possible to include some nonlinearities not considered on previous works, like the function $f(x) = \|x\|_\infty^p$. Also, this nonlienarity can depend on the derivatives of the solution at most at an arbitrary order, determined by the regularity of the Green function of the problem.*

Also, it's introduced arbitrary boundary conditions, which can depend on the derivatives of fractional order of the solution, which at our best knowledge, weren't studied on other works, at least at that generality.

All of this serve to generalize several other works, and to extend its results for example, on the research of Sun and Wang [25], they consider the problem

$$\begin{cases} {}^C D^\alpha u(t) + f(t,u(t),u'(t)) = 0 \quad t \in (0,1) \\ u(0) = u''(0) = 0, \quad u(1) = \lambda \int_0^1 u(s)ds, \end{cases} \tag{4.3.4}$$

where $2 < \alpha < 3$ and $0 < \lambda < 2$. Then, they state that if $f : [0,1]\times[0,\infty)\times\mathbb{R} \to \mathbb{R}$ is a continuous function satisfying:

A1) $f(t,u,v) \leq \frac{d}{K}$, *for* $(t,u,v) \in [0,1] \times [0,d] \times [-d,d]$*;*

A2) $f(t,u,v) > Nb$, *for* $(t,u,v) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times [b,c] \times [-d,d]$*;*

A3) $f(t,u,v) < Za$, *for* $(t,u,v) \in [0,1] \times [0,a] \times [-d,d]$*.*

for some constants $0 < a < b < c < d$, $c = \frac{8\alpha}{\lambda(\alpha-2)N}$ *where* K, N *and* Z *are constants related with* α, λ *and the Green function of the problem, which satisfy conditions* ***G1*** *and* ***G2****, then the problem* (4.3.4) *has at least three positive solutions.*

Hypothesis ***H3*** *and* ***H4*** *don't enter in conflict with hypothesis* ***A1****,* ***A2*** *and* ***A3****, so there exist functions satisfying all of them, like the function*

$$f(t,u,v) = \begin{cases} \frac{Z}{2}\| |u|, \min\{|v|,1\}\|_\infty, & (u,v) \in [-1,1] \times [-d,d] \\ \frac{Z}{2} + 2N(u-1), & 1 < |u| \leq c \quad \text{and } v \in [-d,d] \\ \frac{Z}{2} + 2N(c-1), & c < |u| \leq d \quad \text{and } v \in [-d,d] \\ \left[\frac{Z}{2} + 2N(c-1)\right] e^{\| |u|-d, |v|-d\|_\infty}, & |u| > d \quad \text{and } |v| > d \end{cases}$$

where

$$c = \max\left\{2, \frac{16\alpha}{\lambda(\alpha-2)N}\left[2N - \frac{Z}{2}\right]\right\},$$
$$d = \max\left\{4, K\left[\frac{Z}{2} + 2N(c-1)\right]\right\}$$

Other example of a function satisfying conditions ***H3****,* ***H4****,* ***A1****,* ***A2*** *and* ***A3*** *is the function*

$$f(t,u,v) = C_1 \log(|u|+1) + |u|^p(\sin(x) + 1.1) + C_2|v|^p$$

where $p > 1$, $C_1 > 0$ *is big enough and* $C_2 > 0$ *is small enough.*

However, there are some functions that satisfy conditions ***H3*** *and* ***H4****, but not* ***A1****,* ***A2*** *and* ***A3****, like the function* $C(e^{\|u,v\|_\infty} - 1)$ *for some constant* $C > 0$ *big enough.*

Then, this research complement their hypothesis, and extend their results. As like this problem, our research can be very useful to be applied on general contexts, since it suffices to check some properties on the Green function, which is explicitly given, to conclude on the properties of the problem.

Chapter 5

Linking to Higher Dimensionality Problems and other Fractional operators

The objective of this chapter is to link the results proved before to a problem of higher dimensionality. To do that, radial fractional operator will be constructed over annular domains that extends the Riemann-Liouville fractional derivative to several variables. Then, problem will be proposed, related to the said operator, and will be looking for radial solutions.

5.1 Building a radial fractional operator

First, consider a domain $\Omega \subset \mathbb{R}^n$, where $n \geq 2$. Since the operator is wanted to be radial and to coincide with the Rieman-Liouville fractional operators, the two following definitions are proposed.

Definition 5.1.1. *Let $\Omega \subset \mathbb{R}^d$ be a star shaped set with respect of 0, with $d \in \mathbb{N}$. For each $x \in \Omega$ and $\alpha > 0$ the radial Riemann-Liouville fractional integral of order α is defined as*

$$J^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_0^{|x|} (|x| - \tau)^{\alpha-1} u\left(\frac{\tau x}{|x|}\right) d\tau$$

On the other hand, if $\Omega = B_R \setminus B_r$ is not star shaped, but an anullus, with $0 < r < R$, then for all $x \in \Omega$ alternatively it's defined

$$J^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_r^{|x|} (|x| - \tau)^{\alpha-1} u\left(\frac{\tau x}{|x|}\right) d\tau$$

Definition 5.1.2. *Let $\Omega \subset \mathbb{R}^d$ be either, a star shaped set or an anullus, with $d \in \mathbb{N}$. For each $x \in \Omega$, $\alpha > 0$ and $i \in \{1, \ldots, d\}$, define the radial Riemann-Liouville fractional* *derivative of order* α *as*

$$D_i^\alpha u(x) = \frac{\partial^{[\alpha]}}{\partial x_i^{[\alpha]}} J^{[\alpha]-\alpha} u(x)$$

It's worth mentioning that the two latter definitions are original, and model phenomena that have "memory" on the segment between the origin and the studied point. That is, perturbations propagate radially from the origin.

Onward will be assumed that $d \geq 2$, $\alpha > 1$, $\Omega = B_{R_0} \setminus \overline{B_{r_0}}$, with $0 < r_0 < R_0$, and that $u(x) = v(r)$ is a radial function, where $r = |x|$ and $\theta = \frac{x}{|x|}$. Furthermore, onwards will be considered the fractional operator ∇^α defined as

$$\nabla^\alpha u(x) = (D_1^\alpha u(x), \dots, D_d^\alpha u(x))$$

Under these assumptions one has that

$$J^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_{r_0}^{r} (r-\tau)^{\alpha-1} v(\tau)\, d\tau = J^\alpha v(r)$$

Note that, for $\alpha \in (0,1)$,

$$D_i^\alpha u(x) = \frac{\partial}{\partial x_i} J^{1-\alpha} u(x) = \frac{\partial r}{\partial x_i} \frac{\partial}{\partial r} J^{1-\alpha} v(r)$$

where

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{|x|}$$

So that

$$D_i^\alpha u(x) = \frac{x_i}{|x|} D^\alpha v(r)$$

On the other hand, for $\alpha \in (1,2)$ one has

$$\begin{aligned} D_i^\alpha u(x) &= \frac{\partial}{\partial x_i} D_i^{\alpha-1} u(x) \\ &= \frac{\partial}{\partial x_i} \left(\frac{x_i}{|x|} D^{\alpha-1} v(r) \right) \\ &= \left(\frac{1}{|x|} - \frac{x_i^2}{|x|^3} \right) D^{\alpha-1} v(r) + \frac{x_i}{|x|} \frac{\partial}{\partial x_i} D^{\alpha-1} v(r) \\ &= \left(\frac{1}{|x|} - \frac{x_i^2}{|x|^3} \right) D^{\alpha-1} v(r) + \frac{x_i}{|x|} \frac{\partial r}{\partial x_i} \frac{\partial}{\partial r} D^{\alpha-1} v(r) \\ &= \left(\frac{1}{|x|} - \frac{x_i^2}{|x|^3} \right) D^{\alpha-1} v(r) + \frac{x_i^2}{|x|^2} D^\alpha v(r) \end{aligned}$$

Also, for $\alpha \in (2,3)$,

$$\begin{aligned} D_i^\alpha u(x) &= \frac{\partial}{\partial x_i} D_i^{\alpha-1} u(x) \\ &= \frac{\partial}{\partial x_i}\left(\left(\frac{1}{|x|} - \frac{x_i^2}{|x|^3}\right) D^{\alpha-2} v(r) + \frac{x_i^2}{|x|^2} D^{\alpha-1} v(r)\right) \\ &= \left(-3\frac{x_i}{|x|^3} + 3\frac{x_i^3}{|x|^5}\right) D^{\alpha-2} v(r) + \left(3\frac{x_i}{|x|^2} - 3\frac{x_i^3}{|x|^4}\right) D^{\alpha-1} v(r) + \frac{x_i^3}{|x|^3} D^\alpha v(r) \end{aligned}$$

Finally,

$$\frac{\partial}{\partial x_i} u(x) = \frac{\partial r}{\partial x_i}\frac{\partial}{\partial r} v(r) = \frac{x_i}{|x|} v'(r)$$

So that, if $\alpha \in (2,3)$ y $\Omega = B_1$, the problem

$$\begin{cases} g_0(x)\cdot\nabla^\alpha u(x) + g_1(x)\cdot\nabla^{\alpha-1}u(x) + g_2(x)\nabla^{\alpha-2}u(x) = \lambda c(|x|) f(u(x)), & r_0 < |x| < R_0 \\ u(x) = 0, & |x| = r_0 \\ \dfrac{x}{|x|}\cdot\nabla u(x) = 0, & |x| = R_0 \\ \dfrac{x}{|x|}\cdot\nabla u(x) = 0, & |x| = r_0 \end{cases} \tag{5.1.1}$$

transforms in

$$\begin{cases} rD^\alpha v(r) = \lambda c(r) f(v(r)) & r_0 < r < R_0 \\ v(r_0) = 0, \quad v'(r_0) = 0, \quad v'(R_0) = 0 \end{cases} \tag{5.1.2}$$

where $c : \Omega = B_{R_0} \setminus \overline{B_{r_0}} \to \mathbb{R}$ is a radially symmetric function, $f : \mathbb{R} \to \mathbb{R}$ is continuous and

$$\begin{aligned} g_0(x) &= |x|\left(\frac{1}{x_1}, \dots, \frac{1}{x_d}\right) \\ g_1(x) &= \frac{3}{|x|}(1,\dots,1) - 3|x|(x_1^2,\dots,x_d^2) \\ g_2(x) &= 3x|x| - 3\frac{1}{|x|^3}(x_1^3,\dots,x_d^3) \end{aligned}$$

Finally, doing the variable changes $s = \frac{\tau - r_0}{R_0 - r_0}$ and $w(t) = v\left((R_0 - r_0)t + r_0\right)$, $t \in [0,1]$, then

$$J^\alpha v(r) = \frac{1}{\Gamma(\alpha)}\int_{r_0}^r (r-\tau)^{\alpha-1} v(\tau)d\tau = \frac{R_0 - r_0}{\Gamma(\alpha)}\int_0^1 (t-s) w(s) ds = (R_0 - r_0) J^\alpha w(t)$$

By other side, for $\alpha \in (1,2)$,

$$D^\alpha v(r) = \frac{\partial}{\partial r}\left(\frac{\partial t}{\partial r}\frac{\partial}{\partial t}\left((R_0 - r_0) J^\alpha w(t)\right)\right)$$

$$
\begin{aligned}
&= (R_0 - r_0)\frac{\partial}{\partial r}\left(\frac{1}{R_0 - r_0} D^{\alpha-1} w(t)\right) \\
&= \frac{\partial t}{\partial r}\frac{\partial}{\partial t} D^{\alpha-1} w(t) \\
&= \frac{1}{R_0 - r_0} D^{\alpha} w(t)
\end{aligned}
$$

Meanwhile

$$
v'(r) = w'(t)\frac{\partial t}{\partial r} = \frac{1}{R_0 - r_0} w'(t)
$$

So that (5.1.2) is equivalent to the problem

$$
\begin{cases} D^{\alpha} w(t) = \lambda \hat{c}(t) f(w(t)), t \in (0,1) \\ w(0) = 0, \quad w'(0) = 0, \quad w'(1) = 0 \end{cases}
$$

where $\hat{c}(t) = \frac{(R_0 - r_0)}{(R_0 - r_0)t + r_0} c\left((R_0 - r_0)t + r_0\right)$. In this way, from corollary 4.3.1, the following result is obtained.

Main Theorem 5.1.1. *Suppose that* $\frac{c(r)}{r} \in C(0,1) \cap L(0,1)$ *is non negative, with* $c(r)$ *different from the null function. Also suppose that* $f \in C(\mathbb{R})$ *is a locally lipschitz, non negative function over* $[0, \infty)$*, and such that verifies* ***(H1)*** *and* ***(H2)****. Then the problem* (5.1.1) *has at least one positive solution* $u \in C(\overline{\Omega})$*.*

Chapter 6

Conclusions and Future Work

6.1 Conclusions

In this work several results and procedures are set that could be useful for other researchers to study the existence of solutions of several physical models, and to research on the multiplicity of solutions. In particular, those were used to prove existence of infinitely many positive solutions of the problem (4.1.1), set as

$$\begin{cases} -\mathcal{D}v(t) = \lambda c(t) f(t, v(t), \ldots, D^n v(t)), & 0 < t < 1 \\ D^k v(0) = 0, & 0 \le k \le m \\ \mathcal{A}_i(D^{\gamma_{i,j}} v(c_{i,j}))_{j=m+1}^{[\alpha]-1} = 0, & \forall\, i \in \{m+1, \ldots, [\alpha]-1\} \end{cases}$$

under the hypothesis of chapter 4. The existence problem was solved using the well known fixed point method, via the Krasnosel'skii and Banach fixed point theorems, while the multiplicity was atacked with a new procedure. Also, it was proved that , under the hypothesis that lead to the multiplicity result, then all the positive solutions $v_\lambda \to 0$ when $\lambda \to \infty$.

Problem (4.1.1) complements and extends several problems previously studied, introducing other types of nonlinearities and setting more general boundary conditions. For example, problems (1.4.2), (1.4.3) and (1.4.6) have the same structure of problem (4.1.1), but there are some nonlinearities included in this work that coldn't be included in the ones previously mentioned and viceversa. In that sense, theorem 4.1.1 complement the researchs developed on [13], [26] and [25], and theorem 4.2.1 extends them.

By other side, the research developed in this text on Green functions can be useful in other problems not studied yet, related to the operators studied here, since fixed point methods are a very common tool when studying existence results, where Green functions play a fundamental rol to build fixed point operators.

Finally, the new procedure developed here, consisting in looking for bounds for the norm of the solutions, optimizing them, and manipulating them in a convinient form to prove that the fixed point operator is ismetric on a certain domain, to then apply the Krasnoselskii theorem, proved to be very useful on the search for multiplicity results, like in theorem 4.2.1.

Because of all the above, this reaserch should serve to enrich and facilitate the work of many other reaserchers.

6.2 Future work

After the publishing of this document, there are several lines of research that can be followed using this text as a start point. On this section, will be presented the lines that will be followed by the author.

6.2.1 Extension to more dimensions

In chapter 2 was noted that several fractional operators can be extended to a suitable distributional space, meanwhile in chapter 4 one can note that in the proof of theorem 4.1.1, the key argumentes used are the hypothesis on f, the properties of the Green function and boundness of the domain. Then, is expected that this theorem can be extended to more general problems,and that theorem 4.2.1 can be replicated for those cases.

The goal then is to set suitable conditions on the operator and the problem so theorems 4.1.1 and 4.2.1 still hold true.

6.2.2 Relation with space fractional heat equation

It's known that in the heat equation $v_t - \Delta v = |v|^{p-1}v$, the solution of it is related in some sense to the stationary state of the equation (see for example [16, 20, 28]). In this direction, one can conjecture that a similar thing happens to the fractional space heat equation $v_t - D^\alpha v = |v|^{p-1}v$, so is of interest to stidy this subject.

Appendix A

Some technical results

In this appendix some technical,useful results will be shown for the proof of theorem 4.1.1 on chapters 4 and 5.

A.1 Hypothesis of Krasnosel'skii theorem

One of the hypothesis of Krasnosel'skii theorem is that the working set is a cone. In this text, all the work is developed over the space $C^n[0,1]$ for some $n \in \mathbb{N}_0$, and is asserted that the set

$$\mathcal{C} := \left\{ v \in C^n[0,1] \,:\, v \geq 0,\ \min_{a \leq t \leq b} v(t) \geq \sigma \|v\|_{C^n[0,1]} \right\}$$

where $0 < a < b \leq 1$, $0 < \sigma < 1$, and

$$\|v\|_{C^n[0,1]} = \max_{0 \leq k \leq n} \|D^k v\|_\infty$$

so that $C^n[0,1]$ is a Banach space with the said norm. So, one has the following lemma.

Lemma A.1.1. *The set*

$$\mathcal{C} := \left\{ v \in C^n[0,1] \,:\, v \geq 0,\ \min_{a \leq t \leq b} v(t) \geq \sigma \|v\|_{C^n[0,1]} \right\}$$

is a cone on $C^n[0,1]$.

Proof. It's necessary to prove that $\mathcal{C}$ satisfies the properties of a cone.

1. Note that for every $c \in \mathbb{R}_{\geq 0}$ $c \in \mathcal{C}$. Indeed, c satisfies that $c \in C^n[0,1]$, $c \geq 0$ on $[0,1]$ and

   $$\sigma c = \sigma \min_{a \leq t \leq b} c < c = \|c\|_{C^n[0,1]}$$

   so that $c \in \mathcal{C}$ for all $c \geq 0$, and then $\mathcal{C} \neq \emptyset$ and $\mathcal{C} \neq \{0\}$.

2. Let $\{v_n\} \subset \mathcal{C}$ be a sequence such that $v_n \to v$ for some $v \in C^n[0,1]$. It will be shown that $v \in \mathcal{C}$, so that $\mathcal{C}$ is closed.

   First, note that, since $v_n \to v$ on $C^n[0,1]$, then

$$\|v_n - v\|_{C^n[0,1]} \to 0 \quad \text{when } n \to \infty$$

   In particular, by continuity, for all $0 \le k \le n$ and all $x \in [0,1]$, $v_n(x) \to v(x)$. Then

$$\lim_{n\to\infty} \min_{a \le t \le b} v_n(t) = \min_{a \le t \le b} v(t)$$

   On the other side, since $\|\cdot\|_{C^n[0,1]}$ is lower semicontinuous, then

$$\sigma\|v\|_{C^n[0,1]} \le \sigma \liminf_{n\to\infty} \|v_n\|_{C^n[0,1]} \le \lim_{n\to\infty} \min_{a \le t \le b} v_n(t) = \min_{a \le t \le b} v(t)$$

   and then $v \in \mathcal{C}$.

3. Let $v_1, v_2 \in \mathcal{C}$ and $\lambda \in [0,1]$. Then $\lambda v_1 + (1-\lambda)v_2 \ge 0$ and

$$\begin{aligned}
\sigma\|\lambda v_1 + (1-\lambda)v_2\|_{C^n[0,1]} &\le \sigma\left(\lambda\|v_1\|_{C^n[0,1]} + (1-\lambda)\|v_2\|_{C^n[0,1]}\right) \\
&= \lambda\left(\sigma\|v_1\|_{C^n[0,1]}\right) + (1-\lambda)\left(\sigma\|v_2\|_{C^n[0,1]}\right) \\
&\le \lambda \min_{a \le t \le b} v_1(t) + (1-\lambda)\min_{a \le t \le b} v_2(t) \\
&= \min_{a \le t \le b}\left(\lambda v_1(t) + (1-\lambda)v_2(t)\right)
\end{aligned}$$

   so $\lambda v_1 + (1-\lambda)v_2 \in \mathcal{C}$, concluding that $\mathcal{C}$ is convex.

4. Let $\lambda \ge 0$ and $v \in \mathcal{C}$. Then $\lambda v \ge 0$ and

$$\sigma\|\lambda v\|_{C^n[0,1]} = \lambda\sigma\|v\|_{C^n[0,1]} \le \lambda \min_{a \le t \le b} v(t) = \min_{a \le t \le b} \lambda v(t)$$

   so that $\lambda v \in \mathcal{C}$, and then $\lambda\mathcal{C} \subset \mathcal{C}$.

5. Suppose that $v \in \mathcal{C} \cap (-\mathcal{C})$. Then $v \ge 0$ and $v \le 0$, so that $v = 0$. Since $0 \in \mathcal{C}$, then $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$.

All the above leads to the conclusion that $\mathcal{C}$ is a cone on $C^n[0,1]$. □

A.2 Properties of Green functions

A fundamental argumemnt to prove theorem 4.1.1 is the fact that the family $\{G(\cdot, s)\}_{s\in[0,1]}$ is uniformly equicontinuous, where G is the Green function associated to the operator $\mathcal{D}$. Here this affirmation is proven in all the three possible cases.

First, the property is proven for the Riemann-Liouville case, then for the concatenated case, and finally for the Caputo case.

Lemma A.2.1. *Suppose that G is defined as in lemma 3.2.2. Then the family $\{D^l G(\cdot, s)\}_{s\in[0,1]}$ is uniformly equicontinuous on $C[0,1]$ for all $0 \le l \le m$.*

Proof. Since $D^lG(\cdot,s) \in C^1[0,1]$ for all $s \in [0,1]$ and for all $0 \le l \le m-1$, by the mean value theorem, for all $a, b \in [0,1]$, with $a < b$, there exists $c \in (a,b)$ such that

$$D^lG(a,s) - D^lG(b,s) = D^{l+1}G(c,s)(b-a), \quad \forall\, 0 \le l \le m-1$$

and since $D^{l+1}G(\cdot,s) \in C[0,1]$,

$$|D^lG(a,s) - D^lG(b,s)| \le \max_{t\in[0,1]} D^{l+1}G(t,s)|b-a| = K|b-a|, \quad \forall\, s \in [0,1], \forall\, 0 \le l \le m-1$$

where $K = \max_{t\in[0,1]} D^{l+1}G(t,s)$. So taking $\epsilon > 0$ and $\delta = \frac{\epsilon}{K}$, then for all $s \in [0,1]$ and for all $a, b \in [0,1]$ such that $|b-a| < \delta$

$$|D^lG(a,s) - D^lG(b,s)| < K\frac{\epsilon}{K} = \epsilon, \quad \forall\, 0 \le l \le m-1$$

and so that $\{D^lG(\cdot,s)\}_{s\in[0,1]}$ is uniformly equicontinuous on $C[0,1]$ for all $0 \le l \le m-1$.

Now assume that $l = m$. For all $s \in [0,1]$ define

$$g_{0,s}(t) = \frac{1}{\Gamma(\alpha-m)}(t-s)^{\alpha-m-1}\mu(t-s)$$

and for each $m+1 \le k \le [\alpha]-1$ define

$$C_k(s) = \frac{\Gamma(k+\alpha+1-[\alpha])}{\Gamma(k+\alpha+1-[\alpha]-m)} \sum_{i=m+1}^{[\alpha]-1} \gamma_{i,k} \left(d_i + \sum_{j=m+1}^{[\alpha]-1} \frac{\nu_{i,j}}{\Gamma(\alpha-n_{i,j})}(c_{i,j}-s)^{\alpha-n_{i,j}-1}\mu(c_{i,j}-s) \right)$$

and

$$g_{k,s}(t) = C_k(s)t^{k+\alpha-[\alpha]-m}$$

then

$$D^mG(t,s) = \sum_{k=m+1}^{[\alpha]-1} g_{k,s}(t) - g_{0,s}(t)$$

Note that, since $0 \le s \le 1$,

$$\left| d_i + \sum_{j=m+1}^{[\alpha]-1} \frac{\nu_{i,j}}{\Gamma(\alpha-n_{i,j})}(c_{i,j}-s)^{\alpha-n_{i,j}-1}\mu(c_{i,j}-s) \right| \le |d_i| + \sum_{j=m+1}^{[\alpha]-1} \left| \frac{\nu_{i,j}}{\Gamma(\alpha-n_{i,j})} \right| = K_i$$

for all $s \in [0,1]$, and so that

$$\left| \sum_{i=m+1}^{[\alpha]-1} \gamma_{i,k} \left(d_i + \sum_{j=m+1}^{[\alpha]-1} \frac{\nu_{i,j}}{\Gamma(\alpha-n_{i,j})}(c_{i,j}-s)^{\alpha-n_{i,j}-1}\mu(c_{i,j}-s) \right) \right|$$

$$\leq \sum_{i=m+1}^{[\alpha]-1} |\gamma_{i,k}| K_i$$
$$= M_k$$

In particular, there exists $M > 0$ such that

$$|C_k(s)| \leq M, \quad \forall\, m+1 \leq k \leq [\alpha]-1, \quad \forall\, s \in [0,1]$$

where M is taken as a generic constant. Since $k + Re(\alpha) - [\alpha] - m > 0$ for all $k \geq m+1$, and $Re(\alpha) - m - 1 > 0$ since $m \leq [\alpha] - 2$, one have, for all $a, b \in [0,1]$, with $a < b$, that

$$|D^m G(b,s) - D^m G(a,s)|$$
$$\leq M \sum_{k=m+1}^{[\alpha]-1} |b^{k+\alpha-[\alpha]-m} - a^{k+\alpha-[\alpha]-m}| + \left|\frac{1}{\Gamma(\alpha-m)}\right| \left|(b-s)^{\alpha-m-1}\mu(b-s) - (a-s)^{\alpha-m-1}\mu(a-s)\right|$$
$$= M \sum_{k=m+1}^{[\alpha]-1} \left|(k+\alpha-[\alpha]-m)\int_a^b t^{k+\alpha-[\alpha]-m-1}dt\right| + \left|\frac{(\alpha-m-1)}{\Gamma(\alpha-m)}\right| \left|\int_a^b (t-s)^{\alpha-m-2}\mu(t-s)dt\right|$$

since $k + Re(\alpha) - [\alpha] - m - 1 > -1$ for all $k \geq m+1$, then one of following is satisfied.

- If $-1 < k + Re(\alpha) - [\alpha] - m - 1 < 0$, then $t^{k+\alpha-[\alpha]-m-1} \in L^p(0,1)$ for all $1 \leq p < \frac{1}{|k+Re(\alpha)-[\alpha]-m-1|}$.
- If $0 \leq k + Re(\alpha) - [\alpha] - m - 1$, then $t^{k+\alpha-[\alpha]-m-1} \in L^p(0,1)$ for all $1 \leq p \leq \infty$.

In any case, there exists $1 < p < \infty$ such that $t^{k+\alpha-[\alpha]-m-1} \in L^p(0,1)$ for all $k \geq m+1$. Then, since $1 \in L^q(0,1)$ for all $1 \leq q \leq \infty$, by Holder inequality

$$\int_a^b t^{k+\alpha-[\alpha]-m-1}dt \leq (a-b)^{\frac{p-1}{p}} \|t^{k+\alpha-[\alpha]-m-1}\|_{L^p(a,b)} \leq (a-b)^{\frac{p-1}{p}} \|t^{k+\alpha-[\alpha]-m-1}\|_{L^p(0,1)}$$

On the same way, since $Re(\alpha) - m - 2 > -1$, then there exists $1 < p < \infty$ such that $(t-s)^{\alpha-m-2}\mu(t-s) \in L^p(0,1)$, so that

$$\begin{aligned}\int_a^b (t-s)^{\alpha-m-2}\mu(t-s)dt &\leq (b-a)^{\frac{p-1}{p}} \|(t-s)^{\alpha-m-2}\mu(t-s)\|_{L^p(a,b)} \\ &\leq (b-a)^{\frac{p-1}{p}} \|(t-s)^{\alpha-m-2}\mu(t-s)\|_{L^p(0,1)} \\ &\leq (b-a)^{\frac{p-1}{p}} \|t^{\alpha-m-2}\|_{L^p(0,1)}\end{aligned}$$

Thus

$$|D^m G(b,s) - D^m G(a,s)|$$
$$\leq M \sum_{k=m+1}^{[\alpha]-1} \left|(k+\alpha-[\alpha]-m)\int_a^b t^{k+\alpha-[\alpha]-m-1}dt\right| + \left|\frac{(\alpha-m-1)}{\Gamma(\alpha-m)}\right| \left|\int_a^b (t-s)^{\alpha-m-2}\mu(t-s)dt\right|$$

$$\leq M\left(\int_a^b t^{k+\alpha-[\alpha]-m-1}dt + \int_a^b (t-s)^{\alpha-m-2}\mu(t-s)dt\right)$$
$$\leq M(b-a)^{\frac{p-1}{p}}$$

so that $\{D^m G(\cdot,s)\}_{s\in[0,1]}$ is also uniformly equicontinuous, which prove the lemma.

□

Lemma A.2.2. *Suppose that G is defined as in lemma 3.2.4, and assume that $Re(\alpha) > 1$. Then the family $\{D^l G(\cdot,s)\}_{s\in[0,1]}$ is uniformly equicontinuous on $C[0,1]$ for all $0 \leq l \leq m$.*

Proof. As in the proof of lemma A.2.1, since $G \in C([0,1]\times[0,1])$ with $G(\cdot,s) \in C^m[0,1]$ for all $s \in [0,1]$, by the mean value theorem the family $\{D^l G(\cdot,s)\}_{s\in[0,1]}$ is uniformly equicontinuous for all $0 \leq l \leq m-1$.

For $l = m \leq \left[\alpha - \frac{1}{p}\right] - 1 < Re(\alpha)$ one has that

$$\begin{aligned}
&D^m G(t,s)\\
&= \sum_{k=m+1}^{[\alpha]-1} \frac{\Gamma(k+\alpha-[\alpha]+1)}{\Gamma(k+\alpha-[\alpha]+1-m)} t^{k+\alpha-[\alpha]-m} \left(\sum_{i=m+1}^{[\alpha]+[\beta]-1} \gamma_{i,k} \cdot\right.\\
&\left.\cdot \left(d_i + \sum_{j=m+1}^{[\alpha]+[\beta]-1} \frac{\nu_{i,j}}{\Gamma(\beta)\Gamma(\alpha)} D^{n_{i,j}} Q(c_{i,j},s)\right)\right)\\
&+ \sum_{k=0}^{[\beta]-1} J^{\alpha-m}\left(\frac{t^{k+\beta-[\beta]}}{q(t)}\right)\left(\sum_{i=m+1}^{[\alpha]+[\beta]-1} \delta_{i,k}\left(d_i + \sum_{j=m+1}^{[\alpha]+[\beta]-1} \frac{\nu_{i,j}}{\Gamma(\beta)\Gamma(\alpha)} D^{n_{i,j}} Q(c_{i,j},s)\right)\right)\\
&- \frac{1}{\Gamma(\beta)\Gamma(\alpha)} D^m Q(t,s)
\end{aligned}$$

Note that if $m = [\alpha]-1$, then

$$\begin{aligned}
&\sum_{k=m+1}^{[\alpha]-1} \frac{\Gamma(k+\alpha-[\alpha]+1)}{\Gamma(k+\alpha-[\alpha]+1-m)} t^{k+\alpha-[\alpha]-m} \left(\sum_{i=m+1}^{[\alpha]+[\beta]-1} \gamma_{i,k} \cdot\right.\\
&\left.\cdot \left(d_i + \sum_{j=m+1}^{[\alpha]+[\beta]-1} \frac{\nu_{i,j}}{\Gamma(\beta)\Gamma(\alpha)} D^{n_{i,j}} Q(c_{i,j},s)\right)\right)\\
&= 0
\end{aligned}$$

so this term needs to be checked only in the case $m \leq [\alpha]-2$. Also like the proof of lemma A.2.1, since $D^{n_{i,j}} Q(c_{i,j},s) \in C([0,1]\times[0,1])$ for all $i,j = m+1,\ldots,[\alpha]+[\beta]-1$, and if $m \leq [\alpha]-2$, then given $a, b \in [0,1]$ such that $a < b$,

$$\left| \sum_{k=m+1}^{[\alpha]-1} \frac{\Gamma(k+\alpha-[\alpha]+1)}{\Gamma(k+\alpha-[\alpha]+1-m)} (b^{k+\alpha-[\alpha]-m} - a^{k+\alpha-[\alpha]-m}) \left(\sum_{i=m+1}^{[\alpha]+[\beta]-1} \gamma_{i,k} \cdot \right.\right.$$
$$\left.\left. \cdot \left(d_i + \sum_{j=m+1}^{[\alpha]+[\beta]-1} \frac{\nu_{i,j}}{\Gamma(\beta)\Gamma(\alpha)} D^{n_{i,j}} Q(c_{i,j}, s) \right) \right) \right|$$
$$\leq M \sum_{k=m+1}^{[\alpha]-1} |b^{k+\alpha-[\alpha]-m} - a^{k+\alpha-[\alpha]-m}|$$

Defining $g(x) = x^{\gamma_k} - (x-(b-a))^{\gamma_k}$, with $\gamma_k = k+\alpha-[\alpha]-m$ for all $k \geq m+1$, so that $Re(\gamma_k) > 0$ for all $k \geq m+1$, follows that

$$g'(x) = \gamma_k \left(x^{\gamma_k - 1} - (x-(b-a))^{\gamma_k - 1} \right)$$

where

- $g'(x) < 0$ for all $x > (b-a)$, if $0 < Re(\gamma_k) < 1$, so that
$$\sup_{x \in [(b-a),1]} g(x) = g(b-a) = (b-a)^{\gamma_k}$$
- $g'(x) \geq 0$ for all $x > (b-a)$, if $Re(\gamma_k) > 1$, so that
$$\sup_{x \in [(b-a),1]} g(x) = g(1) = 1 - (1-(b-a))^{\gamma_k}$$

since $g(b) = b^{\gamma_k} - a^{\gamma_k}$, then

$$\left| \sum_{k=m+1}^{[\alpha]-1} \frac{\Gamma(k+\alpha-[\alpha]+1)}{\Gamma(k+\alpha-[\alpha]+1-m)} (b^{k+\alpha-[\alpha]-m} - a^{k+\alpha-[\alpha]-m}) \left(\sum_{i=m+1}^{[\alpha]+[\beta]-1} \gamma_{i,k} \cdot \right.\right.$$
$$\left.\left. \cdot \left(d_i + \sum_{j=m+1}^{[\alpha]+[\beta]-1} \frac{\nu_{i,j}}{\Gamma(\beta)\Gamma(\alpha)} D^{n_{i,j}} Q(c_{i,j}, s) \right) \right) \right|$$
$$\leq M \sum_{k=m+1}^{[\alpha]-1} |b^{k+\alpha-[\alpha]-m} - a^{k+\alpha-[\alpha]-m}|$$
$$\leq M \sum_{k=m+1}^{[\alpha]-1} \left(1 - (1-(b-a))^{k+\alpha-[\alpha]-m} \right)$$
$$\leq M \max\left\{ \left(1 - (1-(b-a))^{\alpha-m-1} \right), (b-a)^{\alpha-m-1} \right\}$$

in the same way,

$$\left| \sum_{i=m+1}^{[\alpha]+[\beta]-1} \delta_{i,k} \left(d_i + \sum_{j=m+1}^{[\alpha]+[\beta]-1} \frac{\nu_{i,j}}{\Gamma(\beta)\Gamma(\alpha)} D^{n_{i,j}} Q(c_{i,j}, s) \right) \right| \leq K < \infty$$

and thus for all $a, b \in [0,1]$ such that $a < b$, follows that

$$\left| \sum_{k=0}^{[\beta]-1} \left(J^{\alpha-m} \left(\frac{t^{k+\beta-[\beta]}}{q(t)} \right) |_{t=b} - J^{\alpha-m} \left(\frac{t^{k+\beta-[\beta]}}{q(t)} \right) |_{t=a} \right) \cdot \right.$$
$$\left. \cdot \left(\sum_{i=m+1}^{[\alpha]+[\beta]-1} \delta_{i,k} \left(d_i + \sum_{j=m+1}^{[\alpha]+[\beta]-1} \frac{\nu_{i,j}}{\Gamma(\beta)\Gamma(\alpha)} D^{n_{i,j}} Q(c_{i,j}, s) \right) \right) \right|$$
$$\leq K \sum_{k=0}^{[\beta]-1} \left| J^{\alpha-m} \left(\frac{t^{k+\beta-[\beta]}}{q(t)} \right) |_{t=b} - J^{\alpha-m} \left(\frac{t^{k+\beta-[\beta]}}{q(t)} \right) |_{t=a} \right|$$

and since $J^{\alpha-m}\left(\frac{t^{k+\beta-[\beta]}}{q(t)}\right) \in C[0,1]$ not depends on s, the family $\{J^{\alpha-m}\left(\frac{t^{k+\beta-[\beta]}}{q(t)}\right)\}_{s\in[0,1]}$ of a single element is uniformly equicontinuous on $C[0,1]$. Indeed, by the uniform continuity over $[0,1]$, and by the non dependence from s, then for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $s \in [0,1]$ and for all $a, b \in [0,1]$ such that $|b-a| < \delta$,

$$\left| J^{\alpha-m} \left(\frac{t^{k+\beta-[\beta]}}{q(t)} \right) |_{t=b} - J^{\alpha-m} \left(\frac{t^{k+\beta-[\beta]}}{q(t)} \right) |_{t=a} \right| < \epsilon$$

Finally, note that, since $m < Re(\alpha)$, then

$$D^m Q(t,s) = D^m J^\alpha \left(\frac{(t-s)^{\beta-1}}{q(t)} \mu(t-s) \right) = J^{\alpha-m} \left(\frac{(t-s)^{\beta-1}}{q(t)} \mu(t-s) \right)$$

so that, for $a, b \in [0,1]$ such that $a < b$,

$$\begin{aligned}
&|D^m Q(b,s) - D^m Q(a,s)| \\
&= \left| \int_0^b (b-\tau)^{\alpha-m-1} \frac{(\tau-s)^{\beta-1}}{q(\tau)} \mu(\tau-s) d\tau - \int_0^a (a-\tau)^{\alpha-m-1} \frac{(\tau-s)^{\beta-1}}{q(\tau)} \mu(\tau-s) d\tau \right| \\
&= \left| \int_0^a \left((b-\tau)^{\alpha-m-1} - (a-\tau)^{\alpha-m-1} \right) \frac{(\tau-s)^{\beta-1}}{q(\tau)} \mu(\tau-s) d\tau \right. \\
&\left. + \int_a^b (b-\tau)^{\alpha-m-1} \frac{(\tau-s)^{\beta-1}}{q(\tau)} \mu(\tau-s) d\tau \right| \\
&\leq \int_0^a \left| (b-\tau)^{\alpha-m-1} - (a-\tau)^{\alpha-m-1} \right| \frac{(\tau-s)^{\beta-1}}{q(\tau)} \mu(\tau-s) d\tau \\
&+ \int_a^b (b-\tau)^{\alpha-m-1} \frac{(\tau-s)^{\beta-1}}{q(\tau)} \mu(\tau-s) d\tau
\end{aligned}$$

Now note that, since $Re(\alpha) - m - 1 > \frac{1}{p} - 1 = -\frac{1}{p'}$, then $(b-\tau)^{\alpha-m-1} \in L^{p'}(0,b)$ for all $b > 0$. Also, since

$$\sup_{s\in(0,1)} \left\| \frac{(\cdot - s)^{\beta-1}}{q(\cdot)} \right\|_{L^p(s,1)} < \infty$$

then

$$\begin{aligned}
&\int_a^b (b-\tau)^{\alpha-m-1}\frac{(\tau-s)^{\beta-1}}{q(\tau)}\mu(\tau-s)d\tau \\
&\leq \|(b-\tau)^{\alpha-m-1}\|_{L^{p'}(a,b)}\left\|\frac{(\tau-s)^{\beta-1}}{q(\tau)}\mu(\tau-s)\right\|_{L^p(a,b)} \\
&= \left(\frac{(b-a)^{(\alpha-m-1)p'+1}}{(\alpha-m-1)p'+1}\right)^{\frac{1}{p'}}\left\|\frac{(\tau-s)^{\beta-1}}{q(\tau)}\right\|_{L^p(s,b)} \\
&\leq \left(\frac{(b-a)^{(\alpha-m-1)p'+1}}{(\alpha-m-1)p'+1}\right)^{\frac{1}{p'}}\sup_{s\in(0,1)}\left\|\frac{(\tau-s)^{\beta-1}}{q(\tau)}\right\|_{L^p(s,1)} \\
&= M(b-a)^{(\alpha-m)-\frac{1}{p}}
\end{aligned}$$

By other side, it also follows that

$$\begin{aligned}
&\int_0^a \left|(b-\tau)^{\alpha-m-1}-(a-\tau)^{\alpha-m-1}\right|\frac{(\tau-s)^{\beta-1}}{q(\tau)}\mu(\tau-s)d\tau \\
&\leq \|(b-\tau)^{\alpha-m-1}-(a-\tau)^{\alpha-m-1}\|_{L^{p'}(0,a)}\left\|\frac{(\tau-s)^{\beta-1}}{q(\tau)}\mu(\tau-s)\right\|_{L^p(0,a)} \\
&= \left(\frac{1}{(\alpha-m-1)p'+1}\left((b-a)^{(\alpha-m-1)p'+1}-b^{(\alpha-m-1)p'+1}+a^{(\alpha-m-1)p'+1}\right)\right)^{\frac{1}{p'}}\left\|\frac{(\tau-s)^{\beta-1}}{q(\tau)}\right\|_{L^p(s,a)} \\
&\leq \left(\frac{1}{(\alpha-m-1)p'+1}\left((b-a)^{(\alpha-m-1)p'+1}-b^{(\alpha-m-1)p'+1}+a^{(\alpha-m-1)p'+1}\right)\right)^{\frac{1}{p'}}\sup_{s\in(0,1)}\left\|\frac{(\tau-s)^{\beta-1}}{q(\tau)}\right\|_{L^p(s,1)} \\
&= M\left((b-a)^{(\alpha-m-1)p'+1}-b^{(\alpha-m-1)p'+1}+a^{(\alpha-m-1)p'+1}\right)^{\frac{1}{p'}}
\end{aligned}$$

Combining all that one gets that

$$|D^mQ(b,s)-D^mQ(a,s)|\to 0, \quad \text{when } a\to b$$

Thus, $\{Q(\cdot,s)\}_{s\in[0,1]}$ is uniformly equicontinuous.

Combining all the above, $\|D^mG(a,s)-D^mG(b,s)\|_\infty\to 0$ when $a\to b$ or $b\to a$, and so that $\{D^m(\cdot,s)\}_{s\in[0,1]}$ is uniformly equicontinuous on $C[0,1]$, which proves the lemma.

□

Lemma A.2.3. *Suppose that G is defined as in lemma 3.2.6. Then the family $\{D^kG(\cdot,s)\}_{s\in[0,1]}$ is uniformly equicontinuous on $C[0,1]$ for all $0\leq k\leq[\alpha]-2$.*

Proof. Since $G\in C([0,1]\times[0,1])$ and $G(\cdot,s)\in C^{[\alpha]-2}[0,1]$ for all $s\in[0,1]$, by the mean value theorem one has that $\{D^kG(\cdot,s)\}_{s\in[0,1]}$ is uniformly equicontinuous on $C[0,1]$ for all $0\leq k\leq[\alpha]-3$. For $k=[\alpha]-2$ follows that if $[\alpha]-2\in I$, then

$$D^{[\alpha]-2}G(t,s)=([\alpha]-2)!\sum_{i\in I}\delta_{i,[\alpha]-2}\left(d_i+\sum_{j\in I}\frac{\nu_{i,j}}{\Gamma(\alpha-n_{i,j})}(c_{i,j}-s)^{\alpha-n_{i,j}-1}\mu(c_{i,j}-s)\right)$$

$$- \frac{1}{\Gamma(2+\alpha-[\alpha])}(t-s)^{1+\alpha-[\alpha]}\mu(t-s)$$

In other case, if $[\alpha]-2 \notin I$, then

$$D^{[\alpha]-2}G(t,s) = -\frac{1}{\Gamma(2+\alpha-[\alpha])}(t-s)^{1+\alpha-[\alpha]}\mu(t-s)$$

In any case, if $t_1, t_2 \in [0,1]$ are such that $t_1 < t_2$, since $1+\alpha-[\alpha] \in (0,1]$,

$$\begin{aligned}
&\left|D^{[\alpha]-2}G(t_2,s) - D^{[\alpha]-2}G(t_1,s)\right| \\
&= \frac{1}{\Gamma(2+\alpha-[\alpha])}\left((t_2-s)^{1+\alpha-[\alpha]}\mu(t_2-s) - (t_1-s)^{1+\alpha-[\alpha]}\mu(t_1-s)\right) \\
&\leq \frac{1}{\Gamma(2+\alpha-[\alpha])}\|(t_2-\cdot)^{1+\alpha-[\alpha]}\mu(t_2-\cdot) - (t_1-\cdot)^{1+\alpha-[\alpha]}\mu(t_1-\cdot)\|_\infty \\
&= \frac{(t_2-t_1)^{1+\alpha-[\alpha]}}{\Gamma(2+\alpha-[\alpha])}
\end{aligned}$$

Thus, for all $\epsilon > 0$, if one takes $\delta = (\epsilon\Gamma(2+\alpha-[\alpha]))^{\frac{1}{1+\alpha-[\alpha]}}$, it follows that if $t_1, t_2 \in [0,1]$ are such that $|t_1 - t_2| < \delta$, then

$$\left|D^{[\alpha]-2}G(t_2,s) - D^{[\alpha]-2}G(t_1,s)\right| < \epsilon, \quad \forall s \in [0,1]$$

so that $\{D^{[\alpha]-2}G(\cdot,s)\}_{s\in[0,1]}$ is uniformly continuous on $C[0,1]$. □

A.3 Other results related to Green functions

Properties **G1** and **G2** are essential for the proof of the main theorems on this text. Despite showing some examples of Green functions that has both preperties in section 4.3, there is still unclear whether it's possible or not for the case $1 < \alpha \leq 2$. The lemmas proven in this section have as goal to clarify this question for both, the Riemann-Liouville and the Caputo case.

First is shown a lemma for the Riemann-Liouville, and then the version for the Caputo case. Also, in the Riemann-Liouville case, a remark is given with some examples on this topic.

Lemma A.3.1. *Assume that $1 < \alpha$ and $G \in C([0,1]\times[0,1])$. Consider the conditions on G:*

G1) *$G(t,s) > 0$ for all $(t,s) \in (0,1)\times(0,1)$,*

G2) *For each $0 \leq k \leq m$ there exists a measurable function $h_k : (0,1) \to (0,1)$ such that $\max_{t\in[0,1]} |D^kG(t,s)| = D^kG(h_k(s),s)$ for all $s \in (0,1)$, and there exist $0 < a < b \leq 1$ and $\sigma = \sigma(a,b,\alpha)$ such that for all $0 \leq k \leq m$ $D^kG(t,s) \geq \sigma D^kG(h_k(s),s)$ for all $(t,s) \in [a,b]\times(0,1)$.*

*If $1 < \alpha \leq 2$, then condition **G1** implies condition **G2**.*

Remark A.3.1. *1. Given $1 < \alpha < 2$ and $0 \leq \beta \leq \alpha - 1$, since $(1-s)^{\alpha-1-\beta} \geq (1-s)^{\alpha-1}$ for all $s \in (0,1)$, the Green function*

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \left((1-s)^{\alpha-1-\beta} t^{\alpha-1} - (t-s)^{\alpha-1} \mu(t-s) \right)$$

*satisfying the boundary conditions $G(0,s) = 0$ and $D^\beta G(1,s) = 0$ satisfies condition **G1**, and therefore satisfies condition **G2**.*

2. If $\alpha = 2$, given $K \in (0,1]$, then the Green function

$$G(t,s) = \frac{1}{K\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha)}(t-s)\mu(t-s)$$

*satisfying the conditions $G'(1,s) = -KG(0,s)$ and $G'(0,s) = 0$ for all $s \in [0,1]$, also satisfies condition **G1**, and therefore it satisfies condition **G2**.*

Proof. If $\alpha = 2$, from lemma 3.2.2, it's known that

$$G(t,s) = a_0(s) + a_1(s)t - \frac{1}{\Gamma(\alpha)}(t-s)\mu(t-s)$$

for some $a_0, a_1 \in C[0,1]$. The condition $G(t,s) > 0$ for all $(t,s) \in (0,1) \times (0,1)$ and the continuity of G imply that $G(0,s) = a_0(s) \geq 0$ for all $s \in [0,1]$. Also, for $0 < t \leq s < 1$, $G(t,s) > 0$ iff

$$a_1(s) > -\frac{a_0(s)}{t}, \quad \forall\, 0 < t \leq s$$

which is equivalent to

$$a_1(s) > -\frac{a_0(s)}{s}, \quad \forall\, s \in (0,1)$$

and for $0 < s < t < 1$, $G(t,s) > 0$ iff

$$a_1(s) > \frac{t-s}{\Gamma(\alpha)t} - \frac{a_0(s)}{t}, \quad \forall\, 0 < s < t < 1$$

where the function $\dfrac{t-s}{\Gamma(\alpha)t} - \dfrac{a_0(s)}{t}$ reach its maximum on $t \in [s,1]$ at $t = 1$. Since for all $0 < s < 1$ one has that

$$-\frac{a_0(s)}{s} \geq \frac{1-s}{\Gamma(\alpha)} - a_0(s) \quad \Leftrightarrow \quad a_0(s)\left[\frac{1}{s} - 1\right] \leq \frac{s-1}{\Gamma(\alpha)} \quad \Rightarrow \quad a_0(s) < 0$$

since $a_0(s) \geq 0$ for all $s \in (0,1)$, then

$$-\frac{a_0(s)}{s} < \frac{1-s}{\Gamma(\alpha)} - a_0(s), \quad \forall\, s \in (0,1)$$

so $G(t,s) > 0$ for all $(t,s) \in (0,1) \times (0,1)$ iff $a_1(s) \geq \frac{1-s}{\Gamma(\alpha)} - a_0(s)$ for all $s \in (0,1)$.

Now, note that $G(\cdot, s) \in C^1(s,1)$ for all $s \in [0,1]$ and

$$G'(t,s) = a_1(s) - \frac{\mu(t-s)}{\Gamma(\alpha)}$$

so, $G(t,s)$ is non decreasing on $t \in [0,1]$ iff $a_1(s) \geq \frac{\mu(t-s)}{\Gamma(\alpha)}$. Thus, defining $h : [0,1] \to [0,1]$ as

$$h(s) = \begin{cases} 0, & a_1(s) \leq 0 \\ s, & 0 < a_1(s) \leq \frac{1}{\Gamma(\alpha)} \\ 1, & \frac{1}{\Gamma(\alpha)} < a_1(s) \end{cases}$$

then

$$\max_{t \in [0,1]} G(t,s) = G(h(s), s), \quad \forall\, s \in [0,1]$$

Now note that, since $a_1(s) \geq \frac{(1-s)}{\Gamma(\alpha)} - a_0(s)$, by continuity of a_0 and a_1 one has that $a_1(1) \geq -a_0(1)$, so if $a_1(1) > -a_0(1)$, then for $0 < t < 1$

$$\begin{aligned} \lim_{s \to 1^-} \frac{G(t,s)}{G(h(s),s)} &= \lim_{s \to 1^-} \frac{a_0(s) + a_1(s)t}{a_0(s) + a_1(s)h(s) - \frac{1}{\Gamma(\alpha)}(h(s) - s)\mu(t-s)} \\ &= \frac{a_0(1) + a_1(1)t}{a_0(1) + a_1(1)h(1)} \\ &> \frac{(1-t)a_0(1)}{a_0(1) + a_1(1)h(1)} \\ &> 0 \end{aligned}$$

and if $a_1(1) = -a_0(1)$, since $a_1(s) \geq \frac{1-s}{\Gamma(\alpha)} - a_0(s)$ for all $s \in [0,1]$, in particular $a_1'(1) \leq -\frac{1}{\Gamma(\alpha)} - a_0'(1)$. Moreover, if $a_0(1) = -a_1(1) > 0$ then $a_1(1) < 0$, so that $h(s) = 0$ for s near 1, so that

$$\begin{aligned} \lim_{s \to 1^-} \frac{G(t,s)}{G(h(s),s)} &= \lim_{s \to 1^-} \frac{a_0(s) + a_1(s)t}{a_0(s)} \\ &= \frac{a_0(1) + a_1(1)t}{a_0(1)} \\ &> 0 \end{aligned}$$

Finally, if $a_0(1) = -a_1(1) = 0$, then $a_1(s) \leq \frac{1}{\Gamma(\alpha)}$ near 1, so that $h(s) \leq s$, and then

$$\begin{aligned} \lim_{s \to 1^-} \frac{G(t,s)}{G(h(s),s)} &= \lim_{s \to 1^-} \frac{a_0(s) + a_1(s)t}{a_0(s) + a_1(s)h(s)} \\ &= \lim_{s \to 1^-} 1 + \frac{a_1(s)(t - h(s))}{a_0(s) + a_1(s)h(s)} \end{aligned}$$

If $a_1(s) \geq 0$ near 1, then $h(s) = s$ near 1, and since $a_0(s) \geq 0$, then for $0 < t < 1$, there exists $\delta > 0$ such that $t - h(s) < 0$ for all $1 - \delta < s \leq 1$, so that

$$\begin{aligned}
\lim_{s\to 1^-} 1 + \frac{a_1(s)(t-h(s))}{a_0(s)+a_1(s)h(s)} &= \lim_{s\to 1^-} 1 + \frac{a_1(s)(t-s)}{a_0(s)+a_1(s)s} \\
&\geq \lim_{s\to 1^-} 1 + \frac{a_1(s)(t-s)}{a_1(s)s} \\
&= 1 + (t-1) = t > 0
\end{aligned}$$

Otherwise, if $a_1(s) \leq 0$ near 1, then $h(s) = 0$, and since $a_0(s) \geq 0$ and $a_1(s) \geq \frac{(1-s)}{\Gamma(\alpha)} - a_0(s) \geq -a_0(s)$, then

$$\lim_{s\to 1^-} 1 + \frac{a_1(s)(t-h(s))}{a_0(s)+a_1(s)h(s)} = \lim_{s\to 1^-} 1 + \frac{a_1(s)t}{a_0(s)} \geq \lim_{s\to 1^-} 1 - \frac{a_0(s)t}{a_0(s)} = 1 - t > 0$$

in any case, follows that

$$\lim_{s\to 1^-} \frac{G(t,s)}{G(h(s),s)} > 0, \quad \forall\, 0 < t < 1$$

Furthermore, since a_0, a_1 are locally lipschitz on a neighbourhood of 0, we can consider three cases:

1. There exists $\epsilon > 0$ such that $a_1(s) \leq 0$ for all $0 \leq s \leq \epsilon$, or
2. There exists $\epsilon > 0$ such that $0 < a_1(s) \leq \frac{1}{\Gamma(\alpha)}$ for all $0 < s < \epsilon$, or
3. There exists $\epsilon > 0$ such that $\frac{1}{\Gamma(\alpha)} < a_1(s)$ for all $0 < s < \epsilon$, or

On the first case, one has that $h(s) = 0$ for all $0 \leq s \leq \epsilon$, and since $a_1(s) \geq \frac{1-s}{\Gamma(\alpha)} - a_0(s)$, then in particular $a_0(0) \geq \frac{1}{\Gamma(\alpha)} - a_1(0) \geq \frac{1}{\Gamma(\alpha)}$. Thus, for $0 < t < 1$,

$$\begin{aligned}
\lim_{s\to 0^+} \frac{G(t,s)}{G(h(s),s)} &= \lim_{s\to 0^+} \frac{a_0(s) + a_1(s)t - \frac{1}{\Gamma(\alpha)}(t-s)}{a_0(s)} \\
&= \frac{a_0(0) + a_1(0)t - \frac{1}{\Gamma(\alpha)}t}{a_0(0)} \\
&\geq \frac{(1-t)\left(\frac{1}{\Gamma(\alpha)} - a_1(0)\right)}{a_0(0)} \\
&> 0
\end{aligned}$$

Also, on the second case, since $0 < a_1(s) \leq \frac{1}{\Gamma(\alpha)}$ for all $0 < s < \epsilon$, then $h(s) = s$ for all $0 < s < \epsilon$, so that

$$\lim_{s\to 0^+} \frac{G(t,s)}{G(h(s),s)} = \lim_{s\to 0^+} \frac{a_0(s) + a_1(s)t - \frac{1}{\Gamma(\alpha)}(t-s)}{a_0(s) + a_1(s)s}$$

Note that if $a_0(0) > 0$, since $a_0(0) \geq \frac{1}{\Gamma(\alpha)} - a_1(0)$ then for $0 < t < 1$ it's satisfied that

$$\begin{aligned}
\lim_{s\to 0^+} \frac{G(t,s)}{G(h(s),s)} &= \frac{a_0(0) + a_1(0)t - \frac{t}{\Gamma(\alpha)}}{a_0(0)} \\
&= \frac{a_0(0) - t\left(\frac{1}{\Gamma(\alpha)} - a_1(0)\right)}{a_0(0)}
\end{aligned}$$

so, if $a_1(0) = \frac{1}{\Gamma(\alpha)}$, then

$$\lim_{s\to 0^+} \frac{G(t,s)}{G(h(s),s)} = 1$$

and if $a_1(0) < \frac{1}{\Gamma(\alpha)}$, then

$$\begin{aligned}
\lim_{s\to 0^+} \frac{G(t,s)}{G(h(s),s)} &= \frac{a_0(0) - t\left(\frac{1}{\Gamma(\alpha)} - a_1(0)\right)}{a_0(0)} \\
&\geq \frac{(1-t)\left(\frac{1}{\Gamma(\alpha)} - a_1(0)\right)}{a_0(0)} \\
&> 0.
\end{aligned}$$

In other case, if $a_0(0) = 0$, since $a_0(s) \geq 0$ for all $s \in [0,1]$, then $a_0'(0) \geq 0$. Furthermore, since $a_1(0) \geq \frac{1}{\Gamma(\alpha)} - a_0(0) = \frac{1}{\Gamma(\alpha)}$ and $a_1(0) \leq \frac{1}{\Gamma(\alpha)}$, then $a_1(0) = \frac{1}{\Gamma(\alpha)}$ and $a_1'(0) \geq \frac{d}{ds}\left(\frac{1-s}{\Gamma(\alpha)} - a_0(s)\right)|_{s=0} = -\frac{1}{\Gamma(\alpha)} - a_0(0)$. From all that one has for all $0 < t < 1$ that

$$\begin{aligned}
\lim_{s\to 0^+} \frac{G(t,s)}{G(h(s),s)} &= \lim_{s\to 0^+} \frac{a_0(s) + a_1(s)t - \frac{1}{\Gamma(\alpha)}(t-s)}{a_0(s) + a_1(s)s} \\
&= \lim_{s\to 0^+} \frac{a_0'(s) + a_1'(s)t + \frac{1}{\Gamma(\alpha)}}{a_0'(s) + a_1(s) + sa_1'(s)} \\
&= \frac{a_0'(0) + a_1'(0)t + \frac{1}{\Gamma(\alpha)}}{a_0'(0) + a_1(0)} \\
&\geq \frac{(1-t)\left(a_0'(0) + \frac{1}{\Gamma(\alpha)}\right)}{a_0'(0) + \frac{1}{\Gamma(\alpha)}} \\
&> 0
\end{aligned}$$

Finally, in the third case, since $a_1(s) > \frac{1}{\Gamma(\alpha)}$ then $h(s) = 1$, so that

$$\lim_{s\to 0^+} \frac{G(t,s)}{G(h(s),s)} = \lim_{s\to 0^+} \frac{a_0(s) + a_1(s)t - \frac{1}{\Gamma(\alpha)}(t-s)}{a_0(s) + a_1(s) - \frac{1}{\Gamma(\alpha)}(1-s)}$$

Since $a_1(0) \geq \frac{1}{\Gamma(\alpha)}$, if $a_0(0) > 0$, it follows that

$$\begin{aligned}
\lim_{s\to 0^+} \frac{G(t,s)}{G(h(s),s)} &= \lim_{s\to 0^+} \frac{a_0(s) + a_1(s)t - \frac{1}{\Gamma(\alpha)}(t-s)}{a_0(s) + a_1(s) - \frac{1}{\Gamma(\alpha)}(1-s)} \\
&\geq \frac{a_0(0)}{a_0(0) + a_1(0) - \frac{1}{\Gamma(\alpha)}} \\
&> 0
\end{aligned}$$

Otherwise, if $a_1(0) > \frac{1}{\Gamma(\alpha)}$ and $a_0(0) = 0$, then

$$\lim_{s\to 0^+} \frac{G(t,s)}{G(h(s),s)} > \frac{a_0(0)}{a_0(0) + a_1(0) - \frac{1}{\Gamma(\alpha)}} \geq 0$$

and if $a_1(0) = \frac{1}{\Gamma(\alpha)}$ and $a_0(0) = 0$, since $a_0(s) \geq 0$ and $a_1(s) \geq \frac{1-s}{\Gamma(\alpha)} - a_0(s)$ for all $s \in [0,1]$, then $a_0'(0) \geq 0$ and $a_1'(0) \geq 0$, so that

$$\begin{aligned}
\lim_{s\to 0^+} \frac{G(t,s)}{G(h(s),s)} &= \lim_{s\to 0^+} \frac{a_0(s) + a_1(s)t - \frac{1}{\Gamma(\alpha)}(t-s)}{a_0(s) + a_1(s) - \frac{1}{\Gamma(\alpha)}(1-s)} \\
&= \lim_{s\to 0^+} \frac{a_0'(s) + a_1'(s)t + \frac{1}{\Gamma(\alpha)}}{a_0'(s) + a_1'(0) + \frac{1}{\Gamma(\alpha)}} \\
&= \frac{a_0'(0) + a_1'(0)t + \frac{1}{\Gamma(\alpha)}}{a_0'(0) + a_1'(0) + \frac{1}{\Gamma(\alpha)}} \\
&> 0
\end{aligned}$$

In any case,

$$\lim_{s\to 0^+} \frac{G(t,s)}{G(h(s),s)} > 0, \quad \forall\, 0 < t < 1$$

and

$$\lim_{s\to 1^-} \frac{G(t,s)}{G(h(s),s)} > 0, \quad \forall\, 0 < t < 1$$

and since $G(t,s) > 0$ for all $(t,s) \in (0,1)\times(0,1)$, then for all $0 < a < b < 1$ there exists $\sigma = \sigma(\alpha, a, b)$ such that $G(t,s) \geq \sigma G(h(s),s)$ for all $(t,s) \in [a,b]\times[0,1]$.

Now, if $1 < \alpha < 2$, then from lemma 3.2.2, it's known that

$$G(t,s) = a_1(s)t^{\alpha-1} - \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}\mu(t-s)$$

for some $a_1 \in C[0,1]$. Since $G(t,s) > 0$ for all $(t,s) \in (0,1)\times(0,1)$, in particular $0 < t \leq s < 1$ imply that $a_1(s) > 0$ for all $s \in (0,1)$. Furthermore, for $s \in (0,1)$ such that $0 < s < t < 1$ one has that

$$0 < G(t,s) = a_1(s)t^{\alpha-1} - \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} \quad \Leftrightarrow \quad a_1(s) > \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)t^{\alpha-1}} =: \frac{g(t)}{\Gamma(\alpha)}, \quad \forall\, 0 < s < t < 1$$

since $g \in C^1(s,1)$. In particular, given $0 < s < t < 1$,

$$g'(t) = \frac{(\alpha-1)\left[(t-s)^{\alpha-2}t^{\alpha-1} - (t-s)^{\alpha-1}t^{\alpha-2}\right]}{t^{2(\alpha-1)}} = \frac{(\alpha-1)[t-(t-s)](t-s)^{\alpha-2}}{t^{\alpha}} > 0$$

so $g \in C[s,1]$ is increasing, so that $\max_{s\le t\le 1} g(t) = g(1)$, and then

$$a_1(s) \geq \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, \quad \forall\, s \in (0,1)$$

By other side, note that $G(\cdot,s) \in C^1((s,1])$ for all $s \in [0,1)$, and that G is increasing on $0 < t < s < 1$. Furthermore, if $0 < s < t < 1$, then

$$G'(t,s) = (\alpha-1)a_1(s)t^{\alpha-2} - \frac{(\alpha-1)}{\Gamma(\alpha)}(t-s)^{\alpha-2}$$

so, $G'(t,s) < 0$ iff

$$a_1^{\frac{1}{\alpha-2}}(s)t > \frac{1}{\Gamma(\alpha)^{\frac{1}{\alpha-2}}}(t-s)$$

or equivalently,

$$t\left(1 - [a_1(s)\Gamma(\alpha)]^{\frac{1}{\alpha-2}}\right) = t\left(\frac{[a_1(s)\Gamma(\alpha)]^{\frac{1}{2-\alpha}} - 1}{[a_1(s)\Gamma(\alpha)]^{\frac{1}{2-\alpha}}}\right) < s$$

so, given $0 < s < t < 1$,

$$G'(t,s) < 0 \quad \Leftrightarrow \quad \begin{cases} t > \dfrac{s[a_1(s)\Gamma(\alpha)]^{\frac{1}{2-\alpha}}}{[a_1(s)\Gamma(\alpha)]^{\frac{1}{2-\alpha}} - 1}, & a_1(s) < \frac{1}{\Gamma(\alpha)} \\ \hfill 0 < s, & a_1(s) = \frac{1}{\Gamma(\alpha)} \\ t < \dfrac{s[a_1(s)\Gamma(\alpha)]^{\frac{1}{2-\alpha}}}{[a_1(s)\Gamma(\alpha)]^{\frac{1}{2-\alpha}} - 1}, & a_1(s) > \frac{1}{\Gamma(\alpha)} \end{cases}$$

since the condition $a_1(s) < \frac{1}{\Gamma(\alpha)}$ implies that

$$\frac{s[a_1(s)\Gamma(\alpha)]^{\frac{1}{2-\alpha}}}{[a_1(s)\Gamma(\alpha)]^{\frac{1}{2-\alpha}} - 1} \leq 0$$

in this case $G'(t,s) < 0$ for all $0 < s < t < 1$.

By other side, the condition $a_1(s) > \frac{1}{\Gamma(\alpha)}$ implies that

$$\frac{s[a_1(s)\Gamma(\alpha)]^{\frac{1}{2-\alpha}}}{[a_1(s)\Gamma(\alpha)]^{\frac{1}{2-\alpha}} - 1} \geq s$$

also, the following equivalence is true,

$$\begin{aligned}\frac{s[a_1(s)\Gamma(\alpha)]^{\frac{1}{2-\alpha}}}{[a_1(s)\Gamma(\alpha)]^{\frac{1}{2-\alpha}}-1}\geq 1 \quad &\Leftrightarrow \quad s\,[a_1(s)\Gamma(\alpha)]^{\frac{1}{2-\alpha}}\geq [a_1(s)\Gamma(\alpha)]^{\frac{1}{2-\alpha}}-1\\ &\Leftrightarrow \quad a_1(s)\leq \frac{1}{\Gamma(\alpha)(1-s)^{2-\alpha}}\end{aligned}$$

so if $a_1(s)\leq \frac{1}{\Gamma(\alpha)(1-s)^{2-\alpha}}$, then $G'(t,s)<0$ for all $0<s<t<1$. Finally, if $a_1(s)>\frac{1}{\Gamma(\alpha)(1-s)^{2-\alpha}}$, then there exists $t_0\in(s,1)$ such that $G'(t,s)<0$ for all $0<s<t\leq t_0<1$, and $G'(t,s)>0$ for all $t_0<t<1$. In particular, $G(\cdot,s)$ reachs its maximum on either, $t=s$ or $t=1$. Note that

$$\begin{aligned}G(s,s)\geq G(1,s) \quad &\Leftrightarrow \quad a_1(s)s^{\alpha-1}\geq a_1(s)-\frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1}\\ &\Leftrightarrow \quad a_1(s)\leq \frac{1}{\Gamma(\alpha)}\left[\frac{(1-s)^{\alpha-1}}{1-s^{\alpha-1}}\right]\end{aligned}$$

Then, defining $h:[0,1]\to[0,1]$ as

$$h(s)=\begin{cases} s, & a_1(s)\leq \dfrac{1}{\Gamma(\alpha)}\left[\dfrac{(1-s)^{\alpha-1}}{1-s^{\alpha-1}}\right]\\ 1, & a_1(s)> \dfrac{1}{\Gamma(\alpha)}\left[\dfrac{(1-s)^{\alpha-1}}{1-s^{\alpha-1}}\right]\end{cases}$$

one has that $\max_{0\leq t\leq 1}G(t,s)=G(h(s),s)$ for all $s\in[0,1]$.

From this,

$$\lim_{s\to 0^+}\frac{G(t,s)}{G(h(s),s)}=\lim_{s\to 0^+}\frac{a_1(s)t^{\alpha-1}-\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}\mu(t-s)}{a_1(s)h(s)^{\alpha-1}-\frac{1}{\Gamma(\alpha)}(h(s)-s)^{\alpha-1}}$$

Also, since a_1 is locally lipschitz on 0, then there exist $\epsilon>0$ such that $a_1\in C^1[0,\epsilon)$ and

- $a_1(s)\leq \dfrac{1}{\Gamma(\alpha)}\left[\dfrac{(1-s)^{\alpha-1}}{1-s^{\alpha-1}}\right]$ for all $s\leq\epsilon$, or
- $a_1(s)> \dfrac{1}{\Gamma(\alpha)}\left[\dfrac{(1-s)^{\alpha-1}}{1-s^{\alpha-1}}\right]$ for all $0<s\leq\epsilon$.

In the first case $h(s)=s$ for all $s\leq\epsilon$, and since $a_1(s)\geq\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}$, then $a_1(0)=\frac{1}{\Gamma(\alpha)}$. From the previous also follows that $a_1'(0)\geq-\frac{\alpha-1}{\Gamma(\alpha)}$, so for $0<t<1$ follows that

$$\begin{aligned}\lim_{s\to 0^+}\frac{G(t,s)}{G(h(s),s)}&=\lim_{s\to 0^+}\frac{a_1(s)t^{\alpha-1}-\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}}{a_1(s)s}\\ &=\lim_{s\to 0^+}\frac{a_1'(s)t^{\alpha-1}+\frac{\alpha-1}{\Gamma(\alpha)}(t-s)^{\alpha-2}}{a_1(s)+sa_1'(s)}\end{aligned}$$

$$\begin{aligned}
&\geq \frac{\frac{\alpha-1}{\Gamma(\alpha)}\left(-t^{\alpha-1}+t^{\alpha-2}\right)}{\frac{1}{\Gamma(\alpha)}}\\
&> 0, \quad \forall\, 0<t<1
\end{aligned}$$

Otherwise, in the second case, since $a_1(s) > \dfrac{1}{\Gamma(\alpha)}\left[\frac{(1-s)^{\alpha-1}}{1-s^{\alpha-1}}\right]$ for all $0 < s \leq \epsilon$, then $h(s) = 1$ for all $0 < s \leq \epsilon$. Then, if $a_1(0) > \frac{1}{\Gamma(\alpha)}$ it follows that

$$\begin{aligned}
\lim_{s\to 0^+}\frac{G(t,s)}{G(h(s),s)} &= \lim_{s\to 0^+}\frac{a_1(s)t^{\alpha-1}-\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}}{a_1(s)}\\
&= \frac{a_1(0)t^{\alpha-1}-\frac{1}{\Gamma(\alpha)}t^{\alpha-1}}{a_1(0)}\\
&> 0, \quad \forall\, 0<t<1
\end{aligned}$$

In other case, if $a_1(0) = \frac{1}{\Gamma(\alpha)}$, since $a_1(s) > \dfrac{1}{\Gamma(\alpha)}\left[\frac{(1-s)^{\alpha-1}}{1-s^{\alpha-1}}\right]$ for all $0 < s \leq \epsilon$, then $a_1'(0) = \infty$, which is a contradiction with the fact that $a_1 \in C^1[0,\epsilon]$.

Finally, since $s \leq h(s) \leq 1$ for all $s \in [0,1]$, then $\lim_{s\to 1^-} h(s) = 1$. Thus, for $0 < t < 1$

$$\lim_{s\to 1^-}\frac{G(t,s)}{G(h(s),s)} = \lim_{s\to 1^-}\frac{a_1(s)t^{\alpha-1}}{a_1(s)h(s)-\frac{1}{\Gamma(\alpha)}(h(s)-s)^{\alpha-1}\mu(h(s)-s)}$$

so, if $a_1(1) > 0$, then

$$\lim_{s\to 1^-}\frac{G(t,s)}{G(h(s),s)} = t^{\alpha-1} > 0, \quad \forall\, 0<t<1$$

and if $a_1(1) = 0$, since $\lim_{s\to 1^-}\frac{(1-s)^{\alpha-1}}{1-s^{\alpha-1}} = \infty$, then there exists $\epsilon > 0$ such that $a_1(s) < \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-s^{\alpha-1})}$, for all $1-\epsilon < s \leq 1$, so that $h(s) = s$ near 1. Therefore, given $0 < t < 1$,

$$\begin{aligned}
\lim_{s\to 1^-}\frac{G(t,s)}{G(h(s),s)} &= \lim_{s\to 1^-}\frac{a_1(s)t^{\alpha-1}}{a_1(s)s^{\alpha-1}}\\
&= t^{\alpha-1}\\
&> 0, \quad \forall\, 0<t<1
\end{aligned}$$

so, since $G(t,s) > 0$ for all $(t,s) \in (0,1)\times(0,1)$, then for all $0 < a < b < 1$ there exists $\sigma > 0$ such that

$$\frac{G(t,s)}{G(h(s),s)} \geq \sigma, \quad \forall\,(t,s)\in[a,b]\times[0,1]$$

so the proof is complete. □

Proof. If $\alpha = 2$, the Green function in the Caputo case is the same as in the Riemann-Liouville case, so the conclusion follows from lemma A.3.1.

If $1 < \alpha < 2$, then G has the form

$$G(t,s) = a_1(s)t - \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}\mu(t-s)$$

If $0 < t \leq s < 1$, then the condition $G(t,s) > 0$ is equivalent to $a_1(s) > 0$. Otherwise, if $0 < s < t < 1$, then the condition $G(t,s) > 0$ is equivalent to

$$a_1(s) > \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)t} =: g(t), \quad \forall\, 0 < s < t < 1$$

since $g \in C^1(s,1)$, with

$$g'(t) = \frac{(\alpha-1)t(t-s)^{\alpha-2} - (t-s)^{\alpha-1}}{t^2} = \frac{(t-s)^{\alpha-2}[s-(2-\alpha)t]}{t^2}$$

then $g'(t) > 0$ for all $t < \frac{s}{2-\alpha}$, so that

- $\max_{t\in[s,1]} g(t) = g(1) = (1-s)^{\alpha-1}$ if $s \geq 2-\alpha$.
- $\max_{t\in[s,1]} g(t) = g\left(\frac{s}{2-\alpha}\right) = \frac{\left(\frac{s}{2-\alpha}-s\right)^{\alpha-1}}{\frac{s}{2-\alpha}} = (\alpha-1)^{\alpha-1}\left[\frac{(2-\alpha)}{s}\right]^{2-\alpha}$ if $s < 2-\alpha$.

Anyway, defining the function $H : (0,1) \to \mathbb{R}$ as

$$H(s) := \begin{cases} \dfrac{(\alpha-1)^{\alpha-1}}{\Gamma(\alpha)}\left[\frac{(2-\alpha)}{s}\right]^{2-\alpha} & 0 < s < 2-\alpha \\ \dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} & s \geq 2-\alpha \end{cases}$$

one has that

$$a_1(s) > \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)t} \quad \forall\, 0 < s < t < 1 \qquad \Leftrightarrow \qquad a_1(s) > H(s), \quad \forall\, 0 < s < t < 1$$

so $G(t,s) > 0$ for all $(t,s) \in (0,1)\times(0,1)$ if and only if $a_1(s) > H(s)$ for all $s \in (0,1)$. However, $a_1(s) > H(s)$ for all $s \in (0,1)$ in particular implies that

$$\lim_{s\to 0^+} a_1(s) \geq \lim_{s\to 0^+} H(s) = \lim_{s\to 0^+} \frac{(\alpha-1)^{\alpha-1}}{\Gamma(\alpha)}\left[\frac{(2-\alpha)}{s}\right]^{2-\alpha} = \infty$$

so a_1 is not continuous in $s = 0$, and then $G \notin C([0,1]\times[0,1])$. In the rest of the proof, it will be assumed that $G \notin C([0,1]\times[0,1])$, but $G \in C([0,1]\times(0,1))$.

Now note that $G(t,s)$ is increasing on $t \in [0,s]$ for all $s \in (0,1)$, and

$$G'(t,s) = a_1(s) - \frac{(\alpha-1)}{\Gamma(\alpha)}(t-s)^{\alpha-2}\mu(t-s)$$

so, if $0 < s < t < 1$ then

$$G'(t,s) = a_1(s) - \frac{(\alpha-1)}{\Gamma(\alpha)}(t-s)^{\alpha-2}$$

Moreover, it follows that $G'(t,s) \leq 0$ if and only if

$$a_1(s)^{\frac{1}{\alpha-2}} \geq \left[\frac{\alpha-1}{\Gamma(\alpha)}\right]^{\frac{1}{\alpha-2}}(t-s)$$

or, equivalently,

$$t \leq \left[\frac{\alpha-1}{a_1(s)\Gamma(\alpha)}\right]^{\frac{1}{2-\alpha}} + s =: g_2(s)$$

since $a_1(s) > 0$ and $\alpha > 1$, then $g_2(s) > s$ for all $s \in (0,1)$. Furthermore,

$$g_2(s) \geq 1 \quad \Leftrightarrow \quad a_1(s) \leq \frac{\alpha-1}{\Gamma(\alpha)}\left[\frac{1}{1-s}\right]^{2-\alpha}$$

and since $a_1(s) \geq \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}$, then

$$\begin{aligned} g_2(s) \geq 1 \quad &\Leftrightarrow \quad \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \leq a_1(s) \leq \frac{\alpha-1}{\Gamma(\alpha)}\left[\frac{1}{1-s}\right]^{2-\alpha} \\ &\Rightarrow \quad 1-s \leq \alpha-1 \\ &\Leftrightarrow \quad s \geq 2-\alpha > 0 \end{aligned}$$

so, if $g_2(s) \geq 1$, then $G(\cdot,s)$ is non increasing on $[0,1]$, and if $g_2(s) < 1$, then there exists $t_0 \in (s,1)$ such that $G(\cdot,s)$ is non increasing on $[s,t_0]$ and non decreasing on $[t_0,1]$, which leads to conclude that $G(\cdot,s)$ reach its maximum on either, $t = s$ or $t = 1$, and for $s > 0$ small enough, $g_2(s) < 1$. Note that

$$\begin{aligned} G(s,s) \geq G(1,s) \quad &\Leftrightarrow \quad (1-s)a_1(s) - \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \leq 0 \\ &\Leftrightarrow \quad a_1(s) \leq \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} \end{aligned}$$

so, defining $h : (0,1) \to (0,1)$ as

$$h(s) = \begin{cases} s, & a_1(s) \leq \dfrac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} \\ 1, & \dfrac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} < a_1(s) \end{cases}$$

then from the previous calculus one concludes that $G(t,s) \leq G(h(s),s)$ for all $(t,s) \in [0,1] \times (0,1)$.

Now, since $\lim_{s\to 0^+} a_1(s) = \infty$ and $\dfrac{(1-s)^{\alpha-2}}{\Gamma(\alpha)}$ is bounded near $s=0$, then there exists $\epsilon > 0$ such that $a_1(s) > \dfrac{(1-s)^{\alpha-2}}{\Gamma(\alpha)}$ for all $0 < s < \epsilon$, so that $h(s) = 1$ for all $0 < s < \epsilon$.

In particular, it follows that

$$\begin{aligned}\lim_{s\to 0^+} \frac{G(t,s)}{G(h(s),s)} &= \lim_{s\to 0^+} \frac{a_1(s)t - \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}}{a_1(s)} \\ &= t\end{aligned}$$

so it follows that

$$\lim_{s\to 0^+} \frac{G(t,s)}{G(h(s),s)} > 0, \quad \forall\, 0 < t < 1$$

Finally, note that since $h(s) \geq s$ for all $s \in [0,1]$, in particular $\lim_{s\to 1^-} h(s) = 1$. Then, for $0 < t < 1$ follows that

$$\lim_{s\to 1^-} \frac{G(t,s)}{G(h(s),s)} = \lim_{s\to 1^-} \frac{a_1(s)t}{a_1(s)h(s)} = t > 0$$

and since $G(t,s) > 0$ for all $(t,s) \in (0,1)\times(0,1)$, then for all $0 < a < b < 1$ there exists $\sigma = \sigma(\alpha, a, b)$ such that $G(t,s) \geq \sigma G(h(s),s)$ for all $(t,s) \in [a,b]\times(0,1)$.

$\square$

Bibliography

[1] S. Kesavan. *Topics in functional analysis and applications. School of Mathematics Tata Institute of Fundamental Research Bangalore, India.*

[2] G. Doetsch. *Introduction to the Theory and Application of the Laplace Transformation. Springer-Verlag Berlin Heidelberg New York 1974.*

[3] L. Grafakos. *Classical Fourier Analysis, Third Edition. Springer New York Heidelberg Dordrecht London.*

[4] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer New York Dordrecht Heidelberg London.*

[5] L. Evans. *Partial Differential Equations. Springer-Verlag Berlin and Heidelberg GmbH & Co. K. 2007. Berlin.*

[6] S. Kesavan. *Measure and Integration. Springer Texts and Readings in Mathematics 77.*

[7] A. Granas, J. Dugundji. *Fixed Point Theory. Springer-Verlag New York 2003.*

[8] R. Agarwal, M. Meehan, D. O'Regan. *Fixed Point Theory and Applications. Cambridge University Press 2004.*

[9] T. Sandev, Z. Tomovski. *Fractional Equations and Models: Theory and Applications. Springer Nature Switzerland AG 2019.*

[10] D. Andrica, T. Rassias. *Differential and Integral Inequalities. Springer Nature Switzerland AG 2019.*

[11] B. Guo, X. Pu, F. Huang. *Fractional Partial Differential Equations and their Numerical Solutions. World Scientific Publishing 2015.*

[12] A. Lischke, G. Pang, M. Gulian, F. Song, C. Glusa, X. Zheng, Z. Mao, W. Cai, M. Meerschaert, M Ainsworth, G. Karniadakis. *What is the fractional Laplacian? A comparative review with new results. Journal of Computational Physics 404 (2020) 109009.*

[13] Z. Bai and H. Lü, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl. **311** (2005) 495—505

[14] O. P. Agrawal, *Fractional variational calculus and the transversality conditions*, J. Phys. A: Math. Gen. **39** (2006) 10375—10384

[15] L. Yang, X. Liu, M. Jia, *Multiplicity results for second-order m-point boundary value problem*, J. Math. Anal. Appl. **324** (2006) 532–542

[16] T. Cazenave, F. Dickstein and F. B. Weissler, *On the structure of global solutions of the nonlinear heat equation in a ball*, J. Math. Anal. Appl., **360** (2009), n.2, 537–547

[17] Z. Shuqin, *Monotone iterative method for initial value problem involving Riemann-Liouville fractional derivatives*, Nonlinear Analysis **71** (2009) 2087–2093

[18] D. Jiang and C. Yuan, *The positive properties of the Green function for Dirichlet-type boundary value problems of nonlinear fractional differential equations and its application*, Nonlinear Analysis **72** (2010) 710–719

[19] R. P. Agarwal, D. O'Regan and S. Stanek, *Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations*, J. Math. Anal. Appl. **371** (2010) 57—68

[20] T. Cazenave, F. Dickstein and F. B. Weissler, *Spectral properties of stationary solutions of the nonlinear heat equation*, Publications Mathèmatiques, **55** (2011), n.1, 185–200

[21] Y. Liu, Z. Weiguo and L. Xiping, *A sufficient condition for the existence of a positive solution for a nonlinear fractional differential equation with the Riemann–Liouville derivative*, Applied Mathematics Letters **25** (2012) 1986—1992

[22] J. R. Graef, L. Kong, Q. Kong and M. Wang, *Existence and uniqueness of solutions for a fractional boundary value problem with Dirichlet boundary condition*, Electronic Journal of Qualitative Theory of Differential Equations (2013), **55**, 1-11

[23] G. L. Karakostas, *Non-existence of solutions for two-point fractional and third-order boundary-value problems*, Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 152, pp. 1–19.

[24] Yongping Sun and Yan Sun, *Positive solutions and monotone iterative sequences for a fractional differential equation with integral boundary conditions*, Sun and Sun Advances in Difference Equations 2014, 2014:29

[25] W. Sun and Y. Wang, *Multiple positive solutions of nonlinear fractional differential equations with integral boundary value conditions*, Fractional Calculus & Applied Analysis Volume 17, Number 3 (2014)

[26] G. Chai and S. Hu, *Existence of positive solutions for a fractional high-order three-point boundary value problem*, Chai and Hu Advances in Difference Equations 2014, 2014:90

[27] L. Yang, C. Shen and D. Xie, *Multiple positive solutions for nonlinear boundary value problem of fractional order differential equation with the Riemann-Liouville derivative*, Advances in Difference Equations 2014, 2014:284

[28] V. Marino, F. Pacella and B. Sciunzi, *Blow up of solutions of semilinear heat equations in general domains*, Communications in Contemporary Mathematics Vol. 17, No. 2 (2015) 1350042 (17 pages)

[29] J. Tan and C. Cheng, *Fractional boundary value problems with Riemann-Liouville fractional derivatives*, Advances In Difference Equations (2015) 2015:80

[30] P. Li, H. Wang and Z. Li, *In*

nitely many solutions to boundary value problems for a coupled system of fractional differential equations, J. Nonlinear Sci. Appl. **9** (2016), 3433–3444

[31] D. Zhao and Y. Liu, *Positive solutions for a class of fractional differential coupled system with integral boundary value conditions*, J. Nonlinear Sci. Appl. **9** (2016), 2922–2942

[32] F. Bensidhoum and H. Dib, *On some regular fractional Sturm-Liouville problems with generalized Dirichlet conditions*, Journal of Integral Equations and Applications Volume 28, Number 4, Winter 2016

[33] A. Dhifli and B. Khamessi, *Existence and boundary behavior of positive solution for a Sturm–Liouville fractional problem with p-laplacian*, J. Fixed Point Theory Appl., Springer International Publishing AG 2017

[34] Zagharide Zine El Abidine, *Multiple Positive Solutions for a Coupled System of Nonlinear Fractional Differential Equations on the Half-line*, Mediterr. J. Math., Springer International Publishing 2017

[35] K. Saoudi, P. Agarwal, P. Kumam, A. Ghanmi and P. Thounthong, *The Nehari manifold for a boundary value problem involving Riemann–Liouville fractional derivative*, Advances In Difference Equations (2018) 2018:263

[36] V. Hedayati and M. E. Samei, *Positive solutions of fractional differential equation with two pieces in chain interval and simultaneous Dirichlet boundary conditions*, Boundary Value Problems (2019) 2019:141

[37] K. B. Oldham and J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, Academic Press (1974)

[38] L. Barreira and C. Valls, *Ordinary Differential Equations Qualitative Theory*, Graduate Studies in Mathematics, Volume 137, American Mathematical Society 2012

[39] L. Grafakos, *Classical Fourier Analysis, Third Edition*, Graduate Texts in Mathematics 249, Springer New York Heidelberg Dordrecht London (2014)

[40] H. A. Fallahgoul, S. M. Focardi and F. J. Fabozzi, *Fractional Calculus and Fractional Processes with Applications to Financial Economics: Theory and Application*, Academic Press (2017)

[41] A. Atangana, *Fractional Operators with Constant and Variable Order with Application in Geo-Hydrology*, Academic Press (2018)

[42] T. Sandev and Z. Tomovski, *Fractional Equations and Models: Theory and Applications*, Developments in Mathematics Volume 61, Springer Nature Switzerland AG 2019

[43] J. Hristov and R. Bennacer, *Heat Conduction: Methods, Applications and Research*, Nova Science Publishers, New York 2019

[44] B. Ahmad, J. Henderson and R. Luca, *Boundary value problems for fractional differential equations and systems*, Trends in Abstract and Applied Analysis Volume 9. (2021)