Exercise 6 - Thermodynamic Systems

6.1 Internal Energy

The variation of internal energy for a **closed system** is

$$\mathrm{d}U = \delta W + \delta Q,$$

or rewritten

$$\Delta U = W + Q,$$

where

- U is the internal energy,
- W is the mechanical energy (work of external forces to the system),

$$\delta W = -P_{\text{ext}} \cdot \mathrm{d}V.$$

- Q is the thermal energy exchanged with the surrounding.

We distinguish three special cases:

- Adiabatic process: no heat transfer, $\Delta Q = 0$, $\Delta U = \Delta W$.
- Isochoric process: no volume change, $\Delta W = 0$, $\Delta U = \Delta Q$.
- Isolated systems: $\Delta U = 0$

For the special case of incompressible systems we have

$$U(T) = m \cdot c \cdot T.$$

Remark.

- These assumptions are also valid for compressible gases if temperature variations are not too large.
- We consider $U = 0$ at 0 Kelvin.

6.2 Enthalpy

The enthalpy is defined as

$$H = U + pV.$$

Moreover, it holds

$$\begin{aligned}
\mathrm{d}H &= \mathrm{d}U + P \cdot \mathrm{d}V + V \cdot \mathrm{d}P \\
&= \delta W + \delta Q + P \cdot \mathrm{d}V + V \cdot \mathrm{d}P \\
&= -P \cdot \mathrm{d}V + m \cdot c_p \cdot \mathrm{d}T + k \cdot \mathrm{d}P + P \cdot \mathrm{d}V + V \cdot \mathrm{d}P \\
&= m \cdot c_p \cdot \mathrm{d}T + (k + V) \cdot \mathrm{d}P,
\end{aligned}$$

where the factor k takes into account non-isobaric processes, i.e. $k = 0$ for isobaric processes. Thanks to the enthalpy we can now formulate the energy balance for a open system:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \sum \dot{H}_{\text{in}} - \sum \dot{H}_{\text{out}} + \dot{Q} + \dot{W},$$

where

- U is the internal energy,
- $\dot{H}$ is the enthalpy-flow,
- $\dot{W}$ is the mechanical energy (power of external forces to the system),
- $\dot{Q}$ is the thermal energy exchanged with the surrounding.

6.3 Ideal Gases

Recall that for an ideal gas we have

$$pV = n\bar{R}T = mRT,$$

where $\bar{R}$ is the universal gas constant and $R = \bar{R}/M_{\text{gas}}$, where M_{gas} is the molar mass. For ideal gases both the internal energy and the enthalpy are function of the temperature, i.e.

$$H(T) = m \cdot c_p \cdot T,$$
$$U(T) = m \cdot c_v \cdot T,$$

where c_v and c_p are the specific heats at constant volume and pressure, respectively. They are related by $R = c_p - c_v$.

6.4 Heat Transfer

There are three kinds of heat transfer:

Conduction: Here we use the Fourier's law. For the one-dimensional case we have

$$\dot{Q} = \frac{\kappa A}{l}(T_1 - T_2),$$

where κ is the thermal conductivity A is the cross-sectional area, and l is the length.

Heat Convection: Here we use the Newton's law. The heat transfer between a solid body with contact surface A and temperature T_1 and the surrounding fluid with temperature T_2 is

$$\dot{Q} = kA(T_1 - T_2),$$

where k is the heat transfer coefficient.

Heat radiation: Here we use the Stefan-Bolzmann's law. The heat radiation from a body with surface A and temperature T_1 to the surrounding at temperature T_2 is

$$\dot{Q} = \varepsilon\sigma A(T_1^4 - T_2^4),$$

where $\epsilon \in [0, 1]$ is the emissivity and σ is the Stefan-Bolzmann constant.

6.5 Example

In order to extend your businness and attract new customers to your SpaghETH, you decide to recreate the typical "Ticino Experience"and produce the real Risotto[1] and serve it on the Polyterrasse. Everybody knows (or should know!) that a keypoint for a good risotto is the so called *Mantecatura*[2]. This ancient procedure consists in mixing butter and cheese (any kind of cheese is ok, the more the best) with the hot rice. Let's assume that the pot where you cook the rice is isoltated. We model the hot rice as a unique element with density ρ_{r}. We opt for a traditional recipe, using butter and parmigiano Reggiano. Both the butter and the parmigiano Reggiano can be modeled to have a cubic form. The butter cubes have side length x_{b} and mass m_{b}, while the parmigiano cubes have side length x_{p} and mass m_{p}. We assume that the butter has constant temperature T_{b} and the parmigiano T_{p}. The hot rice with mass m_{r} and T_{r} is continuously mixed with the ingredients with a wooden spoon. While you are "mantecating", Mr. Balerna throws some peperonata with density ρ_{pep} and specific heat c_{pep} into the pot at rate $\overset{*}{V}_{\mathrm{pep}}$. Since the pot is assumed to be well isolated, the heat exchange occurs only with the air in the environment (temperature T_{air} on the hot rice surface A_{r}). The heat exchange coefficient α_{r} is assumed to be known. A sketch of the system is depicted in Figure 1.
For the modeling of the system, please take into account following assumptions:

- The butter and the parmigiano remain cubic and don't melt completely.
- The heat exchange coefficient between hot rice and the butter reads α_{b}.
- The heat exchange coefficient between hot rice and the parmigiano reads α_{p}.
- The melting enthalpy and the melting temperature of the butter are L_{b} and $T_{\mathrm{s,b}}$. The specific heat of the butter is c_{b}.
- The melting enthalpy and the melting temperature of the parmigiano Reggiano are L_{p} and $T_{\mathrm{s,p}}$. The specific heat of the Parmigiano Reggiano is c_{p}.
- The whole process occurs at ambient pressure.

1. List all the reservoirs with the corresponsing level variables of the system.
2. Find the differential equations which describe the given system, assuming that we are throwing one cube of parmigiano Reggiano and one cube of butter.

[1]Note that in ticino dialect one says "*Ul risott ticines*"
[2]For the lovers of cooking, have a look at ...

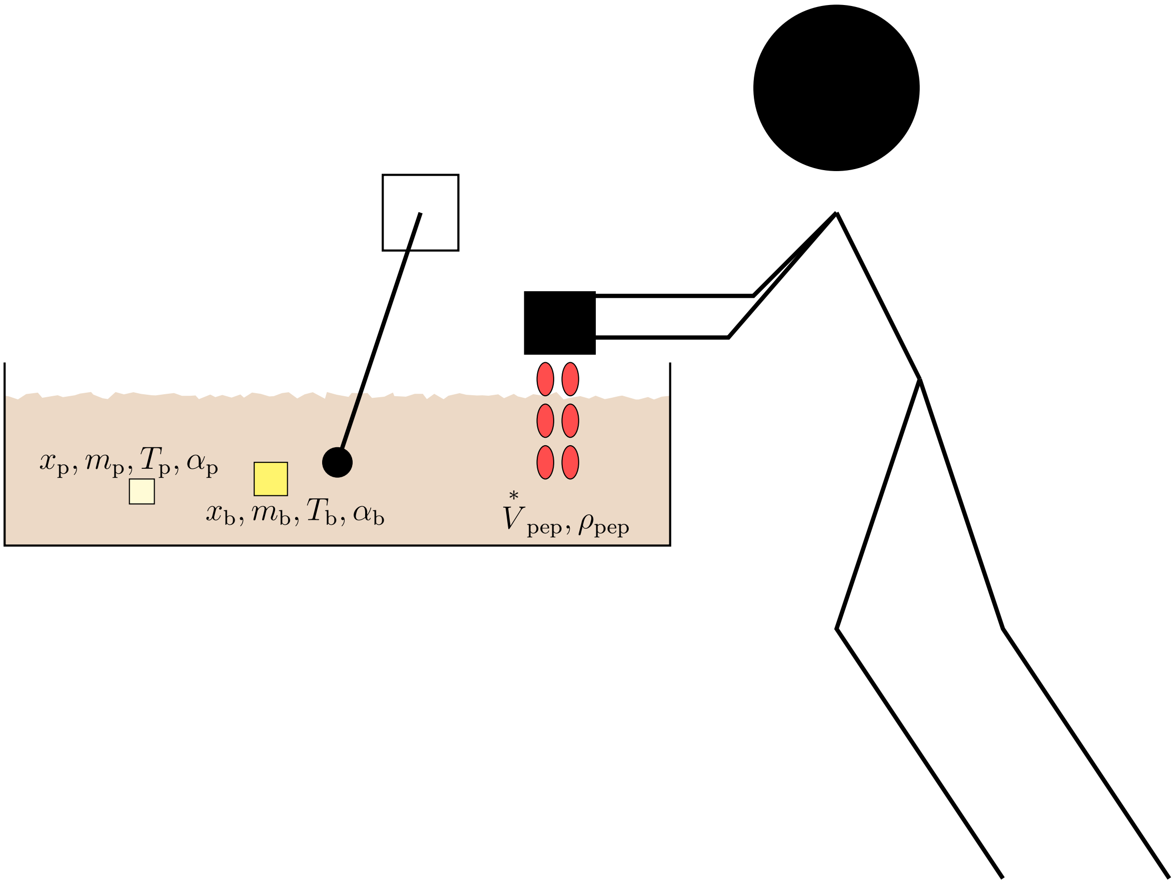

Figure 1: Sketch of the system.

Solution.

1. The reservoirs of the system are the butter mass m_b, the parmigiano mass m_p, the hot rice mass m_r, and the internal energy of the hot rice U_r (with level variable $T_\mathrm{r}(t)$).

2. The melting butter mass flow $\overset{*}{m}_\mathrm{b}$ can be computed as

$$\overset{*}{m}_\mathrm{b} = \frac{\alpha_\mathrm{b} \cdot A_\mathrm{b}}{L_\mathrm{b} + c_\mathrm{b} \cdot (T_\mathrm{s,b} - T_\mathrm{b})} \cdot (T_\mathrm{r}(t) - T_\mathrm{b}),$$

where the butter surface $A_\mathrm{b}(t)$ depends on the remaining mass of butter and reads

$$A_\mathrm{b}(t) = 6x_\mathrm{b}^2 = 6 \cdot \left(\frac{m_\mathrm{b}(t)}{\rho_\mathrm{b}}\right)^{\frac{2}{3}}.$$

Analogously, it the melting parmigiano mass flow $\overset{*}{m}_\mathrm{p}$ can be computed as

$$\overset{*}{m}_\mathrm{p} = \frac{\alpha_\mathrm{b} \cdot A_\mathrm{p}}{L_\mathrm{p} + c_\mathrm{p} \cdot (T_\mathrm{s,p} - T_\mathrm{p})} \cdot (T_\mathrm{r}(t) - T_\mathrm{p}),$$

where the parmigiano surface $A_\mathrm{p}(t)$ depends on the remaining mass of butter and reads

$$A_\mathrm{p}(t) = 6x_\mathrm{p}^2 = 6 \cdot \left(\frac{m_\mathrm{p}(t)}{\rho_\mathrm{p}}\right)^{\frac{2}{3}}.$$

The change in mass for the butter reads

$$\frac{\mathrm{d}}{\mathrm{d}t} m_{\mathrm{b}}(t) = -\overset{*}{m}_{\mathrm{b}}(t).$$

The change in mass for the parmigiano reads

$$\frac{\mathrm{d}}{\mathrm{d}t} m_{\mathrm{p}}(t) = -\overset{*}{m}_{\mathrm{p}}(t).$$

The change in mass for the hot rice reads

$$\frac{\mathrm{d}}{\mathrm{d}t} m_{\mathrm{r}}(t) = \rho_{\mathrm{pep}} \cdot \overset{*}{V}_{\mathrm{pep}}(t) + \overset{*}{m}_{\mathrm{b}}(t) + \overset{*}{m}_{\mathrm{p}}(t).$$

The change in internal energy of the hot rice can be computed with the first law of thermodynamics for open system as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} U_{\mathrm{r}}(t) &= \overset{*}{Q} + \sum_i \overset{*}{m}_{\mathrm{i}} \cdot h_i \\
\frac{\mathrm{d}}{\mathrm{d}t} \left(m_{\mathrm{r}}(t) \cdot c_{\mathrm{r}} \cdot T_{\mathrm{r}}(t) \right) &= \alpha_{\mathrm{r}} \cdot A_{\mathrm{r}} \cdot (T_{\mathrm{air}} - T_{\mathrm{r}}(t)) \\
&\quad - (L_{\mathrm{b}} + c_{\mathrm{b}} \cdot (T_{\mathrm{s,b}} - T_{\mathrm{b}})) \cdot \overset{*}{m}_{\mathrm{b}}(t) - (L_{\mathrm{p}} + c_{\mathrm{p}} \cdot (T_{\mathrm{s,p}} - T_{\mathrm{p}})) \cdot \overset{*}{m}_{\mathrm{p}}(t) \\
&\quad + \overset{*}{m}_{\mathrm{b}}(t) \cdot c_{\mathrm{b}} \cdot T_{\mathrm{s,b}} + \overset{*}{m}_{\mathrm{p}}(t) \cdot c_{\mathrm{p}} \cdot T_{\mathrm{s,p}} + \rho_{\mathrm{pep}} \cdot \overset{*}{V}_{\mathrm{pep}}(t) \cdot c_{\mathrm{pep}} \cdot T_{\mathrm{pep}}(t).
\end{aligned}$$