Lösung zu den Integralübungen

1. **a)** $\int \frac{x^2+2x}{(x+1)^2}\,dx = \int \frac{x^2+2x+1}{(x+1)^2}\,dx - \int \frac{1}{(x+1)^2}\,dx = \int dx + \frac{1}{x+1}$
$= x + \frac{1}{x+1} + D = \frac{x^2+x+1}{x+1} + D = \frac{x^2}{x+1} + 1 + D = \underline{\underline{\frac{x^2}{x+1} + C}}$,
wobei $D \in \mathbb{R}$ und $C := 1 + D$.

b) $$\int \frac{dx}{x^2-x+1} = \int \frac{dx}{x^2-x+\frac{1}{4}+\frac{3}{4}} = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} = \underline{\underline{\frac{2}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + C}}.$$

c) $\int \frac{dx}{\sqrt{x}+x^2}$. Subst. $x = u^2$. Dann ist $dx = 2u\,du$.

$$\begin{aligned}
\int \frac{dx}{\sqrt{x}+x^2} &= \int \frac{2u\,du}{u^4+u} = \int \frac{2\,du}{u^3+1} = \int \left(\frac{\frac{2}{3}}{u+1} + \frac{-\frac{2}{3}u+\frac{4}{3}}{u^2-u+1}\right) du \\
&= \int \left(\frac{2}{3}\cdot\frac{1}{u+1} - \frac{1}{3}\cdot\frac{2u-1}{u^2-u+1} + \frac{1}{u^2-u+1}\right) du \\
&= \frac{2}{3}\log(u+1) - \frac{1}{3}\log(u^2-u+1) + \frac{2}{\sqrt{3}}\arctan\frac{2u-1}{\sqrt{3}} + C \\
&= \underline{\underline{\frac{2}{3}\log(\sqrt{x}+1) - \frac{1}{3}\log(x-\sqrt{x}+1) + \frac{2}{\sqrt{3}}\arctan\frac{2\sqrt{x}-1}{\sqrt{3}} + C}}
\end{aligned}$$

d) Sei $u = x^2$, und also $du = 2xdx$, oder auch $xdx = \frac{1}{2}du$. Daher haben wir

$$\int \frac{x\,dx}{x^4+3} = \frac{1}{2}\int \frac{du}{u^2+3} = \frac{1}{2}\int \frac{du}{3((\frac{u}{\sqrt{3}})^2+1)} = \frac{1}{2\sqrt{3}}\int \frac{\frac{1}{\sqrt{3}}du}{(\frac{u}{\sqrt{3}})^2+1} =$$

$$= \frac{1}{2\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right) + C = \frac{\sqrt{3}}{6}\arctan\left(\frac{x^2\sqrt{3}}{3}\right) + C.$$

e) $\int \tanh^2 x\,dx = \int dx - \int (1-\tanh^2 x)\,dx = \underline{\underline{x - \tanh x + C}}$.

f) Seien $v(x) = \ln x$ und $u'(x) = \frac{1}{x^2}$. Mit einer partiellen Integration haben wir

$$\int \frac{\ln(x)}{x^2}\,dx = -\frac{1}{x}\ln x + \int \frac{1}{x}\cdot\frac{1}{x}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C = \underline{\underline{-\frac{\ln(x)+1}{x} + C}}.$$

Bitte wenden!

2. **a)** Substituiere $t = \tan\frac{x}{2}$. Dann ist

$$dt = \left(1 + \tan^2 \frac{x}{2}\right) \frac{1}{2}\, dx = \frac{1}{2\cos^2 \frac{x}{2}}\, dx = \frac{dx}{1+\cos x}$$

und somit

$$\int_0^{\pi/2} \frac{dx}{1+\cos x} = \int_0^1 dt = 1.$$

b) Eine partielle Integration liefert

$$\begin{aligned}\int_{-1}^{1} \frac{x^2\, dx}{(1+x^2)^2} &= \int_{-1}^{1} \frac{x}{2} \cdot \frac{2x}{(1+x^2)^2}\, dx = \left[\frac{x}{2} \cdot \frac{-1}{1+x^2}\right]_{-1}^{1} + \frac{1}{2}\int_{-1}^{1} \frac{dx}{1+x^2} \\ &= -\frac{1}{2} + \frac{1}{2}\left[\arctan x\right]_{-1}^{1} = \frac{\pi}{4} - \frac{1}{2}.\end{aligned}$$

3. **a)** Partialbruchzerlegung:

$$\begin{aligned}\frac{x}{x^3+x^2-x-1} &= \frac{x}{(x-1)(x+1)^2} \overset{!}{=} \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1} \\ &= \frac{(A+C)x^2 + (B+2C)x + (-A-B+C)}{(x-1)(x+1)^2}\end{aligned}$$

Durch Koeffizientenvergleich erhalten wir $A = -\frac{1}{4}$, $B = \frac{1}{2}$ und $C = \frac{1}{4}$. Somit ist

$$\begin{aligned}\int \frac{x}{x^3+x^2-x-1}\,\mathrm{d}x &= -\frac{1}{4}\int \frac{1}{x+1}\,\mathrm{d}x + \frac{1}{4}\int \frac{1}{x-1}\,\mathrm{d}x + \frac{1}{2}\int \frac{1}{(x+1)^2}\,\mathrm{d}x \\ &= -\frac{1}{4}\ln(x+1) + \frac{1}{4}\ln(x-1) - \frac{1}{2}\frac{1}{x+1} \\ &= \frac{1}{4}\ln\frac{x-1}{x+1} - \frac{1}{2}\frac{1}{x+1}\end{aligned}$$

b) Da x^2+1 das Nennerpolynom teilt, ist $(x^4+2x^3+3x^2+2x+2) : (x^2+1) = (x^2+2x+2)$ (Polynomdivision). Das Polynom x^2+2x+2 hat keine reelle Nullstellen, da dessen Diskriminante -4 ist. Wir betrachten daher

$$\frac{2x^3+5x^2+4x+1}{x^4+2x^3+3x^2+2x+2} = \frac{Ax+C}{x^2+1} + \frac{Bx+D}{x^2+2x+2}$$

Siehe nächstes Blatt!

$$= \frac{(A+B)x^3 + (2A+C+D)x^2 + (2A+B)x + 2C + D}{(x^2+1)(x^2+2x+2)}$$

Koeffizientenvergleich liefert $A = 2$, $B = C = 0$ und $D = 1$. Also

$$\begin{aligned} \int \frac{2x^3+5x^2+4x+1}{x^4+2x^3+3x^2+2x+2}\,\mathrm{d}x &= 2\int \frac{x}{x^2+1}\,\mathrm{d}x + \int \frac{1}{x^2+2x+2}\,\mathrm{d}x \\ &= \ln(x^2+1) + \int \frac{1}{(x+1)^2+1}\,\mathrm{d}x \\ &= \ln(x^2+1) + \arctan(x+1) \end{aligned}$$

c) Es ist

$$\begin{aligned} \frac{x+2}{x^2(x^2+2)} &\overset{!}{=} \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+2} \\ &= \frac{(A+C)x^3 + (B+D)x^2 + 2Ax + 2B}{x^2(x^2+2)}, \end{aligned}$$

woraus folgt: $A = \frac{1}{2}$, $B = 1$, $C = -\frac{1}{2}$ und $D = -1$. Somit

$$\begin{aligned} \int \frac{x+2}{x^2(x^2+2)}\,\mathrm{d}x &= \frac{1}{2}\int \frac{1}{x}\,\mathrm{d}x + \int \frac{1}{x^2}\,\mathrm{d}x - \frac{1}{2}\int \frac{x+2}{x^2+2}\,\mathrm{d}x \\ &= \frac{1}{2}\ln x - \frac{1}{x} - \frac{1}{2}\int \frac{x}{x^2+2}\,\mathrm{d}x - \int \frac{1}{x^2+2}\,\mathrm{d}x \\ &= \frac{1}{2}\ln x - \frac{1}{x} - \frac{1}{4}\ln(x^2+2) - \frac{1}{2}\int \frac{1}{\left(\frac{x}{\sqrt{2}}\right)^2+1}\,\mathrm{d}x \\ &= \frac{1}{2}\ln x - \frac{1}{x} - \frac{1}{4}\ln(x^2+2) - \frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}} \end{aligned}$$

4. a) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos^3 x}{\sin^4 x}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{(1-\sin^2 x)\cos x}{\sin^4 x}\,dx \overset{*}{=} \int_{\frac{1}{2}}^{1} \frac{1-t^2}{t^4}\,dt = \int_{\frac{1}{2}}^{1} (\frac{1}{t^4} - \frac{1}{t^2})\,dt =$
$\displaystyle(-\frac{1}{3t^3} + \frac{1}{t})\Big|_{\frac{1}{2}}^{1} = (-\frac{1}{3\sin^3 x} + \frac{1}{\sin x})\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \underline{\underline{\frac{4}{3}}}.$

(*) Hier wurde die Substitution $t = \sin x$ gemacht. Es gilt also: $dt = \cos x dx$.

b) Substitution: Sei $x = 4\sinh z$. Dann ist $dx = 4\cosh z\,dz$.
$\displaystyle\int \sqrt{x^2+16}\,dx = \int 16\cosh^2 z\,dz = 8\int (\cosh 2z + 1)\,dz = 4\sinh 2z + 8z + C$

Bitte wenden!

$$= 8\sinh z\cosh z + 8z + C = 2x\cosh\left(\operatorname{arsinh}\tfrac{x}{4}\right) + 8\operatorname{arsinh}\tfrac{x}{4} + C$$

$$= \underline{\underline{\frac{1}{2}\,x\sqrt{x^2+16} + 8\operatorname{arsinh}\left(\frac{x}{4}\right) + C}}$$

c) $\displaystyle\int_3^4 \frac{dx}{x^2-7x+10} = \int_3^4 \frac{dx}{(x-5)(x-2)} = \int_3^4 \left(\frac{\frac{1}{3}}{x-5} - \frac{\frac{1}{3}}{x-2}\right) dx = \underline{\underline{-\frac{2}{3}\log 2}}$

Hier wurde die folgende Partialbruchzerlegung gemacht: Gesucht sind $A, B \in \mathbb{R}$ so, dass

$$\frac{1}{x^2-7x+10} = \frac{A}{x-5} + \frac{B}{x-2} = \frac{A(x-2)+B(x-5)}{(x-5)(x-2)} = \frac{x(A+B)-2A-5B}{(x-5)(x-2)}.$$

Nach einem Koeffizientenvergleich gilt

$$\begin{cases} A+B=0 \\ -2A-5B=1, \end{cases}$$

und also $A = \frac{1}{3}, B = -\frac{1}{3}$.

d) $\displaystyle\int_3^4 \frac{x\,dx}{x^2-4x+4} = \int_3^4 \frac{x\,dx}{(x-2)^2} = \int_1^2 \frac{u+2}{u^2}\,du = \underline{\underline{1+\log 2}}$

e) $\displaystyle\int_3^4 \frac{dx}{x^2-2x+5} = \int_3^4 \frac{dx}{(x-1)^2+4} = \int_2^3 \frac{du}{u^2+4} = \int_1^{\frac{3}{2}} \frac{\frac{1}{2}\,dv}{v^2+1}$

$$= \frac{1}{2}\arctan\left(\frac{3}{2}\right) - \frac{1}{2}\arctan(1) = \underline{\underline{\frac{1}{2}\arctan\left(\frac{3}{2}\right) - \frac{\pi}{8}}}$$

5. ...