Exercise 5 - Hydraulic Turbines and Electromagnetic Systems

5.1 Hydraulic Turbines

Whole courses are dedicated to the analysis of gas turbines. For the aim of modeling hydraulic systems, we analyze here the simpler case of hydraulic turbines. The model we use to analyze them is the *Pelton Turbine (Allan Pelton, 1880)*. A sketch of the system is depicted in Figure 1. The water flow flows through a nozzle which converts potential

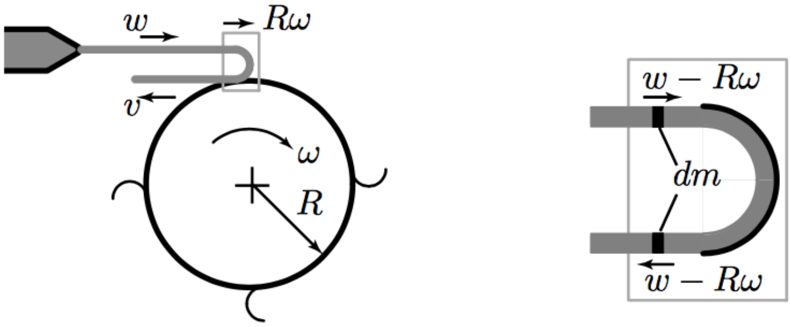

Figure 1: Sketch of a pelton turbine.

energy (essentially the pressure) to kinetic energy (essentially rotational energy). The nozzle is modeled with a mobile pin, whose position can be controlled with a control system for different purposes. We assume that the turbine turns at constant speed for a given time. The main features we want to model are

- The mean force acting on the turbine $F_\mathrm{T} = f(w, \omega, \overset{*}{V})$.
- The resulting wheel torque $T_\mathrm{T} = g(w, \omega, \overset{*}{V})$.
- The power transferred from the fluid to the turbine.

5.1.1 The Mean Force F_T

Since the power transferred from the fluid to the turbine is a consequence of momentum exchange, let's try to model this. The mass of fluid which is allowed to pass is controlled via the defined pin. Moreover, if the flow has a velocity w, one should consider the loss in velocity due to the rotation of the wheel $-R \cdot \omega$. With these informations the momentum equation reads

$$\begin{aligned} \mathrm{d}B &= 2 \cdot (w - R \cdot \omega) \cdot \mathrm{d}m \\ \textit{with } (\mathrm{d}m = \overset{*}{V} \cdot \rho \cdot \mathrm{d}t) \qquad &= 2 \cdot (w - R \cdot \omega) \cdot \overset{*}{V} \cdot \rho \cdot \mathrm{d}t. \end{aligned}$$

In order to obtain the force, we differentiate over time and get

$$\begin{aligned} F_\mathrm{T} &= \frac{\mathrm{d}B}{\mathrm{d}t} \\ &= 2 \cdot \rho \cdot (w - R \cdot \omega) \cdot \overset{*}{V}. \end{aligned}$$

5.1.2 The Wheel Torque T_{T}

The torque reads now easily

$$\begin{aligned} T_{\mathrm{T}} &= F_{\mathrm{T}} \cdot R \\ &= 2 \cdot \rho \cdot R \cdot (w - R \cdot \omega) \cdot \overset{*}{V}. \end{aligned}$$

5.1.3 Optimal Energy Transfer

Considering the velocity of the water after leaving the nozzle

$$v = w - 2 \cdot R \cdot \omega,$$

one can image different scenarios:

- If the turbine is at rest, the end velocity of the water will be w (without considering losses like friction,...). In this case, no power is transferred from the water to the wheel.
- If the turbine turns at $R \cdot \omega = w$, the water undergoes no deceleration. In this case, no power is transferred from the water to the wheel.
- If the wheel speed would be $R \cdot \omega = \frac{1}{2}w$, then the power transfer would be maximal. In fact, at this speed the water is completely decelerated to $v = 0$, meaning that all of its kinetic energy is transferred to the wheel.

5.2 Electromagnetic Systems

One can generalize this type of systems with *RLC* networks, with:

- Resistances R.
- Inductances L.
- Capacitances C.

The reservoirs encountered with this kind of systems are:

- The **magnetic energy**, stored in magnetic fields B.
- The **electric energy**, stored in electric fields E.

These systems are best described through the parameters listed in Table 1.

	Capacitance C	Inductance L
Energy	$W_{\mathrm{E}} = \frac{1}{2}C \cdot U^2(t)$	$W_{\mathrm{M}} = \frac{1}{2}L \cdot I^2(t)$
Level Variable	$U(t)$ (voltage)	$I(t)$ (current)
Conservation Law	$C \cdot \frac{\mathrm{d}}{\mathrm{d}t}U(t) = I(t)$	$L \cdot \frac{\mathrm{d}}{\mathrm{d}t}I(t) = U(t)$

Table 1: Linear Electric Elements.

In general, the electrical power can be derived as

$$\begin{aligned} P(t) &= \frac{\mathrm{d}}{\mathrm{d}t}W_{\mathrm{E}}(t) \\ &= C \cdot U(t) \cdot \frac{\mathrm{d}}{\mathrm{d}t}U(t) \\ &= U(t) \cdot I(t). \end{aligned}$$

Remark. The same derivation can be performed with the inductance L.

In order to work with *RLC* networks the *Kirchoff's laws* are crucial:

(I) The algebraic sum of all currents in each network node is zero.

(II) The algebraic sum of all voltages following a closed network loop is zero.

Remark. The Kirchoff law can be used instead of the energy conservation laws. They state the same thing.

5.2.1 The Electric Oscillator

An electric oscillator is an electronic circuit which produces a periodic, oscillating electronic signal (often a sine wave or a square wave). Typical applications are filters in signal-processing applications. A simplified version of an *RLC* oscillator is depicted in Figure 2.
Let's analyze the system:

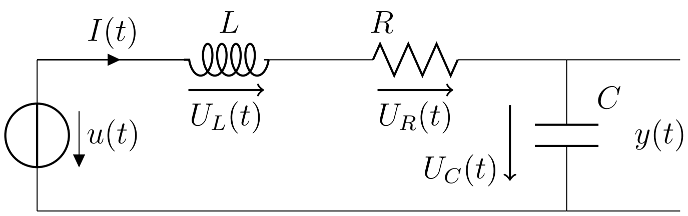

Figure 2: RLC oscillator structure.

Inputs/Outputs:

- The input of the system is the input voltage $u(t)$.
- The output of the system is the output voltage $y(t)$.

Reservoirs:

As stated before, the two relevant reservoirs are:

- The magnetic energy stored in L.
- The electric energy stored in C.

Energy balance:

As stated before, the Kirchoff's laws are analogous to the energy balance. For the system, the second Kirchoff law reads

$$U_L(t) + U_R(t) + U_C(t) = u(t).$$

Differential Equations:

We use the two differential equations which describe the capacitance C and the inductance L. The differential equation for the inductance L reads

$$U_L(t) = L \cdot \frac{\mathrm{d}}{\mathrm{d}t} I(t).$$

The differential equation for the capacitance reads

$$I(t) = C \cdot \frac{\mathrm{d}}{\mathrm{d}t} U_C(t).$$

Ohm's law reads

$$U_R(t) = R \cdot I(t).$$

Result

By considering that the voltage $y(t)$ coincides with the voltage through C, i.e. $y(t) = U_C(t)$, and considering the current $I(t)$ as $I(t) = \frac{\mathrm{d}}{\mathrm{d}t} Q(t)$, one can write:

$$\begin{aligned} I(t) &= C \cdot \frac{\mathrm{d}}{\mathrm{d}t} U_C(t) \\ &= C \cdot \frac{\mathrm{d}}{\mathrm{d}t} y(t). \end{aligned}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t} I(t) = C \cdot \frac{\mathrm{d}^2}{\mathrm{d}t^2} y(t).$$

By plugging this result into the voltage balance, one gets:

$$\begin{aligned} U_L(t) + U_R(t) + U_C(t) &= u(t) \\ L \cdot C \cdot \frac{\mathrm{d}^2}{\mathrm{d}t^2} y(t) + R \cdot C \cdot \frac{\mathrm{d}}{\mathrm{d}t} y(t) + y(t) &= u(t). \end{aligned}$$

5.3 Electromechanical Systems

Electric motors are widely used in many control applications. Most of them are rotational motors, which can be classified as follows:

- **Classical DC drives:** Motors with mechanical commutation of the current in the rotor coils and constant (permanent magnet) or time-varying stator fields (external excitation).
- **Modern brushless DC drives:** Motors which have an electronic commutation of the stator current and permanent magnet on the rotor (i.e., no brushes).
- **AC drives:** Motors which have an electronic commutation of the stator current and use self-inductance to build up the rotor fields.

Remark. For more accurate visual explanations please refer to:
`https://www.youtube.com/watch?v=LAtPHANEfQo`
`https://www.youtube.com/watch?v=bCEiOnuODac`
`https://www.youtube.com/watch?v=LtJoJBUSe28`

5.3.1 Modeling of a DC motor

The circuit which decribes a DC-motor is depicted in Figure 3. We assume that the motor

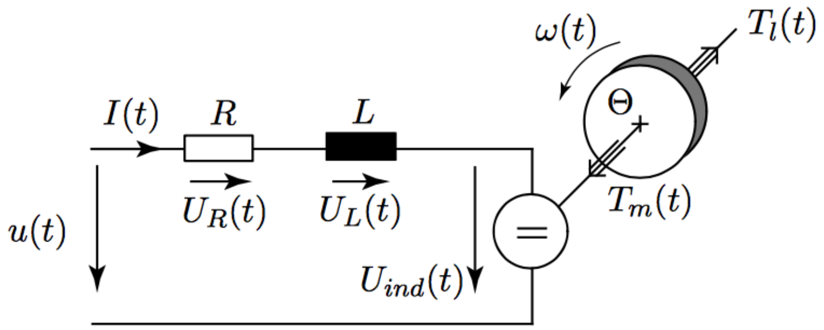

Figure 3: Sketch of a DC motor circuit.

is permanently excited and that parameters κ in the motor and in the generator laws are constant. Furthermore, we assume that the mechanical part of the motor undergoes friction losses $T_{\text{loss}} = d \cdot \omega(t)$. The important laws that are useful to analyze this type of systems are the **Lorentz law**

$$\vec{F} = I \cdot \left(\vec{l} \times \vec{B}\right) = q \cdot \left(\vec{v} \times \vec{B}\right),$$

the **Faraday law**

$$\vec{U} = -v \cdot \left(\vec{l} \times \vec{B}\right),$$

the **motor law**

$$F(t) = \kappa_{\text{mot}} \cdot I(t),$$

and the **generator law**

$$U_{\text{ind}}(t) = \kappa_{el} \cdot \omega(t)$$

Let's analyze the system:

Inputs/Outputs:

- The input of the system are the armature voltage $u(t)$ (control input for motors) and the load torque $T_l(t)$ (in control systems considered as a disturbance).
- The output of the system is the measurement of the rotor speed $\omega(t)$.

Reservoirs:

Two relevant reservoirs are present: As stated before, the two relevant reservoirs are: Two relevant reservoirs are present:

- The magnetic energy stored in the rotor coil. This has level variable $I(t)$.
- The kinetic energy stored in the rotor. This has level variable $\omega(t)$.

Energy Conservation Laws:

We deal with two energy conservation laws. The conservation of the magnetic energy reads (using Kirchoff's second law)

$$L \cdot \frac{\mathrm{d}}{\mathrm{d}t} I(t) = -R \cdot I(t) - U_{\mathrm{ind}}(t) + u(t).$$

The conservation of kinetic energy of the rotor reads (momentum balance)

$$\Theta \cdot \frac{\mathrm{d}}{\mathrm{d}t} \omega(t) = T_{\mathrm{m}}(t) - T_{\mathrm{l}}(t) - d \cdot \omega(t).$$

Differential Equations:

The power of the electric part of the motor should be equal to the power of the mechanical part of the motor. This reads

$$\begin{aligned} P_{\mathrm{elec}}(t) &= P_{\mathrm{mech}}(t) \\ U_{\mathrm{ind}}(t) \cdot I(t) &= T_{\mathrm{m}}(t) \cdot \omega(t) \\ \kappa \cdot \omega(t) \cdot I(t) &= T_{\mathrm{m}}(t) \cdot \omega(t) \\ \Rightarrow T_{\mathrm{m}}(t) &= \kappa \cdot I(t). \end{aligned}$$

Plugging this information into the energy conservation equations we get

$$\begin{aligned} L \cdot \frac{\mathrm{d}}{\mathrm{d}t} I(t) &= -R \cdot I(t) - \kappa \cdot \omega(t) + u(t) \\ \Theta \cdot \frac{\mathrm{d}}{\mathrm{d}t} \omega(t) &= \kappa \cdot I(t) - T_{\mathrm{l}}(t) - d \cdot \omega(t). \end{aligned}$$

5.4 Example

In order to reduce the costs of electricity of your SpaghETH, you decide to augment your systems with a turbine, in order to recover energy from the pot filling process. Assume water, having the density ρ, gets into the pot with a user-defined mass-flow $\dot{m}(t)$ and the constant velocity w. The water hits a turbine rotating with angular velocity $\omega(t)$. The turbine is mechanically linked to a shaft with moment of inertia Θ. To the shaft it is also attached an electric generator with generator/motor (equal for the two) constant κ. Frictions losses are known to be of the form $T_{\text{f}} = \beta\omega(t)$. The electric circuit is equipped with an inductance L. The external loads are modeled as a constant resistance R. Since the goal of the exercise is to control the power delivered to the load, you start by formulating a model of the system. A sketch of the system with the relative parameters is shown in Figure 4.

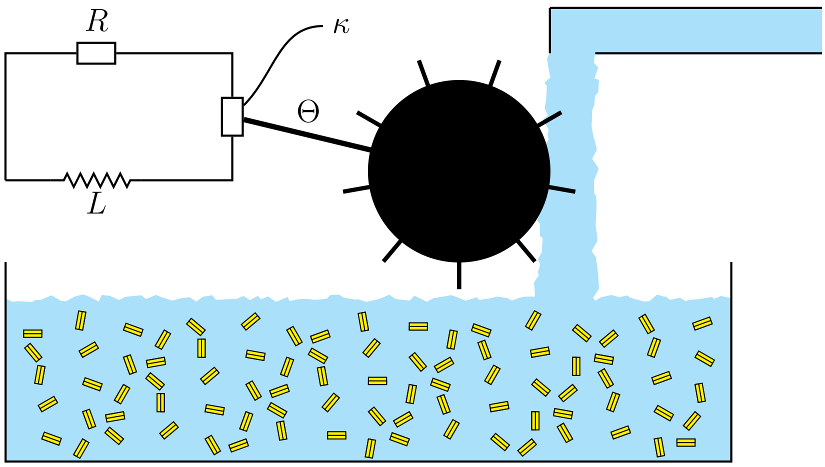

Figure 4: Sketch of the system.

1. What are the input(s) and output(s) to the system?
2. List all the reservoirs and the relative state variables.
3. Draw the causality diagram of the system.
4. Give the algebraic or differential equation for each block of the causality diagram.
5. Show that changing the inductance influences the power delivered by the turbine.

Solution.

1. The input to the system is $\dot{m}(t)$. The output is the power delivered to the load. We will denote the signal with $P(t)$.

2. The reservoirs of the system and their relative level variables are:
   - Kinetic energy of shaft, with level variable $\omega(t)$.
   - Electrical energy, with level variable $I(t)$.

3. The causality diagram is shown in Figure 5.

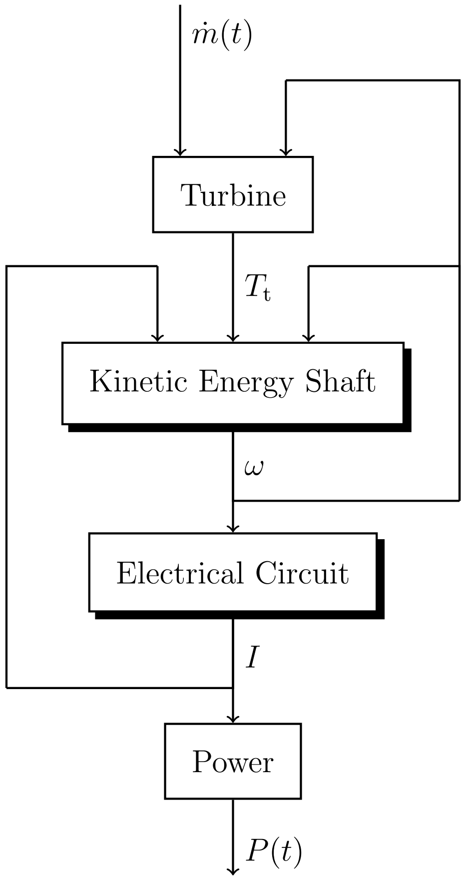

Figure 5: Causality diagram.

4. The torque delivered by the turbine is described by

$$T_{\mathrm{t}} = 2 \cdot R \cdot (w - R \cdot \omega) \cdot \dot{m}(t)$$

Conservation of mechanical energy for the shaft gives

$$\Theta \frac{\mathrm{d}\omega(t)}{\mathrm{d}t} = T_{\mathrm{t}} - \kappa I(t) - \beta\omega(t).$$

The circuit is then described by

$$L \frac{\mathrm{d}I}{\mathrm{d}t} = -RI(t) + \kappa\omega(t).$$

The resulting power is then

$$P(t) = I(t)U(t) = RI(t)^2.$$

5. See causality diagram.