Exercise 2 - Mechanical Systems

2.1 Mechanical Systems

The first bunch of systems we analize, are mechanical systems. In order to perform the modeling of the system as learned the last week, we need to define more specifically how to determine the energies of these systems.

2.1.1 Kinetic Energy

The kinetic energy of a system is represented with T. Moreover, one can express the total kinetic energy as the sum of the translational and the rotational kinetic energy. This reads

$$\begin{aligned} T_{\text{tot}}(t) &= T_{\text{t}}(t) + T_{\text{r}}(t) \\ &= \frac{1}{2}m\dot{\vec{r}}(t)^2 + \frac{1}{2}\Theta\omega(t)^2, \end{aligned} \tag{2.1}$$

where $\vec{r}$ is the position vector of the center of mass, Θ is the moment of inertia with respect to the center of mass and ω is the angular velocity of the system.

2.1.2 Potential Energy

The potential energy is defined as a function of $\vec{r}(t)$ and reads

$$U(t) = U(\vec{r}(t)). \tag{2.2}$$

Practical examples are the gravitational potential energy

$$U_{\text{g}} = mgh, \tag{2.3}$$

and the spring potential energy

$$U_{\text{spring}} = \frac{1}{2}kx^2. \tag{2.4}$$

2.2 The Euler Method

By defining the total energy of the system as the sum of its total kinetic energy and total potential energy

$$E_{\text{tot}} = T_{\text{tot}} + U_{\text{tot}}, \tag{2.5}$$

one can define the **mechanical power balance** as

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = \sum_{i=1}^{k} P_i(t), \tag{2.6}$$

where $P_i(t)$ are the mechanical powers acting on the system. Furthermore, we distinguish between the power of a force

$$P_F = \vec{F} \cdot \vec{v}, \tag{2.7}$$

and the power of a torque is

$$P_T = \vec{T} \cdot \vec{\omega}. \tag{2.8}$$

Useful forces that generates losses of power are the **rolling friction**, defined as

$$F_{\mathrm{r}} = c_{\mathrm{r}} m g \tag{2.9}$$

with c_{r} rolling friction coefficient and the **aerodynamic drag**

$$F_{\mathrm{a}}(t) = \frac{1}{2} \rho c_{\mathrm{w}} A v(t)^2 \tag{2.10}$$

with the drag coefficient c_{w} and the apparent system's surface A.

2.3 Example

Since your SpaghETH is going well, you decide to improve the service you are offering and to buy crane from your colleagues CranETH, so that you can distribute the pots with hot water efficiently while driving. A sketch is shown in Figure 1.

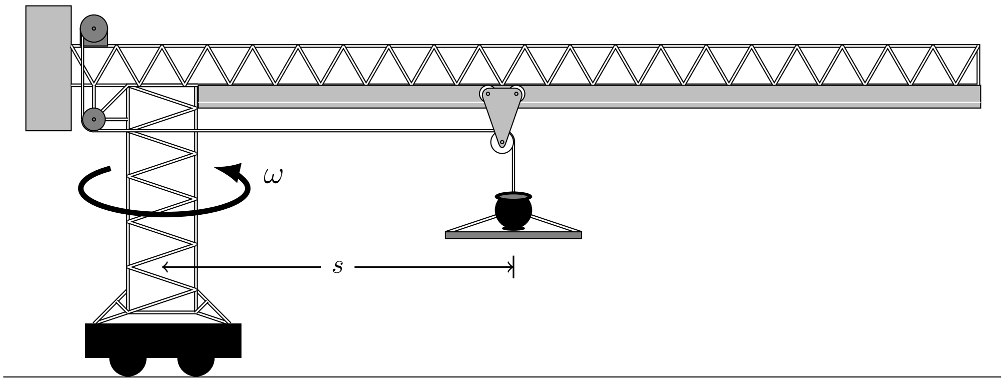

Figure 1: Sketch of the system.

The crane platform has negliglible mass. The crane itself has mass m_{c} and moment of inertia with respect to the vertical axis Θ. The crane has the front surface A. The density of air is known and is ρ, the aerodynamic coefficient c_{w}, and the rolling friction coefficient c_{r}. The rotational friction torque can be expressed as $T_{\mathrm{fric}} = \beta\omega$.
Experiments have shown that the aerodynamic drag coeffiecient is a function of the rotational velocity of the crane. The crane is carrying a pot of mass m_{p}, which is attached at distance s from the vertical axis. You may treat the pot as a point mass. Further, assume that the center of mass does of the system does not move as the mass m_{p} rotates. The propulsive force acting horizontally on the crane and the propulsive torque acting on the crane vertical axes are given by

$$F_{\mathrm{p}}(\phi_1) = F_{\max} \cdot (1 - \exp(-c_1\phi_1))$$
$$T_{\mathrm{p}}(\phi_2) = T_{\max} \cdot (1 - \exp(-c_2\phi_2))$$

where $\phi_1(t)$ and $\phi_2(t)$ are the normalized actuators positions. The constants $P_{\max}, T_{\max}, c_1$, and c_2 are known.

1. Determine the inputs and the outputs of the system.
2. List the reservoir(s) and the corresponding level variable(s).
3. Draw a causality diagram of the system.
4. Formulate the differential/algebraic equations needed to describe the system.
5. Is the system linear or nonlinear? Explain.

Solution.

1. The inputs are the propulsive force F_p, the propulsive torque T_p, and the distance s of the pot from the central axes. The output are the translational and rotational velocities of the system.

2. The system has two reservoirs:
   - the kinetic translational energy of the system E_tr, whose level variable is the velocity v of the system;
   - the kinetic rotational energy of the system E_rot, whose level variable is the rotational velocity ω of the system.

3. The causality diagram is shown in Figure 2.

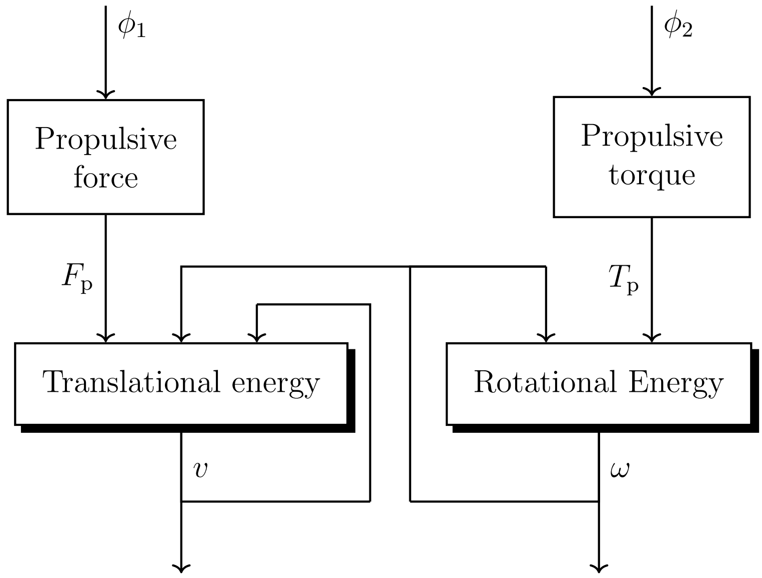

Figure 2: Causality diagram of the system.

4. The differential equation for the translational energy of the truck reads

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\mathrm{tr} = P_+ - P_-,$$

which reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\mathrm{tr} = F_\mathrm{p}v - \frac{1}{2}\rho c_\mathrm{w}(\omega)Av^3 - c_\mathrm{r}(m_\mathrm{c}+m_\mathrm{p})gv.$$

This leads to the differential equation

$$(m_\mathrm{c}+m_\mathrm{p})v\dot{v} = F_\mathrm{p}v - \frac{1}{2}\rho c_\mathrm{w}(\omega)Av^3 - c_\mathrm{r}(m_\mathrm{c}+m_\mathrm{p})gv$$

which simplifies to

$$\dot{v} = \frac{1}{(m_\mathrm{c}+m_\mathrm{p})}\cdot\left(F_\mathrm{p} - \frac{1}{2}\rho c_\mathrm{w}(\omega)Av^2 - c_\mathrm{r}(m_\mathrm{c}+m_\mathrm{p})g\right).$$

The differential equation for the rotational energy of the crane reads

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{rot}} = P_{+} - P_{-},$$

which reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{rot}} = T_{\mathrm{p}}\omega - \beta\omega^2.$$

The rotational energy of the system is

$$E_{\mathrm{rot}} = \frac{1}{2}(\Theta + m_{\mathrm{p}}s^2)\omega^2.$$

Hence, the differential equation reads

$$(\Theta + m_{\mathrm{p}}s^2)\omega\dot{\omega} + m_{\mathrm{p}}\omega^2 s\dot{s} = T_{\mathrm{p}}\omega - \beta\omega^2,$$

which simplifies to

$$\dot{\omega} = \frac{1}{(\Theta + m_{\mathrm{p}}s^2)} \cdot \left(T_{\mathrm{p}} - \beta\omega - m_{\mathrm{p}}\omega s\dot{s}\right).$$

5. The system is nonlinear.