Introduction to Quantum Mechanics **Spring 2013**

Final Exam: Monday, August 12, 2013

<u>THIS IS A CLOSED BOOK EXAM</u>*

EXAM RULES: All notes, as well as all books, notebooks, note cards, *etc.* must be inside closed and latched briefcases or inside closed and zippered backpacks. Briefcases and backpacks must be under your desk. Students should be well separated.

You may write your answers in English or German.

*You are allowed three A4 pages with notes on both sides. You must hand these sheets in with your exam. They are worth points!

Write your name in the upper right hand corner of each page.

<u>SHOW YOUR WORK</u>!!

You have 120 minutes to complete the exam.

Name: Solution Key

Student Number: ____________

"Cheat Sheets":	______/10
Problem 1:	______/50
Problem 2:	______/40
Problem 3:	______/35
Problem 4:	______/25
Problem 5:	______/40
Extra Credit:	______/4
Total:	______/200

Mark: ______

Name: Solution Key

1. Short Answer:

(a) Circle True or False (30 points possible)
(Do not guess: +2 for correct answer, 0 for no answer, -2 for incorrect answer)

T		My name and student number are on the front page of this exam.
T	F	Two Hermitian operators will always commute.
T	F	The Dirac delta function is not in Hilbert space.
T	F	If a particle is described by a wave packet, its energy is always well defined.
T	F	If an electron approaches a potential barrier of any finite height, it can always be reflected.
T	F	If a particle is in a non-stationary state, the measurement of its energy must yield one of several values.
T	F	$\frac{d}{dt}\int_{-\infty}^{\infty}\lvert\Psi(x,t)\rvert^2 dx = 0$ arises only for unphysical solutions to the 1D Schrödinger equation.
T	F	$\int_{-\infty}^{\infty}\lvert\Psi(x,t)\rvert^2 dx = 0$ arises only for unphysical solutions to the 1D Schrödinger equation.
T	F	Given a system of two identical particles, the exchange operator $\hat{P}$ always commutes with the Hamiltonian operator $\hat{H}$.
T	F	Given a specific shell in a hydrogenic atom, all of the subshells have the same energy.
T	F	Every classical observable can be represented in quantum mechanics by a Hermitian operator.
T	F	In metals at T=0K, the electronic states are filled up to the Fermi level.
T	F	If a quantum mechanical observable is measured, one possible result is always the expectation value of the observable.
T	F	An uncertainty relation will exist for any two observables that have operators that do not commute.
T	F	Electronic bands exist in semiconductor and insulators, but not metals.

Name: Solution Key

(b) If a particle is placed in a sphere of radius ρ where the potential $V(r)$ is described by

$$V(r) = \begin{cases} 0 & \text{for } r \leq \rho \\ \infty & \text{for } r > \rho \end{cases}$$

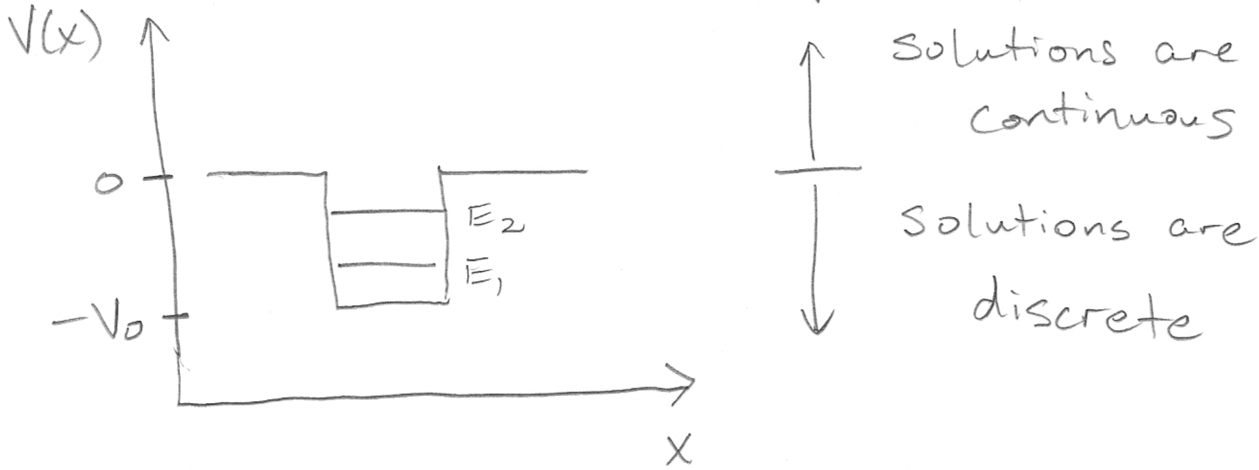
where r is the radial coordinate. Write down the angular components to the particle's wave function? (5 points)

Because the potential is spherically symmetric, we know that the angular components are described by the spherical harmonics

$$\Rightarrow Y_l^m(\theta, \varphi)$$

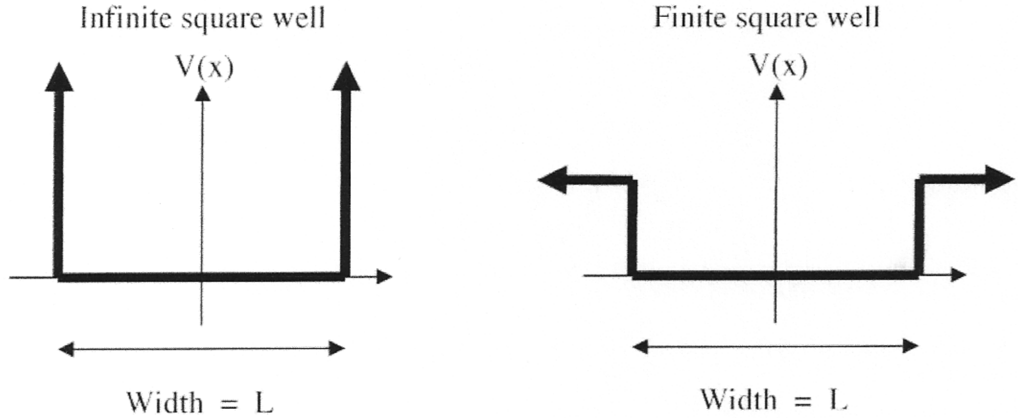
(c) Give an example of a quantum mechanical system that has both a discrete and continuous part to its spectrum. (5 points)

A finite 1D square well

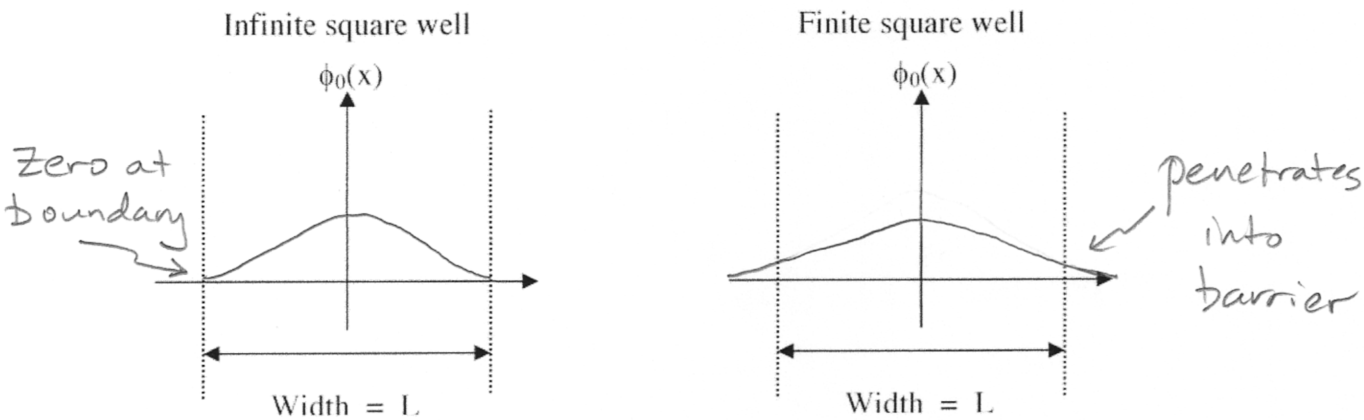


Name: Solution Key

(d) Given the two square wells as shown below with the same width,



sketch the wavefunction for the ground state on each of the corresponding plots below (5 points).



(e) In part (d), do you expect the energy of the ground state of the finite square well to be less than, equal to, or greater than the energy of the ground state of the infinite square well? Explain your answer. (5 points)

The energy of the ground state for the finite square well should be lower than for the infinite square well. The "size" of the box is slightly larger for the particle in the finite square well. This lowers the energy since $E_1 \propto \frac{1}{(\text{size})^2}$

Name: Solution Key

2. Short Answer:

(a) Write the hydrogen eigenstate $\psi_{32\pm1}$. Explain any constants in your answer. (10 points)

Using the tables in the "useful information"

$$\psi_{32\pm1} = R_{32}(r)\, Y_2^{\pm1}(\theta,\varphi)$$

$$= \frac{4}{81\sqrt{30}}\, a^{-3/2}\left(\frac{r}{a}\right)^2 \exp\left[\frac{-r}{3a}\right]\left[\mp\left(\frac{15}{8\pi}\right)^{1/2} \sin\theta \cos\theta \exp(\pm i\varphi)\right]$$

$$\boxed{\psi_{32\pm1} = \mp \frac{1}{81\sqrt{\pi}}\, \frac{1}{a^{3/2}}\left(\frac{r}{a}\right)^2 \exp\left[\frac{-r}{3a}\right] \sin\theta \cos\theta \exp\left[\pm i\varphi\right]}$$

where a is the Bohr radius

(b) What is the electronic configuration and term symbol for the ground state of Al? (10 points)

Al is in the third row of the periodic table. The electronic configuration is:

$$\boxed{[Ne]\, 3s^2\, 3p^1}$$

For the term symbol, we only need to consider the unfilled p-subshell

$\Rightarrow$ One p electron $\quad s = \frac{1}{2} \quad l = 1$

$\Rightarrow \quad J = 3/2$ or $1/2$

$^{2S+1}L_J \quad \Rightarrow \quad {}^2P_{3/2}$ or $\boxed{{}^2P_{1/2}}$

Ground state according to Hund's Rules
Less than half full
$\Rightarrow$ Lowest J

Name: Solution Key

(c) Find the ground-state energy for a system of N noninteracting identical particles that are confined to a one-dimensional infinite square well when the particles are (i) bosons and (ii) spin 1/2 fermions. (10 points) Let $a \equiv$ well width

Because the particles are non-interacting, we can use 1-particle infinite-square-well solutions $\Rightarrow E_n = \frac{n^2\hbar^2\pi^2}{2ma^2}$

(i) N Bosons $\Rightarrow$ ground state has all N particles in $n=1$

$$E_{\text{ground state}} = \frac{N(1)^2\hbar^2\pi^2}{2ma^2} = \boxed{\frac{N\hbar^2\pi^2}{2ma^2}}$$

Bosons

(ii) N Spin $\frac{1}{2}$ Fermions $\Rightarrow$ ground state has 2 particles in each level up to $n = N/2$

$$E_{\text{ground state}} = 2\sum_{n=1}^{N/2} \frac{n^2\hbar^2\pi^2}{2ma^2} = \boxed{\frac{\hbar^2\pi^2}{ma^2}\sum_{n=1}^{N/2} n^2}$$

Spin $\frac{1}{2}$ Fermions

Optional: If N is large

$$\sum_{n=1}^{N/2} n^2 \simeq \frac{1}{3}\left(\frac{N}{2}\right)^3 \quad \text{OR} \quad E_{\text{ground state}} = \frac{N^3\hbar^2\pi^2}{24ma^2}$$

(d) For the N bosons in part (c), write down the ground-state time-independent wave function. (10 points)

Again using 1-particle solutions: $\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$

For the ground state time-independent wave function all N particles are in $\psi_1(x)$

$$\psi_{N\text{ Bosons}} = \prod_{i=1}^{N} \sqrt{\frac{2}{a}}\sin\left(\frac{\pi}{a}x_i\right)$$

where i labels each Boson

$$\therefore \boxed{\psi_{N\text{ Bosons}} = \sqrt{\frac{2^N}{a^N}}\sin\left(\frac{\pi}{a}x_1\right)\sin\left(\frac{\pi}{a}x_2\right)\cdots\sin\left(\frac{\pi}{a}x_N\right)}$$

Name: Solution Key

3. An operator $\hat{A}$, representing observable *A*, has two normalized eigenstates ψ_1 and ψ_2 with eigenvalues a_1 and a_2, respectively. Operator $\hat{B}$, representing observable *B*, has two normalized eigenstates ϕ_1 and ϕ_2 with eigenvalues b_1 and b_2, respectively. The eigenstates are related by

$$\psi_1 = (3\phi_1 + 4\phi_2)/5$$

$$\psi_2 = (4\phi_1 - 3\phi_2)/5$$

(a) Observable *A* is measured, and the eigenvalue a_2 is obtained. What is the state of the system immediately after this measurement? (5 points)

After A is measured and a_2 is obtained, the system is in

$$\boxed{\psi_2}$$

(b) If observable *B* is now measured [after the first measurement in part (a)], what are the possible results, and what are their probabilities? (10 points)

If B is measured, the possible outcomes are $\boxed{b_1 \text{ and } b_2}$

Since the system is in $\psi_2 = \frac{4}{5}\varphi_1 - \frac{3}{5}\varphi_2$

Coefficients C_{φ_1} C_{φ_2}

the probabilities are $|C_n|^2$ with

$$\boxed{\text{Probability of } b_1 = \frac{16}{25} \text{ and } b_2 = \frac{9}{25}}$$

Name: Solution Key

(c) Right after the measurement of B in part (b), A is measured again. You were not told the outcome of the B measurement before the A measurement. What is the probability of getting a_2 after the three measurements (A, B, and then A)? (15 points)

To answer this we need φ_1 and φ_2 in terms of ψ_1, ψ_2

Solving: $\varphi_1 = \frac{3}{5}\psi_1 + \frac{4}{5}\psi_2 \qquad \varphi_2 = \frac{4}{5}\psi_1 - \frac{3}{5}\psi_2$

Probability of getting a_2

$$= \frac{16}{25} \cdot \frac{16}{25} + \frac{9}{25} \cdot \frac{9}{25} = \boxed{0.54}$$

probability that B measurement yields φ_1

probability that second measurement of A yields ψ_2

probability that measurement of B yields ψ_2

probability that second Measurement of A yields ψ_2

(d) Would your answer change in part (c) if you were told the outcome of the B measurement? How? (5 points)

Yes, it would change. If we were told that the outcome of the measurement of B was b_1 then the probability to measure a_2 would be $\frac{16}{25}$ and if b_2 then $\frac{9}{25}$

If outcome of B was:	
b_1 $\Rightarrow$	$16/25$
b_2 $\Rightarrow$	$9/25$

Name: Solution Key

4. We prepare a simple harmonic oscillator with the following wavefunction:

$$\Psi(x,0) = \left(\frac{9\beta^2}{\pi}\right)^{\frac{1}{4}} e^{-9(\beta x)^2/2}$$

$$\text{where } \beta \equiv \sqrt{\frac{m\omega}{\hbar}}.$$

We then immediately measure the energy of the oscillator in this state. What is the probability of getting the ground state energy? (25 points)

To answer this question we need the expansion coefficient C_0 $\Rightarrow$ $C_0 = \int \psi_0^*(x)\,\psi(x)\,dx$

where $\psi(x)$ is given above and ψ_0 is the ground state of the harmonic oscillator

From the "useful information" sheet

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[\frac{-m\omega x^2}{2\hbar}\right]$$

OR using $\beta = \sqrt{\frac{m\omega}{\hbar}}$

$$\psi_0(x) = \left(\frac{\beta^2}{\pi}\right)^{1/4} \exp\left[\frac{-(\beta x)^2}{2}\right]$$

$$\Rightarrow C_0 = \int_{-\infty}^{\infty} \left(\frac{\beta^2}{\pi}\right)^{1/4} \exp\left[\frac{-(\beta x)^2}{2}\right] \left(\frac{9\beta^2}{\pi}\right) \exp\left[\frac{-9(\beta x)^2}{2}\right] dx$$

$$= \left(\frac{3^2\beta^4}{\pi^2}\right)^{1/4} \int_{-\infty}^{\infty} \exp\left[-5(\beta x)^2\right] dx$$

let $u = \sqrt{5}\,\beta x$

$du = \sqrt{5}\,\beta\,dx$

$dx = \frac{du}{\sqrt{5}\,\beta}$

$$= \sqrt{\frac{3}{5\pi}} \int_{-\infty}^{\infty} \exp\left[-u^2\right] du$$

Name: Solution Key

Using the integral tables provided

$$C_0 = \sqrt{\frac{3}{5\pi}}\, 2 \cdot \int_0^{\infty} e^{-u^2}\, du = \sqrt{\frac{3}{5\pi}} \cdot \sqrt{\pi}$$

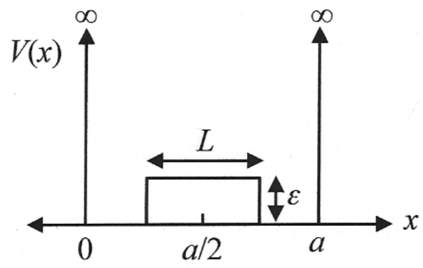
$$C_0 = \sqrt{\frac{3}{5}}$$

$\therefore$ The probability of measuring the ground state energy, E_0, is

$$|c_0|^2 = \boxed{\frac{3}{5}}$$

Name: Solution Key

5. A one-dimensional infinite square well has a potential step centered in the middle as shown below.



We need to use perturbation theory to solve this problem with

$$\begin{cases} E^0_{n=1} = \dfrac{\pi^2 \hbar^2}{2ma^2} \\ \psi^0_{n=1} = \sqrt{\dfrac{2}{a}} \sin\left(\dfrac{\pi x}{a}\right) \end{cases}$$

(a) Calculate the energy of the ground state to first order. Then evaluate it for L=a/10. (25 points)

Energy Correction to first order $\left\{ E'_{n=1} = \int_0^a (\psi_1^0)^* \hat{H}' \psi_1^0 \, dx \right.$

where $\hat{H}' = \varepsilon$ from $\frac{1}{2}(a-L)$ to $\frac{1}{2}(a+L)$

0 otherwise

$$\Rightarrow E'_{n=1} = \frac{2}{a} \varepsilon \int_{\frac{1}{2}(a-L)}^{\frac{1}{2}(a+L)} \sin^2\left(\frac{\pi x}{a}\right)$$

Using integral table provided

$$= \frac{2\varepsilon}{a} \left[\frac{1}{2} x - \frac{a}{4\pi} \sin \frac{2\pi x}{a} \right] \Bigg|_{\frac{1}{2}(a-L)}^{\frac{1}{2}(a+L)}$$

$$= \frac{2\varepsilon}{a} \left[\frac{1}{4}(a+L) - \frac{1}{4}(a-L) - \frac{a}{4\pi} \sin \frac{\pi}{a}(a+L) + \frac{a}{4\pi} \sin \frac{\pi}{a}(a-L) \right]$$

$$= \frac{2\varepsilon}{a} \left[\frac{L}{2} - \frac{a}{4\pi} \left(\sin\left(\pi + \frac{\pi L}{a}\right) - \sin\left(\pi - \frac{\pi L}{a}\right) \right) \right]$$

$$= \varepsilon \left[\frac{L}{a} - \frac{1}{2\pi} \left(\overset{0}{\cancel{\sin \pi}} \cos \frac{\pi L}{a} + \sin \frac{\pi L}{a} \overset{-1}{\cancel{\cos \pi}} - \underset{0}{\cancel{\sin \pi}} \cos \frac{\pi L}{a} + \sin \frac{\pi L}{a} \underset{-1}{\cancel{\cos \pi}} \right) \right]$$

Name: Solution Key

$$E'_{n=1} = \varepsilon \left[\frac{L}{a} - \frac{1}{2\pi} \left(-2 \sin \frac{\pi L}{a} \right) \right]$$

$$= \varepsilon \left[\frac{L}{a} + \frac{1}{\pi} \sin \frac{\pi L}{a} \right]$$

$$\therefore \boxed{E_{n=1} = \frac{\pi^2 \hbar^2}{2ma^2} + \varepsilon \left[\frac{L}{a} + \frac{1}{\pi} \sin \frac{\pi L}{a} \right]}$$

To first order

Evaluating for $L = a/10$

$$\Rightarrow E_{n=1} = \frac{\pi^2 \hbar^2}{2ma^2} + \varepsilon \left[\frac{1}{10} + \frac{1}{\pi} \sin \frac{\pi}{10} \right]$$

$$\boxed{E_{n=1} = \frac{\pi^2 \hbar^2}{2ma^2} + 0.1983\, \varepsilon}$$

Name: Solution Key

(b) Using first-order perturbation theory, determine how much of the n=2 eigenstate from the standard infinite square well (*i.e.* without the potential step in the middle) is contained in the lowest energy eigenstate of the perturbed infinite square well from (a). (15 points)

To first order $\psi_n = \psi_n^0 + \psi_n'$

where $$\psi_n' = \sum_{m \neq n} \left[\frac{\int (\psi_m^0)^* \hat{H}' \psi_n^0 \, dx}{E_n^0 - E_m^0} \right] \psi_m^0$$

We are considering the perturbed ground state

$$\psi_1' = \sum_{m \neq 1} \left[\frac{\int (\psi_m^0)^* \hat{H}' \psi_1^0 \, dx}{E_1^0 - E_m^0} \right] \psi_m^0$$

This is an expansion of ψ_1' in terms of the unperturbed wavefunctions ψ_m^0

So to determine how much of ψ_2^0 is in ψ_1' we need to set $m=2$ and calculate the term in the square brackets.

But note that the integral is

$$\int_{\frac{1}{2}(a-L)}^{\frac{1}{2}(a+L)} \underbrace{\underset{\text{even}}{(\psi_2^0)^*} \; \underset{\text{constant}}{\varepsilon} \; \underset{\text{odd}}{\psi_1^0}}_{\text{odd}} \, dx = 0$$

$\therefore$ The perturbed ground state contains no $\psi_{n=2}^0$ to first order

Name: Solution Key

Extra Credit: (1 point each)

The following questions are about computer operating systems:

1. What year was the Linux kernel first released?

1991

2. What year was Microsoft Windows 1.0 first released?

1985

3. What year was the Apple Macintosh, which utilized the earliest version of Mac OS, released?

1984

4. What year was Android 1.0 released?

2008

Name: ________________________