

# Exercise 5 - Hydraulic Turbines and Electromagnetic Systems

## 5.1 Hydraulic Turbines

Whole courses are dedicated to the modeling of gas turbines. For the sake of simplicity, we restrict our analysis to hydraulic turbines, in particular the *Pelton Turbine* (*Allan Pelton*, 1880). A sketch of the system is depicted in Figure 1. The water flows through

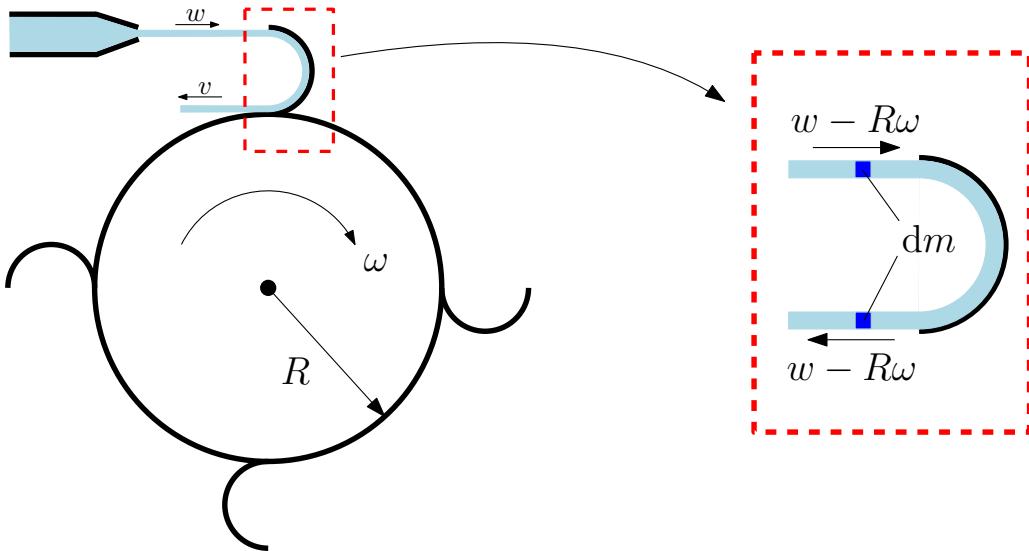


Figure 1: Sketch of a pelton turbine.

a nozzle which converts potential energy (essentially pressure) into kinetic energy. We assume that the turbine turns at constant speed. The quantities we want to model are

- The mean force acting on the turbine  $F_T = f(w, \omega, V^*)$ ;
- the resulting wheel torque  $T_T = g(w, \omega, V^*)$ ;
- the power transferred from the fluid to the turbine,

where  $w$  is the absolute velocity of the water,  $\omega$  is the rotational velocity of the turbine, and  $V^*$  is the water volume flow relative to the turbine blade.

### 5.1.1 The Mean Force $F_T$

The power transfer from the fluid to the turbine is accomplished by momentum exchange, itself consequence of the velocity difference between the fluid and the turbine. Hence, the momentum of the fluid is given by

$$\begin{aligned} dB &= 2 \cdot (w - R \cdot \omega) \cdot dm \\ &= 2 \cdot (w - R \cdot \omega) \cdot V^* \cdot \rho \cdot dt, \end{aligned}$$

where we used  $dm = \overset{*}{V} \cdot \rho \cdot dt$ . In order to obtain the force, we differentiate over time and get

$$\begin{aligned} F_T &= \frac{dB}{dt} \\ &= 2 \cdot \rho \cdot (w - R \cdot \omega) \cdot \overset{*}{V}. \end{aligned}$$

### 5.1.2 The Wheel Torque $T_T$

The torque reads now easily

$$\begin{aligned} T_T &= F_T \cdot R \\ &= 2 \cdot \rho \cdot R \cdot (w - R \cdot \omega) \cdot \overset{*}{V}. \end{aligned}$$

### 5.1.3 The Wheel Power Output $P_T$

The power acting on the turbine is thus

$$\begin{aligned} P_T &= T_T \cdot \omega \\ &= 2 \cdot \rho \cdot R \cdot \overset{*}{V} \cdot (w - R \cdot \omega) \cdot \omega \end{aligned}$$

### 5.1.4 Optimal Energy Transfer

Considering the velocity of the water after leaving the turbine's cup

$$v = w - 2 \cdot R \cdot \omega,$$

one can imagine different scenarios:

- If the turbine is at rest, the end velocity of the water will be  $w$ . In this case, no power is transferred from the water to the wheel.
- If the turbine turns at  $R \cdot \omega = w$ , the water undergoes no deceleration. In this case, no power is transferred from the water to the wheel.
- Setting  $w = 3 \cdot R \cdot \omega$  results in the maximal power output of the turbine.
- Setting  $w = 2 \cdot R \cdot \omega$  results in the maximum energy conversion possible since  $v = 0$ .

*Remark.* A more rigorous approach to the derivation of these equations is presented in the Appendix at the end of the document.

## 5.2 Electromagnetic Systems

One can generalize electromagnetic systems as *RLC* networks, composed of:

- resistances  $R$ ;
- inductances  $L$ ;
- capacitances  $C$ .

The reservoirs encountered with this kind of systems are:

- The **magnetic energy**, stored in magnetic fields  $\mathbf{B}$ .
- The **electric energy**, stored in electric fields  $\mathbf{E}$ .

These systems are best described through the parameters listed in Table 1.

|                  | Capacitance $C$                   | Inductance $L$                    |
|------------------|-----------------------------------|-----------------------------------|
| Energy           | $W_E = \frac{1}{2}C \cdot U^2(t)$ | $W_M = \frac{1}{2}L \cdot I^2(t)$ |
| Level Variable   | $U(t)$ (voltage)                  | $I(t)$ (current)                  |
| Conservation Law | $C \cdot \frac{d}{dt}U(t) = I(t)$ | $L \cdot \frac{d}{dt}I(t) = U(t)$ |

Table 1: Linear Electric Elements.

*Remark.* Notice that resistances are *not* reservoirs since they do not have any dynamics. In fact, changes in voltage result in instantaneous changes in current according to Ohm's law:  $U(t) = R \cdot I(t)$ .

In general, the electrical power can be computed for all components as

$$P = U(t) \cdot I(t).$$

In order to work with *RLC* networks, *Kirchoff's laws* are crucial:

- (I) The algebraic sum of all currents in each network node is zero.
- (II) The algebraic sum of all voltages following a closed network loop is zero.

*Remark.* The conservation laws for the capacitance and inductance are an equivalent formulation of the reservoir based approach  $\frac{d}{dt}W = P(t)$ . For instance, for a capacitance  $C$  we have

$$\begin{aligned} P(t) &= \frac{d}{dt}W_E(t) \\ &= C \cdot U(t) \cdot \frac{d}{dt}U(t) \\ &= U(t) \cdot I(t). \end{aligned}$$

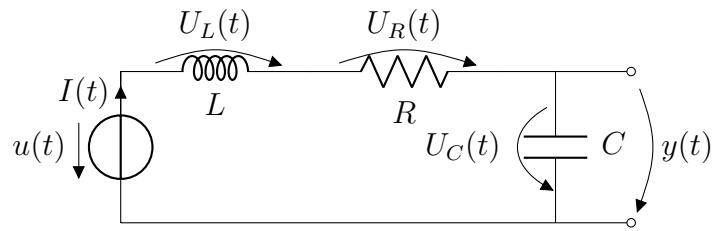


Figure 2: RLC oscillator structure.

### 5.2.1 The Electric Oscillator

An electric oscillator is a type of circuit which produces a periodic, oscillating electronic signal (often a sine wave or a square wave). Typical applications are filters in signal-processing applications. An *RLC* oscillator is depicted in Figure 2.

Let us analyze the system:

#### Inputs/Outputs:

- The input of the system is the input voltage  $u(t)$ .
- The output of the system is the output voltage  $y(t)$ .

#### Reservoirs:

As stated before, the two relevant reservoirs are:

- The magnetic energy stored in  $L$ .
- The electric energy stored in  $C$ .

#### Energy balance:

As stated before, the Kirchoff's laws are analogous to the energy balance. For the system, the second Kirchoff law reads

$$U_L(t) + U_R(t) + U_C(t) = u(t).$$

#### Differential Equations:

We use the two differential equations which describe the capacitance  $C$  and the inductance  $L$ . The differential equation for the inductance  $L$  reads

$$U_L(t) = L \cdot \frac{d}{dt} I(t).$$

The differential equation for the capacitance reads

$$I(t) = C \cdot \frac{d}{dt} U_C(t).$$

Finally, Ohm's law reads

$$U_R(t) = R \cdot I(t).$$

**Result:**

By considering that the voltage  $y(t)$  coincides with the voltage through  $C$ , i.e.  $y(t) = U_C(t)$ , and considering the current  $I(t)$  as  $I(t) = \frac{d}{dt}Q(t)$ , one can write:

$$\begin{aligned} I(t) &= C \cdot \frac{d}{dt}U_C(t) \\ &= C \cdot \frac{d}{dt}y(t). \end{aligned}$$

Hence,

$$\frac{d}{dt}I(t) = C \cdot \frac{d^2}{dt^2}y(t).$$

By plugging this result into the voltage balance, we get

$$\begin{aligned} U_L(t) + U_R(t) + U_C(t) &= u(t) \\ L \cdot C \cdot \frac{d^2}{dt^2}y(t) + R \cdot C \cdot \frac{d}{dt}y(t) + y(t) &= u(t). \end{aligned}$$

### 5.3 Electromechanical Systems

Electric motors are widely used in many control applications. Most of them are rotational motors, which can be classified as follows:

- **Classical DC drives:** Motors with mechanical commutation of the current in the rotor coils and constant (permanent magnet) or time-varying stator fields (external excitation).
- **Modern brushless DC drives:** Motors which have an electronic commutation of the stator current and permanent magnet on the rotor (i.e., no brushes).
- **AC drives:** Motors which have an electronic commutation of the stator current and use self-inductance to build up the rotor fields.

*Remark.* For more accurate visual explanations please refer to:

<https://www.youtube.com/watch?v=LAtPHANEfQo>

<https://www.youtube.com/watch?v=bCEi0nu0Dac>

<https://www.youtube.com/watch?v=LtJoJBUSe28>

#### 5.3.1 Modeling of a DC motor

The circuit which describes a DC-motor is depicted in Figure 3. We assume that the motor is permanently excited and that parameters  $\kappa$  in the motor and in the generator laws are constant (see below). Furthermore, we assume that the mechanical part of the motor undergoes friction losses  $T_{\text{loss}} = d \cdot \omega(t)$ . The important laws that are useful to analyze these types of systems are the **Lorentz law**

$$\mathbf{F} = \int_L I d\mathbf{L} \times \mathbf{B}$$

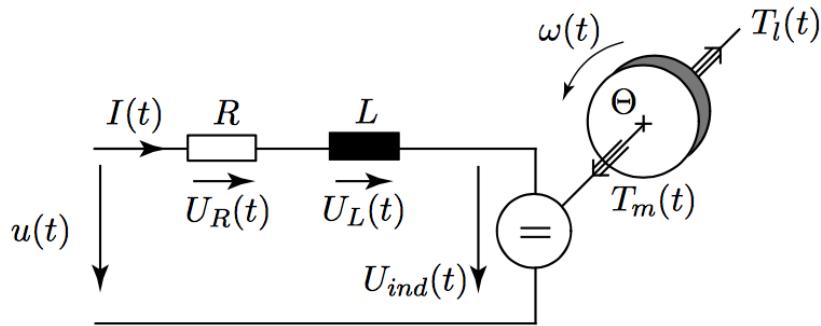


Figure 3: Sketch of a DC motor circuit.

and the **Faraday law**

$$U = -\frac{d}{dt} \int_S \mathbf{B} d\mathbf{S}.$$

These two laws allow us to couple the mechanical and electric part of an electromechanical system. For example, the force generated by a coil immersed in a magnetic field can be written as

$$F(t) = \kappa_{mot} \cdot I(t) \quad (\text{motor law})$$

and the induced EMF as

$$U_{ind}(t) = \kappa_{el} \cdot \omega(t) \quad (\text{generator law})$$

Let us analyze the system:

### Inputs/Outputs:

- The input of the system are the armature voltage  $u(t)$  (control input for motors) and the load torque  $T_l(t)$  (in control systems considered as a disturbance).
- The output of the system is the measurement of the rotor speed  $\omega(t)$ .

### Reservoirs:

Two relevant reservoirs are present:

- The magnetic energy stored in the rotor coil. This has level variable  $I(t)$ .
- The kinetic energy stored in the rotor. This has level variable  $\omega(t)$ .

### Energy Conservation Laws:

As we have two relevant reservoirs, we deal with two energy conservation laws. The conservation of the magnetic energy reads (using Kirchoff's second law)

$$L \cdot \frac{d}{dt} I(t) = -R \cdot I(t) - U_{ind}(t) + u(t).$$

The conservation of kinetic energy of the rotor reads

$$\Theta \cdot \frac{d}{dt} \omega(t) = T_m(t) - T_l(t) - d \cdot \omega(t).$$

## Differential Equations:

The power of the electric part of the motor should equal the power of the mechanical part of the motor. This reads

$$\begin{aligned} P_{\text{elec}}(t) &= P_{\text{mech}}(t) \\ U_{\text{ind}}(t) \cdot I(t) &= T_m(t) \cdot \omega(t) \\ \kappa \cdot \omega(t) \cdot I(t) &= T_m(t) \cdot \omega(t) \\ \Rightarrow T_m(t) &= \kappa \cdot I(t). \end{aligned}$$

Plugging this information into the energy conservation equations we get

$$\begin{aligned} L \cdot \frac{d}{dt} I(t) &= -R \cdot I(t) - \kappa \cdot \omega(t) + u(t) \\ \Theta \cdot \frac{d}{dt} \omega(t) &= \kappa \cdot I(t) - T_l(t) - d \cdot \omega(t). \end{aligned} \tag{5.1}$$

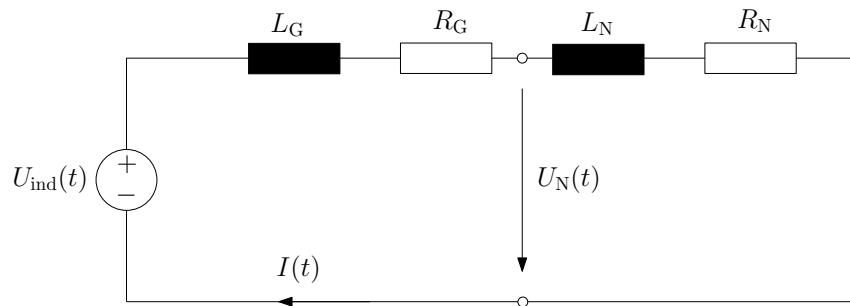
### 5.3.2 Motor vs Generator

Motors and generators have essentially the same physical structure, but are used to achieve different things. A motor's task is to convert electrical energy into kinetic energy, which means that one drives the current of the circuit to produce an angular velocity of the motor's shaft, an effect which in turn reduces the current. Hence, in (5.1) the coupling terms  $\kappa I$  and  $\kappa \omega$  come with a positive and negative sign respectively.

As far as generators are concerned, their goal is to convert mechanical energy into electrical energy (think of manually rotating the shaft), which implies that an increasing current is produced by an increment in  $\omega$ , i.e. the signs are now flipped.

## 5.4 Tips

You may model the circuit as shown below.



## 5.5 Example

In order to reduce the costs of electricity of your SpaghETH, you decide to augment your systems with a turbine, in order to recover energy from the pot filling process. Assume water, having the density  $\rho$ , gets into the pot with a user-defined absolute velocity  $w$  and area  $S$ . The water hits a turbine rotating with angular velocity  $\omega(t)$ . The turbine is mechanically linked to a shaft with moment of inertia  $\Theta$ . To the shaft it is also attached an electric generator with generator/motor (equal for the two) constant  $\kappa$ . Frictions losses are known to be of the form  $T_f = \beta\omega(t)$ . The electric circuit is equipped with an inductance  $L$ . The external loads are modeled as a constant resistance  $R$ . Since the goal of the exercise is to control the power delivered to the load, you start by formulating a model of the system. A sketch of the system with the relative parameters is shown in Figure 4.

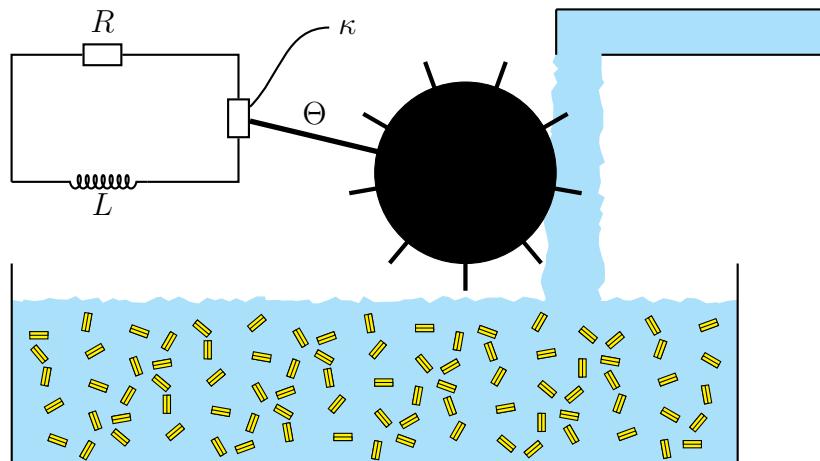


Figure 4: Sketch of the system.

1. What are the input(s) and output(s) to the system?
2. List all the reservoirs and the relative state variables.
3. Draw the causality diagram of the system.
4. Give the algebraic or differential equation for each block of the causality diagram.
5. Show that changing the inductance influences the power delivered by the turbine.

**Solution.**

1. The input to the system is water velocity  $w(t)$ . The output is the power delivered to the load. We will denote the signal with  $P(t)$ .
2. The reservoirs of the system and their relative level variables are:
  - Kinetic energy of shaft, with level variable  $\omega(t)$ .
  - Electrical energy, with level variable  $I(t)$ .
3. The causality diagram is shown in Figure 5.

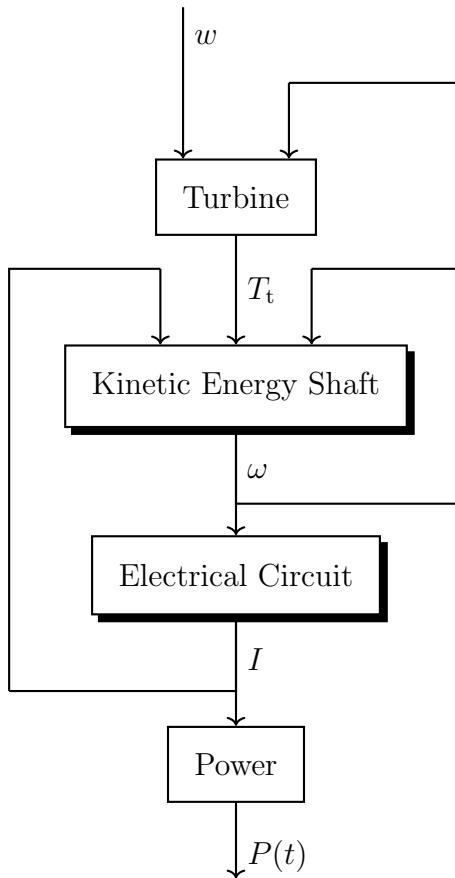


Figure 5: Causality diagram.

4. The torque delivered by the turbine is described by

$$T_t = 2 \cdot \rho \cdot R \cdot S \cdot (w - R \cdot \omega)^2.$$

Conservation of mechanical energy for the shaft gives

$$\Theta \frac{d\omega(t)}{dt} = T_t - \kappa I(t) - \beta \omega(t).$$

The circuit is then described by

$$L \frac{dI}{dt} = -RI(t) + \kappa \omega(t).$$

The resulting power is then

$$P(t) = I(t)U(t) = RI(t)^2.$$

5. See causality diagram.

## A Mean Force Acting on the Pelton Turbine

We consider a *moving* control volume that contains the blade of the turbine, see Figure 6.

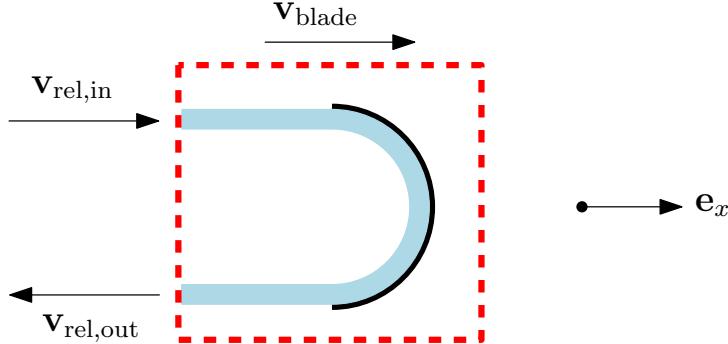


Figure 6: Moving control volume.

The absolute velocities of the fluid entering and exiting the control volume are given by

$$\mathbf{v}_{\text{in}} = w \cdot \mathbf{e}_x, \quad \mathbf{v}_{\text{out}} = -v \cdot \mathbf{e}_x.$$

The relative velocities (w.r.t. the control volume) can then be computed via

$$\mathbf{v}_{\text{rel}} = \mathbf{v} - \mathbf{v}_{\text{blade}} = \mathbf{v} - R\omega \cdot \mathbf{e}_x.$$

Now, our first goal is to find  $v$  as a function of the given quantities. In order to achieve this, we apply the (stationary) mass conservation law on the control volume

$$\int_S \rho \cdot \mathbf{v}_{\text{rel}} \cdot \mathbf{n} dS = 0$$

where  $S$  is the enclosing surface of the control volume and  $\mathbf{n}$  is a unit vector perpendicular to it, pointing outwards. The only parts of the control volume crossed by fluid are the inlet and the outlet, therefore we have

$$\underbrace{\int_{S_{\text{in}}} \rho \cdot \mathbf{v}_{\text{rel,in}} \cdot \mathbf{n}_{\text{in}} dS_{\text{in}}}_{I_1} + \underbrace{\int_{S_{\text{out}}} \rho \cdot \mathbf{v}_{\text{rel,out}} \cdot \mathbf{n}_{\text{out}} dS_{\text{out}}}_{I_2} = 0$$

where

$$I_1 = \int_{S_{\text{in}}} \rho(w - R\omega)(-1) dS_{\text{in}} = -\rho S_{\text{in}}(w - R\omega),$$

and similarly

$$I_2 = -\rho S_{\text{out}}(v - R\omega).$$

It follows that

$$v = \frac{S_{\text{in}}}{S_{\text{out}}}(w - R\omega) - R\omega,$$

which, for identical areas, simplifies to  $v = w - 2R\omega$ .

To find the force acting on the blade we apply conservation of momentum

$$\int_S \rho \cdot \mathbf{v} \cdot (\mathbf{v}_{\text{rel}} \cdot \mathbf{n}) dS = -\mathbf{F}_T$$

where we neglect influences of friction, gravity, surface tension and pressure gradients. Similarly to the previous calculation we obtain

$$\rho S_{\text{in}} \begin{bmatrix} w \\ 0 \end{bmatrix} (-1)(w - R\omega) + \rho S_{\text{out}} \begin{bmatrix} -v \\ 0 \end{bmatrix} (-1)(-v - R\omega) = - \begin{bmatrix} F_{T,x} \\ F_{T,y} \end{bmatrix}.$$

Looking at the  $x$ -direction we have

$$\rho S(w(R\omega - w) - v(v + R\omega)) = -F_{T,x}.$$

By inserting our previous result for  $v$  we can simplify the above expression as

$$F_{T,x} = 2\rho S(w - R\omega)^2,$$

and, inserting  $\overset{*}{V} = S(w - R\omega)$ , we can finally write

$$F_T = 2\rho \overset{*}{V}(w - R\omega).$$