Computation and continuum electrodynamics: Continuum electrodynamics 3

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Green functions are introduced for the perspective of linear operators and algebraic properties.

§3.1 CONCEPTUAL MOTIVATION FOR GREEN FUNCTIONS

The primary motivation for Green functions can be understood as a generalization of the inverse of a matrix. Supposing some finite dimensional system of equations

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{1}$$

one of the central problems of linear algebra is to determine $\mathbf{x}$ supposing that $\mathbf{A}$ and $\mathbf{b}$ are known. If the rank of $\mathbf{A}$ is full, then the question is generally solved by determining of the inverse of $\mathbf{A}$, $\mathbf{A}^{-1}$, which is algebraically determined by the property that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{Id}, \tag{2}$$

where $\mathbf{Id}$ is the identity operator [1]. Once $\mathbf{A}^{-1}$ is known, then, for any $\mathbf{b}$, the definition $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ solves Eq. (1).

As has been suggested in the previous lectures, differential equations can be conceived in much the same way, and, in fact, this connection lies at the foundation of what makes computational modelling possible. All concepts of finite dimensional linear algebra boil down to ordered combinations of multiplication and addition. These operations can be implemented in incredibly short periods of time with extremely high fidelity for rational numbers by complex electric circuits. Therefore, whenever we can faithfully reduce the physical dynamics of some system to linear algebra we can model it on a computer. Specifically, the following definitions are seen to apply equal well to functions as to columns of numbers.

Definition A *vector space* over the real or complex numbers.

A *vector space* $\mathtt{V}$ over the real or complexes numbers, which we will refer to collectively as $\mathbb{F}$, consists of a set $\mathcal{V}$ and two operations—*addition* $+_{\mathtt{v}} : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and scalar multiplication $\cdot_{\mathtt{v}} : \mathbb{F} \times \mathcal{V} \to \mathcal{V}$—respecting the following axioms.

$+_{\mathtt{v}}$ *is group operation on* $\mathcal{V}$—$(\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V})$ the map defined by $+_{\mathtt{v}}$ is *associative*, $\mathbf{u} +_{\mathtt{v}} (\mathbf{v} +_{\mathtt{v}} \mathbf{w}) = (\mathbf{u} +_{\mathtt{v}} \mathbf{v}) +_{\mathtt{v}} \mathbf{w}$ and *commutative*, $\mathbf{u} +_{\mathtt{v}} \mathbf{v} = \mathbf{v} +_{\mathtt{v}} \mathbf{u}$. Moreover, there is an *identity* element $\mathbf{0}_{\mathtt{v}} \in \mathcal{V}$ such that $(\forall v \in \mathcal{V})\ \mathbf{v} +_{\mathtt{v}} \mathbf{0}_{\mathtt{v}} = \mathbf{0}_{\mathtt{v}}$ and for each $\mathbf{v} \in \mathcal{V}$ there is an inverse, $-\mathbf{v}$, such that $\mathbf{v} +_{\mathtt{v}} -\mathbf{v} = \mathbf{0}_{\mathtt{v}}$.

$\cdot_{\mathtt{v}}$ *is compatible with* $\mathbb{F}$—$\forall a, b \in \mathbb{F} \wedge \mathbf{v} \in \mathcal{V}$, $a \cdot_{\mathtt{v}} (b \cdot_{\mathtt{v}} \mathbf{v}) = (ab) \cdot_{\mathtt{v}} \mathbf{v}$; taking 1 to be the multiplicative identity of $\mathbb{F}$, $1 \cdot_{\mathtt{v}} \mathbf{v} = \mathbf{v}$.

distributivity $\forall \mathbf{u}, \mathbf{v} \in \mathcal{V} \wedge a, b \in \mathbb{F}$, $a \cdot_{\mathtt{v}} (\mathbf{u} +_{\mathtt{v}} \mathbf{v}) = (a \cdot_{\mathtt{v}} \mathbf{u}) +_{\mathtt{v}} (a \cdot_{\mathtt{v}} \mathbf{v})$ and $(a +_{\mathtt{v}} b) \cdot_{\mathtt{v}} \mathbf{u} = (a \cdot_{\mathtt{v}} \mathbf{u}) +_{\mathtt{v}} (b \cdot_{\mathtt{v}} \mathbf{u})$

Typically, the operation $\cdot_{\mathtt{v}}$ is simply denoted by juxtaposition and $+_{\mathtt{v}} -\mathbf{v}$ is represented as $-\mathbf{v}$.

$\hookrightarrow$ Take $\mathcal{F}$ to be the collection of all functions $\mathbb{R}^4 \to \mathbb{F}$. $\forall \mathbf{f}, \mathbf{g} \in \mathcal{F} \wedge a \in \mathbb{F} \wedge \mathbf{x} \in \mathbb{R}^4$, take $\mathbf{f} +_{\mathtt{v}} \mathbf{g}$ to be defined by $\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})$, and $a \cdot_{\mathtt{v}} \mathbf{f}$ to be defined by $a\mathbf{f}(\mathbf{x})$. Then, $\mathcal{F}$ is a vector space over $\mathbb{F}$.

Definition—A *linear operator* or *linear map.*

A mapping between two real (resp. complex) vector spaces $\mathbf{A} : \mathtt{V} \to \mathtt{W}$, where both $\mathtt{V}$ and $\mathtt{W}$ are vector space over the real or complex numbers, is said to be a *linear operator*, or *linear map*, if its associations are compatible with the vector space structure of both spaces: $\forall \mathbf{v}_1, \mathbf{v}_2 \in \mathtt{V}$ and $a \in \mathbb{F}$, $\mathbf{A}(\mathbf{v}_1 +_{\mathtt{v}} \mathbf{v}_2) = \mathbf{A}(\mathbf{v}_1) +_{\mathtt{w}} \mathbf{A}(\mathbf{v}_2)$ and $\mathbf{A}(a \cdot_{\mathtt{v}} \mathbf{v}_1) = a \cdot_{\mathtt{w}} \mathbf{A}(\mathbf{v_1})$.

$\hookrightarrow$ $\mathcal{F}$ remains a vector space when restricted to the subset of continuous, smoothly differentiable (infinitely differentiable), functions. In this setting, a linear partial differential operator is a linear operator, and a linear partial differential equation is a linear equation.

With these results in mind, the Green function is "simply" the generalization of Eq. (2) to some appropriately defined vector space of functions. That is, for some differential operator $\Box_{\mathbf{A}}$ we seek an algebraic inverse $\mathbf{G}$,

$$\Box_{\mathbf{A}}\mathbf{G} = \mathbf{G}\,\Box_{\mathbf{A}} = \mathbf{Id}, \tag{3}$$

so that for any system of the form

$$\Box_{\mathbf{A}}\mathbf{x} = \mathbf{b} \tag{4}$$

the solution for $\mathbf{x}$ is $\mathbf{G}\mathbf{b}$.

§3.2 THE VACUUM GREEN FUNCTION IN RECIPROCAL SPACE

In week one of the course, we came to the conclusion that the Maxwell equations could be reduced to

$$\begin{aligned}\boldsymbol{\nabla_d}\epsilon_\circ\ \vec{\mathbf{e}}_{||} &= \mathbf{p}, \qquad (5)\\ \boldsymbol{\partial_t}\epsilon_\circ\ \vec{\mathbf{e}}_{||} &= -\vec{\mathbf{j}}_{||}\\ \boldsymbol{\nabla_d}\ \vec{\mathbf{b}}_{||} &= 0,\\ \boldsymbol{\nabla_c}\ \vec{\mathbf{e}}_{\perp} &= -\boldsymbol{\partial_t}\vec{\mathbf{b}}_{\perp},\\ \boldsymbol{\nabla_c}\mu_\circ^{-1}\ \vec{\mathbf{b}}_{\perp} &= \boldsymbol{\partial_t}\epsilon_\circ\ \vec{\mathbf{e}}_{\perp} + \vec{\mathbf{j}}_{\perp},\end{aligned}$$

with $||$ and $\perp$ subscripts representing transverse and longitudinal parts. Taking Fourier transforms in both space and time Eq. (5) becomes

$$\begin{aligned}-i\vec{\mathbf{k}}\cdot\vec{\mathbf{e}}_{||} &= \mathbf{p}/\epsilon_\circ, \qquad (6)\\ i\omega\boldsymbol{\epsilon_\circ}\vec{\mathbf{e}}_{||} &= -\vec{\mathbf{j}}_{||}\\ \vec{\mathbf{b}}_{||} &= 0,\\ i\vec{\mathbf{k}}\times\vec{\mathbf{e}}_{\perp} &= i\omega\vec{\mathbf{b}}_{\perp},\\ -i\vec{\mathbf{k}}\times\vec{\mathbf{b}}_{\perp} &= i\frac{\omega}{c^2}\vec{\mathbf{e}}_{\perp} + \mu_\circ\vec{\mathbf{j}}_{\perp}.\end{aligned}$$

Noticing that a solution for $\vec{\mathbf{e}}_\perp$ defines the complete "photonic" dynamics of the system, by taking the cross product of the fourth relation with $i\vec{\mathbf{k}}$ and then using the fifth relation we find

$$\begin{aligned}-\vec{\mathbf{k}}\times\boldsymbol{\mu_\circ^{-1}}\vec{\mathbf{k}}\times\vec{\mathbf{e}}_{\perp} &= i\omega\left(-i\omega\boldsymbol{\epsilon_\circ}\ \vec{\mathbf{e}}_{\perp} + \vec{\mathbf{j}}_{\perp}\right)\\ \left(-\vec{\mathbf{k}}\times\vec{\mathbf{k}}\times\ \ - k_\circ^2\mathbf{Id}\right)\vec{\mathbf{e}}_{\perp} &= -i\omega\mu_\circ\vec{\mathbf{j}}_{\perp}, \qquad (7)\end{aligned}$$

where $k_\circ = \omega/c$. Working in Cartesian coordinates,

$$\begin{aligned}\vec{\mathbf{k}}\times &= \delta\left(\tilde{\vec{\mathbf{k}}}-\vec{\mathbf{k}}\right)\begin{bmatrix}0 & -k_z & k_y\\ k_z & 0 & -k_x\\ -k_y & k_x & 0\end{bmatrix}\Rightarrow\\ -\vec{\mathbf{k}}\times\vec{\mathbf{k}}\times &= \delta\left(\tilde{\vec{\mathbf{k}}}-\vec{\mathbf{k}}\right)\begin{bmatrix}k_y^2+k_z^2 & -k_xk_y & -k_xk_z\\ -k_yk_x & k_x^2+k_z^2 & -k_yk_z\\ -k_zk_x & -k_zk_y & k_x^2+k_z^2\end{bmatrix},\end{aligned} \qquad (8)$$

and so the first operator becomes

$$\left(\left(k^2-k_o^2\right)\mathbf{Id} - k^2\hat{\mathbf{k}}\otimes\hat{\mathbf{k}}\right). \qquad (9)$$

Recalling the definition of the transverse projection operator in reciprocal space

$$\perp = \delta\left(\tilde{\vec{\mathbf{k}}}-\vec{\mathbf{k}}\right)\left(\mathbf{Id}-\hat{\mathbf{k}}\otimes\hat{\mathbf{k}}\right),$$

the transverse Green function in reciprocal space, at each $\left\langle\vec{\mathbf{k}},\omega\right\rangle$, is thus defined by the relation

$$\left(\mathbf{Id} - \frac{k^2}{(k^2-k_\circ^2)}\hat{\mathbf{k}}\otimes\hat{\mathbf{k}}\right)\mathbf{G}_\perp = \frac{-i\omega\mu_\circ}{k^2-k_\circ^2}\left(\mathbf{Id}-\hat{\mathbf{k}}\hat{\mathbf{k}}\right). \qquad (10)$$

Introducing $\hat{\mathbf{s}}$ and $\hat{\mathbf{p}}$ as a completion for a right-handed coordinate system beginning with $\hat{\mathbf{k}}$, so that in polar coordinates in reciprocal space

$$\begin{aligned}\hat{\mathbf{s}} &= \langle -s(\phi), c(\phi), 0\rangle, \qquad (11)\\ \hat{\mathbf{p}} &= \langle -c(\theta)c(\phi), -c(\theta)s(\phi), s(\theta)\rangle,\\ \hat{\mathbf{k}} &= \langle s(\theta)c(\phi), s(\theta)s(\phi), c(\theta)\rangle, \qquad (12)\end{aligned}$$

Eq. (10) simplifies to

$$\mathbf{G}_\perp = \frac{-i\omega\mu_\circ}{k^2-k_\circ^2}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}, \qquad (13)$$

showing that in coordinate free form, for reciprocal space,

$$\begin{aligned}&\mathbf{G}_\perp\left(\left\langle\vec{\mathbf{k}},\omega\right\rangle,\left\langle\tilde{\vec{\mathbf{k}}},\tilde{\omega}\right\rangle\right) = \qquad (14)\\ &\frac{-i\omega\mu_\circ}{k^2-k_\circ^2}\left(\hat{\mathbf{s}}\otimes\hat{\mathbf{s}}+\hat{\mathbf{p}}\otimes\hat{\mathbf{p}}\right)\delta\left(\left\langle\vec{\mathbf{k}},\omega\right\rangle - \left\langle\tilde{\vec{\mathbf{k}}},\tilde{\omega}\right\rangle\right).\end{aligned}$$

Accordingly, in real space

$$\begin{aligned}&\int d\tilde{\mathbf{r}}\ \mathbf{G}_\perp\left(\langle\mathbf{r},\omega\rangle,\langle\tilde{\mathbf{r}},\tilde{\omega}\rangle\right) = \delta(\omega-\tilde{\omega}) \qquad (15)\\ &\frac{-i\omega\mu_\circ}{(2\pi)^3}\int d\tilde{\mathbf{r}}\int d\vec{\mathbf{k}}\ e^{-i\vec{\mathbf{k}}\cdot(\mathbf{r}-\tilde{\mathbf{r}})}\frac{\hat{\mathbf{s}}\otimes\hat{\mathbf{s}}+\hat{\mathbf{p}}\otimes\hat{\mathbf{p}}}{k^2-k_\circ^2}.\end{aligned}$$

Relatedly, the full Green function, including the longitudinal part is given by

$$\begin{aligned}&\int d\tilde{\mathbf{r}}\ \mathbf{G}\left(\langle\mathbf{r},\omega\rangle,\langle\tilde{\mathbf{r}},\tilde{\omega}\rangle\right) = \delta(\omega-\tilde{\omega}) \qquad (16)\\ &\frac{-i\omega\mu_\circ}{(2\pi)^3}\int d\tilde{\mathbf{r}}\int d\vec{\mathbf{k}}\ e^{-i\vec{\mathbf{k}}\cdot(\mathbf{r}-\tilde{\mathbf{r}})}\left(\frac{\hat{\mathbf{s}}\otimes\hat{\mathbf{s}}+\hat{\mathbf{p}}\otimes\hat{\mathbf{p}}}{k^2-k_\circ^2} - \frac{\hat{\mathbf{k}}\otimes\hat{\mathbf{k}}}{k_\circ^2}\right).\end{aligned}$$

§3.3 THE ELECTROMAGNETIC GREEN FUNCTION IN REAL SPACE

Jordan's lemma—Suppose that $f:\mathbb{C}\to\mathbb{C}$ is analytic at all points in the upper half-plane

$$\mathrm{U} = \{z\in\mathbb{C}\mid \text{Im}\ z\geq 0\},$$

and take

$$\mathrm{C}_R = \left\{z\in\mathbb{C}\mid z = Re^{i\theta}\wedge\theta\in[0,\pi]\right\},$$

with $R>0$ to denote a semi-circle of radius R confined to the upper half plane. If for each $R>0$ there is a positive constant M_R such that $z\in\mathrm{C}_R\Rightarrow\|f(z)\|\leq\mathrm{M}_R$ and $\mathrm{M}_R\to 0$ as $R\to\infty$, then

$$\lim_{R\to\infty}\int_{\mathrm{C}_R} dz\ f(z)\,e^{i\alpha z}$$

whenever $\alpha>0$.

Proof. To begin, notice that $\sin\theta$ is a concave function on $\theta \in [0,\pi]$, and that $\sin\frac{\pi}{2}=1$. As, such

$$(\forall\alpha\in[0,1])\ \ \sin\left(\alpha\frac{\pi}{2}\right)\geq\alpha\Rightarrow$$
$$\left(\forall\theta\in\left[0,\frac{\pi}{2}\right]\right)\ \ \sin(\theta)\geq\frac{2}{\pi}\theta.$$

Thus, recalling that exp is a monotonic function on $\mathbb{R}$, if $R>0$, this inequality implies that $\exp(-R\sin\theta)\leq\exp(-2R\theta/\pi)$ so that

$$\int_0^{\pi/2} d\theta\ e^{-R\sin\theta}\leq\int_0^{\pi/2} d\theta\ e^{-2R\theta/\pi}=\frac{\pi}{2R}\left(1-e^{-R}\right)\leq\frac{\pi}{2R}.$$

Transforming to polar coordinates

$$\int_{\mathrm{C}_R} dz\ f(z)\,e^{i\alpha z}=$$
$$R\int_0^{\pi} d\theta\ ie^{i\theta}f\left(Re^{i\theta}\right)\exp\left[i\alpha R\left(\cos\theta+i\sin\theta\right)\right].$$

Therefore, because $\|ie^{i\theta}f\left(Re^{i\theta}\right)exp\left(i\alpha R\cos\theta\right)\|\leq\mathrm{M}_R$,

$$\int_{\mathrm{C}_R} dz\ f(z)\,e^{i\alpha z}\leq\frac{\pi}{\alpha}\mathrm{M}_R.$$

Having established this result, the limit that $R\to\infty$ can be taken, proving the lemma. □

To bring the integral part of Eq. (16) into workable form, which we will call $\mathbf{H}$, we begin by making a switch to polar coordinates aligned with $\mathbf{s}=\mathbf{r}-\mathbf{r}'$ so that

$$\mathbf{H}=$$
$$\int d\tilde{\mathbf{r}}\int_0^{\infty} dk\ k^2\int_0^{\pi} d\theta\ \sin\theta\ \frac{e^{-iks\cos\theta}}{(k^2-k_\circ^2)}\left(\mathbf{Id}-\frac{1}{k_\circ^2}\ \vec{\mathbf{k}}\vec{\mathbf{k}}^{\dagger}\right).$$

Although we will not full justify why the following step is valid at the present moment, we may rewrite the above using the "*dyadic* divergence" $\boldsymbol{\nabla}^{\boldsymbol{1}}_{\boldsymbol{d}}\otimes\boldsymbol{\nabla}^{\boldsymbol{1}}_{\boldsymbol{d}}$, as

$$\mathbf{H}=\int d\tilde{\mathbf{r}}\left[\mathbf{Id}+\frac{1}{k_\circ^2}\boldsymbol{\nabla}^{\boldsymbol{1}}_{\boldsymbol{d}}\otimes\boldsymbol{\nabla}^{\boldsymbol{1}}_{\boldsymbol{d}}\right]$$
$$\int_0^{\infty} dk\ k^2\int_0^{\pi} d\theta\ \sin\theta\ \frac{e^{-iks\cos\theta}}{(k^2-k_\circ^2)},\tag{17}$$

with $s=\|s\|$. Continuing from this point

$$\int_0^{\infty} dk\ k^2\int_0^{\pi} d\theta\ \sin\theta\ \frac{e^{-iks\cos\theta}}{(k^2-k_\circ^2)}=$$
$$\int_0^{\infty} dk\ k^2\int_{-1}^{1} du\ \frac{e^{-iksu}}{(k^2-k_\circ^2)}=$$
$$\int_0^{\infty} dk\ \frac{ik^2}{ks\,(k^2-k_\circ^2)}\left(e^{-iks}-e^{iks}\right)=\int_{-\infty}^{\infty} dk\ \frac{ik}{s\,(k^2-k_\circ^2)},e^{-iks},$$

and so

$$\mathbf{H}=\int d\tilde{\mathbf{r}}\left[\mathbf{Id}+\frac{1}{k_\circ^2}\boldsymbol{\nabla}^{\boldsymbol{1}}_{\boldsymbol{d}}\otimes\boldsymbol{\nabla}^{\boldsymbol{1}}_{\boldsymbol{d}}\right]\int_{-\infty}^{\infty} dk\ \frac{ik}{s\,(k^2-k_\circ^2)}e^{-iks}.$$

If this last integrated is deformed into the complex plane by supposing a tiny imaginary contribution in $k_\circ$, then Jordan's lemma can be applied to show that

$$\int_{-\infty}^{\infty} dk\ \frac{ik}{s\,(k^2-k_\circ^2)}e^{-iks}=\pi\frac{e^{-ik_\circ s}}{s}$$

Therefore,

$$\int d\tilde{\mathbf{r}}\ \mathbf{G}\left(\langle\mathbf{r},\omega\rangle,\langle\tilde{\mathbf{r}},\tilde{\omega}\rangle\right)=\delta\left(\omega-\tilde{\omega}\right)$$
$$\frac{-i\omega\mu_\circ}{4\pi}\int d\tilde{\mathbf{r}}\left[\mathbf{Id}+\frac{1}{k_\circ^2}\boldsymbol{\nabla}^{\boldsymbol{1}}_{\boldsymbol{d}}\otimes\boldsymbol{\nabla}^{\boldsymbol{1}}_{\boldsymbol{d}}\right]\frac{e^{-ik_\circ s}}{s}.\tag{18}$$

Redefining all lengths ($\tilde{\mathbf{r}}$ and $\mathbf{s}$) with respect to $k_\circ$, application of $\boldsymbol{\nabla}^{\boldsymbol{1}}_{\boldsymbol{d}}\otimes\boldsymbol{\nabla}^{\boldsymbol{1}}_{\boldsymbol{d}}$ then finally shows that

$$\int d\tilde{\mathbf{r}}\ \mathbf{G}\left(\langle\mathbf{r},\omega\rangle,\langle\tilde{\mathbf{r}},\tilde{\omega}\rangle\right)=-i\frac{Z}{k_\circ}\ \delta\left(\omega-\tilde{\omega}\right)\tag{19}$$
$$\int d\tilde{\mathbf{r}}\ \frac{e^{-is}}{4\pi s}\left[\left(1+\frac{is-1}{s^2}\right)\mathbf{Id}-\left(1+3\frac{is-1}{s^2}\right)\hat{\mathbf{s}}\otimes\hat{\mathbf{s}}\right],$$

where $Z=\sqrt{\mu_\circ/\epsilon_\circ}$ is the impedance of free space.

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[1] Note1. The left and right inverses of $\mathbf{A}$ appearing in the algebraic definition of the inverse must be the same operator because if $\tilde{\mathbf{A}}\mathbf{A}=\mathbf{Id}$, then $\tilde{\mathbf{A}}\mathbf{Id}=\mathbf{A}$.