

Finite representation of real numbers

Floating-point numbers

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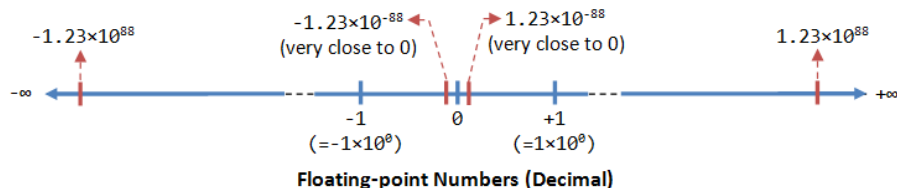
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A floating-point number can represent a very large or a very small value, positive and negative.



A floating-point number is typically expressed in the scientific notation in the form of

$$(-1)^S \times F \times r^E,$$

where,

- S , sign bit.
- F , fraction.
- E , **biased** exponent.
- r , certain radix. $r = 2$ for binary; $r = 10$ for decimal.

IEEE Standard P754 Format

Bit	31	30	29	28	27	26	25	24	23	22	21	20	...	2	1	0
	S	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0	2^{-1}	2^{-2}	2^{-3}	...	2^{-21}	2^{-22}	2^{-23}
Sign (s)	← Exponent (e) →									← Fraction (f) →						

IBM Format

Bit	31	30	29	28	27	26	25	24	23	22	21	20	...	2	1	0
	S	2^6	2^5	2^4	2^3	2^2	2^1	2^0	2^{-1}	2^{-2}	2^{-3}	2^{-4}	...	2^{-22}	2^{-23}	2^{-24}
Sign (s)	← Exponent (e) →									← Fraction (f) →						

DEC (Digital Equipment Corp.) Format

Bit	31	30	29	28	27	26	25	24	23	22	21	20	...	2	1	0
	S	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0	2^{-2}	2^{-3}	2^{-4}	...	2^{-22}	2^{-23}	2^{-24}
Sign (s)	← Exponent (e) →									← Fraction (f) →						

MIL-STD 1750A Format

Bit	31	30	29	...	11	10	9	8	7	6	5	4	3	2	1	0
	2^0	2^{-1}	2^{-2}	...	2^{-20}	2^{-21}	2^{-22}	2^{-23}	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
← Fraction (f) →									← Exponent (e) →							

Modern computers adopt the IEEE 754 standard for representing floating-point numbers at the FPU.

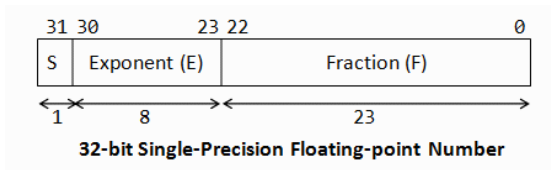
First version was published in 1985. Last version in July 2019 (IEEE 754-2019).

IEEE 754 standard defines several arithmetic formats.

Parameter	Binary formats ($B = 2$)				Decimal formats ($B = 10$)		
	Binary 16	Binary 32	Binary 64	Binary 128	Decimal 132	Decimal 164	Decimal 128
p , digits	$10 + 1$	$23 + 1$	$52 + 1$	$112 + 1$	7	16	34
e_{max}	+15	+127	+1023	+16383	+96	+384	+16,383
e_{min}	-14	-126	-1022	-16382	-95	-383	-16,382
Common name	Half precision	Single precision	Double precision	Quadruple precision			

IEEE 754 standard also defines:

- Rounding rules.
- Arithmetic operations, trigonometric functions.
- Exception handling.

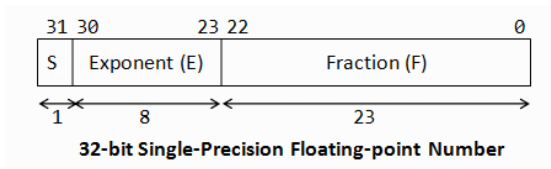


$$(-1)^S \times F \times r^{(E-bias)}$$

- S, sign bit. **0** for positive numbers and **1** for negative numbers.
- F, 23-bits fraction: $[2^{-1} \ 2^{-2} \ \dots \ 2^{-23}]$
- We need to represent both positive and negative exponents.
- E, 8-bits exponent, **no sign bit**.
 - $E = [1, 254]$, $bias = 127$; $-126 \leq E - bias \leq 127$.
 - $E = 0$ and $E = 255$ are reserved.

IEEE 754 standard

Normalized Form



$$(-1)^S \times F \times r^{(E-bias)}$$

- Representation of a floating point number may not be unique:
- For example, the number 13.25 can be represented as
$$1101.01_2 \cdot (2^0) = 110.101_2 \cdot (2^1) = 11.0101_2 \cdot (2^2) = 1.10101_2 \cdot (2^3)$$
- A floating point number is normalized when the integer part of its mantissa is forced to be exactly 1 and its fraction is adjusted accordingly.
- The leading 1 is **implicit**. It is not part of the 32 bits number.
- $1.F = 1. [2^{-1} \ 2^{-2} \dots 2^{-23}]$.

IEEE 754 standard

Example 1

Represent 3215.020002_{10}

Decimal Value Entered: 3215.020002

Single precision (32 bits):

Binary: Status: normal

Bit 31 Sign Bit	Bits 30 - 23 Exponent Field	Bits 22 - 0 Significand
0	100 0101 0	1.100 1000 1111 0000 0101 0010
0: + 1: -	Decimal value of exponent field and exponent 138 - 127 = 11	Decimal value of the significand 1.5698340

Hexadecimal: 4548F052 Decimal: 3215.0200

<http://babbage.cs.qc.cuny.edu/IEEE-754.old/Decimal.html>

Represent $3215.020002_{10} \times 2 = 6430.040004_{10}$

Decimal Value Entered:

Single precision (32 bits):

Binary: Status:

Bit 31 Sign Bit	Bits 30 - 23 Exponent Field	Bits 22 - 0 Significand
<input type="text" value="0"/>	<input type="text" value="10001011"/>	<input type="text" value="1.10010001111000001010010"/>
0: + 1: -	Decimal value of exponent field and exponent <input type="text" value="139"/> - 127 = <input type="text" value="12"/>	Decimal value of the significand <input type="text" value="1.5698340"/>

Hexadecimal: Decimal:

Represent $3215.020002_{10}/4 = 803.7550005_{10}$

Decimal Value Entered:

Single precision (32 bits):

Binary: Status:

Bit 31 Sign Bit	Bits 30 - 23 Exponent Field	Bits 22 - 0 Significand
<input type="text" value="0"/>	<input type="text" value="10001000"/>	<input type="text" value="1.10010001111000001010010"/>
0: + 1: -	Decimal value of exponent field and exponent <input type="text" value="136"/> - 127 = <input type="text" value="9"/>	Decimal value of the significand <input type="text" value="1.5698340"/>

Hexadecimal: Decimal:

- To multiply and divide by 2 is easy using floating-point numbers.
- Not every real number can be represented with floating-point format.
- Floating-point numbers are auto range !

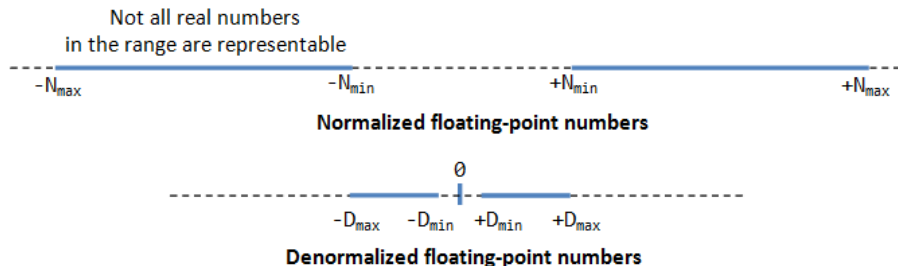
IEEE 754 standard

Why auto range?

2^E	MIN F 1..000000000000000000000000 (b2) 1 (b10)	MAX F 1..111111111111111111111111 (b2) 1.999999881 (b10)
2 ⁻¹²⁶	1.1755E-38	2.3510E-38
2 ⁻³⁰	9.3132E-10	1.8626E-09
2 ⁻²⁰	9.5367E-07	1.9073E-06
2 ⁻¹⁰	9.7656E-04	1.9531E-03
2 ⁻³	0.12500000	0.24999999
2 ⁻²	0.25000000	0.49999997
2 ⁻¹	0.50000000	0.99999994
2⁰	1.00000000	1.99999988
2 ¹	2.00000000	3.99999976
2 ²	4.00000000	7.99999952
2 ³	8.00000000	15.99999905
2 ¹⁰	1.0240E+03	2.0480E+03
2 ²⁰	1.0486E+06	2.0972E+06
2 ³⁰	1.0737E+09	2.1475E+09
2 ¹²⁷	1.7014E+38	3.4028E+38

IEEE 754 standard

De-normalized Form



- Normalized form has a serious problem.
- The number zero cannot be represent with an implicit leading 1!
- De-normalized form is devised to represent zero and small numbers.
- $E = 0 \Rightarrow E - bias = -127$
- Implicit leading $0.F = 0.$ $[2^{-1} \ 2^{-2} \dots 2^{-23}]$.

Represent $-3.4\text{E-}39_{10}$ Decimal Value Entered: Single precision (32 bits):Binary: Status:

Bit 31 Sign Bit	Bits 30 - 23 Exponent Field	Bits 22 - 0 Significand
<input type="text" value="1"/>	<input type="text" value="00000000"/>	<input type="text" value="0.1001010000010111010001"/>
0: + 1: -	Decimal value of exponent field and exponent <input type="text" value="0"/> - 127 = <input type="text" value="-127"/>	Decimal value of the significand <input type="text" value="0.5784800"/>

Hexadecimal: Decimal:

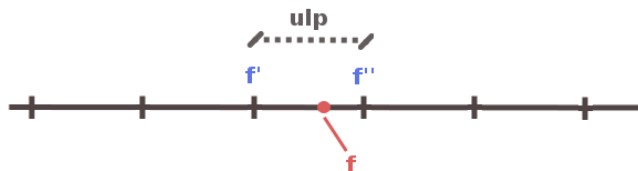
- **Zero:** $E = 0$, $F = 0$. Two representations: **+0** ($S = 0$) and **-0** ($S = 1$).
- **Inf** (Infinity): $E = 0xFF$, $F = 0$. Two representations: **+Inf** ($S = 0$) and **-Inf** ($S = 1$).
- **NaN** (Not a Number): $E = 0xFF$, $F \neq 0$. A value that cannot be represented as a real number (e.g. $0/0$).

MATLAB

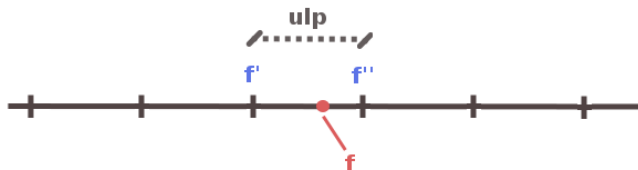
```
1 » a = 1/0
2 » a = Inf
3 » b = exp(1000)
4 » b = Inf
5 » c = log(0)
6 » c = -Inf
```

MATLAB

```
1 » d = -1/0
2 » d = -Inf
3 » e = 0/0
4 » d = NaN
5 » f = Inf/Inf
6 » d = NaN
```



- ulp (unit of least precision). In MATLAB, `eps()`.
- f , significant, $f = 1.F$.
- f' and f'' being two successive multiples of ulp .
- Assume that $f' < f < f''$.
- $f'' = f' + ulp$.
- Then, the rounding function $round(f)$ associates to f either f' or f'' , according to some rounding strategy.



Rounding schemes are:

- ❶ *Truncation* (also called *round toward 0* or *chopping*):
 - if f is positive, $\text{round}(f) = f'$.
 - if f is negative, $\text{round}(-f) = f''$.
- ❷ *Round toward plus infinity*: $\text{round}(f) = f''$.
- ❸ *Round toward minus infinity*: $\text{round}(f) = f'$.
- ❹ *Round to nearest* (default):
 - if $f < f' + \text{ulp}/2$, $\text{round}(f) = f'$.
 - if $f \geq f' + \text{ulp}/2$, $\text{round}(f) = f''$.

Dynamic range is defined as,

$$DR_{db} = 20 \log_{10} \left(\frac{\text{largest possible word value}}{\text{smallest possible word value}} \right) \quad [\text{dB}]$$

Dynamic range for floating-point numbers is defined as,

$$DR_{dB} \approx 6.02 \cdot 2^{b_E}$$

where b_E is the number of bits of E .

For single precision (32-bits):

$$DR_{dB} \approx 6.02 \cdot 2^8 \approx 1541 \text{ dB}$$

For 32-bits fixed point:

$$DR_{dB} \approx 6.02 \cdot 31 \approx 186 \text{ dB}$$

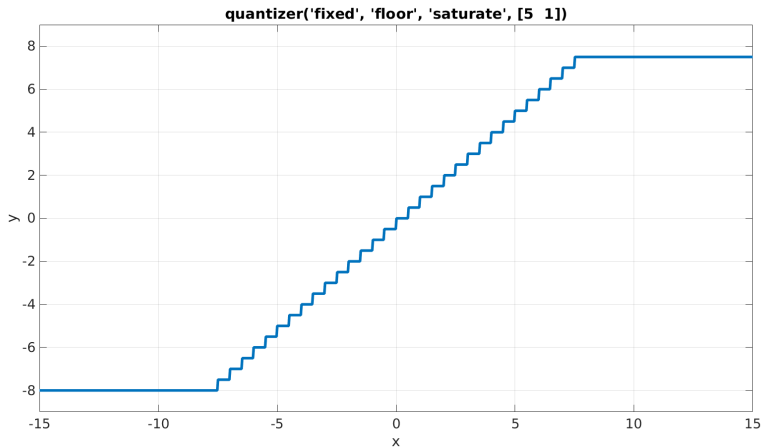
Precision

Fixed-point precision

- Precision is 2^{-n} , where n is the number of bits for the fraction part.
- Precision is constant throughout all fixed-point numbers' range.

MATLAB

```
1 » % Fixed-point quantizer
2 » q = quantizer('fixed','floor','saturate',[5 1]);
3 % [wordlength fractionlength]
4 » u = linspace(-15,15,1000);
5 » y1 = quantize(q,u);
6 » plot(u,y1); title(tostring(q))
```



Precision

Floating-point precision

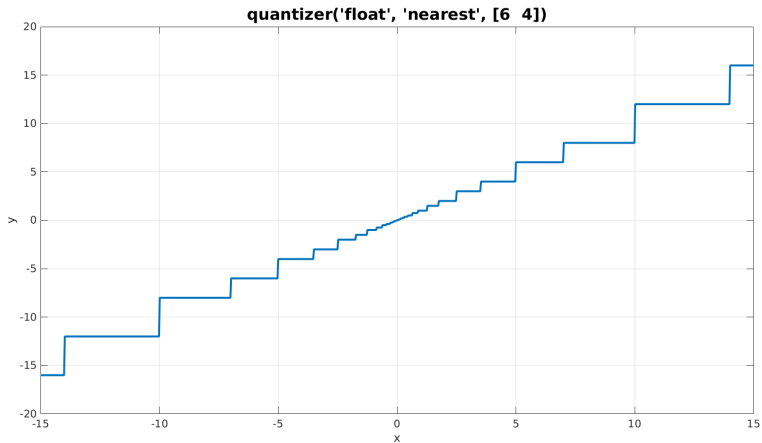
- Precision is $2^E \cdot 2^{-23}$ for single precision.
- Precision is **not** constant throughout all floating-point numbers' range.
- As the numbers get larger, the precision gets larger as well.

MATLAB

```
1 » % Floating-point quantizer
2 » q = quantizer([6 4], 'float', 'nearest');
3 % [wordlength exponentlength]
4 » y2 = quantize(q,u);
5 » plot(u,y2); title(tostring(q))
```

Precision

Floating-point precision



Precision

Precision and ranges

2^E	MIN F 1..000000000000000000000000 (b2) 1 (b10)	MAX F 1..111111111111111111111111 (b2) 1.999999881 (b10)	PRECISION 2 ^E * 2 ⁻²³
2 ⁻¹²⁶	1.1755E-38	2.3510E-38	1.4013E-45
2 ⁻³⁰	9.3132E-10	1.8626E-09	1.1102E-16
2 ⁻²⁰	9.5367E-07	1.9073E-06	1.1369E-13
2 ⁻¹⁰	9.7656E-04	1.9531E-03	1.1642E-10
2 ⁻³	0.12500000	0.24999999	1.4901E-08
2 ⁻²	0.25000000	0.49999997	
2 ⁻¹	0.50000000	0.99999994	5.9605E-08
2⁰	1.00000000	1.99999988	1.1921E-07
2 ¹	2.00000000	3.99999976	2.3842E-07
2 ²	4.00000000	7.99999952	4.7684E-07
2 ³	8.00000000	15.99999905	9.5367E-07
2 ¹⁰	1.0240E+03	2.0480E+03	1.2207E-04
2 ²⁰	1.0486E+06	2.0972E+06	1.2500E-01
2 ³⁰	1.0737E+09	2.1475E+09	1.2800E+02
2 ¹²⁷	1.7014E+38	3.4028E+38	2.0282E+31

Precision

Precision problems

When calculations involve large and small numbers at the same time, the loss of precision affects the small number and the result.

MATLAB

```
1 » a = (2^53 + 1) - 2^53
2 » a = 0
3 » if (a == 0)
4 »     disp('Turn off nuclear reactor')
5 » else
6 »     disp('Do not turn off nuclear reactor')
7
8 » x = 0;
9 » t = tan(x) - sin(x)/cos(x)
10 » t = 0
11 » x = 1;
12 » t = tan(x) - sin(x)/cos(x)
13 » t = 2.2204e-16 % eps(1)
```

Sum of two floating-point numbers

Sum of floating-point numbers in similar range

Perform $0.5 + (-0.4375)$ using 4 bits for the mantissa.

$$0.5_{10} = 0.1000_2 \times 2^0 = 1.0000_2 \times 2^{-1} \text{ (normalised)}$$

$$-0.4375_{10} = -0.0111_2 \times 2^0 = -1.1100_2 \times 2^{-2} \text{ (normalised)}$$

- 1 Match exponents to the bigger one.

Apply n right shifts to -0.4375 where $n = (\text{exponent1} - \text{exponent2}) = (-1 + 2) = 1$.

$$-0.4375 = -1.1100_2 \times 2^{-2} = -\mathbf{0.1110_2} \times \mathbf{2^{-1}}$$

- 2 Add the mantissas.

$$(1.0000_2 - 0.1110_2) \times 2^{-1} = 0.0010_2 \times 2^{-1}$$

- 3 Normalise the sum, checking for overflow/underflow:

$$0.0010_2 \times 2^{-1} = 1.0000_2 \times 2^{-4} = \mathbf{0.0625}$$

$$-126 < -4 < 127, \text{ no overflow or underflow}$$

- 4 Round the sum.

The sum fits in 4 bits so rounding is not required

Sum of two floating-point numbers

Sum of floating-point numbers in very different range

Perform $1\text{e}10 + 1500$ using IEEE-754 single precision.

$$10,000,000,000 = 1.00101010000001011111001_2 \times 2^{33} \text{ (normalised)}$$

$$1500 = 1.01110111000000000000000_2 \times 2^{10} \text{ (normalised)}$$

- 1 Match exponents to the bigger one.

Apply n right shifts to 1500 where $n = (\text{exponent1} - \text{exponent2}) = (33 - 10) = 23$

$$1500 \approx 0.000000000000000000000001_2 \times 2^{33} = (1.0 \times 2^{-23}) \times 2^{33} = 1.0 \times 2^{10} = 1024$$

- 2 Add the mantissas.

$$(1.00101010000001011111001_2 + 0.000000000000000000000001_2) \times 2^{33}$$

- 3 Normalise the sum, checking for overflow/underflow:

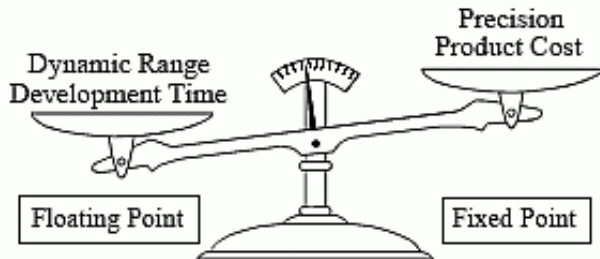
$$1.010101000000101111010_2 \times 2^{33} = 1.16415333710 \times 2^{33} = 10,000,001,024$$

$-126 < 33 < 127$, no overflow or underflow

- 4 Round the sum.

The sum fits in 23 bits so rounding is not required

Fixed-point vs floating-point



- 1 IEEE-SA Standards Board. IEEE Standard for Floating-Point Arithmetic. ISBN 978-0-7381-5752-8. Approved 12 June 2008. New York, NY, USA. [Link](#).
- 2 Jean-Pierre Deschamps, Gustavo D. Sutter, and Enrique Cantó. Floating Point Arithmetic. Guide to FPGA Implementation of Arithmetic Functions, Chapter 12. Springer, 2012.