Finite representation of real numbers

Fixed-point numbers

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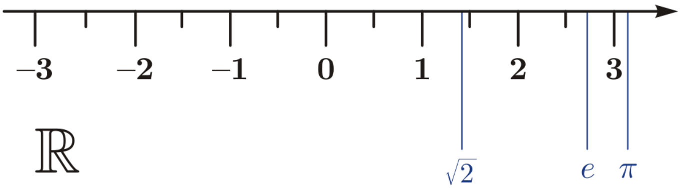
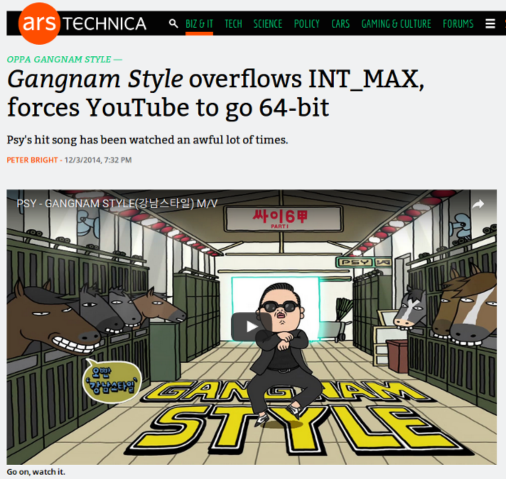
Summary

1. Motivation
   - Finite representation of real numbers in computers
   - Gangnam Style problem
   - Patriot Missile System problem
2. Integers
   - Integer representation
3. Fixed point
   - Fixed-point representation
   - Qm.n notation
   - Conversion to and from fixed point
   - Scale factor
   - Dynamic range
   - Precision and Dynamic range examples

- It is impossible to represent infinite numbers in computers.
- Programmers must choose the best real number representation for a particular application.





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OPPA GANGNAM STYLE —

Gangnam Style overflows INT_MAX, forces YouTube to go 64-bit

Psy's hit song has been watched an awful lot of times.

PETER BRIGHT - 12/3/2014, 7:32 PM

PSY - GANGNAM STYLE(강남스타일) M/V

Go on, watch it.

https://arstechnica.com

- The radar of a Patriot missile system is designed to detect an incoming missile twice in order to avoid false alarms.
- Time is stored to an accuracy of 1/10th of a second in a 24-bit register.
- It results in 0.0001110111001100110011001100110l... with an infinite number of bits.
- The error of representing 1/10th in 24-bit register is 0.000000095 decimal of seconds.
- After 100 hours of operation, cumulative error gives 0.000000095 $\times$100 $\times$ 60 $\times$ 60 $\times$ 10 = 0.34 s.
- A SCUD travels at about 1,676 m/s. In 0.34 s, it travels about 600 meters.
- This error in the time calculation caused the Patriot system to expect an incoming missile at a wrong location for the second detection, causing it to consider the first detection as false alarm.
- On February 25th, 1991, a Patriot Missile system at Dhahran, Saudi Arabia failed to intercept a SCUD missile, killing 28 American soldiers.

More information at https://blog.penjee.com/famous-number-computing-errors/

Unsigned integers:

- An N-bit binary word can represent a total of 2^N separate values.
- Range: 0 to $2^N - 1$
- $n_{10} = 2^{N-1} b_{N-1} + 2^{N-2} b_{N-2} + \cdots + 2^1 b_1 + 2^0 b_0$

Two's complement signed integers:

- Range: -2^{N-1} to $2^{N-1} - 1$.
- $n_{10} = -b_{N-1} 2^{N-1} + \sum_{i=0}^{N-2} b_i 2^i$

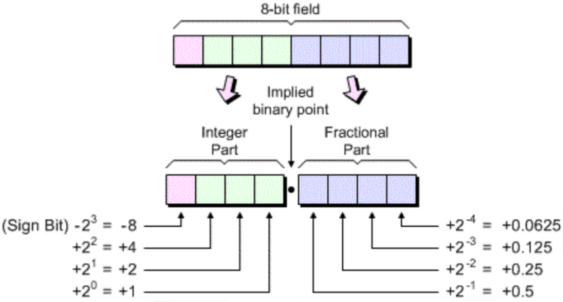
Bit Pattern	Unsigned	2's Complement
0000 0000	0	0
0000 0001	1	1
0000 0010	2	2
•	•	•
•	•	•
0111 1110	126	126
0111 1111	127	127
1000 0000	128	-128
1000 0001	129	-127
•	•	•
•	•	•
1111 1110	254	-2
1111 1111	255	-1

in C:

- 8 bits (`char, int8_t`): [-128, 127]]
- 16 bits (`short, int16_t`): [-32768, 32767]
- 32 bits (`int, long, int32_t`): [-2147483648, 2147483647]

In fixed-point representation, a real number x is represented by an integer X with $N = m + n + 1$ bits, where:

- N is the wordlength.
- m represents the number of integer bits (to the left of the binary point).
- n represents the number of fractional bits (to the right of the binary point).
- One extra bit for the sign bit.
- The weights of bits to the right of the binary point are negative powers of 2: $2^{-1} = \frac{1}{2}$, $2^{-2} = \frac{1}{4}$ ... , etc.
- $n_{10} = -b_m\, 2^m + \left(\sum_{i=0}^{m-1} b_i\, 2^i + \sum_{i=1}^{n} b_i\, 2^{-i}\right)$.
- Precision: 2^{-n}.
- Range: -2^m to $2^m - 2^{-n}$.
- What happens if $n = 0$?



This naming convention does not take the MSB of the number (sign bit) into account.

For instance:

- Q0.15 (Q15)
    - 16 bits;
    - Range: -1 to 0.99996948;
    - Precision: 1/32768 (2^{-15}).
- Q3.12
    - 16 bits;
    - Range: -8 to 7.9998;
    - Precision: 1/4096 (2^{-12}).
- Q0.31 (Q31)
    - 32 bits;
    - Range: -1 to 0.999999999534339;
    - Precision: 4.6566129e-10 (2^{-31}).

Defining:

- Unit: $z = 1 << n = 1 \cdot 2^n$.
  Example: $n = 4 \implies z = 1.0000_2$.
- One half (1/2): $z = 1 << (n-1) = 1 \cdot 2^{(n-1)}$.
  Example: $n = 4 \implies z = 0.1000_2$.

Conversion from floating-point number x ("real") to fixed-point number X using casting:

$$X := (int)(x \cdot (1 << n)) \tag{1}$$
$$X := (int)(x \cdot 2^n) \tag{2}$$

Conversion from fixed-point number X to floating-point number x ("real") using casting:

$$x := (float)(X)/(1 << n) \tag{3}$$
$$x := (float)(X) \cdot 2^{-n} \tag{4}$$

Example 1: Represent x = 13.4 with Q4.3 format using round() function:

$$X = round(13.4 \cdot 2^3) = 107\ (01101011_2)$$

Example 2: Represent x = 0.052246 with Q4.11 format using round() function:

$$X = round(0.052246 \cdot 2^{11}) = 107\ (000000001101011_2)$$

- There is no difference at the CPU level (ALU) between both fixed-point and integer numbers.
- The difference is based on the concept of *scale factor*, which is completely in the head of the programmer.
- Fixed-point numbers in Qm.n notation can be seen as a signed integer simply multiplied by 2^{-n}, the precision.
- In fact, the scale factor can be an arbitrary scale that may not be a power of two.
- **Example:** We want 16-bit numbers between 8000H and 7FFFH to represent decimal values between –5 and +5.
  1. Integer: -32768 to 32767 (8000H - 7FFFH).
  2. $(-32768 \cdot 2^{-15})$ to $(32767 \cdot 2^{-15}) => -1$ to 0.99996948242.
  3. $(-1 \cdot 5)$ to $(0.99996948242 \cdot 5) => -5$ to 4.99984741211.

     The scale factor and precision are the same, $(5 \cdot 2^{-15})$.

Dynamic range is defined as,

$$DR_{db} = 20\ log_{10}\left(\frac{\text{largest possible word value}}{\text{smallest possible word value}}\right) \quad [\text{dB}]$$

For N-bit signed integers,

$$DR_{dB} = 20\ log_{10}\left[\frac{2^{(N-1)} - 1}{1}\right] \quad [\text{dB}]$$
$$DR_{dB} \approx 20\ [(N-1) log_{10}(2)]$$
$$DR_{dB} \approx 20\ log_{10}(2) \cdot (N-1)$$
$$DR_{dB} \approx 6.02 \cdot (N-1) \quad [\text{dB}]$$

Fixed point

Precision and Dynamic range examples

Format (N.M)		Largest positive value (0x7FFF)	Least negative value (0x8000)	Precision (0x0001)		DR(dB)
0	15	0,999969482421875	-1	3,05176E-05	2^-15	90,30873362
1	14	1,99993896484375	-2	6,10352E-05	2^-14	90,30873362
2	13	3,9998779296875	-4	0,00012207	2^-13	90,30873362
3	12	7,999755859375	-8	0,000244141	2^-12	90,30873362
4	11	15,99951171875	-16	0,000488281	2^-11	90,30873362
5	10	31,99902344	-32	0,000976563	2^-10	90,30873362
6	9	63,99804688	-64	0,001953125	2^-9	90,30873362
7	8	127,9960938	-128	0,00390625	2^-8	90,30873362
8	7	255,9921875	-256	0,0078125	2^-7	90,30873362
9	6	511,984375	-512	0,015625	2^-6	90,30873362
10	5	1023,96875	-1024	0,03125	2^-5	90,30873362
11	4	2047,9375	-2048	0,0625	2^-4	90,30873362
12	3	4095,875	-4096	0,125	2^-3	90,30873362
13	2	8191,75	-8192	0,25	2^-2	90,30873362
14	1	16383,5	-16384	0,5	2^-1	90,30873362
15	0	32767	-32768	1	2^-0	90,30873362

Bibliography

1. Richard G. Lyons. *Understanding Digital Signal Processing, 3rd Ed*. Prentice Hill. 2010. Chapter 12.
2. Bruno Paillard. An Introduction To Digital Signal Processors, Chapter 5.