Frequency-resolved structural sensitivity

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We look at how uncertainty in interactions (structure) changes recovery trajectories.

Consider a stable linear system around equilibrium.

$$T\dot{x} = Ax$$

where T is diagonal with positive entries (time scales), and A is the interaction matrix with $A_{ii} = -1$. The generalized resolvent is

$$R(\omega) = (i\omega T - A)^{-1}$$

We also choose a weighting matrix C, which in the biomass-weighted version is

$$C = \mathrm{diag}(u^2)$$

which relates to treating deviations in species with larger equilibrium biomasses as more important in the quadratic energy.

1. The distance-to-equilibrium object

Given an initial displacement x_0, the trajectory is $x(t) = e^{Jt}x_0$ where $J = T^{-1}A$.

The weighted trajectory energy is

$$E(t) = x(t)^\top C x(t)$$

To remove dependence on the particular direction of x_0, we consider an ensemble of initial conditions with covariance $\mathbb{E}[x_0 x_0^\top] = C$.

$$\mathbb{E}[E(t)] = \mathrm{tr}(e^{Jt} C e^{J^\top t})$$

This gives us a normalized "remaining fraction":

$$y(t) = \frac{\mathrm{tr}(e^{Jt} C e^{J^\top t})}{\mathrm{tr}(C)}$$

This is exactly what the biomass-weighted median return rate is, the log-slope summary of this remaining fraction:

$$r_{med}(t) = -\frac{1}{2t}\Big(\log y(t)\Big) = -\frac{1}{2t}\Big(\log \mathrm{tr}(e^{Jt} C e^{J^\top t}) - \log \mathrm{tr}(C)\Big)$$

Now, we define the recovery time t_{95} implicitly from $y(t)$ by

$$y(t_{95}) = 0.05$$

We can also use normalized time $\tau = t/t_{95}$ when we want comparability across systems.

2. The effective return rate

Instead of $r_{med}(t)$, let's now look at the distance-to-equilibrium object itself. We can call it R_{eff} and it equals the total integrated trajectory energy:

$$\int_0^\infty y(t)\,dt$$

This gives us a single number, that we'll call "effective return rate" and is defined by

$$\frac{1}{R_{eff}} = 2\int_0^\infty y(t)\,dt, \qquad \text{so that} \qquad R_{eff} = \frac{1}{2\int_0^\infty y(t)\,dt}$$

I believe this means that if the whole decay was a single exponential $y(t) = e^{-2rt}$, then

$$2\int_0^\infty e^{-2rt}dt = \frac{1}{r}$$

so $R_{eff} = r$.

Thus, R_{eff} is a scalar summary of the entire decay curve that does not retain the full time dependence like $r_{med}(t)$.

3. The generalized resolvent

The main benefit of moving to frequency is that integrals of squared trajectory magnitude can be written as integrals of squared transfer magnitude (Parseval's theorem)

If we write the system as a forced linear system,

$$T\dot{x} = Ax + \xi(t)$$

its Fourier transform satisfies

$$\hat{x}(\omega) = R(\omega)\hat{\xi}(\omega)$$

If ξ is matches the same C we used in time, then we get an identity of the form

$$\int_0^\infty \mathbb{E}[x(t)^\top Cx(t)]\,dt \;\propto\; \int_{-\infty}^\infty \mathrm{tr}\big(R(\omega)\,C_\xi\,R(\omega)^\dagger\big)\,d\omega$$

where $\dagger$ is conjugate transpose and C_ξ is the forcing covariance in frequency.

This is why the generalized resolvent matters. It converts an integrated time-domain energy statement into an integrated frequency-domain magnitude statement.

4. Structural uncertainty and the role of R(ω) and R(ω)PR(ω)

We model interaction uncertainty as

$$A \mapsto A + \varepsilon P$$

where P is a direction (uncertainty pattern) and ε sets its magnitude.

A first-order expansion gives us

$$R_\varepsilon(\omega) - R(\omega) \approx \varepsilon\,R(\omega)PR(\omega)$$

Thus, the frequency-resolved sensitivity to uncertainty direction P is naturally based on the operator RPR. Because we're also considering the biomass-weighted C, sensitivity at a certain ω can be defined as

$$S(\omega; P) = \varepsilon^2 \, \frac{\|R(\omega)PR(\omega)U\|_F^2}{\mathrm{tr}(C)}, \qquad \text{with} \qquad UU^\top = C$$

If $C = \mathrm{diag}(u^2)$ as it has been so far, then $U = \mathrm{diag}(u)$.

Given $S(\omega; P)$, can now define two sensitivities

- Typical sensitivity: average $S(\omega; P)$ over an ensemble of P (noise uncertainty).
- Worst-case sensitivity: maximize over P, which yields a worst-case bound that will be controled by the singular values of $R(\omega)$ and $R(\omega)U$.

5. The cutoff frequency ω_c

Let's consider a cutoff frequency linked to the minimal time to get divergence. Or in other words, the minimal time at which structural information may matter.

We can define it in two different ways

5.1 Picking ω_c from $\rho(A_\omega)$ (better)

The more mechanistic way to define the cutoff is to identify when indirect interaction pathways become important.

We can factor the resolvent as

$$R(\omega) = (i\omega T - A)^{-1} = (I - A_\omega)^{-1}(i\omega T + I)^{-1}$$

with

$$A_\omega = A(i\omega T + I)^{-1}$$

The term $(i\omega T + I)^{-1}$ behaves like a classic low-pass filter.

And the term $(I - A_\omega)^{-1}$ expands as a Neumann series when $\rho(A_\omega) < 1$:

$$(I - A_\omega)^{-1} = I + A_\omega + A_\omega^2 + \cdots$$

This means:

- I is the direct (no-interaction) response.
- A_ω is one-step interaction (direct effects).
- $A_\omega^2, A_\omega^3, \ldots$ are the indirect effects.

A frequency cutoff can therefore be defined from a "collectivity" index such that $K(\omega) = \rho(A_\omega)$. The cutoff ω_c is where indirect effects stop being negligible (for example, when $K(\omega)$ crosses 1), and then $t_c = 1/\omega_c$ is the earliest time at which indirect pathways can matter.

5.2 Picking ω_c from $S(\omega)$ (old idea)

We can also do the following:

- Define a relevant frequency threshold $\omega_{95} = 1/t_{95}$.
- Look at how sensitivity accumulates over frequencies above ω_{95}.
- Define ω_q such that a fraction q of the relevant sensitivity is accumulated by ω_q:

$$\frac{\int_{\omega_{95}}^{\omega_q} S(\omega)\,d\omega}{\int_{\omega_{95}}^{\infty} S(\omega)\,d\omega} = q$$

  Then bring it back to time-domain by $t_q = 1/\omega_q$ and $\tau_q = t_q/t_{95}$.

Why is this useful? If most sensitivity sits at high frequencies, then ω_q is large and t_q is small, meaning trajectories are expected to separate early (in normalized time). If sensitivity sits at lower frequencies, separation is expected later, maybe in slow, not-so-relevant times.

6. Formal link between $r_{med}(t)$ and R_{eff}

Here I give an exact mathematical relation between $r_{med}(t)$ and R_{eff}.

6.1 Common backbone: weighted energy $y(t)$

Let J be the stable Jacobian, $E(t) = e^{tJ}$ and a positive diagonal weight matrix C. The normalised weighted trajectory energy is

$$y(t) \;=\; \frac{\mathrm{tr}\big(E(t)\,C\,E(t)^{\top}\big)}{\mathrm{tr}(C)}$$

Then $y(0) = 1$, $y(t) > 0$, and for stable J, $y(t) \to 0$ as $t \to \infty$.

Thus, our biomass-weighted median return rate is exactly

$$r_{med}(t) \;=\; -\frac{1}{2t}\,\log\big(y(t)\big)$$

6.2 Instantaneous energy decay rate

The instantaneous energy decay rate is

$$r_{inst}(t) \;=\; -\frac{1}{2}\,\frac{d}{dt}\log\big(y(t)\big)$$

Because $y(0) = 1$, we can integrate $d\log y/dt$ from 0 to t to obtain

$$\log y(t) \;=\; -2\int_0^t r_{inst}(s)\,ds$$

Substituting into the definition of $r_{med}(t)$ yields an exact identity:

$$r_{med}(t) \;=\; \frac{1}{t}\int_0^t r_{inst}(s)\,ds$$

Thus, $r_{med}(t)$ is the uniform in time average of the instantaneous decay rate over the interval $[0, t]$.

3.3 Defining R_{eff} from the same $y(t)$

In Section 2, I already showed that

$$R_{eff} \;=\; \frac{1}{2\int_0^\infty y(t)\,dt}$$

6.4 Exact relation

Using $r_{inst}(t) = -(1/2)\, d\log y(t)/dt$, we have

$$\frac{d}{dt}y(t) \;=\; -2\, r_{inst}(t)\, y(t)$$

Integrating both sides from 0 to ∞ gives

$$\int_0^\infty r_{inst}(t)\, y(t)\, dt \;=\; -\frac{1}{2}\int_0^\infty y'(t)\, dt \;=\; \frac{1}{2}\big(y(0) - y(\infty)\big) \;=\; \frac{1}{2}$$

Therefore,

$$R_{eff} \;=\; \frac{1}{2\int_0^\infty y(t)\,dt} \;=\; \frac{\int_0^\infty r_{inst}(t)\, y(t)\, dt}{\int_0^\infty y(t)\, dt}$$

Thus, R_{eff} is a single scalar that averages across all times, with weights proportional to $y(t)$ whereas $r_{med}(t)$ is a function of time and depends on a finite averaging horizon $[0, t]$

6.5 When are they approximately equal?

If $y(t)$ is approximately a single exponential,

$$y(t) \approx e^{-2rt}$$

then $r_{inst}(t) = r$ is constant, and

$$r_{med}(t) = r \quad \text{for all } t, \qquad R_{eff} = \frac{1}{2\int_0^\infty e^{-2rt}dt} = r$$

Hence, exact equality between R_{eff} and $r_{med}(t)$ will only occur when the energy decay is effectively a single rate.