

Frequency-resolved Framework Summary

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2026-01-05

Frequency-resolved structural sensitivity of recovery

We want a consistent way to quantify when and how interaction structure matters for recovery after a pulse, without relying on comparisons between unrelated systems. The central object is a frequency-resolved sensitivity spectrum derived from a generalized resolvent. The output is a set of regime-level diagnostics (band sensitivities and ratios), not a single predicted bump time.

Baseline linear system

We consider a stable linearized community dynamics

$$\dot{x}(t) = Jx(t),$$

with species time-scale vector $u \in \mathbb{R}_+^S$. We write

$$J = \text{diag}(u) A, \quad A_{ii} = -1,$$

and define

$$T = \text{diag}(1/u).$$

Here, A captures interaction structure and T captures intrinsic time scales.

Biomass-weighted recovery metric

We focus on the biomass-weighted return-rate curve $r_{\text{med}}(t)$. This corresponds to weighting perturbations by

$$C = \text{diag}(u^2).$$

This weighting is carried through the frequency-domain formulation by choosing a consistent input channel (defined below).

Forced formulation and channels

The resolvent becomes a literal transfer operator only after specifying how forcing enters. We use the time-scale form

$$T \dot{x}(t) = A x(t) + B u(t).$$

Taking Fourier transforms gives

$$\hat{x}(\omega) = R(\omega) B \hat{u}(\omega), \quad R(\omega) = (i\omega T - A)^{-1}.$$

To align with the biomass-weighted metric, we choose the input channel

$$B = U, \quad U = \text{diag}(u),$$

so that forcing $w(t)$ with unit covariance is mapped through U into the state. This is the reason we use $R(\omega)U$ rather than $R(\omega)$ alone: it is the channel consistent with the biomass-weighted setting $C = U^2$.

Structural uncertainty model

We model imperfect knowledge of interactions by perturbing the interaction matrix:

$$A_\varepsilon = A + \varepsilon P,$$

where ε is a fixed uncertainty magnitude (set relative to the baseline system) and P is a direction of uncertainty. The definition of P is the key modeling choice and represents an uncertainty ensemble $\mathbb{P}(P)$ (for example, rewiring-based directions, constrained perturbations, or other ecologically motivated error models). Directions P are normalized (e.g. $\|P\|_F = 1$) so that ε controls the magnitude.

Frequency-resolved sensitivity spectrum

For a given direction P , small- ε perturbation theory yields that the first-order change in the input-state mapping $R(\omega)B$ is controlled by $R(\omega)PR(\omega)B$. With $B = U$, we define a biomass-consistent sensitivity spectrum

$$S(\omega; P) = \varepsilon^2 \frac{\|R(\omega) P R(\omega) U\|_F^2}{\text{tr}(C)}, \quad C = U^2.$$

This spectrum decomposes sensitivity to interaction uncertainty by Fourier frequency ω .

Typical versus worst-case sensitivity

We consider two complementary summaries.

Typical (ensemble) sensitivity

In the typical view, P is random with $P \sim \mathbb{P}(P)$. We summarize with

$$\bar{S}(\omega) = \mathbb{E}_P[S(\omega; P)],$$

and/or high quantiles over P . This answers: under a specified uncertainty model, which frequency bands are typically most sensitive?

Worst-case (capacity) sensitivity

In the worst-case view, we quantify the maximum sensitivity over allowed directions:

$$S_{\max}(\omega) = \sup_{P \in \mathcal{P}} S(\omega; P),$$

where \mathcal{P} encodes constraints such as $\|P\|_F = 1$ and any ecological restrictions. This answers: what is the intrinsic capacity for extreme sensitivity if uncertainty aligns with the most amplifying direction?

Both are reported because typical sensitivity is realism-driven (depends on $\mathbb{P}(P)$), while worst-case sensitivity is fragility-driven (depends on constraints defining \mathcal{P}).

Regime diagnostics from band sensitivities

We convert $\bar{S}(\omega)$ or $S_{\max}(\omega)$ into regime-level metrics by integrating over frequency bands:

$$S_{\text{low}} = \int_0^{\omega_L} S(\omega) d\omega, \quad S_{\text{mid}} = \int_{\omega_L}^{\omega_H} S(\omega) d\omega, \quad S_{\text{high}} = \int_{\omega_H}^{\infty} S(\omega) d\omega.$$

A primary diagnostic is the ratio

$$\rho = \frac{S_{\text{mid}}}{S_{\text{low}}},$$

interpreted as the relative importance of intermediate-frequency (collective, potentially non-normal) sensitivity versus slow-mode (resilience-like) sensitivity.

Time-domain connection and standardization

We keep time-domain recovery summaries (such as $r_{\text{med}}(t)$) as secondary validation. Raw time t is not comparable across communities, so any comparison should be done in standardized time (e.g. using a recovery time such as t_{95} computed from $r_{\text{med}}(t)$). The framework does not require that a unique bump time exists; instead it predicts whether structural uncertainty is concentrated in slow versus intermediate regimes, which should be reflected statistically in standardized-time recovery differences.

Workflow summary

- 1) Specify baseline (A, T) and compute $R(\omega) = (i\omega T - A)^{-1}$ on a frequency grid.
- 2) Specify an uncertainty model $\mathbb{P}(P)$ and a magnitude ε (normalized directions).
- 3) Compute typical spectrum $\bar{S}(\omega)$ and worst-case envelope $S_{\max}(\omega)$ (subject to constraints).
- 4) Reduce spectra to band sensitivities and ratios (e.g. $S_{\text{mid}}/S_{\text{low}}$).
- 5) Use standardized time to relate these diagnostics to time-domain recovery summaries as a secondary check.