Relevant-window

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Our goal is to quantify when and why network structure (beyond single-species time scales and beyond asymptotic resilience) can generate meaningful deviations in recovery behaviour. We do not attempt to locate a unique "mid-time" peak. Instead, we quantify the amount of structure-driven discrepancy accumulated during the only time interval that is intrinsically interpretable across systems: the pre-recovery window up to a recovery threshold (here 95 percent recovery).

1. Dynamics, recovery metric, and structural discrepancy

Consider a linearised community dynamics around equilibrium,

$$\frac{dx}{dt} = Jx,$$

with a stable Jacobian J. Let u be the vector of species time scales (biomass weights), and define the biomass-weighted return-rate curve $r_{med}(t)$ from the propagator $\Phi(t) = \exp(Jt)$ as

$$r_{med}(t) = -\frac{1}{2t}\Big[\log\,\mathrm{tr}(\Phi(t)\,C\,\Phi(t)^{\top}) - \log\,\mathrm{tr}(C)\Big],$$

where

$$C = \mathrm{diag}(u^2).$$

This is the weighted r_{med} used throughout.

Given a baseline system and a structurally perturbed system (same u, modified interactions), define the instantaneous structural discrepancy

$$\Delta(t) = \big|r_{med,\,base}(t) - r_{med,\,pert}(t)\big|.$$

2. Defining the relevant time window without arbitrary cutoffs

A recovery threshold defines an interpretable time scale. Let

$$y(t) = \exp\big(-r_{med,\,base}(t)\,t\big),$$

which is the predicted remaining fraction of displacement under the baseline r_{med} at time t. Define t_{95} implicitly as the earliest time such that

$$y(t) \le 0.05.$$

In practice, t_{95} is obtained by scanning the $r_{med}(t)$ grid and interpolating when $\exp(-r_{med}(t)\,t)$ crosses 0.05.

The interval

$$[0,\,t_{95}]$$

is the relevant window: it is the time before the system has essentially recovered. Short-time behaviour dominated purely by time scales may exist, but its duration can be extremely small and depends on u. Therefore, we do not partition $[0, t_{95}]$ into "early–mid–late" sub-windows, since the location of collective (non-normal) effects within the relevant window is not known *a priori* and may vary across systems.

3. Relevant-window structural error (time-domain target)

We summarise discrepancy over the relevant window via an integral that is robust to large dynamic ranges of time. Two options are useful.

Option A (scale-fair, equal weight per decade of time):

$$\mathrm{Err}_{rel} = \int_{t_{min}}^{t_{95}} \Delta(t)\, d\log t,$$

that is,

$$\int \Delta(t)\, \frac{dt}{t}.$$

Numerically this is evaluated using a trapezoidal rule on a log-spaced t grid.

Option B (literal time accumulation):

$$\mathrm{Err}^{(t)}_{rel} = \int_{t_{min}}^{t_{95}} \Delta(t)\, dt.$$

The lower bound t_{min} can be chosen as the smallest available time in the grid. If one wishes to exclude an ultra-short regime dominated by the diagonal (time scales), a data-driven alternative is to set t_{min} as the first time at which $r_{med,\,base}(t)$ departs from its smallest-time value by a small relative amount; this exclusion can be checked for robustness.

The main analysis uses Err_{rel} (Option A), and we verify that conclusions do not hinge on the precise handling of the ultra-short regime.

4. Generalised resolvent and sensitivity spectrum (frequency-domain predictor)

Write the system in scaled form

$$J = \mathrm{diag}(u)\, \bar{A},$$

where $\bar{A}$ has diagonal entries

$$\bar{A}_{ii} = -1.$$

Define the time-scale matrix

$$T = \mathrm{diag}(1/u).$$

For purely imaginary $z = i\omega$, the generalised resolvent is

$$R(\omega) = \left(i\omega T - \bar{A}\right)^{-1}.$$

Structural uncertainty is represented by a perturbation of $\bar{A}$,

$$\bar{A} \to \bar{A} + \varepsilon P,$$

where P is a direction matrix with

$$\mathrm{diag}(P) = 0, \qquad \|P\|_F = 1,$$

and ε is a magnitude. For biomass-weighted responses, the natural forcing-to-state factor is $\mathrm{diag}(u)$, because

$$C = \mathrm{diag}(u^2)$$

corresponds to $\mathrm{diag}(u)$ acting as a square-root weighting.

A frequency-resolved sensitivity spectrum for a given P is

$$S(\omega; P) = \varepsilon^2\, \frac{\|R(\omega)\, P\, R(\omega)\, \mathrm{diag}(u)\|_F^2}{\sum_i u_i^2}.$$

We study both:

1. typical sensitivity, by sampling many P from a chosen uncertainty distribution (e.g. iid Gaussian noise on off-diagonals) and averaging $S(\omega; P)$ across P;
2. worst-case sensitivity, by using a singular-value-based upper envelope that captures the most amplifiable perturbation direction at each frequency.

5. Matching frequency content to the relevant window

To avoid arbitrary frequency bands, we anchor frequency cutoffs to the same recovery-derived time scale. Define a recovery frequency

$$\omega_{95} = \frac{1}{t_{95}}.$$

We then define the relevant sensitivity mass as the integral of sensitivity over frequencies corresponding to time scales at or faster than t_{95}:

$$\mathrm{Sens}_{rel} = \int_{\omega_{95}}^{\infty} S(\omega)\, d\omega,$$

computed numerically on a finite frequency grid. If an ultra-short-time exclusion is used in time, the matching upper frequency cutoff is

$$\omega_{max} = \frac{1}{t_{min}},$$

and the integral is taken over

$$[\omega_{95},\, \omega_{max}].$$

This construction ensures that both Err_{rel} and Sens_{rel} refer to the same interpretable regime: the pre-recovery window.

6. Main testable claim

Across heterogeneous baseline systems (different structures) under a fixed uncertainty model for P, we test whether

$$\log \mathrm{Sens}_{rel} \text{ correlates with } \log \mathrm{Err}_{rel}.$$

This is an energy-like link between frequency-resolved structural sensitivity and accumulated structural discrepancy over the relevant time window. Importantly, the framework does not require defining or predicting a unique mid-time peak; it only claims that systems with more sensitivity content at frequencies relevant before recovery tend to accumulate larger deviations in recovery behaviour before recovery is achieved.

7. Optional: Jeff's cutoff frequency as a minimal divergence time scale

To obtain a time-scale statement without a mid-time window, define the cumulative sensitivity mass above ω_{95}:

$$C(\Omega) = \frac{\int_{\omega_{95}}^{\Omega} S(\omega)\, d\omega}{\int_{\omega_{95}}^{\infty} S(\omega)\, d\omega}.$$

For a chosen fraction q (e.g. 0.5 or 0.8), define ω_q as the smallest Ω such that

$$C(\Omega) = q,$$

and set

$$t_q = \frac{1}{\omega_q}.$$

This yields an interpretable minimal time scale at which a chosen fraction of the relevant sensitivity budget is accumulated, without partitioning the relevant window into arbitrary sub-windows.