When does network matters?

Nico, Núria & Jeff

2025-04-15

Why we see these results?

In our trophic model, the Jacobian at equilibrium splits neatly into a diagonal matrix $\mathbf{D}$ (encoding each species' self-regulation) and an off-diagonal interaction part $\mathbf{M}$:

$$J \;=\; D\,M, \quad D = \mathrm{diag}(d_1, \ldots, d_S), \quad M = -I + A^*,$$

where for each species

$$d_i = \begin{cases} \frac{r_i}{K_i}\,B_i^* & \text{(resource)} \\ \frac{m_i}{\xi_i}\,B_i^* & \text{(consumer).} \end{cases}$$

When interaction strengths are moderate—so that the typical off-diagonal magnitude $\sigma = \mathrm{std}(M_{ij})$ is much smaller than the smallest d_i—**first-order perturbation theory** shows the leading eigenvalue of $J = -D + \sigma G$ shifts by

$$\Delta\lambda \approx \sigma\,G_{ii} \approx 0$$

in expectation. Thus the most unstable eigenvalue (governing resilience, return-times, reactivity, etc.) is set almost entirely by the **diagonals** d_i. Randomizing or erasing A (our ladder steps 9–11) leaves D unchanged, so **community-level metrics forget the network**.

When network structure does matter?

If interaction scale grows so that

$$\sigma \sim \min_i d_i,$$

the $O(\sigma)$ corrections no longer vanish and off-diagonals begin to shift the leading eigenvalue. Ecologically, **very strong** competition or predation restores a network imprint on resilience and return-times. Similarly, metrics probing non-local dynamics (multistability, cycles, stochastic variance) will re-inject network effects.

The special role of Self-Regulation Loss (SL)

For each species,

$$\mathrm{SL}_i \;=\; \frac{B_i^*}{K_i}$$

measures its realized equilibrium relative to its monoculture carrying capacity. Since

$$\mathbf{B}^* = (I - A)^{-1}\,\mathbf{K} = \big(I + A + A^2 + \cdots\big)\,\mathbf{K},$$

SL_i depends on **all orders** of A. Even after randomizing A, recalculating B^* via the Neumann series ensures SL_i remains **first-order sensitive** to the exact interaction pattern. In contrast, bulk metrics hinge only on D.

How to prove it formally

1. **Eigenvalue perturbation**

- Write $J = -D + \sigma G$, with G zero-mean. Use standard matrix perturbation to show

$$\lambda_{\max}(J) = -\min_i d_i + O(\sigma^2/\Delta d).$$

2. **Neumann-series expansion**

- Expand $(I - A)^{-1} = \sum_{n \geq 0} A^n$ to express

$$\frac{B_i^*}{K_i} = 1 + \sum_j A_{ij} \frac{K_j}{K_i} + O(\sigma^2).$$

- Show that randomizing A alters the first-order term in SL_i but only second-order in resilience.

Implications

- **Predicting return-times**: measure each species' demography $(r_i, K_i, m_i, \xi_i, B_i^*)$ and you can forecast bulk stability without mapping the full network.
- **Mapping coexistence**: to know *which* species are suppressed or facilitated, you must recover SL_i via the full interaction matrix.
- **Threshold of network relevance**: by tuning interaction strength or connectance until $\sigma \approx \min d_i$, you can pinpoint when network architecture reclaims control over community dynamics.