

Exact Cover Problems

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1 Regular Exact Cover problem

Use backtracking to solve the following exact cover problem, that is: **is there a set of rows containing exactly one 1 in each column?**

We use the following recursive procedure, where A is a Boolean matrix, and S is the solution set of rows:

```

EXACTCOVER(A, S)
1  if A is empty
2      return Success
3  if A has a 0 column
4      return Fail
5  Select column c with least number of 1s
6  for each row r such that A[r, c] == 1
7      Add row r to solution S
8      Cover all columns that are selected by row r, as well as all rows that conflict with r
      // Recursively search reduced matrix A
9      EXACTCOVER(A)
      // Partial solutions that include row r have been explored:
      // backtracking before trying another row
10     Restore previously deleted rows and columns
11     Remove row r from solution S
      // Solutions obtained by selecting column c have been explored:
      // backtracking before trying another column
12 Restore column c in matrix A
    
```

$$\begin{array}{c}
 \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{array}
 \begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{array}
 \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
 \end{array}
 \quad
 \begin{array}{c}
 \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{array}
 \begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{array}
 \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
 \end{array}
 \quad
 \begin{array}{c}
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 \begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{array}
 \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
 \end{array}$$

2 Generalized Exact Cover problem

Solve this generalized exact cover problem, that is: Is there a set of rows containing *exactly one* 1 in each primary column (c_1 through c_5), and *at most one* 1 in each secondary column (c_6 and c_7)?

Use the EXACTCOVER procedure above, but change line 3 to

if A has a 0 primary column...

	c_1	c_2	c_3	c_4	c_5	c_6	c_7
r_1	1	0	1	0	0	0	1
r_2	1	0	1	0	0	0	0
r_3	0	0	1	1	0	0	1
r_4	0	1	0	1	1	0	0
r_5	0	1	0	0	1	1	1
r_6	0	0	0	0	0	1	1

	c_1	c_2	c_3	c_4	c_5	c_6	c_7
r_1	1	0	1	0	0	0	1
r_2	1	0	1	0	0	0	0
r_3	0	0	1	1	0	0	1
r_4	0	1	0	1	1	0	0
r_5	0	1	0	0	1	1	1
r_6	0	0	0	0	0	1	1

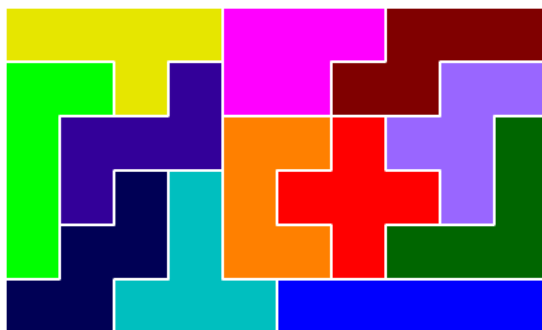
	c_1	c_2	c_3	c_4	c_5	c_6	c_7
r_1	1	0	1	0	0	0	1
r_2	1	0	1	0	0	0	0
r_3	0	0	1	1	0	0	1
r_4	0	1	0	1	1	0	0
r_5	0	1	0	0	1	1	1
r_6	0	0	0	0	0	1	1

3 An application: Pentomino

The Pentomino is a tiling problem involving

- 12 different tiles, each of them covering 5 cells
- a 6x10 grid

The following is one of the many solutions to the Pentomino problem:



Design the exact cover matrix for the Pentomino problem:

- what should it be represented by its columns?
- what should each row of the matrix represent?
- what is a solution to the problem?

Hint: Pentomino is a regular exact cover problem.