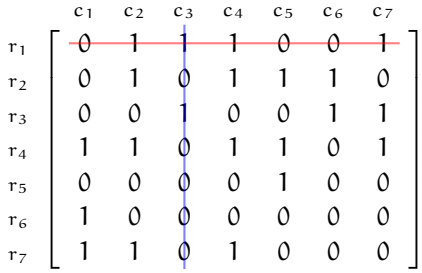
Homework #8: Exact Cover Problems: key

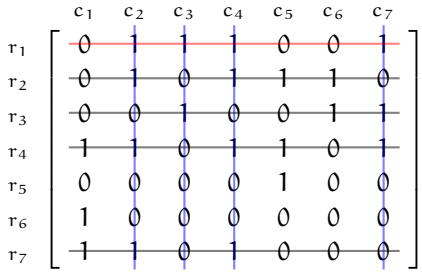
October 12, 2017

1 Regular Exact Cover problem

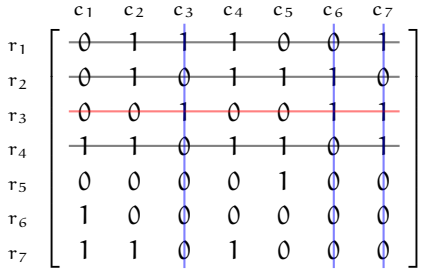
Use backtracking to solve the following exact cover problem, that is: **is there a set of rows containing exactly one 1 in each column?**



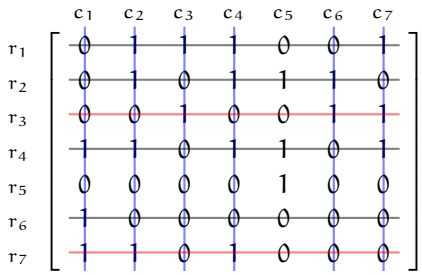
Initial move: choose column 3, select row 1 (as part of the solution)



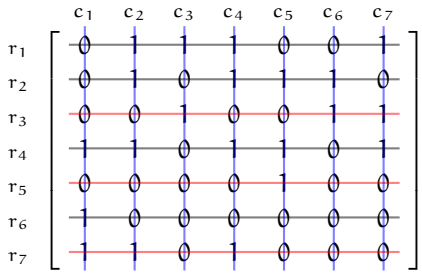
Cover additional columns set in row 1 (cols 2, 4, and 7), as well as conflicting rows 2, 3, 4, and 7.
DEAD END! Column 6 has not been set yet, but all rows have been removed. Backtracking...



Deselect row 1. From column 3, select row 3 instead.
Cover additional columns set in row 3 (cols 6 and 7), as well as conflicting rows 1, 2, and 4.
Recursing...



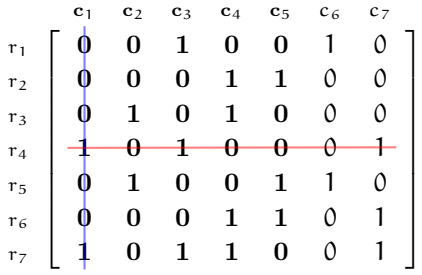
Chose column 2, select row 7.
Cover additional columns set in row 4 (cols 1 and 4), as well as conflicting row 6.
Recursing...



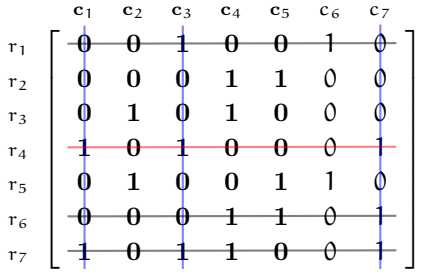
Chose column 5, add row 5 to the solution.
SUCCESS!

2 Generalized Exact Cover problem

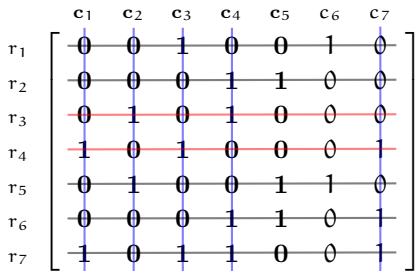
Solve this generalized exact cover problem, that is: **Is there a set of rows containing *exactly one* 1 in each primary column** (c_1 through c_5), and ***at most one* 1 in each secondary column** (c_6 and c_7)?



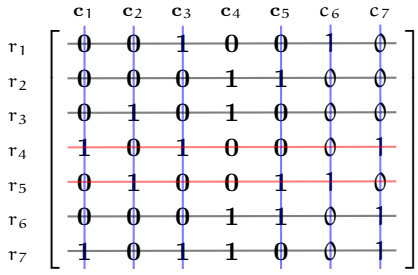
Initial move: choosing column 1, and selecting row 4, as part of the solution.



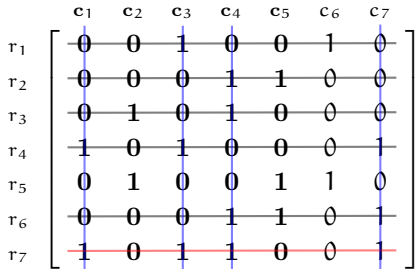
Cover additional columns that are set in row 4 (i.e. cols 3, and 7), as well as rows 1, 6, and 7 (conflict with row 4).



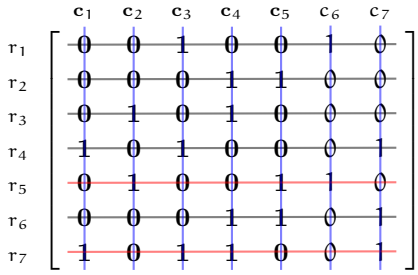
Recursing: choosing column 2, and selecting row 3 (as part of the solution)
Cover additional columns that are set in row 3 (i.e. cols 2 and 4), as well as rows 2 and 5 (conflict with row 3)
DEAD END! Column 5 should be chosen now, but all rows in it have been deleted. Backtracking...



Deselect row 3 (and uncover corresponding columns and conflicting rows), select row 5 instead.
Cover additional columns that are set in row 5 (cols 2, 5, and 6), as well as conflicting rows 2 and 3.
DEAD END! Column 4 should be chosen now, but all rows in it have been deleted. Backtracking...



Deselect rows 5 and 4, and from column 1, select row 7 instead.
Cover additional columns that are set in row 7 (cols 1, 3, 4, and 7), as well as conflicting rows 1, 2, 3, 4, 6.



Recursing: choose column 2, and select row 5 (as part of the solution).
Cover additional columns that are set in row 5 (cols 5 and 6).
SUCCESS!

3 A generalized exact cover problem: 8-Queen Problem

1. Encode the 8-Queen Problem into a matrix of 0s and 1s suitable for a generalized exact cover solution:

- how many rows?
- how many primary columns?
- how many secondary columns?

Answer: The 8-Queen problem is a generalization of the standard exact cover problem. The matrix has 2 sets of columns:

- The first 16 columns represent rank (columns 1 through 8) and file (columns 9 through 16) positions for a given queen
- The next 30 columns represent the up (first 15 columns) and down (next 15 columns) diagonals covered by a queen in the position coded by this row

2. Show that the selection of rows that is a solution to the exact cover problem for your matrix is also a solution for the 8-Queen Problem.

Answer: Why is this a generalized exact cover problem? Each row in the matrix represents a way to place a single queen on the board. The solution set of rows has 8 elements, one for each queen. The first 16 columns are *primary* columns, to which the usual constraint applies: since the solution set of rows contains *exactly* one 1 for each of these columns, we are ensured that each queen is on its own row and column. The 30 diagonals columns are *secondary* columns: for those, the row selection constraint can be relaxed, so that *at most* one 1 can be selected for each column. This ensures that no two queens in different positions cover the same diagonal, without introducing the extra, undue requirement that each diagonal be covered by a queen: the algorithm might terminate with some diagonal columns still uncovered.