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The largest gap between zeros of entire L-functions is less than 41.54

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ABSTRACT

Using suitable feasible pairs and convex combinations of Selberg minorant functions, the upper bound under GRH and the Ramanujan hypothesis on the largest gap between consecutive zeros of an entire L-function in Bober, Conrey, Farmer, Fujii, Koutsoliotas, Lemurell, Rubinstein and Yoshida [2] is improved from 45.3236 to 41.54. An application about nonexistence of certain entire L-functions is also provided.

1. Introduction

The Generalized Riemann Hypothesis (GRH) states that the zeros of any L-function are on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. We may then list the zeros as $\frac{1}{2} + i\gamma_n$ for $n \in \mathbb{Z} \setminus \{0\}$, where

$$\ldots < \gamma_{-2} < \gamma_{-1} \leq 0 < \gamma_1 < \gamma_2 < \ldots$$

and we refer to $\frac{1}{2} + i\gamma_1$, or simply γ_1, as the "first" zero of an L-function.

Miller [12] originally studied the location of the first zero for automorphic L-functions with a real archimedean type. He observed that L-functions with small real spectral parameters and conductor do not usually have a small first zero. Using Weil's explicit formula, he proved that on the GRH there is at least a zero of every such L-functions in the interval $[-\gamma_{1,\zeta}, \gamma_{1,\zeta}]$, where $\gamma_{1,\zeta} \approx 14.13472$ is the imaginary part of the first zero of the Riemann zeta-function.

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Things seem to behave differently if we allow the spectral parameter to be a complex number. In [2], Bober, Conrey, Farmer, Fujii, Koutsoliotas, Lemurell, Rubinstein and Yoshida exhibited a particular degree 4 L-function coming from a GL(4) Maass form discovered in [5] with first zero greater than the first zero of $\zeta(s)$. A possible motivation for this surprising discovery is that the imaginary part of the first trivial zeros of the corresponding L-function appear close to a minima of the (Hardy) Z-function [17]

$$Z(t) = e^{i\theta(t)} L\left(\frac{1}{2} + it\right),$$

where $\theta(t)$ is such that $Z(t)$ is real. These particular trivial zeros seem to suppress the appearance of a nearby nontrivial zero, thereby forcing the first zero to have a higher imaginary part. Miller was also able to show the same result for GL(2) Maass forms of any level.

Taking into account the counterexample provided for a specific L-function, the question they raised was if there exists another higher absolute upper bound for the largest gaps between zeros of general L-functions, including L-functions coming from GL(d) Maass forms. In the same paper [2] they proved an upper bound of 45.3236 for the length of the interval containing at least one zero, which is weaker than Miller's upper bound of $2 \cdot \gamma_{1,\zeta} \approx 28.26944$ for L-functions having a real archimedean type. This new upper bound is obtained using a different set of functions introduced by Selberg [15] whose properties are described in [4]. These are called Selberg's functions.

In this paper we improve the striking result of [2] regarding the upper bound of the largest gap between zeros of general entire L-functions from 45.3236 to 41.54 using suitable feasible pairs and convex combinations of Selberg minorant functions under GRH and Ramanujan hypothesis.

Bober [2] suggested that the lowest upper bound should be around 36. However, no indication is given of how this number is obtained. A future paper of Bober [1] mentioned in [2] will clarify this point.

Based on the properties of feasible sets (see Definition 5.1 below), one arrives at a more detailed upper bound. Furthermore, a natural question that arises would be to see whether this bound is optimal, in the sense that either there would exist at least one L-function with exactly this largest gap, or if it would be an accumulation point of differences between imaginary parts of zeros of general L-functions.

At the end of the paper, another useful application of Selberg's functions is provided. We use the convention

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right),$$

where Γ is the standard Euler gamma function.

Assuming the GRH and the Ramanujan hypothesis for the first Dirichlet coefficient $b(2)$, no entire L-function with completed L-function of the form

$$Q^s \Gamma_{\mathbb{R}}(s)^d L(s),$$

where the conductor (sometimes called level) $N = Q^2$ is an integer, exists for certain values of Q which are too small (to be quantified in §6). In particular, for degree $d = 4$, there is no entire L-function of this form having conductor $N < 324$, while for $d = 5$ there are no such L-functions having $N < 1375$.

2. Automorphic L-functions

Let Ω be the set of entire L-functions attached to irreducible, unitary, cuspidal automorphic representations of GL(n) over $\mathbb{Q}$, and such that the Generalized Riemann Hypothesis holds for L.

If $L \in \Omega$, then it satisfies the following well-known properties (see [2] and also [16]).

(1) It can be written as a formal Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}, \tag{2.1}$$

which is absolutely convergent for $\mathrm{Re}(s) = \sigma > \sigma_0$, where $\sigma_0 \geq 1$.

(2) It satisfies a functional equation of the form

$$\Lambda(s) = \varepsilon \overline{\Lambda}(1-s), \tag{2.2}$$

where

$$\Lambda(s) = Q^s \prod_{j=1}^{d} \Gamma_{\mathbb{R}}(s+\mu_j) L(s) = Q^s \gamma(s, \{\mu_j\}) L(s), \tag{2.3}$$

with $Q \geq 1$ and $Q^2 = N$ a positive integer, $\mathrm{Re}(\mu_j) \geq 0$ and $|\varepsilon| = 1$. Here $\overline{f}(s) = \overline{f(\overline{s})}$, and d is the degree of the L-function.

(3) It has an Euler product of the form,

$$L(s) = \prod_p L_p(p^{-s})^{-1}, \tag{2.4}$$

where $L_p(x)$ is a polynomial of degree d except for finitely many primes such that $p|N$. Moreover, for $p \nmid N$ (the "good primes") one has

$$L_p(x) = \sum_{j=1}^{d} (1 - \alpha_{j,p} x), \tag{2.5}$$

where $\alpha_{j,p}$ are the Satake parameters.

We also denote by Ω_* the subclass of Ω consisting of entire, automorphic L-functions such that the *Petersson–Ramanujan conjecture* holds,

$$b(n) \ll_{L,\epsilon} n^{\epsilon}, \quad \text{for every } \epsilon > 0. \tag{2.6}$$

If (2.6) holds, then the corresponding Satake parameters (2.5) satisfy $|\alpha_{j,p}| = 1$.

Definition 2.1 *(Largest gap).* For each $L \in \Omega$ let Gap_L be the largest gap between consecutive zeros of $L(s)$. In what follows, by the *largest gap between zeros of entire automorphic L-functions* we mean the quantity

$$\mathrm{Gap}_{\Omega} := \sup_{L \in \Omega} \mathrm{Gap}_L .$$

Similarly, we define the *largest gap between zeros of entire automorphic L-functions satisfying the Ramanujan bound* (2.6) as

$$\mathrm{Gap}_{\Omega_*} := \sup_{L \in \Omega_*} \mathrm{Gap}_L .$$

By definition, we have $\mathrm{Gap}_{\Omega_*} \leq \mathrm{Gap}_{\Omega}$.

For an L-function of degree d and full level $N = 1$ coming from a $\mathrm{SL}(d, \mathbb{Z})$ Maass form, the Satake parameters appearing in (2.5) satisfy the bounds

$$|\alpha_{j,p}| \leq \begin{cases} 1, & \text{if } L \in \Omega_*, \\ p^{\frac{1}{2} - \frac{1}{d^2+1}}, & \text{if } L \in \Omega, \end{cases} \tag{2.7}$$

for $1 \leq j \leq d$. This is proved, in more generality, by Luo–Rudnick–Sarnak [11]. Moreover, the Dirichlet coefficients at all primes p satisfy

$$|b(p)| \leq \begin{cases} d, & \text{if } L \in \Omega_*, \\ dp^{\frac{1}{2} - \frac{1}{d^2+1}}, & \text{if } L \in \Omega. \end{cases} \tag{2.8}$$

For an L-function $L \in \Omega$ of degree 2 its Satake parameters at good primes satisfy the bounds

$$|\alpha_{j,p}| \leq p^{7/64}, \; j = 1, 2. \tag{2.9}$$

Moreover, $|b(p)| \leq 2p^{7/64}$. This bound, which currently holds the world record, is due to Kim and Sarnak [9]. See also [6] for more details.

We will use the following notation for the logarithmic derivative of an L-function,

$$-\frac{L'}{L}(s) = \sum_{n=1}^{\infty} \frac{\Lambda_L(n)}{n^s}, \tag{2.10}$$

where $\Lambda_L(n)$ is the corresponding generalized von Mangoldt function. Note that since $L(s)$ has an Euler product, it follows that $\Lambda_L(n)$ is supported on prime powers only, and is given by

$$\Lambda_L(n) = \begin{cases} (\log p) \sum_{j=1}^{d'} \alpha_{j,p}^k, & \text{if } n = p^k \text{ for some prime } p \text{ and positive integer } k, \\ 0, & \text{otherwise}, \end{cases} \tag{2.11}$$

where $b(n)$ are the Dirichlet coefficients and $\alpha_{j,p}$ the Satake parameters associated to the local Euler factor at p. If p is good prime $p \nmid N$ then $d' = d$, if it is a bad prime $p|N$ then $d' < d$. In particular, the corresponding factor is $\Lambda_L(p) = b(p) \log p$ at primes p.

If k is a positive integer, then at all good primes $p \nmid N$:

$$|\Lambda_L(n)| \leq \begin{cases} d \log p, & \text{if } L \in \Omega_* \text{ and } n = p^k, \\ dp^{\frac{k}{2} - \frac{k}{d^2+1}} \log p, & \text{if } L \in \Omega \text{ and } n = p^k, \\ 2p^{7/64} \log p, & \text{if } L \in \Omega \text{ for } d \leq 2 \text{ and } n = p^k. \end{cases} \tag{2.12}$$

The main result of the paper is the following.

Theorem 2.1. *We have*

$$\mathrm{Gap}_{\Omega_*} \leq 41.54,$$

$$\mathrm{Gap}_{\Omega} \leq 43.41. \tag{2.13}$$

3. Weil explicit formula

The key analytic tool used in this paper is the Weil explicit formula, which we will apply to certain test functions defined in §4. Suppose that $L(s)$ has a Dirichlet series expansion (2.1) which continues to an entire function with functional equation (2.2), and suppose that

$$L(\sigma + it) \ll |t|^A,$$

for $A > 0$ uniformly in t and bounded σ. Let $f(s)$ be holomorphic in a horizontal strip $-(1/2+\delta) < \mathrm{Im}(s) < 1/2+\delta$ with $f(s) \ll \min(1, |s|^{-(1+\epsilon)})$ in this region and suppose that $f(x)$ is real valued for real x. Suppose that the Fourier transform of f, defined by

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(u) e^{-2\pi i u x} du,$$

is such that

$$\sum_{n=1}^{\infty} \left(\frac{-\Lambda_L(n)}{\sqrt{n}} \hat{f}\left(\frac{\log n}{2\pi}\right) + \frac{-\bar{\Lambda}_L(n)}{\sqrt{n}} \hat{f}\left(-\frac{\log n}{2\pi}\right) \right)$$

is absolutely convergent. Then we have

$$\begin{aligned} \sum_{\gamma} f(\gamma) = \frac{\hat{f}(0)}{\pi} \log Q + \frac{1}{2\pi} \sum_{j=1}^{d} \ell(\mu_j, f) \\ + \frac{1}{2\pi} \sum_{n=1}^{\infty} \left(\frac{-\Lambda_L(n)}{\sqrt{n}} \hat{f}\left(\frac{\log n}{2\pi}\right) + \frac{-\bar{\Lambda}_L(n)}{\sqrt{n}} \hat{f}\left(-\frac{\log n}{2\pi}\right) \right), \end{aligned} \tag{3.1}$$

where the sum $\sum_\gamma$ runs over the non-trivial zeros of $L(s)$, and

$$\ell(\mu, f) = \mathrm{Re}\left(\int_{-\infty}^{\infty} \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{it}{2} + \mu\right) f(t) dt \right) - \hat{f}(0) \log \pi. \tag{3.2}$$

The proof can be found in [8, §5.5].

Following the remarkably clever idea put forward in [2], we apply Weil's formula to a specific entire function f such that it approximates the characteristic function $\chi_{[\alpha,\beta]}(x)$ for x real, and such that the support of $\hat{f}$ is compact and located within a certain region. The last term of (3.1) would then be a finite sum that can be bounded using conditional or unconditional bounds on the Dirichlet coefficients.

4. Beurling function and Selberg's functions

This section and relative results are summarized from [4, §2] and [2, §4]. In the 1930s, Beurling studied the real entire function

$$B(z) = 1 + 2\left(\frac{\sin(\pi z)}{\pi}\right)^2 \left(\frac{1}{z} - \sum_{n=1}^{\infty} \frac{1}{(n+z)^2}\right).$$

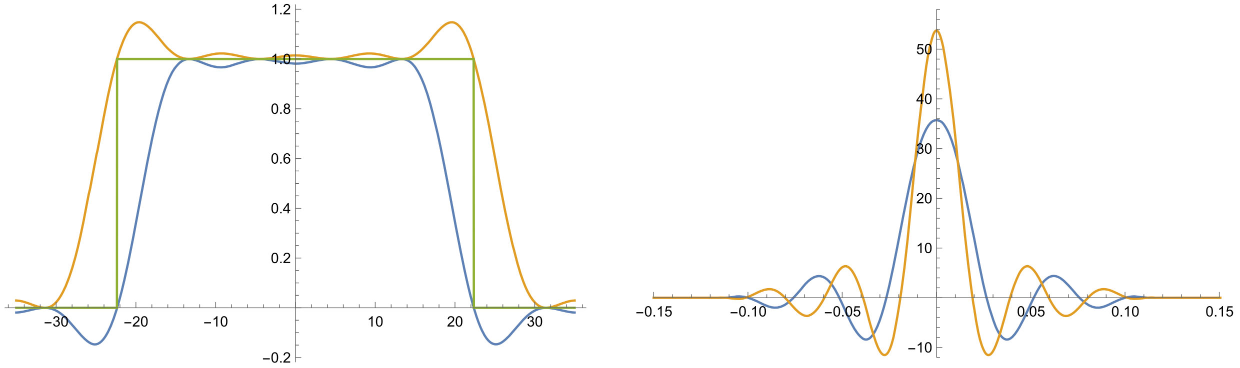

Fig. 4.1. On the left hand side: the Selberg minorant and majorant functions in blue and orange respectively, with $\beta = 22.36$, $\delta = 2.5/\beta$. The green line is the characteristic function of the interval $[-\beta, \beta]$. The second graph shows their Fourier transforms. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

This function is a good smooth approximation of the sign function

$$\operatorname{sgn}(x) = \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0, \end{cases}$$

and it is a majorant of $\operatorname{sgn}(x)$, meaning that $\operatorname{sgn}(x) \leq B(x)$ for all $x \in \mathbb{R}$. For all real x, the Fourier transform $\hat{B}(x)$ is supported on $[-1, 1]$ and satisfies

$$\int_{-\infty}^{\infty} |B(x) - \operatorname{sgn}(x)| \, dx = 1.$$

Moreover, Beurling showed that $B(x)$ minimizes the $L^1(\mathbb{R})$-distance to $\operatorname{sgn}(x)$. Selberg used the Beurling function to define two entire functions in the following way.

Definition 4.1 *(Selberg minorant and majorant functions).* For a parameter $\delta > 0$, the Selberg minorant and majorant functions are defined in the interval $[\alpha, \beta]$ to be

$$S^-(z) = S^-_{\alpha,\beta,\delta}(z) := -\frac{1}{2} \left(B(\delta(\alpha - z)) + B(\delta(z - \beta)) \right), \tag{4.1}$$

and

$$S^+(z) = S^+_{\alpha,\beta,\delta}(z) := \frac{1}{2} \left(B(\delta(-\alpha + z)) + B(\delta(\beta - z)) \right). \tag{4.2}$$

Selberg observed that $S^-(x) \leq \chi_{[\alpha,\beta]}(x) \leq S^+(x)$ for all real x and that their Fourier transforms have support in $[-\delta, \delta]$, as shown in Fig. 4.1.

The next lemma summarizes the main properties of $S^\pm(z)$.

Lemma 4.1 *(Properties of $S^\pm(z)$). Let $S^\pm(z)$ be the Selberg minorant and majorant functions in the interval $[\alpha, \beta]$ and parameter $\delta > 0$. One has*

(1) $S^-(x) \leq \chi_{[\alpha,\beta]}(x) \leq S^+(x)$ *for all real* x.
(2) $\hat{S}^\pm(0) = \int_{-\infty}^{\infty} S^\pm(x) dx = \beta - \alpha \pm \frac{1}{\delta}$.

(3) $\hat{S}^{\pm}(x) = 0$ *for* $|x| > \delta$.
(4) *For any* $\epsilon > 0$, $S^{\pm}(z) \ll_{\delta,\alpha,\beta,\epsilon} \min(1, 1/|z|^2)$ *for* $\mathrm{Im}(z) \leq \epsilon$.
(5) $\hat{S}^{\pm}(z) = \frac{\sin \pi(\beta-\alpha)z}{\pi z} + O\left(\frac{1}{\delta}\right)$ *for* $|z| \leq \delta$.

Proof. This is a specialization of [7, Lemma 2]. For a more detailed proof, see the survey articles of Montgomery [13], Selberg [15] and Vaaler [18]. □

5. Feasible pairs

From now on, we may assume that $\alpha = -\beta$ for simplicity, so that the Selberg minorant function depends on the two parameters β and δ. Therefore, we will write $S^{-}_{-\beta,\beta,\delta}(z) = S^{-}_{\beta,\delta}(z)$.

Definition 5.1 *(Feasible pair).* We say that (β, δ) is a feasible pair for $S^{\pm}_{\beta,\delta}(z)$ if

$$\ell(\mu, S^{\pm}_{\beta,\delta}) := \mathrm{Re}\left(\int_{-\infty}^{\infty} \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{it}{2} + \mu\right) S^{\pm}_{\beta,\delta}(t)dt\right) - \hat{S}^{\pm}_{\beta,\delta}(0)\log\pi > 0,$$

for every $\mu \in \mathbb{C}$ with $\mathrm{Re}(\mu) \geq 0$. We also define the function

$$\begin{aligned} \eta^{\pm}(\beta,\delta) &= \inf_{\substack{\mu\in\mathbb{C} \\ \mathrm{Re}(\mu)\geq 0}} \left[\mathrm{Re}\left(\int_{-\infty}^{\infty} \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{it}{2} + \mu\right) S^{\pm}_{\beta,\delta}(t)dt\right) - \hat{S}^{\pm}_{\beta,\delta}(0)\log\pi\right] \\ &= \inf_{\substack{\mu\in\mathbb{C} \\ \mathrm{Re}(\mu)\geq 0}} \ell(\mu, S^{\pm}_{\beta,\delta}), \end{aligned} \tag{5.1}$$

so that $\eta^{\pm}(\beta, \delta) > 0$ if and only if (β, δ) is a feasible pair.

Moreover, if (β, δ) is a feasible pair, we call $S^{\pm}_{\beta,\delta}(x)$ feasible Selberg minorant/majorant function. Fig. 5.1 shows the two dimensional surface $\ell(\mu, S^{-}_{\beta,\delta})$ for the variables $(\mathrm{Re}(\mu), \mathrm{Im}(\mu))$.

Definition 5.2 *(Feasible set).* The set

$$\begin{aligned} G^{\pm} &= \left\{(\beta,\delta) \in (0,+\infty) \times \left(\frac{\log 2}{2\pi}, +\infty\right) \ : \ (\beta,\delta) \text{ is a feasible pair}\right\} \\ &= \left\{(\beta,\delta) \in (0,+\infty) \times \left(\frac{\log 2}{2\pi}, +\infty\right) \ : \ \eta^{\pm}(\beta,\delta) > 0\right\} \end{aligned} \tag{5.2}$$

is called a feasible set (see Fig. 5.2).

We define for each integer $m > 1$ the feasible region up to $\delta \leq \frac{\log m}{2\pi}$ by

$$G^{\pm}_m = \left\{(\beta,\delta) \in G^{\pm} \ : \ \delta \leq \frac{\log m}{2\pi}\right\}. \tag{5.3}$$

5.1. Linear combinations of Selberg minorant functions

Certain linear combinations of Selberg's functions may be used to improve the upper bound. The idea is to use a specific linear combination that makes the Fourier transform disappear at each, or certain points,

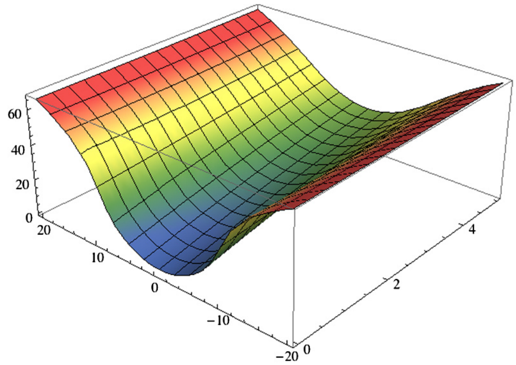

Fig. 5.1. 3D Plot of $\ell(\mu, S^{-}_{\beta,\delta})$ for $(\mathrm{Re}(\mu), \mathrm{Im}(\mu))$, where $\mathrm{Re}(\mu) \geq 0$ for (β, δ) a feasible pair.

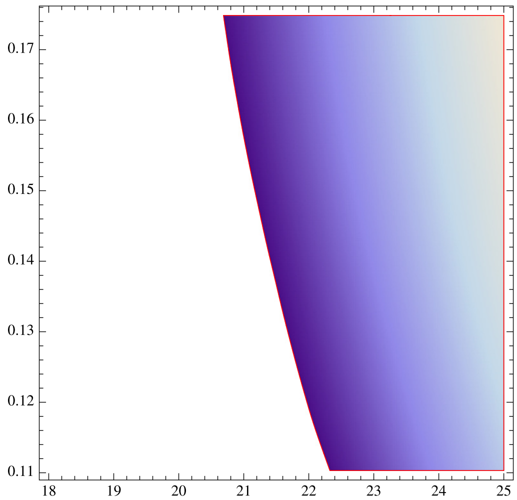

Fig. 5.2. The region of feasible pairs (β, δ) for $18 \leq \beta \leq 25$ and $\frac{\log 2}{2\pi} \leq \delta \leq \frac{\log 3}{2\pi}$.

$\frac{\log m}{2\pi}$ where the Weil explicit formula (3.1) contributes in the last sum. The following well-known fact will be used later.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be n vectors in $\mathbb{R}^k$, and consider the convex hull H containing all these points. Then there exists a linear combination with non-negative coefficients which gives the zero vector if and only if $\mathbf{0}$ is contained in H.

We now state an abstract theorem for finding a possible upper bound using linear combinations of Selberg's minorant functions. The idea of the theorem will be used later in the paper for a straightforward linear combination of two terms (this is due to an exponential increase of computational effort when we increase the terms to three or more).

Theorem 5.1. *For any positive integer m, denote by k_m the number of prime powers strictly smaller than m. Consider the map*

$$\psi_m : G^{-}_m \to \mathbb{R}^{k_m}$$

$$\psi_m\left((\beta, \delta)\right) := \left(\hat{S}^{-}_{\beta,\delta}\left(\frac{\log 2}{2\pi}\right), \ldots, \hat{S}^{-}_{\beta,\delta}\left(\frac{\log p^{k'}}{2\pi}\right)\right),$$

where $p^{k'}$ is the highest prime power less than m, to be the vector with components given by the values of the Fourier transform of $S^-_{\beta,\delta}(t)$ at $\frac{\log 2}{2\pi}$, $\frac{\log 3}{2\pi}$, ... and so on for all the prime powers less than m. For each positive real number B, let

$$G^-_{m,B} = \{(\beta, \delta) \in G^-_m \ : \ \beta \leq B\}$$

and let $X_{m,B} = \psi_m(G^-_{m,B}) \subset \mathbb{R}^{k_m}$ be the image of $G_{m,B}$ through the map ψ_m. Therefore, if we let

$$\beta_\infty = \inf_m \inf\{B > 0 \ : \ \mathbf{0} \in \operatorname{hull}(X_{m,B})\},$$

then $\mathrm{Gap}_\Omega \leq 2\beta_\infty$.

The above theorem states that in certain cases we can find a linear combination of Selberg's functions that make each Fourier transform of $S^-_{\beta,\delta}(t)$ disappear at each $\frac{\log m}{2\pi}$ if m is a prime power. In fact, if we keep m constant and increase B, then the set $X_{m,B}$ increases and so does its convex hull, which therefore has a better chance of containing the origin of $\mathbb{R}^{k_m}$.

If such B exists, then any larger B will automatically have the same desired property. In that case there will exist a smallest B with that property (that is, the convex hull of $X_{m,B}$ contains the origin of $\mathbb{R}^{k_m}$). If we denote this smallest β (which depends on m only) by β_m

$$\beta_m = \inf\{B > 0 \ : \ \mathbf{0} \in \operatorname{hull}(X_{m,B})\},$$

then its lowest value over all m must satisfy the requirements as well. The proof reads as follows.

Proof. If β_∞ is attained in $\{\beta_j \ : \ j > 1\}$, then consider that value of m such that β_m is the minimum. Otherwise, for any $\varepsilon > 0$ there exists an m such that $\beta_\infty + \varepsilon = \beta_m$. Fix $\varepsilon > 0$ and take the corresponding m. Consider the corresponding set $X_{m,\beta_m} \subset \mathbb{R}^{k_m}$, then by the above construction there are vectors $\mathbf{x}_1, \dots, \mathbf{x}_{k_m} \in X_{m,\beta_m}$ such that their convex hull contains $\mathbf{0}$. By the fact of previous page regarding the convex hull, there are then non-negative constants $c_1, \dots, c_{k_m}$ such that

$$c_1\mathbf{x}_1 + \ldots + c_{k_m}\mathbf{x}_{k_m} = \mathbf{0}. \tag{5.4}$$

Consider now the function

$$S(t) := \left(\sum_{j=1}^{k_m} c_j\right)^{-1} \sum_{j=1}^{k_m} c_j S^-_{\psi_m^{-1}(\mathbf{x}_j)}(t).$$

As ψ_m may not be injective, we will take the element in the set $\psi_m^{-1}(\mathbf{x}_j)$ such that β is minimal. Then applying Weil' explicit formula to $S(t)$ and $L \in \Omega$ we obtain

$$\begin{aligned}\#\{\text{zeros in } (-\beta_m, \beta_m)\} \geq \sum_\gamma S(\gamma) &= \frac{\hat{S}(0)}{\pi}\log Q + \frac{1}{2\pi}\sum_{j=1}^{d} \ell(\mu_j, S) \\ &\geq \frac{\hat{S}(0)}{\pi}\log Q + \frac{d}{2\pi}\left(\sum_{j=1}^{k_m} c_j\right)^{-1} \sum_{j=1}^{k_m} c_j \eta(\psi_m^{-1}(\mathbf{x}_j)),\end{aligned} \tag{5.5}$$

because (5.4) makes all the terms which are not prime powers vanish. Now, all the terms on the RHS of (5.5) are positive because of the construction of $S(t)$. Since this is true for any $\varepsilon > 0$ such that $\beta_m = \beta_\infty + \varepsilon$, it must be true for β_∞ as well. Now let $t_0 \in \mathbb{R}$, then any $S^-_{-\beta_\infty+t_0,\beta_\infty+t_0,\delta}(z)$ is a minorant of the characteristic function in the range $(-\beta_\infty + t_0, \beta_\infty + t_0)$. Its Fourier transform at 0 is given by Lemma 4.1:

$$\hat{S}^-_{-\beta_\infty+t_0,\beta_\infty+t_0,\delta}(0) = 2\beta_\infty - \frac{1}{\delta},$$

since the interval difference $\beta - \alpha$ remains constant. For the same β_m chosen at the beginning and for any $t_0 \neq 0$ the feasible set is such that

$$G_m^{-,t_0} := \left\{ (\beta, \delta) \in (0, +\infty) \times \left(\frac{\log 2}{2\pi}, \frac{\log m}{2\pi} \right) \;:\; \inf_{\substack{\mu \in \mathbb{C} \\ \mathrm{Re}(\mu) \geq 0}} \ell(\mu, S^-_{-\beta_m+t_0,\beta_m+t_0,\delta}) > 0 \right\} \supset G_m^-.$$

Thus, $\psi_m(G_m^{-,t_0}) := X^{t_0}_{m,\beta_m} \supset X_{m,\beta_m}$ has a convex hull that contains $\mathbf{0}$. The linear combination of vectors can be thus found as well, and hence the RHS of (5.5) must be positive. $\square$

5.2. Bounding the Dirichlet coefficients

An important idea in this note is to use the bounds on the coefficients (Ramanujan bound and Luo–Rudnick–Sarnak) to obtain improved bounds. The first theorem illustrates the idea under the assumption of the Ramanujan bound.

Theorem 5.2. *Let $L \in \Omega_*$ and $(\beta, \delta) \in G_M^-$ be a feasible pair for $S^-_{\beta,\delta}(t)$ such that*

$$\delta \in \left(\frac{\log(M-1)}{2\pi}, \frac{\log M}{2\pi} \right]$$

for a given $M \geq 2$, and suppose that

$$\eta(\beta, \delta) - \sum_p \sum_{\substack{k \\ p^k < M}} \frac{2 \log p}{\sqrt{p^k}} \left| \hat{S}^-_{\beta,\delta} \left(\frac{k \log p}{2\pi} \right) \right| > 0. \tag{5.6}$$

Then $L(s)$ has a nontrivial zero in every vertical interval of length 2β, i.e. $\mathrm{Gap}_{\Omega_*} \leq 2\beta$.

Proof. By the construction of (β, δ), we have $\eta(\beta, \delta) > 0$. By the Weil explicit formula, the RHS of (3.1) equals

$$\Psi(Q, \mu_1, \ldots, \mu_d) := \frac{\hat{S}^-_{\beta,\delta}(0)}{\pi} \log Q + \frac{1}{2\pi} \sum_{j=1}^{d} \ell(\mu_j, S^-_{\beta,\delta}) - \frac{1}{\pi} \sum_p \sum_{\substack{k \\ p^k < M}} \left(\frac{\mathrm{Re}(\Lambda_L(p^k))}{\sqrt{p^k}} \hat{S}^-_{\beta,\delta} \left(\frac{k \log p}{2\pi} \right) \right).$$

As long as this term is positive for some (β, δ), the statement of the theorem holds. Since this is true for any $Q \geq 1$ and any spectral parameters μ_j, then it must be positive for its infimum,

$$\inf_{\substack{Q \geq 1 \\ \mu_j \in \mathbb{C} \\ \mathrm{Re}(\mu_j) \geq 0}} \Psi(Q, \mu_1, \ldots, \mu_d) \geq \frac{d}{2\pi} \eta(\beta, \delta) - \frac{1}{2\pi} \sum_p \sum_{\substack{k \\ p^k < M}} \left(\frac{2 \,\mathrm{Re}\left(\sum_{j=1}^d \alpha^k_{j,p} \right) \log p}{\sqrt{p^k}} \hat{S}^-_{\beta,\delta} \left(\frac{k \log p}{2\pi} \right) \right).$$

But for that to be positive, because of Equation (2.12), we need

$$\sum_{p} \sum_{\substack{k \\ p^k < M}} \frac{2 \log p}{\sqrt{p^k}} \left| \hat{S}^{-}_{\beta,\delta} \left(\frac{k \log p}{2\pi} \right) \right| < \eta(\beta, \delta),$$

which is exactly equation (5.6). □

The next theorem illustrates a sufficient condition to be satisfied in the case where the Ramanujan bound is not assumed, i.e. in the scenario where the unconditional result of Luo–Rudnick–Sarnak [11] is employed.

Theorem 5.3. *Let $L \in \Omega$ and $(\beta, \delta) \in G^{-}_{M}$ be a feasible pair for $S^{-}_{\beta,\delta}(t)$ such that*

$$\delta \in \left(\frac{\log(M-1)}{2\pi}, \frac{\log M}{2\pi} \right]$$

for a given $M \geq 2$, and suppose that

$$\eta(\beta, \delta) - \sum_{p} \sum_{\substack{k \\ p^k < M}} 2 \log p \left| \hat{S}^{-}_{\beta,\delta} \left(\frac{k \log p}{2\pi} \right) \right| > 0. \tag{5.7}$$

Then $L(s)$ has a nontrivial zero in every vertical interval of length 2β, i.e. $\mathrm{Gap}_{\Omega} \leq 2\beta$.

Proof. Proceed as in the previous proof but here the bound is unconditional and thus we have

$$\left| \frac{\log p \,\mathrm{Re} \left(\sum_{j=1}^{d} \alpha_{j,p}^{k} \right)}{p^{k/2}} \right| \leq \frac{d p^{k/2} \log p}{p^{k/2}} = d \log p,$$

because of (2.12). □

In our next theorem, the goal is to combine the theoretical idea put forward in Theorem 5.1 with the result of Theorem 5.2 for a linear combination of two Selberg minorant functions.

Theorem 5.4. *Let $(\beta_1, \delta_1), (\beta_2, \delta_2) \in G^{-}_{M}$ be two feasible pairs such that*

$$\delta \in \left(\frac{\log(M-1)}{2\pi}, \frac{\log M}{2\pi} \right]$$

for a given $M \geq 2$. Suppose that for an $m \geq 2$ which is a prime power the following inequalities hold,

$$\hat{S}^{-}_{\beta_1,\delta_1} \left(\frac{\log m}{2\pi} \right) > 0 \text{ and } \hat{S}^{-}_{\beta_2,\delta_2} \left(\frac{\log m}{2\pi} \right) < 0,$$

or

$$\hat{S}^{-}_{\beta_1,\delta_1} \left(\frac{\log m}{2\pi} \right) < 0 \text{ and } \hat{S}^{-}_{\beta_2,\delta_2} \left(\frac{\log m}{2\pi} \right) > 0,$$

and set them to be c_1 and $-c_2$ such that either $c_1, c_2 > 0$ or $c_1, c_2 < 0$. Define the new function

$$S(t) := \frac{1}{c_1 + c_2} (c_2 S^{-}_{\beta_1,\delta_1}(t) + c_1 S^{-}_{\beta_2,\delta_2}(t))$$

so that $\hat{S}(\frac{\log m}{2\pi}) = 0$. *If*

$$\frac{1}{c_1+c_2}(c_2\eta^-(\beta_1,\delta_1)+c_1\eta^-(\beta_2,\delta_2)) - \sum_p \sum_{\substack{k \\ p^k<M \\ p^k\neq m}} \frac{2\log p}{\sqrt{p^k}}\left|\hat{S}\left(\frac{k\log p}{2\pi}\right)\right| > 0 \tag{5.8}$$

when $L \in \Omega_*$, *or*

$$\frac{1}{c_1+c_2}(c_2\eta^-(\beta_1,\delta_1)+c_1\eta^-(\beta_2,\delta_2)) - \sum_p \sum_{\substack{k \\ p^k<M \\ p^k\neq m}} 2\log p\left|\hat{S}\left(\frac{k\log p}{2\pi}\right)\right| > 0 \tag{5.9}$$

when $L \in \Omega$, *then either* $\mathrm{Gap}_{\Omega_*} \leq 2\max\{\beta_1,\beta_2\}$, *or* $\mathrm{Gap}_{\Omega} \leq 2\max\{\beta_1,\beta_2\}$.

Proof. In this case, by the linearity of $\ell(\mu_j, f)$ in f, the RHS of (3.1) has infimum over conductor and spectral parameters greater or equal than

$$\frac{d}{2\pi}\left(\frac{1}{c_1+c_2}(c_2\eta^-(\beta_1,\delta_1)+c_1\eta^-(\beta_2,\delta_2)) - \sum_p \sum_{\substack{k \\ p^k<M}} \left(\frac{2\,\mathrm{Re}\left(\sum_{j=1}^d \alpha_{j,p}^k\right)\log p}{\sqrt{p^k}}\hat{S}\left(\frac{k\log p}{2\pi}\right)\right)\right),$$

which is positive whenever (5.8) holds, and the term in the sum corresponding to $\hat{S}(\frac{\log m}{2\pi})$ disappears because by the construction of S. □

5.3. *Proof of Theorem 2.1*

We are now ready to prove our main results, namely Theorem 2.1. As mentioned earlier, the key to obtaining the two bounds is to use a convex linear combination of Selberg's minorant functions as in Theorem 5.4 that makes its Fourier transform disappear at the point $\frac{\log 4}{2\pi}$ in the case of the Ramanujan bound, and at the point $\frac{\log 3}{2\pi}$ in the case of the Luo–Rudnick–Sarnak bound.

Proof of Theorem 2.1. *On the Ramanujan bound.*

For $M = 7$ and the pairs

$$(\beta_1,\delta_1) = \left(18.2, \frac{\log 7}{2\pi}\right) \quad \text{as well as} \quad (\beta_2,\delta_2) = \left(20.770, \frac{\log 7}{2\pi}\right),$$

and $m = 4$ one can show that the conditions of Theorem 5.4 are satisfied and

$$\begin{aligned}&\frac{1}{c_1+c_2}(c_2\eta^-(\beta_1,\delta_1)+c_1\eta^-(\beta_2,\delta_2))\\ &-\frac{2\log 2}{\sqrt{2}}\left|\hat{S}\left(\frac{\log 2}{2\pi}\right)\right| - \frac{2\log 3}{\sqrt{3}}\left|\hat{S}\left(\frac{\log 3}{2\pi}\right)\right| - \frac{2\log 5}{\sqrt{5}}\left|\hat{S}\left(\frac{\log 5}{2\pi}\right)\right|\end{aligned} \tag{5.10}$$

is positive, as shown in Fig. 5.3. Specifically, for $\beta_2 = 20.770$ it gives the value of 0.000825207.

Theorem 5.4 also holds for $\max\{\beta_1,\beta_2\} = \beta_2 = 20.770$.

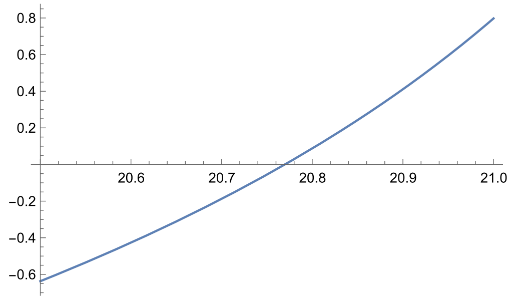

Fig. 5.3. Plot of the function (5.10) for $\beta_1 = 18.2$ and $\beta_2 \in (20.5, 21.0)$ assuming the Ramanujan hypothesis.

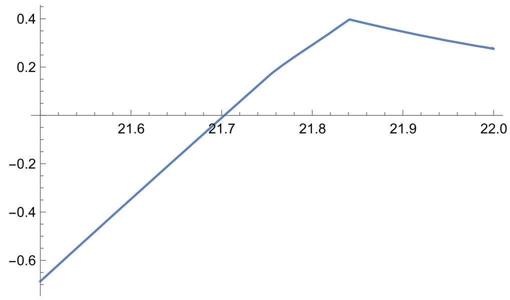

Fig. 5.4. Plot of the function (5.11) for $\beta_1 = 19.7$ and $\beta_2 \in (21.5, 22)$ on the Luo–Rudnick–Sarnak bound.

On the Luo–Rudnick–Sarnak bound.

For $M = 7$ we use again Theorem 5.4 applied to the two pairs

$$(\beta_1, \delta_1) = \left(19.7, \frac{\log 7}{2\pi}\right) \quad \text{as well as} \quad (\beta_2, \delta_2) = \left(21.705., \frac{\log 7}{2\pi}\right),$$

and with $m = 3$. The given choice satisfies the requirements of Theorem 5.4 and the condition (5.9) reduces to

$$\begin{aligned} &\frac{1}{c_1 + c_2}(c_1\eta^-(\beta_1, \delta_1) + c_2\eta^-(\beta_2, \delta_2)) \\ &- 2\log 2 \left|\hat{S}\left(\frac{\log 2}{2\pi}\right)\right| - 2\log 2 \left|\hat{S}\left(\frac{\log 4}{2\pi}\right)\right| - 2\log 5 \left|\hat{S}\left(\frac{\log 5}{2\pi}\right)\right|, \end{aligned} \tag{5.11}$$

which is positive for the given pair from $\beta_2 \geq 21.705$ as shown in Fig. 5.4. Specifically, for $\beta_2 = 21.705$ it gives the value of 0.00655263.

This ends the proof. □

5.4. The lowest bound

The example of L-function shown in [2, §3] shows that there are L-functions whose largest gaps between zeros are greater than the one of the Riemann zeta-function. This is the case where the L-function has non-real spectral parameters, because Miller [12] proved that the largest gap between zeros of the Riemann zeta-function is greater than any other among L-functions of real spectral parameters.

Therefore, there must be a general lowest upper bound which is higher than the largest gap between zeros of the Riemann zeta-function. One could try to guess what this lowest bound might be.

Open Problem 5.1. *Is it true that at least a zero γ of $L \in \Omega$ must be contained in $[t_0, t_0 + 2\beta_-]$ for any $t_0 \in \mathbb{R}$, where*

$$\beta_- := \inf\{\beta > 0 \ : \ \eta(\beta,\delta) = 0\} \approx 17.845 \text{ ?}$$

In other words, is it true that $\mathrm{Gap}_\Omega \le 2\beta_-$*?*

Does the assumption of the Ramanujan bound make any difference?

Open Problem 5.2. *Is it true that* $\mathrm{Gap}_\Omega = \mathrm{Gap}_{\Omega_*}$*, even in the hypothetical case that* $\Omega_* \subsetneq \Omega$*?*

In [2], a lowest upper bound of 36 for Gap_Ω was suggested. The slightly lower value of $2\beta_- \approx 35.69$ might also be plausible since it is the lowest possible β of the feasible region where the positivity of the Γ-terms is guaranteed. The fact that it is close to the value of 36 of [2] is curious and they might have reasoned along similar lines.

We also raise the following problem.

Open Problem 5.3. *Is the value* $2\beta_-$ *an optimal upper bound for the largest gap between zeros among entire L-functions, i.e. is* $\mathrm{Gap}_\Omega = 2\beta_-$*? In other words, is it true that for any* $\varepsilon > 0$ *there is an entire L-function* $L \in \Omega$ *whose largest gap* Gap_L *is either*

$$\mathrm{Gap}_L = 2\beta_-,$$

or

$$\mathrm{Gap}_L \ge 2\beta_- - \varepsilon \text{ ?}$$

The first case above might be too optimistic, the second option might be more probable. In fact, the lowest upper bound could be an accumulation point of differences between imaginary parts of consecutive zeros of certain L-functions with non-real spectral parameters.

6. Nonexistence of certain entire *L*-functions of low conductor

There are other interesting applications of the Weil explicit formula. For instance, by using the Selberg majorant function for a $\delta \in \left(\frac{\log 2}{2\pi}, \frac{\log 3}{2\pi}\right]$, one can bound the real part of the first Dirichlet coefficient $b(2)$ in the following way.

Theorem 6.1. *Let* $L \in \Omega$ *and* $\beta \ge 0$ *and suppose that* $\delta \in \left(\frac{\log 2}{2\pi}, \frac{\log 3}{2\pi}\right]$*. If* $\hat{S}^+_{\beta,\delta}\left(\frac{\log 2}{2\pi}\right) > 0$ *for some* δ *in that interval, then*

$$\mathrm{Re}(b(2)) \le \frac{\sqrt{2}}{\log 2}\left(\hat{S}^+_{\beta,\delta}(0)\log Q + \frac{1}{2}\sum_{j=1}^{d}\ell(\mu_j, S^+_{\beta,\delta})\right)\hat{S}^+_{\beta,\delta}\left(\frac{\log 2}{2\pi}\right)^{-1}.$$

Proof. Take the Selberg majorant function $S^+_{\beta,\delta}(z)$, for $\delta \in \left(\frac{\log 2}{2\pi}, \frac{\log 3}{2\pi}\right]$. Thus, since $S^+_{\beta,\delta}(x) \ge 0$ for all real x,

$$0 \le \#\{\text{ zeros in } (-\beta,\beta)\} \le \sum_\gamma S^+_{\beta,\delta}(\gamma),$$

which translates to

$$0 \le \frac{\hat{S}^+_{\beta,\delta}(0)}{\pi}\log Q + \frac{1}{2\pi}\sum_{j=1}^{d}\ell(\mu_j, S^+_{\beta,\delta}) - \frac{1}{\pi}\left(\frac{\mathrm{Re}(b(2)\log 2)}{\sqrt{2}}\hat{S}^+_{\beta,\delta}\left(\frac{\log 2}{2\pi}\right)\right).$$

That is,

$$\frac{\mathrm{Re}(b(2)\log 2)}{\sqrt{2}}\hat{S}^+_{\beta,\delta}\left(\frac{\log 2}{2\pi}\right) \leq \hat{S}^+_{\beta,\delta}(0)\log Q + \frac{1}{2}\sum_{j=1}^{d}\ell(\mu_j, S^+_{\beta,\delta}).$$

Then, because $\hat{S}^+_{\beta,\delta}\left(\frac{\log 2}{2\pi}\right) > 0$,

$$\mathrm{Re}(b(2)) \leq \frac{\sqrt{2}}{\log 2}\left(\hat{S}^+_{\beta,\delta}(0)\log Q + \frac{1}{2}\sum_{j=1}^{d}\ell(\mu_j, S^+_{\beta,\delta})\right)\hat{S}^+_{\beta,\delta}\left(\frac{\log 2}{2\pi}\right)^{-1},$$

and the proof is now finished. □

Under the Ramanujan hypothesis, we can use this last result to prove that certain entire L-functions having specific functional equations cannot exist.

Corollary 6.1. *If, for some (β, δ) as in Theorem 6.1, the condition*

$$\frac{\sqrt{2}}{\log 2}\left(\hat{S}^+_{\beta,\delta}(0)\log Q + \frac{1}{2}\sum_{j=1}^{d}\ell(\mu_j, S^+_{\beta,\delta})\right)\hat{S}^+_{\beta,\delta}\left(\frac{\log 2}{2\pi}\right)^{-1} < -d$$

holds for some spectral parameter choice, then there is no $L \in \Omega_*$ of degree d having these spectral parameters.

Corollary 6.2. *Let (β, δ) be such that the inequality of Corollary 6.1 is satisfied. There is no $L \in \Omega_*$ such that it has d copies of $\Gamma_{\mathbb{R}}(s)$ terms in the functional equation if*

$$Q < \exp\left(\left(-\frac{d\log 2}{\sqrt{2}}\hat{S}^+_{\beta,\delta}\left(\frac{\log 2}{2\pi}\right) - \frac{d}{2}\ell(0, S^+_{\beta,\delta})\right)\left(2\beta + \frac{1}{\delta}\right)^{-1}\right). \tag{6.1}$$

In particular, for L-functions with degree up to 5, no $L \in \Omega_*$ with functional equation of the form

$$\Lambda(s) = Q^s\Gamma_{\mathbb{R}}(s)^d L(s) = \varepsilon\bar{\Lambda}(1-s) \tag{6.2}$$

exists for all conductors less than the values in the following table:

d	2	3	4	5
Q	< 4.24	< 8.73	< 17.9	< 37.0
N	< 17	< 77	< 324	< 1375

Proof. Apply Corollary 6.1 to the specific case,

$$\left(2\beta + \frac{1}{\delta}\right)\log Q + \frac{d}{2}\ell(0, S^+_{\beta,\delta}) < -\frac{d\log 2}{\sqrt{2}}\hat{S}^+_{\beta,\delta}\left(\frac{\log 2}{2\pi}\right)$$

and solve for Q. The particular cases follow from evaluating (6.1) with the parameters $(\beta, \delta) = (0.5, \frac{\log 3}{2\pi})$ using Mathematica. □

For the particular point $(\beta, \delta) = (0.5, \frac{\log 3}{2\pi})$, the term $\ell(0, S^+_{\beta,\delta})$ is negative and this shows the exponential increase which depends on the degree of the upper bound.

In [19, Table 3], Voight listed the number of totally real fields F with root discriminant less or equal than 14 for each degree $n = [F : \mathbb{Q}] \leq 9$, and moreover he provided lower bounds for their discriminant d_F – see also Odlyzko [14] and the refinements due to Brueggeman and Doud [3]. Their corresponding Dedekind zeta-functions have functional equations of the type (6.2) with degree $d = n$ and conductor $N = |d_F|$. The lower bounds for totally real number fields obtained by Voight [19] are more general because Dedekind zeta-functions are non-entire, while in Corollary 6.2 we are assuming holomorphicity and therefore they cannot be fully compared to ours.

Our bounds are not making substantial contributions for higher degrees. In fact, already for $d = 4$, the more general lower bounds from [19] state that $N \geq 725$, while we proved $N \geq 324$, and the difference increases as the degree increases.

Good candidates of L-functions belonging to our class Ω_* and satisfying (6.2) are irreducible factors of Dedekind zeta-functions of a totally real field. A Dedekind zeta function always has a pole at $s = 1$ (the residue of which gives the analytic class number formula). If one divides out by the Riemann zeta-function, then the resulting quotient is conjecturally entire (Dedekind's conjecture; a consequence of Artin's conjecture; known for Galois extensions by the Aramata–Brauer theorem, and for any field whose Galois closure is solvable).

For $d = 2$, for example, comparing our result to [19, Table 3] shows that there could be only non-entire L-functions satisfying (6.2) with $5 \leq N < 17$. By looking at the LMFDB database [10] we can also discover that our result could not be pushed beyond $N = 145$ because there exists an example of an Artin L-function for which $N = 145$ (see [10, L-function for Artin representation 2.5_29.4t3.1c1]).

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