ON A LIOUVILLE ANALOGUE OF THE SELBERG FORMULA AND A GENERALIZATION OF THE WEIL EXPLICIT FORMULA

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Abstract. Some recent results of Knafo on asymptotic formulae in arithmetic progressions are studied in the context of Liouville analogues. Explicit formulae of Weil type for new generalizations von Mangoldt functions are also provided.

1. Introduction

Let $\Lambda(n) = \log p$ if $n = p^m$ for some prime p and integer m, and 0 otherwise, denote the von Mangoldt function. The Chebyschev function is the partial summation up to x of $\Lambda(n)$,

$$\psi(x) := \sum_{n \leqslant x} \Lambda(n). \tag{1.1}$$

The explicit formula for $\Lambda(n)$ was derived by Riemann and later by von Mangoldt [7, equation (12.6)]. It is given by

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log(1 - x^{-2}), \tag{1.2}$$

for $x > 1$ and $x \neq p^m$ where p is again a prime and m is an integer. The sum over ρ is performed over all non-trivial zeros of Riemann zeta-function. This is a cornerstone result in the *analytic* proof of the prime number theorem, see e.g. [4, 7, 22].

Consider the following arithmetical function, which we will call the generalized von Mangoldt function,

$$\Lambda_k(n) := \sum_{d|n} \mu(d) \log^k \frac{n}{d}. \tag{1.3}$$

This function had already been studied in certain detail by Ivić in [8, 9]. Clearly $\Lambda_1(n) = \Lambda(n)$.

In [20, 23], Siegel and Walfisz separately obtained the following result. The asymptotic formula

$$\sum_{\substack{n \leq x \\ (a,q)=1 \\ n \equiv a \bmod q}} \Lambda(n) \sim \frac{x}{\phi(q)} \tag{1.4}$$

holds uniformly in the range $q < \log^A x$, for any fixed A. This is a uniform version of the prime number theorem in arithmetic progressions. This theorem is in general non-effective. The asymptotic formula is only known to be effective if A is chosen to be smaller than 2.

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In [21], the following result was proved by Selberg. The asymptotic formula

$$\sum_{\substack{n\le x \\ (a,q)=1 \\ n\equiv a \bmod q}} \Lambda_2(n) \sim \frac{2x}{\phi(q)} \log x \tag{1.5}$$

holds uniformly in the range $\log q < \varepsilon(x)\log x$, where $\varepsilon(x)$ be a fixed positive function, tending to 0 as $x \to \infty$. Here $\phi(n)$ denotes the Euler totient function. This theorem was used by Selberg to provide an *elementary* proof of the prime number theorem.

In [10, 11], Knafo generalized Selberg's result as follows. One has that

$$\sum_{\substack{n\le x \\ (a,q)=1 \\ n\equiv a \bmod q}} \Lambda_k(n) \sim \frac{kx}{\phi(q)} \log^{k-1} x \tag{1.6}$$

holds uniformly in the range $\log q < \varepsilon(x)\log x/\log\log x$, where $\varepsilon(x)$ be a fixed positive function, tending to 0 as $x \to \infty$. It should be remarked that the above result holds for a slightly smaller range of $\log q$.

In this paper we consider a different type of von Mangoldt function. Let $k, n \geq 1$ be integers, then new version of the von Mangoldt function, denoted as L_k, will be defined by

$$L_k(n) := \sum_{d|n} \lambda(d) \log^k \frac{n}{d}. \tag{1.7}$$

Moreover, we shall also consider In [18, 19], a generalization of the Möbius function is considered by Sastry. Let $j, n \geq 1$ be integers, then the Sastry analogue of the Möbius function is

$$\mu_j(n) := \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n \text{ contains a } (j+1)\text{th power divisor}, \\ \lambda(n), & \text{otherwise.} \end{cases} \tag{1.8}$$

Here $\lambda(n)$ is the Liouville function. If $j = 1$, then this function coincides with the Möbius function $\mu(n)$; and if $j \to \infty$ then this function coincides with the Liouville function $\lambda(n)$. Moreover, let us set

$$\Lambda_{j,k}(n) := \sum_{d|n} \mu_j(d) \log^k \frac{n}{d}. \tag{1.9}$$

It is clear that if $j = k = 1$, then we recover the usual von Mangoldt function $\Lambda(n)$. If $j = 1$ and $k \geq 1$, then this is the function considered by Ivić, i.e. $\Lambda_{1,k}(n) = \Lambda_k(n)$.

Let us define a one-parameter and a two-parameter generalizations of the Chebyschev function in arithmetic progressions as

$$\vartheta_k(x; q, a) := \sum_{\substack{n\le x \\ (a,q)=1 \\ n\equiv a \bmod q}} L_k(n) \tag{1.10}$$

for $k \geq 1$ integer.

$$\psi_{j,k}(x;q,a) := \sum_{\substack{n\leq x\\(a,q)=1\\n\equiv a \bmod q}} \Lambda_{j,k}(n) \tag{1.11}$$

for $j,k \geq 1$ integers. We will prove the following result which generalizes (1.4), (1.5) and (1.6).

Theorem 1.1. *Let $j,k \geq 1$ be integers. Then the asymptotic formula*

$$\psi_{j,k}(x;q,a) \sim \frac{k}{\phi(q)} c(j,q) x \log^{k-1} x$$

holds uniformly in

$$\log q < (\varepsilon(x))^{\frac{1}{k-1}} \frac{\log x}{\log\log x},$$

where $\varepsilon(x)$ be a fixed positive function, such that $\varepsilon(x) = o(1)$ as $x \to \infty$. Here

$$c(j,q) = \begin{cases} (-1)^{\frac{j+3}{2}} \frac{(j+1)!}{12 B_{j+1}(2\pi)^{j-1}} \prod_{p|q} \frac{1-1/p^2}{1-1/p^{j+1}}, & \text{if } j \geq 1 \text{ odd,}\\ (-1)^j \frac{3}{\pi^4} \frac{\zeta(j+1)(2(j+1))!}{B_{2(j+1)}(2\pi)^{2j}} \prod_{p|q} \frac{1}{(1-1/p^2)(1+1/p^{j+1})}, & \text{if } j \geq 2 \text{ even,} \end{cases}$$

and B_n is the $(n+1)$th Bernoulli number. Here p denotes a prime.

Theorem 1.2. *Let $k \geq 1$ be an integer. Then the asymptotic formula*

$$\vartheta_k(x;q,a) \sim \frac{k}{\phi(q)} c(q) x \log^{k-1} x$$

holds uniformly in

$$\log q < (\varepsilon(x))^{\frac{1}{k-1}} \frac{\log x}{\log\log x},$$

where $\varepsilon(x)$ be a fixed positive function, such that $\varepsilon(x) = o(1)$ as $x \to \infty$. Here

$$c(q) = \frac{\pi^2}{6} \prod_{p|q} \left(1 - \frac{1}{p^2}\right). \tag{1.12}$$

Here p denotes a prime.

Recently the generalized von Mangoldt functions Λ_k have played a key role in the mollification of $\zeta(s)+\zeta'(s)$ in [5, 14, 16], and in the mollification of $\zeta(s)+\zeta'(s)+\zeta''(s)+\cdots+\zeta^{(d)}(s)$ for arbitrary $d \in \mathbb{N}$. The present authors, along with Zaharescu [17], have been able to exploit the properties of $\Lambda_k(n)$, along with the latest technology of bound of trilinear Kloosterman sums [1, 2], to increase the proportion of zeros of the Riemann zeta-function on the critical line from 41.07% to 41.34%. The importance of these functions in the mollification of the Riemann zeta-function provides us with motivation to study further properties of Λ_k, specially their explicit formulae.

The Weil explicit formula [3, 15] for the Riemann zeta function and for the von Mangoldt function is as follows. Suppose that f is an integrable function satisfying some growth conditions to be specified below and let $F(s) = \int_0^\infty f(x)x^{s-1}dx$ be the Mellin transform of f. The set $\mathcal{P}$ will denote the set of primes $\{2, 3, 5, 7, 11, \cdots\}$. Weil [24] proved that

$$\sum_\rho F(\rho) + \sum_{n\in\mathbb{N}} F(-2n) = F(1) + \sum_{n\in\mathbb{N}} \Lambda(n)f(n). \tag{1.13}$$

This is in fact a specific case of a more general set up studied by Weil [24] who treated general L-functions associated with Grössencharakter χ. These are representations of the group of idéle-classes of an algebraic number field $\mathbb{K}$ into the multiplicative group of non-zero complex numbers. See also [3, §1.2] for elaborations.

Now we provide the Weil explicit formula analogue of the function $\Lambda_k(n)$.

Theorem 1.3. *Suppose that k is a positive integer. Let $a > 1$ be a real number. Let $\hat{h}(s)$ be a holomorphic function in the vertical strip $a < \sigma < b$. Let $\hat{h}(s)$ be non-zero and satisfy the bound*

$$\hat{h}(s) \ll e^{-(\frac{1}{2}+\varepsilon)|s|} \tag{1.14}$$

for fixed $\varepsilon > 0$ as $|s| \to \infty$. Let $h(s)$ be the inverse Mellin transform of $\hat{h}(s)$. Then we have

$$\sum_{n=1}^\infty \Lambda_k(n)h(n) = \frac{(-1)^k\hat{h}(1)}{(k-1)!} + (-1)^k \sum_\rho \frac{\zeta^{(k)}}{\zeta'}(\rho)\hat{h}(\rho) + (-1)^k \sum_{m=1}^\infty \frac{\zeta^{(k)}}{\zeta'}(-2m)\hat{h}(-2m).$$

The privileged position on $\Lambda(n)$ in the Weil explicit formula comes from the logarithmic derivative of the zeta function [4]

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{2}\gamma - 1 + \log 2\pi - \frac{1}{s-1} - \frac{1}{2}\frac{\Gamma'(\frac{1}{2}s+1)}{\Gamma(\frac{1}{2}s+1)} + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right), \tag{1.15}$$

since it shows that for $k = j = 1$ in the choice of $\Lambda_{j,k}(n)$ the zeros yield simple poles with residue equal to 1. Thus $k = 1$ is the natural choice. However, once we let $k > 1$ the structure of the logarithmic derivative above is lost and it forces us to consider powers of logarithms of primes. Since $\Lambda_{j,k}(n)$ is already a different arithmetical function from $\Lambda(n)$, one may then ask what would happen if we were to choose a completely different arithmetical function such as the Möbius $\mu(n)$ function, the Euler totient $\phi(n)$ function, or the Liouville $\lambda(n)$ function?

Let us define a Dirichlet series by

$$\sum_{n=1}^\infty \frac{\theta(n)}{n^s} := \frac{\zeta^k(a+bs)}{\zeta(c+ds)} \tag{1.16}$$

for $\operatorname{Re}(s) > \max\{\frac{1-a}{b}, \frac{1-c}{d}\}$. Here a and c are reals, b and d are positive and k is a positive integer. Then an analogue of Weil explicit formula for $\theta(n)$ can be given as

Theorem 1.4. *If h satisfies the conditions of Theorem 1.3 and $\theta(n)$ is given by (1.16), then we have*

$$\sum_n \theta(n)h(n) = \Phi(a,b,c,d,k) + \sum_\rho \frac{\zeta^k(a+\frac{b}{d}(\rho-c))}{d\zeta'(\rho)}\hat{h}(\tfrac{\rho-c}{d})$$

$$+\sum_{m=1}^{\infty}\frac{\zeta^k(a+\frac{b}{d}(-2m-c))}{d\zeta'(-2m)}\hat{h}(\tfrac{-2m-c}{d}), \tag{1.17}$$

where we define

$$\Phi(a,b,c,d,k):=\lim_{s\to(1-a)/b}\frac{(-1)^k}{(k-1)!}\frac{d^{k-1}}{ds^{k-1}}\left((s-\tfrac{1-a}{b})^k\frac{\zeta^k(a+bs)}{\zeta(c+ds)}\hat{h}(s)\right).$$

Now we obtain some simple consequences of the above theorem as special cases. We state them as a corollary.

Corollary 1.1. *The following holds for h as in Theorem 1.3.*

i) *For the Möbius function $\mu(n)$ we have that*

$$\sum_n \mu(n)h(n)=\sum_\rho\frac{\hat{h}(\rho)}{\zeta'(\rho)}+\sum_{m=1}^{\infty}\frac{\hat{h}(-2m)}{\zeta'(-2m)}. \tag{1.18}$$

ii) *Moreover, for the Euler totient $\phi(n)$ function we have that*

$$\sum_n \phi(n)h(n)=\frac{\hat{h}(2)}{\zeta(2)}+\sum_\rho\frac{\zeta(\rho-1)}{\zeta'(\rho)}\hat{h}(\rho-1)+\sum_{m=1}^{\infty}\frac{\zeta(-2m-1)}{\zeta'(-2m)}\hat{h}(-2m-1). \tag{1.19}$$

iii) *Finally, for the Liouville function $\lambda(n)$*

$$\sum_n \lambda(n)h(n)=\frac{\hat{h}(\frac{1}{2})}{2\zeta(\frac{1}{2})}+\sum_\rho\frac{\zeta(2\rho)}{\zeta'(\rho)}\hat{h}(\rho). \tag{1.20}$$

2. Preliminary results

First, we recall Lemma α from [22, Sect. 3.9].

Lemma 2.1. *If $f(s)$ is regular, and*

$$\left|\frac{f(s)}{f(s_0)}\right|<e^M$$

for $M>1$ in the circle $|s-s_0|\le r$, then

$$\left|\frac{f'}{f}(s)-\sum_\rho\frac{1}{s-\rho}\right|<\frac{AM}{r}\qquad(|s-s_0|\le\tfrac{1}{4}r)$$

where ρ runs through the zeros of $f(s)$ such that $|\rho-s_0|\le\frac{1}{2}r$.

A key inequality for dealing with two zeta functions in the denominator of the integrands is provided by the following result.

Lemma 2.2. *There is an absolute constant Θ such that each interval $(T,T+1)$ contains a value of t for which*

$$|\zeta(s)\zeta(\theta s)|>t^{-\Theta}$$

for real θ and holding in the interval $-1\le\sigma\le2$.

Proof. Take

$$f(s) = \zeta(s)\zeta(ks) \tag{2.1}$$

where k is a positive integer. Since the number of non-trivial zeros on the upper side of the critical strip up to height T of $\zeta(s)$ is given by

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

we deduce that the number of non-trivial zeros $\tilde{\rho}$ of $f(s)$ up to height T in the critical strip is given by

$$N_f(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{kT}{2\pi}\log\frac{kT}{2\pi} - \frac{kT}{2\pi} + O(\log kT).$$

Next, take $s_0 = 2 + iT$ and $r = 12$ in Lemma 2.1. In this case $M = A\log kT$ and we obtain

$$\frac{f'}{f}(s) = \sum_{|\rho - s_0| \leq 6} \frac{1}{s - \tilde{\rho}} + O(\log kT) \tag{2.2}$$

for $|s - s_0| \leq 3$, and thus in the range $-1 \leq \sigma \leq 2$ and $t = T$. Thus, if we replace T by t in the error term of (2.2) we obtain that (2.2) holds with a new error term of $O(\log kt)$ and in the range $-1 \leq \sigma \leq 2$

$$\frac{f'}{f}(s) = \sum_{|\rho - s_0| \leq 6} \frac{1}{s - \tilde{\rho}} + O(\log kt) \tag{2.3}$$

We can replace the range of summation to $|t - \tilde{\gamma}|$ where $\tilde{\gamma}$ is the imaginary part of $\tilde{\rho}$ so that when we write

$$\frac{f'}{f}(s) = \sum_{|t - \tilde{\gamma}| \leq 1} \frac{1}{s - \tilde{\rho}} + O(\log kt) \tag{2.4}$$

any term that takes place in (2.3) but not in (2.4) is $O(1)$ and the number of such terms does not exceed

$$N_f(t+6) - N_f(t-6) \ll \log kt.$$

Thus, we are left with

$$\frac{f'}{f}(s) = \sum_{|t - \gamma| \leq 1} \frac{1}{s - \tilde{\rho}} + O(\log kt) \tag{2.5}$$

holding uniformly for $-1 \leq \sigma \leq 2$.

Let us now integrate from s to $2 + it$ under the assumption that t is not equal to the ordinate of any non-trivial zero. Thus,

$$\log f(s) - \log f(2 + it) = \sum_{|t - \tilde{\gamma}| \leq 1} \{\log(s - \tilde{\rho}) - \log(2 + it - \tilde{\rho})\} + O(\log kt).$$

Now, we have $\log f(2 + it) \ll 1$ and $\log(2 + it - \rho) \ll 1$ and there are $O(\log kt)$ such terms. Their sum is therefore $O(\log kt)$. Thus, by continuity for all values of s in the critical strip other than the non-trivial zeros we get

$$\log f(s) = \sum_{|t - \tilde{\gamma}| \leq 1} \log(s - \tilde{\rho}) + O(\log kt) \tag{2.6}$$

uniformly for $-1 \leq \sigma \leq 2$, where $\log f(s)$ has its usual branch meaning, and $-\pi < \mathrm{Im}(\log(s-\tilde{\rho})) \leq \pi$. Now we can prove the theorem. Taking real parts in (2.6) we have

$$\log|f(s)| = \sum_{|t-\tilde{\gamma}|\leq 1} \log|s-\tilde{\rho}| + O(\log kt) \geq \sum_{|t-\tilde{\gamma}|\leq 1} \log|t-\tilde{\gamma}| + O(\log kt).$$

Integrating yields

$$\begin{aligned} \int_T^{T+1} \sum_{|t-\tilde{\gamma}|\leq 1} \log|t-\tilde{\gamma}|dt &= \sum_{T-1\leq\tilde{\gamma}\leq T+2} \int_{\max(\tilde{\gamma}-1,T)}^{\min(\tilde{\gamma}+1,T+1)} \log|t-\tilde{\gamma}|dt \\ &\geq \sum_{T-1\leq\tilde{\gamma}\leq T+2} \int_{\tilde{\gamma}-1}^{\tilde{\gamma}+1} \log|t-\tilde{\gamma}|dt \\ &= \sum_{T-1\leq\tilde{\gamma}\leq T+2} (-2) > -\Theta \log T. \end{aligned} \tag{2.7}$$

Therefore, we are left with

$$\sum_{|t-\tilde{\gamma}|\leq 1} \log|t-\gamma| > -\Theta \log T$$

for some t in $(T, T+1)$. This proves Lemma 2.2. □

Equipped with this result we can prove the following bounds.

Lemma 2.3. *For each real number $T \geq 2$ there is a T_1, $T \leq T_1 \leq T+1$, such that*

$$\frac{\zeta^{(k)}}{\zeta}(\sigma + iT_1) \ll_k \log^{k+1} T.$$

Proof. To prove this, we will need the bound for $k = 1$, which is given by [15, Lemma 12.2]

$$\frac{\zeta'}{\zeta}(\sigma + iT_1) \ll \log^2 T,$$

uniformly for $-1 \leq \sigma \leq 2$, as well as an application of the Faà di Bruno formula which allows us to write

$$\frac{f^{(n)}}{f}(s) = n! \sum_{(\mu_1,\cdots,\mu_n)\in P(n)} \prod_{i=1}^{n} \frac{1}{\mu_i!(i!)^{\mu_i}} \left(\left(\frac{f'}{f}\right)^{(i-1)} (s) \right)^{\mu_i},$$

where $P(n) = \{(\mu_1, \cdots, \mu_n) | \mu_1, \cdots, \mu_n$ are nonnegative integers such that $\sum_{i=1}^{n} i\mu_i = n\}$. Setting $f(s) = \zeta(s)$ yields the desired results. See [12, p. 30 and p. 44] for further details. □

Lemma 2.4. *Let $\mathcal{A}$ denote the set of those points $s \in \mathbb{C}$ such that $\sigma \leq -1$ and $|s + 2m| \geq \frac{1}{4}$ for every positive integer m. Then*

$$\frac{\zeta^{(k)}}{\zeta}(s) \ll_k \log^k(|s| + 1)$$

uniformly for $s \in \mathcal{A}$

Proof. This proceeds along the same lines as the previous lemma, except that now we use [15, Lemma 12.4], which states that

$$\frac{\zeta'}{\zeta}(s) \ll \log(|s| + 1)$$

uniformly for $s \in \mathcal{A}$. $\square$

Let $q > 1$ and χ be a character modulo q. Let $L(s,\chi)$ be the Dirichlet L-function. From [4, Sect. 14], there is a positive absolute constant A_4 (which we take to be less that $\frac{1}{12}$), such that if,

$$\sigma \geq 1 - \frac{A_4}{\log(q(|t|+1))}, \tag{2.8}$$

then there is at most one zero of $L(s,\chi)$. If such a zero exists, then it is real and simple, χ is real and non-principal, and $L(s,\psi)$ has no zeros in the above region for any other character ψ of the modulus q.

If such a zero β exists and if $\beta > 1 - \frac{A_4}{9\log 2q}$, then we shall call χ an exceptional character, β an exceptional zero, and the modulus q an exceptional modulus. The following combinatorial objects are introduced in [12].

Definition 2.1. We define the constants $a_n(q)$ by

$$a_n(q) := \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n (n+1)! \sum_{(\theta_1,\cdots,\theta_n)\in Q(n)} \frac{(c_0(q))^{\theta_1}\cdots(c_{n-1}(q))^{\theta_n}}{\theta_1!(1!)^{\theta_1}\cdots\theta_n!(n!)^{\theta_n}} & \text{if } n \geq 1, \end{cases}$$

where

$$Q(n) = \left\{ (\theta_1,\cdots,\theta_n)|\theta_1,\cdots,\theta_n \in \mathbb{N}^+ \text{ such that } 0 \leqslant \sum_{i=1}^n i\theta_i \leqslant n \right\}$$

for $n \geq 1$ and

$$c_j(q) = g_j + \sum_{p|q} \frac{A_j(p)\log^{j+1} p}{(p-1)^{j+1}}$$

for $j \geq 0$. Here $A_j(x)$ denotes the Eulerian polynomial of degree j and g_j is the constant such that

$$g_j = (j+1)! \sum_{(\theta_1,\cdots,\theta_{j+1})\in P(k-1)} \frac{(\theta_1 + \cdots + \theta_{j+1} - 1)!}{\theta_1!(0)!^{\theta_1}\cdots\theta_{j+1}!(j!)^{\theta_{j+1}}} \gamma_0^{\theta_1}\cdots\gamma_j^{\theta_{j+1}}$$

where γ_i are the Stieltjes constants and

$$P(m) = \left\{ (\theta_1,\cdots,\theta_m)|\theta_1,\cdots,\theta_m \in \mathbb{N}^+ \text{ such that } \sum_{i=1}^m i\theta_i = m \right\}$$

for $m \geq 1$.

Next, from [10] we have the following lemma.

Lemma 2.5. *Let $k \geq 1$ be an integer. Let $2 \leq T \leq x$ and define the contour $\mathcal{C}_\chi$ to consist of $\sigma = 1 - \frac{A_4}{B\log(q(|t|+2))}$, $t \leq |T|$, together with the line segments*

$$t = \pm T, \quad 1 - \frac{A_4}{B\log(q(|T|+2))} \leq \sigma \leq 1 + \frac{1}{\log x},$$

where we take $B = 8$ if χ is exceptional and $B = 10$ otherwise. We define $\mathcal{L} = \log(q(|t|+2))$. For each character χ of modulus q, on the contour $\mathcal{C}_\chi$,

$$(-1)^k \frac{L^{(k)}}{L}(s,\chi) \ll_k \mathcal{L}^{k+4}.$$

Next we prove a lemma which plays a key role in the proof of Theorem 1.1. Note that this lemma is a generalization of Lemma 2 of [10] which itself generalizes Lemma 3.2 of Friedlander [6].

Lemma 2.6. *For χ_q^0 the principal character modulus q, the residue of*

$$(-1)^k \frac{L^{(k)}}{L}(s,\chi_q^0)\frac{x^s}{s}F(s),$$

at $s=1$, where F is analytic at $s=1$ and $F^{(m)}(1) \ll_q 1$ for $m \geq 0$, is given by

$$R(j,k,x) = x\log^{k-1}x \sum_{m=0}^{k-1}\binom{k}{m}F^{(m)}(1)\sum_{n=0}^{k-m-1}\binom{k-m}{k-m-n-1}\frac{a_n(q)}{\log^{m+n}x}, \tag{2.9}$$

where $a_n(q)$ is as in Definition 2.1. In particular, since $a_n(q) \ll_n (\log\log q)^n$, this implies that the residue above is

$$x\log^{k-1}x\left(F(1)+O_{k,q,j}\left(\sum_{m=0}^{k-1}\sum_{\substack{n=0\\(m,n)\neq(0,0)}}^{k-m-1}\binom{k}{m}\binom{k-m}{k-m-n-1}\frac{(\log\log q)^n}{\log^{m+n}}\right)\right)$$

for $k \geq 2$. For χ_q exceptional, the corresponding residue at the exceptional zero β is $g_{k,q}\frac{x^\beta}{\beta}F(\beta)$ where

$$g_{k,q}(\beta) = \begin{cases} -1, & \text{if } k=1,\\ (-1)^k k! \sum_{(\theta_1,\cdots,\theta_{k-1})\in P(k-1)} \frac{(c_0(\beta))^{\theta_1}\cdots(c_{k-2}(\beta))^{\theta_{k-1}}}{\theta_1!1^{\theta_1}\cdots\theta_{k-1}!(k-1)^{\theta_{k-1}}}, & \text{if } k\geq 2.\end{cases}$$

Moreover, for $k \geq 2$, we have

$$g_{k,q}(\beta) \ll_k (\log q\log\log q)^{k-1}.$$

Proof. By the use of the fact that

$$\left.\frac{d^n}{ds^n}\frac{x^s}{s}\right|_{s=1} = n!x\sum_{i=0}^{n}\frac{(-1)^{n-i}\log^i x}{i!}$$

we can write

$$\begin{aligned}\frac{x^s}{s}F(s) &= \sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^n}{ds^n}\left(\frac{x^s}{s}F(s)\right)\bigg|_{s=1}(s-1)^n \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{l=0}^{n}\binom{n}{l}\left(\frac{d^{n-l}}{ds^{n-l}}\frac{x^s}{s}\right)\left(\frac{d^l}{ds^l}F(s)\right)\right)\bigg|_{s=1}(s-1)^n \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{l=0}^{n}\binom{n}{l}x(n-l)!\sum_{i=0}^{n-l}\frac{(-1)^{n-l-i}\log^i x}{i!}F^{(l)}(1)(s-1)^n \\ &= x\sum_{n=0}^{\infty}\sum_{l=0}^{n}\frac{(-1)^l F^{(l)}(1)}{l!}\sum_{i=0}^{n-l}\frac{(-1)^{n-i}\log^i x}{i!}(s-1)^n.\end{aligned} \tag{2.10}$$

Next, we recall Fact 4.17 of [12] which states that if $k \geq 1$, then the principal part of

$$(-1)^k\frac{L^{(k)}}{L}(s,\chi_q^0)$$

is given by

$$\frac{a_{-k,k,q}}{(s-1)^k} + \frac{a_{-(k-1),k,q}}{(s-1)^{k-1}} + \cdots + \frac{a_{-j,k,q}}{(s-1)^j} + \cdots + \frac{a_{-1,k,q}}{s-1} \tag{2.11}$$

where

$$a_{-k,k,q} = k! \tag{2.12}$$

and

$$a_{-j,k,q} = (-1)^{k-j} k! \sum_{\theta_1,\cdots,\theta_{k-j} \in P(k-j)} \frac{(c_0(q))^{\theta_1} \cdots (c_{k-j-1}(q))^{\theta_{k-j}}}{\theta_1!(1!)^{\theta_1} \cdots \theta_{k-j}!((k-j)!)^{\theta_{k-j}}}. \tag{2.13}$$

From (2.10) and (2.11), we deduce that

$$R(j,k,x) = x \sum_{n=1}^{k} a_{-n,k,q} \sum_{i=0}^{n-1} \frac{(-1)^i F^{(i)}(1)}{i!} \sum_{l=0}^{n-1-i} \frac{(-1)^{n-1-l} \log^l x}{l!}. \tag{2.14}$$

Inverting the order of summation and re-arrenging we have

$$R(j,k,x) = x \sum_{m=0}^{k-1} \frac{(-1)^m}{m!} F^{(m)}(1) \sum_{n=0}^{k-m-1} (-1)^n \frac{\log^n x}{n!} \sum_{l=m+n+1}^{k} (-1)^{l-1} a_{-l,k,q}.$$

Bringing in (2.13) we obtain

$$\begin{aligned} R(j,k,x) &= x \sum_{m=0}^{k-1} \frac{(-1)^m}{m!} F^{(m)}(1) \sum_{n=0}^{k-m-1} (-1)^n \frac{\log^n x}{n!} \sum_{l=m+n+1}^{k} (-1)^{l-1} (-1)^{k-l} k! \\ &\quad \times \sum_{\theta_1,\cdots,\theta_{k-l} \in P(k-l)} \frac{(c_0(q))^{\theta_1} \cdots (c_{k-l-1}(q))^{\theta_{k-l}}}{\theta_1!(1!)^{\theta_1} \cdots \theta_{k-l}!((k-l)!)^{\theta_{k-l}}} \\ &= x \sum_{m=0}^{k-1} \frac{k!(-1)^{k-m-1}}{m!} F^{(m)}(1) \sum_{n=0}^{k-m-1} (-1)^n \frac{\log^n x}{n!} \\ &\quad \times \sum_{\theta_1,\cdots,\theta_{k-m-n-1} \in Q(k-m-n-1)} \frac{(c_0(q))^{\theta_1} \cdots (c_{k-m-n-2}(q))^{\theta_{k-m-n-1}}}{\theta_1!(1!)^{\theta_1} \cdots \theta_{k-m-n-1}!((k-m-n-1)!)^{\theta_{k-m-n-1}}} \end{aligned} \tag{2.15}$$

By Definition 2.1, (2.15) can now be written as

$$\begin{aligned} R(j,k,x) &= x \sum_{m=0}^{k-1} \binom{k}{m} F^{(m)}(1) \sum_{n=0}^{k-m-1} \binom{k-m}{n} a_{k-m-n-1}(q) \log^n x \\ &= x \log^{k-1} x \sum_{m=0}^{k-1} \binom{k}{m} F^{(m)}(1) \sum_{n=0}^{k-m-1} \binom{k-m}{k-m-n-1} \frac{a_n(q)}{\log^{m+n} x} \end{aligned} \tag{2.16}$$

Since $a_n(q) \ll_n (\log\log q)^n$ and $F^{(m)}(1) \ll_{q,j} 1$ we have

(2.17)
$$R(j,k,x) = x\log^{k-1} x\left(kF(1) + O_{k,q,j}\sum_{m=0}^{k-1}\sum_{\substack{n=0\\(m,n)\neq(0,0)}}^{k-m-1}\binom{k}{m}\binom{k-m}{k-m-n-1}\frac{(\log\log q)^n}{\log^{m+n}}\right),$$

where we used the fact $a_0(q)=1$. This proves the first part of the theorem.

Next we consider the case when χ_q is an exceptional character. From [11, p. 338] we have

$$(-1)^k\frac{L^{(k)}}{L}(s,\chi_q)\frac{x^s}{s}F(s) = (-1)^k\frac{E_\beta^{(k)}(s)+kE_\beta^{(k-1)}(s)}{(s-\beta)E_\beta(s)}\frac{x^s}{s}F(s), \tag{2.18}$$

where $E_\beta(s)$ is such that

$$L(s,\chi_q) = (s-\beta)E_\beta(s), \quad E_\beta(s) = a_0(\beta)+a_1(\beta)(s-\beta)+a_2(\beta)(s-\beta)^2+\cdots.$$

Now we compute the residue of the expression in the left-hand side of (2.18). Since $s=\beta$ is a simple pole of the expression of the left hand-side of (2.18). then the residue is given by

$$(-1)^k\frac{E_\beta^{(k-1)}}{E_\beta}(\beta)\frac{x^\beta}{\beta}F(\beta).$$

Since for $m\geq 0$ we have

$$E_\beta^{(m)}(s) = \sum_{n=m}^{\infty}(n)_m a_n(\beta)(s-\beta)^{n-m}$$

so that $E_\beta^{(k-1)}(\beta) = (k-1)!a_{k-1}(\beta)$, we can now write that for $k\geq 1$

$$(-1)^k k\frac{E_\beta^{(k-1)}}{E_\beta}(\beta)\frac{x^\beta}{\beta}F(\beta) = (-1)^k k\frac{a_{k-1}(\beta)}{a_0(\beta)}\frac{x^\beta}{\beta}F(\beta).$$

Here $(k)_n$ denotes the falling factorial. This is equal to

$$\begin{cases} -\frac{x^\beta}{\beta}F(\beta), & \text{if } k=1,\\ (-1)^k k!\sum_{(\theta_1,\cdots,\theta_{k-1})\in P(k-1)}\frac{(c_0(\beta))^{\theta_1}\cdots(c_{k-2}(\beta))^{\theta_{k-1}}}{\theta_1!1^{\theta_1}\cdots\theta_{k-1}!(k-1)^{\theta_{k-1}}}\frac{x^\beta}{\beta}F(\beta), & \text{if } k\geq 2.\end{cases}$$

Using the estimate $|c_j(\beta)| \ll_j \log^{j+1} q\log\log q$ in [11, page 339], we have for $k\geq 2$ that

$$g_{k,q}(\beta) = (-1)^k k!\sum_{(\theta_1,\cdots,\theta_{k-1})\in P(k-1)}\frac{(c_0(\beta))^{\theta_1}\cdots(c_{k-2}(\beta))^{\theta_{k-1}}}{\theta_1!1^{\theta_1}\cdots\theta_{k-1}!(k-1)^{\theta_{k-1}}}\ll_k(\log q\log\log q)^{k-1}$$

as claimed. □

3. Proof of Theorem 1.1

First we prove the following theorem.

Theorem 3.1. *Let $j,k\geq 1$ be integers. Let $2\leq T\leq x$, $q\leq x$, and A_4 be as in (2.8). For each character χ* mod q, *we have*

$$\varphi_k(x,\chi) = \sum_{n\leqslant x}\chi(n)L_k(n) \tag{3.1}$$

$$= W_k(x,\chi) + O_{k,q}\left(\frac{x}{T}\log^{k+3} x\right) + O_{k,q}\left(x \log^{k+5}(qT)\exp\left(-\frac{A_4 \log x}{20 \log qT}\right)\right).$$

Here $W_k(x,\chi)$ is given as follows:

(1) *if χ is principal character, then*

$$W_k(x,\chi) = x\log^{k-1} x\left(kF(1) + O_{k,q}\left(\sum_{m=0}^{k-1}\sum_{\substack{n=0\\(m,n)\neq(0,0)}}^{k-m-1}\binom{k}{m}\binom{k-m}{k-m-n-1}\frac{(\log\log q)^n}{\log^{m+n}}\right)\right),$$

for all $k \geq 2$,

(2) *if χ is exceptional character, then*

$$W_k(x,\chi) = g_{k,q}(\beta) \ll_k (\log q \log\log q)^{k-1}$$

for all $k \geq 2$, and

$$W_k(x,\chi) = 0$$

otherwise.

Proof. By Lemma 3.12 of [22] with $c = 1 + \frac{1}{\log x}$ and $\alpha = k+1$ we obtain

$$\sum_{n\leq x}\chi(n)L_k(n) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}(-1)^k\frac{L^{(k)}}{L}(s,\chi)L(2s,\chi)\frac{x^s}{s}ds + O_{k,q}\left(\frac{x}{T}\log^{k+1} x\right).$$

Now applying Lemma 2.5 for the contour C_χ and encounter the residues at the poles we have

(3.2)
$$\begin{aligned}\sum_{n\leq x}\chi(n)L_k(n) &= W_k(x,\chi) + \frac{1}{2\pi i}\int_{C_\chi}(-1)^k\frac{L^{(k)}}{L}(s,\chi)L(2s,\chi)\frac{x^s}{s}ds + O_{k,q}\left(\frac{x}{T}\log^{k+1} x\right)\\
&= W_k(x,\chi) + O_{k,q}\left(\frac{x}{T}\log^{k+1} x\right)\\
&\quad + O_{k,q}\left(\frac{x}{T}\log^{k+4}(q(T+2))\left(\frac{1}{\log q(T+2)} + \frac{1}{\log x}\right)\right)\\
&\quad + O_{k,q}\left(x\log^{k+5}(qT)\exp\left(-\left(\frac{A_4}{20}\frac{\log x}{\log qT}\right)\right)\right),\end{aligned}$$

where by Lemma 2.6

(3.3)
$$W_k(x,\chi) = x\log^{k-1} x\left(kF(1) + O_{k,q,j}\left(\sum_{m=0}^{k-1}\sum_{\substack{n=0\\(m,n)\neq(0,0)}}^{k-m-1}\binom{k}{m}\binom{k-m}{k-m-n-1}\frac{(\log\log q)^n}{\log^{m+n}}\right)\right),$$

If χ is principal character,

$$W_k(x,\chi) = g_{k,q}(\beta) \ll_k (\log q\log\log q)^{k-1}, \tag{3.4}$$

if χ is exceptional character and

$$W_{j,k}(x,\chi) = 0, \tag{3.5}$$

otherwise. □

Finally, we can now prove the new analogue of Selberg formula. This follows from the orthogonal relation of Dirichlet characters and for $x = \log^5 T$

$$\vartheta_k(x;q,a) = \sum_{\substack{m\leqslant x\\ m\equiv a \bmod q}} L_k(m) = \frac{1}{\phi(q)}\sum_{\chi}\bar{\chi}(a)\sum_{m\leq x}\chi(m)L_k(m).$$

Thus by Theorem 3.1

$$\vartheta_k(x;q,a) \sim \frac{k}{\phi(q)}xF(1,\chi_q^0)\log^{k-1}x,$$

and since $(\chi * \lambda * \log^k)(n)$ has generating series

$$\sum_{n=1}^{\infty}\frac{(\chi*\lambda*\log^k)(n)}{n^s} = \frac{L(2s,\chi)}{L(s,\chi)}(-1)^k L^{(k)}(s)$$

for $\mathrm{Re}(s) > 1$, we see that

$$F(s,\chi) = L(2s,\chi), \quad \text{so that} \quad F(1,\chi) = L(2,\chi_q^0)$$

Here χ_q^0 denote the principal character modulo q. Now consider the fact that

$$L(s,\chi_q^0) = \zeta(s)\prod_{p|q}\left(1-\frac{1}{p^s}\right),$$

for $\mathrm{Re}(s) > 1$ and

$$\zeta(2n) = (-1)^{n+1}\frac{B_{2n}(2\pi)^{2n}}{2(2n)!},$$

where B_n denotes the Bernoulli numbers and n is a non-negative integer. Consequently,

$$F(1,\chi_q^0) = \frac{\pi^2}{6}\prod_{p|q}\left(1-\frac{1}{p^2}\right).$$

4. Proof of Theorems 1.3 and 1.4

Let us start with the proof of Theorem 1.3. In this case $j = 1$ so that $F(1,s) = 1$ and we set $\Lambda_{1,k}(n) = \Lambda_k(n)$. First, for a large positive number T let T_1 be the number supplied by Lemma 2.3 and consider the positively oriented contour $\mathcal{C}$ determined by the line segments $[c-iT_1, c+iT_1], [c+iT_1, \lambda+iT_1], [-K+iT_1, -K-iT_1], [-K-iT_1, c+iT_1]$ with K is some positive large integer in order to capture the trivial zeros. Let us assume that the horizontal line segments do not through any poles of $\zeta^{(k)}/\zeta$. For $\mathrm{Re}(s) = c > a$, one can write

$$h(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\hat{h}(s)x^{-s}ds. \tag{4.1}$$

Now we observe from (4.1) that

$$h(x) \ll |x|^{-a}.$$

Therefore

$$\begin{aligned}\sum_{n=1}^{\infty}\Lambda_k(n)h(n) &= \sum_{n=1}^{\infty}\Lambda_k(n)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\hat{h}(s)n^{-s}ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\hat{h}(s)\sum_{n=1}^{\infty}\frac{\Lambda_k(n)}{n^s}ds\\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(-1)^k\frac{\zeta^{(k)}}{\zeta}(s)\hat{h}(s)ds, \qquad (4.2)\end{aligned}$$

with $c > 1$ so that the interchange is justified. The poles of the integrand are located at $s = 1$, $s = \rho$ where ρ is a non-trivial zero of $\zeta(s)$ and at $s = -2m$ where $m = 1, 2, 3, \cdots$. By Cauchy's theorem we have

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} (-1)^k \frac{\zeta^{(k)}}{\zeta}(s)\hat{h}(s)ds = R_1(k) + \sum_{-T<\operatorname{Im}\rho<T} R_2(k,\rho) + \sum_{m=1}^{K} R_3(k,m),$$

where R_1, R_2 and R_3 are the residues at $s = 1$, $s = \rho$ and $s = -2m$ respectively, i.e.

$$R_1(k) = \operatorname*{res}_{s=1} (-1)^k \frac{\zeta^{(k)}}{\zeta}(s) = \lim_{s\to 1} \frac{(-1)^k}{(k-1)!}\frac{d^{k-1}}{ds^{k-1}}\left((s-1)^k \frac{\zeta^{(k)}}{\zeta}(s)\hat{h}(s)\right) =: \Phi(k),$$

as well as

$$R_2(k,\rho) = \operatorname*{res}_{s=\rho} (-1)^k \frac{\zeta^{(k)}}{\zeta}(s) = (-1)^k \frac{\zeta^{(k)}}{\zeta'}(\rho)\hat{h}(\rho),$$

and finally,

$$R_3(k,m) = \operatorname*{res}_{s=-2m} (-1)^k \frac{\zeta^{(k)}}{\zeta}(s) = (-1)^k \frac{\zeta^{(k)}}{\zeta'}(-2m)\hat{h}(-2m).$$

Next, we can make horizontal and far-left integrals tend to zero as $K \to \infty$ and $T \to \infty$ using the well-chosen sequence that T_1 obeys. In particular,

$$\int_{-1\pm iT_1}^{c\pm iT_1} (-1)^k \frac{\zeta^{(k)}}{\zeta}(s)\hat{h}(s)ds \ll_\varepsilon \frac{\log^{k+1} T}{T},$$

and

$$\int_{-1\pm iT_1}^{-K\pm iT_1} (-1)^k \frac{\zeta^{(k)}}{\zeta}(s)\hat{h}(s)ds \ll_\varepsilon \frac{\log^{k} T}{T}.$$

by the use of Lemmas 2.3 and 2.4. For the vertical line at the far left, we have

$$\int_{-K-iT_1}^{-K+iT_1} (-1)^k \frac{\zeta^{(k)}}{\zeta}(s)\hat{h}(s)ds \ll_\varepsilon \frac{T\log^{k}(KT)}{K} \to 0$$

as $K \to \infty$, by Lemma 2.4. Thus we are left with

$$\sum_{n=1}^{\infty} \Lambda_k(n)h(n) = \Phi(k) + (-1)^k \sum_{\rho} \frac{\zeta^{(k)}}{\zeta'}(\rho)\hat{h}(\rho) + (-1)^k \sum_{m=1}^{\infty} \frac{\zeta^{(k)}}{\zeta'}(-2m)\hat{h}(-2m),$$

as it was to be shown. The first few values of k are given by

$$\Phi(1) = \hat{h}(1), \quad \Phi(2) = -2(\gamma_0\hat{h}(1) - \hat{h}'(1)),$$

and

$$\Phi(3) = 3(2\gamma_0^2\hat{h}(1) + 2\gamma_1\hat{h}(1) - 2\gamma_0\hat{h}'(1) + \hat{h}''(1)).$$

If $k = 1$, then we obtain the Weil explicit formula (1.13).

Finally, we prove Theorem 1.4. Suppose that h satisfies the conditions of Theorem 1.3. By following the same strategy as in the proof of Theorem 1.3, we see that

$$\sum_{n} \theta(n)h(n) = \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{\zeta^k(a+bs)}{\zeta(c+ds)}\hat{h}(s)ds.$$

Next, we can establish the contour of integration $\mathcal{C}$ so that

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{\zeta^k(a+bs)}{\zeta(c+ds)}\hat{h}(s)ds = R_1(a,b,c,d,k) + \sum_{-T<\operatorname{Im}\rho<T} R_2(a,b,c,d,k).$$

The residues inside $\mathcal{C}$ are as follows. For the leading term, we have

$$R_1(a,b,c,d,k) = \operatorname*{res}_{s=(1-a)/b} \frac{\zeta^k(a+bs)}{\zeta(c+ds)}\hat{h}(s) = \Phi(a,b,c,d,k),$$

where

$$\Phi(a,b,c,d,k) = \lim_{s\to(1-a)/b} \frac{(-1)^k}{(k-1)!}\frac{d^{k-1}}{ds^{k-1}}\Big((s-\tfrac{1-a}{b})^k \frac{\zeta^k(a+bs)}{\zeta(c+ds)}\hat{h}(s)\Big).$$

This is a computable polynomial of degree $k-1$ in the derivatives of the Riemann zeta-function and involving the Stieltjes' constants, not unlike the ones encountered in Theorem 1.3. The first few values of k are given by

$$\Phi(a,b,c,d,1) = \frac{\hat{h}(\frac{1-a}{b})}{b\zeta(c+\frac{d(1-a)}{b})}, \quad \Phi(a,b,c,d,2) = \frac{A(a,b,c,d)}{b^2\zeta(c+\frac{d(1-a)}{b})},$$

where

$$A(a,b,c,d) = -d\hat{h}(\tfrac{1-a}{b})\zeta'(c+\tfrac{d(1-a)}{b}) + \zeta(c+\tfrac{d(1-a)}{b})(2\gamma\hat{h}(\tfrac{1-a}{b}) + \hat{h}'(\tfrac{1-a}{b})).$$

For the oscillating term associated with the non-trivial zeros we have

$$R_2(a,b,c,d,k) = \operatorname*{res}_{s=(\rho-c)/d} \frac{\zeta^k(a+bs)}{\zeta(c+ds)}\hat{h}(s) = \frac{\zeta^k(a+\frac{b}{d}(\rho-c))}{d\zeta'(\rho)}\hat{h}(\tfrac{\rho-c}{d}).$$

From [4] we have

$$\log\zeta(\sigma+it) \geq \sum_{|t-\gamma|<1} \log|t-\gamma| + O(\log t), \tag{4.3}$$

for sufficient large t. We choose a suitable large T so that

$$|T-\gamma| > e^{\frac{-A_1\gamma}{\log\gamma}} \tag{4.4}$$

for any ordinate γ of a zero of $\zeta(s)$ and for a sufficiently small constant A_1. Then for above admissible value of T one can find from (4.3) that

$$\frac{1}{\zeta(\sigma+iT)} < e^{A_2 T}. \tag{4.5}$$

Choose A_1 small enough so that $A_2 \leq 1/2(j+2)$. Since for all $k\geq 1$, $\zeta^{(k)}(s) \ll |t|^A$ as $|t|$ tends to infinity (see Lemma 12.4 of [7]) then from (1.14), and

$$I_{\pm} = \int_{\lambda\pm iT}^{c\pm iT} (-1)^k \frac{\zeta^k(a+bs)}{\zeta(c+ds)}\hat{h}(s)ds,$$

we can find $I_{\pm}\to 0$ as T tends to infinity through a sequence of values which satisfies (4.4).

By following the same technique as in the proof of Theorem 1.3, we can show that the horizontal integrals and far left vertical integral tend to zero through a sequence of values which satisfies (4.4). By doing this, we pick the poles associated with the trivial zeros. Their residues are given by

$$R_3(a,b,c,d,k) = \operatorname*{res}_{s=(-2m-c)/d} \frac{\zeta^k(a+bs)}{\zeta(c+ds)}\hat{h}(s) = \frac{\zeta^k(a+\frac{b}{d}(-2m-c))}{d\zeta'(-2m)}\hat{h}(\tfrac{-2m-c}{d})$$

where $m = 1, 2, 3, \cdots$. Summing over $m \geq 1$ leads to the result.

Corollary 1.1 now follows easily by considering different values of a, b, c, d and k.

(1) If $a = b = c = k = 0$ and $d = 1$, then $\theta(n) = \mu(n)$ and $\Phi(0,0,0,1,0) = 0$ so that (1.18) follows.
(2) If $a = -1$, $b = d = k = 1$ and $c = 0$, then $\theta(n) = \phi(n)$ and $\Phi(-1,1,0,1,1) = \frac{\hat{h}(2)}{\zeta(2)}$, which proves (1.19).
(3) if $a = c = 0$, $b = 2$ and $d = k = 1$, then $\theta(n) = \lambda(n)$ and $\Phi(0,2,0,1,1) = \frac{\hat{h}(\frac{1}{2})}{2\zeta(\frac{1}{2})}$. Furthermore, we note that for the trivial zeros we have $\zeta(-4m) = 0$ in the numerator of $R_3(0,2,0,1,1)$ so that there is no contribution coming from this term. This establishes (1.20).

5. On the Sastry-Möbius function

We shall need the following auxiliary results.

Lemma 5.1. *There exists a positive constant c such that for all $n \in \mathbb{N}$ there exists T_n where $n \leq T_n \leq n+1$ for which*

$$\frac{1}{\zeta(s)}\frac{\zeta(2s)}{\zeta((1+j)s)} \ll T_n^c$$

in the interval $-\frac{1}{2} \leq \sigma \leq \frac{3}{2}$.

Proof. Use convexity bound in the numerator and Lemma 2.2 for the denominator. □

Lemma 5.2. *There exists a positive constant c such that for all $n \in \mathbb{N}$ there exists T_n where $n \leq T_n \leq n+1$ for which*

$$\frac{1}{\zeta(s)}\frac{\zeta(2s)\zeta(2(1+j)s)}{\zeta((1+j)s)} \ll T_n^c$$

in the interval $-\frac{1}{2} \leq \sigma \leq \frac{3}{2}$.

Proof. Use convexity bound in the numerator and Lemma 2.2 for the denominator. □

The explicit formula for the μ_j function is as follows.

Theorem 5.1. *If h satisfies the conditions of Theorem 1.3, then*

$$\sum_n \mu_j(n)h(n) = \frac{\hat{h}(\frac{1}{2})}{2\zeta(\frac{1}{2})\zeta(\frac{1}{2}(j+1))} + \sum_\rho \frac{\zeta(2\rho)}{\zeta'(\rho)\zeta((j+1)\rho)}\hat{h}(\rho)$$

$$+ \sum_\rho \frac{\zeta(\frac{2\rho}{1+j})}{(1+j)\zeta(\frac{\rho}{1+j})\zeta'(\rho)}\hat{h}(\tfrac{\rho}{1+j}) + \sum_m \frac{\zeta(-\frac{4m}{1+j})}{(1+j)\zeta(-\frac{2m}{1+j})\zeta'(-2m)}\hat{h}(-\tfrac{2m}{1+j}), \tag{5.1}$$

when $j \geq 1$ is odd; and

$$\sum_n \mu_j(n)h(n) = \frac{\zeta(\frac{1}{2}(1+j))\hat{h}(\frac{1}{2})}{2\zeta(\frac{1}{2})\zeta(\frac{1}{2}(j+1))} + \frac{\zeta(\frac{2}{1+j})\hat{h}(\frac{1}{1+j})}{(1+j)\zeta(2)\zeta(\frac{1}{1+j})} + \sum_m \frac{\zeta(-m)\zeta(\frac{-2m}{1+j})\hat{h}(\frac{-m}{1+j})}{2(1+j)\zeta(\frac{-m}{1+j})\zeta'(-2m)}$$

$$+\sum_{\rho}\frac{\zeta((1+j)\rho)\zeta(2\rho)}{\zeta'(\rho)\zeta(2(j+1)\rho)}\hat{h}(\rho)+\sum_{\rho}\frac{\zeta(\frac{\rho}{2})\zeta(\frac{\rho}{1+j})\hat{h}(\frac{\rho}{2(1+j)})}{2(1+j)\zeta(\frac{\rho}{2(1+j)})\zeta'(\rho)},$$

when $j \geq 2$ is even.

Proof. By convention the integer 1 will be counted as an odd integer. It is convenient to define

$$\mathfrak{W}(j,s) := \begin{cases} 1/\zeta(s), & \text{if } j \geq 1 \text{ is odd,} \\ \zeta(s)/\zeta(2s), & \text{if } j \geq 2 \text{ is even.} \end{cases} \tag{5.2}$$

In [19] the Dirichlet series for $\mu_j(n)$ is given. For $\mathrm{Re}(s) > 1$ we have

(5.3)
$$\sum_{n=1}^{\infty}\frac{\mu_j(n)}{n^s}=\frac{\zeta(2s)}{\zeta(s)}\mathfrak{W}(j,(j+1)s)=\frac{1}{\zeta(s)}F(j,s), \quad \text{where} \quad F(j,s):=\zeta(2s)\mathfrak{W}(j,(j+1)s).$$

The proof of Theorem 5.1 follows the same guidelines as in the proof of Theorem 1.3, therefore, it is only necessary to give the main terms. Suppose first that $j \geq 1$ is odd. One has

$$\sum_{n}\mu_j(n)h(n)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\zeta(2s)}{\zeta(s)\zeta((j+1)s)}\hat{h}(s)ds.$$

with $c > 1$. The poles are now at $s = \frac{1}{2}$, $s = \rho$, $s = \rho/(j+1)$ and $s = -2m/(j+1)$ with $m = 1, 2, 3, \cdots$. The residues are as follows. For the leading term, we have

$$M_1(j)=\operatorname*{res}_{s=1/2}\frac{\zeta(2s)}{\zeta(s)\zeta((j+1)s)}\hat{h}(s)=\frac{\hat{h}(\frac{1}{2})}{2\zeta(\frac{1}{2})\zeta(\frac{1}{2}(j+1))}.$$

For the non-trivial zeros, we have

$$M_2=\sum_{\rho}\operatorname*{res}_{s=\rho}\frac{\zeta(2s)}{\zeta(s)\zeta((j+1)s)}\hat{h}(s)=\sum_{\rho}\frac{\zeta(2\rho)}{\zeta'(\rho)\zeta((j+1)\rho)}\hat{h}(\rho),$$

and for the re-scaled non-trivial zeros

$$M_3=\sum_{\rho}\operatorname*{res}_{s=\rho/(1+j)}\frac{\zeta(2s)}{\zeta(s)\zeta((j+1)s)}\hat{h}(s)=\sum_{\rho}\frac{\zeta(\frac{2\rho}{1+j})}{(1+j)\zeta(\frac{\rho}{1+j})\zeta'(\rho)}\hat{h}(\tfrac{\rho}{1+j}).$$

Finally, the trivial zeros yield

$$M_4=\sum_{m}\operatorname*{res}_{s=-2m/(1+j)}\frac{\zeta(2s)}{\zeta(s)\zeta((j+1)s)}\hat{h}(s)=\sum_{m}\frac{\zeta(-\frac{4m}{1+j})}{(1+j)\zeta(-\frac{2m}{1+j})\zeta'(-2m)}\hat{h}(-\tfrac{2m}{1+j}).$$

Constructing the same contour $\mathcal{C}$ as before will yield the proof. The case where j is even is identical, except that

$$\sum_{n=1}^{\infty}\mu_j(n)h(n)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\zeta(2s)\zeta((j+1)s)}{\zeta(s)\zeta(2(j+1)s)}\hat{h}(s)ds,$$

with $c > 1$. The poles are at $s = 1/2$, $s = 1/(j+1)$, $s = \rho$, $s = \rho/2(1+j)$ and at $s = -m/(1+j)$. The residues can be found to be the ones in the statement of the theorem. In this case, we use Lemmas 5.1 and 5.2 to show that the horizontal integrals tend to zero through the suitable sequence of T_n's. Finally, we apply the functional equation of $\zeta(s)$ to make the vertical integral on the far left tend to zero as per [22, §14.27] or as per [13, §3]. $\square$

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