An Analysis and Preliminary Study on the Convexity Adjustment for Zero Coupon Swaps

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§0.0 Abstract

This analysis extends my previous studies centered on convexity adjustments for STIR futures and focuses on the convexity adjustment required when considering a portfolio consisting of a zero coupon swap (ZCS) hedged with vanilla interest rate swaps (IRS). Traditionally, and for a number of years, ZCS have existed alongside IRS with no adjustment and with no observable risk or profit and loss implications. This empirical evidence suggests that any results presented in this analysis should tend to zero when older curve models are used as a basis for valuation. The analysis presented here provides a closed form solution in order to model ZCS convexity adjustments subject to assumed covariance of relevant rates and supports the above expectation. Specifically this analysis demonstrates a contribution to the ZCS convexity adjustment resulting from gamma considerations, as well as a larger cross gamma contribution which is highly dependent upon a number of assumptions based on correlation, mean reversion and hedging costs.

§1.0 Introduction & Contents

The appendix gives a comprehensive review of notation and some of the useful and relevant mathematical formulas that are repeatedly required. Descriptively the yield curve is divided up into contiguous forward (LIBOR) forecast rates, typically referred to here as 'outright yields' with contiguous forward spreads between the forecast rates and the discount rates, typically referred to by traders as LIBOR-OIS basis and in this document as 'discounting basis spreads'. Occasionally, equivalent continuously compounded rates are substituted where it is of an advantage for taking derivatives.

The proposed method of determining the results required is to create a portfolio, and then study the evolution of the profit and loss of the portfolio under market moves. The portfolio created consists of a ZCS hedged with single period forward IRS, defined in such a way that initially the ZCS will value to zero (i.e. be on market), and each of the single period IRS are also on market. The notionals on the IRS are defined in such a way as to perfectly hedge the delta on the ZCS. This is demonstrated in §2.

In order to complete the analysis it will be demonstrated separately, that the construction above is sufficient to produce the results and that consideration of off market trades has no impact upon the result. This is shown in §3. Given this section has no bearing on the overall conclusion the reader may choose to omit.

§4 outlines an iterative and practical method of employing the results of §2 in order to obtain some estimates of the ZCS convexity adjustment, subject to a number of specified assumptions.

§5 provides a summary of the results and suggested further study.

§2.0 Construction of a Total Portfolio for Analysis

§2.1 Formulation of Zero Coupon Swap

This section introduces the construction of the ZCS and concludes with the result of a second order polynomial describing the net present value of the ZCS, given deviations to the forward rate yields and deviations to the forward discounting basis spreads.

Consider the ZCS component forming part of the total portfolio. The fixed rate is chosen such that initially, i.e. at time zero, the ZCS is on market and has a net present value of precisely zero. The direction on this ZCS is such that a positive notional is interpreted as having paid fixed payments in order to receive floating payments. The formula for the net present value of the ZCS is thus;

$$\psi_n(r_1,..,r_n,s_1,..,s_n) = N_0 v_n \left(\prod_{k=1}^{n} (1 + d_k r_k) - \prod_{k=1}^{n} (1 + d_k R_k) \right)$$

for $R_1, \dots, R_n$ constants.

Choose to express, $r_k(t) = R_k + \xi_k(t)$, and, $s_k(t) = S_k + \eta_k(t)$, for $S_1, \dots, S_n$ constants, where $\xi_k(t)$ and $\eta_k(t)$ are expected to be small deviations and take initial values, $\xi_k(0) = 0$, and, $\eta_k(0) = 0$. The above is then re-expressed as;

$$\psi_n(\xi_1,..,\xi_n,\eta_1,..,\eta_n) = N_0 \frac{\left(\prod_{k=1}^{n}(1 + d_k R_k + d_k \xi_k) - \prod_{k=1}^{n}(1 + d_k R_k)\right)}{\prod_{k=1}^{n}(1 + d_k R_k + d_k S_k + d_k \xi_k + d_k \eta_k)}$$

Clearly the ZCS is on market initially since, $\psi_n|_0 = 0$. In order to simplify the algebra again let;

$$A_k = 1 + d_k R_k, \quad B_k = 1 + d_k R_k + d_k S_k, \quad x_k = d_k \xi_k, \quad y_k = d_k \eta_k$$

Then after substitution the above reduces to;

$$\psi_n(x_1,..,x_n,y_1,..,y_n) = N_0 \frac{(A_1 + x_1) \dots (A_n + x_n) - A_1 \dots A_n}{(B_1 + x_1 + y_1) \dots (B_n + x_n + y_n)}$$

$$\psi_n = N_0\left(g(x_1,..,x_n,y_1,..,y_n) - f(x_1,..,x_n,y_1,..,y_n)\right)$$

[using §6.4, §6.5]

$$\psi_n \sim N_0 \; \frac{A_1 \dots A_n}{B_1 \dots B_n}\left(\sum_{k=1}^{n}\left[\frac{x_k}{A_k} - \frac{1}{2}\frac{x_k x_k}{A_k A_k}\right] + \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}\left[\left(\frac{1}{A_k A_l} - \frac{1}{A_k B_l} - \frac{1}{B_k A_l}\right) x_k x_l - \frac{x_k y_l}{A_k B_l} - \frac{x_l y_k}{A_l B_k}\right]\right)$$

[using §6.6]

This, the result of §2.1, is the second order approximation for the net present value of the ZCS.

§2.2 Construction of Interest Rate Swap Hedges

Having established the ZCS component of the portfolio this section introduces the on market, single period, IRS hedges that are included in the total portfolio. The notional on these hedges will be chosen such that the total portfolio is completely delta hedged initially. In this section a positive notional is interpreted as having received fixed to pay floating payments on the IRS. We begin by considering the net present value of the j'th IRS hedge, which as in §2.1 is shown to be precisely zero initially;

$$\theta_j = \bar{N}_j d_j v_j (R_j - r_j) = -\bar{N}_j d_j v_j \xi_j \,, \quad \Rightarrow \quad \theta_j\big|_0 = 0$$

$$\frac{\partial \theta_j}{\partial s_k} = -\alpha_{kj} d_k \frac{\partial \bar{s}_k}{\partial s_k}\theta_j \,, \quad \frac{\partial \theta_j}{\partial r_k} = -\delta_{kj}\bar{N}_j d_j v_j + \frac{\partial \theta_j}{\partial s_k}$$

$$\Rightarrow \; \frac{\partial \theta_j}{\partial s_k}\bigg|_0 = 0, \quad \frac{\partial \theta_j}{\partial r_k}\bigg|_0 = -\delta_{kj}\bar{N}_j d_j v_j\big|_0 = -\frac{\delta_{kj}\bar{N}_j d_j}{B_1 \dots B_j}$$

In addition observe that, developing on formulas from §2.1;

$$\frac{\partial \psi_n}{\partial s_k} = -\alpha_{kn} d_k \frac{\partial \bar{s}_k}{\partial s_k}\psi_n \,, \quad \frac{\partial \psi_n}{\partial r_k} = \frac{N_0 v_n d_k}{(1 + d_k r_k)}\prod_{l=1}^{n}(1 + d_l r_l) + \frac{\partial \psi_n}{\partial s_k}$$

$$\Rightarrow \quad \frac{\partial \psi_n}{\partial s_k}\bigg|_0 = 0, \quad \frac{\partial \psi_n}{\partial r_k}\bigg|_0 = \frac{N_0 d_k}{A_k}\;\frac{A_1 \dots A_n}{B_1 \dots B_n}$$

Then the total portfolio is initially delta hedged with respect to discount basis spreads since;

$$\left.\frac{\partial \psi_n}{\partial s_k}\right|_0 + \sum_{j=1}^{n} \left.\frac{\partial \theta_j}{\partial s_k}\right|_0 = 0, \quad \forall\, k = 1, \ldots, n$$

In order that the total portfolio is initially delta hedged with respect to outright rates;

$$\left.\frac{\partial \psi_n}{\partial r_k}\right|_0 + \sum_{j=1}^{n} \left.\frac{\partial \theta_j}{\partial r_k}\right|_0 = \frac{N_0 d_k}{A_k}\ \frac{A_1 \ldots A_n}{B_1 \ldots B_n} - \frac{\bar{N}_k d_k}{B_1 \ldots B_k} = 0\,, \ \ \forall\, k = 1, \ldots, n$$

$$\Rightarrow \quad \bar{N}_k = \frac{N_0}{A_k}\ \frac{A_1 \ldots A_n}{B_{k+1} \ldots B_n}\,, \ \ \forall\, k = 1, \ldots, n$$

Thus the net present value of each IRS hedge can be expressed as;

$$\theta_j = -\frac{N_0}{A_j}\ \frac{A_1 \ldots A_n}{B_{j+1} \ldots B_n} d_j \xi_j v_j$$

$$\Rightarrow \quad \theta_j \sim -N_0\ \frac{A_1 \ldots A_n}{B_{j+1} \ldots B_n} \frac{x_j}{A_j} \frac{1}{B_1 \ldots B_j} \left(1 - \sum_{l=1}^{j} \frac{(x_l + y_l)}{B_l} \right)$$

[using modified §6.4]

Lastly we obtain the result of §2.2, the second order approximation of the sum of all IRS hedges contained within the total portfolio;

$$\sum_{k=1}^{n} \theta_k \ \sim\ N_0 \frac{A_1 \ldots A_n}{B_1 \ldots B_n} \left(-\sum_{k=1}^{n} \frac{x_k}{A_k} + \sum_{k=1}^{n} \sum_{l=1}^{k} \left[\frac{x_k x_l}{A_k B_l} + \frac{x_k y_l}{A_k B_l} \right] \right)$$

§2.3 Analysis of the Total Portfolio

The previous sections have systematically constructed the components which form part of the total portfolio, *P*, that being an on market ZCS hedged with a series of on market, single period, IRS and have demonstrated closed form equations approximating their net present value. Here we consider the evolution of the net present value of the total portfolio and infer from the results implications for the calculation of the ZCS convexity adjustment;

$$\Delta P(t) = P(t) - P(0) = \psi_n(t) - \psi_n(0) + \sum_{k=1}^{n} \theta_k\,(t) - \sum_{k=1}^{n} \theta_k\,(0)$$

$$
\begin{aligned}
\Delta P(t) &= \psi_n + \sum_{k=1}^{n} \theta_k \\
&\sim\ N_0\,\frac{A_1 \dots A_n}{B_1 \dots B_n}\left(\sum_{k=1}^{n}\left[\frac{x_k}{A_k} - \frac{1}{2}\frac{x_k x_k}{A_k A_k}\right]\right. \\
&\qquad + \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}\left[\left(\frac{1}{A_k A_l} - \frac{1}{A_k B_l} - \frac{1}{B_k A_l}\right)x_k x_l - \frac{x_k y_l}{A_k B_l} - \frac{x_l y_k}{A_l B_k}\right] \\
&\qquad \left. - \sum_{k=1}^{n}\frac{x_k}{A_k} + \sum_{k=1}^{n}\sum_{l=1}^{k}\left[\frac{x_k x_l}{A_k B_l} + \frac{x_k y_l}{A_k B_l}\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
=\ N_0\,\frac{A_1 \dots A_n}{B_1 \dots B_n}&\left(\sum_{k=1}^{n}\left[-\frac{1}{2}\frac{x_k x_k}{A_k A_k} + \frac{x_k x_k}{A_k B_k} + \frac{x_k y_k}{A_k B_k}\right.\right. \\
&\quad \left. + \frac{1}{2}\left(\frac{1}{A_k A_k} - \frac{1}{A_k B_k} - \frac{1}{B_k A_k}\right)x_k x_k - \frac{1}{2}\frac{x_k y_k}{A_k B_k} - \frac{1}{2}\frac{x_k y_k}{A_k B_k}\right] \\
&\quad + \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{k-1}\left[\left(\frac{1}{A_k A_l} - \frac{1}{A_k B_l} - \frac{1}{B_k A_l}\right)x_k x_l - \frac{x_k y_l}{A_k B_l} - \frac{x_l y_k}{A_l B_k}\right] \\
&\quad + \frac{1}{2}\sum_{k=1}^{n}\sum_{l=k+1}^{n}\left[\left(\frac{1}{A_k A_l} - \frac{1}{A_k B_l} - \frac{1}{B_k A_l}\right)x_k x_l - \frac{x_k y_l}{A_k B_l} - \frac{x_l y_k}{A_l B_k}\right] \\
&\quad \left. + \sum_{k=1}^{n}\sum_{l=1}^{k-1}\left[\frac{x_k x_l}{A_k B_l} + \frac{x_k y_l}{A_k B_l}\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
=\ N_0\,\frac{A_1 \dots A_n}{B_1 \dots B_n}&\left(\frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{k-1}\left[\left(\frac{1}{A_k A_l} + \frac{1}{A_k B_l} - \frac{1}{B_k A_l}\right)x_k x_l + \frac{x_k y_l}{A_k B_l} - \frac{x_l y_k}{A_l B_k}\right]\right. \\
&\quad + \frac{1}{2}\sum_{k=1}^{n}\sum_{l=k+1}^{n}\left[\left(\frac{1}{A_k A_l} - \frac{1}{A_k B_l} - \frac{1}{B_k A_l}\right)x_k x_l - \frac{x_k y_l}{A_k B_l}\right. \\
&\quad \left.\left. - \frac{x_l y_k}{A_l B_k}\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
=\ N_0\,\frac{A_1 \dots A_n}{B_1 \dots B_n}&\left(\frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{k-1}\left[\left(\frac{1}{A_k A_l} + \frac{1}{A_k B_l} - \frac{1}{B_k A_l}\right)x_k x_l + \frac{x_k y_l}{A_k B_l} - \frac{x_l y_k}{A_l B_k}\right]\right. \\
&\quad \left. + \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{k-1}\left[\left(\frac{1}{A_k A_l} - \frac{1}{A_k B_l} - \frac{1}{B_k A_l}\right)x_k x_l - \frac{x_k y_l}{A_k B_l} - \frac{x_l y_k}{A_l B_k}\right]\right)
\end{aligned}
$$

[using §6.7]

$$= \; N_0 \, \frac{A_1 \dots A_n}{B_1 \dots B_n} \left(\sum_{k=1}^{n} \sum_{l=1}^{k-1} \left[\left(\frac{1}{A_k A_l} - \frac{1}{B_k A_l} \right) x_k x_l - \frac{x_l y_k}{A_l B_k} \right] \right)$$

Thus we arrive at the principle result of §2 which is the second order approximation of the change in the net present value of the total portfolio stated explicitly for each move in outright yields and discounting basis spread changes.

$$\Delta P(t) \; \sim \; N_0 \, \frac{A_1 \dots A_n}{B_1 \dots B_n} \left(\sum_{k=1}^{n} \sum_{l=1}^{k-1} \left[\left(\frac{1}{A_k} - \frac{1}{B_k} \right) \frac{x_k x_l}{A_l} - \frac{x_l y_k}{A_l B_k} \right] \right)$$

There are a number of encouraging elements contained within this formula that support its validity and accuracy that are highlighted;

1) The equation contains no first order terms, by design, since the total portfolio was delta hedged with respect to outright yields and discount basis spreads.

2) Observe that in the limit as the initial discounting basis spreads tend to zero and remain at zero indefinitely then,

$$S_k, \eta_k(t) \to 0 \quad \Rightarrow \quad y_k \to 0, \qquad B_k \to A_k \quad \forall k = 1..n$$

and hence $\Delta P(t) \to 0$ regardless of the changes in outright yields. This is in agreement with empirical evidence set out the introduction which states ZCS hedged with IRS experience no gamma where the discount curve is equivalent to the forecast yield curve.

3) The contribution of the “cross gamma” term has the potential to be much larger than that of the “gamma” term, due to their respective coefficients. This is also in agreement with empirical evidence where risk changes due to cross gamma vastly exceed those due to gamma for given curve moves and hence typically result in greater changes to the net present value.

This concludes §2.

§3.0 Other Considerations and Subsequent Evolutions of the Total Portfolio

The analysis so far has considered only an on market ZCS hedged with a series of on market, single period IRS. This is impractical in the first instance since the precise execution of this combination of instruments is at any time impossible in a financial market. In the second instance when the impact of any market moves is considered, and the portfolio re-hedged, there then remains the question how the net present value of the total portfolio evolves with subsequent market moves, where in that subsequent analysis the ZCS and any number of the IRS hedges are considered to be initially off market.

This section is included to demonstrate explicitly that any off market element of the ZCS and IRS hedges has no impact on the overall result of this analysis.

§3.1 Decomposition of an Off Market Zero Coupon Swap

Any arbitrary off market ZCS, indicated below, can be decomposed into an on market ZCS minus a fixed cash flow which accounts for the off market element;

$$\Psi_n = N_0 v_n \left(\prod_{k=1}^{n} (1 + d_k r_k) - \prod_{k=1}^{n} (1 + d_k T_k) \right)$$

$$\Psi_n = N_0 v_n \left(\prod_{k=1}^{n} (1 + d_k r_k) - \prod_{k=1}^{n} (1 + d_k R_k) \right) - Z_n v_n$$

$$\Psi_n = \psi_n - Z_n v_n$$

§3.2 Decomposition of a Series of Off Market Interest Rate Swaps

Similarly any set of off market, single period IRS can be decomposed as above, as indeed can any set of off market multi period IRS (provided reset dates and accrual periods overlap which is always assumed in this analysis). Consider an arbitrary off market IRS;

$$\Theta_{mn} = \bar{N}_{mn} \sum_{k=m}^{n} d_k v_k \left(T_j - r_j \right)$$

$$\Theta_{mn} = \bar{N}_{mn} \sum_{k=m}^{n} d_k v_k (R_j - r_j) + \sum_{k=m}^{n} Z_k v_k$$

$$\Theta_{mn} = \sum_{k=m}^{n} \frac{\bar{N}_{mn}}{\bar{N}_k} \theta_k + \sum_{k=m}^{n} Z_k v_k$$

By extension any linear combination of IRS can be decomposed in the above manner to yield a series of on market, single period IRS with fixed payment cash flows accounting for the off market component, with the determining variable being the notional attached to each IRS. This of course also allows for the possibility of hedging ZCS with par IRS as is common practice and these can then be represented above using the decomposition.

§3.3 Decomposition of the Evolved Total Portfolio

Using the above results we state that a portfolio containing an off market ZCS, and series of off market IRS hedges can be decomposed into an on market ZCS, on market, single period IRS hedges, and a series of fixed cash flows;

$$P = \Psi_n + \sum_{k=1}^{N} \Theta_k = \psi_n + \sum_{k=1}^{n} C_k \theta_k + \sum_{k=1}^{n} Z_k v_k$$

In the above C_k represent constants determined by the notional on the collection of IRS hedges. If the hedges are constructively chosen, akin to §2.2, then C_k will revert to unity and the analysis reverts to that of §2, with the additional element of fixed cash flows the only remaining consideration.

§3.4 Hedging of Fixed Cash Flows

Fixed cash flows can be hedged by entering into a series of on market, single period IRS with a floating index that equates to that of the discount curve. The net present value of the *j*'th IRS is expressed as;

$$\varphi_j = \bar{\bar{N}}_j d_j v_j (R_j + S_j - r_j - s_j) = -\bar{\bar{N}}_j d_j v_j (\xi_j + \eta_j), \quad \Rightarrow \quad \varphi_j\big|_0 = 0$$

$$\frac{\partial \varphi_j}{\partial r_k} = \frac{\partial \varphi_j}{\partial s_k} = -\alpha_{kj} d_k \frac{\partial \bar{s}_k}{\partial s_k} \varphi_j - \delta_{kj} \bar{\bar{N}}_j d_j v_j$$

$$\Rightarrow \quad \left.\frac{\partial \varphi_j}{\partial r_k}\right|_0 = \left.\frac{\partial \varphi_j}{\partial s_k}\right|_0 = -\frac{\delta_{kj}\overline{\overline{N}}_j d_j}{B_1 \dots B_j}$$

Also consider the corresponding equations for a single fixed cash flow;

$$F_n = Z_n v_n, \qquad \frac{\partial F_n}{\partial r_k} = \frac{\partial F_n}{\partial s_k} = -\alpha_{kn} d_k \frac{\partial \bar{s}_k}{\partial s_k} F_n$$

$$\Rightarrow \quad \left.\frac{\partial F_n}{\partial r_k}\right|_0 = \left.\frac{\partial F_n}{\partial s_k}\right|_0 = -\alpha_{kn} d_k \frac{1}{B_k} \frac{Z_n}{B_1 \dots B_n}$$

Comparing the above analyzes it is seen that a single cash flow's outright and discounting delta can be initially hedged by setting;

$$\overline{\overline{N}}_k = -\frac{Z_n}{B_k \dots B_n}, \quad \forall\, k = 1 \dots n$$

§3.5 A Hedged Portfolio Containing the Simplest Single Cash Flow

We consider a portfolio, Q_1,consisting of a single cash flow, Z_1, payable at the earliest maturity, hedged with a single IRS against a floating index equivalent to the discount curve, as defined above, and consider the change in the portfolio as outright yields and discounting basis spreads evolve with time;

$$\Delta Q = Z_1 v_1 - Z_1 v_1|_0 + \varphi_1 - \varphi_1|_0$$

$$\sim \frac{Z_1}{B_1}\left(1 - \frac{x_1 + y_1}{B_1} + \frac{x_1 x_1 + 2x_1 y_1 + y_1 y_1}{B_1 B_1}\right) - \frac{Z_1}{B_1} + \frac{Z_1}{B_1}(x_1 + y_1)\frac{1}{B_1}\left(1 - \frac{x_1 + y_1}{B_1}\right)$$

$$= \frac{Z_1}{B_1}\left(-\frac{x_1 + y_1}{B_1} + \frac{x_1 x_1 + 2x_1 y_1 + y_1 y_1}{B_1 B_1}\right) + \frac{Z_1}{B_1}\left(\frac{x_1 + y_1}{B_1} - \frac{x_1 x_1 + 2x_1 y_1 + y_1 y_1}{B_1 B_1}\right)$$

$$= 0$$

Thus this result demonstrates the simplest cash flow initially hedged exhibits no profit or loss due to any gamma effects since the change in value of the portfolio to second order reduces to zero.

§3.6 A Hedged Portfolio Containing Any Cash Flow

We extend section §3.5 to demonstrate that any cash flow, regardless of its maturity, initially hedged, exhibits no profit or loss due to gamma effects, measured to second order, and therefore by extension nor does any linear combination of cash flows and collection of hedges. This is shown by induction;

Assume for $1 \leq k < n$ that a portfolio, Q_k, consisting of a cash flow, Z_k, payable at maturity, *k,* and a series of on market, single period IRS hedges, as defined above, has no change in value, measured to second order, as outright yields and discount basis spreads evolve.

Consider the addition of two cash flows to portfolio Q_{n-1}, $-Z_{n-1}$, and $Z_{n-1}B_n$, payable at maturities, *n-1* and *n* respectively, and observe;

$$\begin{aligned}\frac{\partial(-Z_{n-1}v_{n-1}+Z_{n-1}B_n v_n)}{\partial r_k} &= \frac{\partial(-Z_{n-1}v_{n-1}+Z_{n-1}B_n v_n)}{\partial s_k} \\ &= -d_k\frac{\partial \bar{s}_k}{\partial s_k}(-\alpha_{kn-1}Z_{n-1}v_{n-1}+\alpha_{kn}Z_{n-1}B_n v_n)\end{aligned}$$

$$\begin{aligned}\left.\frac{\partial(-Z_{n-1}v_{n-1}+Z_{n-1}B_n v_n)}{\partial r_k}\right|_0 &= \left.\frac{\partial(-Z_{n-1}v_{n-1}+Z_{n-1}B_n v_n)}{\partial s_k}\right|_0 \\ &= \begin{cases} -d_k\dfrac{1}{B_k}\left(-\dfrac{Z_{n-1}}{B_1 \dots B_{n-1}}+\dfrac{Z_{n-1}B_n}{B_1 \dots B_n}\right)=0\,, & k \leq n-1 \\ -d_k\dfrac{1}{B_k}\dfrac{Z_{n-1}B_n}{B_1 \dots B_n}\,, & k=n \end{cases}\end{aligned}$$

Thus the above shows that given the additional cash flows to the portfolio there is no adjustment needed to the existing IRS hedges within portfolio, Q_{n-1}, to maintain a state of zero initial risk. There simply needs to be the inclusion in the portfolio of one additional IRS hedge;

$$\varphi_n, \text{ where, } \frac{\bar{\bar{N}}_n d_n}{B_1 \dots B_n} = -d_n\frac{Z_{n-1}}{B_1 \dots B_n} \quad \Rightarrow \quad \bar{\bar{N}}_n = -Z_{n-1}$$

Hence;

$$\Delta Q = Z_{n-1}v_{n-1} - Z_{n-1}v_{n-1}|_0 + \sum_{k=1}^{n-1}\varphi_k - \sum_{k=1}^{n-1}\varphi_k|_0 - Z_{n-1}v_{n-1} + Z_{n-1}v_{n-1}|_0 + Z_{n-1}B_n v_n - Z_{n-1}B_n v_n|_0 + \varphi_n - \varphi_n|_0$$

$$\Delta Q = -Z_{n-1}v_{n-1} + Z_{n-1}B_n v_n + \varphi_n$$

[using original assumption]

$$\begin{aligned}
\Delta Q \sim & -\frac{Z_{n-1}}{B_1 \dots B_{n-1}}\left(1 - \sum_{k=1}^{n-1}\frac{(x_k + y_k)}{B_k} + \frac{1}{2}\sum_{k=1}^{n-1}\frac{(x_k x_k + 2x_k y_k + y_k y_k)}{B_k B_k} + \frac{1}{2}\sum_{k=1}^{n-1}\sum_{l=1}^{n-1}\frac{(x_k x_l + x_k y_l + x_l y_k + y_k y_l)}{B_k B_l}\right) \\
& + \frac{Z_{n-1}}{B_1 \dots B_{n-1}}\left(1 - \sum_{k=1}^{n}\frac{(x_k + y_k)}{B_k} + \frac{1}{2}\sum_{k=1}^{n}\frac{(x_k x_k + 2x_k y_k + y_k y_k)}{B_k B_k} + \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{(x_k x_l + x_k y_l + x_l y_k + y_k y_l)}{B_k B_l}\right) \\
& + \frac{Z_{n-1}}{B_1 \dots B_n}\left(1 - \sum_{k=1}^{n}\frac{(x_k + y_k)}{B_k}\right)(x_n + y_n)
\end{aligned}$$

$$\begin{aligned}
= & \frac{Z_{n-1}}{B_1 \dots B_{n-1}}\left(-\frac{(x_n + y_n)}{B_n} + \frac{(x_n x_n + 2x_n y_n + y_n y_n)}{B_n B_n} + \frac{(x_n + y_n)}{B_n}\sum_{k=1}^{n-1}\frac{(x_k + y_k)}{B_k}\right) \\
& + \frac{Z_{n-1}}{B_1 \dots B_{n-1}}\left(\frac{(x_n + y_n)}{B_n} - \frac{(x_n + y_n)}{B_n}\sum_{k=1}^{n}\frac{(x_k + y_k)}{B_k}\right)
\end{aligned}$$

$$= 0$$

Therefore, by induction, any cash flow at any maturity hedged initially experiences no gamma or cross gamma, since there is no change in net present value of the portfolio, measured to second order, as outright yields and discounting basis spreads evolve. By extension this also applies to any set of cash flows and set of hedges.

§4.0 Iterative Modeling to Calculate the Convexity Adjustment

§2 develops a theory of a portfolio containing a ZCS hedged with IRS which concludes with a closed form second order polynomial, in terms of outright yield and discounting basis spread changes, describing the net present value of that portfolio. The aim of this section is to extend upon that formula and discuss a method for calculating the convexity adjustment for the ZCS.

We begin by presenting an arbitrage free argument and then considering this over the life of the portfolio, accounting for re-hedging over appropriate time periods, and the evolution of the trades contained within the portfolio.

§4.1 Expectation of the Net Present Value of the Total Portfolio

Recall from §2.3;

$$\Delta P(t) \sim N_0 \frac{A_1 \dots A_n}{B_1 \dots B_n}\left(\sum_{k=1}^{n}\sum_{l=1}^{k-1}\left[\left(\frac{1}{A_k}-\frac{1}{B_k}\right)\frac{x_k x_l}{A_l}-\frac{x_l y_k}{A_l B_k}\right]\right)$$

Then;

$$E[\Delta P] \sim N_0 \frac{A_1 \dots A_n}{B_1 \dots B_n}\left(\sum_{k=1}^{n}\sum_{l=1}^{k-1}\left[\left(\frac{1}{A_k}-\frac{1}{B_k}\right)\frac{E[x_k x_l]}{A_l}-\frac{E[x_l y_k]}{A_l B_k}\right]\right)$$

$$= N_0 \frac{A_1 \dots A_n}{B_1 \dots B_n}\left(\sum_{k=1}^{n}\sum_{l=1}^{k-1}\left[\left(\frac{1}{A_k}-\frac{1}{B_k}\right)\frac{Cov(x_k,x_l)+E[x_k]E[x_l]}{A_l} - \frac{Cov(x_l,y_k)+E[x_l]E[y_k]}{A_l B_k}\right]\right)$$

We make the assumption that $E[x_k], E[y_k]$ are zero for all *k*, thus;

$$E[\Delta P] \sim N_0 \frac{A_1 \dots A_n}{B_1 \dots B_n}\left(\sum_{k=1}^{n}\sum_{l=1}^{k-1}\left[\left(\frac{1}{A_k}-\frac{1}{B_k}\right)\frac{Cov(x_k,x_l)}{A_l}-\frac{Cov(x_l,y_k)}{A_l B_k}\right]\right)$$

Given the total portfolio is delta hedged it is not expected to gain or lose in value. We suppose to use the above equation to calculate the gain or loss in the portfolio after a single period, and discount this back to the initial time in order to demonstrate the first convexity adjustment required.

§4.2 Convexity Adjustment of the Total Portfolio when *n*=2

Consider the simplest case in which a convexity adjustment is required, that when *n=2,* in which there are two periods. Then the result of §4.1 reduces to;

$$E[\Delta P] \sim N_0 \frac{A_1 A_2}{B_1 B_2}\left(\left(\frac{1}{A_2}-\frac{1}{B_2}\right)\frac{Cov(x_2,x_1)}{A_1}-\frac{Cov(x_1,y_2)}{A_1 B_2}\right)$$

After one period it is observed that the first IRS hedge, θ_1, will have matured, the two period ZCS will have been reduced to a single period ZCS, with a new, increased, fixed notional. At this point the expected gain or loss in the portfolio is crystallized and the portfolio is re-hedged, assumed to be at zero cost, according to the methodology set out in §2 and §3. Over the final period the expected change in value of the re-hedged portfolio is zero according to §4.1.

The crystallized gain or loss at time, *t=1*, is thus the convexity adjustment which needs to be taken upfront, and therefore discounted back to the present, in order to satisfy the arbitrage free principle.

$$CxAdj[\psi_2] \sim \frac{1}{B_1}\ N_0 \frac{A_1 A_2}{B_1 B_2}\left(\left(\frac{1}{A_2}-\frac{1}{B_2}\right)\frac{Cov(x_2,x_1)}{A_1}-\frac{Cov(x_1,y_2)}{A_1 B_2}\right)$$

§4.3 Convexity Adjustment of the Total Portfolio in General

In order to calculate total convexity adjustments for general *n* it is necessary to make further assumptions;

1) Initial curves represent the expected and likely path of future interest rates,
2) The calculated and assumed covariance are expected to roll, which we express mathematically as;
   $Cov(x_k, x_l)$ evaluated at $t = j$, is; $Cov_j(x_k, x_l) = Cov(x_{k-j}, x_{l-j})$
   [technically the day count fraction should affect the above but its omission indirectly assumes each period's day count fraction is identical and is insignificant in any case given the broad underlying assumption]

Then the total convexity adjustment is the sum of all convexity adjustments as the portfolio evolves through time, after each period an IRS hedge maturing, re-hedging to maintain delta neutrality, and maturity of the ZCS decreasing, with its notional accreting;

$$
\begin{aligned}
CxAdj[\psi_n] \sim{}& \frac{1}{B_1}\, N_0\, \frac{A_1 \dots A_n}{B_1 \dots B_n}\left(\sum_{k=1}^{n}\sum_{l=1}^{k-1}\left[\left(\frac{1}{A_k}-\frac{1}{B_k}\right)\frac{Cov(x_k,x_l)}{A_l}\right.\right.\\
&\left.\left.-\frac{Cov(x_l,y_k)}{A_l B_k}\right]\right)\\
&+\frac{1}{B_1 B_2}\, A_1 N_0\, \frac{A_2 \dots A_n}{B_2 \dots B_n}\left(\sum_{k=2}^{n}\sum_{l=2}^{k-1}\left[\left(\frac{1}{A_k}-\frac{1}{B_k}\right)\frac{Cov_1(x_k,x_l)}{A_l}\right.\right.\\
&\left.\left.-\frac{Cov_1(x_l,y_k)}{A_l B_k}\right]\right)+\cdots\\
&+\frac{1}{B_1 \dots B_n}\, A_1 \dots A_{n-1} N_0\, \frac{A_n}{B_n}\left(\sum_{k=n}^{n}\sum_{l=n}^{k-1}\left[\left(\frac{1}{A_k}\right.\right.\right.\\
&\left.\left.\left.-\frac{1}{B_k}\right)\frac{Cov_{n-1}(x_k,x_l)}{A_l}-\frac{Cov_{n-1}(x_l,y_k)}{A_l B_k}\right]\right)
\end{aligned}
$$

Thus we arrive at the principle formula of this document, which after making the final assumption of the values of the covariance allows for a preliminary estimate of the convexity adjustment for a ZCS;

$$
\begin{aligned}
CxAdj[\psi_n] \sim{}& N_0 \frac{A_1 \dots A_n}{B_1 \dots B_n}\sum_{j=1}^{n}\left[\frac{1}{B_j}\sum_{k=j}^{n}\sum_{l=j}^{k-1}\left[\left(\frac{1}{A_k}-\frac{1}{B_k}\right)\frac{Cov_{j-1}(x_k,x_l)}{A_l}\right.\right.\\
&\left.\left.-\frac{Cov_{j-1}(x_l,y_k)}{A_l B_k}\right]\right]
\end{aligned}
$$

§5.0 Results and Further Study

Results are omitted in preliminary draft ahead of doing the calculations.

An extension to this considering the expected discounting risk change and suggested hedging costs will lead to an additional adjustment required.

§6.0 Appendix

§6.1 Notation

i, j, k, l	counting indices
t	time, commonly used to depict evolutions in rates
m_i	specific date /maturity
d_i	day count fraction between dates m_{i-1} and m_i
$r_i(t)$	discrete outright yield between dates m_{i-1} and m_i
$s_i(t)$	discrete discounting basis spread between dates m_{i-1}
-	and m_i
$\bar{r}_i(t)$	cont. compounded outright yield between dates m_{i-1}
-	and m_i
$\bar{s}_i(t)$	cont. compounded discounting basis spread between
-	dates m_{i-1} and m_i
$v_i(t)$	discount factor attributed to discount curve at date m_i
$w_i(t)$	discount factor attributed to outright curve at date m_i
N_0	notional attributed to having paid fixed on a ZCS
$\bar{N}_i$	notional attributed to having received fixed on the *i* 'th
-	IRS hedge
$\bar{\bar{N}}_i$	notional attributed to the *i* 'th discounting IRS hedge
$\psi_n(t)$	net present value of an initially on market ZCS with *n*
-	periods starting at date m_0 and maturing at date m_n
$\theta_i(t)$	net present value of the i 'th initially on market single
-	period IRS hedge starting at date m_{i-1} and maturing at
-	date m_i
$\varphi_i(t)$	net present value of the i 'th initially on market single
-	period discounting IRS hedge starting at date m_{i-1} and
-	maturing at date m_i
$\Psi_n(t)$	net present value of an initially off market ZCS with *n*
-	periods starting at date m_0 and maturing at date m_n
$\Theta_i(t)$	net present value of the i 'th initially off market single
-	period IRS hedge starting at date m_{i-1} and maturing at
-	date m_i
Z_i	an arbitrary cashflow receivable at date m_i

$$\delta_{jk} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

$$\alpha_{jk} = \begin{cases} 1 & j \leq k \\ 0 & j > k \end{cases}$$

other useful notation developed within the document in particular in §2.1

§6.2 Useful Derivatives

Omitted in preliminary draft

§6.3 Multivariate Taylor series expansion about zero;

$$F(x_1, \dots, x_n) \sim F|_0 + \sum_{k=1}^{n} x_k \left.\frac{\partial F}{\partial x_k}\right|_0 + \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n} x_k\, x_l \left.\frac{\partial^2 F}{\partial x_k \partial x_l}\right|_0$$

§6.4 – Multivariate Taylor series expansion about zero in relation to discount factors;

Let,

$$f(x_1, \dots, x_n, y_1, \dots, y_n) = \frac{A_1 \dots A_n}{(B_1 + x_1 + y_1) \dots (B_n + x_n + y_n)}$$

for $A_1, \dots, A_n, B_1, \dots, B_n$ constant.

$$\Rightarrow \quad f\Big|_0 = \frac{A_1 \dots A_n}{B_1 \dots B_n}$$

$$\frac{\partial f}{\partial x_k} = \frac{\partial f}{\partial y_k} = -\frac{f}{(B_k + x_k + y_k)}$$

$$\Rightarrow \quad \left.\frac{\partial f}{\partial x_k}\right|_0 = \left.\frac{\partial f}{\partial y_k}\right|_0 = -\frac{1}{B_k} f|_0$$

$$\frac{\partial^2 f}{\partial x_k \partial x_l} = \frac{\partial^2 f}{\partial x_k \partial y_l} = \frac{\partial^2 f}{\partial y_k \partial x_l} = \frac{\partial^2 f}{\partial y_k \partial y_l}$$
$$= -\frac{1}{(B_k + x_k + y_k)}\frac{\partial f}{\partial x_l} + \delta_{kl}\frac{f}{(B_k + x_k + y_k)^2}$$

$$\Rightarrow \quad \left.\frac{\partial^2 f}{\partial x_k \partial x_l}\right|_0 = \left.\frac{\partial^2 f}{\partial x_k \partial y_l}\right|_0 = \left.\frac{\partial^2 f}{\partial y_k \partial x_l}\right|_0 = \left.\frac{\partial^2 f}{\partial y_k \partial y_l}\right|_0 = \frac{1}{B_k B_l} f|_0 + \frac{\delta_{kl}}{B_k B_l} f|_0$$

Thus,

$$f \sim \frac{A_1 \dots A_n}{B_1 \dots B_n}\left(1 - \sum_{k=1}^{n}\frac{(x_k + y_k)}{B_k} + \frac{1}{2}\sum_{k=1}^{n}\frac{(x_k x_k + 2x_k y_k + y_k y_k)}{B_k B_k}\right.$$
$$\left. + \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{(x_k x_l + x_k y_l + x_l y_k + y_k y_l)}{B_k B_l}\right)$$

§6.5 Multivariate Taylor series expansion about zero in relation to compounded LIBOR legs;

Let,

$$g(x_1, \dots, x_n, y_1, \dots, y_n) \;=\; \frac{(A_1 + x_1)\dots(A_n + x_n)}{(B_1 + x_1 + y_1)\dots(B_n + x_n + y_n)}$$

for $A_1, \dots, A_n, B_1, \dots, B_n$ constant.

$$\Rightarrow \quad g\Big|_0 \;=\; f|_0 \;= \frac{A_1 \dots A_n}{B_1 \dots B_n}$$

$$\frac{\partial g}{\partial x_k} = g\left(\frac{1}{(A_k + x_k)} - \frac{1}{(B_k + x_k + y_k)}\right),\;\; \frac{\partial g}{\partial y_k} = g\left(-\frac{1}{(B_k + x_k + y_k)}\right)$$

$$\Rightarrow \quad \frac{\partial g}{\partial x_k}\bigg|_0 \;=\; \left(\frac{1}{A_k} - \frac{1}{B_k}\right) g|_0, \qquad \frac{\partial g}{\partial y_k}\bigg|_0 \;=\; -\frac{1}{B_k} g|_0$$

$$\frac{\partial^2 g}{\partial x_k \partial x_l} = g\left(\frac{1}{(A_k + x_k)} - \frac{1}{(B_k + x_k + y_k)}\right)\left(\frac{1}{(A_l + x_l)} - \frac{1}{(B_l + x_l + y_l)}\right) - \delta_{kl} g\left(\frac{1}{(A_k + x_k)^2} - \frac{1}{(B_k + x_k + y_k)^2}\right)$$

$$\frac{\partial^2 g}{\partial x_k \partial y_l} = g\left(\frac{1}{(A_k + x_k)} - \frac{1}{(B_k + x_k + y_k)}\right)\left(-\frac{1}{(B_l + x_l + y_l)}\right) - \delta_{kl} g\left(-\frac{1}{(B_k + x_k + y_k)^2}\right)$$

$$\frac{\partial^2 g}{\partial y_k \partial y_l} = g\left(-\frac{1}{(B_k + x_k + y_k)}\right)\left(-\frac{1}{(B_l + x_l + y_l)}\right) - \delta_{kl} g\left(-\frac{1}{(B_k + x_k + y_k)^2}\right)$$

$$\Rightarrow \quad \frac{\partial^2 g}{\partial x_k \partial x_l}\bigg|_0 \;=\; g|_0 \left(\frac{1}{A_k} - \frac{1}{B_k}\right)\left(\frac{1}{A_l} - \frac{1}{B_l}\right) - \delta_{kl}\, g|_0 \left(\frac{1}{A_k^2} - \frac{1}{B_k^2}\right)$$

$$\Rightarrow \quad \frac{\partial^2 g}{\partial x_k \partial y_l}\bigg|_0 \;=\; g|_0 \left(\frac{1}{A_k} - \frac{1}{B_k}\right)\left(-\frac{1}{B_l}\right) - \delta_{kl}\, g|_0 \left(-\frac{1}{B_k^2}\right)$$

$$\Rightarrow \quad \frac{\partial^2 g}{\partial y_k \partial y_l}\bigg|_0 \;=\; g|_0 \left(-\frac{1}{B_k}\right)\left(-\frac{1}{B_l}\right) - \delta_{kl}\, g|_0 \left(-\frac{1}{B_k^2}\right)$$

Thus,

$$g \sim \frac{A_1 \dots A_n}{B_1 \dots B_n}\left(1 + \sum_{k=1}^{n}\left[\left(\frac{1}{A_k} - \frac{1}{B_k}\right)x_k - \frac{1}{B_k}y_k\right]\right.$$
$$+ \frac{1}{2}\sum_{k=1}^{n}\left[\left(\frac{1}{B_kB_k} - \frac{1}{A_kA_k}\right)x_kx_k + \frac{2}{B_kB_k}x_ky_k + \frac{1}{B_kB_k}y_ky_k\right]$$
$$+ \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}\left[\left(\frac{1}{A_kA_l} - \frac{1}{A_kB_l} - \frac{1}{B_kA_l} + \frac{1}{B_kB_l}\right)x_kx_l\right.$$
$$+ \left(-\frac{1}{A_kB_l} + \frac{1}{B_kB_l}\right)x_ky_l + \left(-\frac{1}{A_lB_k} + \frac{1}{B_lB_k}\right)x_ly_k$$
$$\left.\left. + \left(\frac{1}{B_kB_l}\right)y_ky_l\right]\right)$$

§6.6 Explicit linear combination of §6.4 and §6.5;

$$g - f \sim \frac{A_1 \dots A_n}{B_1 \dots B_n}\left(1 + \sum_{k=1}^{n}\left[\left(\frac{1}{A_k} - \frac{1}{B_k}\right)x_k - \frac{1}{B_k}y_k\right]\right.$$
$$+ \frac{1}{2}\sum_{k=1}^{n}\left[\left(\frac{1}{B_kB_k} - \frac{1}{A_kA_k}\right)x_kx_k + \frac{2}{B_kB_k}x_ky_k + \frac{1}{B_kB_k}y_ky_k\right]$$
$$+ \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}\left[\left(\frac{1}{A_kA_l} - \frac{1}{A_kB_l} - \frac{1}{B_kA_l} + \frac{1}{B_kB_l}\right)x_kx_l\right.$$
$$+ \left(-\frac{1}{A_kB_l} + \frac{1}{B_kB_l}\right)x_ky_l + \left(-\frac{1}{A_lB_k} + \frac{1}{B_lB_k}\right)x_ly_k$$
$$\left. + \left(\frac{1}{B_kB_l}\right)y_ky_l\right] - 1 + \sum_{k=1}^{n}\frac{(x_k + y_k)}{B_k}$$
$$- \frac{1}{2}\sum_{k=1}^{n}\frac{(x_kx_k + 2x_ky_k + y_ky_k)}{B_kB_k}$$
$$\left. - \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{(x_kx_l + x_ky_l + x_ly_k + y_ky_l)}{B_kB_l}\right)$$

$$= \frac{A_1 \dots A_n}{B_1 \dots B_n}\left(\sum_{k=1}^{n}\left[\frac{x_k}{A_k} - \frac{1}{2}\frac{x_kx_k}{A_kA_k}\right]\right.$$
$$\left. + \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}\left[\left(\frac{1}{A_kA_l} - \frac{1}{A_kB_l} - \frac{1}{B_kA_l}\right)x_kx_l - \frac{x_ky_l}{A_kB_l} - \frac{x_ly_k}{A_lB_k}\right]\right)$$

§6.7 Result for §2.3 through use of symmetry;

$$\sum_{k=1}^{n}\sum_{l=k+1}^{n}\left[\left(\frac{1}{A_kA_l}-\frac{1}{A_kB_l}-\frac{1}{B_kA_l}\right)x_kx_l-\frac{x_ky_l}{A_kB_l}-\frac{x_ly_k}{A_lB_k}\right]=$$

$$=\sum_{k=1}^{n}\sum_{l=1}^{n}\left[(1-\alpha_{lk})\left(\frac{1}{A_kA_l}-\frac{1}{A_kB_l}-\frac{1}{B_kA_l}\right)x_kx_l-(1-\alpha_{lk})\left(\frac{x_ky_l}{A_kB_l}+\frac{x_ly_k}{A_lB_k}\right)\right]$$

$$=\sum_{l=1}^{n}\sum_{k=1}^{n}\left[(1-\alpha_{lk})\left(\frac{1}{A_lA_k}-\frac{1}{A_lB_k}-\frac{1}{B_lA_k}\right)x_lx_k-(1-\alpha_{lk})\left(\frac{x_ly_k}{A_lB_k}+\frac{x_ky_l}{A_kB_l}\right)\right]$$

$$=\sum_{l=1}^{n}\sum_{k=1}^{l-1}\left[\left(\frac{1}{A_lA_k}-\frac{1}{A_lB_k}-\frac{1}{B_lA_k}\right)x_lx_k-\frac{x_ly_k}{A_lB_k}-\frac{x_ky_l}{A_kB_l}\right]$$