Exploration Numérique 1

MAP-433 - statistiques

Gr 5

Guilherme Vieira, João Andreotti, Nicolas Lopes

1.

Comme S est une application continue et différenciable sur $\mathcal{R}$², on cherche con minimum global en étudiant les minimums locales, qui satisfont l'equation:

$$\nabla_{\beta_1,\beta_2} S = \vec{0}$$

On établit la notation $\vec{\beta} = [\beta_1, \beta_2]^T$ et $\tilde{t} = [\vec{1}, \vec{t}]$. Alors, on a:

$$\nabla_{\vec{\beta}} S(\hat{\vec{\beta}}) = -2\tilde{t}^T(\vec{X} - \tilde{t}\,\vec{\beta}) = \vec{0}$$

Comme $\tilde{t}^T\tilde{t}$ est une matrice carré inversible (en supposant que la famille $\{\vec{t}, \vec{1}\}$ est libre), alors:

$$\hat{\vec{\beta}} = \left(\tilde{t}^T\tilde{t}\right)^{-1}\tilde{t}^T\vec{X}$$

On peut montrer que cette solution est unique et correspond au minimum globale de la fonction S.

En dévelopant les expressions on arrive a montrer que:

$$\hat{\beta}_2 = \frac{\sum_{i=1}^{n}(t_i - \bar{t})X_i}{\sum_{i=1}^{n}(t_i - t)^2}$$

$$\hat{\beta}_1 = \bar{X} - \hat{\beta}_2\bar{t}$$

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In [ ]:

```python
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt

from scipy.stats import chi2
from scipy.stats import t as student


from typing import Tuple
```

In [ ]:

```python
# Reading the data into a pandas dataframe
GLB_data = pd.read_csv("GLB.Ts+dSST.csv", header=1)
GLB_data = GLB_data[1:-2] # removing years with missing values
GLB_data["J-D"] = GLB_data["J-D"].astype(float) # Converts means to floats
GLB_data
```

Out[ ]:

	Year	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	J-D	D-N	DJF
1	1881	-0.19	-0.14	0.04	0.05	0.07	-0.18	0.01	-.03	-.15	-.22	-.18	-.07	-0.08	-.09	-.17
2	1882	0.16	0.14	0.05	-0.16	-0.13	-0.22	-0.16	-.07	-.14	-.23	-.17	-.36	-0.11	-.08	.08
3	1883	-0.29	-0.37	-0.12	-0.19	-0.18	-0.07	-0.07	-.14	-.22	-.11	-.24	-.11	-0.18	-.20	-.34
4	1884	-0.13	-0.08	-0.37	-0.40	-0.34	-0.35	-0.31	-.28	-.27	-.25	-.33	-.31	-0.28	-.27	-.11
5	1885	-0.58	-0.33	-0.26	-0.42	-0.45	-0.44	-0.34	-.31	-.29	-.24	-.24	-.10	-0.33	-.35	-.41
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
136	2016	1.17	1.37	1.36	1.10	0.95	0.80	0.84	1.02	.90	.88	.91	.86	1.01	1.04	1.23
137	2017	1.02	1.14	1.16	0.94	0.91	0.72	0.82	.87	.77	.90	.88	.93	0.92	.92	1.01
138	2018	0.82	0.84	0.88	0.89	0.82	0.77	0.82	.76	.80	1.01	.82	.92	0.85	.85	.86
139	2019	0.93	0.95	1.17	1.01	0.85	0.91	0.94	.94	.92	1.01	.99	1.09	0.98	.96	.93
140	2020	1.16	1.24	1.17	1.13	1.02	0.92	0.90	.87	.98	.88	1.10	.81	1.01	1.04	1.16

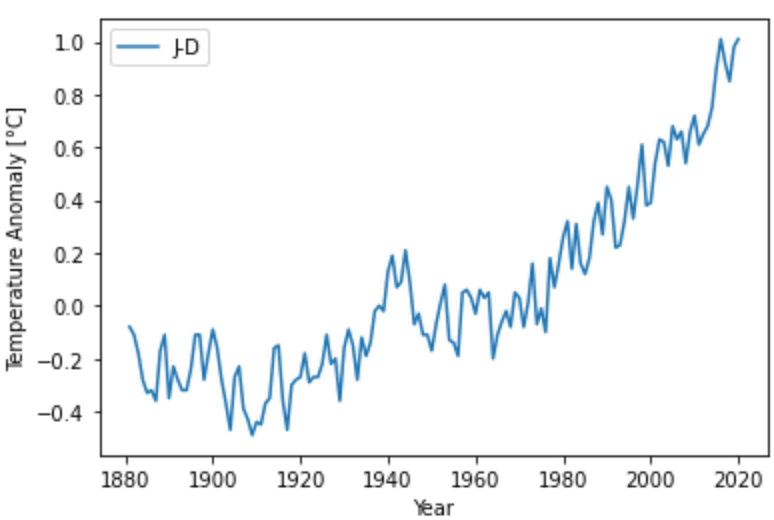
140 rows × 19 columns

In [ ]:

```python
# Plotting and visualizing the data
ax = GLB_data.plot(
    x='Year',
    y='J-D')

ax.set_ylabel("Temperature Anomaly [°C]")
```

Out[ ]: `Text(0, 0.5, 'Temperature Anomaly [°C]')`



3

Estimation des paramètres

In [ ]:
```
# We convert the data to numpy arrays
t = np.array(GLB_data['Year'], dtype=int)
X = np.array(GLB_data['J-D'], dtype=float)
one = np.ones_like(t)
```

In [ ]:
```
# Funciton to calculate estimators
def calculate_estimators(t: np.array, X: np.array) -> Tuple[float, float, float]:
    t_bar = t.mean()
    X_bar = X.mean()
    n = len(t)

    beta_2_est = (t - t_bar) @ X / ((t - t_bar) @ (t - t_bar))
    beta_1_est = X_bar - beta_2_est * t_bar
    X_hat = beta_1_est + beta_2_est * t
    epsilon_hat = X - X_hat
    sigma_2_hat = epsilon_hat @ epsilon_hat / (n - 2)

    return beta_1_est, beta_2_est, sigma_2_hat
```

In [ ]:
```
delta_t = 30
t_len = 10
```

In [ ]:
```
# calculating the estimators every 10 years in a 30 years interval
t0 = t[0]
tf = t[-1]

ranges = [
    range(time, time + delta_t)
    for time in range(0, len(t) - delta_t, t_len)
]
ranges = np.array(ranges)
times = ranges + t0
estimators = np.array([
    calculate_estimators(t[r], X[r])
    for r in ranges
])

estimators
```

Out[ ]:
```
array([[ 1.17521869e+01, -6.34037820e-03,  1.12774861e-02],
       [ 8.06885206e+00, -4.38932147e-03,  1.22446194e-02],
       [-9.95107453e+00,  5.03893215e-03,  1.02644319e-02],
       [-2.24428024e+01,  1.15461624e-02,  8.35584830e-03],
       [-2.04927208e+01,  1.05361513e-02,  1.33537121e-02],
       [-4.42806452e+00,  2.25806452e-03,  1.43103687e-02],
       [ 6.02434038e+00, -3.08787542e-03,  1.06317909e-02],
       [-1.01532614e+01,  5.16573971e-03,  1.00866344e-02],
```

```
[-2.86208951e+01,  1.45383760e-02,  1.06376868e-02],
[-3.27925488e+01,  1.66340378e-02,  1.01454701e-02],
```

$$[\hat{\beta}_1, \hat{\beta}_2, \hat{\sigma}^2]$$

4

I) Déterminons d'abord pour σ^2. Nous avons que :

$$\frac{n-2}{\sigma^2}\hat{\sigma}^2 \sim \chi^2_{(n-2)}$$

Nous cherchons avoir un intervale de confiance avec α. En utilisant les quantiles d'une loi de $\chi^2_{(n-2)}$ nous pouvons écrire que:

$$\mathbb{P}\left[q^{\chi^2_{n-2}}_{\alpha/2} \leq \frac{n-2}{\sigma^2}\hat{\sigma}^2 \leq q^{\chi^2_{n-2}}_{1-\alpha/2}\right] = 1-\alpha$$

Donc:

$$\mathbb{P}\left[\frac{(n-2)\hat{\sigma}^2}{q^{\chi^2_{(n-2)}}_{1-\alpha/2}} \leq \sigma^2 \leq \frac{(n-2)\hat{\sigma}^2}{q^{\chi^2_{(n-2)}}_{\alpha/2}}\right] = 1-\alpha$$

Nous avons donc que l'intervale de confiance pour σ^2 avec α est:

$$\boxed{I_{1-\alpha}(\sigma^2) = \left[\frac{(n-2)\hat{\sigma}^2}{q^{\chi^2_{(n-2)}}_{1-\alpha/2}}, \frac{(n-2)\hat{\sigma}^2}{q^{\chi^2_{(n-2)}}_{\alpha/2}}\right]}$$

II) Soit $j \in \{1,2\}$. Déterminons l'intervale de confience α pour β_j. Nous avons que :

$$\frac{\hat{\beta}_j - \beta_j}{\hat{\sigma}_j} \sim T_{n-2}$$

Nous cherchons avoir un intervale de confiance avec α. En utilisant les quantiles d'une loi de $\chi^2_{(n-2)}$ nous pouvons écrire que:

$$\mathbb{P}\left[q^{T_{n-2}}_{\alpha/2} \leq \frac{\hat{\beta}_j - \beta_j}{\hat{\sigma}_j} \leq q^{T_{n-2}}_{1-\alpha/2}\right] = 1-\alpha$$

Donc:

$$\mathbb{P}\left[\hat{\beta}_j - \hat{\sigma}_j q^{T_{n-2}}_{1-\alpha/2} \leq \beta_j \leq \hat{\beta}_j - \hat{\sigma}_j q^{T_{n-2}}_{\alpha/2}\right] = 1-\alpha$$

Nous avons donc que l'intervale de confiance pour β_j avec α est:

$$\boxed{I_{1-\alpha}(\beta_j) = \left[\hat{\beta}_j - \hat{\sigma}_j q_{1-\alpha/2}^{T_{n-2}}, \hat{\beta}_j - \hat{\sigma}_j q_{\alpha/2}^{T_{n-2}}\right]}$$

III)

L'intervalle de confiance autour de la droite de régression est determiné par le résidu de prédiction:

$$\hat{\varepsilon}_i = X_i - \hat{X}_i = X_i - \hat{\beta}_2 t - \hat{\beta}_1$$

Nous avons, donc que:

$$\frac{\hat{\varepsilon}_i}{\hat{\sigma}_1} \sim T_{n-2}$$

Or, cela implique que:

$$\boxed{I_{1-\alpha}(X_j) = \left[\hat{X}_j - \hat{\sigma}_1 q_{1-\alpha/2}^{T_{n-2}}, \hat{X}_j - \hat{\sigma}_1 q_{\alpha/2}^{T_{n-2}}\right]}$$

Obtenons numériquement leur valeur pour $\alpha = 0.05$

In [ ]:

```
def sigma_2_confidence_interval(t: np.array, X: np.array, n: int, alpha: float):
    beta_1_hat, beta_2_hat, sigma_2_hat = calculate_estimators(t, X)
    lower_bound = (n - 2) * sigma_2_hat / chi2.ppf(1 - alpha / 2, n - 2)
    upper_bound = (n - 2) * sigma_2_hat / chi2.ppf(alpha / 2, n - 2)
    return lower_bound, upper_bound
```

In [ ]:

```
def beta_1_confidence_interval(t: np.array, X: np.array, n: int, alpha: float):
    beta_1_hat, beta_2_hat, sigma_2_hat = calculate_estimators(t, X)
    sigma_2_j_hat = sigma_2_hat * (t**2).sum() / (n * ((t - t.mean())**2).sum())
    sigma_j_hat = np.sqrt(sigma_2_j_hat)
    lower_bound = beta_1_hat - sigma_j_hat * student.ppf(1 - alpha / 2, n - 2)
    upper_bound = beta_1_hat - sigma_j_hat * student.ppf(alpha / 2, n - 2)
    return lower_bound, upper_bound
```

In [ ]:

```
def beta_2_confidence_interval(t: np.array, X: np.array, n: int, alpha: float):
    beta_1_hat, beta_2_hat, sigma_2_hat = calculate_estimators(t, X)
    sigma_2_j_hat = sigma_2_hat / ((t - t.mean())**2).sum()
    sigma_j_hat = np.sqrt(sigma_2_j_hat)
    lower_bound = beta_2_hat - sigma_j_hat * student.ppf(1 - alpha / 2, n - 2)
    upper_bound = beta_2_hat - sigma_j_hat * student.ppf(alpha / 2, n - 2)
    return lower_bound, upper_bound
```

In [ ]:

```
def prediction_uncertainty(t: np.array, X: np.array, n: int, alpha: float):
    _, _, sigma_2_hat = calculate_estimators(t, X)
    return student.ppf(1 - alpha / 2, n - 2) * np.sqrt(sigma_2_hat)
```

In [ ]:

```
alpha = 0.05
n = len(ranges[0])

confidence_intervals = [
    (
        beta_1_confidence_interval(t[r], X[r], n, alpha),
        beta_2_confidence_interval(t[r], X[r], n, alpha),
        sigma_2_confidence_interval(t[r], X[r], n, alpha),
        prediction_uncertainty(t[r], X[r], n, alpha)
    )
    for r in ranges
]

confidence_intervals
```

Out[ ]:

```
[((3.0545574932863424, 20.44981625532322),
  (-0.010928897693893522, -0.001751858702102027),
  (0.00710220393377239, 0.020627938802648297),
  0.2175316387139444),
 ((-1.0418639664336187, 17.17956808211771),
  (-0.009170545312657163, 0.0003919023760609467),
  (0.00771127390122603, 0.02239694714601972),
  0.22666732893952182),
 ((-18.336410734135857, -1.565738320369146),
  (0.0006613541765281101, 0.009416510117131512),
  (0.006464214458780436, 0.0187749354293873),
  0.20753136394637806),
 ((-30.047984434606093, -14.837620311407996),
  (0.007596484279764446, 0.01549584052557482),
  (0.005262248886136066, 0.015283896238369787),
  0.1872455713082802),
 ((-30.15689952771755, -10.828542074062213),
  (0.005543083805717633, 0.015529218752680589),
  (0.008409745356877338, 0.024425616919081964),
  0.23671037045548984),
 ((-14.484113388846628, 5.627984356588561),
  (-0.0029107606463023663, 0.0074268896785604326),
  (0.009012217417444684, 0.026175462025133083),
  0.24504265674697231),
 ((-2.687937492484565, 14.73661824888056),
  (-0.007543100438659266, 0.001367349604398976),
  (0.006695565515366034, 0.019446881135491645),
  0.21121244023033292),
 ((-18.68262894354497, -1.6238938595696037),
  (0.0008262409366423477, 0.009505238484937185),
  (0.006352243185354862, 0.018449721369439787),
  0.20572611283046585),
 ((-37.42471668582555, -19.817073451363925),
  (0.010081915777548524, 0.018994836169058824),
  (0.006699278610140873, 0.01945766560359451),
  0.21127099712950825),
 ((-41.433798097935636, -24.15129941782263),
  (0.012281901212759874, 0.02098617442683969),
  (0.006389296075051177, 0.018557339335390093),
  0.20632524475119873),
 ((-42.52189026858127, -26.794025193042778),
  (0.013638736741149286, 0.0215203288873268),
  (0.005238607970971887, 0.01521523257342962),
  0.18682449282023306)]
```

$$((\beta_{1,min}, \beta_{1,max}), (\beta_{2,min}, \beta_{2,max}), (\sigma^2_{min}, \sigma^2_{max}), \hat{\sigma}_1 q^{T_{n-2}}_{1-\alpha/2})$$

5

In [ ]:

```
def plot_interval(X, t, plot_range,  beta_1_hat, beta_2_hat, epsilon_ci):
    plot_lower, plot_upper = plot_range[0], plot_range[-1]
    plt.plot(plot_range, beta_1_hat + beta_2_hat * plot_range)
    max_left = beta_1_hat + beta_2_hat * plot_lower + epsilon_ci
    min_left = beta_1_hat + beta_2_hat * plot_lower - epsilon_ci
    min_right = beta_1_hat + beta_2_hat * plot_upper - epsilon_ci
    max_right = beta_1_hat + beta_2_hat * plot_upper + epsilon_ci
    plt.fill_between([plot_lower, plot_upper],
        [max_left, max_right],
        [min_left, min_right],
        alpha=0.35,
    )
```

Pour changer les périodes pour lesquelles la droite de régression est visualisée, changer la variable "*plot_periods*"

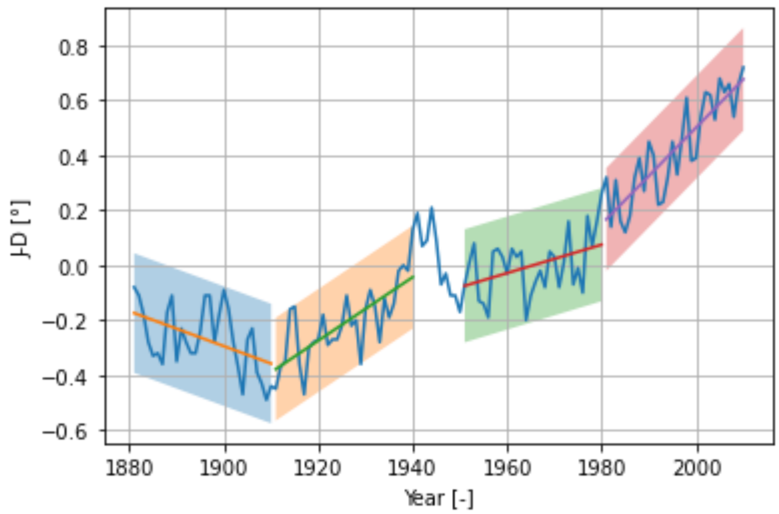
In [ ]:

```
plot_periods = [0, 3, 7, 10] # Change this value to change plotted periods [0, 10]

lost_pts = t[-1] - times[-1][-1]
plt.plot(t[:-lost_pts], X[:-lost_pts])
for time_interval in plot_periods:
    beta_1_hat, beta_2_hat, _, = estimators[time_interval]
    beta_1_interval, beta_2_interval, _, prediction_uncertainty = confidence_interval
    time = times[time_interval]
    rang = ranges[time_interval]

    plot_interval(
        X,
        t,
        time,
        beta_1_hat,
        beta_2_hat,
        prediction_uncertainty
    )

plt.xlabel("Year [-]")
plt.ylabel("J-D [°]")
plt.grid()
plt.show()
```



6

In [ ]:

```
# Calculating the p-value
def p_value(t, X, n):
    beta_1_hat, beta_2_hat, sigma_2_hat = calculate_estimators(t, X)
    sigma_2_j_hat = sigma_2_hat / ((t - t.mean()) @ (t - t.mean()))
    return 2 * (1 - student.cdf(np.abs(beta_2_hat) / np.sqrt(sigma_2_j_hat), n - 2))
```

In [ ]:

```
p_values = [
    p_value(t[r], X[r], n)
    for r in ranges
]
p_values
```

Out[ ]:

```
[0.008505785813796463,
 0.07047873226794543,
 0.025598638615577318,
 1.8893823596766168e-06,
 0.00017604854704034167,
 0.3784841191076409,
 0.16672427700696524,
 0.021350839327671567,
 2.9776871546260963e-07,
 1.5792273933001866e-08,
 6.776170735633968e-10]
```

7

Conclusion

Les p-valeurs indiquent que pendant plusieurs périodes de 30 ans le coefficient angulaire de la régression est différent de zéro (à niveau $\alpha = 0.05$). Notamment, pour toutes périodes s'amorçant après 1960 le coefficient β_2 est positif et les p-valeurs sont inférieures à 10^{-7}. Cela est indicatif d'un réchauffement de la planète pendant les dernières décennies qui correspondent à celles où les émissions de gaz à effet de serre ont été les plus élevées [1]. Cela dit, l'analyse réalisée ne permet pas d'établir un lien de causalité entre ces deux quantités (quoique cette relation ait été établie par d'autres études).

[1] - https://ourworldindata.org/co2-dataset-sources