Trabajo Práctico N° 3

Problema 1

a) $$G(x) = \sum_{k=0}^{} \binom{N-1}{k} p^k (1-p)^{N-1-k} x^k$$

$$= \sum_{k=0}^{} \binom{N-1}{k} (1-p)^{N-1-k} (px)^k$$

Esta fórmula es similar a la del binomio de Newton

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

con $n = N-1$, $a = 1-p$ y $b = px$

$$\Rightarrow G(x) = (1-p+px)^{N-1}$$

b) $$\langle k \rangle = x G'(x) \Big|_{x=1}$$

$$= x \frac{d}{dx} (1-p+px)^{N-1} \Big|_{x=1} = x (N-1)(1-p+px)^{N-2} p \Big|_{x=1}$$

$$= p(N-1)$$

b) Primero, notemos que $\langle k(k-1)\rangle = \langle k^2\rangle - \langle k\rangle$

$$\langle k(k-1)\rangle = \sum_{k=0} k(k-1)\, P(k)$$

$$= \sum_{k=0} k^2 P(k) - \sum_{k=0} k\, P(k) = \langle k^2\rangle - \langle k\rangle$$

Ahora debemos calcular $\langle k^2\rangle$

$$\langle k^2\rangle = \left(x\frac{d}{dx}\right)^2 G(x)\Big|_{x=1} = \left(x\frac{d}{dx}\right)\left(x\,G'(x)\right)\Big|_{x=1}$$

$$= \underbrace{x\,G'(x)\Big|_{x=1}}_{=\langle k\rangle} + x^2 G''(x)\Big|_{x=1}$$

Por lo tanto, $\langle k(k-1)\rangle = x^2 G''(x)\Big|_{x=1}$

$$= x^2 \frac{d^2}{dx^2}\left((1-p+px)^{N-1}\right)\Big|_{x=1}$$

$$= x^2 \frac{d}{dx}\left((1-p+px)^{N-2}(N-1)\,p\right)\Big|_{x=1}$$

$$= x^2 (N-2)(N-1)\,p^2\,(1-p+px)^{N-3}\Big|_{x=1}$$

$$= (N-2)(N-1)\,p^2$$

Repetimos los incisos para la distribución de Poisson

$$P_{(k)} = \frac{1}{k!} c^k e^{-c}$$

$$G(x) = \sum_{k=0}^{\infty} P_{(k)} x^k = \sum_{k=0}^{\infty} \frac{1}{k!} e^{-c} (cx)^k$$

$$= e^{-c} \sum_{k=0}^{\infty} \frac{1}{k!} (cx)^k = e^{-c} e^{cx} = e^{c(x-1)}$$

$$\langle k \rangle = x\, G'(x) \Big|_{x=1} = x\, e^{c(x-1)} c \Big|_{x=1} = c \checkmark$$

$$\langle k(k-1) \rangle = x^2 G''(x) \Big|_{x=1} = x^2 \frac{d}{dx} \left(e^{c(x-1)} c \right) \Big|_{x=1} = x^2 c^2 e^{c(x-1)} \Big|_{x=1}$$

$$= c^2 \checkmark$$