Guía 7

Problema 1

a) $$\begin{cases} \dfrac{d\rho_k}{dt} = -\rho_k + \lambda k (1-\rho_k)\, \Theta_{(t)} \\[2ex] \Theta_{(t)} = \dfrac{1}{\langle k \rangle} \displaystyle\sum_k k P_{(k)} \rho_k \end{cases}$$

$$\frac{d\Theta}{dt} = \frac{1}{\langle k \rangle} \sum_k k P_{(k)} \frac{d\rho_k}{dt} = \frac{1}{\langle k \rangle} \sum_k k P_{(k)} \left[-\rho_k + \lambda k (1-\rho_k)\, \Theta_{(t)} \right]$$

$$\frac{d\Theta}{dt} = -\sum_k k P_{(k)} \rho_k + \sum_k \lambda k^2 \Theta_{(t)} P_{(k)} - \sum_k P_{(k)} \lambda k^2 \rho_k$$

$$\frac{d\Theta}{dt} = -\Theta_{(t)} + \frac{\lambda}{\langle k \rangle} \langle k^2 \rangle \Theta_{(t)} - \underline{\lambda \Theta_{(t)} \sum k^2 P_{(k)} \rho_k}$$

Este término es de 2^{do} orden en ρ_k

$$\frac{d\Theta}{dt} \approx \left(\frac{\lambda \langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} \right) \Theta(t)$$

Esta ecuación diferencial tiene solución general de la forma

$$\boxed{\Theta_{(t)} = \Theta_0\, e^{t/\tau}}$$

donde en este caso $$\boxed{\tau = \frac{\langle k \rangle}{\lambda \langle k^2 \rangle - \langle k \rangle}}$$

(b) $\frac{d\rho_k}{dt} = -\rho_k + \lambda k (1-\rho_k)\,\theta(\lambda)$ → Asumo que conozco θ

$$\frac{d\rho_k}{dt} = \lambda k\,\theta(\lambda) - \rho_k(1+\lambda\theta(\lambda))$$

En el estado estacionario, $\frac{d\rho_k}{dt} = 0$

$$\Rightarrow \boxed{\rho_k = \frac{\lambda k\,\theta(\lambda)}{1+\lambda k\,\theta(\lambda)}}$$

(c)

$$\theta(\lambda) = \frac{1}{\langle k\rangle}\sum_k k\,P(k)\,\rho_k$$

Con la expresión de ρ_k hallada en (b) para el estado estacionario,

$$\boxed{\theta(\lambda) = \frac{1}{\langle k\rangle}\sum_k k\,P(k)\,\frac{k\lambda\,\theta(\lambda)}{1+k\lambda\,\theta(\lambda)}}$$

(d) La epidemia es endémica si el tiempo característico τ es positivo

$$\tau = \frac{\langle k\rangle}{\lambda\langle k^2\rangle - \langle k\rangle} > 0 \Rightarrow \lambda\langle k^2\rangle - \langle k\rangle > 0$$

$$\boxed{\lambda > \frac{\langle k\rangle}{\langle k^2\rangle}}$$

El valor crítico para este cambio de régimen es el valor de λ que hace que τ sea 0, es decir

$$\boxed{\lambda_c = \frac{\langle k\rangle}{\langle k^2\rangle}}$$

(e)

$$\theta = \frac{1}{\langle k \rangle} \sum_k \frac{k P_{(k)} k \lambda \theta}{1 + k\lambda\theta}$$

En la aproximación al continuo, y tomando $P_{(k)} = 2m k^{-3}$ (BA),

$$1 = \frac{\lambda}{\langle k \rangle} \int_m^\infty 2m^2 k^{-3} \frac{k^2\, dk}{1 + \lambda k \theta(\lambda)}$$

$$1 = m\lambda \int_m^\infty \frac{dk}{k(1 + \theta(\lambda)\lambda k)}$$

$$1 = m\cdot\lambda \ln\left[\frac{k}{1 + \lambda k \theta}\right]\Bigg|_m^\infty$$

$$1 = m\lambda \ln\left[\frac{1 + \lambda m \theta(\lambda)}{m\lambda\theta(\lambda)}\right]$$

$$e^{1/m\lambda} = \frac{1 + \lambda m \theta(\lambda)}{m\lambda\theta(\lambda)} = 1 + \frac{1}{m\lambda\theta(\lambda)}$$

$$\left(e^{1/m\lambda} - 1\right)^{-1} = m\lambda\,\theta(\lambda)$$

$$\boxed{\theta(\lambda) = \frac{1}{m\lambda} e^{-1/m\lambda}\left(1 - e^{-1/m\lambda}\right)^{-1}}$$

$$\rho = \int_m^\infty P_{(k)} \rho_k\, dk = \int_m^\infty 2m^2 k^{-3} \frac{\lambda k \theta(\lambda)}{1 + \lambda k \theta_{(\lambda)}}\, dk$$

$$\rho = \int_m^\infty 2m^2\lambda\theta \frac{dk}{k^2(1 + k\lambda\theta(\lambda))}$$

$$\rho = 2m^2\lambda\theta_{(\lambda)}\left[\lambda\theta \ln\left(\frac{1 + \lambda\theta k}{k}\right) - \frac{1}{k}\right]\Bigg|_m^\infty$$

$$\rho = 2m^2\lambda\theta\left[\underbrace{\lambda\theta \ln\left(\frac{m\lambda\theta}{1 + m\lambda\theta}\right)}_{O(\theta^2)} + \underbrace{\frac{1}{m}}_{O(\theta)}\right]$$

Despreciamos términos de 2do orden..

$$\rho \approx 2m\lambda\theta = 2m\lambda\left[\frac{1}{m\lambda}\, e^{-1/m\lambda}\, \underbrace{\left(1-e^{-1/m\lambda}\right)^{-1}}_{\approx 1}\right]$$

$$\boxed{\rho \approx 2e^{-1/m\lambda}}$$

Problema 2

(a) $g_k^c \overset{?}{=} 1 - \frac{1}{k\lambda}$

$$\frac{d\rho}{dt} = 0 \Rightarrow -\rho + \tilde{\lambda}(1-\rho)\,\theta(\lambda) \Rightarrow \rho = \frac{\tilde{\lambda}\,\theta(\lambda)}{\tilde{\lambda}\,\theta(\lambda)}$$

Por lo tanto,

$$\Theta(\lambda) = \frac{1}{\langle k\rangle}\sum_k \frac{\overbrace{k P(k)}^{\langle k\rangle}\,\tilde{\lambda}\,\theta(\lambda)}{1+\tilde{\lambda}\,\theta(\lambda)}$$

$$\Theta(\lambda) = \frac{\tilde{\lambda}\,\theta(\lambda)}{1+\tilde{\lambda}\,\theta(\lambda)}$$

Ahora pedimos que $\left.\frac{d\theta}{d\theta}\right|_{\theta\to 0} = 1$

$$1 = \frac{d}{d\theta}\left[\frac{\tilde{\lambda}\,\theta}{1+\tilde{\lambda}\,\theta}\right]$$

$$1 = \left.\frac{\tilde{\lambda}}{(1+\theta\tilde{\lambda})^2}\right|_{\theta\to 0}$$

$$\Rightarrow \tilde{\lambda} = 1 \Rightarrow \lambda_c(1-g_k^c)\,k = 1$$

$$\boxed{g_k^c = 1 - \frac{1}{k\lambda_c}}$$

(b) $$\rho_c = \sum_{k>\frac{1}{\lambda}}^{\infty} \left(1-\frac{1}{k\lambda}\right) P(k)$$

$$\rho_c \approx \int_{1/\lambda}^{\infty} \left(1-\frac{1}{k\lambda}\right) 2m^2 k^{-3}\, dk$$

$$\rho_c = \int_{1/\lambda}^{\infty} 2m^2 k^{-3}\, dk - \int_{1/\lambda}^{\infty} \frac{2m^2}{\lambda k^{-4}}$$

$$\boxed{\rho_c = m^2\lambda^2 - \frac{2}{3} m^2\lambda^2 = \frac{m^2\lambda^2}{3}}$$

Problema 3

(a) $$G(x) = \sum_k P_g(k)\, x^k = \sum_k \sum_{q\geq k} P(q) \binom{q}{k} (1-p)^k p^{q-k} x^k$$

$$= \sum_q \sum_{k=0}^{q} P(q) \binom{q}{k} (1-p)^k x^k p^{q-k}$$

$$= \sum_q P(q) \left[p + (1-p)x \right]^q$$ → Función generatriz de la distribución modificada

$$\langle k \rangle_g = x \frac{dG}{dx}\Big|_{x=1} = \sum_q (1-p)\, q\, P(q) = (1-p) \sum_q q\, P(q)$$

$$\boxed{\langle k \rangle_g = (1-p) \langle k \rangle_t}$$

$$\langle k^2 \rangle_g = \left(x\frac{d}{dx}\right)^2 G(x)\Big|_{x=1} = \left(\frac{x\, dG}{dx} + x^2 \frac{d^2G}{dx^2}\right)\Big|_{x=1}$$

$$\langle k^2 \rangle_g = \sum_q q P(q) (1-p) + \sum_q q(q-1) P(q) (1-p)^2$$

$$= (1-p)\langle k \rangle_t + (1-p)^2 \langle k^2 \rangle_t - (1-p)^2 \langle k \rangle_t$$

$$\boxed{\langle k^2 \rangle_t = (1-p)^2 \langle k^2 \rangle_t + p(1-p)\langle k \rangle_t}$$

(b) $\lambda_c = \dfrac{\langle k\rangle_g}{\langle k^2\rangle_g} \Rightarrow$ (ec. 10 de Pastor-Satorras 2002b)

expresando esto en función de $\langle k\rangle_t$ y $\langle k^2\rangle_t$

$$\lambda_c^{-1} = \frac{\langle k^2\rangle_g}{\langle k\rangle_g} = \frac{(1-p)^2\langle k^2\rangle_t + p(1-p)\langle k\rangle_t}{(1-p)\langle k\rangle_t}$$

$$\boxed{\lambda_c^{-1} = (1-p(g_c))\frac{\langle k^2\rangle_t}{\langle k\rangle_t} + p(g_c)}$$

(c) $$g = \int_{k_t}^{\infty} P(k)\,dk = \int_m^{\infty} P(k)\,dk - \int_m^{k_t} P(k)\,dk$$

$$g = 1 - \int_m^{k_t} P(k)\,dk$$

$$g = 1 - \int_m^{k_t} 2m^2 k^{-3}\,dk$$

$$= 1 - \left(1 - \frac{m}{k_t^2}\right) = \frac{m^2}{k_t^2},$$

$$\Rightarrow \boxed{k_t = m\, g^{-1/2}}$$

(d) $$p(g) = \frac{1}{\langle k\rangle}\int_{k_t}^{\infty} k\, 2m^2 k^{-3}\,dk$$

$$= \frac{1}{2m}\int_{m g^{-1/2}}^{\infty} 2m^2 k^{-2}\,dk$$

$$\boxed{p(g) = \frac{1}{\cancel{2m}}\left(\cancel{2}\sqrt{g}\,\cancel{m}\right) = \sqrt{g}}$$

(e) Pastor-Satorras 2002b nos da dos resultados para la red BA en la aprox. al continuo:

$$\checkmark \langle k \rangle_t = 2m$$

$$\checkmark \langle k^2 \rangle = 2m^2 \ln\left(\rho^{-1/2}\right) = -m^2 \ln(\rho)$$

Vamos a partir de la ecuación de λ,

$$\lambda_c^{-1} = \frac{(1-\rho)\langle k^2 \rangle_t}{\langle k \rangle_t} + \rho,$$

y asumir que $\rho << 1$

$$\Rightarrow \lambda_c^{-1} \approx \frac{\langle k^2 \rangle_t}{\langle k \rangle_t} = \frac{-\cancel{m^2} \ln(\rho)}{2\cancel{m}} = -\frac{m}{2} \ln(\rho)$$

$$\boxed{\rho_c = e^{-2/m\lambda}}$$