Higher Rank Series and Root Puzzles for Plumbed 3-Manifolds

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Goal Construct invariants of 3-manifolds

up to orientation preserving homeomorphisms

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Guiding Problem Categorify WRT invariants

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Recent Progress

graded Euler char.

$\left.\begin{array}{l}\text{[Gukov - Putrov - Vafa, 2017]} \\ \text{[Gukov - Pei - Putrov - Vafa, 2020]}\end{array}\right] \Rightarrow$ a physically defined homology theory $\Rightarrow$ $\boxed{\text{a Laurent series } \hat{Z}(q)}$

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① $\lim\limits_{q \to \text{root of } 1} \hat{Z}(q) \text{ "=" } \mathrm{WRT}$

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Gukov-Manolescu For negative definite plumbed 3-manifolds,

2021

$\hat{Z}(q)$ is an invariant.

Plumbings

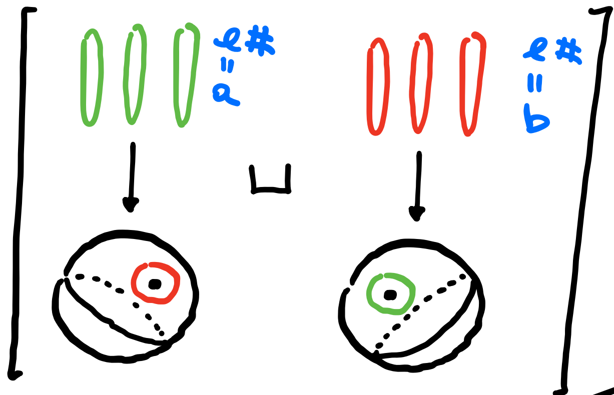
Plumbing Trees $\longrightarrow$ closed oriented 3-mfds

$\Gamma \longmapsto M(\Gamma)$

a —— b

framing matrix

$$B = \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}$$



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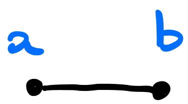

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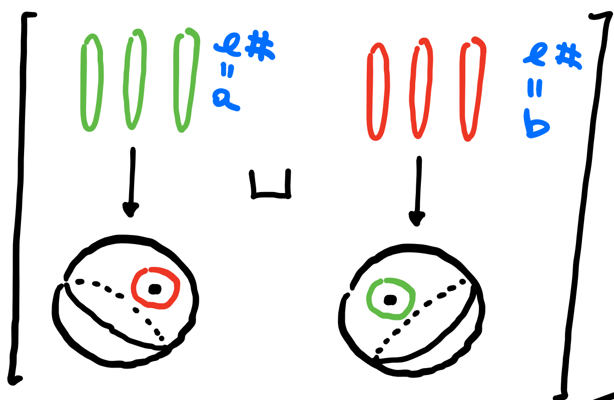

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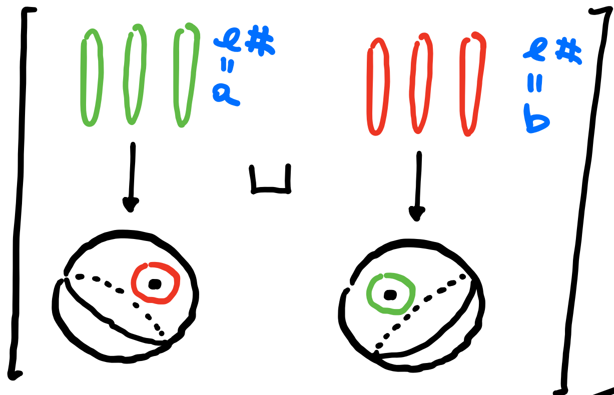

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Def. M a neg. def. plumbed 3-mfd

$\Longleftrightarrow M \cong M(\Gamma)$ for some plumbing tree Γ with B neg. def.

Neumann
1981

$M(\Gamma) \cong M(\Gamma')$ orientation preserving homeomorphism

$\Leftrightarrow$ the graphs Γ, Γ' are related by a series of moves.

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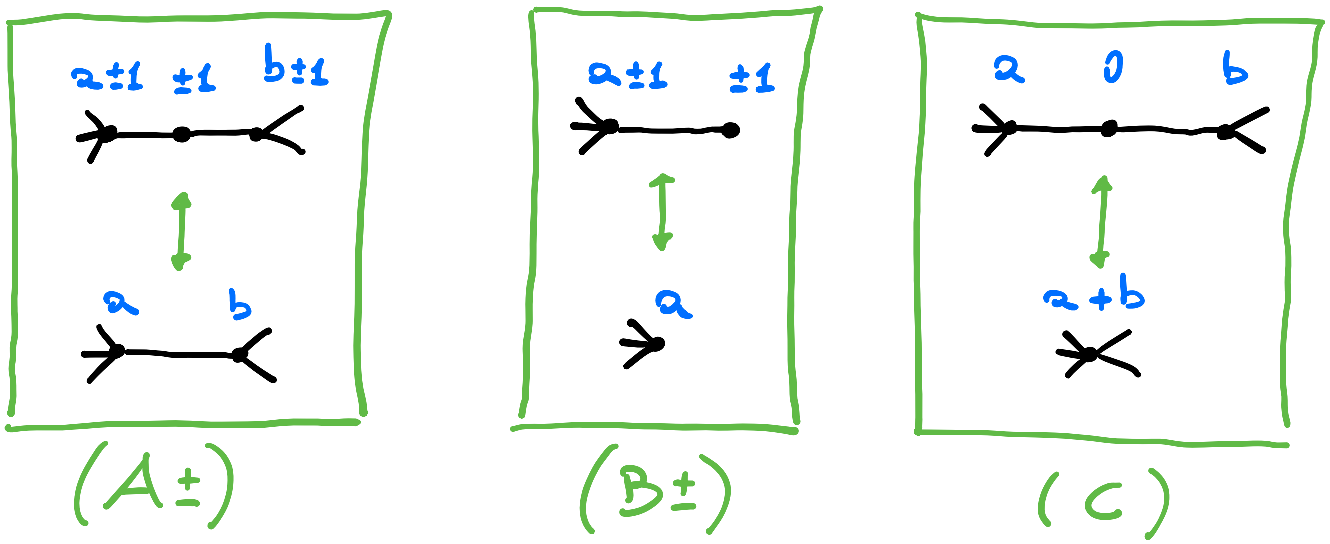

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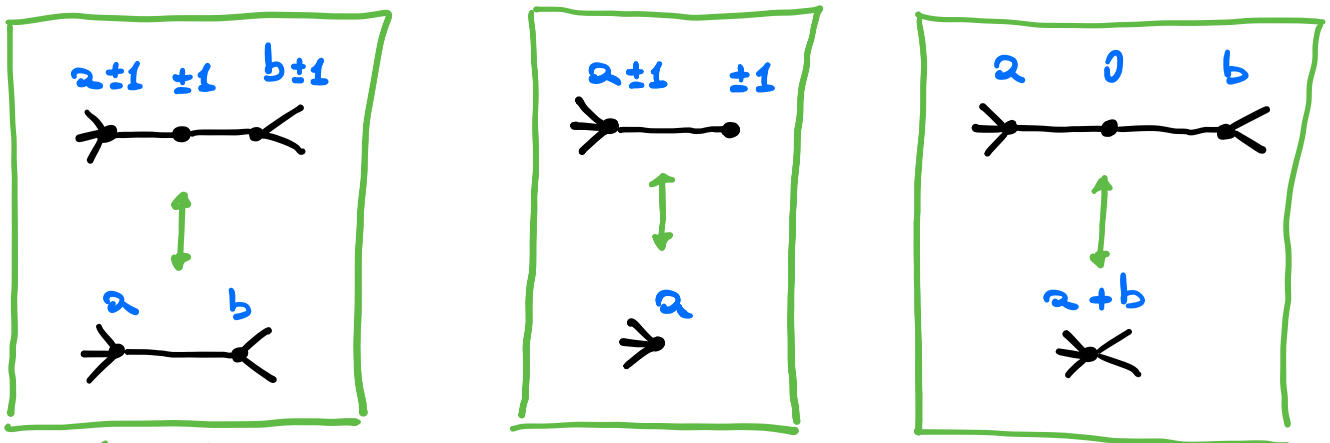

(A±) $\qquad$ (B±) $\qquad$ (C)

Moreover $\hat{Z}_s(q)$ depends on $s \in Spin^c(M) \cong_{aff.} H_1(M, \mathbb{Z})$

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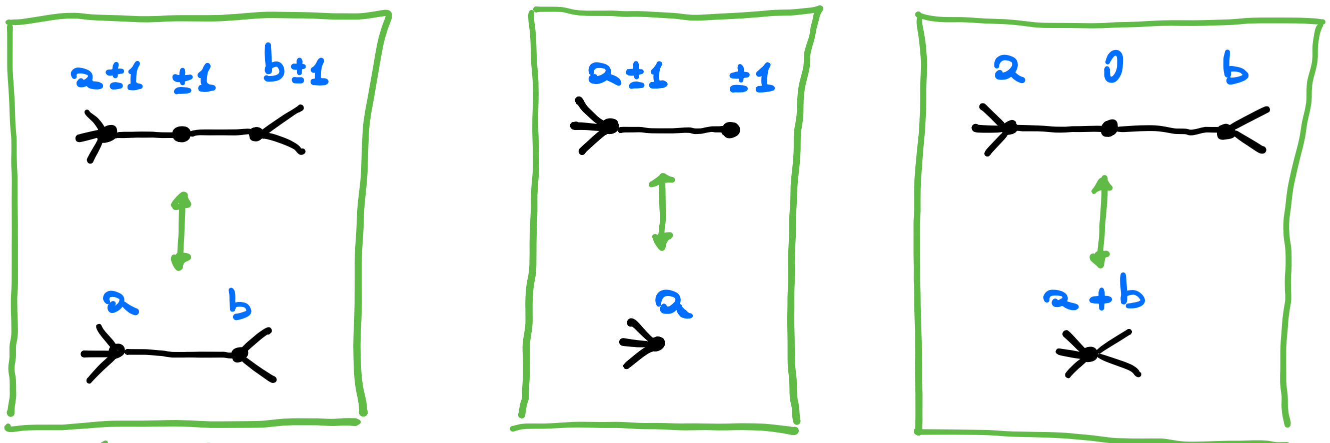

(A±) (B±) (C)

Theorem $\hat{Z}_s(q)$ depends on $s \in Spin^c(M) \cong_{aff.} H_1(M, \mathbb{Z})$

Gukov-Manolescu 2021 For M neg. def. and $s \in Spin^c(M)$,

$\hat{Z}_s(q)$ is:

1. invariant under the 5 Neumann moves
2. invariant up to $s \mapsto -s$.

Park 2020

$$(\Gamma, Q, s) \longrightarrow \hat{Z}_s(q)$$

Q: root lattice

s: generalised Spin^c-str.

$\hat{Z}_s(q)$: invariant series

$\boxed{\text{[GPPV]-[GM] is the case } Q = A_1}$

Park 2020

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Q: root lattice; $\mathfrak{s}$: generalised Spin^c-str.; $\hat{Z}_{\mathfrak{s}}(q)$: invariant series

[GPPV]-[GM] is the case $Q = A_1$

Akhmechet - Johnson - Krushkal 2023

$\hat{Z}(q)$ fits in an ∞ family of series invariant under moves (A-) and (B-).

$Q = A_1$ here

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root lattice — generalised $Spin^c$-str. — invariant series

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Thm 1 (Moore-T., 2024)

$$\underbrace{(\Gamma,\ Q,\ s,}_{\text{as in Park}}\ \underset{\text{root puzzle}}{P}) \longrightarrow Y_{P,s}(q)$$

$Y_{P,s}(q)$ is:
1. invariant under the 5 Neumann moves.
2. invariant up to action of Weyl gp on s.

Thm 2 (Moore-T.)

1. $Y_{P,s}(q) = \hat{Z}_s(q)$ for $Q = A_1$ (unique!)

2. $\exists\ \infty$ many inv. series $Y_{P,s}(q)$ for irred. Q of rank ≥ 2.

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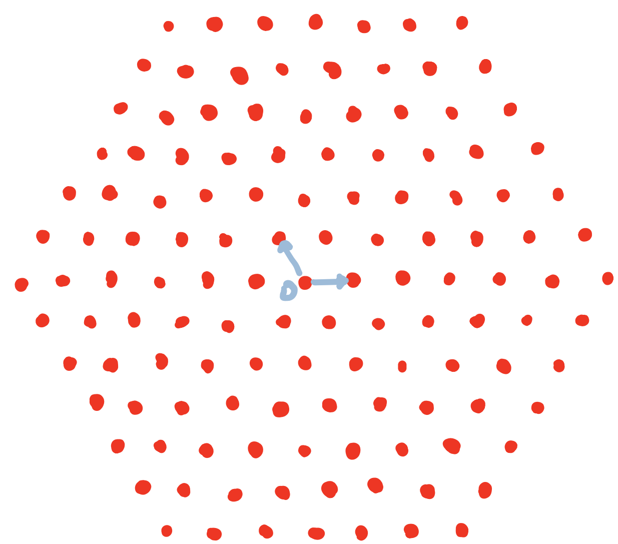
Ex Consider the root lattice A_2

= vertex arrangement of the tiling of the Euclidean plane by equilateral triangles



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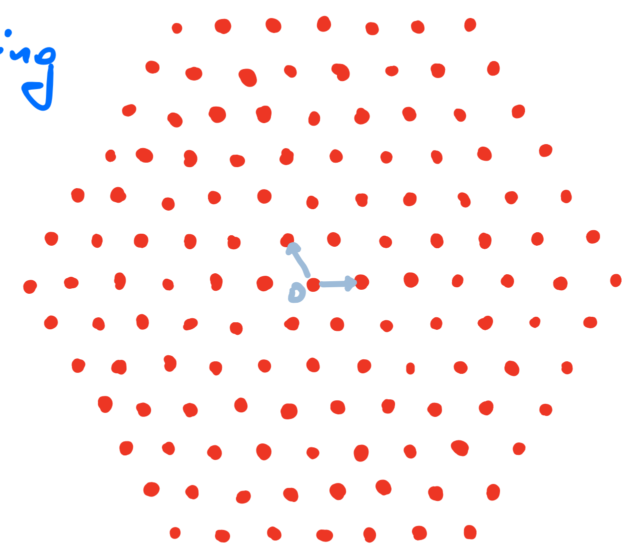
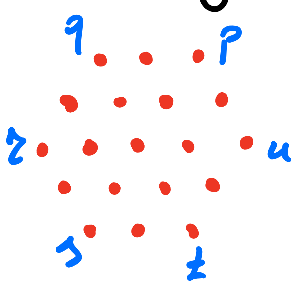
a fct
$$c : \Lambda A_2 \longrightarrow R \quad \text{a commutative ring}$$
$$\alpha \longmapsto c(\alpha)$$
s.t.:



① $\forall \alpha,\ c(n\alpha) = 0$ for $n >> 0$ or $n << 0$

② $\forall$ hexagon one has



$$-p+q-r+s-t+u = \begin{cases} 1 & \text{if hex. is centered at } 0 \\ 0 & \text{otherwise.} \end{cases}$$

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a fct $c : 2A_2 \longrightarrow R$ a commutative ring

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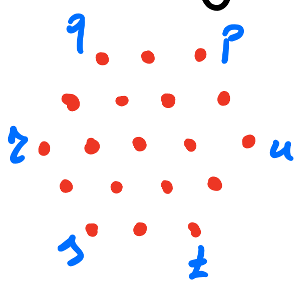

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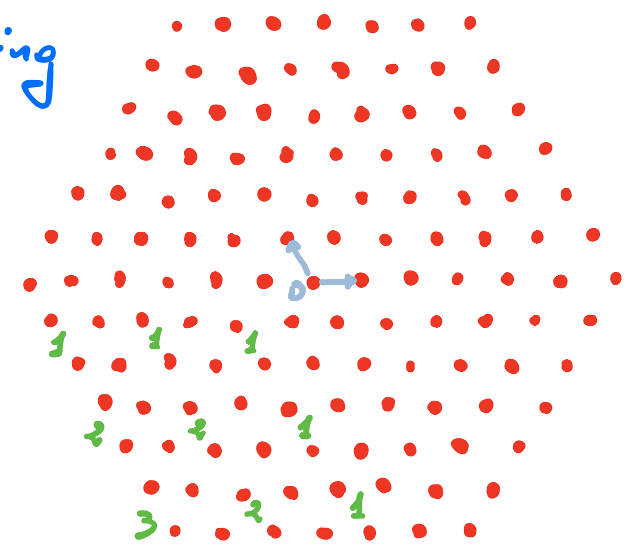

Kostant partition fct

Thm 2 (Moore–T.)

(1) $Y_{P,s}(q) = \hat{Z}_s(q)$ for $Q = A_1$ (unique!)

(2) $\exists$ ∞ many inv. series $Y_{P,s}(q)$ for irred. Q of rank ≥ 2.

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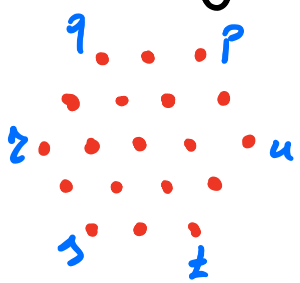

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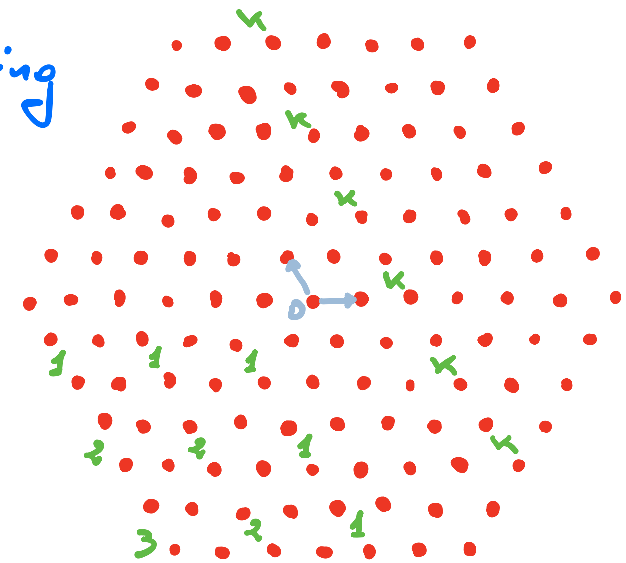

for arbitrary $\kappa \in R$

$\Rightarrow$ ∞'ly many variations!

The invariant series

$$\hat{Y}_{P,s}(q) := (-1)^{|\Delta^+|\pi} \, q^{\frac{1}{2}(3\sigma - \operatorname{tr} B)\langle \rho,\rho\rangle} \sum_{\ell \in s + 2B\mathbb{Z}^{V(\Gamma)} \otimes Q} c_\Gamma(\ell) \, q^{-\frac{1}{8}\langle \ell, \ell \rangle}$$

where

$$c_\Gamma(\ell) := \frac{1}{|\Xi|} \sum_{\xi \in \Xi} \prod_{v \in V(\Gamma)} \left[P^{\xi_v}_{\deg v}(z_v) \right]_{\ell_v}$$

for "reduced" Γ (a non-restrictive assumption)

with

$$P^x_n(z) := \begin{cases} \left(\sum_{w \in W} (-1)^{\ell(w)} z^{2w(\rho)} \right)^{2-n} & \text{if } n = 0,1,2 \\ \left((-1)^{\ell(x)} \sum_{\alpha \in 2\rho + 2Q} c(\alpha) \, z^{x(\alpha)} \right)^{n-2} & \text{if } n \geq 3 \end{cases}$$

for $x \in W$

$s \in (\deg v_1, \deg v_2, \ldots) \otimes 2\rho + 2\mathbb{Z}^{V(\Gamma)} \otimes_{\mathbb{Z}} Q$

Open Problems

* Define a homology theory with graded Euler char $= Y_{p,s}(q)$

* Extend $Y_{p,s}(q)$ to all 3-mfds