

Higher Rank Series and Root Puzzles

Palermo,
July 2024

for Plumbed 3-Manifolds J.W.W. Allison Moore
(VCU)

Goal Construct invariants of 3-manifolds ! arXiv:2405.14972
up to orientation preserving homeomorphisms

Guiding Problem Categorify WRT invariants

Recent Progress

$\left[\begin{array}{l} \text{Gukov-Putrov-Vafa, 2017} \\ \text{Gukov-Pei-Putrov-Vafa, 2020} \end{array} \right] \Rightarrow$ a physically defined
homology theory
 \Downarrow graded Euler char.

Expected Properties

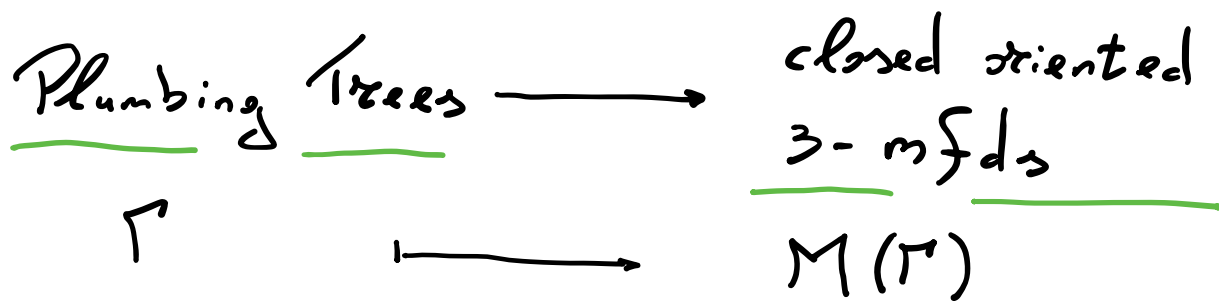
a Laurent series $\hat{Z}(q)$

① $\lim_{q \rightarrow \text{root of } 1} \hat{Z}(q) = \text{WRT}$

② $\hat{Z}(q)$ is a quantum modular form, à la Zagier

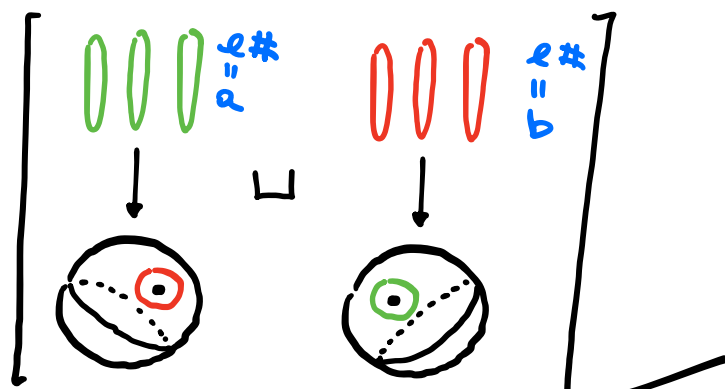
Gukov-Moore For negative definite plumbed 3-manifolds,
2021 $\hat{Z}(q)$ is an invariant.

Plumbing



$\begin{matrix} a & b \\ \bullet & \text{---} & \bullet \end{matrix}$
 \hookrightarrow framing matrix

$$B = \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}$$



Idea $M = \mathcal{O}X$ for a 4-mf X
 and B represents the intersection form
 on $H_2(X, \mathbb{Z})$

$$\begin{aligned}
 S^1 \times S^1 &\cong S^1 \times S^1 \\
 (x, y) &\mapsto (y, x)
 \end{aligned}$$

Def. ① Γ (weakly) neg. def.

$$\Leftrightarrow \left\{ \begin{array}{l} B \text{ invertible and} \\ B^{-1} \text{ neg. def. (on subspace spanned} \\ \text{by vertices of } \Gamma \text{ of deg } \geq 3) \end{array} \right.$$

② M a neg. def. plumbed 3-mf

$$\Leftrightarrow M = M(\Gamma) \text{ for some w. neg. def. plumbing tree } \Gamma$$

Neumann

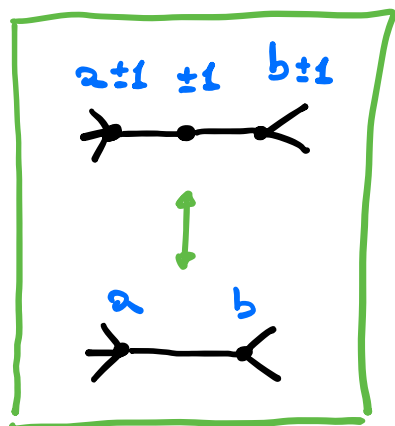
1981

two arbitrary plumbed 3-mfds

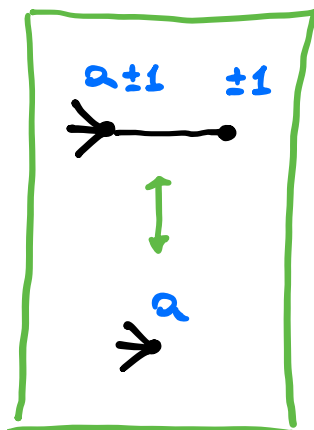
$$M(\Gamma) \cong M(\Gamma') \quad \text{orientation preserving homeomorphism}$$

\Leftrightarrow the graphs Γ, Γ' are related by a series of moves.

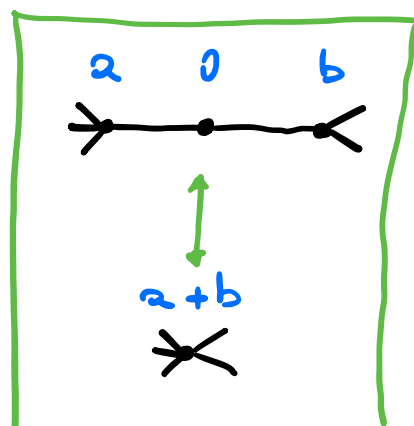
Rmk The 5 moves between plumbing trees are:



(A \pm)



(B \pm)



(C)

Strategy Proving invariance = Checking invariance under the Neumann moves

However $\hat{\mathbb{Z}}_{\rightarrow}(q)$ depends on $\rightarrow \in \text{Spin}^c(M) \cong_{\text{aff.}} H_2(M, \mathbb{Z})$

Gukov-Manolescu For M neg. def. and $\rightarrow \in \text{Spin}^c(M)$,

- 2021 $\hat{\mathbb{Z}}_{\rightarrow}(q)$ is:
- ① invariant under the 5 Neumann moves
 - ② invariant up to $\rightarrow \mapsto -\rightarrow$.

Alex Machot - Johnson - Krushkal

2023

For neg. def. Γ .

$\hat{Z}(q)$ fits in an ∞ family of invariant series

Note For neg. def. Γ , one needs only (A-) and (B-)

Parke $(\Gamma, Q, \rightsquigarrow) \longrightarrow \hat{Z}_\rightsquigarrow(q)$
2020 w. neg. def. root lattice generalised Spin^c-str. invariant series

E* [GPPV] - [GM] is the case $Q = A_1$.

Q For a w. neg. def. Γ and an arbitrary root lattice Q , does $\hat{Z}_\rightsquigarrow(q)$ fit in a larger family of inv. series?

Thm 1 (Moore-T., 2024)

$(\Gamma, Q, \rightsquigarrow, P) \longmapsto \gamma_{P, \rightsquigarrow}(q)$
as in Parke root puzzle

$\gamma_{P, \rightsquigarrow}(q)$ is:
(1) invariant under the 5 Neumann moves.
(2) invariant up to action of Weyl gp on \rightsquigarrow .

Thm 2 (Moore-T.)

- ① $\gamma_{p,n}(q) = \hat{Z}_n(q)$ for $Q = A_1$ (unique!)
- ② \exists ∞ many inv. series $\gamma_{p,n}(q)$
for irred. Q of rank ≥ 2 .

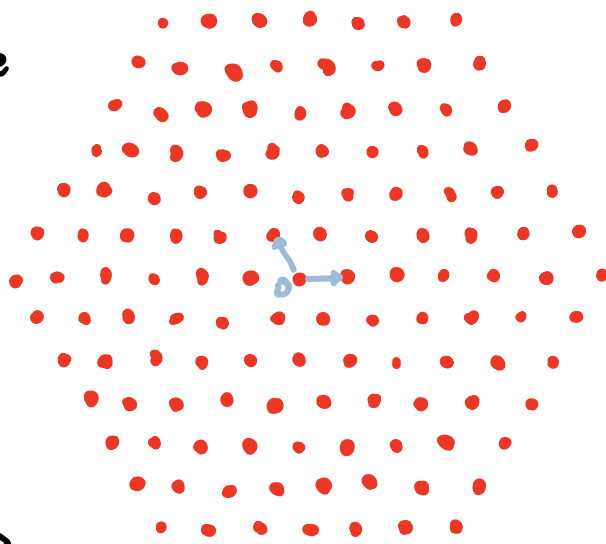
These new invariant series depend on what we call a

Root Puzzle

||

solution of a combinatorial problem on the root lattice

Ex Consider the root lattice A_2
= vertex arrangement
of the tiling
of the Euclidean plane
by equilateral triangles



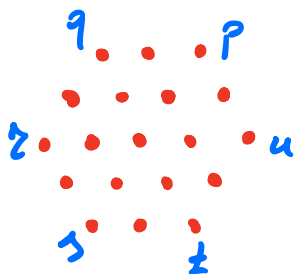
A root puzzle for A_2 is ...

... a fct $c: {}_2A_2 \longrightarrow \mathbb{R}$ a commutative ring
 $\alpha \longmapsto c(\alpha)$

such that:

① $\forall \alpha, \quad c(n\alpha) = 0$ for $n \gg 0$ or $n \ll 0$

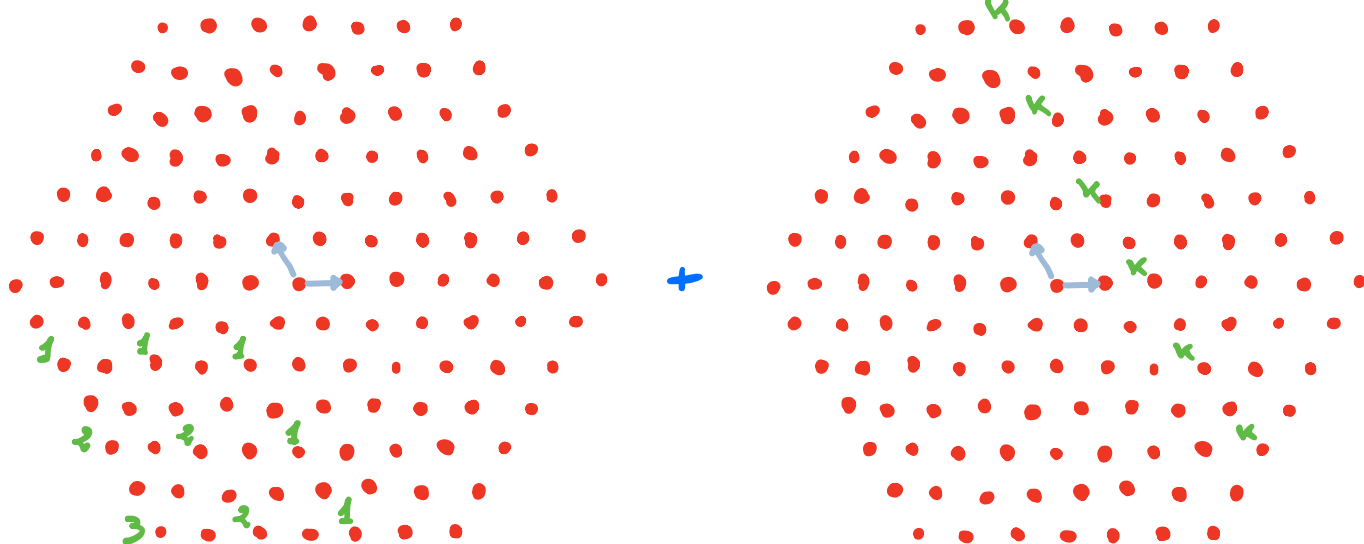
② \forall hexagon



one has

$$-p + q - r + s - t + u = \begin{cases} 1 & \text{if hex. is centered at } 0 \\ 0 & \text{otherwise.} \end{cases}$$

E_*



cfr. Kostant partition fct

for arbitrary $x \in \mathbb{R}$

\Rightarrow ∞ 'ly many variations!

The invariant series

For

- ① a reduced w. neg. def. plumbing tree Γ
- ② a root lattice Q ,
- ③ a gen. Spin-str. $\rightarrow \in (\deg \tau_1, \deg \tau_2, \dots) \otimes_{\mathbb{Z}} \mathbb{Z}^{V(\Gamma)} \otimes_{\mathbb{Z}} Q$
- ④ and a root puzzle $c: \mathbb{Z} \otimes Q \longrightarrow \mathbb{R}$

The series:

$$\chi_{p, \gamma}(q) := (-1)^{|\Delta^+|/\pi} q^{\frac{1}{2}(30 - t_2 B) \langle p, p \rangle} \sum_{\ell \in \gamma + \mathbb{Z} B \mathbb{Z}^{V(\Gamma)} \otimes Q} c_{\Gamma}(\ell) q^{-\frac{1}{8} \langle \ell, \ell \rangle}$$

where

$$c_{\Gamma}(\ell) := \frac{1}{|\square|} \sum_{\xi \in \square} \prod_{j \in V(\Gamma)} \left[P_{\deg j}^{\xi_j}(\mathbb{Z}_j) \right]_{\ell_j}$$

with

$$P_n^x(\mathbb{Z}) := \begin{cases} \left(\sum_{w \in W} (-1)^{\ell(w)} \mathbb{Z}^{x(w)} \right)^{2-n} & \text{if } n=0,1,2 \\ \left((-1)^{\ell(x)} \sum_{\alpha \in \mathbb{Z} \otimes Q} c(\alpha) \mathbb{Z}^{x(\alpha)} \right)^{n-2} & \text{if } n \geq 3. \end{cases}$$

for $x \in W$

Open Problems

- * Define a homology theory
with graded Euler char = $\chi_{\text{P.L.}}(q)$
- * Extend $\chi_{\text{P.L.}}(q)$ to all 3-mflds