Higher Rank Series and Root Puzzles for Plumbed 3-Manifolds

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Goal Construct invariants of 3-manifolds up to orientation preserving homeomorphisms

Guiding Problem Categorify WRT invariants

Recent Progress

[Gukov-Putrov-Vafa, 2017]
[Gukov-Pei-Putrov-Vafa, 2020] $\Rightarrow$ a physically defined homology theory

$\Downarrow$ graded Euler char.

$\boxed{\text{a Laurent series } \hat{Z}(q)}$

Expected Properties

1. $\lim_{q \to \text{root of } 1} \hat{Z}(q)$ "=" WRT

2. $\hat{Z}(q)$ is a quantum modular form, à la Zagier

Gukov-Manolescu 2021 For negative definite plumbed 3-manifolds, $\hat{Z}(q)$ is an invariant.

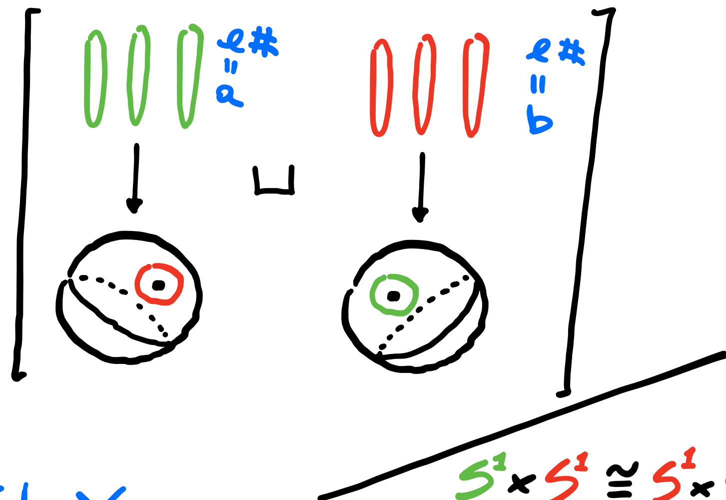
Plumbings

Plumbing Trees $\longrightarrow$ closed oriented 3-mfds

$\Gamma \longmapsto M(\Gamma)$

framing matrix

$B = \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}$



$S^1 \times S^1 \cong S^1 \times S^1$

$(x,y) \mapsto (y,x)$

<u>Idea</u> $M = \partial X$ for a 4-mfd X

and B represent the intersection form

on $H_2(X, \mathbb{Z})$

<u>Def.</u> ① Γ (weakly) neg. def.

$\Longleftrightarrow$ $\begin{cases} B \text{ invertible and} \\ B^{-1} \text{ neg. def. (on subspace spanned} \\ \text{by vertices of } \Gamma \text{ of deg} \geq 3) \end{cases}$

② M a neg. def. plumbed 3-mfd

$\Longleftrightarrow$ $M = M(\Gamma)$ for some w. neg. def. plumbing tree Γ

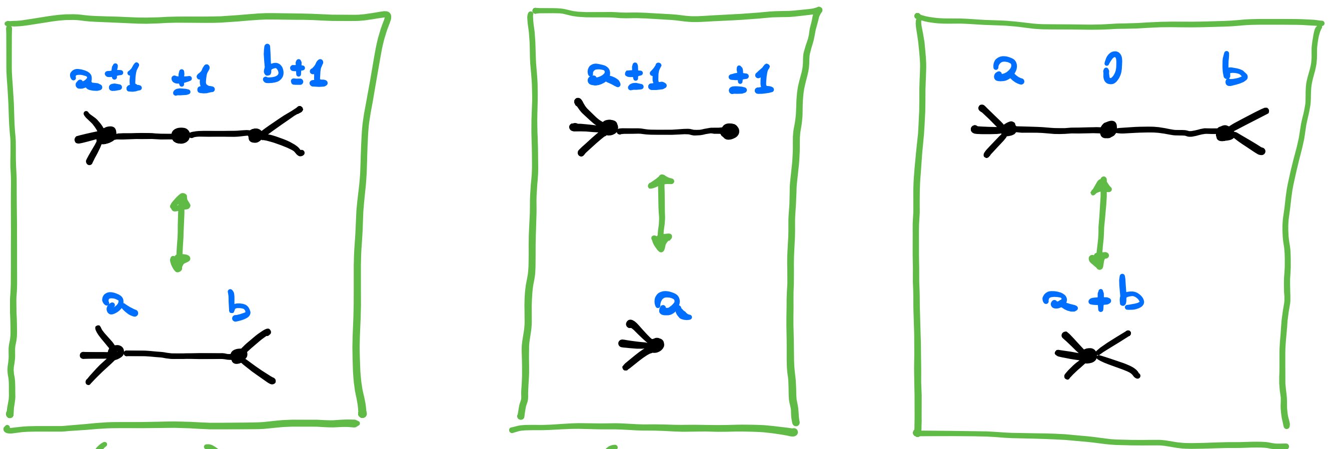
Neumann 1981

two arbitrary plumbed 3-mfds

$$M(\Gamma) \cong M(\Gamma')$$ orientation preserving homeomorphism

$\Leftrightarrow$ the graphs Γ, Γ' are related by a series of moves.

Rmk The 5 moves between plumbing trees are:



(A±) (B±) (C)

Strategy Proving invariance = Checking invariance under the Neumann moves

Moreover $\hat{Z}_s(q)$ depends on $s \in Spin^c(M) \cong_{aff.} H_1(M, \mathbb{Z})$

Gukov-Manolescu 2021

For M neg. def. and $s \in Spin^c(M)$, $\hat{Z}_s(q)$ is:

1. invariant under the 5 Neumann moves
2. invariant up to $s \mapsto -s$.

Akhmechet - Johnson - Krushkal

2023 For neg. def. Γ,

$\hat{Z}(q)$ fits in an ∞ family of invariant series

Note For neg. def. Γ, one needs only (A-) and (B-)

Park $(\Gamma, Q, s) \longrightarrow \hat{Z}_s(q)$

2020 w. neg. def. — root lattice — generalized Spin^c-str. — invariant series

Ex [GPPV] - [GM] is the case $Q = A_1$.

Q For a w. neg. def. Γ and an arbitrary root lattice Q, does $\hat{Z}_s(q)$ fit in a larger family of inv. series?

Thm 1 (Moore - T., 2024)

$(\Gamma, Q, s, P) \longrightarrow Y_{P,s}(q)$

as in Park — root puzzle

$Y_{P,s}(q)$ is:

1. invariant under the 5 Neumann moves.
2. invariant up to action of Weyl gp on s.

Thm 2 (Moore - T.)

① $Y_{p,s}(q) = \hat{Z}_s(q)$ for $Q = A_1$ (unique!)

② $\exists$ ∞ many inv. series $Y_{p,s}(q)$ for irred. Q of rank ≥ 2.

These new invariant series depend on what we call a

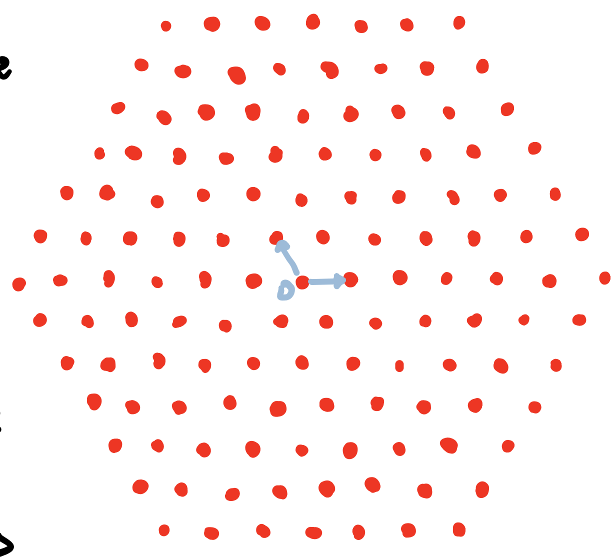
$\boxed{\text{Root Puzzle}}$

$\|$

$\boxed{\text{solution of a combinatorial problem on the root lattice}}$

Ex Consider the root lattice A_2

= vertex arrangement of the tiling of the Euclidean plane by equilateral triangles



A root puzzle for A_2 is ...

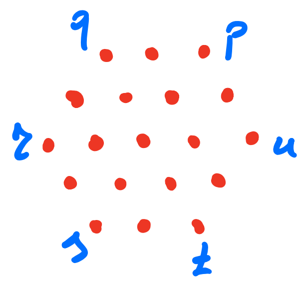
... a fct $c: \mathbb{Z}A_2 \longrightarrow R$ a commutative ring

$$\alpha \longmapsto c(\alpha)$$

such that:

1. $\forall \alpha$, $c(n\alpha)=0$ for $n \gg 0$ or $n \ll 0$

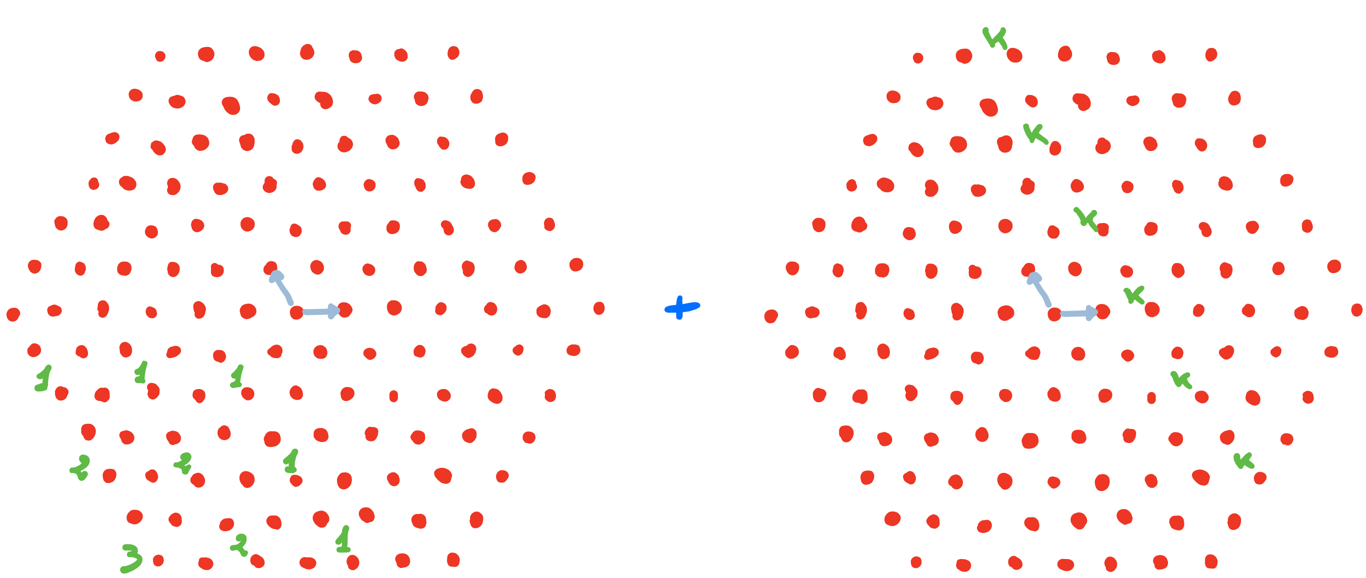
2. $\forall$ hexagon



one has

$$-p+q-r+s-t+u = \begin{cases} 1 & \text{if hex. is centered at } 0 \\ 0 & \text{otherwise.} \end{cases}$$

Ex



cfr. Kostant partition fct

for arbitrary $\kappa \in R$

$\Rightarrow$ ∞'ly many variations!

The invariant series

For

1. a reduced w. neg. def. plumbing tree Γ
2. a root lattice Q,
3. a gen. Spin^c-str. $s \in (\deg v_1, \deg v_2, \dots) \otimes 2\rho + 2\mathbb{Z}^{V(\Gamma)} \otimes_{\mathbb{Z}} Q$
4. and a root puzzle $c : 2\rho + 2Q \longrightarrow \mathbb{R}$

The series:

$$Y_{\Gamma,s}(q) := (-1)^{|\Delta^+|\pi} q^{\frac{1}{2}(3\sigma - \mathrm{tr} B)\langle\rho,\rho\rangle} \sum_{\ell \in s + 2B\mathbb{Z}^{V(\Gamma)} \otimes Q} c_\Gamma(\ell)\, q^{-\frac{1}{8}\langle \ell, \ell\rangle}$$

where

$$c_\Gamma(\ell) := \frac{1}{|W|} \sum_{\xi \in W} \prod_{v \in V(\Gamma)} \left[P^{\xi_v}_{\deg v}(z_v) \right]_{\ell_v}$$

with

$$P^x_n(z) := \begin{cases} \left(\sum_{w \in W} (-1)^{\ell(w)} z^{2w(\rho)} \right)^{2-n} & \text{if } n = 0,1,2 \\ \left((-1)^{\ell(x)} \sum_{\alpha \in 2\rho + 2Q} c(\alpha)\, z^{x(\alpha)} \right)^{n-2} & \text{if } n \geq 3. \end{cases}$$

for $x \in W$

Open Problems

* Define a homology theory with graded Euler char $= Y_{p,s}(q)$

* Extend $Y_{p,s}(q)$ to all 3-mfds