

COINVARIANTS OF METAPLECTIC REPRESENTATIONS ON MODULI OF ABELIAN VARIETIES

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ABSTRACT. We construct spaces of coinvariants at principally polarized abelian varieties with respect to the action of an infinite-dimensional Lie algebra. We show how these spaces globalize to twisted \mathcal{D} -modules on moduli of principally polarized abelian varieties, and we determine the Atiyah algebra of a line bundle acting on them. We prove analogous results on the universal abelian variety. An essential aspect of our study involves analyzing the Atiyah algebra of the Hodge and canonical line bundles on moduli of abelian varieties and the universal abelian variety.

INTRODUCTION

Spaces of coinvariants have classically been constructed by assigning representations of affine Lie algebras, and more generally, vertex operator algebras, to pointed algebraic curves [TUY, FBZ]. These spaces globalize to quasi-coherent sheaves equipped with a twisted \mathcal{D} -module structure on moduli of curves. Said structure is induced by an equivariant action of the Virasoro algebra on vertex operator algebra modules and on moduli of curves with marked points and formal coordinates. The construction generalizes to produce twisted \mathcal{D} -modules on moduli spaces parametrizing pointed curves with extra features: line bundles, principal bundles, and \mathcal{D} -bundles [FS, FBZ, BZN].

Removing curves out of the picture, Arbarello–De Concini [ADC] constructed a moduli space $\widehat{\mathcal{A}}_g$ of suitable extensions of principally polarized abelian varieties which includes, via an extended Torelli map, the moduli space $\widehat{\mathcal{M}}_g$ of algebraic curves with a marked point and a formal coordinate. They showed that an infinite-dimensional symplectic algebra $\mathfrak{sp}(H')$ acts transitively on $\widehat{\mathcal{A}}_g$, extending the transitive action of the Witt algebra on $\widehat{\mathcal{M}}_g$. Here, H' is the vector space of Laurent series in a formal variable with zero constant term and carries a symplectic form induced from the infinite-dimensional Heisenberg algebra. The oscillator representation which injects the Witt algebra into $\mathfrak{sp}(H')$ thus appears as the differential of the extended Torelli map.

The present project started from the realization that the results of [ADC] offer an opening for an extension of the classical conformal field theory from moduli spaces of curves to moduli spaces of abelian varieties. Removing curves out of the construction of coinvariants, we thus build and study twisted \mathcal{D} -modules on moduli spaces of abelian varieties.

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Our construction proceeds as follows. The symplectic algebra $\mathfrak{sp}(H')$ admits a unique (up to isomorphism) non-trivial central extension, called the *metaplectic algebra* $\mathfrak{mp}(H')$ (this is reviewed in §1.4 and the uniqueness follows from Proposition 5.1). For a representation V of $\mathfrak{mp}(H')$, we define the space of coinvariants of V at a point $a \in \widehat{\mathcal{A}}_g$ as

$$(0.1) \quad \widehat{\mathbb{V}}(V)_a := V / \mathfrak{sp}_F(H') \cdot V,$$

where $\mathfrak{sp}_F(H')$ is the stabilizer of $\mathfrak{sp}(H')$ at a —see (2.9)—and the action of $\mathfrak{sp}_F(H')$ on V is given by a splitting $\mathfrak{sp}_F(H') \subset \mathfrak{mp}(H')$ —see Proposition 5.3. The space (0.1) is the largest quotient of V on which $\mathfrak{sp}_F(H')$ acts trivially. We show that the spaces (0.1) globalize to a sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{A}}_g$ equipped with a twisted \mathcal{D} -module structure (Theorem 8.1). Furthermore, in case V is an admissible $\mathfrak{mp}(H')$ representation, as per Definition 7.1, we show that the sheaf $\widehat{\mathbb{V}}(V)$ descends to the moduli space \mathcal{A}_g of principally polarized, g -dimensional abelian varieties:

Theorem 1. *For an admissible $\mathfrak{mp}(H')$ representation V , the spaces of coinvariants (0.1) give rise to a twisted \mathcal{D} -module $\mathbb{V}(V)$ on \mathcal{A}_g .*

For instance, the theorem applies to Heisenberg vertex algebras of arbitrary rank and even lattice vertex algebras (Lemma 7.3). The metaplectic algebra $\mathfrak{mp}(H')$ acts on $\widehat{\mathcal{A}}_g$ via the projection $\mathfrak{mp}(H') \rightarrow \mathfrak{sp}(H')$ and replaces the Virasoro algebra of the classical conformal field theory. The action of $\mathfrak{mp}(H')$ on V induces the twisted \mathcal{D} -module structure on $\mathbb{V}(V)$. To explicitly determine it, we use the formalism of Atiyah algebras from Beilinson–Schechtman [BS]. Let Λ be the Hodge line bundle on \mathcal{A}_g , and \mathcal{F}_Λ the Atiyah algebra of Λ , i.e., the sheaf of first-order differential operators acting on Λ . We say that an admissible $\mathfrak{mp}(H')$ representation V is of central charge c if the action of $\mathfrak{mp}(H')$ induces an action of the Virasoro algebra on V of central charge c —see Definition 7.2.

Theorem 2. *For an admissible $\mathfrak{mp}(H')$ representation V of central charge c , the sheaf $\mathbb{V}(V)$ on \mathcal{A}_g carries an action of the Atiyah algebra $\frac{c}{2}\mathcal{F}_\Lambda$. This action induces the twisted \mathcal{D} -module structure on $\mathbb{V}(V)$.*

Moreover, we extend these results over the universal abelian variety. It is shown in [ADC] that $\widehat{\mathcal{A}}_g$ admits a universal family $\widehat{\mathcal{X}}_g \rightarrow \widehat{\mathcal{A}}_g$. This extends the universal family \mathcal{X}_g over the stack \mathcal{A}_g . The space $\widehat{\mathcal{X}}_g$ carries a transitive action of the Lie algebra $\mathfrak{sp}(H') \ltimes H'$. The terms of degree at most two in the Weyl algebra $\widetilde{\mathcal{U}}(H)$ give a non-trivial central extension $\widetilde{\mathcal{U}}_2(H)$ of $\mathfrak{sp}(H') \ltimes H'$ (§1.5). For a representation V of $\widetilde{\mathcal{U}}_2(H)$, we define the space of coinvariants of V at $x \in \widehat{\mathcal{X}}_g$ as

$$(0.2) \quad \widehat{\mathbb{V}}(V)_x := V / (\mathfrak{sp}_F(H') \ltimes F) \cdot V,$$

where $\mathfrak{sp}_F(H') \ltimes F$ is the stabilizer of $\mathfrak{sp}(H') \ltimes H'$ at x . We show that the spaces (0.2) globalize to a twisted \mathcal{D} -module on $\widehat{\mathcal{X}}_g$ (Theorem 8.2). In case V is an admissible $\widetilde{\mathcal{U}}_2(H)$ representation, as per Definition 7.1, we show:

Theorem 3. *For an admissible $\widetilde{\mathcal{U}}_2(H)$ representation V , the spaces of coinvariants (0.2) give rise to a twisted \mathcal{D} -module $\mathbb{V}(V)$ on \mathcal{X}_g .*

The twisted \mathcal{D} -module structure is explicitly determined by the action of an appropriate multiple of the Atiyah algebra \mathcal{F}_Ξ of the canonical line bundle Ξ on \mathcal{X}_g (see §2.7):

Theorem 4. *For an admissible $\widetilde{\mathcal{U}}_2(H)$ representation V of central charge c , the sheaf $\mathbb{V}(V)$ on \mathcal{X}_g carries an action of the Atiyah algebra $-\frac{c}{2}\mathcal{F}_\Xi$. This action induces the twisted \mathcal{D} -module structure on $\mathbb{V}(V)$.*

To prove the above statements, we need to show some auxiliary results, which are of independent interest. It is shown in [ADCKP] that there exist canonical homomorphisms

$$\begin{aligned} \bar{\nu}: H^2(\text{Witt}, \mathbb{C}) &\rightarrow H^2(\mathcal{M}_g, \mathbb{C}), \\ \bar{\mu}: H^2(\text{Witt} \ltimes H', \mathbb{C}) &\rightarrow H^2(\mathcal{P}'_{g-1}, \mathbb{C}), \end{aligned}$$

with ν being an isomorphism for $g \geq 3$ [ADCKP, (4.14)], and μ an isomorphism for $g \geq 5$ [ADCKP, (4.15)]. Here \mathcal{M}_g is the moduli space of smooth genus g curves, and \mathcal{P}'_{g-1} is the moduli space of quadruples (C, P, v, \mathcal{L}) , where C is a smooth genus g curve, P is a point of C , v is a nonzero tangent vector to C at P , and \mathcal{L} is a degree $g-1$ line bundle on C . We show that these maps extend to give:

Theorem 5. *There exist canonical homomorphisms*

$$\begin{aligned} \nu: H^2(\mathfrak{sp}(H'), \mathbb{C}) &\rightarrow H^2(\mathcal{A}_g, \mathbb{C}), \\ \mu: H^2(\mathfrak{sp}(H') \ltimes H', \mathbb{C}) &\rightarrow H^2(\mathcal{X}_g, \mathbb{C}), \end{aligned}$$

with ν being an isomorphism for $g \geq 3$, and μ an isomorphism for $g \geq 5$.

As an immediate consequence, we obtain the following result. For simplicity, we omit the notation for the pull-back when referring to the pull-back of the line bundles Λ and Ξ via the natural projections $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$ and $\widehat{\mathcal{X}}_g \rightarrow \mathcal{X}_g$.

Theorem 6. *(i) For $g \geq 3$, the line bundle Λ on $\widehat{\mathcal{A}}_g$ carries an action of $\mathfrak{mp}(H')$ by first-order differential operators extending the transitive action of $\mathfrak{sp}(H')$ on $\widehat{\mathcal{A}}_g$ and with the central element $\mathbf{1} \in \mathfrak{mp}(H')$ acting as multiplication by 2 on the fibers of $\Lambda \rightarrow \widehat{\mathcal{A}}_g$.*

(ii) For $g \geq 5$, the line bundle Ξ on $\widehat{\mathcal{X}}_g$ carries an action of $\widetilde{\mathcal{U}}_2(H)$ by first-order differential operators extending the transitive action of $\mathfrak{sp}(H') \ltimes H'$ on $\widehat{\mathcal{X}}_g$ and with the central element $\mathbf{1} \in \widetilde{\mathcal{U}}_2(H)$ acting as multiplication by -2 on the fibers of $\Xi \rightarrow \widehat{\mathcal{X}}_g$.

Moreover, we study group (ind)-schemes of symplectic automorphisms in §3, generalizing the group (ind)-schemes of ring automorphisms from [BD, FBZ]. Our constructions of the sheaves of coinvariants on \mathcal{A}_g and \mathcal{X}_g are thus variations on the formalism of localization of modules over Harish-Chandra pairs (§4).

The present results are intended to initiate the study of coinvariants on abelian varieties and their moduli. We mention some natural directions in §10 and will return to these in the near future.

The paper is structured as follows. We review the algebraic and geometric background in §§1 and 2, respectively. In §3 we define and study a group scheme which acts transitively on the fibers of the natural map $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$, together with related group (ind-)schemes. In §4 we review the Harish-Chandra localization on quotient stacks from Beilinson–Bernstein [BB]. In §5 we start the study of the cohomology of the Lie algebras $\mathfrak{sp}(H')$ and $\mathfrak{sp}(H') \ltimes H'$. In §6 we prove Theorem 6.1, which establishes two canonical isomorphisms of H^2 spaces, and we show how this implies Theorems 5 and 6. Next, an intermezzo in §7 contains some required properties of metaplectic representations. We construct sheaves of coinvariants on $\widehat{\mathcal{A}}_g$ and $\widehat{\mathcal{X}}_g$ in §8 and descend them to \mathcal{A}_g and \mathcal{X}_g in §9, where we prove Theorems 1–4. We conclude with some final remarks in §10. Throughout, we work over \mathbb{C} .

1. HEISENBERG, METAPLECTIC, AND VIRASORO ALGEBRAS

We review here the infinite-dimensional Lie algebras from [ADCK, ADC] which will play a role in the paper. The complex Lie algebras from these references are here promoted to functors on commutative \mathbb{C} -algebras, as needed for our geometric constructions. Moreover, emphasis is given to the two-cocycles defining the relevant central extensions, later needed in §§5–6.

Here is some notation used throughout. For a topological vector space V , a subset $B \subset V$ is said to *topologically generate* V if finite linear combinations of elements of B form a dense open subspace of V . Also, let $\mathfrak{gl}(V)$ be the Lie algebra of continuous linear endomorphisms of V .

1.1. The algebra of formal Laurent series. Let H be the functor which assigns to a \mathbb{C} -algebra R the R -algebra $H(R) := R((t))$ of formal Laurent series in a parameter t . One has a decomposition

$$H = H_+ \oplus H_-, \quad \text{where} \quad H_+(R) := R[[t]] \quad \text{and} \quad H_-(R) := t^{-1}R[t^{-1}].$$

Also, let H' be the functor which assigns to R the R -module

$$H'(R) := R((t))/Rt^0$$

of Laurent series with zero constant term. One has a decomposition

$$H' := H'_+ \oplus H_- \quad \text{where } H'_+(R) := tR[[t]].$$

The R -module $H(R)$ is endowed with the t -adic topology. This is the topology that has the collection of cosets

$$\{f + t^N H_+(R) \mid f \in H(R) \text{ and } N \geq 1\}$$

as a basis of open subsets. This gives $H(R)$ the structure of a complete topological space, topologically generated over R by $b_i := t^i$ for $i \in \mathbb{Z}$. In the following, we will sometimes abuse notation and write simply H with an implicit choice of a \mathbb{C} -algebra R instead of $H(R)$.

We will repeatedly use the following result from [ADCK]: the Lie algebra $\mathfrak{gl}(H)$ has a canonical two-cocycle given by

$$(1.1) \quad \psi(A, B) := \text{Tr}(\pi_+ A \pi_- B \pi_+ - \pi_+ B \pi_- A \pi_+) \quad \text{for } A, B \in \mathfrak{gl}(H),$$

where $\pi_+: H \rightarrow H_+$ and $\pi_-: H \rightarrow H_-$ are the natural projections.

1.2. The Heisenberg algebra. Consider the symplectic form $\langle \cdot, \cdot \rangle$ on H given by

$$(1.2) \quad \langle f, g \rangle := - \underset{t=0}{\text{Res}} f dg \quad \text{for } f, g \in H.$$

This only depends on df and dg , for $f, g \in H$, and restricts to a nondegenerate symplectic form on H' . The *Heisenberg algebra* is the Lie algebra structure on H given by the Lie bracket $\langle \cdot, \cdot \rangle$. As (1.2) is continuous with respect to the t -adic topology, the Heisenberg algebra H is a complete topological Lie algebra. Moreover, (1.2) is invariant by changes of the parameter t . Note that $\langle f, g \rangle = \psi(f, g)$, where ψ is from (1.1), and $f, g \in H$ act on H by multiplication. Thus, the Heisenberg algebra H is the central extension

$$0 \rightarrow \mathfrak{gl}_1 \mathbf{1} \rightarrow H \rightarrow H' \rightarrow 0$$

given by the restriction of the two-cocycle ψ , where $\mathbf{1} := b_0$, and \mathfrak{gl}_1 is the functor assigning to a \mathbb{C} -algebra R the trivial Lie algebra R .

1.3. The enveloping algebra $\mathcal{U}(H)$. Let $\mathcal{U}(H)$ be the quotient of the universal enveloping algebra of the Heisenberg algebra H by the two-sided ideal generated by $\mathbf{1} - 1$, where $\mathbf{1}$ is the central element of H , and 1 is the unit in the universal enveloping algebra. The algebra $\mathcal{U}(H)$ has a canonical filtration $\mathcal{U}_0(H) \subset \mathcal{U}_1(H) \subset \dots$, where $\mathcal{U}_0(H) = \mathfrak{gl}_1 \mathbf{1}$, and $\mathcal{U}_N(H)$ for $N \in \mathbb{N}$ assigns to a \mathbb{C} -algebra R the R -module generated by the formal products $f_1 \cdots f_m$ with $m \leq N$ and $f_i \in H(R)$ for $i \leq N$.

Remarkably, $\mathcal{U}_2(H)$ is a Lie subalgebra of $\mathcal{U}(H)$ for the Lie bracket given by the commutator. Explicitly, the Lie bracket in $\mathcal{U}_2(H)$ is given by

$$(1.3) \quad \begin{aligned} [f, g] &= \langle f, g \rangle, \\ [f, \mathbf{1}] &= [fg, \mathbf{1}] = 0, \\ [fg, k] &= \langle f, k \rangle g + \langle g, k \rangle f, \\ [fg, kl] &= \langle g, k \rangle fl + \langle f, k \rangle gl + \langle g, l \rangle kf + \langle f, l \rangle kg \end{aligned}$$

for $f, g, k, l \in H'$. One has

$$\mathcal{U}_2(H)/\mathfrak{gl}_1 \mathbf{1} \cong S^2(H') \ltimes H'$$

as Lie algebras, where $S^2(H')$ is the second symmetric power of H' , and the semidirect product is given by the action of $S^2(H')$ on H' induced by

$$(1.4) \quad fg: H' \rightarrow H', \quad k \mapsto \langle f, k \rangle g + \langle g, k \rangle f, \quad \text{for } f, g, k \in H'.$$

Hence, $\mathcal{U}_2(H)$ is a central extension

$$0 \rightarrow \mathfrak{gl}_1 \mathbf{1} \rightarrow \mathcal{U}_2(H) \rightarrow S^2(H') \ltimes H' \rightarrow 0.$$

To identify the corresponding two-cocycle, arguing by cases, one shows that

$$\begin{aligned} [:fg:, :kl:] &= -\frac{1}{2} \psi(fg, kl) \mathbf{1} \\ &\quad + \langle g, k \rangle :fl: + \langle f, k \rangle :gl: + \langle g, l \rangle :kf: + \langle f, l \rangle :kg: \end{aligned}$$

in $\mathcal{U}_2(H)$ is compatible with (1.3), where $: :$ denotes the normal order product giving a specific lift of elements from $S^2(H')$ to $\mathcal{U}_2(H)$, and ψ is the restriction of the two-cocycle in (1.1). Hence, the two-cocycle c defining $\mathcal{U}_2(H)$ is given by

$$c(fg, kl) = -\frac{1}{2} \psi(fg, kl), \quad c(f, g) = \langle f, g \rangle, \quad c(fg, k) = 0$$

for $f, g, k, l \in H'$.

We will need a completion of $\mathcal{U}_2(H)$. For this, we first introduce the completion of $S^2(H')$ given by the symplectic algebra and its corresponding extension.

1.4. Symplectic and metaplectic algebras. The *symplectic algebra* $\mathfrak{sp}(H')$ is the Lie subalgebra of $\mathfrak{gl}(H')$ defined as

$$\mathfrak{sp}(H') := \{X \in \mathfrak{gl}(H') \mid \langle Xa, b \rangle + \langle a, Xb \rangle = 0, \text{ for all } a, b \in H'\}.$$

The action of $S^2(H')$ on H' induced by (1.4) realizes $S^2(H')$ as a dense Lie subalgebra of $\mathfrak{sp}(H')$.

The *metaplectic algebra* $\mathfrak{mp}(H')$ is the central extension

$$(1.5) \quad 0 \rightarrow \mathfrak{gl}_1 \mathbf{1} \rightarrow \mathfrak{mp}(H') \rightarrow \mathfrak{sp}(H') \rightarrow 0$$

defined by the two-cocycle

$$(1.6) \quad -\frac{1}{2} \psi(X, Y) \quad \text{for } X, Y \in \mathfrak{sp}(H')$$

where ψ is the restriction of (1.1).

1.5. The Weyl algebra. The *Weyl algebra* $\widetilde{\mathcal{U}}(H)$ is the completion of the algebra $\mathcal{U}(H)$ with respect to the topology in which the basis of open neighborhoods of 0 is formed by the left ideals of the submodules

$$t^N H_+ \subset H \subset \widetilde{\mathcal{U}}(H) \quad \text{for } N \in \mathbb{Z}$$

[FBZ, §2.1.2]. Elements of $\widetilde{\mathcal{U}}(H)$ can be described as possibly infinite series of the form $A_0 + \sum_{i \geq 1} A_i b_i$ with A_i equal to a finite linear combination of finite formal products of elements in $\{b_j\}_{j \in \mathbb{Z}}$ for $i \geq 0$.

The closure $\widetilde{\mathcal{U}}_2(H)$ of $\mathcal{U}_2(H)$ in $\widetilde{\mathcal{U}}(H)$ is a complete topological Lie subalgebra of $\widetilde{\mathcal{U}}(H)$ for the Lie bracket given by the commutator. Explicitly, consider first the semidirect product $\mathfrak{sp}(H') \ltimes H'$ given by the action

$$[X, f] = X(f) \quad \text{for } X \in \mathfrak{sp}(H') \text{ and } f \in H'.$$

Hence, the Lie bracket for $\mathfrak{sp}(H') \ltimes H'$ is

$$[X + f, Y + g] = XY - YX + X(g) - Y(f)$$

for $X, Y \in \mathfrak{sp}(H')$ and $f, g \in H'$. Then, one has a central extension

$$(1.7) \quad 0 \rightarrow \mathfrak{gl}_1 \mathbf{1} \rightarrow \widetilde{\mathcal{U}}_2(H) \rightarrow \mathfrak{sp}(H') \ltimes H' \rightarrow 0$$

with two-cocycle c given by

$$(1.8) \quad c(X, Y) = -\frac{1}{2} \psi(X, Y), \quad c(f, g) = \langle f, g \rangle, \quad c(X, f) = 0$$

for $X, Y \in \mathfrak{sp}(H')$ and $f, g \in H'$, where ψ is the restriction of (1.1). This is compatible with the Lie bracket for $\mathcal{U}_2(H)$ in (1.3).

The metaplectic algebra $\mathfrak{mp}(H')$ is realized as a Lie subalgebra of $\widetilde{\mathcal{U}}_2(H)$ via the injection

$$(1.9) \quad \mathfrak{mp}(H') \hookrightarrow \widetilde{\mathcal{U}}_2(H), \quad \mathbf{1} \mapsto \mathbf{1}, \quad X \mapsto X \quad \text{for } X \in \mathfrak{sp}(H').$$

1.6. Witt and Virasoro algebras. The *Witt algebra* Witt is the functor assigning to a \mathbb{C} -algebra R the Lie algebra $R((t))\partial_t$ of continuous derivations of $R((t))$. It is topologically generated over R by $L_p := -t^{p+1}\partial_t$ for $p \in \mathbb{Z}$, with relations $[L_p, L_q] = (p - q)L_{p+q}$ for $p, q \in \mathbb{Z}$. The oscillator representation of Witt gives an injection of Lie algebras

$$(1.10) \quad \tau: \text{Witt} \hookrightarrow \mathfrak{sp}(H'), \quad L_p \mapsto \frac{1}{2} \sum_i b_{-i} b_{i+p}$$

where the sum is over $i \in \mathbb{Z} \setminus \{0, -p\}$. Here and throughout, $b_i b_j$ denotes the tensor $b_i \otimes b_j \in S^2(H')$, rather than the product in H .

The *Virasoro algebra* Vir is the central extension

$$0 \rightarrow \mathfrak{gl}_1 \mathbf{1} \rightarrow \text{Vir} \rightarrow \text{Witt} \rightarrow 0$$

with Lie bracket given by

$$[L_p, L_q] = (p - q)L_{p+q} + \frac{1}{12} (p^3 - p) \delta_{p+q, 0} \mathbf{1},$$

$$\begin{array}{ccccc}
& \mathfrak{mp}(H') & \longleftrightarrow & \widetilde{\mathcal{U}}_2(H) & \\
\widehat{\tau} \swarrow & \downarrow & & \searrow \widehat{\sigma} & \downarrow \\
\text{Vir} & \longleftrightarrow & \mathfrak{D} & \longleftrightarrow & \mathfrak{sp}(H') \ltimes H' \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
& \mathfrak{sp}(H') & \longleftrightarrow & & \\
\tau \swarrow & & & \searrow \sigma & \\
\text{Witt} & \longleftrightarrow & \text{Witt} \ltimes H' & &
\end{array}$$

FIGURE 1. The main Lie algebra maps.

where $\delta_{p+q,0} = 1$ when $p + q = 0$, and $\delta_{p+q,0} = 0$ otherwise. The two-cocycle defining this extension is the restriction over Witt of the two-cocycle (1.6). The map τ is lifted by the injection

$$(1.11) \quad \widehat{\tau}: \text{Vir} \hookrightarrow \mathfrak{mp}(H'), \quad L_p \mapsto \frac{1}{2} \sum_i : b_{-i} b_{i+p} :, \quad \mathbf{1} \mapsto \mathbf{1}$$

where the sum is over $i \in \mathbb{Z} \setminus \{0, -p\}$.

1.7. The map σ and the Lie algebra \mathfrak{D} . We will need the Lie algebra injection

$$(1.12) \quad \sigma: \text{Witt} \ltimes H' \hookrightarrow \mathfrak{sp}(H') \ltimes H', \quad L_p \mapsto \frac{1}{2} \sum_i b_{-i} b_{i+p} - \frac{p+1}{2} b_p, \quad b_q \mapsto b_q$$

where the sum is over $i \in \mathbb{Z} \setminus \{0, -p\}$. For $p = 0$, one simply has $L_0 \mapsto \frac{1}{2} \sum_i b_{-i} b_i$, since b_0 vanishes in H' .

Remark 1.1. There are infinitely many ways of injecting $\text{Witt} \ltimes H'$ in $\mathfrak{sp}(H') \ltimes H'$: by replacing in (1.12) the coefficient $-\frac{p+1}{2}$ with $a(p+1)$ for $a \in \mathbb{C}$, one obtains a one-parameter family of such injections, as in the Segal-Sugawara construction [FBZ, 5.2.8]. The case $a = -\frac{1}{2}$ is specifically needed in §2.6.

Finally, let \mathfrak{D} be the central extension

$$0 \rightarrow \mathfrak{gl}_1 \mathbf{1} \rightarrow \mathfrak{D} \rightarrow \text{Witt} \ltimes H' \rightarrow 0$$

defined by the restriction to $\text{Witt} \ltimes H'$ of the two-cocycle ψ from (1.1) such that Witt acts on H as in (1.10) and H' acts on H by multiplication. The map σ is lifted by the injection

$$(1.13) \quad \widehat{\sigma}: \mathfrak{D} \hookrightarrow \widetilde{\mathcal{U}}_2(H), \quad L_p \mapsto \frac{1}{2} \sum_i : b_{-i} b_{i+p} : - \frac{p+1}{2} b_p, \quad b_q \mapsto b_q, \quad \mathbf{1} \mapsto \mathbf{1}$$

where the sum is over $i \in \mathbb{Z} \setminus \{0, -p\}$.

The main Lie algebra maps of this section are summarized by the commutative diagram in Figure 1. There, the four horizontal surjections admit Lie algebra splittings, and the left and right squares are Cartesian.

2. EXTENDED ABELIAN VARIETIES AND THEIR MODULI

First, we review some geometric constructions and results from [ADC] related to suitable extensions of abelian varieties and their moduli. Theorems 2.2 and 2.3 play a key role in later sections. Then, we conclude with a review of the degree-2 cohomology classes on the moduli spaces in consideration.

2.1. Extended abelian varieties. Here we work with coefficients in an arbitrary \mathbb{C} -algebra R and write H for $H(R)$. Recall the symplectic form \langle , \rangle on H' from (1.2). An *extended principally polarized abelian variety* (extended PPAV, for short) of dimension g is a triple (Z, F, L) , where

- Z is a Lagrangian subspace of H' ,
- F is a codimension g subspace of Z , and
- L is a rank $2g$ lattice in F^\perp/F ,

satisfying four suitable conditions. The form \langle , \rangle on H' induces a nondegenerate symplectic form on F^\perp/F , which will still be denoted as \langle , \rangle . The first three conditions imposed on the triple (Z, F, L) are:

- (2.1) $Z \cap H'_+ = 0$,
- (2.2) $L_{\mathbb{R}} \cap A = 0$ in F^\perp/F , where $L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}$ and $A := F^\perp \cap H'_+$,
- (2.3) $\frac{1}{2\pi i} \langle , \rangle$ is unimodular on L .

Condition (2.1) implies the decompositions

$$H' = Z \oplus H'_+ \quad \text{and} \quad F^\perp/F = Z/F \oplus A$$

into maximal isotropic subspaces. It follows that F^\perp/F has dimension $2g$. Condition (2.2) implies that: the projection $F^\perp/F \rightarrow Z/F$ induces a real isomorphism $L_{\mathbb{R}} \cong Z/F$; the real structure on F^\perp/F induced by $L_{\mathbb{R}}$ identifies A with the conjugate of Z/F ; and the form on Z/F given by

$$B(u, v) := \frac{1}{\pi} \langle \bar{u}, v \rangle \quad \text{for } u, v \in Z/F$$

is Hermitian. The fourth and last condition on the triple (Z, F, L) is

- (2.4) the form B on Z/F is positive definite.

A triple (Z, F, L) satisfying the four conditions (2.1)-(2.4) determines an abelian variety X of dimension g , which is the quotient of the g -dimensional space Z/F by the image of L under the projection $F^\perp/F \rightarrow Z/F$, equipped with the polarization B , which is principal by (2.3). More precisely, the triple (Z, F, L) is equivalent to an isomorphism class of extensions

$$(2.5) \quad 0 \rightarrow H'_+ \rightarrow H'/K \rightarrow X \rightarrow 0$$

where K is the preimage of L under the projection $F^\perp \rightarrow F^\perp/F$.

2.2. Moduli of extended PPAVs. Let $\widehat{\mathcal{A}}_g$ be the moduli space of extended PPAVs of dimension g . In [ADC], the corresponding coarse moduli space, here still denoted $\widehat{\mathcal{A}}_g$ for simplicity, is constructed as an infinite-dimensional analytic manifold. Here is a sketch of the construction. For this, let $H = H(\mathbb{C})$. Consider the infinite-dimensional analytic manifold

$$(2.6) \quad \widehat{\mathcal{H}}_g := \widetilde{S}^2(H'_+) \times \mathcal{B}_g(H'_+) \times \mathcal{H}_g$$

where

$$\widetilde{S}^2(H'_+) := \left\{ \varphi: H_- \rightarrow H'_+ \mid \begin{array}{l} \varphi \text{ is linear and symmetric,} \\ \text{i.e., } \langle a, \varphi(b) \rangle = \langle \varphi(a), b \rangle \text{ for all } a, b \in H_- \end{array} \right\},$$

$\mathcal{B}_g(H'_+) :=$ the manifold of frames of $(H'_+)^g$,

$\mathcal{H}_g :=$ the Siegel upper half-space of degree g .

Points of $\mathcal{B}_g(H'_+)$ are the elements (h_1, \dots, h_g) in $(H'_+)^g$ consisting of g linearly independent vectors. One has a map of sets

$$(2.7) \quad \widehat{\mathcal{H}}_g \rightarrow \widehat{\mathcal{A}}_g, \quad (\varphi, (h_1, \dots, h_g), \Omega) \mapsto (Z, F, L)$$

where: (i) the Lagrangian Z is defined as the graph of φ ; (ii) considering h_1, \dots, h_g as linear forms on H_- and thus on Z after composing with the projection $Z \rightarrow H_-$, the isotropic space $F \subset Z$ is defined by $h_1 = \dots = h_g = 0$; and (iii) the lattice L is constructed from Ω .

The symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$ acts freely and properly discontinuously on $\widehat{\mathcal{H}}_g$, hence the quotient $\widehat{\mathcal{H}}_g / \mathrm{Sp}(2g, \mathbb{Z})$ has a natural structure of an infinite-dimensional analytic manifold. Moreover, the map (2.7) factors to a bijection of sets

$$(2.8) \quad \widehat{\mathcal{H}}_g / \mathrm{Sp}(2g, \mathbb{Z}) \rightarrow \widehat{\mathcal{A}}_g.$$

Hence, (2.8) induces on $\widehat{\mathcal{A}}_g$ the structure of an infinite-dimensional analytic manifold [ADC, §3].

One has a commutative diagram of moduli spaces

$$\begin{array}{ccc} \widehat{\mathcal{H}}_g & \xrightarrow{/ \mathrm{Sp}(2g, \mathbb{Z})} & \widehat{\mathcal{A}}_g \\ \downarrow & & \downarrow \\ \mathcal{H}_g & \xrightarrow{/ \mathrm{Sp}(2g, \mathbb{Z})} & \mathcal{A}_g \end{array}$$

where \mathcal{A}_g is the moduli space of principally polarized abelian varieties of dimension g . The vertical maps are the natural projections. The action of $\mathrm{Sp}(2g, \mathbb{Z})$ on \mathcal{H}_g has finite nontrivial stabilizers, hence at least an orbifold structure is required on \mathcal{A}_g . Here we consider \mathcal{A}_g as the resulting quotient stack. This is a separated, smooth Deligne-Mumford stack. From [ADC], $\widehat{\mathcal{A}}_g$ is rational homotopy equivalent to \mathcal{A}_g .

Remark 2.1. In the following, both $\widehat{\mathcal{A}}_g$ and \mathcal{A}_g will be considered as smooth Deligne-Mumford stacks rather than coarse moduli spaces. We will consider families $(Z, F, L) \rightarrow S$ of extended PPAVs of dimension g over a smooth scheme S . We will construct sheaves of coinvariants on such families in §8 and descend them to families of PPAVs in §9, thus yielding sheaves on \mathcal{A}_g .

2.3. Symplectic uniformization of moduli of extended PPAVs. An action of a Lie algebra \mathfrak{g} on a variety X over \mathbb{C} is a homomorphism of Lie algebras $\mathfrak{g} \rightarrow \text{Vect}(X)$, where $\text{Vect}(X)$ is the Lie algebra of regular vector fields on X . The action is said to be *transitive* if for every $x \in X$, the evaluation map $\mathfrak{g} \rightarrow T_x(X)$ is surjective.

Theorem 2.2 (Uniformization of $\widehat{\mathcal{A}}_g$ [ADC]). *The moduli space $\widehat{\mathcal{A}}_g$ carries a transitive action of $\mathfrak{sp}(H')$. For (Z, F, L) in $\widehat{\mathcal{A}}_g$, one has*

$$T_{(Z,F,L)}(\widehat{\mathcal{A}}_g) \cong \mathfrak{sp}_F(H') \setminus \mathfrak{sp}(H')$$

where

$$(2.9) \quad \mathfrak{sp}_F(H') := \left\{ X \in \mathfrak{sp}(H') \mid X(F^\perp) \subseteq F \right\}.$$

We sketch the proof from [ADC]: using the map (2.8), and since the action of $\text{Sp}(2g, \mathbb{Z})$ on $\widehat{\mathcal{H}}_g$ is properly discontinuous, one deduces

$$T_{(Z,F,L)}(\widehat{\mathcal{A}}_g) \cong \widetilde{S}^2(H'_+) \times \text{Hom}(H_-, Z/F) \times S^2(Z/F).$$

Each factor here is the tangent space to the corresponding factor of $\widehat{\mathcal{H}}_g$ in (2.6) at a point mapping to (Z, F, L) under (2.8). Identifying the dual of H_- with H'_+ via the symplectic form \langle , \rangle , one has

$$T_{(Z,F,L)}(\widehat{\mathcal{A}}_g) \cong \widetilde{S}^2(H'_+) \times (H'_+ \widehat{\otimes} Z/F) \times S^2(Z/F).$$

From the splitting $H'/F = H'_+ \oplus Z/F$, one has

$$T_{(Z,F,L)}(\widehat{\mathcal{A}}_g) \cong \widetilde{S}^2(H'/F).$$

Here $\widetilde{S}^2(H'/F)$ is the closure of $S^2(H'/F)$ inside $\mathfrak{sp}(H')$. Finally, since $\mathfrak{sp}_F(H') \cong (H'/F \widehat{\otimes} F) \times S^2(F)$, one has

$$T_{(Z,F,L)}(\widehat{\mathcal{A}}_g) \cong \mathfrak{sp}_F(H') \setminus \mathfrak{sp}(H').$$

We emphasize that the subspace

$$(2.10) \quad \widetilde{S}^2(H'_+) \times (H'_+ \widehat{\otimes} Z/F)$$

is the tangent space to the fiber of the forgetful map $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$ at (Z, F, L) , while the subspace $S^2(Z/F)$ encodes the infinitesimal deformations of the complex structure of the underlying abelian variety.

2.4. The universal extended PPAV. The moduli space $\widehat{\mathcal{A}}_g$ admits a universal family $\widehat{\mathcal{X}}_g \rightarrow \widehat{\mathcal{A}}_g$. The fiber over a point (Z, F, L) in $\widehat{\mathcal{A}}_g$ is

$$(H'/K) \times_G \mathcal{Q}$$

where G is the group of points of order 2 in H'/K , and \mathcal{Q} is the G -torsor of integral quadratic forms q on L satisfying

$$q(a) + q(b) - q(a+b) \equiv \frac{1}{2\pi i} \langle a, b \rangle \pmod{2}.$$

The need for the twist by G becomes clear when one defines the maps in (2.11) below. Points of $\widehat{\mathcal{X}}_g$ are denoted as (Z, F, L, \bar{h}, q) . The map $\widehat{\mathcal{X}}_g \rightarrow \widehat{\mathcal{A}}_g$ has no zero-section, however the fiber over (Z, F, L) in $\widehat{\mathcal{A}}_g$ is non-canonically isomorphic to H'/K . From [ADC], $\widehat{\mathcal{X}}_g$ is rational homotopy equivalent to the universal family \mathcal{X}_g over the stack \mathcal{A}_g . More generally, we will consider families $(Z, F, L, \bar{h}, q) \rightarrow S$ over a smooth scheme S for a fixed g .

2.5. Uniformization of the universal extended PPAV. Recall the Lie algebra $\mathfrak{sp}(H') \ltimes H'$ from §1.5.

Theorem 2.3 (Uniformization of $\widehat{\mathcal{X}}_g$ [ADC]). *The moduli space $\widehat{\mathcal{X}}_g$ carries a transitive action of $\mathfrak{sp}(H') \ltimes H'$. For (Z, F, L, \bar{h}, q) in $\widehat{\mathcal{X}}_g$, one has*

$$T_{(Z, F, L, \bar{h}, q)}(\widehat{\mathcal{X}}_g) \cong \mathfrak{sp}_F(H') \ltimes F \setminus \mathfrak{sp}(H') \ltimes H'.$$

2.6. The case of curves and line bundles. The moduli spaces $\widehat{\mathcal{A}}_g$ and $\widehat{\mathcal{X}}_g$ are naturally extensions of moduli spaces of curves and line bundles. To see this, let $\widehat{\mathcal{M}}_g$ be the moduli space of triples (C, P, t) , where C is a smooth genus g curve, P is a point of C , and t is a formal coordinate at P [ADCKP]. From [ADC], there is an *extended Torelli map* $\widehat{\mathcal{M}}_g \rightarrow \widehat{\mathcal{A}}_g$ mapping (C, P, t) to a triple (Z, F, L) with identifications

$$F \cong H^0(C \setminus P, \mathcal{O}_C), \quad Z/F \cong H^1(C, \mathcal{O}_C), \quad L \cong H^1(C, \mathbb{Z}).$$

The isotropicity of F with respect to (1.2) follows from the residue theorem.

Moreover, let $\widehat{\mathcal{P}}_d \rightarrow \widehat{\mathcal{M}}_g$ be the *relative Picard variety* parametrizing isomorphism classes of quintuples $(C, P, t, \mathcal{L}, \varphi)$ where \mathcal{L} is a line bundle of degree d on C and φ is a trivialization of the restriction of \mathcal{L} to the formal disc around P . For $d = g - 1$, one has a commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{P}}_{g-1} & \longrightarrow & \widehat{\mathcal{X}}_g \\ \downarrow & & \downarrow \\ \widehat{\mathcal{M}}_g & \longrightarrow & \widehat{\mathcal{A}}_g \end{array} \tag{2.11}$$

where $(C, P, t, \mathcal{L}, \varphi)$ in $\widehat{\mathcal{P}}_{g-1}$ is mapped to (Z, F, L, \bar{h}, q) in $\widehat{\mathcal{X}}_g$ such that q is arbitrary in \mathcal{Q} , and \bar{h} is the unique point of H'/K given by the isomorphism class of $(\mathcal{L} \eta^{-1}, \varphi dt^{-\frac{1}{2}})$, where η is the unique theta-characteristic on C

identified by q via Riemann's theorem [ADC, (3.23)]. By the construction of $\widehat{\mathcal{X}}_g$, this map is independent of the choice of q .

Recall the commutative diagram

$$(2.12) \quad \begin{array}{ccc} \text{Witt} \ltimes H' & \xrightarrow{\sigma} & \mathfrak{sp}(H') \ltimes H' \\ \downarrow & & \downarrow \\ \text{Witt} & \xrightarrow{\tau} & \mathfrak{sp}(H') \end{array}$$

from Figure 1. It is shown in [ADC, (5.11)] that the Lie algebras in (2.12) act transitively on the corresponding moduli spaces in (2.11), and the maps in (2.12) induce the differentials of the corresponding maps in (2.11).

2.7. The Hodge and canonical line bundles. Here we review the degree-2 cohomology classes on the moduli spaces in consideration. Let $\Lambda \rightarrow \mathcal{A}_g$ be the *Hodge line bundle* defined by

$$\Lambda := \det(\pi_*(\omega_{\mathcal{X}_g/\mathcal{A}_g}))$$

where $\pi: \mathcal{X}_g \rightarrow \mathcal{A}_g$ is the natural projection and $\omega_{\mathcal{X}_g/\mathcal{A}_g}$ is the relative dualizing sheaf. Let $\Xi \rightarrow \mathcal{X}_g$ be the *canonical line bundle* whose restriction to the fibers of π induces *twice* the principal polarization on each fiber. For simplicity, we omit the notation for the pull-back when referring to the pull-back of Λ and Ξ via the natural projections $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$ and $\widehat{\mathcal{X}}_g \rightarrow \mathcal{X}_g$. Let λ and ξ be the first Chern classes of Λ and Ξ , respectively.

Proposition 2.4. *For $g \geq 2$, one has*

$$(2.13) \quad H^2(\widehat{\mathcal{A}}_g, \mathbb{Q}) = H^2(\mathcal{A}_g, \mathbb{Q}) = \mathbb{Q}\lambda,$$

$$(2.14) \quad H^2(\widehat{\mathcal{X}}_g, \mathbb{Q}) = H^2(\mathcal{X}_g, \mathbb{Q}) = \mathbb{Q}\lambda \oplus \mathbb{Q}\xi.$$

Moreover, the pull-back via the extended Torelli map induces an identification for $g \geq 3$

$$(2.15) \quad H^2(\widehat{\mathcal{A}}_g, \mathbb{Q}) \xrightarrow{\sim} H^2(\widehat{\mathcal{M}}_g, \mathbb{Q}) = H^2(\mathcal{M}_g, \mathbb{Q})$$

and an inclusion for $g \geq 5$

$$(2.16) \quad H^2(\widehat{\mathcal{X}}_g, \mathbb{Q}) \subset H^2(\widehat{\mathcal{P}}_{g-1}, \mathbb{Q}) = H^2(\mathcal{P}'_{g-1}, \mathbb{Q}).$$

The statement combines various results from the literature.

Proof. For $g \geq 2$, one has $H^2(\mathcal{A}_g, \mathbb{Q}) = \mathbb{Q}\lambda$, and for $g \geq 3$, the Torelli map induces an identification $H^2(\mathcal{A}_g, \mathbb{Q}) = H^2(\mathcal{M}_g, \mathbb{Q})$ (e.g., [VDG, 8.1] and [Hai, 17.3–17.4]). The identification $H^2(\mathcal{A}_g, \mathbb{Q}) = H^2(\mathcal{M}_g, \mathbb{Q})$ fails for $g = 2$, since $H^2(\mathcal{M}_2, \mathbb{Q}) = 0$.

Since \mathcal{A}_g and $\widehat{\mathcal{A}}_g$ have the same rational homotopy type [ADC, pg. 16–17], (2.13) follows. Similarly, \mathcal{X}_g and $\widehat{\mathcal{X}}_g$ have the same rational homotopy type [ADC, pg. 17], hence the first isomorphism in (2.14). The second identification in (2.14) follows from [FP].

Next, consider the composition of forgetful maps

$$(2.17) \quad \widehat{\mathcal{M}}_g \rightarrow \mathcal{M}'_g \rightarrow \mathcal{M}_g$$

where \mathcal{M}'_g is the moduli space of triples (C, P, v) , where C is a curve of genus g , P is a point of C , and v is a nonzero tangent vector to C at P . Since $\widehat{\mathcal{M}}_g$ and \mathcal{M}'_g have the same rational homotopy type [ADCKP, 3.3] and

$$H^2(\mathcal{M}'_g, \mathbb{Q}) = H^2(\mathcal{M}_g, \mathbb{Q}) = \mathbb{Q}\lambda$$

[ADCKP, 5.8], the composition of the maps in (2.17) induces an identification $H^2(\widehat{\mathcal{M}}_g, \mathbb{Q}) = \mathbb{Q}\lambda$. Using (2.13), we deduce (2.15). Note that the class of the cotangent line bundle at the marked point vanishes on \mathcal{M}'_g and $\widehat{\mathcal{M}}_g$.

Finally, $\widehat{\mathcal{P}}_{g-1}$ and \mathcal{P}'_{g-1} have the same rational homotopy type and $H^2(\mathcal{P}'_{g-1}, \mathbb{Q})$ for $g \geq 5$ is described in [ADCKP, 5.7]: it has dimension three, and λ and ξ are linearly independent. The inclusion (2.16) follows, hence the statement. \square

3. GROUP (IND-)SCHEMES OF SYMPLECTIC AUTOMORPHISMS

Here we study how to exponentiate the symplectic algebra $\mathfrak{sp}(H')$ and related Lie algebras. We use group ind-schemes as in [FBZ, §A.1.4]. For a topological vector space E , let $\mathrm{GL}(E)$ be the group of continuous linear automorphisms of E . Also, for a topological group G , let G_\circ be its connected component containing the identity.

3.1. The Lie algebra $\mathfrak{sp}^+(H')$. Let $\mathfrak{sp}^+(H')$ be the Lie subalgebra of $\mathfrak{sp}(H')$ which preserves the subspace H'_+ , that is:

$$\mathfrak{sp}^+(H') := \{X \in \mathfrak{sp}(H') \mid X(H'_+) \subseteq H'_+\}.$$

This is topologically generated by $S^2(H'_+)$ and the elements $b_i b_j$ in $S^2(H') \subset \mathfrak{sp}(H')$ such that $i > 0 > j$.

The Lie algebra $\mathfrak{sp}^+(H')$ has the following geometric incarnation. Recall from (2.10) that for (Z, F, L) in $\widehat{\mathcal{A}}_g$, the tangent space to the fiber of the forgetful map $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$ at (Z, F, L) is

$$\mathfrak{sp}_{H'/F}^+(H') := \widetilde{S}^2(H'_+) \times (H'_+ \widehat{\otimes} Z/F).$$

This can also be described as

$$\mathfrak{sp}_{H'/F}^+(H') = \{X \in \mathfrak{sp}^+(H') \mid X(H') \subseteq H'/F\}.$$

Thus one has a natural inclusion

$$(3.1) \quad \mathfrak{sp}_{H'/F}^+(H') \hookrightarrow \mathfrak{sp}^+(H')$$

realizing $\mathfrak{sp}_{H'/F}^+(H')$ as a Lie subalgebra of $\mathfrak{sp}^+(H')$. The transitive action of $\mathfrak{sp}(H')$ on $\widehat{\mathcal{A}}_g$ from §2.3 extends the transitive action of $\mathfrak{sp}_{H'/F}^+(H')$ on

the fiber of $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$ at (Z, F, L) . The stabilizer of $\mathfrak{sp}^+(H')$ at (Z, F, L) is

$$(3.2) \quad \mathfrak{sp}_F^+(H') := \mathfrak{sp}^+(H') \cap \mathfrak{sp}_F(H')$$

which is a Lie subalgebra of $\mathfrak{sp}^+(H')$.

Moreover, the symplectic uniformization of $\widehat{\mathcal{A}}_g$ induces the following double coset realization of the tangent space of \mathcal{A}_g . For $(Z, F, L) \in \widehat{\mathcal{A}}_g$, let $(X, B) \in \mathcal{A}_g$ be the image of (Z, F, L) via $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$ as in (2.5). One has

$$(3.3) \quad T_{(X, B)}(\mathcal{A}_g) \cong \mathfrak{sp}_F(H') \setminus \mathfrak{sp}(H') / \mathfrak{sp}_{H'/F}^+(H').$$

3.2. The group schemes $Sp^+(H')$ and $Sp_F^+(H')$. We will need to exponentiate the Lie algebra $\mathfrak{sp}^+(H')$. For this, consider the group scheme of continuous symplectic automorphisms of H' preserving the subspace H'_+ , and let $Sp^+(H')$ be its connected component of the identity, that is:

$$Sp^+(H') := \left\{ \rho \in \mathrm{GL}(H') \mid \begin{array}{l} \langle \rho(a), \rho(b) \rangle = \langle a, b \rangle \text{ for all } a, b \in H', \\ \rho(H'_+) = H'_+ \end{array} \right\}_\circ.$$

The group scheme $Sp^+(H')$ represents the group functor which assigns to a \mathbb{C} -algebra R the identity component $Sp^+(H'(R))$ of the group of continuous symplectic automorphisms of $H'(R)$ preserving the subspace $H'_+(R) = tR[[t]]$.

One easily checks that

$$\mathrm{Lie}(Sp^+(H')) = \mathfrak{sp}^+(H').$$

Also, since $Sp^+(H')$ is by definition connected, elements in $Sp^+(H')$ are products of exponentials of elements in $\mathfrak{sp}^+(H')$.

Remark 3.1. As the group $\mathrm{GL}(H)$ is not connected [ADCKP, pg. 11], taking the identity component is a nontrivial part of the definition of $Sp^+(H')$.

Moreover, for (Z, F, L) in $\widehat{\mathcal{A}}_g$, let $Sp_F^+(H')$ be the subgroup scheme of $Sp^+(H')$ defined as

$$Sp_F^+(H') := \left\{ \rho \in Sp^+(H') \mid \begin{array}{l} \rho(F^\perp) = F^\perp, \\ \rho(F) = F, \\ \rho|_{F^\perp/F} = \mathrm{Id}_{F^\perp/F} \end{array} \right\}.$$

Here $\rho|_{F^\perp/F}$ stands for the composition of the linear maps

$$F^\perp/F \xrightarrow{\mathrm{id}} F^\perp \xrightarrow{\rho} F^\perp \twoheadrightarrow F^\perp/F$$

with the last map given by the natural projection. One easily checks that

$$\mathrm{Lie}(Sp_F^+(H')) = \mathfrak{sp}_F^+(H'),$$

with $\mathfrak{sp}_F^+(H')$ from (3.2).

Theorem 2.2 has the following extension:

Theorem 3.2. (i) The moduli space $\widehat{\mathcal{A}}_g$ carries a transitive action of $\mathfrak{sp}(H')$ compatible with a transitive action of $\mathcal{Sp}^+(H')$ on the fibers of $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$.
(ii) The stabilizer of $\mathcal{Sp}^+(H')$ at (Z, F, L) in $\widehat{\mathcal{A}}_g$ is $\mathcal{Sp}_F^+(H')$.
(iii) The subgroup schemes $\mathcal{Sp}_F^+(H')$ obtained by varying (Z, F, L) are pairwise conjugated inside $\mathcal{Sp}^+(H')$.

Proof. The natural action of $\mathcal{Sp}^+(H')$ on H' induces a transitive action of $\mathcal{Sp}^+(H')$ on the fibers of $\widehat{\mathcal{H}}_g \rightarrow \mathcal{H}_g$. Explicitly, recall from (2.6) that the fibers of $\widehat{\mathcal{H}}_g \rightarrow \mathcal{H}_g$ are isomorphic to $\widetilde{S}^2(H'_+) \times \mathcal{B}_g(H'_+)$. The space $\widetilde{S}^2(H'_+)$ consists of linear symmetric maps $\varphi: H_- \rightarrow H'_+$. The assignment $\varphi \mapsto Z := \text{graph}(\varphi)$ gives a bijection of $\widetilde{S}^2(H'_+)$ with the set of Lagrangian subspaces Z of H' such that $Z \cap H'_+ = 0$ — this is (2.1). Clearly, $\mathcal{Sp}^+(H')$ acts transitively on the set of such Lagrangians, hence on $\widetilde{S}^2(H'_+)$. Moreover, $\mathcal{Sp}^+(H')$ maps onto the group $\text{GL}(H'_+)$, for which $\mathcal{B}_g(H'_+)$ is a homogeneous space. Hence $\mathcal{Sp}^+(H')$ acts transitively on the fibers of $\widehat{\mathcal{H}}_g \rightarrow \mathcal{H}_g$.

The action of $\text{Sp}(2g, \mathbb{Z})$ on $\mathcal{Sp}^+(H')$ by conjugation and on $\widehat{\mathcal{H}}_g$ induce an action of the semi-direct product $\text{Sp}(2g, \mathbb{Z}) \ltimes \mathcal{Sp}^+(H')$ on $\widehat{\mathcal{H}}_g$. After quotienting by $\text{Sp}(2g, \mathbb{Z})$, the transitive action of $\mathcal{Sp}^+(H')$ on the fibers of $\widehat{\mathcal{H}}_g \rightarrow \mathcal{H}_g$ descends to a transitive action of $\mathcal{Sp}^+(H')$ on the fibers of $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$.

Finally, the transitive action of $\mathfrak{sp}(H')$ on $\widehat{\mathcal{A}}_g$ from Theorem 2.2 extends the transitive action of $\mathfrak{sp}_{H'/F}^+(H')$ on the fiber of $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$ at each (Z, F, L) , and the actions of $\mathfrak{sp}(H')$ and $\mathcal{Sp}^+(H')$ commute. This proves (i).

For part (ii), it is clear that $\mathcal{Sp}_F^+(H')$ is in the stabilizer of $\mathcal{Sp}^+(H')$ at (Z, F, L) . For the converse, it is enough to show that if $\rho \in \mathcal{Sp}^+(H')$ is in the stabilizer, then $\rho|_{F^\perp/F} = \text{Id}_{F^\perp/F}$. For this, it is easy to see that $\rho|_{F^\perp/F}$ must be in the discrete group $\text{Sp}(2g, \mathbb{Z})$, and for ρ in the connected group $\mathcal{Sp}_F^+(H')$, one necessarily has $\rho|_{F^\perp/F} = \text{Id}_{F^\perp/F}$.

Part (iii) is clear from the transitive action of $\mathcal{Sp}^+(H')$ on the fibers of $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$. \square

3.3. The group ind-scheme $\mathcal{Sp}(H')$. More generally, consider the group functor which assigns to a \mathbb{C} -algebra R the connected group $\mathcal{Sp}(H'(R))$ of continuous linear automorphisms of $H'(R)$ of the following type

$$\mathcal{Sp}(H'(R)) := \left\{ \rho \in \text{GL}(H'(R)) \mid \begin{array}{l} \langle \rho(a), \rho(b) \rangle = \langle a, b \rangle \quad \text{for all } a, b \in H', \\ \rho(H'_+(R)) \subseteq H'_+(R) \oplus H_-(R_{\text{nil}}) \end{array} \right\}.$$

where $R_{\text{nil}} \subseteq R$ is the \mathbb{C} -subalgebra consisting of nilpotent elements. The presence of nilpotent elements entails that this functor can only be represented by an ind-scheme, as schemes are insufficient (this is as in [FBZ, §§6.2.3, A.1.2–A.1.4]). Let $\mathcal{Sp}(H')$ be the group ind-scheme representing

this functor. Naturally, $\mathcal{S}p^+(H')$ is a subgroup of $\mathcal{S}p(H')$. One checks that

$$\text{Lie}(\mathcal{S}p(H')) = \mathfrak{sp}(H') .$$

3.4. Derivations and ring automorphisms. Here we remark how the Lie algebras and group (ind-)schemes of this section naturally extend analogous objects studied in [BD, FBZ] and related to the geometry of curves. Namely, let $\text{Der}(\mathcal{O})$ be the functor assigning to a \mathbb{C} -algebra R the Lie algebra

$$\text{Der}(\mathcal{O}(R)) = R[[t]]\partial_t .$$

This is the Lie subalgebra of the Lie algebra $\text{Witt}(R)$ from (1.6) topologically generated over R by L_p for $p \geq -1$. The inclusion $\tau: \text{Witt} \hookrightarrow \mathfrak{sp}(H')$ from (1.10) maps $\text{Der}(\mathcal{O})$ to $\mathfrak{sp}^+(H')$, so that one has a commutative diagram of Lie algebras

$$(3.4) \quad \begin{array}{ccc} \text{Witt} & \xhookrightarrow{\tau} & \mathfrak{sp}(H') \\ \downarrow & & \downarrow \\ \text{Der}(\mathcal{O}) & \xhookrightarrow{\quad} & \mathfrak{sp}^+(H') . \end{array}$$

From [ADCKP], the moduli space $\widehat{\mathcal{M}}_g$ carries a transitive action of Witt. This action is compatible with a simply transitive action of $\text{Der}(\mathcal{O})$ on the fibers of $\widehat{\mathcal{M}}_g \rightarrow \mathcal{M}_g$: the elements L_p with $p \geq 0$ topologically generate the tangent directions to the changes of the formal coordinate at the marked point, and the element L_{-1} spans the tangent space to the curve at the marked point. Hence, for (C, P, t) in $\widehat{\mathcal{M}}_g$, one has

$$(3.5) \quad T_C(\mathcal{M}_g) \cong \text{Vect}(C \setminus P) \setminus \text{Witt} / \text{Der}(\mathcal{O}) .$$

From [ADC], the moduli space $\widehat{\mathcal{A}}_g$ carries a transitive action of $\mathfrak{sp}(H')$ and by quotienting Witt and $\mathfrak{sp}(H')$ by appropriate Lie subalgebras, the inclusion τ from (1.10) induces the differential of the extended Torelli map $\widehat{\mathcal{M}}_g \rightarrow \widehat{\mathcal{A}}_g$. We emphasize how (3.3) extends (3.5).

One has a similar picture for the related group (ind-)schemes. Namely, consider first the group scheme $\text{Aut}(\mathcal{O})$ representing the functor which assigns to a \mathbb{C} -algebra R the group of continuous ring automorphisms of $R[[t]]$ preserving the ideal $tR[[t]]$. Explicitly, this is:

$$\text{Aut}(\mathcal{O}(R)) := \{t \mapsto a_1 t + a_2 t^2 + \dots \mid a_i \in R \text{ and } a_1 \text{ is a unit}\} .$$

Since the symplectic form (1.2) is invariant by changes of the parameter t and $tR[[t]] = H'_+(R)$ as vector spaces, it follows that $\text{Aut}(\mathcal{O})$ is a subgroup of $\mathcal{S}p^+(H')$. However, the Lie algebra of $\text{Aut}(\mathcal{O})$ is smaller than $\text{Der}(\mathcal{O})$: As remarked in [FBZ, §6.2.3], one has

$$\text{Lie}(\text{Aut}(\mathcal{O})) = t \text{Der}(\mathcal{O}) ,$$

and in order to exponentiate $\text{Der}(\mathcal{O})$, one needs to consider a group ind-scheme. For this, let $\underline{\text{Aut}}(\mathcal{O})$ be the group ind-scheme representing the

functor which assigns to a \mathbb{C} -algebra R the group of continuous ring automorphisms of $R[[t]]$:

$$\underline{\text{Aut}}(\mathcal{O}(R)) := \left\{ t \mapsto a_0 + a_1 t + a_2 t^2 + \dots \middle| \begin{array}{l} a_i \in R, a_1 \text{ is a unit,} \\ \text{and } a_0 \text{ is nilpotent} \end{array} \right\}.$$

Similarly, let $\text{Aut}(\mathcal{K})$ be the group ind-scheme representing the functor which assigns to a \mathbb{C} -algebra R the group of continuous ring automorphisms of $R((t))$:

$$\text{Aut}(\mathcal{K}(R)) := \left\{ t \mapsto \sum_{i \geq i_0} a_i t^i \middle| \begin{array}{l} a_i \in R, a_1 \text{ is a unit,} \\ a_i \text{ nilpotent for } i \leq 0 \end{array} \right\}.$$

One has

$$\text{Lie}(\underline{\text{Aut}}(\mathcal{O})) = \text{Der}(\mathcal{O}) \quad \text{and} \quad \text{Lie}(\text{Aut}(\mathcal{K})) = \text{Witt}.$$

We refer to [BD, FBZ] for more on $\text{Der}(\mathcal{O})$, $\underline{\text{Aut}}(\mathcal{O})$ and $\text{Aut}(\mathcal{K})$.

Elements in $\underline{\text{Aut}}(\mathcal{O})$ and $\text{Aut}(\mathcal{K})$ naturally induce elements in $\text{GL}(H)$, and after composing with the projection $H \rightarrow H'$, they induce symplectic elements in $\text{GL}(H')$. Hence, one has a commutative diagram of group (ind)-schemes

$$\begin{array}{ccc} \text{Aut}(\mathcal{K}) & \longrightarrow & \mathcal{S}p(H') \\ \uparrow & & \uparrow \\ \underline{\text{Aut}}(\mathcal{O}) & \longrightarrow & \mathcal{S}p^+(H') \end{array}$$

such that the differentials of these maps restricted to the tangent spaces at the identities are given by (3.4).

3.5. The group ind-scheme $\mathcal{M}p(H')$. We will also consider the group ind-scheme $\mathcal{M}p(H')$ given by the central extension

$$1 \rightarrow \mathbb{G}_m \rightarrow \mathcal{M}p(H') \rightarrow \mathcal{S}p(H') \rightarrow 1$$

corresponding to the metaplectic algebra $\mathfrak{mp}(H')$ from (1.5).

3.6. The group scheme $\mathcal{S}p^+(H') \ltimes \mathcal{O}_1^\times$. Finally, we describe the analogous Lie algebras and group (ind)-schemes for the moduli space $\widehat{\mathcal{X}}_g$. The transitive action of $\mathfrak{sp}(H') \ltimes H'$ on $\widehat{\mathcal{X}}_g$ from §2.5 extends a transitive action of $\mathfrak{sp}_{H'/F}^+(H') \ltimes H'_+$ on the fiber of $\widehat{\mathcal{X}}_g \rightarrow \mathcal{X}_g$ at each (Z, F, L, \bar{h}, q) . Let \mathcal{O}_1^\times be the group scheme of invertible Taylor series with constant term 1. This is a subgroup of the group scheme \mathcal{O}^\times of invertible Taylor series from [FBZ, §20.3.4]. One has $\text{Lie}(\mathcal{O}^\times) = H_+$ and $\text{Lie}(\mathcal{O}_1^\times) = H'_+$. The natural action of $\mathcal{S}p^+(H') \ltimes \mathcal{O}_1^\times$ on H' induces a natural action of $\mathcal{S}p^+(H') \ltimes \mathcal{O}_1^\times$ on the fibers of $\widehat{\mathcal{X}}_g \rightarrow \mathcal{X}_g$. Theorem 2.3 has the following extension:

Theorem 3.3. (i) The moduli space $\widehat{\mathcal{X}}_g$ carries a transitive action of $\mathfrak{sp}(H') \ltimes H'$ compatible with the transitive action of $Sp^+(H') \ltimes \mathcal{O}_1^\times$ on the fibers of $\widehat{\mathcal{X}}_g \rightarrow \mathcal{X}_g$.
(ii) The stabilizer of $Sp^+(H') \ltimes \mathcal{O}_1^\times$ at (Z, F, L, \bar{h}, q) in $\widehat{\mathcal{X}}_g$ is $Sp_F^+(H')$.

4. HARISH-CHANDRA LOCALIZATION

Here we review the formalism of localization of modules over Harish-Chandra pairs first introduced in [BB] and further developed in [BFM, BS] and [FBZ, §17.2]. This will be used in §9 for the constructions of the sheaves of coinvariants on \mathcal{A}_g and \mathcal{X}_g .

4.1. Harish-Chandra pairs and moduli of curves. A *Harish-Chandra pair* (\mathfrak{g}, K) consists of a Lie algebra \mathfrak{g} and a Lie group K equipped with an embedding $\mathfrak{k} := \text{Lie}(K) \hookrightarrow \mathfrak{g}$ and an action Ad of K on \mathfrak{g} . The action Ad is required to restrict to the adjoint action of K on $\mathfrak{k} \subset \mathfrak{g}$, and the differential of Ad at the identity element is required to be equal to the action of \mathfrak{k} on \mathfrak{g} .

Let V be a (\mathfrak{g}, K) -module, i.e., a vector space with compatible actions of \mathfrak{g} and K . The localization functor assigns to V a (possibly twisted) \mathcal{D} -module on a variety or stack identified by the Harish-Chandra pair. For instance, localizations of:

- modules over $(\text{Der}(\mathcal{O}), \text{Aut}(\mathcal{O}))$ yield \mathcal{D} -modules on a smooth curve (see notation from §3.4);
- modules over $(\text{Witt}, \text{Aut}(\mathcal{O}))$ yield \mathcal{D} -modules on the moduli space $\mathcal{M}_{g,1}$ of pointed smooth curves of genus g ; and
- modules over $(\text{Vir}, \text{Aut}(\mathcal{O}))$ yield *twisted* \mathcal{D} -modules on $\mathcal{M}_{g,1}$.

We refer to [FBZ, Ch. 17–18] for a discussion of these and more geometries.

4.2. Harish-Chandra pairs and moduli of abelian varieties. It is easy to verify that

$$(\mathfrak{sp}(H'), Sp^+(H')) \quad \text{and} \quad (\mathfrak{sp}(H') \ltimes H', Sp^+(H') \ltimes \mathcal{O}_1^\times),$$

with notation as in §3, are Harish-Chandra pairs. Also, one has that

$$(\mathfrak{mp}(H'), Sp^+(H')) \quad \text{and} \quad \left(\widetilde{\mathcal{U}}_2(H), Sp^+(H') \ltimes \mathcal{O}_1^\times \right)$$

are Harish-Chandra pairs. Indeed, the main point is to show the required Lie algebra embeddings. For this, it is immediate to see that the two-cocycle on $\mathfrak{sp}(H')$ defining $\mathfrak{mp}(H')$ from (1.6) vanishes on the Lie algebra $\mathfrak{sp}^+(H')$ from §3.1, hence one has a Lie algebra splitting

$$(4.1) \quad \mathfrak{sp}^+(H') \hookrightarrow \mathfrak{mp}(H').$$

Similarly, the two-cocycle on $\mathfrak{sp}(H') \ltimes H'$ defining $\widetilde{\mathcal{U}}_2(H)$ from (1.8) vanishes on $\mathfrak{sp}^+(H') \ltimes H'_+$, hence one has a Lie algebra splitting

$$(4.2) \quad \mathfrak{sp}^+(H') \ltimes H'_+ \hookrightarrow \widetilde{\mathcal{U}}_2(H).$$

We emphasize a difference here with respect to the examples in §4.1 involving moduli spaces of curves. The localization functor mentioned there yields (possibly twisted) \mathcal{D} -modules on $\mathcal{M}_{g,1}$ via descent along the principal $\text{Aut}(\mathcal{O})$ -bundle $\widehat{\mathcal{M}}_g \rightarrow \mathcal{M}_{g,1}$. We will construct (possibly twisted) \mathcal{D} -modules on \mathcal{A}_g via descent along $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$. However, while the group $\text{Aut}(\mathcal{O})$ acts simply transitively on the fibers of $\widehat{\mathcal{M}}_g \rightarrow \mathcal{M}_{g,1}$, the group $Sp^+(H')$ acts transitively, but not simply, on the fibers of $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$, as discussed in §3. Thus \mathcal{A}_g is realized as the quotient stack $[Sp^+(H') \backslash \widehat{\mathcal{A}}_g]$. Similarly, \mathcal{X}_g is realized as the quotient stack $[(Sp^+(H') \ltimes \mathcal{O}_1^\times) \backslash \widehat{\mathcal{X}}_g]$. For this, we need the following result from [BB] on the descent to a quotient stack, more general than the descent along principal bundles treated in [FBZ].

4.3. \mathcal{D} -algebras on quotient stacks. We review a result from [BB] characterizing \mathcal{D} -algebras and their modules on quotient stacks. We start with the definitions of Harish–Chandra Lie algebroids and their modules.

Let S be an arbitrary smooth scheme. A *Lie algebroid* on S is a quasicoherent \mathcal{O}_S -module P equipped with a map of \mathcal{O}_S -modules $\alpha: P \rightarrow \mathcal{T}_S$ and a \mathbb{C} -linear mapping $[,]: P \otimes_{\mathbb{C}} P \rightarrow P$ such that: $[,]$ is a Lie algebra bracket, α is a Lie algebra map, and for local sections $p_1, p_2 \in P$ and $f \in \mathcal{O}_S$, one has

$$[p_1, fp_2] = f[p_1, p_2] + \alpha(p_1)(f)p_2.$$

Examples of Lie algebroids are given by Atiyah algebras of vector bundles.

To a Lie algebroid $\alpha: P \rightarrow \mathcal{T}_S$ on S , one may assign its *universal enveloping \mathcal{D} -algebra* $\mathcal{U}(P)$. This is the associative \mathcal{O}_S -algebra generated by a subalgebra $i: \mathcal{O}_S \rightarrow \mathcal{U}(P)$ and a Lie subalgebra $j: P \rightarrow \mathcal{U}(P)$ subject to the relations

$$j(fp) = i(f)j(p) \quad \text{and} \quad [j(p), i(f)] = i(\alpha(p) \cdot f)$$

for $p \in P$ and $f \in \mathcal{O}_S$. The \mathcal{O}_S -algebra $\mathcal{U}(P)$ is a *\mathcal{D} -algebra*, as defined in [BB, §1.1]. The functor assigning to a Lie algebroid P its universal enveloping \mathcal{D} -algebra $\mathcal{U}(P)$ is a functor from the category of Lie algebroids to the category of \mathcal{D} -algebras and admits a right adjoint functor, see [BB, §1.2.5]. Modules for $\mathcal{U}(P)$ are the same as \mathcal{O}_S -modules with a compatible P -action. Modules for $\mathcal{U}(\mathcal{T}_S)$ are called *\mathcal{D} -modules*, and modules for $\mathcal{U}(P)$ for other Lie algebroids P are called *twisted \mathcal{D} -modules*.

Next we review the equivariant setting. Let K be an algebraic group, and let X be a smooth scheme equipped with a K -action $\mu: K \times X \rightarrow X$. A quasicoherent \mathcal{O}_X -module P is *K -equivariant* if there exists an isomorphism $\mu_P: \mu^*P \rightarrow p_X^*P$ of $\mathcal{O}_{K \times X}$ -modules that satisfies the cocycle condition (see [The, 03LF]). Here $p_X: K \times X \rightarrow X$ is the natural projection.

A *Lie algebroid* on the quotient stack $[K \backslash X]$ is the same as a Lie algebroid on X equipped with a K -equivariant structure as an \mathcal{O}_X -module [BB, §1.7].

A *Harish–Chandra Lie algebroid* on X is a Lie algebroid $\alpha: P \rightarrow \mathcal{T}_X$ equipped with a K -equivariant structure as an \mathcal{O}_X -module and a Lie algebra map $\iota_{\mathfrak{k}}: \mathfrak{k} \rightarrow P$ with $\mathfrak{k} := \text{Lie}(K)$ such that:

- One has $k^*[p_1, p_2] = [k^*p_1, k^*p_2]$ and $k^*(\alpha(p)) = \alpha(k^*p)$ for $k \in K$ and $p_i, p \in P$. Here, k^* is induced by the K -equivariant structure on P ;
- One has $\iota_{\mathfrak{k}}(\text{ad}_k \gamma) = k^* \iota_{\mathfrak{k}}(\gamma)$ for $k \in K$ and $\gamma \in \mathfrak{k}$;
- The action of \mathfrak{k} on P induced from the K -equivariant structure on P coincides with $\text{ad}_{\iota_{\mathfrak{k}}}$.

If X is a point, a Harish–Chandra Lie algebroid on X is the same as a Harish–Chandra pair as in §4.1.

As for Lie algebroids, to a Harish–Chandra Lie algebroid P on X , one may assign its universal enveloping \mathcal{D} -algebra $\mathcal{U}(P)$. This is now a *Harish–Chandra algebra*, i.e., a \mathcal{D} -algebra equipped with a K -action, as in [BB, §1.8.3]. The functor assigning to a Harish–Chandra Lie algebroid P its universal enveloping \mathcal{D} -algebra $\mathcal{U}(P)$ is a functor from the category of Harish–Chandra Lie algebroids to the category of Harish–Chandra algebras and admits a right adjoint functor, see [BB, §1.8.4].

For a Harish–Chandra algebra \mathcal{U} on X , a (\mathcal{U}, K) -module is a \mathcal{U} -module M with a K -equivariant structure as an \mathcal{O}_X -module compatible with the K -equivariant structure on \mathcal{U} (i.e., $k^*(um) = k^*(u)k^*(m)$ for $k \in K$, $u \in \mathcal{U}$ and $m \in M$) and such that the action of \mathfrak{k} on M induced from the K -equivariant structure coincides with the one coming from $\iota_{\mathfrak{k}}: \mathfrak{k} \rightarrow \mathcal{U}$ and the \mathcal{U} -module structure on M .

We will apply the following key result from [BB]:

Lemma 4.1 ([BB, Lemma 1.8.7]). *(i) Harish–Chandra algebras on X are the same as \mathcal{D} -algebras on $[K \backslash X]$.*
(ii) For a Harish–Chandra algebra \mathcal{U} , the (\mathcal{U}, K) -modules are the same as the modules over the corresponding \mathcal{D} -algebra on $[K \backslash X]$.

Example 4.2. When the action of K is free, Lemma 4.1(ii) is the usual descent of (\mathcal{U}, K) -modules along the principal K -bundle $X \rightarrow [K \backslash X]$ to twisted \mathcal{D} -modules on $[K \backslash X]$, see [BB, §1.8.9].

4.4. Harish–Chandra localization on quotient stacks. Let K and X be as in §4.3. Let \mathfrak{g} be a Lie algebra such that $\alpha: \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_X \rightarrow \mathcal{T}_X$ is a Harish–Chandra Lie algebroid on X with α surjective.

Proposition 4.3. *For a \mathfrak{g} -module V , the sheaf*

$$(4.3) \quad \widehat{\mathbb{V}}(V) := V \otimes_{\mathbb{C}} \mathcal{O}_X / \text{Ker } \alpha \cdot (V \otimes_{\mathbb{C}} \mathcal{O}_X)$$

is a \mathcal{D} -module on X . Moreover, if the induced action of $\mathfrak{k} \subset \mathfrak{g}$ can be exponentiated to a K -equivariant structure on $\widehat{\mathbb{V}}(V)$, then $\widehat{\mathbb{V}}(V)$ corresponds to a \mathcal{D} -module $\mathbb{V}(V)$ on $[K \backslash X]$.

Proof. By definition, $\widehat{\mathbb{V}}(V)$ is a quasi-coherent sheaf of \mathcal{O}_X -modules. Moreover, the action of \mathfrak{g} on V and on X induce an action of the Lie algebroid

$\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_X$ on $\widehat{\mathbb{V}}(V)$. Since $\text{Ker } \alpha \subset \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_X$ acts trivially on $\widehat{\mathbb{V}}(V)$ by definition (4.3), the action of $\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_X$ factors to an action of $\mathcal{F}_X = \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_X / \text{Ker } \alpha$ on $\widehat{\mathbb{V}}(V)$. Hence $\widehat{\mathbb{V}}(V)$ is a \mathcal{D} -module on X .

Furthermore, assume that the induced action of $\mathfrak{k} \subset \mathfrak{g}$ can be exponentiated to a K -equivariant structure on $\widehat{\mathbb{V}}(V)$. The fact that the K -equivariant structure is obtained by exponentiating the \mathfrak{k} -action implies that $\widehat{\mathbb{V}}(V)$ is additionally a (\mathcal{U}, K) -module for the Harish-Chandra algebra $\mathcal{U} = \mathcal{U}(\mathcal{F}_X)$. From Lemma 4.1, $\widehat{\mathbb{V}}(V)$ corresponds to a \mathcal{D} -module on $[K \setminus X]$. \square

Furthermore, assume that \mathfrak{g} has a central extension $0 \rightarrow \mathbb{C} \rightarrow \widehat{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow 0$ that admits splittings $\mathfrak{k} \hookrightarrow \widehat{\mathfrak{g}}$ and $\text{Ker } \alpha \hookrightarrow \widehat{\mathfrak{g}} \otimes_{\mathbb{C}} \mathcal{O}_X$.

Proposition 4.4. *For a $\widehat{\mathfrak{g}}$ -module V , the sheaf $\widehat{\mathbb{V}}(V)$ as in (4.3) is a twisted \mathcal{D} -module on X . Moreover, if the induced action of $\mathfrak{k} \subset \widehat{\mathfrak{g}}$ can be exponentiated to a K -equivariant structure on $\widehat{\mathbb{V}}(V)$, then $\widehat{\mathbb{V}}(V)$ corresponds to a twisted \mathcal{D} -module $\mathbb{V}(V)$ on $[K \setminus X]$.*

Proof. The action of $\widehat{\mathfrak{g}}$ on V and on X via the projection $\widehat{\mathfrak{g}} \rightarrow \mathfrak{g}$ induce an action of the Lie algebroid $\widehat{\mathfrak{g}} \otimes_{\mathbb{C}} \mathcal{O}_X$ on $\widehat{\mathbb{V}}(V)$. Since $\text{Ker } \alpha \subset \widehat{\mathfrak{g}} \otimes_{\mathbb{C}} \mathcal{O}_X$ acts trivially on $\widehat{\mathbb{V}}(V)$ by definition (4.3), the action of $\widehat{\mathfrak{g}} \otimes_{\mathbb{C}} \mathcal{O}_X$ factors to an action of the Lie algebroid $\widehat{\mathfrak{g}} \otimes_{\mathbb{C}} \mathcal{O}_X / \text{Ker } \alpha$ on $\widehat{\mathbb{V}}(V)$. Hence $\widehat{\mathbb{V}}(V)$ is a twisted \mathcal{D} -module on X .

Furthermore, assume that the induced action of $\mathfrak{k} \subset \widehat{\mathfrak{g}}$ can be exponentiated to a K -equivariant structure on $\widehat{\mathbb{V}}(V)$. Then $\widehat{\mathbb{V}}(V)$ is additionally a (\mathcal{U}, K) -module for the Harish-Chandra algebra $\mathcal{U} = \mathcal{U}(\widehat{\mathfrak{g}} \otimes_{\mathbb{C}} \mathcal{O}_X / \text{Ker } \alpha)$. From Lemma 4.1, $\widehat{\mathbb{V}}(V)$ corresponds to a twisted \mathcal{D} -module on $[K \setminus X]$. \square

In case the Harish-Chandra Lie algebroid $\widehat{\mathfrak{g}} \otimes_{\mathbb{C}} \mathcal{O}_X / \text{Ker } \alpha$ on X is isomorphic to the Atiyah algebra \mathcal{F}_L of the pull-back of a line bundle L from $[K \setminus X]$, one has:

Proposition 4.5. *For a $\widehat{\mathfrak{g}}$ -module V , if the induced action of $\mathfrak{k} \subset \widehat{\mathfrak{g}}$ can be exponentiated to a K -equivariant structure on $\widehat{\mathbb{V}}(V)$, then the twisted \mathcal{D} -module structure on $\mathbb{V}(V)$ is induced by the action of \mathcal{F}_L .*

Proof. From the proof of Proposition 4.4, $\widehat{\mathbb{V}}(V)$ is a (\mathcal{U}, K) -module for the Harish-Chandra algebra $\mathcal{U} = \mathcal{U}(\mathcal{F}_L)$ on X . Thus from Lemma 4.1, $\mathbb{V}(V)$ is a module for the \mathcal{D} -algebra $\mathcal{U}(\mathcal{F}_L)$ on $[K \setminus X]$. Hence the statement. \square

5. ON THE COHOMOLOGY OF THE LIE ALGEBRAS

Here we study the cohomology of the Lie algebras $\mathfrak{sp}(H')$ and $\mathfrak{sp}(H') \ltimes H'$. We will then show in Theorem 6.1 that their H^2 spaces are canonically isomorphic to $H^2(\mathcal{A}_g, \mathbb{C})$ and $H^2(\mathcal{X}_g, \mathbb{C})$, respectively.

For a topological Lie algebra \mathfrak{g} , let $H^*(\mathfrak{g}, \mathbb{C})$ be its continuous Lie algebra cohomology with complex coefficients.

Recall the two-cocycle ψ of $\mathfrak{gl}(H)$ from (1.1), and consider the two-cocycles on $\mathfrak{sp}(H') \ltimes H'$

$$\begin{aligned}\alpha(X + f, Y + g) &:= \psi(X, Y), \\ \beta(X + f, Y + g) &:= \psi(f, g) = -\text{Res}_{t=0} f dg, \\ \gamma(X + f, Y + g) &:= \psi(X, g) - \psi(Y, f)\end{aligned}$$

for $X, Y \in \mathfrak{sp}(H')$ and $f, g \in H'$, where f, g act on H by multiplication. Hence, $\psi = \alpha + \beta + \gamma$ on $\mathfrak{sp}(H') \ltimes H'$. Restricting to $\text{Witt} \ltimes H'$, one has

$$\begin{aligned}\alpha(f\partial_t + g, h\partial_t + k) &= \frac{1}{6} \text{Res}_{t=0} f dh'', \\ \gamma(f\partial_t + g, h\partial_t + k) &= -\frac{1}{2} \text{Res}_{t=0} (f dk' - h dg').\end{aligned}$$

Here we use the notation $k' := \partial_t k$, and likewise for g' and h'' . Similarly, consider the two-cocycle $\alpha(X, Y) := \psi(X, Y)$ on $\mathfrak{sp}(H')$. It is shown in [ADCKP, 2.1] that $H^1(\text{Witt}, \mathbb{C}) = H^1(\text{Witt} \ltimes H', \mathbb{C}) = 0$, and

$$H^2(\text{Witt}, \mathbb{C}) = \mathbb{C}\bar{\alpha} \quad \text{and} \quad H^2(\text{Witt} \ltimes H', \mathbb{C}) = \mathbb{C}\bar{\alpha} \oplus \mathbb{C}\bar{\beta} \oplus \mathbb{C}\bar{\gamma}$$

where $\bar{\alpha}$ is the class of α , and similarly for $\bar{\beta}$ and $\bar{\gamma}$. Recall $\tau: \text{Witt} \hookrightarrow \mathfrak{sp}(H')$ from (1.10) and $\sigma: \text{Witt} \ltimes H' \hookrightarrow \mathfrak{sp}(H') \ltimes H'$ from (1.12).

Proposition 5.1. *One has*

$$(5.1) \quad H^1(\mathfrak{sp}(H'), \mathbb{C}) = 0,$$

$$(5.2) \quad H^1(\mathfrak{sp}(H') \ltimes H', \mathbb{C}) = 0,$$

$$(5.3) \quad H^2(\mathfrak{sp}(H'), \mathbb{C}) = \mathbb{C}\bar{\alpha},$$

$$(5.4) \quad H^2(\mathfrak{sp}(H') \ltimes H', \mathbb{C}) = \mathbb{C}\bar{\alpha} \oplus \mathbb{C}\bar{\beta}.$$

Moreover, the pull-back via τ induces the identification

$$\tau^*: H^2(\mathfrak{sp}(H'), \mathbb{C}) \xrightarrow{\cong} H^2(\text{Witt}, \mathbb{C}),$$

and the pull-back via σ induces the injection

$$\sigma^*: H^2(\mathfrak{sp}(H') \ltimes H', \mathbb{C}) \hookrightarrow H^2(\text{Witt} \ltimes H', \mathbb{C}), \quad \bar{\alpha} \mapsto \bar{\alpha}, \quad \bar{\beta} \mapsto \frac{1}{2}\bar{\alpha} + \bar{\beta}.$$

Remark 5.2. The two-cocycle defining the central extension $\mathfrak{mp}(H')$ of $\mathfrak{sp}(H')$ in (1.6) is $-\frac{1}{2}\alpha$. Also, the two-cocycle defining the central extension $\widetilde{\mathcal{U}}_2(H)$ of $\mathfrak{sp}(H') \ltimes H'$ in (1.8) is $-\frac{1}{2}\alpha + \beta$.

Proof. Let H'_f be the functor which assigns to a \mathbb{C} -algebra R the R -subalgebra $H'_f(R) := R[t, t^{-1}] \subset H'(R)$. One checks that

$$\begin{aligned}S^2(H'_f) &= [S^2(H'_f), S^2(H'_f)] \quad \text{and} \\ S^2(H'_f) \ltimes H'_f &= [S^2(H'_f) \ltimes H'_f, S^2(H'_f) \ltimes H'_f].\end{aligned}$$

Since H'_f is dense in H' and $S^2(H')$ is dense in $\mathfrak{sp}(H')$, we deduce

$$\begin{aligned}\mathfrak{sp}(H') &= [\mathfrak{sp}(H'), \mathfrak{sp}(H')] \quad \text{and} \\ \mathfrak{sp}(H') \ltimes H' &= [\mathfrak{sp}(H') \ltimes H', \mathfrak{sp}(H') \ltimes H']\end{aligned}$$

hence (5.1) and (5.2) follow.

Next we prove (5.4). Assume δ is an arbitrary two-cocycle on $\mathfrak{sp}(H') \ltimes H'$. In particular, δ satisfies the two-cocycle relation

$$(5.5) \quad \delta(x, [y, z]) + \delta(y, [z, x]) + \delta(z, [x, y]) = 0$$

for all $x, y, z \in \mathfrak{sp}(H') \ltimes H'$. The pull-back of δ via σ^* is a two-cocycle on $\text{Witt} \ltimes H'$. Recall the relations

$$(5.6) \quad [L_p, L_q] = (p - q)L_{p+q}, \quad [L_p, b_q] = -q b_{p+q}, \quad [b_p, b_q] = 0$$

on $\text{Witt} \ltimes H'$, where $L_p = -t^{p+1}\partial_t$ and $b_q = t^q$ for $p, q \in \mathbb{Z}$. Considering terms of type

$$\delta(L_p, [L_q, L_r]), \quad \delta(L_p, [L_q, b_r]), \quad \delta(L_p, [b_q, b_r])$$

and imposing the two-cocycle relation (5.5) for all $x, y, z \in \text{Witt} \ltimes H'$ and the relations (5.6), a direct computation as in the proof of [ADCKP, 2.1] shows that

$$\begin{aligned}\delta(X + f, Y + g) &\equiv A\alpha(X + f, Y + g) + B\beta(X + f, Y + g) \\ &\quad + C\gamma(X + f, Y + g)\end{aligned}$$

modulo coboundaries, for some $A, B, C \in \mathbb{C}$. Next, a further constraint is imposed by elements of $\mathfrak{sp}(H') \ltimes H'$ not in $\text{Witt} \ltimes H'$. Consider the two-cocycle relation for

$$x = b_1 b_1, \quad y = b_1 b_{-2}, \quad \text{and} \quad z = b_{-1}.$$

Observe that

$$[b_1 b_{-2}, b_{-1}] = b_{-2}, \quad [b_{-1}, b_1 b_1] = -2b_1, \quad [b_1 b_1, b_1 b_{-2}] = 0.$$

Since

$$\gamma(b_1 b_1, b_{-2}) = 2 \quad \text{and} \quad \gamma(b_1 b_{-2}, b_1) = 0,$$

it follows that the only non-trivial contribution to the two-cocycle relation is $C\gamma(b_1 b_1, [b_1 b_{-2}, b_{-1}]) = 0$, hence $C = 0$. A direct analysis shows that the two-cocycle relations do not impose any relations on A and B , hence (5.4).

The identification in (5.3) follows similarly from the argument for (5.4).

For the final part of the statement, recall the map $\widehat{\sigma}: \mathfrak{D} \hookrightarrow \widetilde{\mathcal{U}}_2(H)$ from (1.13). Since the two-cocycle on $\text{Witt} \ltimes H'$ defining \mathfrak{D} is ψ , and the two-cocycle on $\mathfrak{sp}(H') \ltimes H'$ defining $\widetilde{\mathcal{U}}_2(H)$ is $-\frac{1}{2}\alpha + \beta$, we deduce $\sigma^*(-\frac{1}{2}\alpha + \beta) = \psi$. A similar argument involving the map $\widehat{\tau}$ from (1.11) shows that $\sigma^*(\alpha) = \tau^*(\alpha) = \alpha$, hence the statement. \square

For a topological Lie algebra \mathfrak{g} , the space $H^2(\mathfrak{g}, \mathbb{C})$ classifies Lie algebra continuous central extensions of \mathfrak{g} up to isomorphism. Thus as a consequence of the previous statement, we have:

Proposition 5.3. *Let Z be a Lagrangian subspace of H' with $Z \cap H'_+ = 0$. Every Lie algebra continuous central extension*

$$0 \rightarrow \mathfrak{gl}_1 \rightarrow \widehat{\mathfrak{sp}(H')} \rightarrow \mathfrak{sp}(H') \rightarrow 0$$

—e.g., $\mathfrak{mp}(H')$ —splits over $\mathfrak{sp}_F(H')$ for all $F \subset Z$. Similarly, every Lie algebra continuous central extension

$$0 \rightarrow \mathfrak{gl}_1 \rightarrow \widehat{\mathfrak{sp}(H')} \ltimes H' \rightarrow \mathfrak{sp}(H') \ltimes H' \rightarrow 0$$

—e.g., $\widetilde{\mathcal{U}}_2(H)$ —splits over $\mathfrak{sp}_F(H') \ltimes F$ for all $F \subset Z$.

Proof. One needs to show that the restriction of $H^2(\mathfrak{sp}(H'), \mathbb{C})$ to $\mathfrak{sp}_F(H')$ (respectively, the restriction of $H^2(\mathfrak{sp}(H') \ltimes H', \mathbb{C})$ to $\mathfrak{sp}_F(H') \ltimes F$) vanishes for all $F \subset Z$. From Proposition 5.1, it is enough to consider the class of the two-cocycle α (resp., the classes of the two-cocycles α and β). Also, after rescaling to $-\frac{1}{2}\alpha$ as in Remark 5.2, we can reduce to the case $\widehat{\mathfrak{sp}(H')} = \mathfrak{mp}(H')$.

For $Z = H_-$ and $F \subset Z$, one has $F \subset H_- \subset F^\perp$, thus for $X \in \mathfrak{sp}_F(H')$, one has

$$\text{Im}(\pi_+ X \pi_-) = \pi_+ X(H_-) \subset \pi_+ X(F^\perp) \subseteq \pi_+ F = 0.$$

It follows that for $F \subset H_-$, one has $\alpha(X, Y) = 0$ for all X, Y in $\mathfrak{sp}_F(H')$. Next, consider an arbitrary Lagrangian subspace Z of H' with $Z \cap H'_+ = 0$ and a subspace $F \subset Z$. From §3.2, the group scheme $\mathcal{Sp}^+(H')$ acts transitively on the space of such pairs (Z, F) , hence there exists $\rho \in \mathcal{Sp}^+(H')$ such that $(Z, F) = \rho(H_-, \overline{F})$ for some $\overline{F} \subset H_-$. Let $\mathfrak{mp}_F(H')$ be the restriction of $\mathfrak{mp}(H')$ over $\mathfrak{sp}_F(H')$ and $\mathfrak{mp}_{\overline{F}}(H')$ the restriction over $\mathfrak{sp}_{\overline{F}}(H')$. The adjoint action of the group ind-scheme $\mathcal{Mp}(H')$ on $\mathfrak{mp}(H')$ factors through $\mathcal{Sp}(H')$, hence

$$\text{Ad } \rho(\mathfrak{sp}_{\overline{F}}(H')) = \mathfrak{sp}_F(H') \quad \text{and} \quad \text{Ad } \rho(\mathfrak{mp}_{\overline{F}}(H')) = \mathfrak{mp}_F(H').$$

Since we have shown that $\mathfrak{mp}_{\overline{F}}(H')$ is a trivial extension, it follows that $\mathfrak{mp}_F(H')$ is a trivial extension, hence $\alpha(X, Y) = 0$ for all X, Y in $\mathfrak{sp}_F(H')$.

The case of α on $\mathfrak{sp}(H') \ltimes H'$ follows from α on $\mathfrak{sp}(H')$, and the case of β follows immediately from F being isotropic with respect to (1.2). \square

6. THE ISOMORPHISM OF SECOND COHOMOLOGY SPACES

Consider a smooth variety X over \mathbb{C} carrying a transitive action of a Lie algebra \mathfrak{g} (as defined in §2.3). For $x \in X$, let $\mathfrak{g}_x := \text{Ker}(\mathfrak{g} \rightarrow T_x(X))$. It is shown in [ADCKP, 4.1] that if $\mathfrak{g}_x = \overline{[\mathfrak{g}_x, \mathfrak{g}_x]}$ for all $x \in X$, then every Lie algebra continuous central extension

$$0 \rightarrow \mathbb{C} \rightarrow \widehat{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow 0$$

which splits over \mathfrak{g}_x for all $x \in X$ yields a continuous extension

$$0 \rightarrow X \times \mathbb{C} \rightarrow E \rightarrow T(X) \rightarrow 0.$$

This defines a canonical map

$$(6.1) \quad H^2(\mathfrak{g}, \mathbb{C})_0 \rightarrow \text{Ext}^1(\mathcal{T}_X, \mathcal{O}_X) = H^1(\Omega_X^1)$$

where $H^2(\mathfrak{g}, \mathbb{C})_0 \subset H^2(\mathfrak{g}, \mathbb{C})$ is the subspace obtained by intersecting the kernels of the restriction maps $H^2(\mathfrak{g}, \mathbb{C}) \rightarrow H^2(\mathfrak{g}_x, \mathbb{C})$ for all $x \in X$. Moreover, one has a canonical homomorphism

$$c: H^1(\mathcal{O}_X^*) \rightarrow H^1(\Omega_X^1)$$

mapping the class of each line bundle L to the extension class given by the sheaf of differential operators of order less than or equal to 1 acting on L .

Combining Theorems 2.2 and 2.3 with Proposition 5.3, the above result from [ADCKP] applies to give canonical maps

$$\begin{aligned} \nu: H^2(\mathfrak{sp}(H'), \mathbb{C}) &\rightarrow H^1(\Omega_{\widehat{\mathcal{A}}_g}^1), \\ \mu: H^2(\mathfrak{sp}(H') \ltimes H', \mathbb{C}) &\rightarrow H^1(\Omega_{\widehat{\mathcal{X}}_g}^1). \end{aligned}$$

We determine these maps explicitly in terms of the basis elements of the H^2 spaces given in Proposition 5.1 and the line bundle classes from §2.7:

Theorem 6.1. *One has*

$$(6.2) \quad \lambda = -\nu[\alpha], \quad \text{for } g \geq 3,$$

$$(6.3) \quad \lambda = -\mu[\alpha], \quad \xi = \mu[\alpha] - 2\mu[\beta], \quad \text{for } g \geq 5.$$

In particular, the image of μ and ν equal the span of the image of the corresponding map c .

Proof. One has a commutative diagram

$$(6.4) \quad \begin{array}{ccccc} H^2(\mathfrak{sp}(H'), \mathbb{C}) & \xrightarrow{\nu} & H^1(\Omega_{\widehat{\mathcal{A}}_g}^1) & \xleftarrow{c} & H^1(\mathcal{O}_{\widehat{\mathcal{A}}_g}^*) \\ \tau^* \downarrow \parallel & & \downarrow & & \downarrow \parallel \\ H^2(\text{Witt}, \mathbb{C}) & \xrightarrow{\bar{\nu}} & H^1(\Omega_{\widehat{\mathcal{M}}_g}^1) & \xleftarrow{\quad} & H^1(\mathcal{O}_{\widehat{\mathcal{M}}_g}^*) \end{array}$$

where the map τ^* is the identification studied in Proposition 5.1, the map $\bar{\nu}$ follows from (6.1), and the two vertical maps on the right-hand side are induced by the pull-back via the extended Torelli map as in Proposition 2.4. From [ADCKP, 4.10.iv], one has $\lambda = -\bar{\nu}[\alpha]$. Hence, (6.2) follows.

Furthermore, one has a commutative diagram

$$(6.5) \quad \begin{array}{ccccc} H^2(\mathfrak{sp}(H') \ltimes H', \mathbb{C}) & \xrightarrow{\mu} & H^1(\Omega_{\widehat{\mathcal{X}}_g}^1) & \xleftarrow{c} & H^1(\mathcal{O}_{\widehat{\mathcal{A}}_g}^*) \\ \sigma^* \downarrow & & \downarrow & & \downarrow \\ H^2(\text{Witt} \ltimes H', \mathbb{C}) & \xrightarrow{\bar{\mu}} & H^1(\Omega_{\widehat{\mathcal{P}}_{g-1}}^1) & \xleftarrow{\quad} & H^1(\mathcal{O}_{\widehat{\mathcal{M}}_g}^*) \end{array}$$

where the map σ^* is the injection studied in Proposition 5.1, the map $\bar{\mu}$ follows from (6.1), and the two vertical maps on the right-hand side are induced by the pull-back via the extended Torelli map as in Proposition 2.4. From [ADCKP, 4.10], one has $\lambda = -\bar{\mu}[\alpha]$ and $\xi = -2\bar{\mu}[\psi]$. Applying the description of σ^* from Proposition 5.1, (6.3) follows. \square

Consequently, we prove Theorem 5 from the intro, here restated:

Theorem 6.2. *The maps ν and μ induce canonical homomorphisms*

$$\begin{aligned}\nu: H^2(\mathfrak{sp}(H'), \mathbb{C}) &\rightarrow H^2(\mathcal{A}_g, \mathbb{C}), \\ \mu: H^2(\mathfrak{sp}(H') \ltimes H', \mathbb{C}) &\rightarrow H^2(\mathcal{X}_g, \mathbb{C}),\end{aligned}$$

with ν being an isomorphism for $g \geq 3$, and μ an isomorphism for $g \geq 5$.

Proof. The argument is similar to [ADCKP, pg. 30]. Consider first the map ν . Since $\widehat{\mathcal{A}}_g$ is not complete, we cannot apply Hodge theory to obtain a natural inclusion $H^1(\Omega_{\widehat{\mathcal{A}}_g}^1) \rightarrow H^2(\widehat{\mathcal{A}}_g, \mathbb{C})$, and it is unclear whether such an inclusion exists. Instead, one proceeds as follows. From Theorem 6.1, the image of ν equals the span of the image of c . Let

$$c_1: H^1(\mathcal{O}_{\widehat{\mathcal{A}}_g}^*) \rightarrow H^2(\widehat{\mathcal{A}}_g, \mathbb{C})$$

be the first Chern class map. For the class of a line bundle L , its image via c_1 is the de Rham class of $c[L]$. From Proposition 2.4, the space $H^2(\widehat{\mathcal{A}}_g, \mathbb{C})$ is generated by λ . Thus mapping the image of ν to its de Rham class yields a canonical isomorphism (still denoted ν)

$$\nu: H^2(\mathfrak{sp}(H'), \mathbb{C}) \rightarrow H^2(\widehat{\mathcal{A}}_g, \mathbb{C}).$$

Since one has $H^2(\widehat{\mathcal{A}}_g, \mathbb{C}) \cong H^2(\mathcal{A}_g, \mathbb{C})$ from Proposition 2.4, the statement follows. The statement about μ follows similarly from Theorem 6.1 and Proposition 2.4. \square

Thus we prove Theorem 6, here restated:

Theorem 6.3. (i) *For $g \geq 3$, the line bundle Λ on $\widehat{\mathcal{A}}_g$ carries an action of $\mathfrak{mp}(H')$ by first-order differential operators extending the transitive action of $\mathfrak{sp}(H')$ on $\widehat{\mathcal{A}}_g$ and with the central element $\mathbf{1} \in \mathfrak{mp}(H')$ acting as multiplication by 2 on the fibers of $\Lambda \rightarrow \widehat{\mathcal{A}}_g$.*
(ii) *For $g \geq 5$, the line bundle Ξ on $\widehat{\mathcal{X}}_g$ carries an action of $\widetilde{\mathcal{U}}_2(H)$ by first-order differential operators extending the transitive action of $\mathfrak{sp}(H') \ltimes H'$ on $\widehat{\mathcal{X}}_g$ and with the central element $\mathbf{1} \in \widetilde{\mathcal{U}}_2(H)$ acting as multiplication by -2 on the fibers of $\Xi \rightarrow \widehat{\mathcal{X}}_g$.*

Proof. From Remark 5.2, the two-cocycle defining the central extension $\mathfrak{mp}(H')$ of $\mathfrak{sp}(H')$ in (1.6) is $-\frac{1}{2}\alpha$. From (6.2), one has $\nu[-\frac{1}{2}\alpha] = \frac{1}{2}\lambda$ for $g \geq 3$. Thus part (i) follows by definition of the map ν .

Part (ii) follows similarly: from Remark 5.2, the two-cocycle defining the central extension $\widetilde{\mathcal{U}}_2(H)$ of $\mathfrak{sp}(H') \ltimes H'$ in (1.8) is $-\frac{1}{2}\alpha + \beta$, and by (6.3), one has $\mu[-\frac{1}{2}\alpha + \beta] = -\frac{1}{2}\xi$ for $g \geq 5$. \square

The spaces and line bundles appearing in Theorem 6 are summarized by the commutative diagram:

$$(6.6) \quad \begin{array}{ccccc} & \Lambda & & \Xi & \\ & \swarrow & \downarrow & \searrow & \\ \Lambda & & \widehat{\mathcal{A}}_g & & \Xi \\ \downarrow & & \longleftarrow & \downarrow & \downarrow \\ & \swarrow & & \searrow & \\ \widehat{\mathcal{M}}_g & & \widehat{\mathcal{P}}_{g-1} & & \widehat{\mathcal{X}}_g \end{array}$$

where the left and right squares are Cartesian. This diagram provides a geometric counterpart to Figure 1. The conclusions of Theorem 6 are summarized by the following diagrams. Let $(Z, F, L) \rightarrow S$ be a family of extended abelian varieties of dimension g over a smooth base S . Recall that Proposition 5.3 gives the splitting $\mathfrak{sp}_F(H') \hookrightarrow \mathfrak{mp}(H')$. Let \mathcal{F}_Λ be the Atiyah algebra of the line bundle Λ on $\widehat{\mathcal{A}}_g$. Theorem 6 implies the following commutative diagram of sheaves of Lie algebras with exact rows and columns:

$$(6.7) \quad \begin{array}{ccccccc} & \mathcal{O}_S & \xrightarrow{\frac{1}{2}} & \mathcal{O}_S & & & \\ & \downarrow & & \downarrow & & & \\ \mathfrak{sp}_F(H'(\mathcal{O}_S)) & \hookrightarrow & \mathfrak{mp}(H'(\mathcal{O}_S)) & \twoheadrightarrow & \mathcal{F}_{\Lambda|S} & & \\ \downarrow \parallel & & \downarrow & & \downarrow & & \\ \mathfrak{sp}_F(H'(\mathcal{O}_S)) & \hookrightarrow & \mathfrak{sp}(H'(\mathcal{O}_S)) & \twoheadrightarrow & \mathcal{T}_S & & \end{array}$$

Similarly, consider a family $(Z, F, L, \bar{h}, q) \rightarrow S$ over a smooth base S with g fixed as in §2.4. Proposition 5.3 gives the splitting $\mathfrak{sp}_F(H') \ltimes F \hookrightarrow \widetilde{\mathcal{U}}_2(H)$. Let \mathcal{F}_Ξ be the Atiyah algebra of the line bundle Ξ on $\widehat{\mathcal{X}}_g$. Theorem 6 implies the following commutative diagram of sheaves of Lie algebras with exact

rows and columns:

$$(6.8) \quad \begin{array}{ccccc} \mathcal{O}_S & \xrightarrow{-\frac{1}{2}} & \mathcal{O}_S \\ \downarrow & & \downarrow \\ \mathfrak{sp}_F(H'(\mathcal{O}_S)) \ltimes F(\mathcal{O}_S) & \hookrightarrow & \widetilde{\mathcal{U}}_2(H(\mathcal{O}_S)) & \twoheadrightarrow & \mathcal{F}_{\Xi|S} \\ \downarrow \| & & \downarrow & & \downarrow \\ \mathfrak{sp}_F(H'(\mathcal{O}_S)) \ltimes F(\mathcal{O}_S) & \hookrightarrow & \mathfrak{sp}(H'(\mathcal{O}_S)) \ltimes H'(\mathcal{O}_S) & \twoheadrightarrow & \mathcal{T}_S. \end{array}$$

7. INTERMEZZO ON METAPLECTIC REPRESENTATIONS

Here we define some properties of representations of the metaplectic algebra which will be used in the following sections.

Definition 7.1. (i) An *admissible $\mathfrak{mp}(H'(\mathbb{C}))$ representation* V is a representation of $\mathfrak{mp}(H'(\mathbb{C}))$ such that:

- (a) the action of $b_{-i} b_i$ on V is diagonalizable with integral eigenvalues for $i \geq 1$, and
- (b) the action of $b_i b_j$ on V is locally nilpotent for i, j not both negative and $i + j \neq 0$.

(ii) An *admissible $\widetilde{\mathcal{U}}_2(H(\mathbb{C}))$ representation* V is a representation of $\widetilde{\mathcal{U}}_2(H(\mathbb{C}))$ which induces an *admissible $\mathfrak{mp}(H'(\mathbb{C}))$ representation* on V and such that the action of $H'_+ \subset \widetilde{\mathcal{U}}_2(H(\mathbb{C}))$ is locally nilpotent on V .

We will also use the following induced characterization. Consider that a representation V of the Virasoro algebra is said to be of central charge $c \in \mathbb{C}$ if the actions of the Virasoro operators on V satisfy

$$[L_p, L_q] = (p - q) L_{p+q} + \frac{c}{12} (p^3 - p) \delta_{p+q, 0} \text{id}_V \quad \text{for } p, q \in \mathbb{Z}.$$

Recall the inclusion (1.11).

Definition 7.2. A representation V of $\mathfrak{mp}(H'(\mathbb{C}))$ (respectively, $\widetilde{\mathcal{U}}_2(H(\mathbb{C}))$) is said to be *of central charge* $c \in \mathbb{C}$ if the action of $\mathfrak{mp}(H'(\mathbb{C}))$ (resp., $\widetilde{\mathcal{U}}_2(H(\mathbb{C}))$) induces an action of Vir on V of central charge c .

7.1. Examples. Examples of admissible $\mathfrak{mp}(H'(\mathbb{C}))$ representations are given by certain vertex operator algebras for which the action of the Virasoro algebra extends to an action of the metaplectic algebra. We refer to [FHL, Kac] for treatments of vertex operator algebras. Briefly, a vertex operator algebra is a $\mathbb{Z}_{\geq 0}$ -graded complex vector space V together with a distinguished degree-0 element $\mathbf{1}$, a distinguished degree-2 element ω , and a linear map $Y(, t): V \rightarrow \text{End}(V)[[t, t^{-1}]]$, satisfying suitable conditions. The Fourier coefficients of $Y(\omega, t)$ realize an action of the Virasoro algebra on V .

Via the inclusion (1.11), the Virasoro element L_p is realized in the metaplectic algebra as $\frac{1}{2} \sum_i :b_{-i} b_{i+p}:$. The element L_0 acts on a vertex operator

algebra V as the grading operator, and the elements L_p with $p > 0$ have negative degree on V . Consequently, the action of $\frac{1}{2} \sum_i : b_{-i} b_i :$ on V is diagonalizable with integral eigenvalues and the action of $\frac{1}{2} \sum_i : b_{-i} b_{i+p} :$ on V with $p > 0$ is locally nilpotent. It follows that the conditions defining admissible $\mathfrak{mp}(H'(\mathbb{C}))$ representations in Definition 7.1 provide a strengthening of these properties.

For instance, we have:

Lemma 7.3. *Heisenberg vertex algebras of arbitrary rank and even lattice vertex algebras are admissible $\widetilde{\mathcal{U}}_2(H(\mathbb{C}))$ representations.*

Proof. Let V be either the rank-one Heisenberg vertex algebra or a rank-one even lattice vertex algebra. The degree of the operator b_i on V is $-i$, hence the degree of $b_i b_j$ on V is $-(i+j)$. In particular, H'_+ and $\widetilde{S}^2(H'_+)$ act on V by operators of negative degree, hence H'_+ and $\widetilde{S}^2(H'_+)$ act locally nilpotently on V .

Next, consider $b_i b_j$ with $i > 0$ and $j < 0$. If $i + j > 0$, then the degree of $b_i b_j$ on V is still negative, hence $b_i b_j$ acts locally nilpotently on V . If $i + j < 0$, then the degree of $b_i b_j$ on V is positive; however, b_i and b_j commute, and the degree of b_i is negative, hence $b_i b_j$ still acts locally nilpotently on V .

Finally, it is easy to see that the action of $b_{-i} b_i$ on V for $i \geq 1$ is diagonalizable with integral eigenvalues.

The case of arbitrary rank is similar. \square

8. COINVARIANTS ON FAMILIES OF EXTENDED ABELIAN VARIETIES

We define here spaces of coinvariants at extended abelian varieties and show how these yield twisted \mathcal{D} -modules on $\widehat{\mathcal{A}}_g$. Using results from §6, we identify a multiple of the Atiyah algebra of the line bundle Λ on $\widehat{\mathcal{A}}_g$ which acts on the sheaves of coinvariants and thus determines their twisted \mathcal{D} -module structure. Similarly, we define and study twisted \mathcal{D} -modules of coinvariants on $\widehat{\mathcal{X}}_g$.

8.1. Spaces of coinvariants at extended PPAVs. Let (Z, F, L) be an extended abelian variety over a \mathbb{C} -algebra R . Consider a representation V of $\mathfrak{mp}(H'(\mathbb{C}))$. The action of $\mathfrak{mp}(H'(\mathbb{C}))$ on V extends R -linearly to an action of $\mathfrak{mp}(H'(R))$ on $V \otimes_{\mathbb{C}} R$. Composing with the splitting $\mathfrak{sp}_F(H') \hookrightarrow \mathfrak{mp}(H')$ from Proposition 5.3, one has an action of $\mathfrak{sp}_F(H'(R))$ on $V \otimes_{\mathbb{C}} R$. We define the *space of coinvariants* of V at (Z, F, L) as

$$\widehat{\mathbb{V}}(V)_{(Z, F, L)} := V \otimes_{\mathbb{C}} R / \mathfrak{sp}_F(H'(R))(V \otimes_{\mathbb{C}} R).$$

8.2. Sheaves of coinvariants on $\widehat{\mathcal{A}}_g$. The spaces of coinvariants induce sheaves of coinvariants as follows. Let $(Z, F, L) \rightarrow S$ be a family of extended abelian varieties of dimension g over a smooth base S . Consider a representation V of $\mathfrak{mp}(H'(\mathbb{C}))$. The action of $\mathfrak{mp}(H'(\mathbb{C}))$ on V extends \mathcal{O}_S -linearly

to an action of the sheaf of Lie algebras $\mathfrak{mp}(H'(\mathcal{O}_S))$ on the sheaf $V \otimes_{\mathbb{C}} \mathcal{O}_S$. This action restricts to an action of the sheaf of Lie algebras $\mathfrak{sp}_F(H'(\mathcal{O}_S))$ on $V \otimes_{\mathbb{C}} \mathcal{O}_S$ via the splitting $\mathfrak{sp}_F(H'(\mathcal{O}_S)) \hookrightarrow \mathfrak{mp}(H'(\mathcal{O}_S))$ as in (6.7). We define the *sheaf of coinvariants* of V on $(Z, F, L) \rightarrow S$ as the quasi-coherent sheaf of \mathcal{O}_S -modules

$$\widehat{\mathbb{V}}(V)_{(Z,F,L) \rightarrow S} := V \otimes_{\mathbb{C}} \mathcal{O}_S / \mathfrak{sp}_F(H'(\mathcal{O}_S))(V \otimes_{\mathbb{C}} \mathcal{O}_S).$$

This gives rise to a quasi-coherent sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{A}}_g$.

Theorem 8.1. *For a representation V of $\mathfrak{mp}(H'(\mathbb{C}))$ of central charge c , the sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{A}}_g$ carries an action of the Atiyah algebra $\frac{c}{2}\mathcal{F}_{\Lambda}$. This action induces a twisted \mathcal{D} -module structure on $\widehat{\mathbb{V}}(V)$.*

Proof. For a family of extended abelian varieties of dimension g over a smooth base S , the action of $\mathfrak{mp}(H'(\mathbb{C}))$ on V and its action on $\widehat{\mathcal{A}}_g$ via the projection $\mathfrak{mp}(H') \rightarrow \mathfrak{sp}(H')$ induce an action of $\mathfrak{mp}(H'(\mathcal{O}_S))$ on the sheaf $\widehat{\mathbb{V}}(V)$. Explicitly, the action of $\mathfrak{mp}(H'(\mathcal{O}_S))$ on $V \otimes_{\mathbb{C}} \mathcal{O}_S$ is given by

$$X \cdot (v \otimes_{\mathbb{C}} f) := (X \cdot v) \otimes_{\mathcal{O}_S} f + v \otimes_{\mathbb{C}} (X \cdot f)$$

for local sections $X \in \mathfrak{mp}(H'(\mathcal{O}_S))$ and $v \otimes_{\mathbb{C}} f \in V \otimes_{\mathbb{C}} \mathcal{O}_S$. While it is not linear in $f \in \mathcal{O}_S$, this action extends the \mathcal{O}_S -linear action of $\mathfrak{sp}_F(H'(\mathcal{O}_S))$ on $V \otimes_{\mathbb{C}} \mathcal{O}_S$ since $\mathfrak{sp}_F(H')$ acts trivially on $\widehat{\mathcal{A}}_g$. Applying the central row of (6.7), the action of $\mathfrak{mp}(H')$ factors to an action of the Atiyah algebra $\alpha\mathcal{F}_{\Lambda}$ on $\widehat{\mathbb{V}}(V)$, for some $\alpha \in \mathbb{C}$. Since the central element $\mathbf{1}$ of $\mathfrak{mp}(H')$ acts on V as multiplication by the central charge c , the first row of (6.7) implies $\alpha = \frac{c}{2}$, hence the statement. \square

8.3. Sheaves of coinvariants on $\widehat{\mathcal{X}}_g$. Let $(Z, F, L, \bar{h}, q) \rightarrow S$ be a family as in §2.4 over a smooth base S with g fixed. Consider a representation V of $\widetilde{\mathcal{U}}_2(H(\mathbb{C}))$. The action of $\widetilde{\mathcal{U}}_2(H(\mathbb{C}))$ on V extends \mathcal{O}_S -linearly to an action of the sheaf of Lie algebras $\widetilde{\mathcal{U}}_2(H(\mathcal{O}_S))$ on the sheaf $V \otimes_{\mathbb{C}} \mathcal{O}_S$. Composing with the splitting

$$\mathfrak{sp}_F(H'(\mathcal{O}_S)) \ltimes F(\mathcal{O}_S) \hookrightarrow \widetilde{\mathcal{U}}_2(H(\mathcal{O}_S))$$

as in (6.8), one has an action of $\mathfrak{sp}_F(H'(\mathcal{O}_S)) \ltimes F(\mathcal{O}_S)$ on $V \otimes_{\mathbb{C}} \mathcal{O}_S$. We define the *sheaf of coinvariants* of V on $(Z, F, L, \bar{h}, q) \rightarrow S$ as the quasi-coherent sheaf of \mathcal{O}_S -modules

$$\widehat{\mathbb{V}}(V)_{(Z,F,L,\bar{h},q) \rightarrow S} := V \otimes_{\mathbb{C}} \mathcal{O}_S / (\mathfrak{sp}_F(H'(\mathcal{O}_S)) \ltimes F(\mathcal{O}_S))(V \otimes_{\mathbb{C}} \mathcal{O}_S).$$

This gives rise to a quasi-coherent sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{X}}_g$.

Theorem 8.2. *For a representation V of $\widetilde{\mathcal{U}}_2(H(\mathbb{C}))$ of central charge c , the sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{X}}_g$ carries an action of the Atiyah algebra $-\frac{c}{2}\mathcal{F}_{\Xi}$. This action induces a twisted \mathcal{D} -module structure on $\widehat{\mathbb{V}}(V)$.*

Proof. For a family as in §2.4 over a smooth base S , the action of $\widetilde{\mathcal{U}}_2(H(\mathbb{C}))$ on V and its action on $\widehat{\mathcal{X}}_g$ via the projection $\widetilde{\mathcal{U}}_2(H) \rightarrow \mathfrak{sp}(H') \ltimes H'$ induce an action of $\widetilde{\mathcal{U}}_2(H(\mathcal{O}_S))$ on the sheaf $\widehat{\mathbb{V}}(V)$. Explicitly, the action of $\widetilde{\mathcal{U}}_2(H(\mathcal{O}_S))$ on $V \otimes_{\mathbb{C}} \mathcal{O}_S$ is given by

$$X \cdot (v \otimes_{\mathcal{O}_S} f) := (X \cdot v) \otimes_{\mathcal{O}_S} f + v \otimes_{\mathbb{C}} (X \cdot f)$$

for local sections $X \in \widetilde{\mathcal{U}}_2(H(\mathcal{O}_S))$ and $v \otimes_{\mathbb{C}} f \in V \otimes_{\mathbb{C}} \mathcal{O}_S$. Applying the central row of (6.8), this action factors to an action of the Atiyah algebra $\alpha \mathcal{F}_{\Xi}$ on $\widehat{\mathbb{V}}(V)$, for some $\alpha \in \mathbb{C}$. Since the central element $\mathbf{1}$ of $\widetilde{\mathcal{U}}_2(H)$ acts on V as multiplication by c , the first row of (6.8) implies $\alpha = -\frac{c}{2}$, hence the statement. \square

9. COINVARIANTS ON FAMILIES OF ABELIAN VARIETIES

Here we construct twisted \mathcal{D} -modules of coinvariants on \mathcal{A}_g by descending the twisted \mathcal{D} -modules $\widehat{\mathbb{V}}(V)$ from §8 along the projection $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$. We proceed similarly on \mathcal{X}_g .

9.1. The action of $\mathcal{Sp}^+(H')$. Let V be a representation of $\mathfrak{mp}(H'(\mathbb{C}))$ of central charge c . From Theorem 8.1, the sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{A}}_g$ carries an action of the Atiyah algebra $\frac{c}{2} \mathcal{F}_{\Lambda}$. This is induced by an action of $\mathfrak{mp}(H'(\mathbb{C}))$ on $\widehat{\mathbb{V}}(V)$. Composing with the inclusion (4.1), we deduce an action of $\mathfrak{sp}^+(H'(\mathbb{C}))$ on the sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{A}}_g$. For a point (Z, F, L) in $\widehat{\mathcal{A}}_g$, recall the stabilizer $\mathcal{Sp}_F^+(H')$ of $\mathcal{Sp}^+(H')$ at (Z, F, L) from §3.2. Next, we show:

Proposition 9.1. *For an admissible $\mathfrak{mp}(H'(\mathbb{C}))$ representation V :*

- (i) *the action of $\mathfrak{sp}^+(H'(\mathbb{C}))$ on the sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{A}}_g$ can be exponentiated to an $\mathcal{Sp}^+(H')$ -equivariant structure on $\widehat{\mathbb{V}}(V)$;*
- (ii) *for (Z, F, L) in $\widehat{\mathcal{A}}_g$, the $\mathcal{Sp}^+(H')$ -equivariant structure on $\widehat{\mathbb{V}}(V)$ induces a trivial action of $\mathcal{Sp}_F^+(H')$ on the fiber of $\widehat{\mathbb{V}}(V)$ at (Z, F, L) .*

Proof. Recall from §3.2 that since $\mathcal{Sp}^+(H')$ is connected, elements in $\mathcal{Sp}^+(H')$ are products of exponentials of elements in $\mathfrak{sp}^+(H')$. Hence it is enough to show that the action on $\widehat{\mathbb{V}}(V)$ of exponentials of elements in $\mathfrak{sp}^+(H')$ is well-defined.

Also, note that for (Z, F, L) in $\widehat{\mathcal{A}}_g$, the stabilizer in $\mathfrak{sp}^+(H')$ at (Z, F, L) is the Lie subalgebra $\mathfrak{sp}_F^+(H') = \text{Lie}(\mathcal{Sp}_F^+(H'))$ from (3.2). Since $\mathfrak{sp}_F^+(H')$ is a Lie subalgebra of $\mathfrak{sp}_F(H')$, it follows that $\mathfrak{sp}_F^+(H')$ acts trivially on the fiber of $\widehat{\mathbb{V}}(V)$ at (Z, F, L) . Hence $\mathcal{Sp}_F^+(H')$ acts trivially on the fiber of $\widehat{\mathbb{V}}(V)$ at (Z, F, L) , proving part (ii).

Next, consider elements in $\mathfrak{sp}^+(H')$. By definition, $\mathfrak{sp}^+(H')$ is topologically generated by $b_i b_j \in S^2(H')$ with i, j not both negative. First consider $b_i b_j$ with $i + j = 0$. Since V is an admissible $\mathfrak{mp}(H'(\mathbb{C}))$ representation, the action of $b_i b_j$ on V is diagonalizable with integral eigenvalues by Definition

7.1. This action can be exponentiated to an action of the multiplicative group scheme \mathbb{G}_m on V by letting $a \in \mathbb{G}_m$ act as multiplication by a^k on the eigenspace of $b_i b_j$ with eigenvalue k .

Next, consider $b_i b_j$ with i, j not both negative and $i + j \neq 0$. By Definition 7.1, the action of $b_i b_j$ on V is locally nilpotent and hence can be exponentiated, since the formula for the action of the exponential of $b_i b_j$ is locally a finite sum.

Finally, an arbitrary element of $\mathfrak{sp}^+(H')$ has only finitely many terms $b_i b_j$ with $i + j \leq 0$ modulo $\mathfrak{sp}_F^+(H')$. This implies that for an arbitrary element of $\mathfrak{sp}^+(H')$, only finitely many terms $b_i b_j$ with $i + j \leq 0$ act locally non-trivially on $\widehat{\mathbb{V}}(V)$. Thus, by the arguments in the previous paragraphs, the formula for the action of the exponential of any element in $\mathfrak{sp}^+(H')$ is locally a finite sum on $\widehat{\mathbb{V}}(V)$. The statement follows. \square

9.2. **Sheaves of coinvariants on \mathcal{A}_g .** Let V be an admissible $\mathfrak{mp}(H'(\mathbb{C}))$ representation of central charge c . We are now ready for:

Proof of Theorems 1 and 2. The statements follow from the Harish-Chandra localization in §4. Namely, recall from Theorem 3.2 that $\widehat{\mathcal{A}}_g$ carries a transitive action of $\mathfrak{sp}(H')$ compatible with a transitive action of $\mathcal{Sp}^+(H')$ on the fibers of $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$. In particular, \mathcal{A}_g is realized as the quotient stack $[\mathcal{Sp}^+(H') \backslash \widehat{\mathcal{A}}_g]$. From Proposition 9.1, the sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{A}}_g$ is $\mathcal{Sp}^+(H')$ -equivariant. It follows from Proposition 4.4 that $\widehat{\mathbb{V}}(V)$ is a twisted \mathcal{D} -module on $\widehat{\mathcal{A}}_g$ which descends to a twisted \mathcal{D} -module $\mathbb{V}(V)$ on \mathcal{A}_g . This proves Theorem 1.

Moreover, from Theorem 8.1 the sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{A}}_g$ carries an action of the Atiyah algebra $\frac{c}{2} \mathcal{F}_\Lambda$. Thus from Proposition 4.5 we deduce that the sheaf $\mathbb{V}(V)$ on \mathcal{A}_g is equipped with an action of $\frac{c}{2} \mathcal{F}_\Lambda$. Hence Theorem 2. \square

Example 9.2. When $c = 0$, the action of the Atiyah algebra factors to an action of the tangent sheaf to \mathcal{A}_g on the sheaf $\mathbb{V}(V)$ on \mathcal{A}_g . In this case, one can simply use Proposition 4.3 in the above proof, instead of Proposition 4.4. Hence for $c = 0$, the sheaf $\mathbb{V}(V)$ is more simply a \mathcal{D} -module on \mathcal{A}_g .

9.3. **The action of $\mathcal{Sp}^+(H') \ltimes \mathcal{O}_1^\times$.** Next, we discuss how to construct similarly sheaves of coinvariants on \mathcal{X}_g . For this, we start describing the action of $\mathcal{Sp}^+(H') \ltimes \mathcal{O}_1^\times$ on the sheaf of coinvariants on $\widehat{\mathcal{X}}_g$.

Let V be a representation of $\widetilde{\mathcal{U}}_2(H(\mathbb{C}))$ of central charge c . From Theorem 8.2, the sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{X}}_g$ carries an action of the Atiyah algebra $-\frac{c}{2} \mathcal{F}_\Xi$. This is induced by an action of $\widetilde{\mathcal{U}}_2(H)$ on $\widehat{\mathbb{V}}(V)$. Composing with the inclusion (4.2), we deduce an action of $\mathfrak{sp}^+(H') \ltimes H'_+$ on the sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{X}}_g$.

Proposition 9.3. *For an admissible $\widetilde{\mathcal{U}}_2(H(\mathbb{C}))$ representation V , the action of $\mathfrak{sp}^+(H') \ltimes H'_+$ on the sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{X}}_g$ can be exponentiated to a $(\mathcal{S}p^+(H') \ltimes \mathcal{O}_1^\times)$ -equivariant structure on $\widehat{\mathbb{V}}(V)$.*

The statement follows similarly to Proposition 9.1, since elements in $\mathcal{S}p^+(H') \ltimes \mathcal{O}_1^\times$ are products of exponentials of elements in $\mathfrak{sp}^+(H') \ltimes H'_+$, and H'_+ acts locally nilpotently on V by Definition 7.1.

9.4. Sheaves of coinvariants on \mathcal{X}_g . Let V be an admissible $\widetilde{\mathcal{U}}_2(H(\mathbb{C}))$ representation of central charge c .

Proof of Theorems 3 and 4. From Theorem 3.3, $\widehat{\mathcal{X}}_g$ carries a transitive action of $\mathfrak{sp}(H') \ltimes H'$ compatible with the transitive action of $\mathcal{S}p^+(H') \ltimes \mathcal{O}_1^\times$ on the fibers of $\widehat{\mathcal{X}}_g \rightarrow \mathcal{X}_g$. In particular, \mathcal{X}_g is realized as the quotient stack $[(\mathcal{S}p^+(H') \ltimes \mathcal{O}_1^\times) \backslash \widehat{\mathcal{X}}_g]$. From Proposition 9.3, the sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{X}}_g$ is $(\mathcal{S}p^+(H') \ltimes \mathcal{O}_1^\times)$ -equivariant. It follows from Proposition 4.4 that $\widehat{\mathbb{V}}(V)$ is a twisted \mathcal{D} -module on $\widehat{\mathcal{X}}_g$ which descends to a twisted \mathcal{D} -module $\mathbb{V}(V)$ on \mathcal{X}_g . This proves Theorem 3.

Moreover, from Theorem 8.2, the sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{X}}_g$ carries an action of the Atiyah algebra $-\frac{c}{2}\mathcal{F}_{\Xi}$. Thus from Proposition 4.5 we deduce that the sheaf $\mathbb{V}(V)$ on \mathcal{X}_g is equipped with an action of $-\frac{c}{2}\mathcal{F}_{\Xi}$. Hence Theorem 4. \square

10. FINAL REMARKS

10.1. Comparison with classical coinvariants on curves. For a vertex operator algebra V , the space of coinvariants at $(C, P, t) \in \widehat{\mathcal{M}}_g$ constructed in [FBZ] is the quotient of V by the action of a Lie algebra $\mathcal{L}_{C \setminus P}(V)$ depending on both V and the open subset $C \setminus P \subset C$. The kernel of the map $\text{Witt} \rightarrow T_{(C, P, t)}(\widehat{\mathcal{M}}_g)$ is a Lie subalgebra of $\mathcal{L}_{C \setminus P}(V)$, but in general is not equal to it. It is shown in [FBZ] how the construction globalizes over $\widehat{\mathcal{M}}_g$ and yields an $\text{Aut}(\mathcal{O})$ -equivariant twisted \mathcal{D} -module on $\widehat{\mathcal{M}}_g$. This thus descends along the principal $\text{Aut}(\mathcal{O})$ -bundle $\widehat{\mathcal{M}}_g \rightarrow \mathcal{M}_{g,1}$. Furthermore, since the resulting fibers at (C, P) and (C, Q) in $\mathcal{M}_{g,1}$ are canonically isomorphic for all $P, Q \in C$, the sheaf is constant along the fibers of $\mathcal{M}_{g,1} \rightarrow \mathcal{M}_g$ and thus descends to a twisted \mathcal{D} -module on \mathcal{M}_g .

For a representation V of $\mathfrak{mp}(H'(\mathbb{C}))$, the space of coinvariants at $(Z, F, L) \in \widehat{\mathcal{A}}_g$ from §8 is the quotient of V by the action of the Lie algebra $\mathfrak{sp}_F(H')$. Contrary to $\mathcal{L}_{C \setminus P}(V)$, the Lie algebra $\mathfrak{sp}_F(H')$ is independent of V and equals the kernel of the analogous map $\mathfrak{sp}(H') \rightarrow T_{(Z, F, L)}(\widehat{\mathcal{A}}_g)$, hence it is the smallest Lie algebra whose coinvariants yield twisted \mathcal{D} -modules on $\widehat{\mathcal{A}}_g$. For this, it is natural to determine a Lie algebra containing as Lie subalgebras both $\mathfrak{sp}_F(H')$ and $\mathcal{L}_{C \setminus P}(V)$ at points in the Jacobian locus. We plan to present such an extension in a follow-up work.

The spaces of coinvariants for $\mathcal{L}_{C \setminus P}(V)$ are known to have finite dimension when V satisfies some finiteness and semisimplicity conditions [AN, DGT2], and thus give rise to vector bundles of coinvariants on \mathcal{M}_g whose Chern classes are determined by their twisted \mathcal{D} -module structure [DGT1]. It would be interesting to determine a similar result on spaces of coinvariants on \mathcal{A}_g . The mentioned extension of $\mathfrak{sp}_F(H')$ will provide a better context for investigating this property, which we intend to explore accordingly.

10.2. Failing descent for arbitrary modules. Spaces of coinvariants for $\mathcal{L}_{C \setminus P}(V)$ have been constructed also for the action of $\mathcal{L}_{C \setminus P}(V)$ on an arbitrary vertex operator algebra module [FBZ]. These give rise to twisted \mathcal{D} -modules on $\mathcal{M}_{g,1}$ [DGT2, §8.7], which in general do not descend to \mathcal{M}_g . In fact, the descent along $\mathcal{M}_{g,1} \rightarrow \mathcal{M}_g$ is only possible after tensoring the sheaf of coinvariants by an appropriate power of the relative cotangent line bundle to offset the variation of the spaces of coinvariants along the fibers of $\mathcal{M}_{g,1} \rightarrow \mathcal{M}_g$ (as in [DGT2, §8.7]). Similarly, the twisted \mathcal{D} -modules of coinvariants from arbitrary vertex operator algebra modules constructed as in §8 in general do not descend to \mathcal{A}_g . As the relative H^2 -space for the map $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$ is trivial (Proposition 2.4), tensoring by a line bundle will not help here. It would be interesting to find a finite-dimensional extension of the moduli space \mathcal{A}_g where the sheaves of coinvariants from arbitrary vertex operator algebra modules could descend from $\widehat{\mathcal{A}}_g$.

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CONFLICT OF INTEREST AND DATA AVAILABILITY STATEMENTS

The author states that there is no conflict of interest. No datasets have been analyzed or generated, as this work proceeds within a theoretical approach.

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