Fluctuation data analysis in Fusion Relevant Plasmas

Extracting information on relevant underlying dynamics

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Motivation & Outline

Diagnostics provides information with different time and spatial resolution

1. Measurements coming from a single point
2. Spatially distributed arrays of measurements (resolving portion of the plasma or entire torus)
3. line integrated measurements (single Line Of Sight (LoS))
4. Arrays of LoS (examples are tomographic reconstruction)

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We will focus on analysis technique suitable for single-point/multi point measurements, extracting information on spatial/temporal dynamics

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Some remarks on basic Fourier Transform and its discrete counterpart the Discrete Fourier Transform are mandatory

Continuous and Discrete Fourier transform 2

The Direct and Inverse fourier transform of a generic function of time $x(t)$ is defined as :

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-2\pi i f t} \mathrm{d}t \qquad x(t) = \int_{-\infty}^{+\infty} X(f) e^{2\pi i f t} \mathrm{d}f$$

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Theorem

Similarity Theorem: If $x(t)$ has the Fourier transform $X(f)$ then $x(at)$ has the Fourier Transform $|a|^{-1}F(f/a)$

This will be useful in the analysis of scaling properties of the fluctuations

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Addition Theorem: *If $x(t)$ and $g(t)$ have FT respectively $X(f)$ and $G(f)$ then $x(t)+g(t)$ has Fourier transform $X(f)+G(f)$*

The linearity of Fourier transform allows an easy treatment of linear equation in the Fourier domain

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Shift theorem: If $x(t)$ and $g(t)$ have FT respectively $X(f)$ then $x(t-a)$ has Fourier transform $e^{-2\pi i a f} X(f)$

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Theorem

Convolution theorem: If $x(t)$ and $g(t)$ have FT respectively equal to $X(f)$ and $G(f)$ the convolution of the two function $h(t) = \int_{-\infty}^{+\infty} x(t') g(t - t') \mathrm{d}t'$ is equal to $X(f)G(f)$

The importance of the convolution equation resides on the fact that it allows treatment of non-linearities as the term $\mathbf{v} \cdot \nabla \mathbf{v}$ in the Navier-Stokes equations

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Rayleigh's Theorem The integral of squared modulus of a function is equalt to the integral of the squared modulus of its spectrum, i.e:

$$\int_{-\infty}^{+\infty} |f(t)|^2 \mathrm{d}t = \int_{-\infty}^{+\infty} |F(f)|^2 \mathrm{d}f$$

This is equivalent to an energy conservation law for the time or frequency domain representation of the signal

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- The *Nyquist Frequency* $f_N = \frac{1}{2\Delta t}$, equivalent to half ot the sampling frequency, thus define the maximum frequency which can be properly resolved, or equivalently given the frequency of the system we would like to investigate, we had to sample at least at twice the values of this frequency.

Discrete Fourier Transform

- Given the discretiness of the sampling, also the corresponding Fourier transform will be discrete in the Fourier space.
- The basic concept of DFT resides on the fact that the algorithm applied on an N-sampled data, will produces an information on N frequencies, assuming that the information is conserved
- The Fourier transform of a digitized N-sampled data will be of the form $f_n = n\Delta f$ with (if N is even) $-\frac{N}{2} \leq n \leq \frac{N}{2}$. The frequency resolution, in order to span all the allowed frequency range will be of the form $\Delta f = \frac{1}{T} = \frac{1}{\Delta t}$
- The Direct and Inverse Discrete Fourier transform are then defined as

$$X_n = \frac{1}{N}\sum_{k=0}^{N-1} x_k e^{-2\pi kn/N}$$

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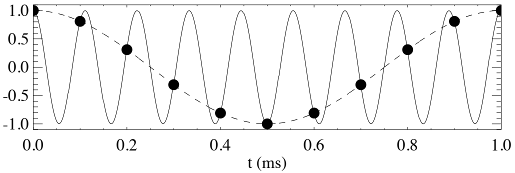
Aliasing, leaking and windowing

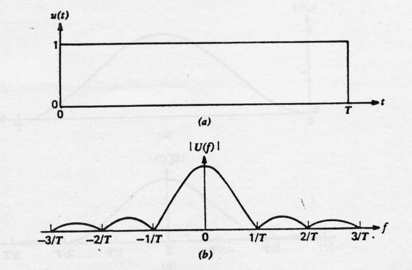
- The presence of frequency higher than the Nyquist frequency may lead to the presence of spurious frequency



- A 9 kHz sine if sampled at 10 kHz exhibits a spurious 1 kHz oscillation

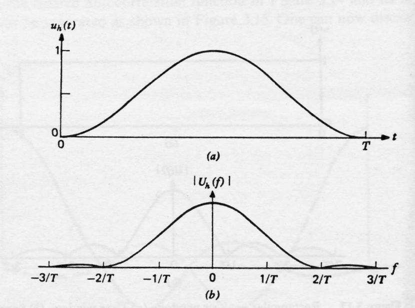
Aliasing, leaking and windowing

- The finite extension of the measurements, acquired for a given period T is equivalent to the convolution of the signal with a box function $G(t)$ with domain $0 \leq t \leq T$, i.e. $G(t) = 1 \mathrm{if} 0 \leq t \leq T$ 0 otherwise
- According the theorems for the Fourier transform (which can be applied also for discrete transform) this is equivalent in the Fouirer space to the moltiplication of the of the fourier transform but the Fourier representation of a box function is $\mathrm{sinc}(x) = \sin(x)/x$ function as shown, whichleaking some power from one frequency bin to the adjacents ones.



Aliasing, leaking and windowing

- Solution to the lakage problem is multiplying data by an appropriate window function which reduces the lobes as the *Hanning window* defined as $u_h(t) = \frac{1}{2}(1 - \cos(2\pi t/T))$ for $0 \leq t \leq T$ and 0 otherwise. Its behavior in real and fourire space is the following



Single Point: the autocorrelation function

- We know from the statistics a random process $x(t)$ is completely described by its moments, which are the average over the probability distribution function

$$E|x(t)| \quad E|x(t_1)x(t_2)| \quad E|x(t_1)x(t_2)x(t_3)| \quad \ldots$$

- The Auto-correlation function, i.e. the second order momentum of the distribution, or the autocovariance function

$$R(\tau) = E|x(t)x(t-\tau)| \qquad C(\tau) = E|(x(t)-m)(x(t-\tau)-m)|$$

being m the average of $x(t)$

- The Auto-correlation coefficient factor is defined as $\rho(\tau) = C(\tau)/C(0)$
- For digitized signals with N samples the estimator of $C(\tau)$ is defined as

$$C_j = \frac{1}{N}\sum_{i=j}^{N-1}(x_i - \overline{x})(x_{i-j} - \overline{x}) \qquad \overline{x} = \frac{1}{N}\sum_{i=0}^{N-1} x_i$$

Auto-correlation: practical use

- Define the *Auto-correlation time* of a turbulent field such as the potential
- **Inserisci figura con autocorrelazione di un potenziale**

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- From a practial point of view, we divide signals into M slices, assumed as indipendent realization of the same stochastic process and we compute

$$\hat{S}(f) = \frac{1}{M}\sum_{k=1}^{M} S^{(k)}(f); \qquad S^{(k)}(f) = \frac{1}{T}|X_T^{(k)}(f)|$$

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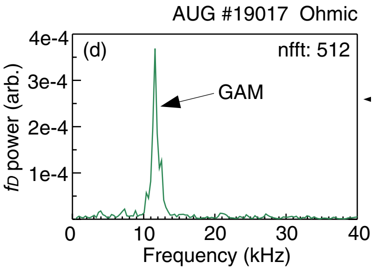
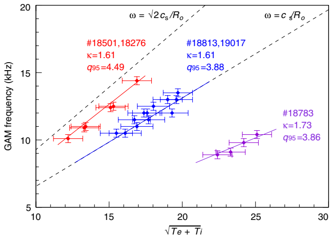
- With digitized signal, the power spectral estimator $\hat{S}_n$ is related to the real power spectrum $S(f_n)$ as

$$\hat{S}_n = \frac{1}{M}\sum_{k=1}^{M} |X_n^{(k)}|^2; \qquad \hat{S}_n \simeq S(f_n)\Delta f$$

Power spectrum: practical use

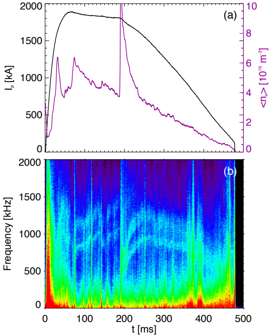


- Mode identification at a given frequency (Conway et al. 2005)



- But information must be completed. In the example, precise identification require scaling of identified mode as a function of ion sound gyroradius

- The same information can be also analyzed in time applying the spectrogram technique which shows how the spectral density of the signal varying in time/frequency space (Spagnolo et al. 2011)



- Again the information must be completed, as in the case proposed where the Alfvénic nature of the observed peaks is revealed by the comparison with the plasma density

Two-points techniques

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- The minimum set includes two measurements $x(t)$ and $y(t)$. We can define the Cross-correlation function, The cross-covariance function and the cross-correlation coefficient function

$$R_{xt}(\tau) = E[y(t)x(t-\tau)]$$

$$C_{xy}(\tau) = E[(y(t)-\overline{y})(x(t-\tau)-\overline{x})]$$

$$\rho_{yx}(\tau) = \frac{C_{yx}(\tau)}{\sqrt{C_{xx}(0)C_{yy}(0)}}$$

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- In the discrete counterpart of the cross-covariance is defined as

$$C_{yx,j} = \frac{1}{N}\sum_{i=j}^{N-1}(y_i-\overline{y})(x_{i-j}-\overline{x})$$

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- In the case of discrete signals with finite temporal length the following definitions hold (in analogy to single point case)

$$\hat{S}_{Y,X,n} = \frac{1}{M}\sum_{k=1}^{M} Y_n^{(k)} X_n^{*(k)} \qquad \hat{S}_{Y,X,n} \simeq S_{YX}(f_n)\Delta f$$

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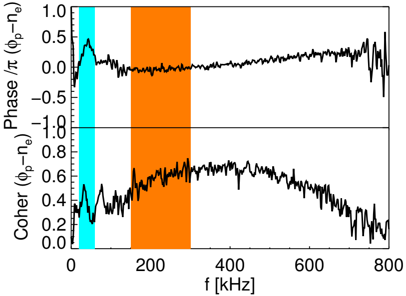

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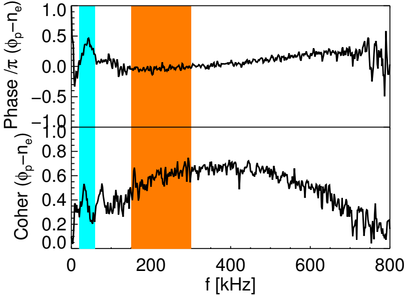

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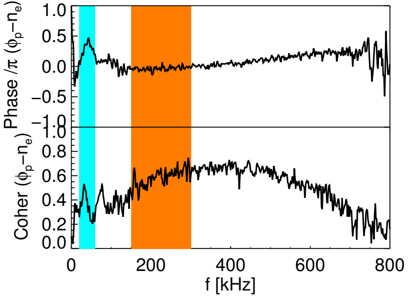

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- Other possibility is the determination of the polarization of magnetic fluctuations frequency resolved

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- Indeed if $k = k(f)$ then at a generic position $\overline{r}$ the function $g(t, \overline{r}) = \int_{-\infty}^{+\infty} G(f) e^{-ik(f)\cdot\overline{r}} e^{i2\pi f} df$

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- If measurements are displaced in the aximuthal direction, i.e. $d = r_p \Delta\Theta$ then the azimuthal mode number is computed as $m = \Theta_{12}/\Delta\Theta$

Wave-vector estimate

- In the case of a reasonably deterministic dispersion relation between k and f, the phase may be used for the determination of k
- Indeed if $k = k(f)$ then at a generic position $\overline{r}$ the function $g(t, \overline{r}) = \int_{-\infty}^{+\infty} G(f) e^{-ik(f)\cdot\overline{r}} e^{i2\pi f} df$
- If measures exhists at two position $g(t, \overline{r}_1)$ and $g(t, \overline{r}_2)$ separated by d at a given frequency the two signals will be phase shifted of $\Theta_{12}(f) = k(f)d$ where $\Theta_{12}(f)$ is computed trough the cross-power spectrum
- If measurements are displaced in the aximuthal direction, i.e. $d = r_p \Delta\Theta$ then the azimuthal mode number is computed as $m = \Theta_{12}/\Delta\Theta$
- The probe distance d must be less than aa wave length, less than a correlation lenght, but far enough the detect a measurable phase difference

Fluctuation-induced particle transport

- Fluctuations induced particle flux is defined as $\Gamma = E[\tilde{n}(t)\tilde{v}(t)] = E[\tilde{n}(t)\tilde{E}(t)]/B$
- According to previous definitions and properties

$$\Gamma = \frac{1}{B} R_{nE}(\tau = 0) = \frac{1}{B}\int_{-\infty}^{+\infty} S_{nE}(f)e^{i2\pi f\tau}df = \frac{2}{B}\int_{0}^{+\infty} \Re[S_{nE}(f)]df$$

- In quasi-static approximation $\tilde{E} = -\nabla\tilde{\phi}$, and considering the finitness of the measurements we end up with the formula

$$\Gamma(f) = \frac{2}{BT}\Im\{E[k(f)N(f)\Phi^*(f)]\}$$

$$\Gamma(f) = \frac{2k(f)}{B}\Im\{S_{n\phi}(f)\} \text{ if } k(f) \text{ is deterministic}$$

Fluctuation-induced particle transport 2

- In practice, considering digitzed signals we have (see for example (Antoni et al. 2000))

$$\Gamma(f) = \frac{1}{M}\sum_{k=1}^{M}\Gamma^{k}(f)$$

$$\Gamma^{(k)}(f) = \frac{2}{B}\Im\{k^{(k)}(f)N^{(k)}(f)\Phi^{*(k)}(f)\} 0 < f < N/2 - 1$$

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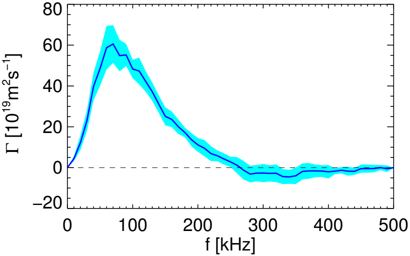

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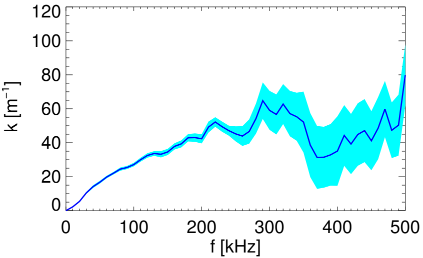

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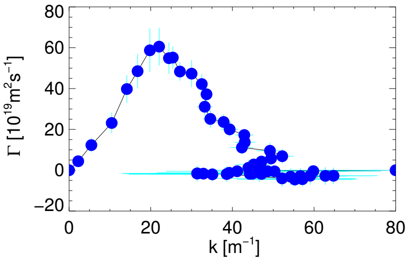

More than transport of particles

- Similar method may be used for the determination of the *Reynolds stress* $\langle \tilde{v}_r \tilde{v}_\perp \rangle$ which play a role in the momentum generation for both Tokamak and RFPs as $\partial_t (V_\phi) \propto -\partial_r \langle \tilde{v}_r \tilde{v}_\phi \rangle + \ldots$ (see e.g. (Vianello et al. 2005a,b, 2006))

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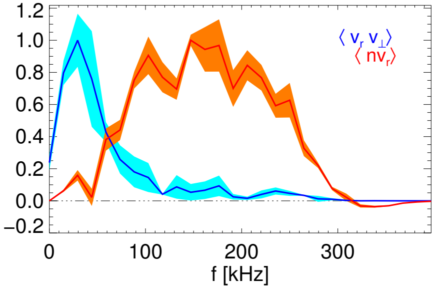

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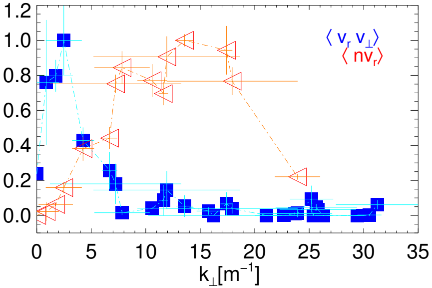

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- Again for finite time length T and finite space L we have an estimate

$$\hat{S}(k, \omega) = \frac{1}{LT} E[G_{LT}(k, \omega) G^*_{LT}(k, \omega)]$$

$$\lim_{L,T \to \infty} \hat{S}(k, \omega) = S(k, \omega)$$

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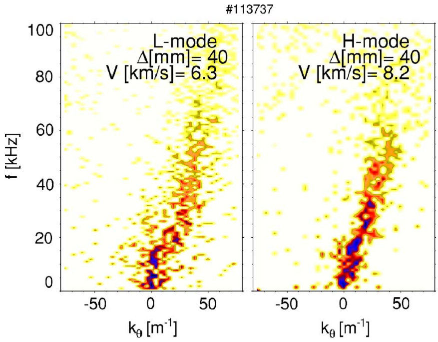
- If, as usual, only 2 points are available, the spectrum is reconstructed on a statistical basis:

$$\hat{S}_L(k, \omega) = \hat{S}_L(p\delta k, 2\pi n \Delta f) = \frac{1}{M} \sum_{j=1}^{M} S_n^{(j)} I_p[k_n^j]$$

$$I_p[k_n^j] = \begin{cases} 1 \text{ for } (p - 1/2)\Delta k < k_n^{(j)} < (p + 1/2)\delta k \\ 0 \text{ elsewhere} \end{cases}$$

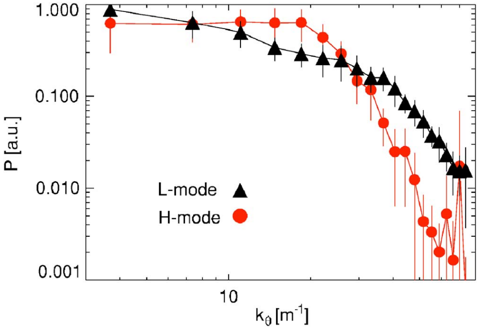
Spectral power density

- Spectral power density from GPI LoS on NSTX (Agostini et al. 2007)



Spectral power density

- We can consequently compute $S(k) = \int_{-\infty}^{+\infty} S(k,\omega)\frac{d\omega}{2\pi}$



Beyond Fourier: Wavelet transform

- The Fourier decomposition uses trigonometric functions as orthogonal basis
- They have the drawback that these functions oscillates forevere, i.e. the information content of a generic function is spread over all the spectral component
- Thus Fourier decomposition is not suitable for processes highly localized in time/space. We can use `Wavelet Transform`(Farge 1992)
- A Wavelet is a function $\psi \in \mathrm{L}^2(\mathbb{R})$ which satisfies the admissibility condition $C_\psi = \int_{-\infty}^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty$ which means that ψ is a zero mean function $\int_{\infty}^{\infty} \psi(t) dt = 0$
- Defining time-frequency atoms as $\psi_{s,\tau} = \frac{1}{\sqrt{\tau}} \psi\left(\frac{t-s}{\tau}\right)$ the `Continuous Wavelet Transform` is defined as

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Choice & type of Wavelet

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  (i) Orthogonal or nonorthogonal, according to the type of basis uses. Nonorthogonal are redunant at large scales, where wavelet spectrum at adjacent times is highly correlated. They are useful in time series analysis

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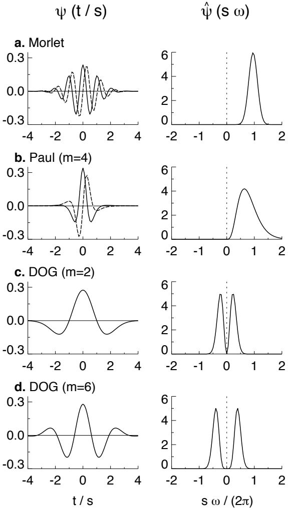
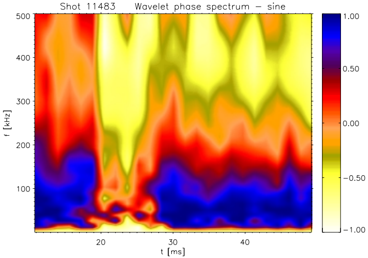
  (iv) Shape: The shape of the wavelet should reflect the use. For example for the studies of variation into the signal a boxcar-like function is more suitable.

- Example of different types of wavelet (Mallat 1999)



Wavelet application

- In analogy to Fourier we can define Wavelet Cross power spectrum and Corresponding phase spectrum (well localized in time/frequency)



- Phase spectrum between density and potential varies because of varioation of the shear → responsible for transport reduction (Antoni et al. 2000)

Wavelet for turbulence analysis

- It can be shown that wavelet coefficient exhibits similar scaling properties as the fluctuations of the signals at a given scales

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- Easy way to compute the `Probability Distribution Function` of normalized flutuations $C(t,\tau) = \frac{w(t,\tau) - \langle w(t,\tau)\rangle}{\sigma_\tau}$. For self-similar fluctuations, these should collapse to a single form

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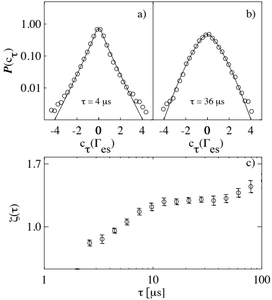
- **Revealing non self-similariy i.e. Intermittency**



Local Intermittency Measurements

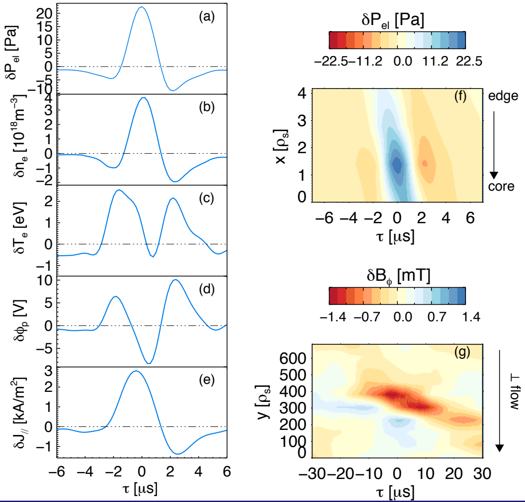
- Intermittency is due to the presence of strong, sporadic fluctuations
- The `Local Intermittency Measurements` is a method, based on wavelet, which identifies in time and scales this fluctuations (Antoni et al. 2001)
- The method is based on the following:

$$\{w(t,\tau)\} = \{w_e(t,\tau)\} \oplus \{w_g(t,\tau)\} \quad \text{with } F(\tau) = \frac{\langle w_g(t,\tau)^4 \rangle}{[\langle w_g(t,\tau)^2]^2} = 3$$

- The typical fluctuations may be deriven using `Conditional Average Procedure`, i.e. averaging different time window of the signal, each centered around the occurrence of an Intermittent Events or blobs

Example of structures

- Conditional average may be applied to different signals using the same trigger



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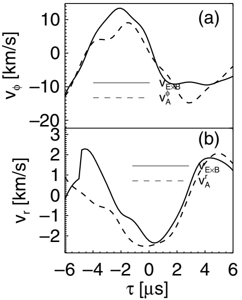

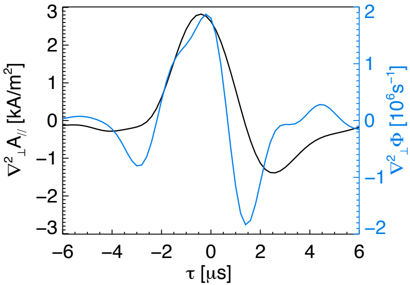

- Example of Drift-Kinetic Alfén vortices (Martines et al. 2009; Vianello et al. 2010)

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