Fluctuation data analysis in Fusion Relevant Plasmas

N. Vianello

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Motivation & Outline

Diagnostics provides information with different time and spatial resolution

1. Localized measurements
   (a) Measurements coming from a single point
   (b) Spatially distributed arrays of measurements (resolving portion of the plasma or entire torus)
2. Line integrated measurements
   (a) Single Line of Sight
   (b) Arrays of LoS

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FOURIER TRANSFORM → SINGLE POINT ANALYSIS → TWO POINT ANALYSIS → ADVANCED

Continuous and Discrete Fourier transform

A

- The Direct and Inverse fourier transform of a generic function of time $x(t)$ is defined as :

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-2\pi i f t} \mathrm{d}t \qquad x(t) = \int_{-\infty}^{+\infty} X(f) e^{2\pi i f t} \mathrm{d}f$$

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$$X_n = \frac{1}{N} \sum_{k=0}^{N-1} x_k e^{-2\pi k n/N} \qquad x_k = \frac{1}{N} \sum_{n=-N/2}^{N/2} X_n e^{2\pi k n/N}$$

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- The `Sampling Theorem` (Bracewell 1999) ensure that a function whose Fourier transform is zero for $f > f_c$ is fully specified by values spaced at equal intervals not exceeding $\frac{1}{2} f_c^{-1}$

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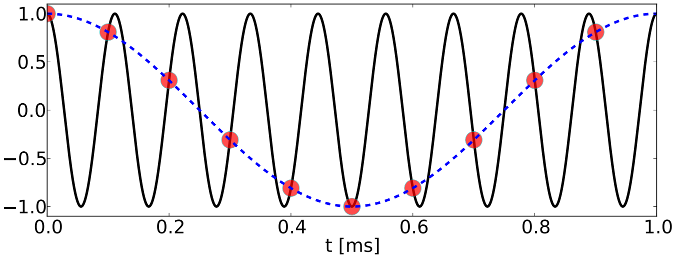
- The `Nyquist Frequency` $f_N = \frac{1}{2\Delta t}$, defines the maximum frequency which can be properly resolved, or, equivalently, given the frequency of the system we would like to investigate, we had to sample at least at twice the values of this frequency.

Aliasing

A

- The presence of frequency higher than the Nyquist frequency may lead to the presence of spurious frequency



- A 9 kHz cosine if sampled at 10 kHz exhibits a spurious 1 kHz oscillation

Properties of Fourier transform

A

- Various theorems may be applied to FT (Bracewell 1999) among which we cite:

Theorem

Convolution theorem: If $x(t)$ and $g(t)$ have FT respectively equal to $X(f)$ and $G(f)$ the convolution of the two functions $h(t) = \int_{-\infty}^{+\infty} x(t')g(t-t')\mathrm{d}t'$ is equal to $X(f)G(f)$

Theorem

Rayleigh's Theorem The integral of squared modulus of a function is equal to the integral of the squared modulus of its spectrum, i.e:

$$\int_{-\infty}^{+\infty} |f(t)|^2 \mathrm{d}t = \int_{-\infty}^{+\infty} |F(f)|^2 \mathrm{d}f$$

Single Point: the autocorrelation function

B

- A random process $x(t)$ is completely described by its moments, i.e. averages over the probability distribution function

$$E[x(t)] \quad E[x(t_1)x(t_2)] \quad E[x(t_1)x(t_2)x(t_3)] \quad \dots$$

- We define the `Auto-correlation function`, i.e. the second order momentum of the distribution, and the `autocovariance function`

$$\boxed{R(\tau) = E|x(t)x(t-\tau)|} \qquad \boxed{C(\tau) = E|(x(t)-m)(x(t-\tau)-m)|}$$

being m the average of $x(t)$

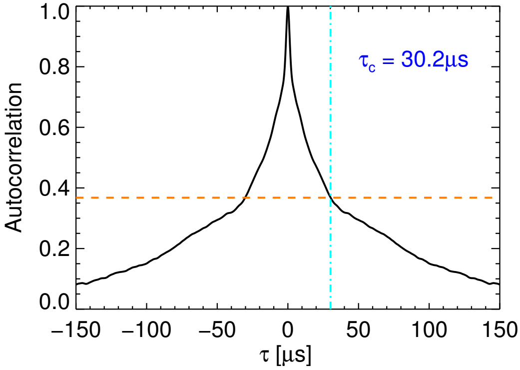
- The `Auto-correlation coefficient factor` is defined as $\rho(\tau) = C(\tau)/C(0)$
- For digitized signals with N samples the estimator of $C(\tau)$ is defined as

$$C_j = \frac{1}{N}\sum_{i=j}^{N-1}(x_i - \overline{x})(x_{i-j} - \overline{x}) \qquad \overline{x} = \frac{1}{N}\sum_{i=0}^{N-1} x_i$$

Auto-correlation: practical use

B

- Define the Auto-correlation time of a turbulent field such as the potential: $R(\tau_c) = \frac{\max(R(\tau))}{e}$



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- It describes the frequency distribution of the *power of the signal.*
- It corresponds to the limit of the `periodogram` of limited signals $x_T(t)$ $\frac{E[FT(x_T(t))^2]}{T} \xrightarrow[T\to\infty]{} S(f)$
- Numerically, the signal is divided into M slices, treated as independent realizations, and we compute the power spectral estimator $\hat{S}_n$ related to the real power spectrum $S(f_n)$ according to

$$\hat{S}_n = \frac{1}{M}\sum_{k=1}^{M} |X_n^{(k)}|^2; \qquad \hat{S}_n \simeq S(f_n)\Delta f$$

Power spectrum: practical use

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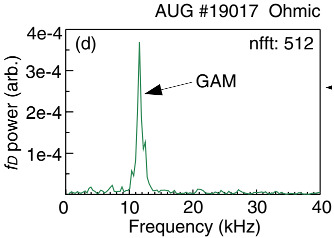
- Mode identification at a given frequency (Conway et al. 2005)



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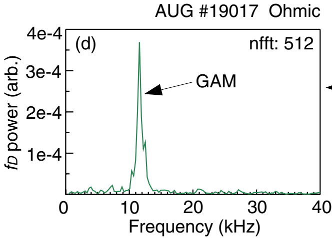
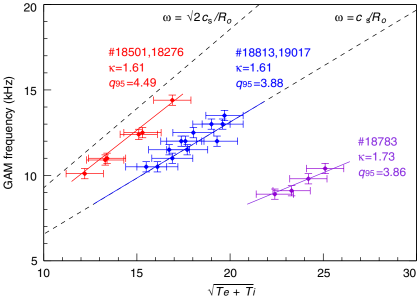


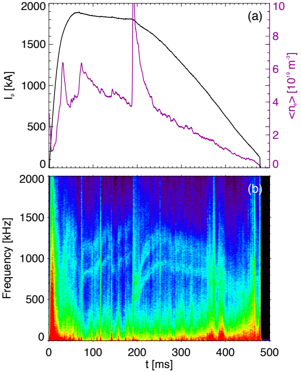


- Information must be completed. In the example Geodesic Acoustic Modes identified considering their scaling with c_s

Power spectrum: The spectrogram

B

- The same information can be also analyzed in time applying the spectrogram technique in the time/frequency space (Spagnolo et al. 2011)



- Alfvénic nature, with ω dependent from the Alfvén velocity $v_A = B/\sqrt{\rho\mu_0}$, revealed by the comparison with the plasma density

Two-points techniques

- Spatially distributed measurements allow access to spatial structure of the fluctuations

Two-points techniques

C

- Spatially distributed measurements allow access to spatial structure of the fluctuations
- The minimum set includes two measurements $x(t)$ and $y(t)$. We can define the `Cross-correlation function`, `The cross-covariance function` and the `cross-correlation coefficient function`

$$R_{xt}(\tau) = E[y(t)x(t-\tau)]$$

$$C_{xy}(\tau) = E[(y(t)-\overline{y})(x(t-\tau)-\overline{x})]$$

$$\rho_{yx}(\tau) = \frac{C_{yx}(\tau)}{\sqrt{C_{xx}(0)C_{yy}(0)}}$$

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- The discrete counterpart of the cross-covariance is defined as

$$C_{yx,j} = \frac{1}{N}\sum_{i=j}^{N-1}(y_i-\overline{y})(x_{i-j}-\overline{x})$$

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- We define the `coherence` as $\gamma_{YX}(f) = \frac{|S_{YX}(f)|}{\sqrt{S_Y(f)S_X(f)}}$
- In the case of discrete signals with finite temporal length the following definitions hold (in analogy to single point case)

$$\hat{S}_{Y,X,n} = \frac{1}{M}\sum_{k=1}^{M} Y_n^{(k)} X_n^{*(k)} \qquad \hat{S}_{Y,X,n} \simeq S_{YX}(f_n)\Delta f$$

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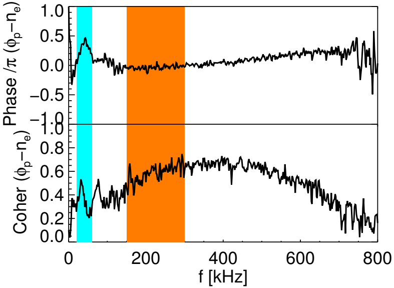

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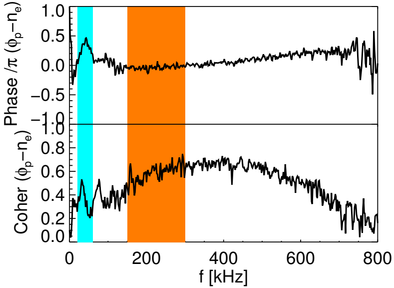

- This allow the possibility to distinguish the frequency where turbulence is Interchange-dominated from that where turbulence is Drift-dominated

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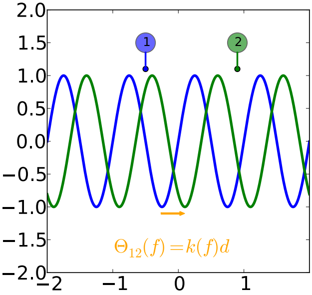

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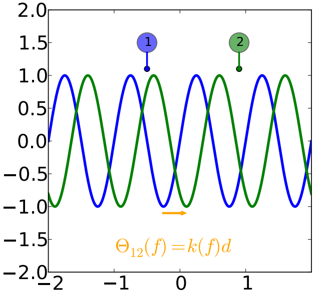

- Distance d must be less than a wave length, less than a correlation length, but far enough the detect a measurable phase difference

Fluctuation-induced particle transport

- Fluctuations induced particle flux is defined as $\Gamma = E[\tilde{n}(t)\tilde{v}(t)] = E[\tilde{n}(t)\tilde{E}(t)]/B$
- According to previous definitions and properties

$$\Gamma = \frac{1}{B} R_{nE}(\tau = 0) = \frac{2}{B} \int_0^{+\infty} \mathrm{Re}[S_{nE}(f)] df$$

- In quasi-static approximation $\tilde{E} = -\nabla\tilde{\phi}$, with finite record length T, and assuming a deterministic dispersion relation

$$\Gamma(f) = \frac{2k(f)}{B} \mathrm{Im}\{S_{n\phi}(f)\}$$

Fluctuation-induced particle transport 2

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- In practice, considering digitized signals we have (see for example (Antoni et al. 2000))

$$\Gamma(f) = \frac{1}{M}\sum_{k=1}^{M}\Gamma^{k}(f) = \frac{2}{BM}\sum_{k=1}^{M-1}\mathrm{Im}\{k^{(k)}(f)N^{(k)}(f)\Phi^{*(k)}(f)\}$$

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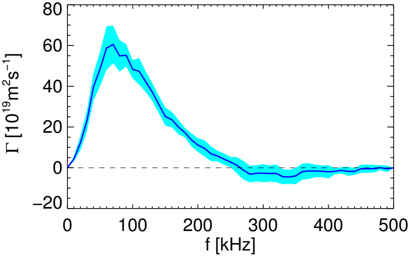

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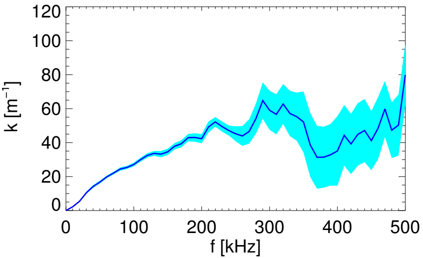

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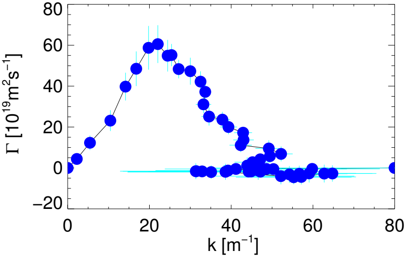
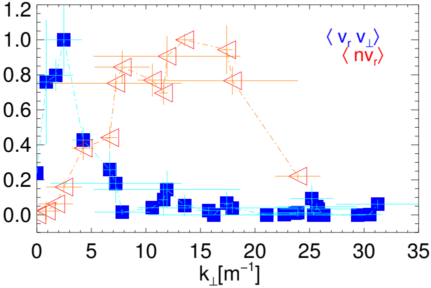


More than transport of particles

- Similar method may be used for the determination of the *Reynolds stress* $\langle \tilde{v}_r \tilde{v}_\perp \rangle$ which play a role in the momentum generation for both Tokamak and RFPs as $\partial_t(V_\phi) \propto -\partial_r \langle \tilde{v}_r \tilde{v}_\phi \rangle + \ldots$ (see e.g. (Vianello et al. 2005a,b, 2006))



The wavenumber-frequency Spectrum

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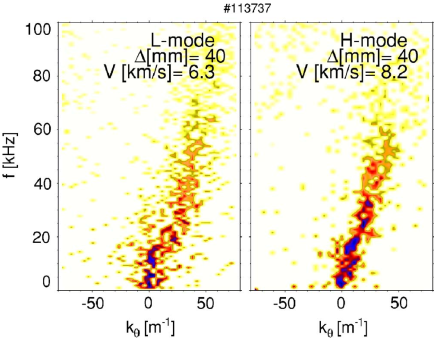
- In the two 2 points cases are available, the spectrum is reconstructed on a statistical basis considering the M different realizations:

$$\hat{S}_L(k,\omega) = \hat{S}_L(p\Delta k, 2\pi n\Delta f) = \frac{1}{M}\sum_{j=1}^{M} S_n^{(j)} I_p[k_n^j]$$

$$I_p[k_n^j] = \begin{cases} 1 \text{ for } (p-1/2)\Delta k < k_n^{(j)} < (p+1/2)\Delta k \\ 0 \text{ elsewhere} \end{cases}$$

Spectral power density

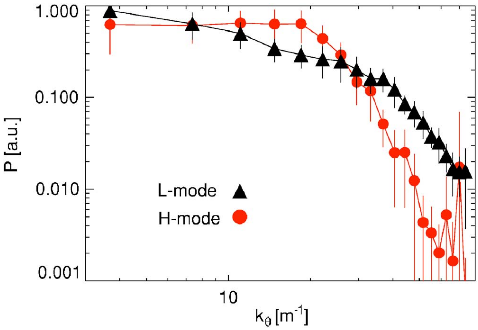
- Spectral power density from GPI LoS on NSTX (Agostini et al. 2007)



Spectral power density

C

- We can consequently compute $S(k) = \int_{-\infty}^{+\infty} S(k,\omega)\frac{d\omega}{2\pi}$



Beyond Fourier: Wavelet transform

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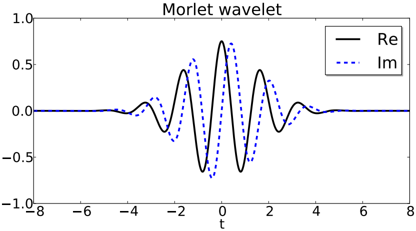
- We need functions localized in time and scales ⇒ `Wavelet Transform` (Farge 1992; Mallat 1999).



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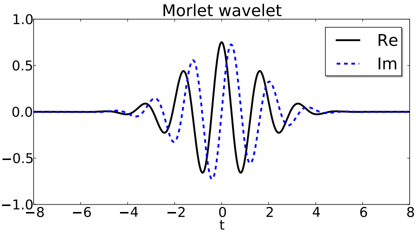
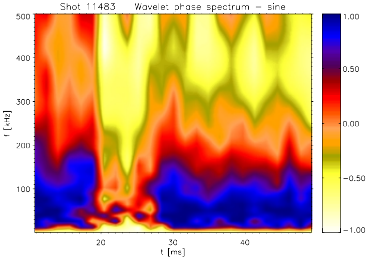


- Defining time-frequency atoms as $\psi_{s,\tau} = \frac{1}{\sqrt{\tau}}\psi\left(\frac{t-s}{\tau}\right)$ the `Continuous Wavelet Transform` is defined as

$$w(s,\tau) = \frac{1}{\sqrt{\tau}}\int_{-\infty}^{+\infty} f(t)\psi^*\left(\frac{t-s}{\tau}\right)dt$$

Wavelet application

- In analogy to Fourier we can define Wavelet Cross power spectrum and Corresponding phase spectrum (well localized in time/frequency)



- Phase spectrum between density and potential varies because of variation of the shear → responsible for transport reduction (Antoni et al. 2000)

Wavelet for turbulence analysis

D

- Wavelet coefficients exhibit similar scaling properties as the fluctuations of the signals at the same

$$\delta_\tau f = f(t+\tau) - f(t) \sim \tau^h \Rightarrow |w(t,\tau)| \sim \tau^{h+1/2}$$

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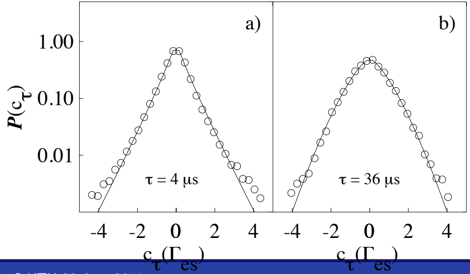
- **Revealing non self-similariy i.e. Intermittency**



Summary

- High temporal and spatial resolution are needed for better characterization of the plasma. But two points still gives a bunch of information
- Fourier transform allows estimate of quantities directly comparable with theories
- Often localized events (in space or time) require more sophisticated tools which maintain the locality of the information
- As much as possible correlation between different diagnostics are generally needed for an appropriate comprehension

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