

Fluctuation data analysis in Fusion Relevant Plasmas

Extracting information on relevant underlying dynamics

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Diagnostics provides information with different time and spatial resolution

1. Measurements coming from a single point
2. Spatially distributed arrays of measurements (resolving portion of the plasma or entire torus)
3. line integrated measurements (single Line Of Sight (LoS))
4. Arrays of LoS (examples are tomographic reconstruction)

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We will focus on analysis technique suitable for single-point/multi point measurements, extracting information on spatial/temporal dynamics

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Some remarks on basic Fourier Transform and its discrete counterpart the Discrete Fourier Transform are mandatory

Continuous and Discrete Fourier transform 2



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Theorem

Similarity Theorem: If $x(t)$ has the Fourier transform $X(f)$ then $x(at)$ has the Fourier Transform $|a|^{-1}F(f/a)$

This will be useful in the analysis of scaling properties of the fluctuations

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Addition Theorem: If $x(t)$ and $g(t)$ have FT respectively $X(f)$ and $G(f)$ then $x(t) + g(t)$ has Fourier transform $X(f) + G(f)$

The linearity of Fourier transform allows an easy treatment of linear equation in the Fourier domain

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Shift theorem: If $x(t)$ and $g(t)$ have FT respectively $X(f)$ then $x(t - a)$ has Fourier transform $e^{-2\pi i a f} X(f)$

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Theorem

Convolution theorem: If $x(t)$ and $g(t)$ have FT respectively equal to $X(f)$ and $G(f)$ the convolution of the two function $h(t) = \int_{-\infty}^{+\infty} x(t') g(t - t') dt'$ is equal to $X(f)G(f)$

The importance of the convolution equation resides on the fact that it allows treatment of non-linearities as the term $\mathbf{v} \cdot \nabla \mathbf{v}$ in the Navier-Stokes equations

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Theorem

Rayleigh's Theorem The integral of squared modulus of a function is equal to the integral of the squared modulus of its spectrum, i.e:

$$\int_{-\infty}^{+\infty} |f(t)|^2 dt = \int_{-\infty}^{+\infty} |F(f)|^2 df$$

This is equivalent to an energy conservation law for the time or frequency domain representation of the signal

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- ▶ A fundamental result of signal analysis is *Sampling Theorem* which states that **A function whose Fourier transform is zero for $f > f_c$ is fully specified by values spaced at equal intervals not exceeding $\frac{1}{2}f_c^{-1}$**
- ▶ **The Nyquist Frequency $f_N = \frac{1}{2\Delta t}$, equivalent to half of the sampling frequency, thus defines the maximum frequency which can be properly resolved, or equivalently given the frequency of the system we would like to investigate, we had to sample at least at twice the values of this frequency.**

- ▶ Given the discreteness of the sampling, also the corresponding Fourier transform will be discrete in the Fourier space.
- ▶ The basic concept of DFT resides on the fact that the algorithm applied on an N -sampled data, will produce an information on N frequencies, assuming that the information is conserved
- ▶ The Fourier transform of a digitized N -sampled data will be of the form $f_n = n\Delta f$ with (if N is even) $-\frac{N}{2} \leq n \leq \frac{N}{2}$. The frequency resolution, in order to span all the allowed frequency range will be of the form $\Delta f = \frac{1}{T} = \frac{1}{\Delta t}$
- ▶ The **Direct** and **Inverse** Discrete Fourier transform are then defined as

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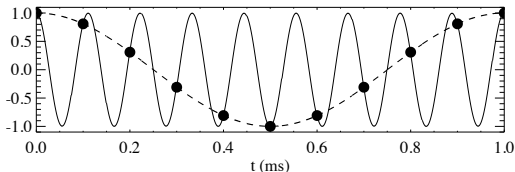
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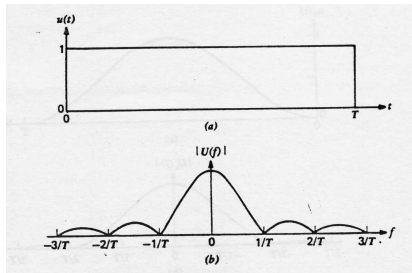
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- ▶ The presence of frequency higher than the Nyquist frequency may lead to the presence of spurious frequency

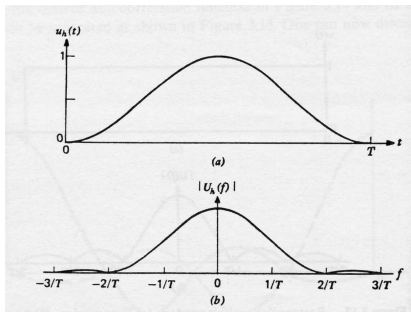


- ▶ A 9 kHz sine if sampled at 10 kHz exhibits a spurious 1 kHz oscillation

- ▶ The finite extension of the measurements, acquired for a given period T is equivalent to the convolution of the signal with a box function $G(t)$ with domain $0 \leq t \leq T$, i.e. $G(t) = 1$ if $0 \leq t \leq T$ 0 otherwise
- ▶ According the theorems for the Fourier transform (which can be applied also for discrete transform) this is equivalent in the Fourier space to the multiplication of the of the fourier transform but the Fourier representation of a box function is $\text{sinc}(x) = \sin(x)/x$ function as shown, which **leaking** some power from one frequency bin to the adjacents ones.



- Solution to the leakage problem is multiplying data by an appropriate window function which reduces the lobes as the *Hanning window* defined as $u_h(t) = \frac{1}{2}(1 - \cos(2\pi t/T))$ for $0 \leq t \leq T$ and 0 otherwise. Its behavior in real and fourire space is the following



- ▶ We know from the statistics a random process $x(t)$ is completely described by its moments, which are the average over the probability distribution function

$$E|x(t)| \quad E|x(t_1)x(t_2)| \quad E|x(t_1)x(t_2)x(t_3)| \quad \dots$$

- ▶ The **Auto-correlation function**, i.e. the second order momentum of the distribution, or the **autocovariance function**

$$R(\tau) = E|x(t)x(t - \tau)|$$

$$C(\tau) = E|(x(t) - m)(x(t - \tau) - m)|$$

being m the average of $x(t)$

- ▶ The **Auto-correlation coefficient factor** is defined as $\rho(\tau) = C(\tau)/C(0)$
- ▶ For digitized signals with N samples the estimator of $C(\tau)$ is defined as

$$C_j = \frac{1}{N} \sum_{i=j}^{N-1} (x_i - \bar{x})(x_{i-j} - \bar{x}) \quad \bar{x} = \frac{1}{N} \sum_{i=0}^{N-1} x_i$$

- ▶ Define the *Auto-correlation time* of a turbulent field such as the potential
- ▶ **Inserisci figura con autocorrelazione di un potenziale**

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- ▶ In the real case, with limited temporal acquisition of a signal $x_T(t)$ the ensemble average of its **periodogram** tends to the power spectrum

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- ▶ From a practical point of view, we divide signals into M slices, assumed as independent realization of the same stochastic process and we compute

$$\hat{S}(f) = \frac{1}{M} \sum_{k=1}^M S^{(k)}(f); \quad S^{(k)}(f) = \frac{1}{T} |X_T^{(k)}(f)|$$

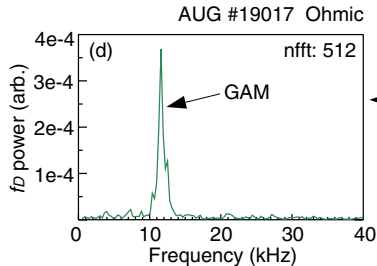
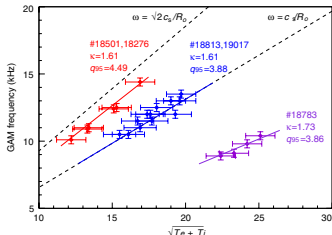
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- ▶ With digitized signal, the power spectral estimator \hat{S}_n is related to the real power spectrum $S(f_n)$ as

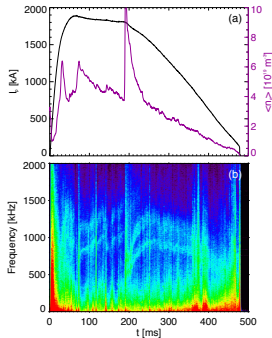
$$\hat{S}_n = \frac{1}{M} \sum_{k=1}^M |X_n^{(k)}|^2; \quad \hat{S}_n \simeq S(f_n) \Delta f$$

- Mode identification at a given frequency (Conway et al. 2005)



- But information must be completed. In the example, precise identification require scaling of identified mode as a function of ion sound gyroradius

- ▶ The same information can be also analyzed in time applying the **spectrogram** technique which shows how the spectral density of the signal varying in time/frequency space (Spagnolo et al. [2011](#))



- ▶ Again the information must be completed, as in the case proposed where the Alfvénic nature of the observed peaks is revealed by the comparison with the plasma density

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- ▶ The minimum set includes two measurements $x(t)$ and $y(t)$. We can define the **Cross-correlation function**, **The cross-covariance function** and the **cross-correlation coefficient function**

$$R_{xt}(\tau) = E[y(t)x(t - \tau)]$$

$$C_{xy}(\tau) = E[(y(t) - \bar{y})(x(t - \tau) - \bar{x})]$$

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- ▶ In the discrete counterpart of the cross-covariance is defined as

$$C_{yx,j} = \frac{1}{N} \sum_{i=j}^{N-1} (y_i - \bar{y})(x_{i-j} - \bar{x})$$

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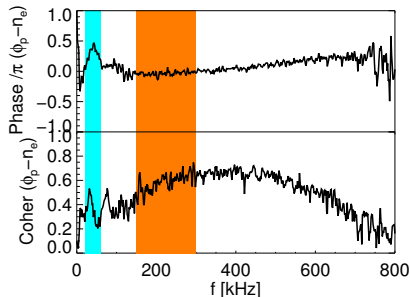
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- ▶ In the case of discrete signals with finite temporal length the following definitions hold (in analogy to single point case)

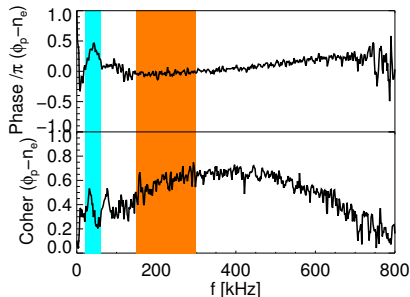
$$\hat{S}_{Y,X,n} = \frac{1}{M} \sum_{k=1}^M Y_n^{(k)} X_n^{*(k)} \quad \hat{S}_{Y,X,n} \simeq S_{YX}(f_n) \Delta f$$

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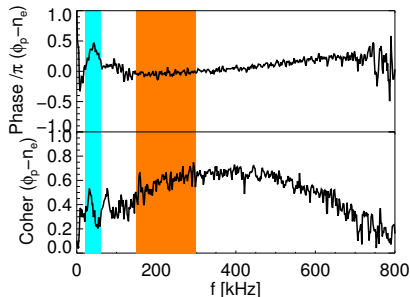


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- ▶ Other possibility is the determination of the polarization of magnetic fluctuations frequency resolved



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- ▶ The probe distance d must be less than aa wave length, less than a correlation lenght, but far enough the detect a measurable phase difference

- ▶ Fluctuations induced particle flux is defined as
$$\Gamma = E[\tilde{n}(t)\tilde{v}(t)] = E[\tilde{n}(t)\tilde{E}(t)]/B$$
- ▶ According to previous definitions and properties

$$\Gamma = \frac{1}{B} R_{nE}(\tau = 0) = \frac{1}{B} \int_{-\infty}^{+\infty} S_{nE}(f) e^{i2\pi f\tau} df = \frac{2}{B} \int_0^{+\infty} \Re[S_{nE}(f)] df$$

- ▶ In quasi-static approximation $\tilde{E} = -\nabla\tilde{\phi}$, and considering the finiteness of the measurements we end up with the formula

$$\Gamma(f) = \frac{2}{BT} \Im\{E[k(f)N(f)\Phi^*(f)]\}$$
$$\Gamma(f) = \frac{2k(f)}{B} \Im\{S_{n\phi}(f)\} \text{ if } k(f) \text{ is deterministic}$$

- In practice, considering digitized signals we have (see for example (Antoni et al. [2000](#)))

$$\Gamma(f) = \frac{1}{M} \sum_{k=1}^M \Gamma^k(f)$$

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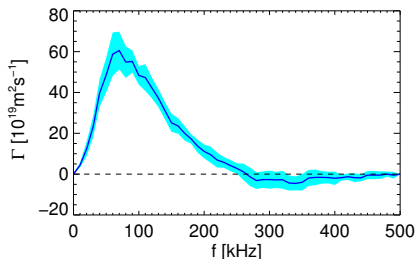
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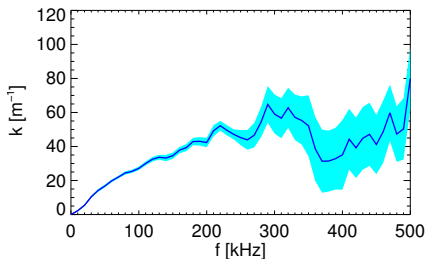


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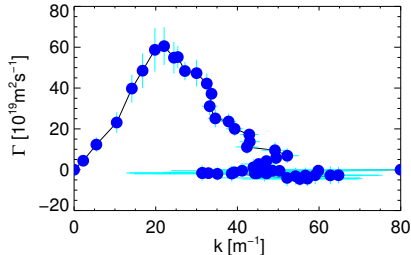


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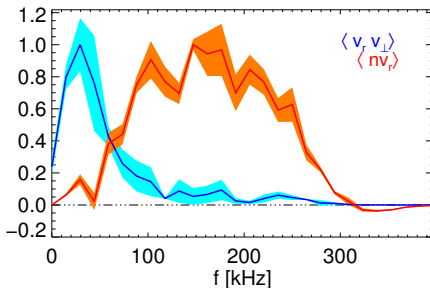
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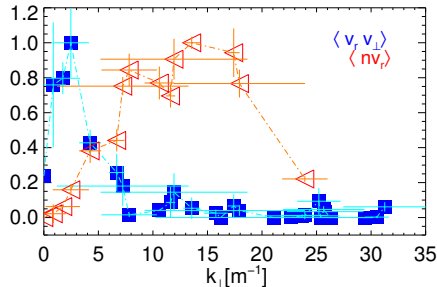


- ▶ Similar method may be used for the determination of the *Reynolds stress* $\langle \tilde{v}_r \tilde{v}_\perp \rangle$ which play a role in the momentum generation for both Tokamak and RFPs as $\partial_t(V_\phi) \propto -\partial_r \langle \tilde{v}_r \tilde{v}_\phi \rangle + \dots$ (see e.g. (Vianello et al. [2005a,b](#), [2006](#)))

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Ricerca Formazione Innovazione

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- ▶ Again for finite time length T and finite space L we have an estimate

$$\hat{S}(k, \omega) = \frac{1}{LT} E[G_{LT}(k, \omega) G_{LT}^*(k, \omega)]$$
$$\lim_{L, T \rightarrow \infty} \hat{S}(k, \omega) = S(k, \omega)$$

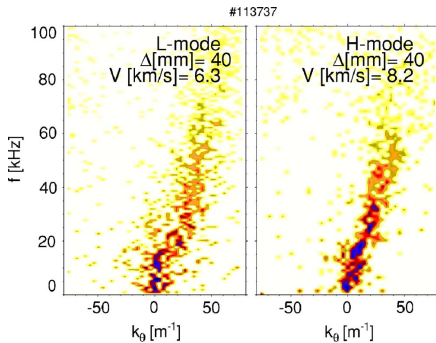
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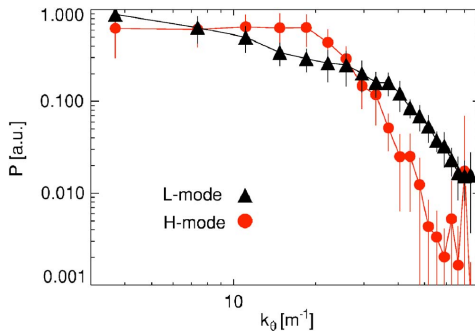
- ▶ If, as usual, only 2 points are available, the spectrum is reconstructed on a statistical basis:

$$\hat{S}_L(k, \omega) = \hat{S}_L(p\delta k, 2\pi n\Delta f) = \frac{1}{M} \sum_{j=1}^M S_n^{(j)} I_p[k_n^j]$$
$$I_p[k_n^j] = \begin{cases} 1 & \text{for } (p - 1/2)\Delta k < k_n^{(j)} < (p + 1/2)\delta k \\ 0 & \text{elsewhere} \end{cases}$$

- Spectral power density from GPI LoS on NSTX (Agostini et al. [2007](#))



- ▶ We can consequently compute $S(k) = \int_{-\infty}^{+\infty} S(k, \omega) \frac{d\omega}{2\pi}$



- ▶ The Fourier decomposition uses trigonometric functions as orthogonal basis
- ▶ They have the drawback that these functions oscillates forever, i.e. the information content of a generic function is spread over all the spectral component
- ▶ Thus Fourier decomposition is not suitable for processes highly localized in time/space. We can use **Wavelet Transform** (Farge 1992)
- ▶ A Wavelet is a function $\psi \in L^2(\mathbb{R})$ which satisfies the admissibility condition $C_\psi = \int_{-\infty}^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty$ which means that ψ is a zero mean function $\int_{-\infty}^{+\infty} \psi(t) dt = 0$
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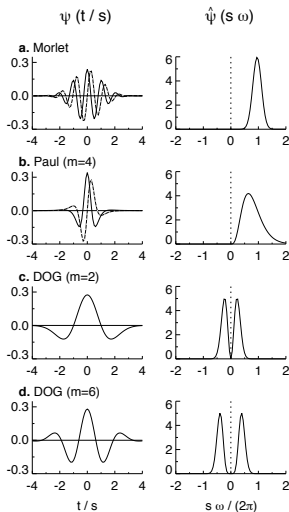
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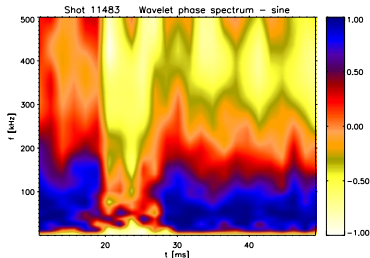
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 - (iv) Shape: The shape of the wavelet should reflect the use. For example for the studies of variation into the signal a boxcar-like function is more suitable.

- ▶ Example of different types of wavelet (Mallat 1999)



- In analogy to Fourier we can define Wavelet Cross power spectrum and Corresponding phase spectrum (well localized in time/frequency)



- Phase spectrum between density and potential varies because of variation of the shear → responsible for transport reduction (Antoni et al. [2000](#))

- ▶ It can be shown that wavelet coefficient exhibits similar scaling properties as the fluctuations of the signals at a given scales

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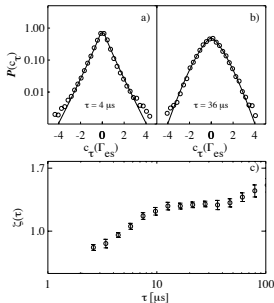
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- ▶ Easy way to compute the **Probability Distribution Function** of normalized fluctuations $C(t, \tau) = \frac{w(t, \tau) - \langle w(t, \tau) \rangle}{\sigma_{\tau}}$. For **self-similar** fluctuations, these should collapse to a single form

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- ▶ **Revealing non self-similarity i.e. Intermittency**



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