Bounds on the spreading radius in droplet impact: the 2D case

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Introduction

- I will look at droplet impact on a smooth surface.
- **Impact, Spread, Retraction**
  In the land of splashes, what the scientist knows as Inertia and Surface Tension are the sculptors in liquids, and fashion from them delicate shapes none the less beautiful because they are too ephemeral for any eye but that of the high-speed camera [Yarin, Annu. Rev. Fluid Mech. (2006)]

Dimensionless numbers

- Reynolds number, Re, represents the ratio between Inertial and Viscous forces, and is defined as follows:

$$\mathrm{Re} = \frac{\rho U_0 R_0}{\mu}$$

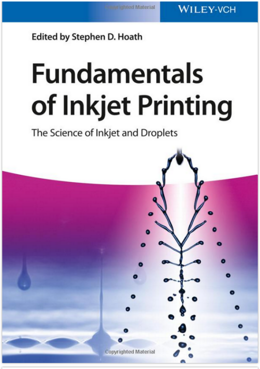
- Weber number, We, refers to ratio between Inertia and Surface Tension, and is defined as follows:

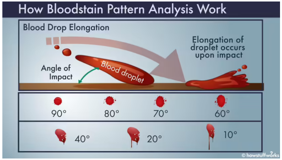
$$\mathrm{We} = \frac{\rho U_0^2 R_0}{\gamma}$$

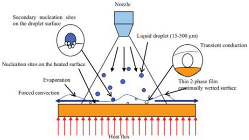
where ρ is the density of the liquid, μ is the dynamic viscosity of the liquid, γ is the surface tension, U_0 is the speed of the droplet before impact, and R_0 is the radius of the droplet prior to impact.

Motivation

- Industry (inkjet printing, cooling, bloodstain pattern analysis)



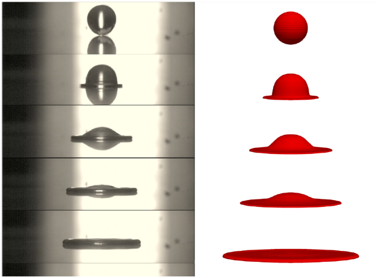




- **Scientific curiosity**...

Below the splash threshold

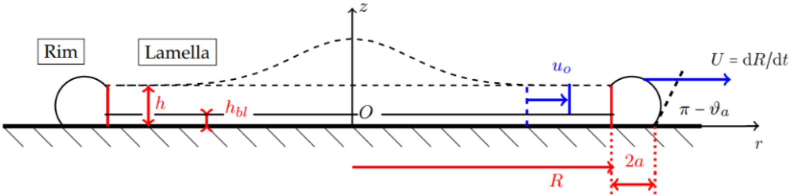
- Droplet spreading (no splash) below **splash threshold**, $K \lesssim 3,000$, where $K = \mathrm{We}\sqrt{\mathrm{Re}}$
- Below this threshold, droplet spreads out into a pancake structure – rim and lamella.
- Of interest is the maximum spreading radius $\mathcal{R}_{max}$ and its dependence on We and Re.



Droplet impact study. Left: high-speed camera. Right: OpenFOAM simulations. Credit: Conor Quigley. Parameters: $\mathrm{Re} = 1700$ and $\mathrm{We} = 20$.

Rim-Lamella Models

Based on 3D droplets:



Context:

- In cases of high-speed droplet impact below the splash threshold, the droplet flattens upon impact into a thin **lamella** with a **rim** on the outer edge.
- The key parameter we will analyse is the maximum spreading radius, $\mathcal{R}_{max}$, influenced by Re, We and ϑ_a, the contact angle.

Point of departure

Instead of looking at 3D axisymmetric rim-lamella structures, in this project we look at 2D Cartesian rim-lamella structures, which occur in ‘cylindrical droplets’.

- Cylindrical droplets cannot exist naturally, however they can be engineered in the laboratory.
- They are generated by confining a liquid bridge between two parallel plates, forming a shape that resembles a cylindrical droplet.

Rim-Lamella Models: 2D

Exact solutions (inviscid)

- The inviscid limit ($\mathrm{Re} = \infty$), yields an exact solution in the 2D case.
- The inviscid limit is obtained by shrinking the boundary layer, h_{bl} to zero.
- This solution correctly predicts the scaling behaviour, $\mathcal{R}_{max}/R_0 \sim \mathrm{We}$, as seen in previously published works, as well as being consistent with energy-budget predictions.

Tractable analysis (viscous)

- Provides rigorous upper and lower bounds for $\beta_{max} = \mathcal{R}_{max}/R_0$.
- We can hence utilise Gronwall's inequality to derive upper and lower bounds on the spreading radius.

Project Aims

Project had two aims:

1. To understand the rim-lamella anlysis in 2D and hence to understand the upper and lower bounds derived by supervisor, recalled here as:

$$k_1 \mathrm{Re}^{\frac{1}{3}} - k_2 \mathrm{Re}^{\frac{1}{2}} \mathrm{We}^{-\frac{1}{2}} (1 - \cos\vartheta_a)^{\frac{1}{2}} \leq \beta_{max} \leq k_1 \mathrm{Re}^{\frac{1}{3}} \tag{1}$$

2. To carry out numerical simulations of two-phase flow to validate the bounds (1)

Structure of talk

1. Summarize 2D rim-lamella model briefly
2. Overview of simulation setup
3. Simulation results

Rim-Lamella Modelling

After impact, a rim-lamella structure forms. Mass and momentum balances:

$$\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) &= 0, \\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} &= 0. \end{aligned}$$

- Valid for $t \geq \tau$ and $x \in (0, R)$.
- Notation: R marks the end of the lamella and the start of the rim.
- Exact solution:

$$u = \frac{x}{t + t_0}.$$

Mass and Momentum Balance in the Rim

Mass and momentum balance in the rim are described by the following ordinary differential equations:

$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= 2\,(\overline{u} - U)\,h(R,t),\\
V\frac{\mathrm{d}U}{\mathrm{d}t} &= 2\Bigg[\underbrace{(\overline{u} - U)^2\,h(R,t)}_{=\text{Inertia}} - \underbrace{\frac{\sigma}{\rho}\,(1-\cos\vartheta_a)}_{=\text{Surface Tension}}\Bigg],\\
\frac{\mathrm{d}R}{\mathrm{d}t} &= U,
\end{aligned}
$$

where:

- V is the rim area,
- $\overline{u}$ is $u_0 = R/(t+t_0)$, corrected for the viscous boundary layer,
- U is the rim velocity,
- ϑ_a is the advancing contact angle

Initial conditions:

$$R(\tau) = R_{init}, \qquad U(\tau) = U_{init}, \qquad V(\tau) = V_{init}.$$

Gronwall's Inequality

Differential form [edit]

Let I denote an interval of the real line of the form $[a, \infty)$ or $[a, b]$ or $[a, b)$ with $a < b$. Let β and u be real-valued continuous functions defined on I. If u is differentiable in the interior I° of I (the interval I without the end points a and possibly b) and satisfies the differential inequality

$$\boxed{u'(t) \le \beta(t)\, u(t), \qquad t \in I^\circ,}$$

then u is bounded by the solution of the corresponding differential *equation* $v'(t) = \beta(t)\, v(t)$:

$$\boxed{u(t) \le u(a) \exp\left(\int_a^t \beta(s)\, \mathrm{d}s \right)}$$

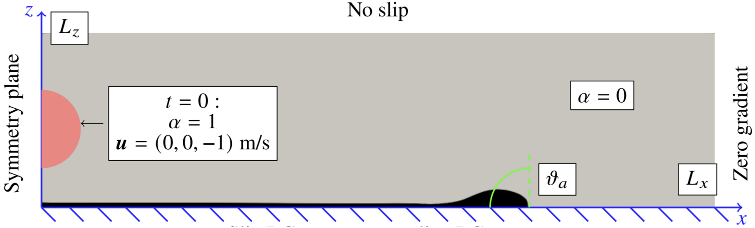
for all $t \in I$.

Remark: There are no assumptions on the signs of the functions β and u.

- Used to put bounds on solutions of ODEs.
- E.g. Regularity of Navier–Stokes, Mixing Efficiency (advection-diffusion),...
- Used here to establish the bounds (1)
- Main focus of my project is validating these bounds with numerical simulations.

Numerical Setup

- The simulations are completed using the CFD software, OpenFOAM.
- The impact of a droplet on a solid, no-slip surface in a two-dimensional Cartesian geometry is simulated.
- We use domain large enough such that the fully spread-out droplet is captured (never leaves the domain).
- We use local mesh refinement (simple grading) near $z = 0$ to fully resolve the fine details of the droplet dynamics.



Volume of Fluid Method

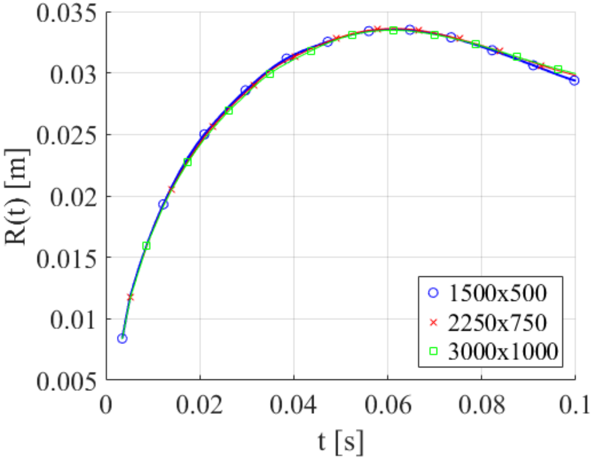
We make use of the Volume of Fluid method (VOF) to simulate the gas-liquid two-phase flow.

What is the Volume of Fluid Method?

- The volume of fluid method tracks the fraction of each computational cell filled with liquid, which allows the evolution of the interface to be tracked over time.
- $\alpha = 1$ indicates a cell filled entirely with liquid.
- $\alpha = 0$ indicates a cell filled entirely with gas.
- The level set $\alpha = \frac{1}{2}$ represents the interface.

Mesh Refinement Study

In the below figure, a fixed case is examined under three levels of mesh refinement. The 2250x750 grid is converges sufficiently to the 3000x100 grid, hence the 2250x770 grid is sufficiently fine. This means that the results of the simulations are independent of the mesh.



Summary results: $\theta = 120^{\circ}$

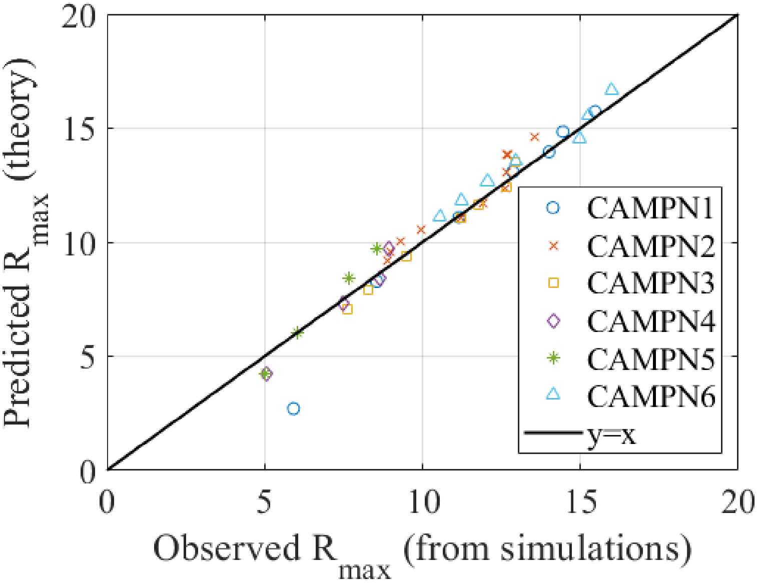
Summary results: $\theta = 60^\circ$

Simulation Campaigns

$$\beta_{max} = k_1 \mathrm{Re}^{1/3} - k_0 (1 - \cos\vartheta_a)^{1/2} (\mathrm{Re}/\mathrm{We})^{1/2}$$

Campaign	**Parameter Varied**
1	Weber Number
2	Contact Angle
3	Reynolds number (via kinematic viscosity)
4	Weber number (higher fixed Reynolds number)
5	Weber number (higher Reynolds number, simulated using Diffuse Interface method)
6	Contact Angle (with increased fixed Reynolds number)

Results 2



Predicted R max (theory)
Observed R max (from simulations)
0
5
10
15
20
0
5
10
15
20
CAMPN1
CAMPN2
CAMPN3
CAMPN4
CAMPN5
CAMPN6
y=x

Conclusions and Future Work

Conclusions

- We have formulated a rim-lamella model of droplet spreading valid for two-dimensional droplets, and derived theoretical bounds on the spreading radius.
- Simulation results using the Volume of Fluid method fell within the theoretical bounds, confirming the prediction of the model.
- Furthermore, results all collapse onto a single curve, confirming the theoretical scaling.

Future Work

- Investigate how the model performs above the splash threshold.

Acknowledgments

- Mr Conor Quigley, for simulations and high-speed video analysis.
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