```
(*3.1a*)
(*Calculate the radius r0and the period T of the limit cycle for μ>0*)

f[r_] = μ * r - r^3;
g[r_] = ω + ν * r^2;
fixedP = Solve[{f[r, θ] == 0}, r]
```

$\{\{r \to 0\}, \{r \to -\sqrt{\mu}\}, \{r \to \sqrt{\mu}\}\}$

```
(*We only want to evaluate r>0(radius is a length)l,
therefore the only relevant fixed point
 is the third which is the radius of the cycle*)
fp = fixedP[[3]]
```

$\{r \to \sqrt{\mu}\}$

```
r0 = fp[[1]][[2]]
```

$\sqrt{\mu}$

```
T = 2 π / g[r, θ] /. fp
```

$$\frac{2\pi}{\mu\nu + \omega}$$

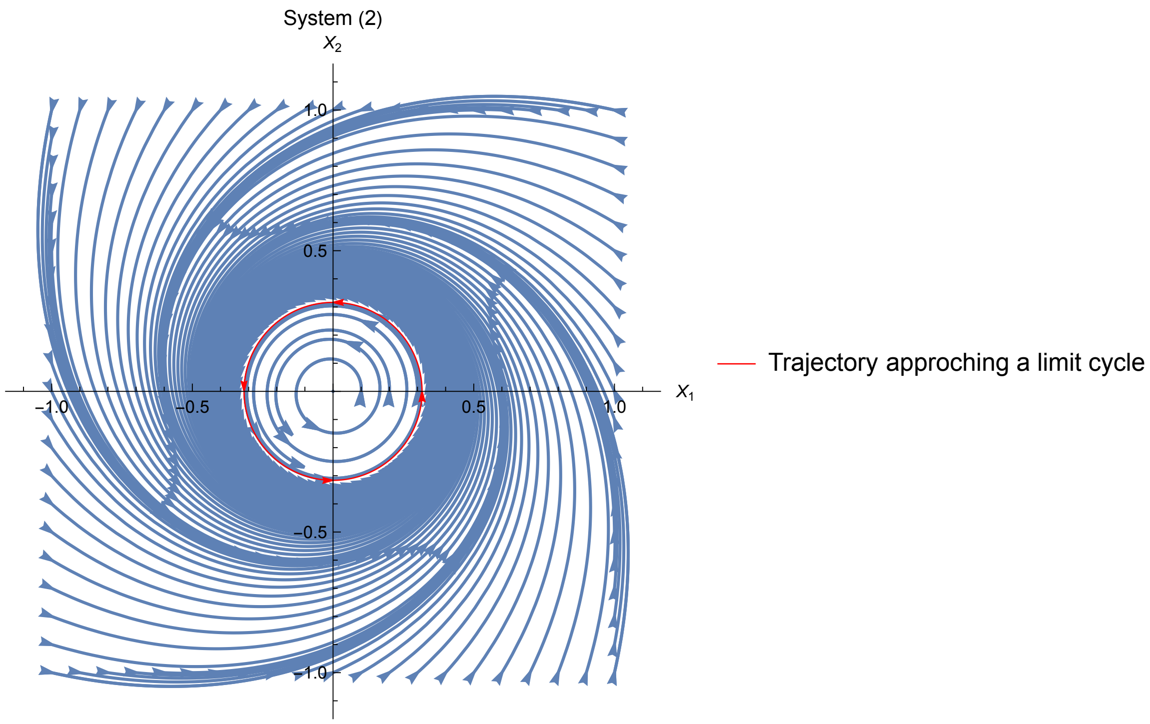
```
(*3.2b*)
(*(b) Make a phase portrait of the dynamical system (2)
 showing a few representative trajectories.In the same figure,
plot the limit cycle using a suitable representative trajectory.*)

(* I begin to solve 3.2c to find the values of μ, ω and ν*)
(* 3.2c *)
(* Convert system 1 to cartesian*)
X1Dot[x, y] = f[r] * Cos[θ] - r * g[r] * Sin[θ] /.
    {r → Sqrt[X1^2 + X2^2], θ → ArcTan[X1, X2]} // ExpandAll
X2Dot[x, y] =
 f[r] * Cos[θ] + r * g[r] * Sin[θ] /. {r → Sqrt[X1^2 + X2^2], θ → ArcTan[X1, X2]} //
  ExpandAll
```

$\mu X_1 - X_1^3 - \omega X_2 - \nu X_1^2 X_2 - X_1 X_2^2 - \nu X_2^3$

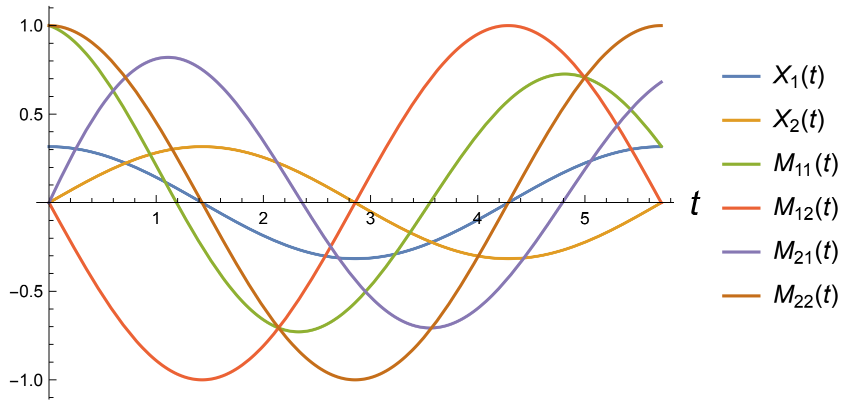
$\mu X_1 - X_1^3 + \omega X_2 + \nu X_1^2 X_2 - X_1 X_2^2 + \nu X_2^3$

```
(* We can compare the output with system 2;*)
(* Streamplot system 2 *)
X1Dot2 = 1 / 10 * X1[t] - X2[t]^3 - X1[t] * X2[t]^2 - X1[t]^2 * X2[t] - X2[t] - X1[t]^3;
X2Dot2 = X1[t] + 1 / 10 * X2[t] + X1[t] * X2[t]^2 + X1[t]^3 - X2[t]^3 - X1[t]^2 * X2[t];
(*From comparison we get values *)
μ = 1 / 10;
ω = 1;
ν = 1;
```

```
(*Circle plot*)
circle = ParametricPlot[{r₀ * Cos[t], r₀ * Sin[t]},
    {t, 0, 2 π}, PlotStyle → {Thickness[0.002], Red},
    PlotLegends → {"Trajectory approching a limit cycle"}] /. Line[x_] :→
    Sequence[Arrowheads[{0.015, 0.015, 0.015, 0.015, 0.015}], Arrow[x]];
(*Outer plot*)
sol1[X10_, X20_] := NDSolve[{X1'[t] == X1Dot2,
    X2'[t] == X2Dot2, X1[0] == X10, X2[0] == X20}, {X1, X2}, {t, 10}];
inits =
  Join[Table[{-1, X2}, {X2, -1, 1, 0.1}], Table[{1, X2}, {X2, -1, 1, 0.1}],
   Table[{X1, -1}, {X1, -1, 1, 0.1}], Table[{X1, 1}, {X1, -1, 1, 0.1}]];

outer = Table[
   ParametricPlot[Evaluate[{X1[t], X2[t]} /. sol1[inits⟦i, 1⟧, inits⟦i, 2⟧]],
     {t, 0, 10}, PlotRange → All] /. Line[x_] :→ Sequence[Arrowheads[
       {0.015, 0.015, 0.015, 0.015, 0.015}], Arrow[x]], {i, Length[inits]}];

(* Plot the inner part *)
sol2[X10_, X20_] := NDSolve[{X1'[t] == X1Dot2,
    X2'[t] == X2Dot2, X1[0] == X10, X2[0] == X20}, {X1, X2}, {t, 10}];
inits = Join[Table[{0, X2}, {X2, 0, 0, 0.1}], Table[{0, X2}, {X2, 0, 0, 0.1}],
   Table[{X1, 0}, {X1, 0, r₀, 0.1}], Table[{X1, 0}, {X1, 0, r₀, 0.1}]];
inner = Table[
   ParametricPlot[Evaluate[{X1[t], X2[t]} /. sol2[inits⟦i, 1⟧, inits⟦i, 2⟧]],
     {t, 0, 10}, PlotRange → All] /. Line[x_] :→ Sequence[
      Arrowheads[{0.02, 0.02, 0.02, 0.02}], Arrow[x]], {i, Length[inits]}];
p4 = Show[outer, inner, circle,
  PlotLabel → "System (2)", AxesLabel → {X1, X2}, ImageSize → Medium]
(*Export["3.2b.jpg",p4]*)
```


System (2)
X_2
1.0
0.5
−0.5
−1.0
−1.0
−0.5
0.5
1.0
X_1
Trajectory approching a limit cycle

```
(*3.2d*)
(*Plot all six quantities as functions
 of tt for one period TT of the limit cycle*)
(*We only consider system 2*)
J = Grad[{X1Dot2, X2Dot2}, {X1[t], X2[t]}];
sol3 = NDSolve[
   {X1'[t] == X1Dot2, X2'[t] == X2Dot2, M11'[t] == J[[1]][[1]] * M11[t] + J[[1]][[2]] * M21[t],
    M12'[t] == J[[1]][[1]] * M12[t] + J[[1]][[2]] * M22[t], M21'[t] ==
     J[[2]][[1]] * M11[t] + J[[2]][[2]] * M21[t], M22'[t] == J[[2]][[1]] * M12[t] + J[[2]][[2]] * M22[t],
    X1[0] == Sqrt[μ], X2[0] == 0, M11[0] == 1, M12[0] == 0, M21[0] == 0, M22[0] == 1},
   {X1[t], X2[t], M11[t], M12[t], M21[t], M22[t]}, {t, 0, T}];

(* Plot with different colors for each line *)
p5 = Plot[Evaluate[{X1[t], X2[t], M11[t], M12[t], M21[t], M22[t]} /. sol3],
  {t, 0, T}, PlotStyle → Automatic,
  PlotLegends → {X1[t], X2[t], M11[t], M12[t], M21[t], M22[t]},
  AxesLabel → {Style[t, FontSize → 18], ""}]
(*Export["3.2d_plot.jpg", p5]*)
```




```
(*3.2e*)
(*Give your numerical M(T) obtained in (d) to 4 relevant digits accuracy*)


M2[t_] = {{M11[t], M12[t]}, {M21[t], M22[t]}} /. sol3;
(*two different ways to use for numerical solutions*)
N[Floor[M2[T] * 10 000], 4] / 10 000
NumberForm[M2[T], 4, ExponentFunction → (Null &)]
{{{0.3190, 0}, {0.6809, 1.000}}}

{{{0.3191, 0.00000002123}, {0.6809, 1.}}}

(*With 4 decimals the numerical result is [0.3191,0],[0.6809,1]*)

(*3.2f*)
(*alculate the stability exponents of separations σ1 and σ2 of the limit
 cycle from the eigenvalues of M(T) to 4 relevant digits accuracy*)
```

```
eigV1 = Eigenvalues[M[T]];
σ = N[Floor[1 / T * Log[eigV1] * 10 000] / 10 000]
```

{0., -0.2}

```
(*So σ1is -0.2 and σ2 is 0 because σh1 ≤ σh2 *)
(*3.2g*)
(*Using what you know from all parts of this problem,
calculate the deformation matrix M(T) analytically.*)
```

```
jacobian = Grad[{f[r], g[r]}, {r, θ}];
J2 = jacobian /. r → r0;
M3 = MatrixExp[T * J2] // FullSimplify;
(* Write in cartesian coord. by multiplying left and right *)
J3 = Grad[{Sqrt[x[t]^2 + y[t]^2], ArcTan[x[t], y[t]]}, {x[t], y[t]}] /. t → T /.
   {x[T] → r0 * Cos[0], y[T] → r0 * Sin[0]};
Mcartesian = Inverse[J3].M3.J3
```

$\{\{e^{-4\pi/11}, 0\}, \{1 - e^{-4\pi/11}, 1\}\}$

```
(*3.2h*)
(*Compute the stability exponents of
 separations σ1 and σ2 of the limit cycle analytically*)
eigV2 = Eigenvalues[Mcartesian];
σh = 1 / T * Log[eigV2]
```

$\left\{0, -\frac{1}{5}\right\}$

```
(*So σ1is -1/5 and σ2 is 0 because σh1 ≤ σh2 *)
```