(* 2.2a*)
(*Give θ_0 in terms of(g,l,γ,m)*)

(* From the task:

$$\ddot{\theta} = -\frac{g}{l}*\sin(\theta) - \frac{\gamma}{m}*\dot{\theta};$$

By substitution $\omega = \dot{\theta}$; and the equations $\theta = \theta_0*x$; $\omega = \omega_0*y$; $t = t_0 t'$;

We obtain:

$$\frac{dx}{dt'} = y; \quad \frac{dy}{dt'} = -\sin(x)-\sigma*y;$$

Start by computing θ_0;

$\omega = \dot{\theta} \leftrightarrow \omega_0*y = \frac{\theta_0}{t_0}*\frac{dx}{dt'}$;

$\frac{dx}{dt'} = \frac{t_0}{\theta_0}*\omega_0*y$;

The similarities from $\frac{dx}{dt'} = y$; we can conclude that:

$$\frac{t_0 w_0}{\theta_0} = 1;$$

$\rightarrow \theta_0 = t_0 w_0$;

Start calulating 2.2b and 2.2c and come back and insert the values.

$\rightarrow \theta_0 = t_0 w_0 = \sqrt{\frac{g}{l}} * \sqrt{\frac{l}{g}} = 1$;

(* 2.2b*)
(*Give ω_0 in terms of(g,l,γ,m)*)

Rewriting the main function and set $\ddot{\theta} = \dot{\omega}$, thus:

$$\dot{\omega} = -\frac{g}{l}*\sin(\theta_0*x) - \frac{\gamma}{m}*w_0*y;$$

Where $\dot{\omega} = \frac{\omega_0}{t_0}*\frac{dy}{dt'}$;

By inserting and rewriting we obtain:

$$\frac{dy}{dt'} = -\frac{t_0}{\omega_0}*\frac{g}{l}*\sin(\theta_0*x) - \frac{t_0}{\omega_0}*\frac{\gamma}{m}*w_0*y;$$

The similarities from $\frac{dy}{dt'} = -\sin(x)-\sigma*y$; we can conclude that:

$\theta_0 = 1$; and $\frac{t_0}{\omega_0}*\frac{g}{l} = 1$;

and by combaning the eqs above and inserting $\theta_0 = 1$ in $\theta_0 = t_0 w_0$; we obtain:

$\rightarrow \omega_0 = \sqrt{\frac{g}{l}}$;

(* 2.2c*)
(*Give t_0 in terms of(g,l,γ,m)*)

With the same equations calculated in 2.2b and inserting $\theta_0 = 1$ in $\theta_0 = t_0 w_0$;
we obtain:

$\rightarrow t_0 = \sqrt{\frac{l}{g}}$;

(* 2.2d*)
(*Give σ in terms of(g,l,γ,m)*)

Now from the second term we can compute σ:

```
t0/ω0 * ϒ/m * w0 * y is similar to σ*y;

We can insert the computed t0 in combination with eliminating w0:
  → σ = Sqrt[l/g] * ϒ/m
*)

(* 2.2e *)
(* For arbitrary σ ≥ 0,
calculate and classify the fixed points as a function of σ*)
(* From the task:
   dx/dt' = 0 → y → 0 and  dy/dt' = 0 →  -sin(x)-σ*0 = 0;
we then obatin that x =
 nπ → the fixed is at (nπ, 0) → investigate it's eigenvalues*)
M = Grad[{y, -Sin[x] - σ * y}, {x, y}];
eig = Eigenvalues[M];
(*n=1, odd number*)
eig /. x → π
```

$$\left\{\frac{1}{2}\left(-\sigma-\sqrt{4+\sigma^2}\right), \frac{1}{2}\left(-\sigma+\sqrt{4+\sigma^2}\right)\right\}$$

```
(*n=2, even number*)
eig /. x → 2 π
```

$$\left\{\frac{1}{2}\left(-\sigma-\sqrt{-4+\sigma^2}\right), \frac{1}{2}\left(-\sigma+\sqrt{-4+\sigma^2}\right)\right\}$$

```
(*For n=2 and σ ≥ 2*)
eig /. {x → 2 π, σ → 3}
```

$$\left\{\frac{1}{2}\left(-3-\sqrt{5}\right), \frac{1}{2}\left(-3+\sqrt{5}\right)\right\}$$

```
(* For odd n →
 theterm under the root will always be greater than the term outside. →
  λ1 will be negative while λ2 will be positive → saddle-point*)
(* For even n →
 term under the root will always be smaller than the other term. →
  both λ1,2 will be negative for σ ≥ 0. *)
(*When σ ≥ 2 → stable real values under the root. When σ < 2 →
  imaginary values under the root. *)
(* We can imagine for the case 2x2 where left side → σ < 2,
right side → σ > 2,top half →
 for odd n and bottom half → for even n. The classification would then be →[[
   Saddle, Saddle], [Stable with oscillation, Stable]] *)
(*See fixed points evaluated with Solve*)
xDot[x_, y_, σ_] := y
yDot[x_, y_, σ_] := -Sin[x] - σ y
Solve[{xDot[x, y, σ] == 0, yDot[x, y, σ] == 0}]
```

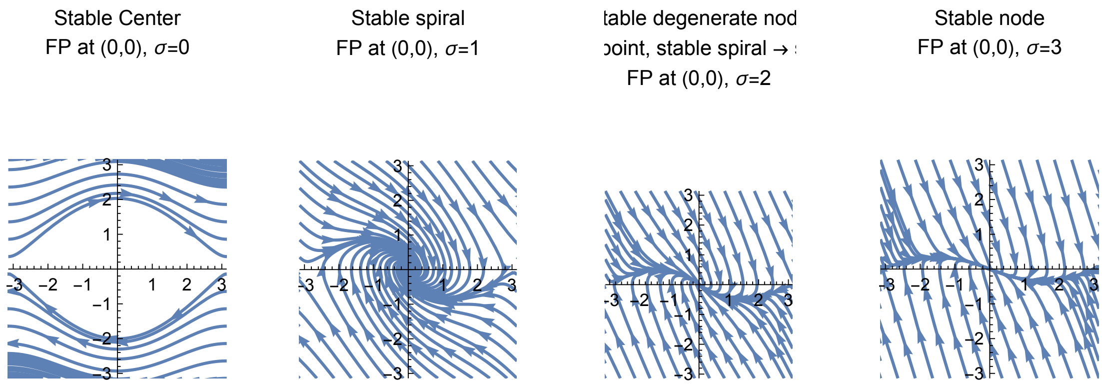
$\{\{y \to 0, x \to 2\pi c_1 \text{ if } c_1 \in \mathbb{Z}\}, \{y \to 0, x \to \pi + 2\pi c_1 \text{ if } c_1 \in \mathbb{Z}\}\}$

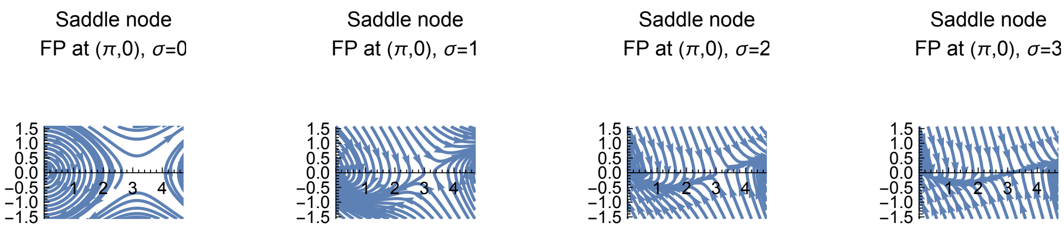
```
(*Plot with different values on σ, the values = 0,1,2,3 *)
minx = -π; miny = -π; maxx = π; maxy = π;
sol1[x0_, y0_] :=
  Table[NDSolve[{D[x[t], t] == y[t], D[y[t], t] == -Sin[x[t]] - σ y[t],
     x[0] == x0, y[0] == y0}, {x[t], y[t]}, {t, 0, 10}], {σ, {0}}];
initialConditions = Join[Table[{minx, y}, {y, miny, maxy, 0.5}],
   Table[{maxx, y}, {y, miny, maxy, 0.5}], Table[{x, miny}, {x, minx, maxx, 0.5}],
   Table[{x, maxy}, {x, minx, maxx, 0.5}]];
p1 = Show[Table[ParametricPlot[Evaluate[
       {x[t], y[t]} /. sol1[initialConditions[[i, 1]], initialConditions[[i, 2]]]],
      {t, 0, 10}, PlotRange → {{minx, maxx}, {miny, maxy}},
      PlotLabel → "Stable Center\nFP at (0,0), σ=0"] /.
     Line[x_] :> {Arrowheads[{0, 0.04, 0.04, 0.04, 0}], Arrow[x]},
    {i, Length[initialConditions]}]];
sol2[x0_, y0_] :=
  Table[NDSolve[{D[x[t], t] == y[t], D[y[t], t] == -Sin[x[t]] - σ y[t],
     x[0] == x0, y[0] == y0}, {x[t], y[t]}, {t, 0, 10}], {σ, {1}}];
initialConditions = Join[Table[{minx, y}, {y, miny, maxy, 0.5}],
   Table[{maxx, y}, {y, miny, maxy, 0.5}], Table[{x, miny}, {x, minx, maxx, 0.5}],
   Table[{x, maxy}, {x, minx, maxx, 0.5}]];
p2 = Show[Table[ParametricPlot[Evaluate[
       {x[t], y[t]} /. sol2[initialConditions[[i, 1]], initialConditions[[i, 2]]]],
      {t, 0, 10}, PlotRange → {{minx, maxx}, {miny, maxy}},
      PlotLabel → "Stable spiral\nFP at (0,0), σ=1"] /.
     Line[x_] :> {Arrowheads[{0, 0.04, 0.04, 0.04, 0}], Arrow[x]},
    {i, Length[initialConditions]}]];
sol3[x0_, y0_] :=
  Table[NDSolve[{D[x[t], t] == y[t], D[y[t], t] == -Sin[x[t]] - σ y[t],
```

```
      x[0] == x0, y[0] == y0}, {x[t], y[t]}, {t, 0, 10}], {σ, {2}}];
initialConditions = Join[Table[{minx, y}, {y, miny, maxy, 0.5}],
   Table[{maxx, y}, {y, miny, maxy, 0.5}], Table[{x, miny}, {x, minx, maxx, 0.5}],
   Table[{x, maxy}, {x, minx, maxx, 0.5}]];
p3 = Show[Table[ParametricPlot[Evaluate[
        {x[t], y[t]} /. sol3[initialConditions⟦i, 1⟧, initialConditions⟦i, 2⟧]],
       {t, 0, 10}, PlotRange → {{minx, maxx}, {miny, maxy}},
       PlotLabel → "Stable degenerate node\nBifurcation point,
          stable spiral → stable node\nFP at (0,0), σ=2"] /.
      Line[x_] ⧴ {Arrowheads[{0, 0.04, 0.04, 0.04, 0}], Arrow[x]},
     {i, Length[initialConditions]}]];
sol4[x0_, y0_] :=
  Table[NDSolve[{D[x[t], t] == y[t], D[y[t], t] == -Sin[x[t]] - σ y[t],
      x[0] == x0, y[0] == y0}, {x[t], y[t]}, {t, 0, 10}], {σ, {3}}];
initialConditions = Join[Table[{minx, y}, {y, miny, maxy, 0.5}],
   Table[{maxx, y}, {y, miny, maxy, 0.5}], Table[{x, miny}, {x, minx, maxx, 0.5}],
   Table[{x, maxy}, {x, minx, maxx, 0.5}]];
p4 = Show[Table[ParametricPlot[Evaluate[
        {x[t], y[t]} /. sol4[initialConditions⟦i, 1⟧, initialConditions⟦i, 2⟧]],
       {t, 0, 10}, PlotRange → {{minx, maxx}, {miny, maxy}},
       PlotLabel → "Stable node\nFP at (0,0), σ=3"] /.
      Line[x_] ⧴ {Arrowheads[{0, 0.04, 0.04, 0.04, 0}], Arrow[x]},
     {i, Length[initialConditions]}]];

minx = 0; miny = -π / 2; maxx = π * 3 / 2; maxy = π / 2;
sol5[x0_, y0_] :=
  Table[NDSolve[{D[x[t], t] == y[t], D[y[t], t] == -Sin[x[t]] - σ y[t],
      x[0] == x0, y[0] == y0}, {x[t], y[t]}, {t, 0, 10}], {σ, {0}}];
initialConditions = Join[Table[{minx, y}, {y, miny, maxy, 0.3}],
   Table[{maxx, y}, {y, miny, maxy, 0.5}], Table[{x, miny}, {x, minx, maxx, 0.3}],
   Table[{x, maxy}, {x, minx, maxx, 0.5}]];
p5 = Show[Table[ParametricPlot[Evaluate[
        {x[t], y[t]} /. sol5[initialConditions⟦i, 1⟧, initialConditions⟦i, 2⟧]],
       {t, 0, 10}, PlotRange → {{minx, maxx}, {miny, maxy}},
       PlotLabel → "Saddle node\nFP at (π,0), σ=0"] /.
      Line[x_] ⧴ {Arrowheads[{0, 0.04, 0.04, 0.04, 0}], Arrow[x]},
     {i, Length[initialConditions]}]];
sol6[x0_, y0_] :=
  Table[NDSolve[{D[x[t], t] == y[t], D[y[t], t] == -Sin[x[t]] - σ y[t],
      x[0] == x0, y[0] == y0}, {x[t], y[t]}, {t, 0, 10}], {σ, {1}}];
initialConditions = Join[Table[{minx, y}, {y, miny, maxy, 0.3}],
   Table[{maxx, y}, {y, miny, maxy, 0.3}], Table[{x, miny}, {x, minx, maxx, 0.3}],
   Table[{x, maxy}, {x, minx, maxx, 0.3}]];
p6 = Show[Table[ParametricPlot[Evaluate[
        {x[t], y[t]} /. sol6[initialConditions⟦i, 1⟧, initialConditions⟦i, 2⟧]],
       {t, 0, 10}, PlotRange → {{minx, maxx}, {miny, maxy}},
       PlotLabel → "Saddle node\nFP at (π,0), σ=1"] /.
```

```
      Line[x_] ⧴ {Arrowheads[{0, 0.04, 0.04, 0.04, 0}], Arrow[x]},
     {i, Length[initialConditions]}]];
sol7[x0_, y0_] :=
  Table[NDSolve[{D[x[t], t] ⩵ y[t], D[y[t], t] ⩵ -Sin[x[t]] - σ y[t],
      x[0] ⩵ x0, y[0] ⩵ y0}, {x[t], y[t]}, {t, 0, 10}], {σ, {2}}];
initialConditions = Join[Table[{minx, y}, {y, miny, maxy, 0.3}],
    Table[{maxx, y}, {y, miny, maxy, 0.3}], Table[{x, miny}, {x, minx, maxx, 0.3}],
    Table[{x, maxy}, {x, minx, maxx, 0.3}]];
p7 = Show[Table[ParametricPlot[Evaluate[
         {x[t], y[t]} /. sol7[initialConditions⟦i, 1⟧, initialConditions⟦i, 2⟧]],
       {t, 0, 10}, PlotRange → {{minx, maxx}, {miny, maxy}},
       PlotLabel → "Saddle node\nFP at (π,0), σ=2"] /.
      Line[x_] ⧴ {Arrowheads[{0, 0.04, 0.04, 0.04, 0}], Arrow[x]},
     {i, Length[initialConditions]}]];
sol8[x0_, y0_] :=
  Table[NDSolve[{D[x[t], t] ⩵ y[t], D[y[t], t] ⩵ -Sin[x[t]] - σ y[t],
      x[0] ⩵ x0, y[0] ⩵ y0}, {x[t], y[t]}, {t, 0, 10}], {σ, {3}}];
initialConditions = Join[Table[{minx, y}, {y, miny, maxy, 0.3}],
    Table[{maxx, y}, {y, miny, maxy, 0.3}], Table[{x, miny}, {x, minx, maxx, 0.3}],
    Table[{x, maxy}, {x, minx, maxx, 0.3}]];
p8 = Show[Table[ParametricPlot[Evaluate[
         {x[t], y[t]} /. sol8[initialConditions⟦i, 1⟧, initialConditions⟦i, 2⟧]],
       {t, 0, 10}, PlotRange → {{minx, maxx}, {miny, maxy}},
       PlotLabel → "Saddle node\nFP at (π,0), σ=3"] /.
      Line[x_] ⧴ {Arrowheads[{0, 0.04, 0.04, 0.04, 0}], Arrow[x]},
     {i, Length[initialConditions]}]];
GraphicsRow[{p1, p2, p3, p4}, ImageSize → Full]
GraphicsRow[{p5, p6, p7, p8}, ImageSize → Full]
```





```
(* 2.2f *)
(*Derive an integral of motion for this dynamical system in terms of(ϕ,ω,τ)*)
(*From the task:
```

$\dot{\phi} = \omega$; and $\dot{\omega} = \sin(\phi)[\cos(\phi)-\tau-1]$;

Rewrite as ODE with $\ddot{\phi} = \dot{\omega}$ and inserting it to the second term;

$\ddot{\phi} = \sin(\phi)[\cos(\phi)-\tau-1] \rightarrow \ddot{\phi} - \sin(\phi)[\cos(\phi)-\tau-1] = 0$;

Multiply both sides by $\frac{d\phi}{dt}$;

$\frac{d\phi}{dt}\frac{d^2\phi}{dt^2} - \frac{d\phi}{dt}(\sin(\phi)[\cos(\phi)-\tau-1]) = 0$;

One can integrate twice(followed an example from the book);

$\frac{1}{2}\frac{d}{dt}\left(\frac{d\phi}{dt}\right)^2 - \frac{d}{dt}(\sin(\phi)[\cos(\phi)-\tau-1]\Big)d\phi = 0; \rightarrow$

$\frac{d}{dt}\left(\omega^2 - \int 2\sin(\phi)[\cos(\phi)-\tau-1]\ d\phi\right) = 0; \rightarrow$

$\omega^2 - \cos(\phi)(-2(1+\tau)+\cos(\phi)) + c = 0$;

With $\omega \rightarrow 0$ and $\phi \rightarrow \frac{\pi}{2}$ we want $-1 \rightarrow c = -1$;

We can now combine and put together the final expression:

Derive an integral of motion $\rightarrow = -1\ \omega^2 - \cos(\phi)*(-2*(1+\tau)+\cos(\phi))$;

```
*)

Integrate[-2 Sin[ϕ] * (Cos[ϕ] - τ - 1), ϕ] // FullSimplify;
ω^2 + Cos[ϕ] * (-2 * (1 + τ) + Cos[ϕ]) + c /. {ω → 0, ϕ → π / 2, c → -1};
ω^2 + Cos[ϕ] * (-2 * (1 + τ) + Cos[ϕ]) - 1
```

$-1 + \omega^2 + \text{Cos}[\phi]\ (-2\ (1 + \tau) + \text{Cos}[\phi])$