CONTINUOS TIME SIGNALS

LTI SYSTEMS
used to approximate NL systems (important math. tool)
mature

Analog case: $\xrightarrow{x(t)} \boxed{T\{\cdot\}} \xrightarrow{y(t)}$ $y(t) = T\{x(t)\}$

T: transformation

- **LINEARITY**: $y_1(t) = T\{x_1(t)\}$
  $y_2(t) = T\{x_2(t)\}$
  transformation 2 generic analogs signal

  LC: $x_3 = ax_1 + bx_2$

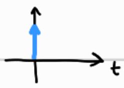
  $y_3 = T\{x_3\} = ay_1 + by_2$ — $\forall$ LINEAR COMB. a,b; $\forall$ couple of x_1, x_2

- **TIME INVARIANCE**: delay in input produces same delay in the output
  $x_1(t) = x(t-t_0)$
  $y(t) = T\{x_1(t)\} = y(t-t_0)$ — $\forall$ time delay t_0; $\forall$ time instant t

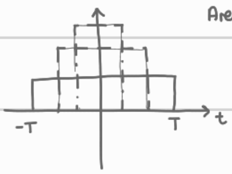
  NB. Math. definition true $\forall t$ but in reality true only for an INTERVAL OF TIME (stochastic signal)

DIRAC DELTA
IMPULSE (input / not a function) generalized function

$$\delta(t) = \begin{cases} \infty & \text{for } t=0 \\ 0 & \text{for } t \neq 0 \end{cases}$$



- $\int_{-\infty}^{+\infty} \delta(t)\,dt = 1$ → approximation: FAMILY OF RECTANGULAR FUNCTION
  [-T, T] interval



Area = 1 = base · height = $2T \cdot \frac{1}{2T}$

$$K_T(t) = \begin{cases} 1/2T & t \in [-T,T] \\ 0 & \text{otherwise} \end{cases}$$

$\delta(t) = \lim_{T\to 0} K_T(t)$ — limit of the family of rect. curves
height → ∞

$$\int_{-\infty}^{+\infty} \lim_{T\to 0} K_T(t)\,dt = \lim_{T\to 0} \underbrace{\int_{-\infty}^{+\infty} K_T(t)\,dt}_{\text{area}=1} = 1$$

- SAMPLING PROPERTY analog into discrete by δ

$$\begin{cases} T=0 \rightarrow \int_{-\infty}^{+\infty} \delta(t)\,\Phi(t)\,dt = \Phi(0) \\ T\neq 0 \rightarrow \int_{-\infty}^{+\infty} \delta(t-T)\,\Phi(t)\,dt = \Phi(T) \end{cases}$$

generalization — sampling at time $t-T=0$

Dirac Delta important for the A/D conversion.
Let see the relationship btw output of LTI system and the input in the time domain .

$\xrightarrow{x_a(t)} \boxed{LTI: T\{\cdot\}} \xrightarrow{y_a(t)}$
generic analog signal

$h_a(t) = T\{\delta(t)\}$ IMPULSE RESPONSE → OUTPUT of ANY SIGNAL IS know if we know $h_a(t)$

$x_a(t) = \int_{-\infty}^{+\infty} x_a(T)\,\delta(t-T)\,dT$ — sampling property ($t=T$)

↳ convolution of a signal with δ (neutral element → obtain signal itself)

output → come T input

$y_a(t) = T\{x_a(t)\} =$ — hp: LINEAR SYSTEM

) sampling with δ (conv)

$= T\left\{ \int_{-\infty}^{+\infty} x_a(T)\,\delta(t-T)\,dT \right\} =$

L: $= \int_{-\infty}^{+\infty} T\{x_a(T)\,\delta(t-T)\,dT\} \overset{L}{=} \int_{-\infty}^{+\infty} x_a(T)\,\underbrace{T\{\delta(t-T)\}}_{h_a(t-T)}\,dT$ — impulse response

TI: $= \int_{-\infty}^{+\infty} x_a(T)\,h_a(t-T)\,dT = y_a(t)$

NB. convolution is COMMUTATIVE

$$\underbrace{\int_{-\infty}^{+\infty} x_a(T)\,\delta(t-T)\,dT}_{\text{conv. of signal with } \delta} = \underbrace{\int_{-\infty}^{+\infty} x_a(t-T)\,\delta(T)\,dT}_{\text{convolution of } \delta \text{ with signal}}$$

riprendo formula finale

CV: $T' = t - T \rightarrow T = t - T'$
$dT' = -dT$

$y_a(t) = \int_{-\infty}^{+\infty} x_a(T)\,h_a(t-T)\,dT =$

CV: $= -\int_{+\infty}^{-\infty} x_a(t-T')\,h_a(T')\,dT' = \int_{-\infty}^{+\infty} x_a(t-T')\,h_a(t)\,dT'$

$T' \to T$: $= \int_{-\infty}^{+\infty} x_a(t-T)\,h_a(T)\,dT$

LAPLACE TRANSFORM

new domain: Laplace Domain

COMPLEX EXP. FUNCTION $x_a(t) = Ae^{st}$ $s \in \mathbb{C}$ $(s = \sigma + j\Omega)$ t: analog time

EIGENFUNCTION for LTI sys. using it as input, same form of the output

$= A e^{\sigma t} e^{j\Omega t}$

ATTENUATION or AMPLIFICATION TERM real term

OSCILLATORY TERM associated to the part (sinusoidal part)

related to euler's formula: $\cos \Omega t = \frac{e^{j\Omega t} + e^{-j\Omega t}}{2}$

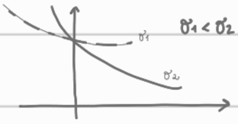
$$y_a(t) = \int_{-\infty}^{+\infty} A e^{s(t-\tau)} h_a(\tau)\, d\tau = A e^{st} \underbrace{\int_{-\infty}^{+\infty} h_a(\tau) e^{-s\tau} d\tau}_{H_a(s)} = A e^{st} H_a(s)$$

$H_a(s)$ complex number
LAPLACE TRANSFORM of IMPULSE RESPONSE $\equiv$ TRANSFER FUNCTION OF A SYSTEM

GENERALIZED DEFINITION:

(Hp)
- $x_a(t) = 0$ for $t < 0$
- $x_a(t)$ continuous or piecewise cont.
- $x_a(t)$ of exp. order
  - if $\exists \sigma \in \mathbb{R}$ (real value) for which $\lim_{t \to \infty} |x_a(t)| e^{-\sigma t} = 0$

σ_c: abscissa of convergence for which lim is valid



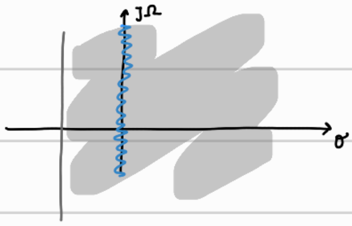
$\sigma_1 < \sigma_2$ $\quad |x_a(t)| e^{-\sigma t}$ real exp (attenuation term)

lim = 0 if $|x_a(t)| \to \infty$ more than exponential

$$\boxed{X_a(s) = \int_{-\infty}^{+\infty} x_a(t) e^{-st} dt \qquad s = \sigma + j\Omega \in \mathbb{C}}$$

3hp satisfied

RIGHT HEMIPLANE defined by σ_c



region of existance of L. transform in s-plane (by 3rd hp)

For any transformation, we have a COUPLE of FORMULAS

TIME DOMAIN convolut. $\xrightarrow{\mathcal{L}}$ LAPLACE DOMAIN product $\xrightarrow{\mathcal{L}^{-1}}$ TIME DOMAIN

if $\sigma > \sigma_c$:

$$\boxed{x_a(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X_a(s) e^{st} dt}$$

$\to$ signal = superimposition ($\int$) of many (∞) elems

integral along each vertical line in the region of convergence

$\to$ we reconstruct the signal using the L. transform as WEIGHT of the building blocks (exp. functions)

$x_a(t) \to$ LTI $\xrightarrow{y_a(t)}$ $\quad Y_a(s) = \underbrace{X_a(s) H(s)}_{\text{product instead of convolution}}$

$\frac{1}{2\pi j} X_a(s) e^{st}$ eigenfunction (complex exp) $\longrightarrow$ LTI $\longrightarrow$ $\frac{1}{2\pi j} X_a(s) e^{st} H_a(s)$

$\underbrace{\frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X_a(s) e^{-st} ds}_{\text{sum of many exp. functions}}$ $\longrightarrow$ LTI $\longrightarrow$ $\frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} \underbrace{H_a(s) X_a(s)}_{Y_a(s) \text{ lapl. transform of the output}} e^{-st} ds$

$x_a(t)$ ANTITRASFORMATION FORMULA

same LC of the individual outputs $\to Y_a(s) = H_a(s) X_a(s)$

$y_a(t)$ ANTITRASFORMATION FORMULA

COUPLE of TRANSFORMATIONS

ANALYSIS FORMULA
$$X_a(s) = \int_{-\infty}^{+\infty} x_a(t) e^{-st} dt$$
$\to$ coeff. to use in synthesis formula

SYNTHESIS FORMULA
$$x_a(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X_a(s) e^{st} dt$$

FOURIER TRANSFORM

Ω: analog case

particular case $s = j\Omega$ $\to$ s restricted to imaginary axis
laplace transform restricted to Im axis

COUPLE OF TRANSFORMATION

ANALYSIS F.
$$X_a(j\Omega) = \int_{-\infty}^{+\infty} x_a(t) e^{-j\Omega t} dt$$

SYNTHESIS F.
$$x_a(t) = \frac{1}{2\pi \cancel{j}} \int_{-\infty}^{+\infty} X_a(j\Omega) e^{j\Omega t} d(\cancel{j}\Omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_a(j\Omega) e^{j\Omega t} dt$$

BASIC FUNCTION $e^{st} = \underbrace{\cancel{e^{\sigma t}}}_{Re = 0} e^{j\Omega t} = e^{j\Omega t}$ sinus/cosinusoidal function $\to$ HARMONIC ANALYSIS

(TH) If signal is finite energy, we can define the Fourier transform
$$E = \int_{-\infty}^{+\infty} |x_a(t)|^2 dt < \infty$$

Let define the relationship btw Laplace and Fourier domain.

Harmonic analysis

$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{+\infty} \delta(t) e^{-j\Omega t} dt = e^{-j\Omega 0} = 1$

Fourier tr. applied to δ

$\underbrace{\delta(t) \neq 0}_{\text{for } t=0}$; $t=0$

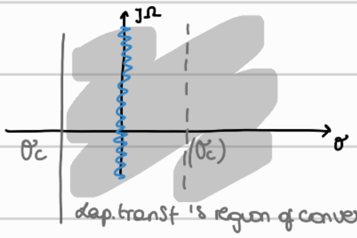
IMPORTANT!

NB. In time domain, $\delta(t)$ defined for $t=0$
In freq. domain, we need every freq to reconstruct the signal
IDEAL IMPULSE : composed by every frequency $\longrightarrow$ coeff. in time domain are all 1

Consider $x_a(t)$ for which $\mathcal{L}$ exists ①
② $s = \sigma + j\Omega$
③ Split real and imagin. parts in the 2 exp. $\longrightarrow$ $\mathcal{F}$ transform recognized

$$X_a(s) \overset{1}{=} \int_{-\infty}^{+\infty} x_a(t) e^{-st} dt \overset{2}{=} \int_{-\infty}^{+\infty} x_a(t) e^{-(\sigma + j\Omega)t} dt \overset{3}{=} \int_{-\infty}^{+\infty} \underbrace{x_a(t) e^{-\sigma t}}_{\text{signal}} e^{-j\Omega t} dt = \mathcal{F}\{\underbrace{x_a(t) e^{-\sigma t}}_{\text{signal} \cdot \text{exp.}}\}$$

$\mathcal{F}$: particular case of $\mathcal{L}$ transform
$\hookrightarrow$ $\mathcal{I}m$ axis
$\hookrightarrow$ hemiplane defined by σ_c



• If $\mathcal{L}$ transform $\exists$ and $\sigma_c < 0$ ($\mathcal{I}m \in$ hemiplane) $\Rightarrow$ $\mathcal{F}$ transform $\exists$
• If $\sigma_c > 0$, $\mathcal{L}$ transform $\exists$ BUT $\mathcal{F}$ transform $\nexists$

FINITE ENERGY SIGNAL
$\Updownarrow$
$\exists$ $\mathcal{L}$ transform
$\Updownarrow$
$\exists$ $\mathcal{F}$ transform
(be sure $\sigma_c < 0$)

$\mathcal{F}$ transform: properties

NB. $X_a(s)$ is complex even if signal is real (for both $\mathcal{F}/\mathcal{L}$ transform)
amplitude and phase

AMPLITUDE SPECTRUM : amplitude of $\mathcal{F}$ as function of Ω
of the coeff. associated to the corresponding sin. funct. to reconstruct signal in time domain

PHASE SPECTRUM : phase of $\mathcal{F}$ as function of Ω

• CONVOLUTION TH. $x_1(t), x_2(t) \longrightarrow \mathcal{F}\{\underbrace{x_1(t) * x_2(t)}_{\substack{\text{convolution} \\ \text{(time domain)}}}\} = \underbrace{\mathcal{F}\{x_1(t)\} \cdot \mathcal{F}\{x_2(t)\}}_{\substack{\text{product} \\ \text{(freq domain)}}}$

dual : convolution in one domain $\equiv$ product in other domain

• PRODUCT TH. $x_1(t), x_2(t) \longrightarrow \mathcal{F}\{x_1(t) \cdot x_2(t)\} = \frac{1}{2\pi} \mathcal{F}\{x_1(t)\} * \mathcal{F}\{x_2(t)\}$

we are using Ω

• SHIFT TH. $\underbrace{\mathcal{F}\{x_a(t-T)\}}_{\mathcal{F}\text{ of shifted signal}} = \underbrace{\mathcal{F}\{x_a(t)\}}_{\mathcal{F}\text{ of unshifted signal}} \underbrace{e^{-jT}}_{\text{PHASE TERM}}$ $\longrightarrow$ shift in time (time domain) $\equiv$ change in phase (freq. domain) $\longrightarrow$ freq. components are the same in amplitude

REAL SIGNAL / FUNCTION
• module is EVEN
• phase is ODD
more informative
• real part is EVEN
• imaginary part is ODD
} symmetries of $\mathcal{F}$ transform

plot for $\Omega > 0$
• EVEN SYMMETRY for module
• ODD SYMMETRY for phase

$e^{j\Omega t}$ as function of cos/sin (EULER'S FORMULAS)

$$\begin{cases} \cos \Omega t = \dfrac{e^{j\Omega t} + e^{-j\Omega t}}{2} \\ \sin \Omega t = \dfrac{e^{j\Omega t} - e^{-j\Omega t}}{2} \end{cases}$$

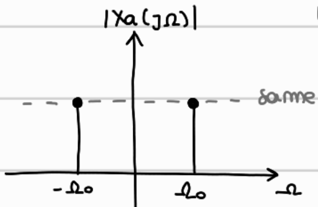
$$X_a(j\Omega) = \int_{-\infty}^{+\infty} x_a(t) e^{-j\Omega t} dt = \int_{-\infty}^{+\infty} x_a(t) \cos(\Omega t) dt - j \int_{-\infty}^{+\infty} x_a(t) \sin(\Omega t) dt$$

If $x_a(t)$ real, sin and cos are real $\longrightarrow$ product is real

$$X_a(-j\Omega) = \int_{-\infty}^{+\infty} x_a(t) \underbrace{\cos(\Omega t)}_{= \cos(-\Omega t)} dt + j \int_{-\infty}^{+\infty} x_a(t) \underbrace{\sin(\Omega t)}_{\sin(-\Omega t) = -\sin(\Omega t)} dt = X_a^*(j\Omega)$$

$\longrightarrow$ HERMITIAN SYMMETRY

$X_a(-j\Omega) = X_a^*(j\Omega)$
$\downarrow$
complex conjugate in Ω



$\mathcal{F}^{-1}$ $\left\{ \begin{array}{l} \text{2 lines} \\ \text{in freq domain} \end{array} \right\}$ ≡ cosine in time domain

Harmonic components

Use $\mathcal{F}^{-1}$ to find the signal. The integral in pulsation is limited to 2 components.

$$X_a(t) = \frac{1}{2\pi}\left(X_a(j\Omega_0) e^{j\Omega_0 t} + X_a(-j\Omega_0) e^{-j\Omega_0 t} \right)$$

freal → symmetry

$$= \frac{1}{2\pi}\left(|X_a(j\Omega_0)| e^{\Phi(X_a(j\Omega_0))} e^{j\Omega_0 t} + |X_a(j\Omega_0)| e^{-\Phi(X_a(j\Omega_0))} e^{-j\Omega_0 t} \right) =$$

EVEN S. module is the same

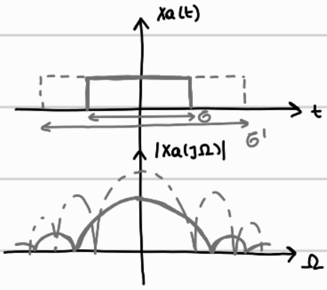
ODD S. phase di $-\Omega_0$ = − phase Ω_0

$$= \frac{1}{\pi} |X_a(j\Omega_0)| \underbrace{\left(\frac{e^{j\Omega_0 t + \Phi(X_a(j\Omega_0))} + e^{-(j\Omega_0 t + \Phi(X_a(j\Omega_0)))}}{2} \right)}_{\text{Euler's formula}} =$$

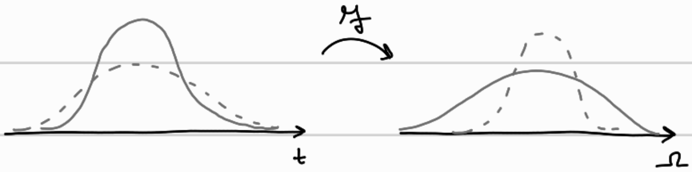
$$= \frac{1}{\pi} |X_a(j\Omega_0)| \cos\left(\Omega_0 t + \Phi(X_a(j\Omega_0))\right)$$

RECTANGULAR PULSE

- if change amplitude in time, change amplitude in freq
- If change rectangular form
  - ↳ wider pulse → $\mathcal{F}$ more concentrated in the origin in freq domain
  - ↳ shrinker pulse → $\mathcal{F}$ larger in freq domain



GAUSSIAN IMPULSE



- wider in time → shorter in freq
- shorter in time → higher in freq
  - ↳ higher freq to reconstruct short signal in time domain

FOURIER SERIES

Analog periodic signals are not energy signals → $\cancel{\mathcal{F}}$ (not finite energy)
power signals

↳ $\tilde{x}_a(t) = \tilde{x}_a(t-T)$ T: period (signal repeats itself every T sec)

(HP) PIECEWISE CONTINUOUS, DERIVABLE signal
→ Fourier series can be defined

→ Σ ∞ terms

$$\boxed{\tilde{x}_a(t) = \sum_{\ell=-\infty}^{+\infty} C_\ell \, e^{j\ell \frac{2\pi}{T} t}}$$

SYNTHESIS FORMULA
→ LC: sum weighted with coeff. of the basic function used to reconstruct the signal

NB. reconstruction is better for ↑ # terms

~ antitransformation formula → ANALOGUE FORMULATIONS
$x_a(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_a(j\Omega) e^{j\Omega t} d\Omega$
(similar interpretation)

BASIC FUNCTION: $e^{j \cdot \text{something} \cdot t}$ (sinusoidal term)

$e^{j\ell \frac{2\pi}{T} t}$

→ $\Omega = \ell \frac{2\pi}{T} = \ell\,\Omega_0$

Ω_0 MAIN/FUNDAMENTAL PULSATION

Remember: $\cos(\Omega t) = \frac{e^{j\Omega t} + e^{-j\Omega t}}{2}$

$\left[\begin{array}{l} \Omega = \pm \ell\,\Omega_0 \\ \text{and so on } \ell = \pm 2, \dots \\ \text{(multiples of } \Omega_0\text{)} \end{array} \right]$

composition of sinusoidal signals
Ω: DISCRETE VARIABLE
→ Fourier series is a sum

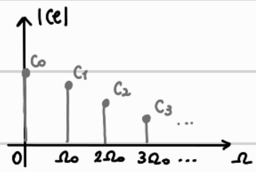
ℓ: integer
- $\ell = 0$, $e^{j0} = 1$ constant term
- $\ell = \pm 1$, $e^{\pm j \frac{2\pi}{T}}$ MAIN HARMONICS
- $\ell = \pm 2$, $e^{\pm j \frac{4\pi}{T}}$ double of main pulsation
- ....

$$\boxed{C_\ell = \frac{1}{T}\int_{-T/2}^{+T/2} \tilde{x}_a(t)\, e^{-j\frac{2\pi}{T}\ell t}\, dt}$$

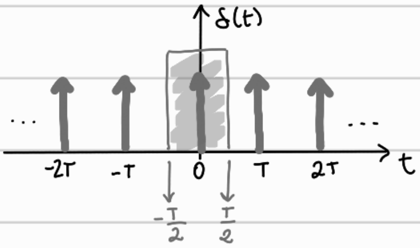
ANALYSIS FORMULA
~ $\mathcal{F}$ formula but within a period (periodic signal)

integral because time domain is continuous

→ C_ℓ complex → MAGNITUDE and PHASE SPECTRA
discrete func $\Omega = \pm\ell\,\Omega_0$ discrete



- REGION OF CONVERGENCE: where HP are satisfied (better convergence for high # of harmonics except for points where there is a discontinuity → series does not converge to the signal
(GIBBS EFFECT)



IMPULSE TRAIN: train of impulse $\delta(t)$ (periodic) $\longrightarrow$ used to sample an analog signal into a discrete one

$$\delta(t) = \sum_{k=-\infty}^{+\infty} \delta(t-kT)$$

$\hookrightarrow$ TIME SHIFT

$$= \sum_{\ell=-\infty}^{+\infty} c_\ell \, e^{j\ell \frac{2\pi}{T} t}$$

$$\boxed{c_\ell} = \frac{1}{T} \int_{-T/2}^{+T/2} \underbrace{\sum_{k=-\infty}^{+\infty} \delta(t-kT)}_{\text{in the interval: just } \delta(t)} e^{-j\ell \frac{2\pi}{T} t} \, dt =$$

$$= \frac{1}{T} \int_{-T/2}^{+T/2} \underbrace{\delta(t)}_{\neq \text{ only for } t=0} \underbrace{e^{-j\ell \frac{2\pi}{T} t}}_{=1 \text{ for } t=0} \, dt = \boxed{\frac{1}{T}}$$

coeff equal for all the terms

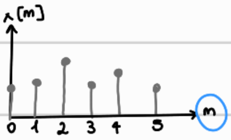
$\mathcal{F}\{\delta(t)\} = 1$ constant independently on the pulsation

$$\boxed{= \frac{1}{T} \sum_{\ell=-\infty}^{+\infty} e^{j\ell \frac{2\pi}{T} t}}$$

DISCRETE-TIME SIGNALS AND SYSTEMS

or SEQUENCES : index ≡ location of sample in time

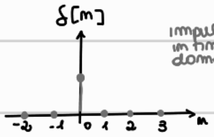
$x[m]$ → discrete time signal
↳ index (discrete time)



m : integer number (otherwise, not defined)
↓
DEFINED NUMBER

DIRAC DELTA
main discrete time signal
normal function
(easier than continuous signal)

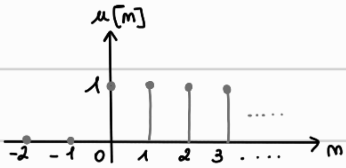
$$\delta[m] = \begin{cases} 1 & m=0 \\ 0 & m\neq 0 \end{cases}$$



impulse in time domain

HEAVISIDE FUNCTION
step function

$$u[m] = \begin{cases} 1 & m \geq 0 \\ 0 & m < 0 \end{cases}$$



SINUSOIDAL FUNCTION → BASIC FUNCTION of Fourier series (also for discrete time domain)
or cosinusoidal function

$x[m] = A \cdot \cos(\omega m + \phi)$ — amplitude (A), pulsation (ω), time (m), phase (ϕ)

class of function

periodic discrete time signal : $\tilde{x}[m] = \tilde{x}[m+N] \quad \forall m$ N: period → smaller integer value for which signal repeat itself

GENERIC COSINUSOIDAL SIGNAL

$\underbrace{X_a(t) = A\cos(\Omega t + \Phi)}_{\text{continuous time analog cosinusoidal function}}$ $\xrightarrow{T_c}$ DISCRETE TIME SIGNAL

T_c SEMPLING STEP (time domain)

$F_c = \frac{1}{T_c}$

$X[m] = X_a(mT_c)$
$= A\cos(\omega m + \Phi)$
$= A\cos(\Omega T_s m + \phi)$ $\boxed{\omega = \Omega T_c}$

After sampling, we have generic cosine related to the continuous one by relationship btw Ω and ω

• PERIODICITY discrete time signals not always periodic

$X[m] = A\cos(\omega m + \Phi) \overset{\text{when?}}{=} X[m+N] = A\cos(\omega[m+N] + \Phi)$ — equal arguments

$\omega \cancel{m} + \cancel{\Phi} = \omega[\cancel{m} + N] + \cancel{\Phi} + 2\pi K$ $K = 0, \pm 1, \pm 2, \dots$ arbitrary
equal or difference of $2K\pi$

→ $\omega = \frac{K}{N} 2\pi$ periodic signal IF this relationship is valid

$\omega = \Omega T_c$ changing T_c, we change pulsation BUT we must satisfy this condition to have periodicity

• $\Omega = 2\pi F$ linear relationship in continuous time domain
if F↑, pulsation Ω↑ → we obtain the same signal

Not true for discrete time domain

$X_1[m] = A\cos(\omega m + \Phi)$
$X_2[m] = A\cos((\omega + 2\pi)m + \Phi)$
} same amplitude and phase BUT different pulsation →

$X_2[m] = A\cos((\omega + 2\pi)m + \Phi) =$ → we increase the pulsation
$= A\cos(\omega m + 2\pi m + \Phi) =$ m: integer
$= A\cos(\omega m + \Phi) = X_1[m]$
↳ freq is the same

⇒ FREQ DOMAIN is periodic of PERIOD 2π for discrete time domain

↳ when Ω↑ from 0 to 2π (Ω equal to continuous pulsation)
{ freq↑ for $\Omega \in [0, \pi]$
{ freq↓ for $\Omega \in [\pi, 2\pi]$

same pulsation
BUT different behavior of the freq
↳ higher close to $\Omega = \pi$
↳ lower close to $\Omega = 0$

higher freq close to $\Omega = \pi$
lower freq close to $\Omega = 0$

Example: $x_a(t) = A\cos(\Omega t + \Phi) =$
"   —   "

For semplicity: $A = 1, \Phi = 0$

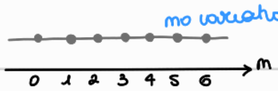
$= \cos(\Omega t)$

$\xrightarrow[\text{sampling}]{T_c = 1s}$ $x[n] = \cos[\omega n]$
"•"

$\omega = \Omega T_c = \Omega$

$\Omega = 0 \rightarrow \cos(0) = 1$
$T = \infty$

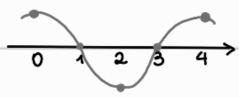
no variation



discrete freq ↑

$\Omega = \frac{\pi}{2} \rightarrow \cos[\frac{\pi}{2} n] = 0, 1, -1, 0, 1$

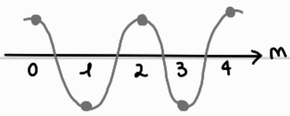
$T = \frac{2\pi}{\Omega} = 4\,sec$



discrete freq ↑

$\Omega = \pi \longrightarrow \cos[\pi n] = 1, -1, 1, -1, 1, \ldots$

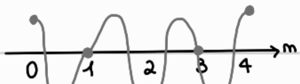
$T = \frac{2\pi}{\Omega} = 2\,sec$



discrete freq ↓

$\Omega = \frac{3}{2}\pi > \pi \longrightarrow \cos(\frac{3}{2}\pi n) = 1, 0, -1, 0, 1$

$T = \frac{2\pi}{\Omega} = \frac{4}{3} sec$

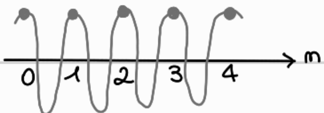


same discrete signal of $\Omega = \frac{\pi}{2}$
but Ω is higher

discrete Ω ↑

$\Omega = 2\pi \longrightarrow \cos(2\pi n) = 1, \forall n$

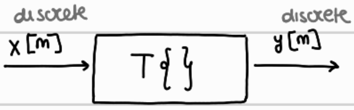
$T = 1\,sec$



same discrete signal of $\Omega = 0$

DIGITAL SIGNALS ⟶ DIGITAL SYSTEMS
(quantization)

SYSTEM
GENERAL CASE



I/O TRANSFORMATION
of discrete time signals

- **WITHOUT MEMORY** — $y[n]$ depends on $x[n]$ at the same time instant (not on previous ones)
  no memory of the past

  Ex: $y[n] = (x[n])^3$

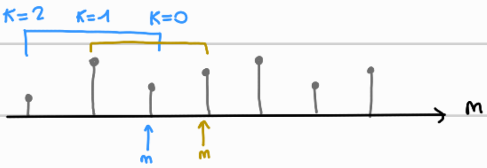
- **WITH MEMORY** — Ex: $y[n] = (x[n])^3 + \underbrace{x[n-1]}_{\text{dependency on the past}}$

MOVING AVERAGE FILTER
average of N samples

$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} x[n-k]$

$N = 1, \quad y[n] = x[n]$ identity — No sum: w/o memory

$N > 1$ with memory



N=3

LTI SYSTEMS

↳ TF describes completely the system
(∃ only for LTI)

IMPORTANT Biological systems are not LTI but
LTI can be an approximation in certain conditions
(NL systems are complex to study)

- **LINEARITY** superposition principle holds

$y_1[n] = T\{x_1[n]\}$
$y_2[n] = T\{\underbrace{x_2[n]}_{\text{2 general signals}}\}$
Transformation

NB. linearity must be proven for any possible LC

1 case of non linearity ⟹ NL system

LC: $x_3[n] = a x_1[n] + b x_2[n]$

$y_3[n] = T\{x_3[n]\} = a y_1[n] + b y_2[n]$

∀ LC a, b
∀ couple of signals x_1, x_2

- **TIME INVARIANCE**: delay in input produces the same delay in output

$x_1[m] = x[m-m_0]$ $\forall m$ (time point)
$y_1[m] = y[m-m_0]$ $\forall m_0$ (delay)

ACCUMULATOR $y[m] = \sum_{k=-\infty}^{m} x[k]$ SUM of input from $-\infty$ to current instant

Linear?

$y_1[m] = \sum_{k=-\infty}^{m} x_1[k]$

$y_2[m] = \sum_{k=-\infty}^{m} x_2[k]$

LC: $x_3[m] = a x_1[m] + b x_2[m]$

$y_3[m] \overset{?}{=} a y_1[m] + b y_2[m]$ $\longrightarrow$ $y_3[m] = \sum_{k=-\infty}^{m} x_3[k] =$

$= \sum_{k=-\infty}^{m} [a x_1[k] + b x_2[k]]$

Σ is linear

$= a \sum_{k=-\infty}^{m} x_1[k] + b \sum_{k=-\infty}^{m} x_2[k]$

$= a y_1[m] + b y_2[m]$ $\forall a,b$ $\forall x_1, x_2$

Time invariant?

$x_1[m] = x[m-m_0]$ generic time signal

$\underbrace{y[m-m_0] = \sum_{k=-\infty}^{m-m_0} x[k]}_{*}$ output shifted in time $\longrightarrow$ $y_1[m] = \sum_{k=-\infty}^{m} x_1[k] =$

$= \sum_{k=-\infty}^{m} x[k-m_0]$

CV $k_1 = k - m_0$: DUMMY VARIABLE

$= \sum_{k_1=-\infty}^{m-m_0} x[k_1] =$

CV $k = k_1$

$\underbrace{= \sum_{k=-\infty}^{m-m_0} x[k]}_{*}$ $\forall m, m_0$

Two formulation are equal

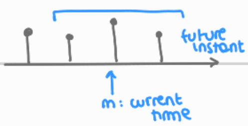
- **CAUSAL SYSTEM** output of discrete time x_0 does not depend on future inputs but just only on past and present ones } real time analysis ex: moving average filter
  ↳ output follows the inputs (effect)

$y[m] = \underbrace{x[m]}_{\text{present sample}} + \underbrace{x[m-m_0]}_{\text{past sample}}$ $m \leq m_0$

- **NON CAUSAL SYSTEM** use of past, present and future data } offline analysis (recorded data)

EX: $y[m] = \frac{1}{2N+1} \sum_{k=-N}^{N} x[m-k]$ moving average filter centered to current time $\rightarrow$ non causal

N=1



We can use it in ~ real time
we must wait for future samples before applying the formula (if possible)
$\rightarrow$ delay in the output

- **BIBO STABILITY** if $\forall x[m]$ such that $\underbrace{|x[m]| \leq B_x}_{\text{bounded input}} \Rightarrow \underbrace{|y[m]| \leq B_y}_{\text{bounded output by a constant}}$ $\forall m$

Bounded input, Bounded output

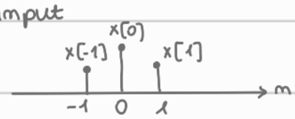
EX: $y[m] = (x[m])^{1000000}$ $|x[m]| \leq B_x \Rightarrow |y[m]| \leq B_x^{1000000}$ BIBO STABLE

$y[m] = \log_{10}(|x[m]|)$ NOT BIBO STABLE $x[m] = 0$, $y[m] = -\infty$

CONVOLUTION

$\delta(t)$ introduced for defining the relationship btw output and input

$x[m]$ can be defined as a function of δ
(sample weighted by δ)
↳ useful to define CONVOLUTION

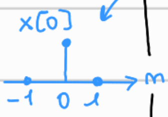


$$x[m] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[m-k]$$ generic signal

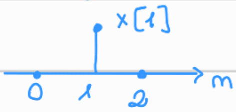
$$= x[0]\delta[m] + x[1]\delta[m-1] + x[-1]\delta[m+1] + \dots$$

Analog signal: $x_a(t) = \int_{-\infty}^{+\infty} x_a(\tau)\delta(t-\tau)d\tau$
δ: neutral term
⟹ we obtain signal itself both in cont / discr time
$\int$ Σ

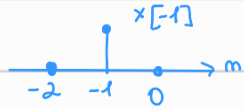
K=0



K=1



K=−1



∞ SUMMATIONS
de composition in many pieces

$= x[m]$

LTI: $T\{\}$ completely define I/O relationship (any)

transformation of a generic discr. time signal

$x[m] \rightarrow \boxed{T\{\}} \rightarrow y[m]$

$y[m] = T\{x[m]\}$ output = transformation of input

$$y[m] = T\{x[m]\} = T\left\{\sum_{k=-\infty}^{+\infty} x[k]\delta[m-k]\right\} =$$ valid for any system

L $= \sum_{k=-\infty}^{+\infty} T\{x[k]\delta[m-k]\} =$ K constant, m time variable

L $= \sum_{k=-\infty}^{+\infty} x[k]\, T\{\underbrace{\delta[m-k]}_{\text{shift in time}}\} =$ $h[m] = T\{\delta[m]\}$ IMPULSE RESPONSE

TI $= \sum_{k=-\infty}^{+\infty} x[k]\, h[m-k] = y[m]$ OUTPUT: convolution of input with impulse response

↓
CONVOLUTION PRODUCT

Def of convolution btw discrete time signals

$$\boxed{y[m] = \sum_{k=-\infty}^{+\infty} x[k]\cdot h[m-k] = \sum_{k=-\infty}^{+\infty} x[m-k]\, h[k]}$$

important relationship

Knowing output of impulse response, we can compute output of any other input signal
ONLY for LTI system

similar definition for cont. time signals
($\int$ instead of Σ)

TH: LTI BIBO STABLE ⟺ $B_h = \sum_{k=-\infty}^{\infty} |h[m]| < \infty$

LTI CAUSAL ⟺ $h[m] = 0$ for $m < 0$

⟺ necessary and sufficient condition

Knowing $h[m]$ impulse response
we know everything about LTI system → INFORMATIVE

Z- TRANSFORM

∃ analogy with Laplace transform → TIME DOMAIN ⇄ OTHER DOMAIN (transform / inverse transform) ~ Laplace domain

PAIR OF TRANSFORMATION → similar interpretation (not the same)

$x[m] = A z^m$ $z \in \mathbb{C}$ EIGENFUNTION for an LTI system

particular discr. time signal
complex exponential

base of complex number is an exponential

↳ we feed it as input of LTI system → output again exp. function
(same form of the input)

→ $x(t) = Ae^{st}$ → $x[m] \overset{\text{sampling}}{=} x(mT_c) = Ae^{smT_c} = Az^m$ $z \in \mathbb{C}$

exp in continuous time
exp in discrete time domain

LTI: $y[m] = \sum_{k=-\infty}^{+\infty} h[k]\, x[m-k]$ — convolution of the input with impulse response

$x[m] = A z^m$

$= \sum_{k=-\infty}^{+\infty} h[k]\, A z^{m-k} =$

L

$= A z^m \underbrace{\sum_{-\infty}^{+\infty} h[k]\, z^{-k}}_{\text{constant wrt } m} =$ → it changes amplitude and phase but not the form

$H(z)$: z-TRANSFORM of impulse response → **TRANSFER FUNCTION** of LTI discrete time system

$= \underbrace{A z^m H(z)}_{\text{proportional term}}$

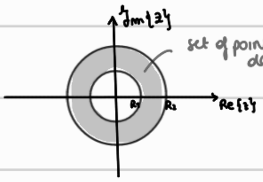
z-TRANSFORM of a generic signal (discrete time): $\boxed{X(z) = x[k]\, z^{-k}}$

↳ "X" upper case

↓ if it ∃:

- z-transform is complex even if signal is real
- It recall concept of POWER SIGNALS: $\sum_{0}^{\infty} a_m z^m$

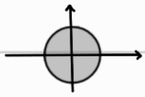
REGION OF CONVERGENCE $z \in$ complex plane



set of points btw 2 circumferences defined by radius and centered in the origin

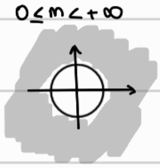
- $x[m]$ **BILATERAL**: defined from $-\infty$ to $+\infty$
- $x[m]$ **LEFT UNILATERAL**: defined for $-\infty < m \leq 0$

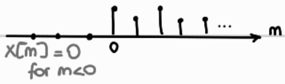
↳ $R_1 \to 0$: region of convergence is a circle



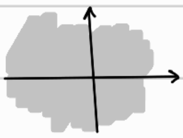
- $x[m]$ **RIGHT UNILATERAL**: defined for $0 \leq m < +\infty$

↳ $R_2 \to \infty$: region of convergence outside the circle





- $R_1 \to 0$, $R_2 \to \infty$: region of convergence is all the plane

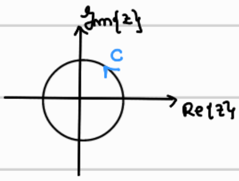


Antitransformation formula makes use of CURVILINEAR INTEGRAL (complex math. tool)

↳ linear operator

CAUCHY INTEGRAL TH. Define CURVE C:
- close
- counter clockwise
- origin is included

$z \in \mathbb{C}$



$$\frac{1}{2\pi j} \oint_C z^{m-1}\, dz = \begin{cases} 1 & \text{if } m=0 \\ 0 & \text{if } m \neq 0 \end{cases}$$

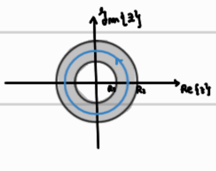
integral always 0 except for $m=0$ $[z^{0-1} = z^{-1} \to 1]$

$$\frac{1}{2\pi j} \oint_C X(z)\, z^{m-1}\, dz = \frac{1}{2\pi j} \oint_C \Big(\underbrace{\sum_{k=-\infty}^{+\infty} x[k]\, z^{-k}}_{X(z) \text{ of a generic signal}} \Big) z^{m-1}\, dz =$$

need to define the curve C

linear

$= \underbrace{\sum_{-\infty}^{+\infty} x[k]}_{\text{dependency on } k} \underbrace{\frac{1}{2\pi j} \oint_C z^{m-k-1}\, dz}_{\text{cauchy int. th.}}$ → $m-k-1 = -1$ when $m=k$ → $\frac{1}{2\pi j}\oint = 1$



$= \begin{cases} x[m] & \text{for } k=m \\ 0 & \text{for } k \neq m \end{cases}$ several terms at 0 and only one $\neq 0$

$x[m]$ curvilinear integral in which there is z-transf.

→ ANTITRANSFORMATION to come back to time domain

→ LC of BASIC FUNCTIONS weighted by z transf = coeff

complex exp EIGENFUNCTIONS

z - TRANSFORM

ANALYSIS F. $X(z) = \sum_{k=-\infty}^{+\infty} x[k]\, z^{-k}$ — time is discrete — from time to z-domain

SYNTHESIS F. $x[m] = \frac{1}{2\pi j} \oint_C X(z)\, z^{m-1}\, dz$ — z is continuous and complex — from z-domain to time

In general: DIRECT TRANSFORM — analysis formulas
INVERSE TRANSFORM — synthesis formulas
→ we are summing basic blocks weighted by coeff.

what differs is the basic function
interpretations is the same

Properties

LTI

TIME DOMAIN: $x[m] \rightarrow \boxed{T\{\}} \rightarrow y[m]$ convolution

Z-DOMAIN: $X(z) \rightarrow \boxed{H(z)} \rightarrow Y(z)$ product — relationship btw input/output

$Y(z) = H(z)X(z)$

- DIRAC DELTA $\delta[m] = \begin{cases} 1 & m=0 \\ 0 & m \neq 0 \end{cases}$

$\mathcal{Z}\{\delta[m]\} = \sum_{k=-\infty}^{+\infty} \underbrace{\delta[k]}_{=1 \text{ for } k=0} z^{-k} =$ def

$= 1$ → no dependency on z

- TIME SHIFTING property / theorem

$x[m]$ generic signal
$y[m] = x[m-N]$ output of a system that delays the input of N samples

relationship of the 2 $\mathcal{Z}\{.\}$?

$$Y(z) = \sum_{k=-\infty}^{+\infty} y[k]\, z^{-k} = \sum_{k=-\infty}^{+\infty} x[k-N]\, z^{-k}$$

$$= \left(\sum_{k=-\infty}^{+\infty} x[k-N]\, z^{-(k-N)}\right) z^{-N}$$

CV $m = k-N$

$$= z^{-N} \underbrace{\sum_{m=-\infty}^{+\infty} x[m]\, z^{-m}}_{\text{def of } \mathcal{Z}\{x\}}$$

$$= z^{-N} X(z)$$

TIME SHIFT of N samples

delays / anticipations are very common in digital filters

DELAYS
- 1 sample $(N=1) \longrightarrow Y(z) = z^{-1}X(z)$
- 2 samples $(N=2) \longrightarrow Y(z) = z^{-2}X(z)$
⋮

ADVANCE
- 1 sample $\longrightarrow Y(z) = z^{+1}X(z)$
- 2 samples $\longrightarrow Y(z) = z^{+2}X(z)$
⋮

LP filter: TF of single pole (laplace domain) ≡ exp. in time domain

- EXPONENTIAL SEQUENCE $x[m] = a^m u[m]$ $a \in \mathbb{C}$
  → 0 for $m<0$
  → exp for $m \geq 0$

$u[m] = \begin{cases} 1 & \text{for } m \geq 0 \\ 0 & \text{for } m < 0 \end{cases}$ STEP UNITY — right unilateral

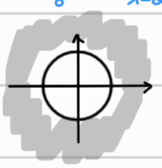
$$X(z) = \sum_{k=-\infty}^{+\infty} a^k \underbrace{u[k]}_{=1 \text{ only for } k\geq 0} z^{-k} =$$

$$= \sum_{k=0}^{+\infty} a^k z^{-k} = \sum_{k=0}^{+\infty} (a \cdot z^{-1})^k$$

$\alpha = az^{-1}$ → GEOMETRICAL SERIES $\sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha}$ $|\alpha|<1$

$$= \frac{1}{1-az^{-1}}$$

with $|az^{-1}|<1 \implies |z|>|a|$



NB circumferences are important

LAPLACE DOMAIN (s)

single pole

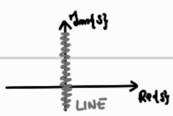
$\frac{1}{s-p} \rightarrow e^{pt}u(t)$ exp. term

attenuation/amplification factor in time domain

ASYMPTOTIC STABILITY : $Re\{p\} < 0$

$e^{pt} \downarrow$ for $t \to \infty$

$Im\{p\}$: oscillatory term (no effect on stability)



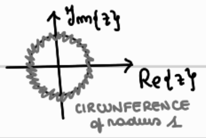
Z - DOMAIN

$z = a$: pole

$\frac{1}{1-az^{-1}} \rightarrow a^m u[m]$ $a \in \mathbb{C}$

$a^m \to 0$ for $m \to \infty$ if $|a|<1$

ASYMPTOTIC STABILITY : $|a|<1$



DTFT and IDTFT

INVERSE DTFT

DISCRETE TIME FOURIER TRANSFORM

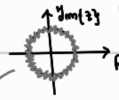
Remember: $\mathcal{F}\{\cdot\} = \mathcal{L}\{\cdot\}$ restricted to imaginary axis: $-\infty < \omega < +\infty$ $\left[Ae^{st} \xrightarrow[s=j\Omega]{} Ae^{j\Omega t} \right]$

$A z^m \xrightarrow[z=e^{j\omega}]{} A e^{j\omega m}$

$|z| = |e^{j\omega}| = 1$

ω: pulsation along circle (radius 1, center in origin)

DTFT = restriction of z-transform $\in \mathbb{C}$



UNIT RADIUS CIRCUMFERENCE

every 2π we are in the starting point

→ sinusoidal signals are periodic in discrete time domain)

changing ω, we change position along circle

DTFT

$$z = e^{j\omega} \longrightarrow \boxed{X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x[k]\, e^{-j\omega k}}$$

important math tool in discrete time signals

$$x[m] = \frac{1}{2\pi j} \oint X(z)\, \underbrace{z^{m-1}}_{z^m \cdot z^{-1}}\, dz =$$

$$= \frac{1}{2\pi \cancel{j}} \oint X(e^{j\omega})\, e^{j\omega m}\, \cancel{e^{-j\omega}}\, \cancel{e^{j\omega}}\, \cancel{j}\, d\omega$$

$$= \frac{1}{2\pi} \underbrace{\int_{-\pi}^{+\pi}}_{1\ \text{period}} X(e^{j\omega})\, e^{j\omega m}\, d\omega$$

⟶ integral of basic functions (~ LC of sinusoidal basic components) weighted by complex amount (z-transform $\in \mathbb{C}$)

EULER FUNCTIONS : $\cos(\omega m) = \dfrac{e^{j\omega m} + e^{-j\omega m}}{2}$

DTFT / IDTFT

ANALYSIS $\quad X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x[k]\, e^{-j\omega k}$

SYNTHESIS $\quad x[m] = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega m}\, d\omega$

$-\pi \le \omega \le \pi$

m: discrete in time domain → Σ

COMPLEX → magnitude + phase spectra

ω: continuous in Fourier domain → $\int$ ⟶ we are not able to implement on a computer because continuous variable

↳ we need a pair of transformation where also freq is discrete

DTFT periodic of 2π $\qquad \omega' = \omega + 2\pi$

$$X(e^{j\omega'}) = \sum_{k=-\infty}^{+\infty} x[k]\, e^{-j\omega' k} = \sum_{k=-\infty}^{+\infty} x[k]\, e^{-j(\omega+2\pi)k} =$$

$$= \sum_{k=-\infty}^{+\infty} x[k]\, e^{-j\omega k}\, \underbrace{\cancel{e^{-j2\pi k}}}_{=1} = X(e^{j\omega})$$

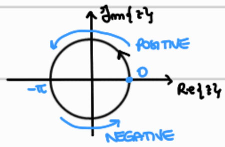
CONDITION OF EXISTANCE

several theorems

• $x[m]$ $\qquad \mathcal{E} = \sum_{k=-\infty}^{+\infty} |x[k]|^2 < \infty \implies \exists$ DTFT ⟶ just check energy

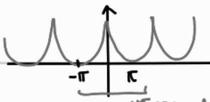
FINITE ENERGY

sufficient but NOT necessary



if finite energy ⟶ we are moving along this circumference ⟶ satisfied condition

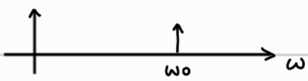
• SPECTRUM is periodic (2π) always



Typical spectrum

1st period repeated ∞ times (shifted by $k2\pi$)

EXAMPLE:



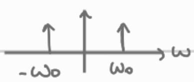
$X(e^{j\omega}) = 2\pi C\, \delta(\omega - \omega_0)$

$$x[m] = \frac{1}{\cancel{2\pi}} \int_{-\pi}^{\pi} \cancel{2\pi} C\, \underbrace{\delta(\omega-\omega_0)}_{\substack{1 \text{ only when } \omega=\omega_0 \\ \text{otherwise is } 0}}\, e^{j\omega m}\, d\omega = \underbrace{C e^{j\omega_0 m}}_{\text{half sinusoidal signal}}$$

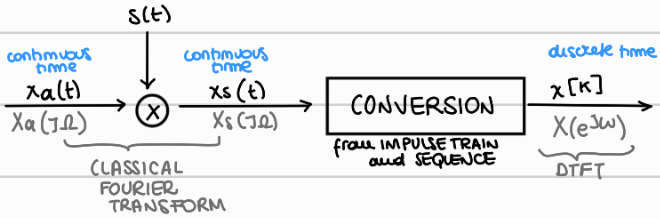
TIME DOMAIN : half sinusoids

SPECTRUM : line

SPECTRUM of a cosine : 2 lines

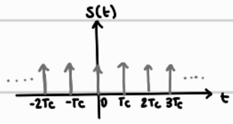


SAMPLING OF A CONT. TIME SIGNAL



Relationship btw continuous time signal $x_a(t)$ and its sampled version $x[k]$ in time and freq. domain

TRAIN of IMPULSES (Dirac Delta)



$$s(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT_c)$$

$x_s(t) = x_a(t)\,s(t) =$ IMPULSE MODULATED SIGNAL

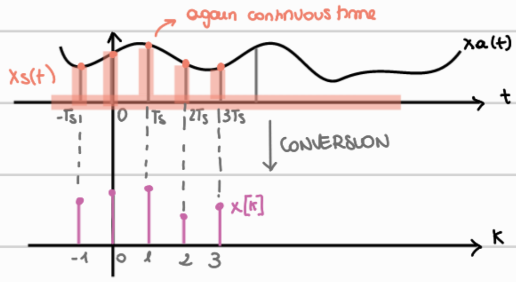
product in time btw x_a and $s(t)$

$$= \sum_{k=-\infty}^{+\infty} x_a(t)\,\underbrace{\delta(t-kT_c)}_{=1 \text{ when } t=kT_c} =$$

$$= \sum_{k=-\infty}^{+\infty} \underbrace{x_a(kT_c)}_{\text{effect of sampling}}\,\delta(t-kT_c) =$$

$$= \sum_{k=-\infty}^{+\infty} x[k]\,\delta(t-kT_c)$$

continuous time signal evaluated at kT_c



- $x_s(t)$: impulse train signal modulated by the continuous time signal
  - continuous time
  - T_c: distance btw samples
  - each sample has value of cont. time signal in that point
  - signal is 0 btw samples

conversion

- $x[k]$ sequence → we extract only coeff. from $x_s(t)$
  - → not defined btw samples

3 signals
- $x_a(t) \to X_a(j\Omega)$ INPUT
- $x_s(t) \to X_s(j\Omega)$
- $x[k] \to X(e^{j\omega})$ OUTPUT

Relationship between Xs and X in frequency domain

$$X_s(j\Omega) = \mathcal{F}\{x_s(t)\} =$$

$$= \mathcal{F}\left\{\sum_{k=-\infty}^{+\infty} x[k]\,\delta(t-kT_c)\right\} =$$

L

$$= \sum_{k=-\infty}^{+\infty} x[k]\,\mathcal{F}\{\delta(t-kT_c)\}$$

$\mathcal{F}\{\delta(t)\} = 1$
SHIFT of kT_c
$\mathcal{F}\{\delta(t-kT_c)\} = 1\,e^{-j\Omega k T_c}$

$$= \sum_{k=-\infty}^{+\infty} x[k]\,e^{-j\Omega k T_c}$$

$\omega = \Omega T_c$

$$= \underbrace{\sum_{k=-\infty}^{+\infty} x[k]\,e^{-j\omega k}}_{\text{Def of DTFT}} = X(e^{j\omega})$$

$$= X(e^{j\Omega T_c})$$

⟶ sequence and impulse modulated signal have the same Fourier transform

$x[k]$ and $x_s(t)$ carry same info both in time and freq.

DTFT periodic of 2π ⟹ $X_s(j\Omega)$ periodic

$\Delta\omega = 2\pi$ (period) $\xrightarrow{\omega = \Omega T_c}$ $\Delta\Omega T_c = 2\pi$

$\Delta\Omega = \frac{2\pi}{T_c} = 2\pi F_c = \Omega_c$ (period) sampling pulsation

$\Delta F = F_c$ sampling frequency

BUT $X_a(j\Omega)$ not periodic

⟹ SAMPLING INTRODUCES THE PERIODICITY

Relationship between Xa and Xs in frequency domain

$$X_s(j\Omega) = \int_{-\infty}^{+\infty} \underbrace{x_s(t)}_{x_a(t)\cdot s(t)}\, e^{-j\Omega t}\, dt = \int_{-\infty}^{+\infty} x_a(t)\, s(t)\, e^{-j\Omega t}\, dt =$$

$$s(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT_c) \qquad T_c: \text{period}$$

$$= \sum_{k=-\infty}^{+\infty} \underbrace{C_\ell}_{1/T_c} e^{j\ell\frac{2\pi}{T_c}t} =$$

$$= \frac{1}{T_c}\sum_{\ell=-\infty}^{+\infty} e^{j\ell\left(\frac{2\pi}{T_c}\right)t}, \quad \frac{2\pi}{T_c} = \Omega_c$$

we use the analog train of impulses to sample $x_a(t)$

$$= \int_{-\infty}^{+\infty} x_a(t)\, \frac{1}{T_c}\sum_{\ell=-\infty}^{+\infty} e^{j\ell\Omega_c t}\, e^{-j\Omega t}\, dt =$$

expanding the terms

$$= \frac{1}{T_c}\sum_{\ell=-\infty}^{+\infty} \underbrace{\int_{-\infty}^{+\infty} x_a(t)\, e^{-j(\Omega-\ell\Omega_c)t}\, dt}_{\mathcal{F}\{x_a(t)\}\text{ evaluated in } (\Omega-\ell\Omega_c)\text{, SHIFT of } \ell\Omega_c} =$$

$$\mathcal{F}\{x_a(t)\} = \int_{-\infty}^{+\infty} x_a(t)\, e^{-j\Omega t}\, dt$$

$$= \frac{1}{T_c}\Big(\underbrace{X_a(j\Omega)}_{\ell=0} + \underbrace{X_a(j(\Omega-\Omega_c))}_{\ell=1} + \underbrace{X_a(j(\Omega-2\Omega_c))}_{\ell=2} + \dots\Big) \rightarrow \infty \text{ terms}$$

$$= \underbrace{\frac{1}{T_c}\sum_{\ell=-\infty}^{+\infty} X_a(j(\Omega-\ell\Omega_c))}_{\text{spectrum of original signal (different representation)}} = X_s(j\Omega) = X(e^{j\omega})$$

RELATIONSHIP btw spectra

$$\boxed{\frac{1}{T_c}\sum_{\ell=-\infty}^{+\infty} X_a(j(\Omega-\ell\Omega_c)) = X_s(j\Omega) = X(e^{j\omega})}$$

we can link them together

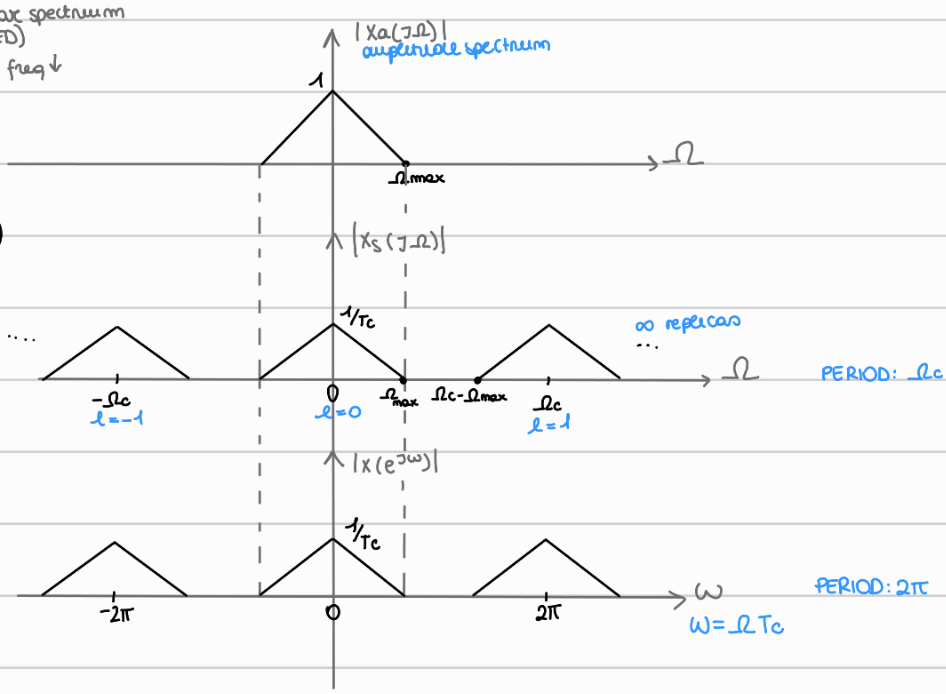
EXAMPLE

Typical triangular spectrum
(BAND LIMITED)
amplitude ↑ when freq ↓

$$X_s(j\Omega) = \frac{1}{T_c}\sum_{\ell=-\infty}^{+\infty} X_a(j(\Omega-\ell\Omega_c))$$

amplitude is attenuated

$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x[k]\, e^{-j\omega k}$$



We would like to sample the signal **without losing information**? In which conditions is it possible so that we can **reconstruct** the original continuous time signal **without error**?

NYQUIST - SHANNON THEOREM → NO ALIASING when $\Omega_s - \Omega_{max} \geq \Omega_{max}$

$\Omega_s \geq 2\,\Omega_{max}$

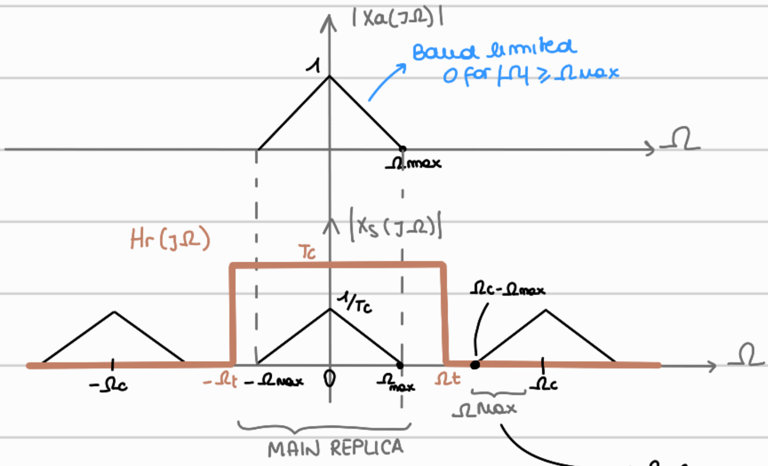
Let $x_a(t)$ a continuos BAND LIMITATED signal ⟶ $X_a(j\Omega) = $ for $|\Omega| \geq \Omega_{max}$ $\begin{cases} \Omega \geq \Omega_{max} \\ -\Omega \leq -\Omega_{max} \end{cases}$

⇒ $x_a(t)$ is UNIQUELY DETERMINED (without any error) by its samples: $x[m] = x_a(mT_c)$ $m = 0, \pm1, \pm2, \dots$ standard sampling

If $\Omega_c = \frac{2\pi}{T_c} \geq \underbrace{2\,\Omega_{max}}_{\text{NYQUIST RATE}}$ → reconstruct the signal without any error if the sampling rate is sufficient

EXACT RELATIONSHIP

Application:



→ Replicas are not overlapped since
$\Omega_c - \Omega_{max} > \Omega_{max}$
$\Omega_c > 2\Omega_{max}$
(we are in the condition of NS th.)

To reconstruct the signal, we apply a RECONSTRUCTION FILTER $H_r(j\Omega)$ to $X_s(j\Omega)$

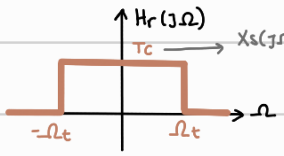
- LP filter → high freq. are removed to remove replicas
- Analog (cont. time signal)
- Best $\Omega_t = \frac{\Omega_c}{2}$ $\quad \Omega_{max} < \Omega_t < \Omega_c - \Omega_{max}$
  (max distance btw Ω_{max} and Ω_c)

hp: NS conditions are satisfied

- $\Omega_t = \frac{\Omega_c}{2} \overset{?}{\geq} \Omega_{max}$ $\quad \Omega_c \overset{?}{\geq} 2\Omega_{max}$ $\quad$ YES, for NS th.
  → Ω_t outside of main replica
- $\Omega_t = \frac{\Omega_c}{2} \overset{?}{\leq} (\Omega_c - \Omega_{max})$ $\quad \Omega_c \overset{?}{\leq} 2\Omega_c - 2\Omega_{max}$
  $\Omega_c \geq 2\Omega_{max}$
  YES, for NS th

IDEAL TRANSFER FUNCTION: $H_r(j\Omega) = \begin{cases} T_c & |\Omega| \leq \Omega_t \\ 0 & |\Omega| > \Omega_t \end{cases}$

phase = 0



T_c → $X_s(j\Omega)$ is attenuated of T_c

In practice: signals are bandlimited → reasonable BUT noise is added to ideal component
→ physical signals can be $\neq 0$ when $\Omega \to \infty$ (limitation of real world)

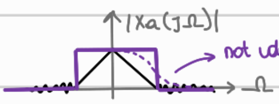
NOISE is always present, usually not bandlimited
→ Spectrum not bandlimited due to noise (BANDLIMITED CONDITION is difficult to be held)

⟹ we apply a NON-ALIASING FILTER before sampling
→ LP filter

we remove high freq. due to noise
→ LP signal

① $X_a(j\Omega) = 0$ for $|\Omega| \geq \Omega_{max}$
→ Known because real signal is known

② Ω_c given by NS th.
(→ F_c, linear relationship)

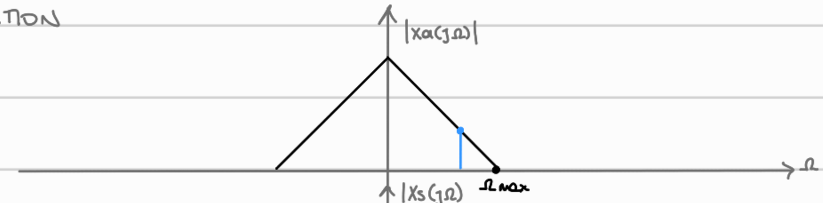


→ not ideal filter
(real transfer function)

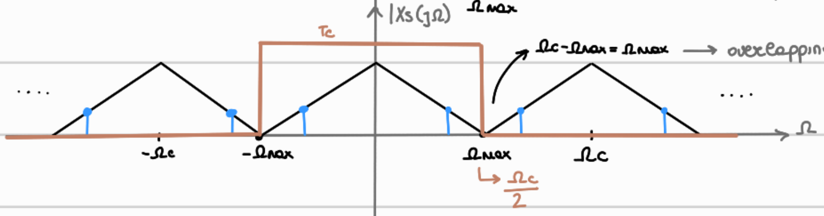
- Ω_{max} not really defined (NOT EXACT)
- freq is reduced, not removed at all

If $\Omega_c \uparrow$, distance btw replicas increases → NO ALIASING → we apply NS th ($\Omega_c \geq 2\Omega_{max}$)
BUT we need to use $\Omega_c \gg 2\Omega_{max}$ (i.e., $5/10\ \Omega_{max}$)
→ replicas far away from main spectrum

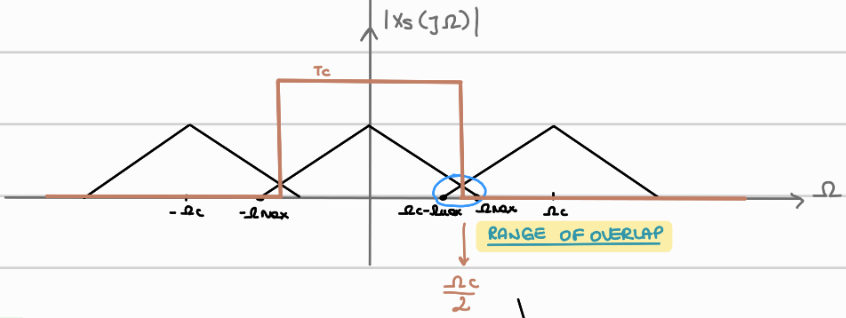
GRAPHICAL REPRESENTATION OF ALIASING



Reduced Ω_c : $\Omega_c = 2\,\Omega_{max}$
NO ALIASING



NS conditions not satisfied
$\Omega_c < 2\Omega_{max}$
→ ALIASING



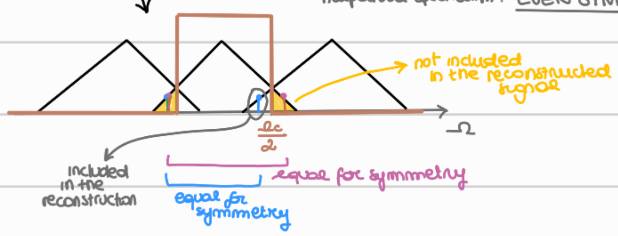
VARIATION OF NS THEOREM

$|x_a(j\Omega)| = 0$ for $|\Omega| > \Omega_{max}$ ⇒ signal ≠ 0 for $|\Omega| = \Omega_{max}$
not ≥

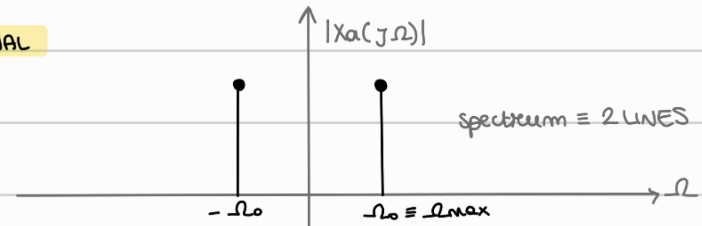
→ $\Omega_c = \frac{2\pi}{T_c} > 2\,\Omega_{max}$ to avoid overlapp when signal ≠ 0 for $|\Omega| = \Omega_{max}$

using the reconstruction formula, we consider freq. belonging to replicas and we do not consider part of main spectrum

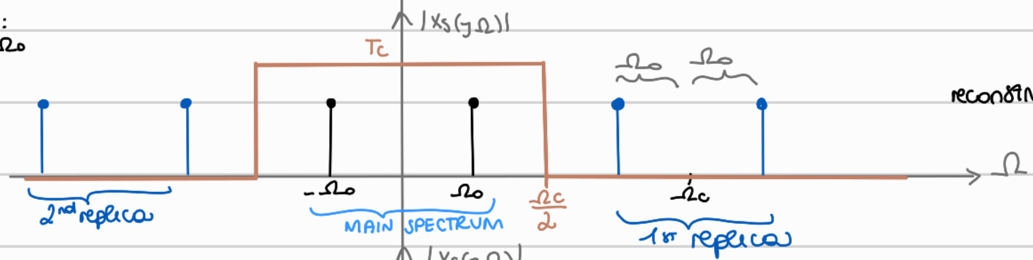
Amplitude spectrum: EVEN SYMMETRY



Example:
REAL COSINUSOIDAL SIGNAL

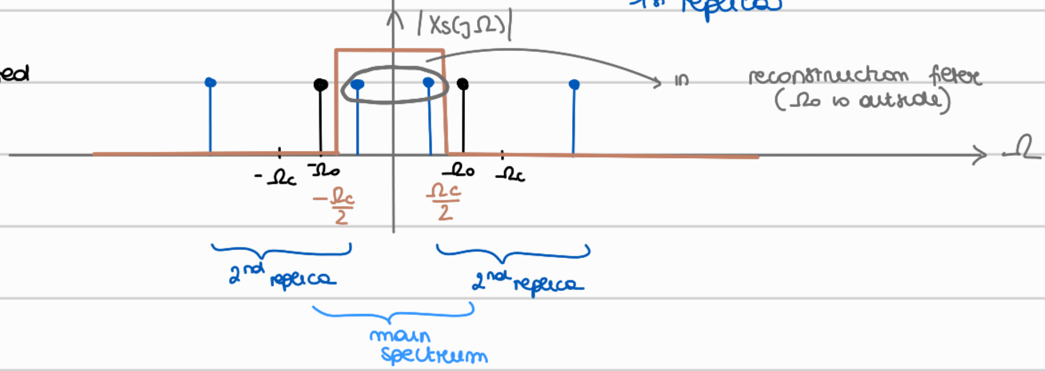


NS TH. satisfied:
$\Omega_c > 2\,\Omega_{max} = 2\Omega_0$
$\frac{\Omega_c}{2} > \Omega_0$



reconstructed signal is OK

NS th. not satisfied
$\Omega_c < 2\Omega_0$
$\frac{\Omega_c}{2} < \Omega_0$



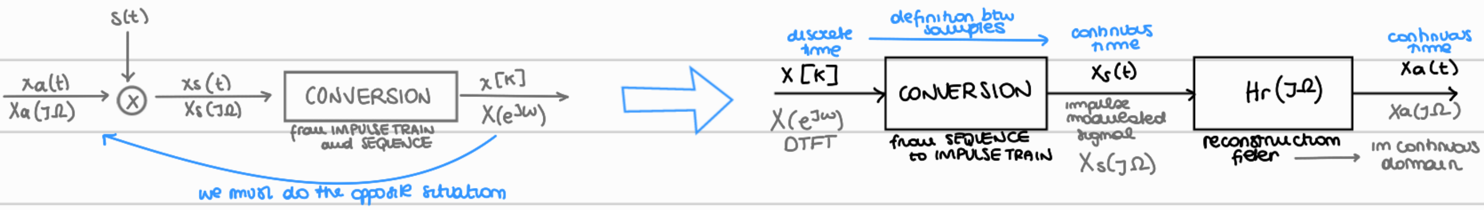
reconstruction filter (Ω_0 is outside)

$F_0 = 10\,Hz$
$F_c = 16\,Hz$
$F_0 - F_c = 6\,Hz$
signal is reconstructed with 6Hz instead of 10Hz

RECONSTRUCTION FORMULA

applied to sequence $x[k]$ to reconstruct original analog signal

under NS th. hypothesis
- ANALOG SIGNAL
- BANDLIMITED
- best situation $\Omega_c \geq 2\Omega_{max}$



Relationship btw sequence x[k] and Xs(t)

- Already proved: $\underbrace{X_s(J\Omega) = X(e^{Jw})}$ where $w = \Omega T_c$ (at sampling time) } same spectrum (Fourier domain)
- Time domain: $X_s(t) = \sum_{k=-\infty}^{+\infty} x[k]\,\delta(t-kT_c)$ Def of impulse modulated signals

Relationship btw sequence Xs(t) and Xa(t)

- We have already defined the transfer function (in Fourier domain) of the filter: $H_r(J\Omega) = \begin{cases} T_c & \text{for } |\Omega| \leq \Omega_c/2 \\ 0 & \text{for } |\Omega| > \Omega_c \end{cases}$ (constant) IDEAL LP FILTER — within the band of the filter (defined by $\frac{\Omega_c}{2}$)
- In time domain (continuous):

$$X_a(t) = T\{X_s(t)\}$$
∀ SYSTEM (even NOT LTI)

$$= T\Big\{\sum_{k=-\infty}^{+\infty} x[k]\,\delta(t-kT_c)\Big\} =$$
t time variable, k constant

RECONSTRUCTION FILTER IS LTI

$$\overset{L}{=} \sum_{k=-\infty}^{+\infty} x[k]\,\underbrace{T\{\delta(t-kT_c)\}}_{\text{IMPULSE RESPONSE}}$$

$$\overset{TI}{=} \sum_{k=-\infty}^{+\infty} x[k]\,h_r(t-kT_c)$$
same shift of the input

RECONSTRUCTION FORMULA but h_r unknown, yet

$\xrightarrow{h_r\ ?}$ IMPULSE RESPONSE of the filter

$$h_r(t) = \mathcal{F}^{-1}\{H_r(J\Omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underline{H_r(J\Omega)}\, e^{J\Omega t}\, d\Omega =$$
$H_r = 0$ outside band $[-\frac{\Omega_c}{2}, \frac{\Omega_c}{2}]$

$$= \frac{1}{2\pi}\int_{-\frac{\Omega_c}{2}}^{+\frac{\Omega_c}{2}} T_c\, e^{J\Omega t}\, d\Omega =$$
const.

$$= \frac{1}{2\pi} T_c \int_{-\frac{\Omega_c}{2}}^{+\frac{\Omega_c}{2}} e^{J\Omega t}\, d\Omega = \begin{cases} t=0 & ① \\ t\neq 0 & ② \end{cases}$$

① $t=0$

$$h_r(0) = \frac{1}{2\pi} T_c \int_{-\frac{\Omega_c}{2}}^{+\frac{\Omega_c}{2}} 1\, d\Omega = \frac{1}{2\pi} T_c \Big[\Omega\Big]_{-\frac{\Omega_c}{2}}^{+\frac{\Omega_c}{2}} =$$

$$= \frac{1}{2\pi} T_c\, \Omega_c = 1$$
$\frac{2\pi}{T_c}$

② $t \neq 0$

$$h_r(t) = \frac{1}{2\pi} T_c \int_{-\frac{\Omega_c}{2}}^{+\frac{\Omega_c}{2}} e^{J\Omega t}\, d\Omega = \frac{1}{2\pi} T_c \left[\frac{e^{J\Omega t}}{Jt}\right]_{-\frac{\Omega_c}{2}}^{+\frac{\Omega_c}{2}} =$$

$$= \frac{1}{2\pi} T_c \left(\frac{e^{J\frac{\Omega_c}{2}t} - e^{J\frac{\Omega_c}{2}t}}{Jt}\right) =$$
switch 2 and t

$$= \frac{1}{\pi t} T_c \left(\frac{e^{J\frac{\Omega_c}{2}t} - e^{-J\frac{\Omega_c}{2}t}}{2J}\right) = \frac{\sin(\frac{\Omega_c}{2}t)}{\frac{\Omega_c}{2}t} =$$
$= \sin(\frac{\Omega_c}{2}t)$

$\Omega_c = 2\pi F_c = \frac{2\pi}{T_c} \longrightarrow \frac{\Omega_c}{2} = \frac{\pi}{T_c} \Rightarrow \frac{T_c}{\pi} = \frac{1}{\frac{\Omega_c}{2}}$

$$= \operatorname{sinc}\left(\frac{\Omega_c}{2}t\right)$$

$\operatorname{sinc}(x) = \frac{\sin(x)}{x}$ ≠ Matlab function $\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$

RECONSTRUCTION FORMULA → exact reconstruction if SN th.

$$\boxed{x_a(t) = \sum_{k=-\infty}^{+\infty} x[k]\, \text{sinc}\left((t - kT_c)\frac{\Omega_c}{2} \right)}$$

→ LC samples weighted by sinc shifted in time by kT_c

SINC FUNCTION

sinc in time domain ≡ rect in freq. domain

and vice versa

$$h_r(t) = \begin{cases} 1 & \text{for } t = 0 \\ \text{sinc} & \text{for } t \neq 0 \end{cases}$$

NB. sinc function not defined for $t = 0$

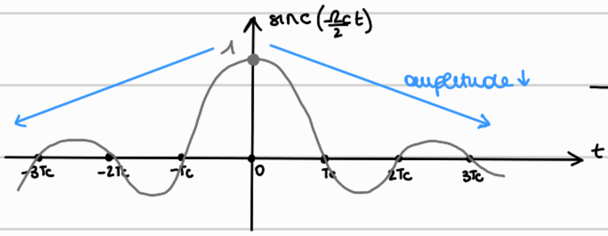
$$\lim_{t\to 0} \text{sinc}\left(\frac{\Omega_c}{2}t\right) = \lim_{t\to 0} \frac{\sin\left(\frac{\Omega_c}{2}t\right)}{\frac{\Omega_c}{2}t} = \frac{0}{0} \xrightarrow{H} \lim_{t\to 0} \frac{\cos\left(\frac{\Omega_c}{2}t\right)\cancel{\frac{\Omega_c}{2}}}{\cancel{\frac{\Omega_c}{2}}} = 1$$

$$\text{sinc}\left(\frac{\Omega_c}{2}t\right) = \frac{\sin\left(\frac{\Omega_c}{2}t\right)}{\frac{\Omega_c}{2}t} = 0 \quad \text{when} \quad \begin{cases} \sin\left(\frac{\Omega_c}{2}t\right) = 0 \longrightarrow \frac{\Omega_c}{2}t = k\pi \\ \frac{\Omega_c}{2}t \neq 0 \longrightarrow t \neq 0 \end{cases}$$

both numerator and denominator are zero

$$\begin{cases} k \neq 0 \\ k = \pm 1, \pm 2, \dots \end{cases} \longrightarrow t_k = \frac{2\pi k}{\Omega_c} = \frac{2\pi k}{\frac{2\pi}{T_c}} = kT_c \qquad \Omega_c = \frac{2\pi}{T_c}$$

PLOT is important to understand the interpretation of the reconstruction formula



→ DUMPED OSCILLATORY FUNCTION : when argument ↑, amplitude ↓

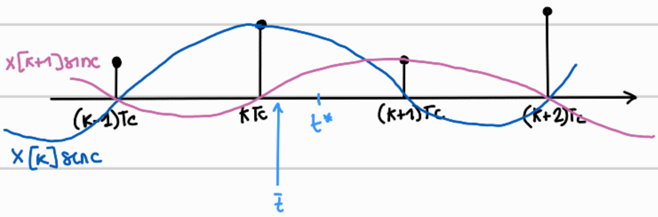
$\frac{1}{\frac{\Omega_c t}{2}}$ HYPERBOLE → 0 when $t \to \infty$ (ATTENUATION TERM)

→ sin (OSCILLATORY TERM)

argument ∝ t

RECONSTRUCTION FORMULA : sinc is the weighting factor

$$x_a(t) = \sum_{k=-\infty}^{+\infty} x[k]\, \text{sinc}\left((t - kT_c)\frac{\Omega_c}{2} \right)$$



we want to reconstruct the signal or a n t* value

$x_a(t)\big|_{t=t^*}$ we replace t with t* value (→ 1 value of x_a) and we have the sum of many components (one for each sample)

- sample k is weighted by a sinc
  - centered in kT_c
  - multiplied by $x[k]$ (value of the sample)
- for k+1 sample is weighted by a sinc
  - centered in $(k+1)T_c$
  - multiplied by $x[k+1]$
- … and so on for all the samples

→ $x_a(t^*) = \sum$ all products for any k

↳ $\sum \infty$ terms if $t^* \neq$ sampling time kT_c, otherwise only 1 term

NB. for a time $\bar{t}$ close to a sample, that sample is the most important (sinc function is higher when centered is the sample, and lower for t far from the sample)

$\bar{t} = kT_s$, sinc is $x[k]$ height, and 0 for the other samples

$x_a(\bar{t}) = x[k]$

WINDOWING

We have started with the analysis of **deterministic signals**. Especially, we have seen that they are mathematical functions (continuous and discrete), and that we can study the harmonic analysis (in the frequency domain), which depends on the characteristic of the signal:

- CONTINUOS TIME
  - PERIODIC: FOURIER SERIES
  - APERIODIC + FINITE ENERGY: FOURIER TRANSFORM
- DISCRETE TIME
  - APERIODIC: DTFT
  - PERIODIC
- Relationship btw continuous and sampling and sequence

In theory, the reconstruction formula is exact, if we satisfy the Shannon theorem. But it is not always possible to follow the Shannon theorem in practice because not all the signals are bandlimited, for example, when we apply a time window. ***What happens in frequency domain when we apply a time window in the time domain?*** Indeed, real signals are not defined from - infinity to + infinity, but we have a time window. *What happens to the spectrum if we apply a time window to a band limited signal?* ⟶ TIME WINDOWED SIGNAL

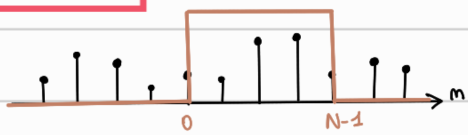
Time window = limited time frame

RECTANGULAR WINDOW

$$\boxed{w[m] = \begin{cases} 1 & 0 \le m \le N-1 \\ 0 & \text{otherwise} \end{cases}}$$

N samples

In time domain: $\underbrace{x[m]}_{\text{WINDOWED SIGNAL}} = \overbrace{x_i[m]}^{\text{original signal }]-\infty,+\infty[} \, w[m]$ simple relationship: PRODUCT



In freq domain: $W(e^{j\omega}) = \overset{\text{DTFT}}{\sum_{k=-\infty}^{+\infty}} \underbrace{w[k]}_{=1 \text{ only for } 0\le m\le N-1} e^{-j\omega k} = \sum_{k=0}^{N-1} 1 e^{-j\omega k} = \sum_{k=0}^{N-1} e^{-j\omega k}$

DTFT of windowed signal? REMEMBER: product in time domain ⟷ convolution in freq. domain (and viceversa) ⟶ dual property

$y[m] = x[m]w[m]$ $\quad Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\theta})\, W(e^{j(\omega-\theta)})\, d\theta$

θ: dummy variable, ω: real pulsation } we are working in freq. domain

we are working in pulsation (rather than freq) BUT const. is not a problem ⟶ 1 period of DTFT

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X_i(e^{j\theta})\, W(e^{j(\omega-\theta)})\, d\theta$$

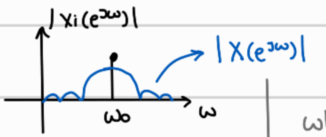
complicated formula

SIMPLE CASE: $x_i[m] = C e^{j\omega_0 m}$ HALF SINUSOID

$X_i(e^{j\omega}) = 2\pi C\, \delta(\omega-\omega_0)$ (1 line in the spectrum) ⟶

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \underbrace{X_i(e^{j\theta})}_{\text{known}} W(e^{j(\omega-\theta)})\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{+\pi} 2\pi C\, \underbrace{\delta(\theta-\omega_0)}_{1 \text{ only for } \theta=\omega_0, \text{ otherwise } 0} W(e^{j(\omega-\theta)})\, d\theta = C\, W(e^{j(\omega-\omega_0)})$$

↳ spectrum of the window centered at ω_0



why?

$$W(e^{j\omega}) = \sum_{k=0}^{N-1} 1 e^{-j\omega k} = \sum_{k=0}^{N-1} (\overset{\alpha}{e^{-j\omega}})^k = \sum_{k=0}^{N-1} \alpha^k =$$

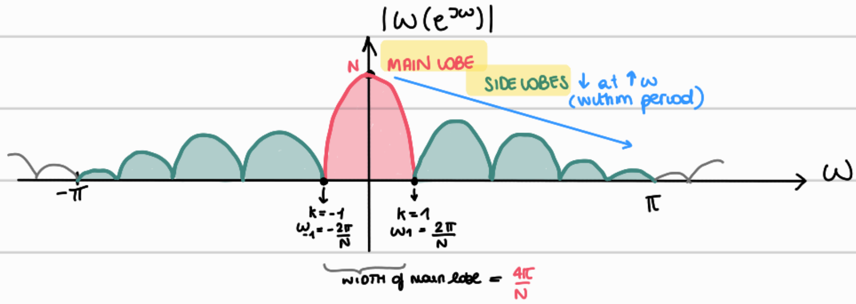
geometrical sum: $\sum_k \alpha = \frac{1-\alpha^N}{1-\alpha}$ for finite sum

$$= \frac{1-\alpha^N}{1-\alpha} = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}} = \frac{e^{-j\omega\frac{N}{2}}}{e^{-j\frac{\omega}{2}}} \frac{(e^{j\omega\frac{N}{2}} - e^{-j\omega\frac{N}{2}})/2j}{(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}})/2j} = e^{-j\omega\frac{N-1}{2}} \cdot \frac{\sin(\omega\frac{N}{2})}{\sin(\frac{\omega}{2})}$$

Try to obtain Euler formulas

$$\longrightarrow |W(e^{j\omega})| = \frac{|\sin(\frac{\omega}{2}N)|}{|\sin(\frac{\omega}{2})|}$$

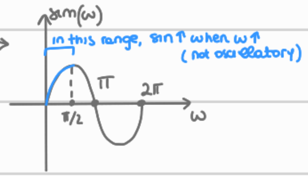
STUDY OF A FUNCTION:



- $\lim_{\omega\to 0} \frac{\sin(\frac{\omega}{2}N)}{\sin(\frac{\omega}{2})} \overset{H}{=} \lim_{\omega\to 0} \frac{\cos(\frac{\omega}{2}N)\frac{N}{2}}{\cos(\frac{\omega}{2})\frac{1}{2}} = N$ positive value $|N|=N$

- $w[n]$ is real: AMPLITUDE SPECTRUM is even (symmetric) and periodic of 2π → study from 0 to π and then derive $[-\pi, 0]$

- sin: oscillatory function → W = ratio btw oscillatory function and a not oscillatory one = OSCILLATORY FUNCTION



$\left[\text{arg.} = \frac{\omega}{2}\right]$

- $|W(e^{j\omega})| = 0$ when $\begin{cases} \sin(\frac{\omega}{2}N) = 0 & \longrightarrow \frac{\omega}{2}N = k\pi \text{ with } k \neq 0 \\ \sin(\frac{\omega}{2}) \neq 0 & \longrightarrow \omega \neq 0 \end{cases}$

  - N even $k = \pm1, \pm2, \dots \pm\frac{N}{2}$ → $\omega_k = \frac{2\pi}{N}k$
    $\omega_{\frac{N}{2}} = \frac{\cancel{2}\pi}{\cancel{N}}\frac{\cancel{N}}{\cancel{2}} = \pi$ (max ω, $k=\frac{N}{2}$ last k number)
  - N odd $k = \pm1, \pm2, \dots \pm\frac{N-1}{2}$ → $\omega_k = \frac{2\pi}{N}k$
    k must be integer
    $\omega_{\frac{N-1}{2}} = \frac{\cancel{2}\pi}{N}\cdot\frac{N-1}{\cancel{2}}$
    $\frac{N-1}{N}\pi < \pi$ (max ω, $k=\frac{N-1}{2}$ last value before π)

ideal signal from $-\infty$ to $+\infty$ ↓ LINE spreads to CENTRAL + MANY SIDE LOBES ← windowed signal

LINE → energy concentrated at ω_0; ω_{max} defined; $\omega_{max} = \omega_0$ (band limited)

CENTRAL + MANY SIDE LOBES → energy is spread on RANGE OF PULSATIONS (frequencies); NOT band-limited; $\nexists\, \omega_{max}$ (NS th. is an approximation) → REAL ANTIALIASING FILTER: $f_c \gg 2f_{max}$ since f_{max} not exact

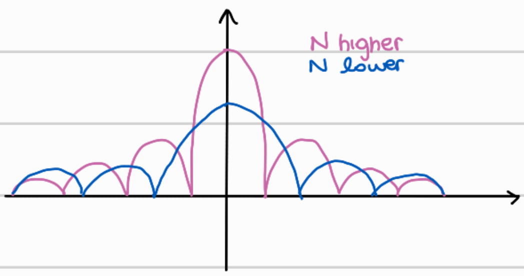
RECONSTRUCTION FORMULA → approximation of original signal

EX: 2 cosines at ω_{01}, ω_{02} → each line spread into spectrum of rect. window

if cont. spectrum → every line is spread (overall effect: COMBINATION)

$N \to \infty$, effect of windowing disappears (both in time and freq)
entire signal

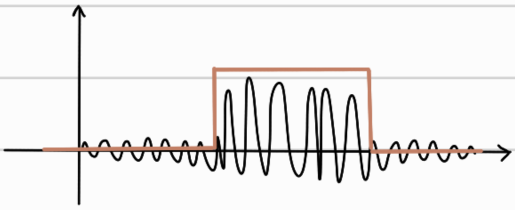
$\begin{cases} \text{width} = \frac{4\pi}{N} \xrightarrow{N\to\infty} 0 \\ \text{height} = N \xrightarrow{N\to\infty} \infty \end{cases}$ ⇒ MAIN LOBE → Dirac Delta (neutral element of convolution) → spectrum approaches to spectrum ideal signal



- Superimposition between original windowed signal and reconstructed signal is better in the center, and lower at the boundaries of the time window (reconstruction formula is an approximation)

- **LOCALIZATION**: We can localize a signal both in time and frequency domains.

General rule: if we use a small time window, we obtain a good localized information in time but a spectrum is very spread in frequency (loss of information in frequency), and vice versa, if we use a large time window, we loose the localization in time gaining in frequency.
There is a simple mathematical formula that describes the relationship between localization in time and in frequency.



- LENGTH T of time window ≡ uncertainty in the localization of a signal in time domain

T in seconds

$T = N T_c$

- WIDTH of the central lobe ≡ uncertainty in the ability to define the real freq

In freq domain, spectrum is spread over a range of ω and energy is mainly spread over the central lobe (neglect side lobes → approximation)

$\Delta\omega = \frac{4\pi}{N}$

$\Delta\omega = \frac{4\pi}{N}$, $\omega = \Omega T_c$

$\rightarrow \Delta\Omega = \frac{\Delta\omega}{T_c} = \frac{4\pi}{N T_c}$ uncert. in pulsat.

$\Omega = 2\pi F$

$\rightarrow \Delta F = \frac{\Delta\Omega}{2\pi} = \frac{\cancel{4}^{2}\pi}{\cancel{2}\pi N T_c} = \frac{2}{N T_c} = \frac{2}{T}$ uncertainty in freq.

$\Rightarrow$ $\Delta F \cdot T = 2$ for RECT. window

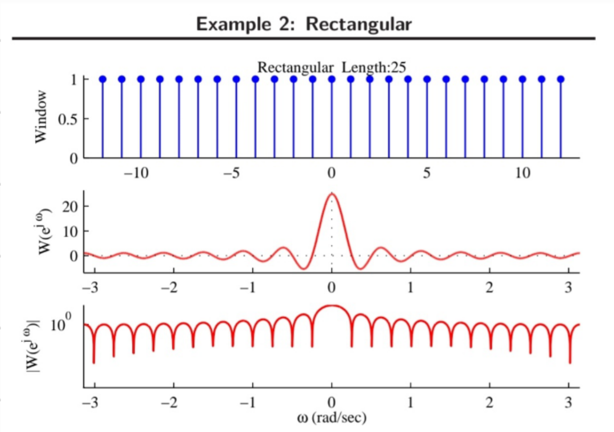
"2" is given by width of main lobe of rect. window

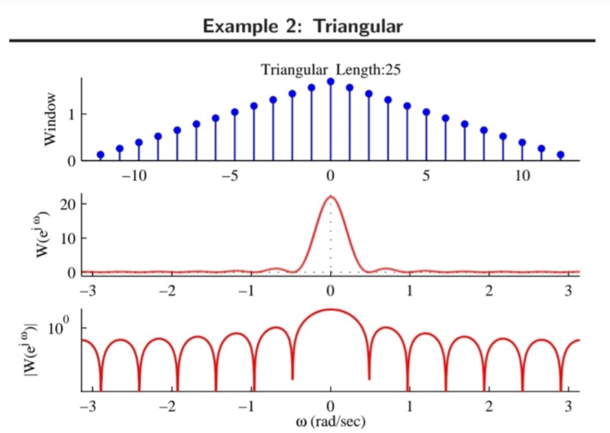
PRODUCT btw width in freq and time domain is CONSTANT : $\Delta F \cdot T = \text{const.}$

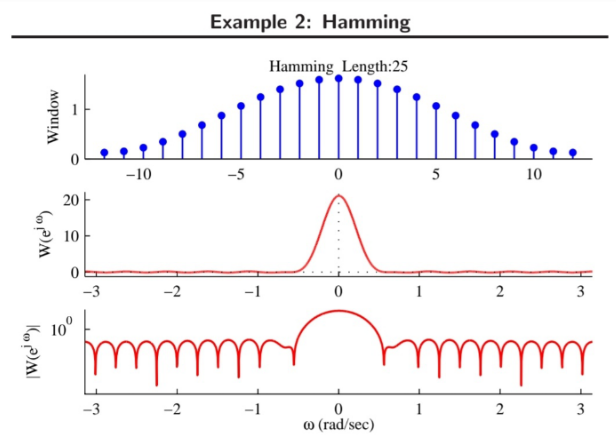
- if T↑, ΔF↓
- if T↓, ΔF↑

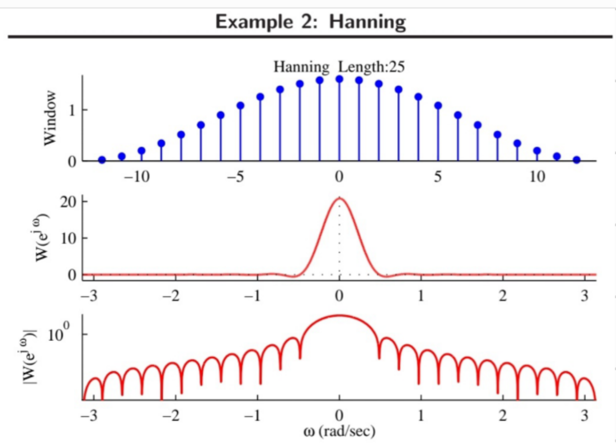
LIMIT of Fourier analysis: we cannot reduce both!

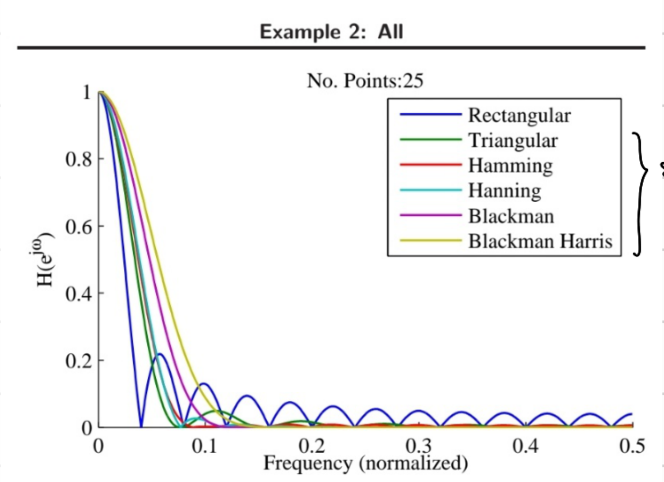
OTHER WINDOWS











softer windows wrt rectangular → • MAIN LOBE: wider but lower
• SIDE LOBES: smaller

TRADE OFF
btw advantages and disadvantages

• SLOPE is higher: peaks of side lobes go to zero more quickly

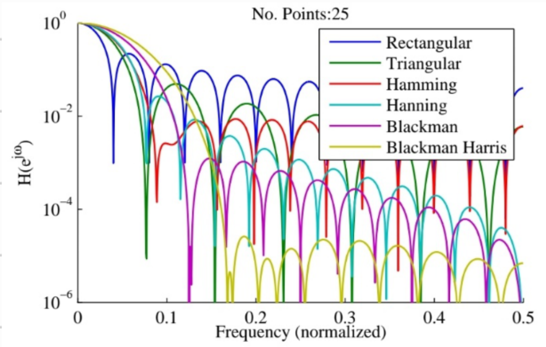
• OSCILLATORY PART visible in dB [20log(|·|)]

how much 1st side lobe is lower than main lobe

N samples

Type of the window	Peak Side-Lobe Amplitude (Relative) (dB)	Approximate Width of Main Lobe	Decay rate (db/decade)
Rettangolare	-13	$4\pi/N$	-20
Bartlett	-25	$8\pi/(N-1)$	-40
Hann	-31	$8\pi/(N-1)$	-60
HAMMING	-41	$8\pi/(N-1)$	-20
BLACKMAN	-57	$12\pi/(N-1)$	-60

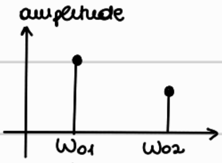
slope of height of side lobe as $\omega\uparrow$
↓
behavior of spectrum at high freq.



NB. use both if we do not know what we are searching

- h of side lobes ↓ when $\omega\uparrow$ (lower than main lobe)
  ↳ just 1st side lobe since highest
- soft windows have larger main lobe and higher decay rate

How to choose the proper window? It depends on the situation!



we need to discriminate the 2 components

if we reduce distance, we need rectangular window since with smallest width of main lobe

- *Lines close in frequency*: the interference is due to the width of the main lobe, thus the rectangular window (shorter width) is the best choice
- *Lines far in frequency*: the interference is due to the first side lobe, thus we use window the higher peak-lobe amplitude in abs value (Blackman window)
- *Lines very far in frequency*: the interference is due to the other side lobes, thus we use the window with the very high slope (Han Blackman)

These theoretical speculations are not always true in practice.
We can use the rectangular window for sure if we need the highest the frequency resolution. In general, Hamming or Han windows are good choices (good trade off between the indications), but we can switch between the windows to obtain more information about the signal.

NB. In all the previous cases, the **variable** describing the *frequency domain* is **continuous**, thus we cannot directly plot the spectrum in a PC. We need a discrete variable both in time and in frequency/pulsation (not till now).

We need the following mathematical tool: DFT.

We are working with deterministic signals, but it is the basis also for stochastic ones.

The DTFT is discrete in time, but continuous in frequency. Thus, we need a sort of samping of the spectrum obtained by the DTFT in order to obtain a spectrum discrete also in the pulsation. But sampling the DTFT, obtaining the array of samples, the discrete time signal is no more the original one. ***What is the discrete time signal which has that discrete spectrum?***

Let introduce the preliminary tool.

PERIODIC DISCRETE TIME SIGNALS: $\tilde{x}[m] = \tilde{x}[m+N]$ $\forall N$ N period, $r=1$; $\forall r$ multiple of N

sum of harmonics with pulsations multiple of the fundamental pulsation: $\frac{2\pi}{N}$

GENERIC HARMONIC: $e_k[m] = e^{j\frac{2\pi}{N}km}$ k: integer

complex exp. ~ half cont sinusoidal signal

each periodic of period N

$$e_k[m+N] = e^{j\frac{2\pi}{N}k(m+N)} = e^{j\frac{2\pi}{N}km}\underbrace{e^{j\frac{2\pi}{N}k\cancel{N}}}_{1} = e^{j\frac{2\pi}{N}km} = e_k[m]$$

NB. we want to represent a periodic disc. time signal → BASIS FUNCTION MUST BE PERIODIC OF THE SAME AMOUNT

FOURIER SERIES

$$\tilde{x}[m] = \frac{1}{N}\sum_k \tilde{X}[k]e^{j\frac{2\pi}{N}km}$$

~ fourier series for cont. time signals

SYNTHESIS F. signal in time = LC of basic functions

BASIC FUNCTION complex periodic exponential

NB. Limits of $\sum_k$ not defined (cont. case: $\sum_{-\infty}^{+\infty}$)

if we consider a generic k and we add N (or ℓN: multiple of N) the 2 basis functions are the same

- $k=0$: $e_0[m] = e_N[m]$
- $k=1$: $e_1[m] = e_{N+1}[m]$
- ⋮
- $k=k+\ell N$: $e_{k+\ell N}[m] = e^{j\frac{2\pi}{N}(k+\ell N)m} = e^{j\frac{2\pi}{N}km}\underbrace{e^{j\frac{2\pi}{\cancel{N}}\ell \cancel{N}m}}_{1} = e_k[m]$

→ Just N different basic function → we can consider k just from 0 to N-1

FOURIER SERIES

$$\boxed{\begin{aligned}\tilde{X}[k] &= \sum_{m=0}^{N-1}\tilde{x}[m]e^{-j\frac{2\pi}{N}km}\\ \tilde{x}[m] &= \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j\frac{2\pi}{N}km}\end{aligned}}$$

DIRECT F. (ANALYSIS) → how to compute the coefficients

INVERSE FORMULA (SYNTHESIS)

FREQUENCY DOMAIN • discrete k: discrete variable (integer) • periodic of period N

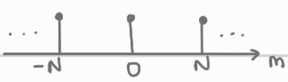
dual situation (same properties in both time and frequency domain)

$$\tilde{X}[k+N] = \sum_{m=0}^{N-1}\tilde{x}[m]e^{-j\frac{2\pi}{N}(k+N)m} = \sum_{m=0}^{N-1}\tilde{x}[m]e^{-j\frac{2\pi}{N}km}\underbrace{e^{-j\frac{2\pi}{\cancel{N}}\cancel{N}m}}_{1} = \tilde{X}[k]$$

DIRAC DELTA TRAIN IMPULSE

$$\tilde{x}[m] = \sum_{\ell=-\infty}^{+\infty}\delta[m-\ell N]$$

sum of ∞ shifted delta



$$\tilde{X}[k] = \sum_{m=0}^{N-1}\tilde{x}[m]e^{-j\frac{2\pi}{N}km} = \sum_{m=0}^{N-1}\sum_{\ell=-\infty}^{+\infty}\delta[m-\ell N]e^{-j\frac{2\pi}{N}km} = \sum_{\ell=-\infty}^{+\infty}\sum_{k=0}^{N-1}\delta[m-\ell N]e^{-j\frac{2\pi}{N}m} = 1$$

→ $\ell=0$: $\sum_{k=0}^{N-1}\delta[m]e^{-j\frac{2\pi}{N}km} = 1$

$\ell=1$: $\sum_{k=0}^{N-1}\delta[m-N]e^{-j\frac{2\pi}{N}} = 0$

1 only for m=N, out of Z

The same for $\forall \ell\neq 0$

All coeff = 1

fourier domain do not depend on ω: impulse in time ≡ const. in freq (disc/cont)

$$\rightarrow \tilde{x}[m] = \frac{1}{N}\sum_{k=0}^{N-1}1\cdot e^{j\frac{2\pi}{N}km}$$

Properties

- **LINEARITY** superposition principle holds

$\tilde{x}_1[m] \longrightarrow \tilde{X}_1[k]$
$\tilde{x}_2[m] \longrightarrow \tilde{X}_2[k]$

LC: $\tilde{x}_3[m] = a\tilde{x}_1[m] + b\tilde{x}_2[m]$

$\underset{\substack{\uparrow \\ \text{same LC} \\ \text{of the single coeff.}}}{\tilde{X}_3[k]} = a\tilde{X}_1[k] + b\tilde{X}_2[k]$ $\quad \forall$ LC (a,b)
$\forall$ couple of signals $\tilde{x}_1, \tilde{x}_2$

- **SHIFT** $\tilde{x}[m] \longrightarrow \tilde{X}[k]$

shift in time of m samples

$\tilde{x}[m-m] \longrightarrow \tilde{X}[k]\, e^{-j\frac{2\pi}{N}km}$ PHASE TERM (amplitude = 1)

- **PERIODIC CONVOLUTION** of discrete time periodic (N) signals

$\left.\begin{array}{l}\tilde{x}_1[m] \longrightarrow \tilde{X}_1[k] \\ \tilde{x}_2[m] \longrightarrow \tilde{X}_2[k]\end{array}\right\}$ $\quad \underbrace{\tilde{X}_3[k]} = \tilde{X}_1[k] \cdot \tilde{X}_2[k]$ product of coeff in freq. domain

which signal in time domain gives this coefficient?

NB. product in freq ≡ convolution in time
(and vice versa)
if we define the periodic convolution

$$\tilde{x}_3[m] = \sum_{m=0}^{N-1} \tilde{x}_1[m] \cdot \tilde{x}_2[m-m] =$$

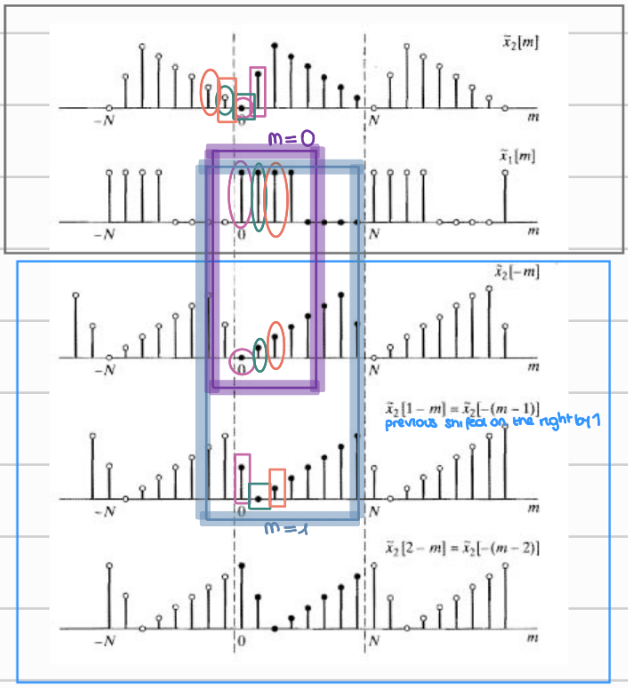
COMMUTATIVE

$$= \sum_{m=0}^{N-1} \tilde{x}_2[m] \cdot \tilde{x}_1[m-m]$$

m: dummy variable

n: real time variable

$$\tilde{x}_3[m] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\, \tilde{x}_2[m-m] \quad \tilde{x}_2[-(m-m)]$$



2 periodic discrete time signals

3 dummy signals useful to compute the periodic convolution for different value of m

m = 0 $\quad \tilde{x}_1[m]\,\tilde{x}_2[-m]$

To multiply term term

$m=0$: $\tilde{x}_1[0]\,\tilde{x}_2[0]$
$m=1$: $\tilde{x}_1[1]\,\tilde{x}_2[-1]$
$m=2$: $\tilde{x}_1[2]\,\tilde{x}_2[-2]$
till $m = N-1$

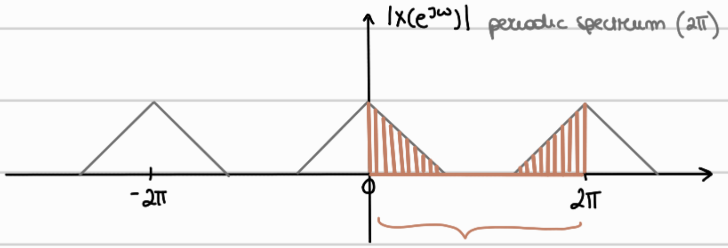
m = 1 $\quad \tilde{x}_1[m] \cdot \tilde{x}_2[1-m] = \tilde{x}_1[m]\,\tilde{x}_2[-(m-1)]$

$m=0$ $\quad \tilde{x}_1[0]\,\tilde{x}_2[1]$
$m=1$ $\quad \tilde{x}_2[1]\,x_2[0]$

We have obtained discrete time and frequency, but we have hypothesized that signal is periodic.
Now, we want to define a discrete frequenency domain also for a *generic sequence (not periodic)*.

$x[n] \xrightarrow{DTFT} X(e^{j\omega})$

GENERIC DISCRETE TIME SIGNAL of any length



N samples (for the sampling)

$\omega_k \longrightarrow$ pulsation uniformly sampled (N sample)

NB: we sample just 1 period since others are replicas

SAMPLED PULSATION $\omega_k = \frac{k}{N} 2\pi \qquad k = 0, 1, \dots, N-1$

SAMPLED DTFT $\tilde{X}[k] = X(e^{j\omega_k}) = \sum_{n=-\infty}^{+\infty} \underbrace{x[n]}_{\text{original signal}} e^{-j\omega_k n} = \sum_{n=-\infty}^{+\infty} x[n] e^{-j 2\pi \frac{k}{N} n}$

sampling DTFT

FREQ DOMAIN: sampled → discrete, periodic

Which is the discrete time signal corresponding to $\tilde{X}[k]$ (discrete and coefficients of Fourier Series)?

INVERSE FOURIER SERIES FORMULA:

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j \frac{2\pi}{N} k n} =$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left(\sum_{m=-\infty}^{+\infty} x[m] e^{-j \frac{2\pi k}{N} m} \right) e^{j \frac{2\pi}{N} k n}$$

$$= \sum_{m=-\infty}^{+\infty} x[m] \underbrace{\frac{1}{N} \sum_{k=0}^{N-1} e^{j \frac{2\pi}{N} k (n-m)}}_{\text{Dirac Delta impulse train } (*)} =$$

$$= \sum_{m=-\infty}^{+\infty} x[m] \sum_{\ell=-\infty}^{+\infty} \delta[(n-m) - \ell N] =$$

similar to $\tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} 1\, e^{j \frac{2\pi}{N} k n} =$

$(*) = \sum_{\ell=-\infty}^{+\infty} \delta[n - \ell N]$ but evaluated in $(n-m)$

$$= \sum_{\ell=-\infty}^{+\infty} \underbrace{\sum_{m=-\infty}^{+\infty} x[m] \delta[(n-m) - \ell N]}_{} =$$

δ: neutral term of convolution (signal is conv. of itself)

$x[n] = \sum_{m=-\infty}^{+\infty} x[m] \delta[n-m]$

$x[n-\ell N] = \sum_{m=-\infty}^{+\infty} x[m] \delta[(n-\ell N) - m]$

$$= \sum_{\ell=-\infty}^{+\infty} x[n - \ell N] = \tilde{x}[n] \qquad \text{PERIODIC SIGNAL}$$

$$= x[n] + \underbrace{x[n-N]}_{\ell=1} + \underbrace{x[n-2N]}_{\ell=2} + \dots + \underbrace{x[n+N]}_{\ell=-1} + \underbrace{x[n+2N]}_{\ell=-2} + \dots$$

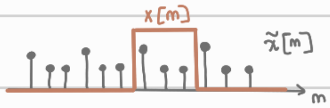
ORIGINAL SIGNAL $+ \infty$ **REPLICAS** of the signal shifted in time by multiples of N

SUM builds up the periodic signal which discrete spectrum obtained by sampling the DTFT

$x[n] \xrightarrow{DTFT} X(e^{j\omega}) \xrightarrow{\text{SAMPLING (N samples)}} \tilde{X}[k] \xrightarrow{\text{INVERSE FOURIER SERIES}} \tilde{x}[n] \xrightarrow{\text{TIME WINDOW}} x[n] = \begin{cases} \tilde{x}[n] & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$

generic — periodic and continuous — periodic — periodic — 1 period

SHANNON CONDITION must be satisfied (no aliasing) to reconstruct the signal



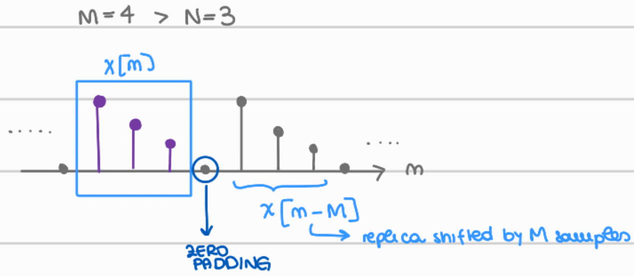
DUALITY IN SAMPLING IN ONE DOMAIN E PERIODICITY IN THE OTHER DOMAIN

- sampling in time ⟶ periodic spectrum
- sampling in freq ⟶ periodic time signal

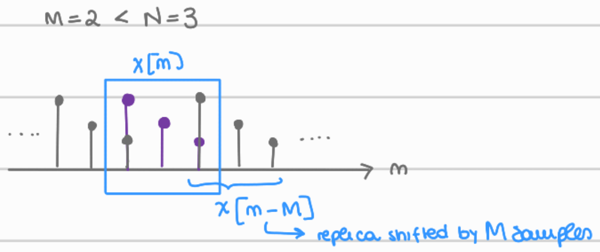
TEMPORAL ALIASING

$x[m]$ of length N BUT sampling with $M \neq N$ sampling

- **M > N** : NO ALIASING but ZERO PADDING
  - we can reconstruct the signal
  - no overlap btw replica and main repetition
  - values at zero coming back to time domain in the margin btw 1st replica and main repetition



- **M < N** : TEMPORAL ALIASING
  - overlap btw samples in reconstructing signal



$\boxed{M \geq N}$ to avoid temporal aliasing

$N < \infty$: LIMITED SIGNAL (what we have in practise)

Now, we can plot the DFT since spectrum is discrete. We can approximate the DTFT at computing (for example using Matlab) sampling the DTFT with a high value of M.

- High M means high resolution (very small difference between two consecutive samples), so a more accurate approximation BUT a higher time for computation.

DFT (DISCRETE FOURIER TRANSFORM) and IDFT (INVERSE)

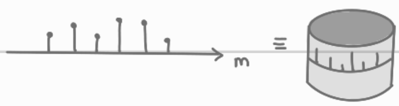
(1) $x[m]$ generic sequence of length N $\longrightarrow$ (2) $\tilde{x}[m]$: associated periodic sequence (period N) — sum of $x[m]$ and ∞ replicas shifted of multiples of N

$$\tilde{x}[m] = \sum_{r=-\infty}^{+\infty} x[m - rN] = x[(m \text{ modulo } N)]$$

- NO TEMPORAL ALIASING — sequence of length N + replicas shifted of rN
- NO ZERO PADDING — replicas shifted of rN

} ∞ copies of the signal closed together w/o overlapping and zero padding

PERIODIC VERSION of $x[m]$ $\quad x[m] = \begin{cases} \tilde{x}[m] & \text{for } 0 \leq m \leq N-1 \\ 0 & \text{otherwise} \end{cases}$



(3) Fourier series on $\tilde{x}[m]$ $\longrightarrow$ $\tilde{X}[k]$ periodic coefficients $\longrightarrow$ Freq domain is discrete

(4) Freq. domain (discrete) of $x[n]$ taking just 1 period of $\tilde{X}[k]$

$$x[m] \longrightarrow \tilde{x}[m] \xrightarrow{\text{FOURIER SERIES}} \tilde{X}[k] \longrightarrow X[k] = \begin{cases} \tilde{X}[m] & 0 \leq k \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

$x[m]$: generic; $\tilde{x}[m]$: periodic version; $\tilde{X}[k]$: coefficients periodic; $X[k]$: spectrum of $x[m]$

we need a periodic signal to find a SPECTRUM • periodic • discrete

DFT / IDFT

"~" omitted because we take only 1 period

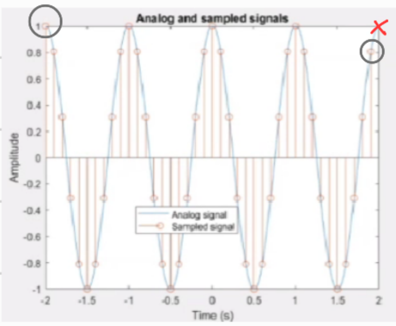
$$X[k] = \begin{cases} \sum_{0}^{N-1} {}_m \, x[m] \, e^{-j\frac{2\pi}{N}km} & \text{for } 0 \leq m \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

DFT : direct formula $\equiv$ 1 period of the Fourier series of $\tilde{x}[m]$ = replicas of $x[m]$

$$x[m] = \begin{cases} \frac{1}{N} \sum_{0}^{N-1} {}_k \, X[k] \, e^{j\frac{2\pi}{N}km} & \text{for } 0 \leq m \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

IDFT : inverse formula $\equiv$ 1 period of inverse Fourier series of $\tilde{X}[k]$

Example:

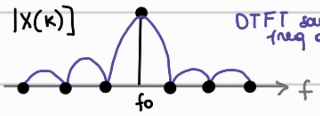


Analog signal : COSINE

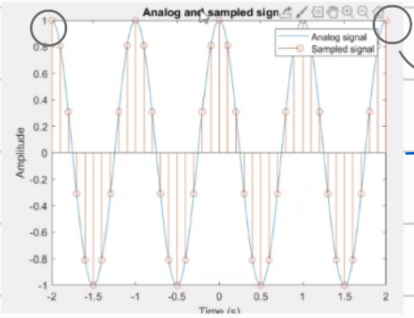
Sampled version (finite length) → DFT: amplitude spectrum is 1 LINE
↳ EVEN SYMMETRY: just freq>0

If we think at cosine in a cylinder, we have from $-\infty$ to $+\infty$

→ we do not see effect of the window

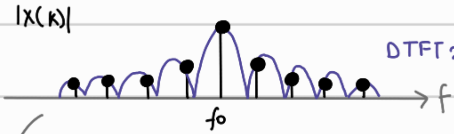


DTFT sampled at real freq of the cosine
All other sampling are zero in the zero point of DTFT



→ If we change something in sampling: spectrum = SEVERAL LINES
NB. windowing is not a problem for DFT

→ we see effect of the window



DTFT ≈ DFT with M>N samples

If this is correct, we have not a cosine from $-\infty$ to $+\infty$

FLAT INTERVAL: 1ST and LAST samples are equal
similar but NOT a cosine
↳ ≡ next sample

Depending on sampling, we see something different in freq domain

DTFT can be approximated by the DFT with a different # samples in freq. domain
↳ M ≥ N
DFT → defined with N samples

Particular case: ZERO PADDING DFT → $X(e^{j\omega_k}) = \sum_{m=0}^{N-1} x[m]\, e^{-j\frac{2\pi}{N}km} = X(k)$

$x[m]$ generic sequence of length N

zero padding version of $x[m]$

$$y[m] = \begin{cases} x[m] & m = 0,1,\dots,N-1 \\ 0 & m = N, N+1, \dots, 2N-1 \end{cases}$$

2N length

DTFT

$$Y(e^{j\omega}) = \sum_{m=0}^{2N-1} \underbrace{y[m]}_{=0 \text{ from N to 2N-1}} e^{-j\omega m} = \sum_{m=0}^{N-1} y[m]\, e^{-j\omega m} = \sum_{m=0}^{N-1} x[m]\, e^{-j\omega m} = X(e^{j\omega})$$

DTFT is the same before and after the zero padding

DFT

$$Y(e^{j\omega_\ell}) = \sum_{m=0}^{2N-1} y[m]\, e^{-j\frac{\cancel{2}\pi}{\cancel{2}N}\ell m} = \sum_{m=0}^{2N-1} x[m]\, e^{-j\frac{\pi}{N}\ell m}$$

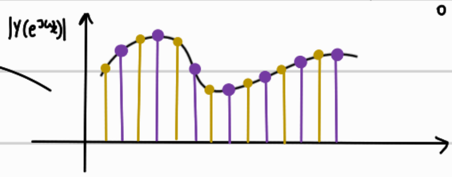
$\ell = 0, \dots, 2N-1$

$y[m]$ has length 2N → DFT has length 2N

• EVEN LINES ℓ: even $\ell = 2k$ $k = 0,1,\dots,N-1$

$$Y(e^{j\omega_\ell}) = \sum_{m=0}^{N-1} x[m]\, e^{-j\frac{2\pi k}{N}m} = X(e^{j\omega_k})$$

DFT is the same for even lines before and after the zero padding



EVEN LINES
ODD LINES interpolation of the DTFT for other points in freq

SPECTRAL INTERPOLATION

samples ↑ in time → # samples ↑ in freq.

DENSER sampling of DTFT: DTFT of $x[m]$ and $y[m]$ is the same but $y[m]$ has higher length
↓
more points

We need to translate mathematical into computationa tools, and DFT has a high computational cost.

FFT (FAST FOURIER TRANSFORM)

It is a set of algorithms that implement the DFT in a fast way (mathematics is the same). FFT is what we implement in Matlab to compute the DFT.
One way to improve the speed ofthe DFT through an FFT is using a constrain: $N =$ power of 2
We can expand the signal with zero (using zero padding) since the DTFT does not change in frequency domain, but we obtain a denser interpolation.

The spectrum of the impulse modulated signal and of the corresponding sequence is the same, thus we can choose how to put in the axis (analog or discrete pulsation or frequency).
Remember the Shannon theorem: $\Omega_c \equiv F_c \equiv 2\pi$ in discrete ω } 3 axis are related
Since the spectrum is periodic, we must choose the period
- $[0, \Omega_c]$ or $[-\Omega_c/2, \Omega_c/2]$
- $[0, F_c]$ or $[-F_c/2, F_c/2]$
- $[0, 2\pi]$ or $[-\pi, \pi]$

Properties

- **LINEARITY** superposition principle holds

$x_1[m]$, $x_2[m]$ } generic sequences of length N

LC: $x_3[m] = a x_1[m] + b x_2[m]$

$$\underbrace{X_3[k] = a X_1[k] + b X_2[k]}_{\text{same LC in the DFT}}$$

$\forall$ LC (a,b)
$\forall$ couple of sequences (x_1, x_2)

- **CIRCULAR SHIFT**

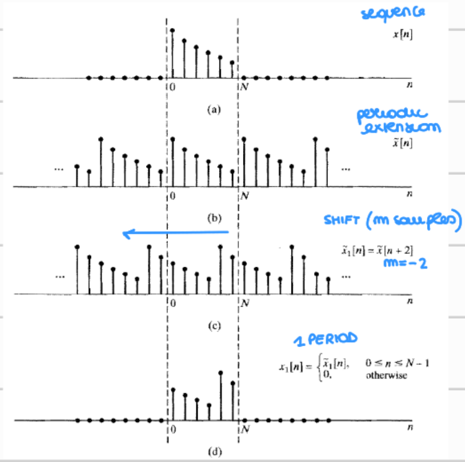
DFT ~ cylinder → shift of the sequence is circular

$x[m]$ generic sequence (length N)

$$x_1[m] = \begin{cases} \tilde{x}[m-m] = x[(m-m) \text{ modulo } N] & 0 \le m \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

- x is extended (periodic version $\tilde{x}$)
- shift of the extension
- take 1 period

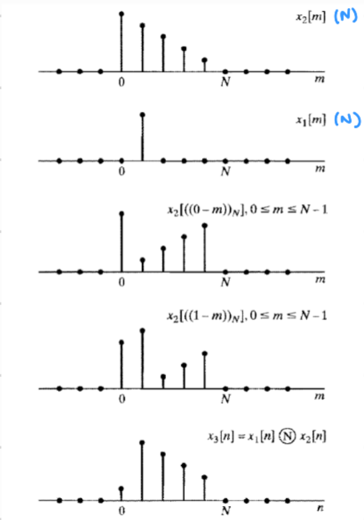
$X[k]e^{-j\frac{2\pi}{N}km}$ PHASE TERM (amplitude = 1)



- **CIRCULAR CONVOLUTION**

$x_1[m] \xrightarrow{DFT} X_1[k]$
$x_2[m] \longrightarrow X_2[k]$ }

$$\underbrace{X_3[k]}_{\text{which signal in time domain gives this coefficient?}} = X_1[k] \cdot X_2[k] \quad \text{product of DFT}$$



$x_2[m]$ (N) → $\tilde{x}_2[m]$
$x_1[m]$ (N) → $\tilde{x}_1[m]$
periodic convolution
↓
$\tilde{x}_3[m]$
↓ 1 period
$x_3[m]$

$$x_3[m] = \sum_{m=0}^{N-1} x_1[m] \cdot x_2[m-m] = \quad 0 \le m \le N-1$$

COMMUTATIVE

$$= \sum_{m=0}^{N-1} \tilde{x}_2[m] \cdot \tilde{x}_1[m-m]$$

period of $\tilde{x}_3[m]$

m: dummy variable
m: real time variable

DIGITAL FILTERS (DESIGN)

We have used analog filters in many previous courses. **Digital filters** are the tool used to *process signals in frequency domain*. The goal is to design **real filter,** that we are able to implement working in the frequency domain. There are many implementations: remove noise, select the portion of interest, ...
A digital filter is a type of *LTI digital system*.

INPUT $x[m]$ (digital sequence) $\longrightarrow$ $\boxed{LTI}$ $\longrightarrow$ OUTPUT $y[m]$ (digital sequence)

$h[m]$ or its DTFT $H(e^{j\omega})$ $\longrightarrow$ TRANSFER FUNCTION

impulse response to characterize the system

pair of transform

IDEAL LP FILTER

the simplest one

DIGITAL $\longrightarrow$ freq domain = DTFT (continuous ω)

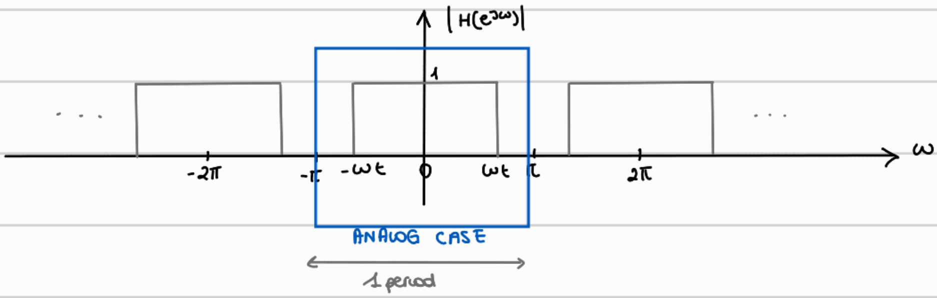
spectrum is periodic

CUTOFF PULSATION

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_t \in (-\pi,\pi) \\ 0 & \text{otherwise in } (-\pi,\pi) \end{cases}$$

$\longrightarrow$ within the band $[0, \omega_t]$

$\omega_t < |\omega| < \pi$



$H(e^{j\omega})$ ideal $\xrightarrow{\text{inverse DTFT}}$

$$h[m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega m}\, d\omega = \frac{1}{2\pi}\int_{-\omega_t}^{\omega_t} 1\cdot e^{j\omega m}\, d\omega =$$

$h[m]$ ideal

1 period

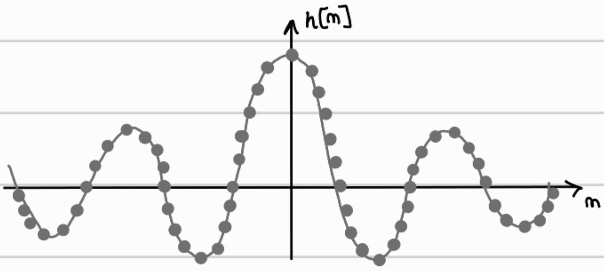
outside band $H=0$

samples = ∞
(no computation on a PC)

$$m=0: \quad = \frac{1}{2\pi}\int_{-\omega_t}^{+\omega_t} 1\, d\omega = \frac{1}{2\pi}\left[\omega\right]_{-\omega_t}^{\omega_t} = \frac{2\omega_t}{2\pi} = \boxed{\frac{\omega_t}{\pi}}$$

$$m\neq 0: \quad = \frac{1}{2\pi}\left[\frac{e^{j\omega m}}{jm}\right]_{-\omega_t}^{\omega_t} = \frac{1}{2\pi}\,\frac{e^{j\omega_t m} - e^{-j\omega_t m}}{jm} = \boxed{\frac{\sin(\omega_t m)}{\pi m}}$$

Euler formula



ANALOG DOMAIN
Among the class of **analog passive filters**, the most common filter is the *RC low pass filter* (1st order). Using more R, L, and C, we can design more complex filters. Another class is the class of **analog active filters**, in which we use instrumentational amplifiers.
We have a network described by *differential equations*; applying the *Laplace transform*, we obtain algebric equations, then we go back to time domain.
DIGITAL DOMAIN
We have *difference equations*, and we apply the *Z transform*.

CAUSAL DIGITAL FILTER

output depends just on previous and actual inputs / data (and not on future ones)

DIFFERENCE EQ OF ORDER N

TIME DOMAIN $\xrightarrow{z}$ FREQ. DOMAIN

length of memory of the output = order of differential eq. in time domain
|||
higher value in the past need to compute the output

$$\underbrace{\sum_0^N {}_k\, a_k\, y[m-k]}_{\text{LC of current } (k=0) \text{ and past } (k>0) \text{ outputs}} = \underbrace{\sum_0^M {}_k\, b_k\, x[m-k]}_{\text{LC of current } (k=0) \text{ and past } (k>0) \text{ inputs}}$$

$k \geq 0$ always
future $\equiv k<0$ NEVER

we can explicit the current output $(k=0)$:

$$a_0\, y[m] + \sum_1^N {}_k\, a_k\, y[m-k] = \sum_0^M {}_k\, b_k\, x[m-k]$$

HP: $a_0 = 1$ no loss of generality

if $a_0 \neq 1$, we divide lhs and rhs by a_0 $(a_0 \neq 0)$

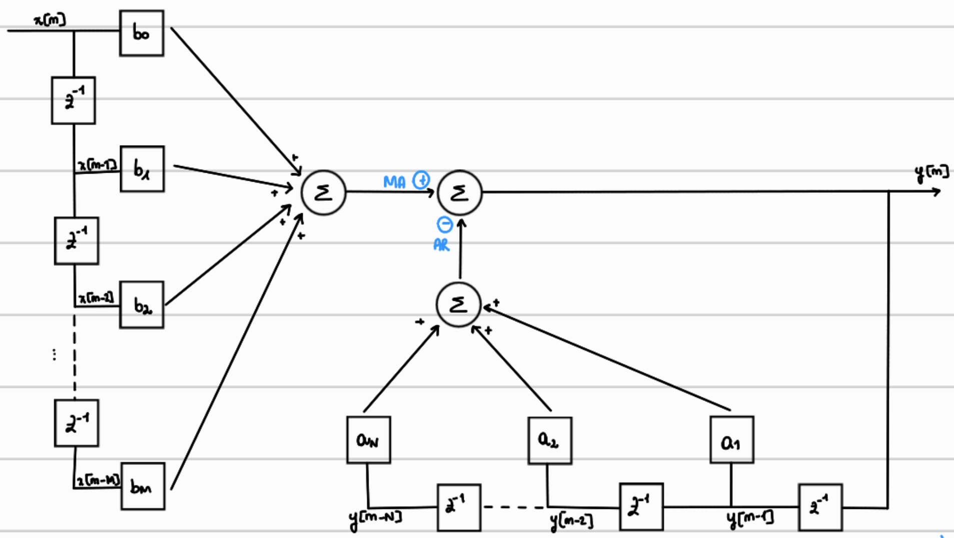
$$y[m] = \boxed{-\sum_1^N {}_k\, a_k\, y[m-k]} + \boxed{\sum_0^M {}_k\, b_k\, x[m-k]}$$

ACTUAL OUTPUT as function of LC of past output and LC of past and current inputs

AR: AUTOREGRESSIVE part
FEEDBACK computation (past outputs to compute actual one)

MA: MOVING AVERAGE
FEEDFORWARD computation (from inputs to outputs)

TIME DOMAIN



MA part (feedforward)
inputs weighted by a coeff. and summed
each time shifted by one sample (z^{-m}, $m=1$ shift property)

AR part (feedback)
outputs shifted by 1 sample each time, weighted and summed

Z DOMAIN

$$z\left\{\sum_0^N {}_k\, a_k\, y[m-k]\right\} = z\left\{\sum_0^M {}_k\, b_k\, x[m-k]\right\}$$

L

$$\sum_0^N {}_k\, a_k\, z\{y[m-k]\} = \sum_0^M {}_k\, b_k\, z\{x[m-k]\}$$

SHIFT PROPERTY

$$\sum_0^N {}_k\, a_k\, Y(z)\, z^{-k} = \sum_0^M {}_k\, b_k\, X(z)\, z^{-k} \rightarrow H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{\sum_0^M {}_k\, b_k\, z^{-k}}{\sum_0^N {}_k\, a_k\, z^{-k}} =$$

hp: $a_0 =$

$$= \frac{\sum_0^M {}_k\, b_k\, z^{-k}}{1 + \sum_1^N {}_k\, a_k\, z^{-k}}$$

polynomials of variable z^{-k}

→ vector of b_k = MA part
→ vector of a_k = AR part

STABILITY related to AR part

NB. study of stability only if AR part is present

some filters can have only MA part (only num)

M ≥ N
order num > den

difference btw orders

$$H(z) = \underbrace{\sum_{0}^{M-N} {}_r B_r z^{-r}}_{\text{QUOTIENT}} + \underbrace{\frac{\sum_0^{N-1} {}_k \bar{b}_k z^{-k}}{\sum_0^{N} {}_k a_k z^{-k}}}_{\text{ratio btw polynomials : rest divided by dem (order num < den)}}$$

MULT=1 of each pole

$$= \sum_0^{M-N} {}_r B_r z^{-r} + \sum_1^{N} {}_k \frac{A_k}{1-d_k z^{-1}}$$

(case M<N) best case order num < dem

d_k : single pole (for which den = 0)
$1-d_k z^{-1}=0$ when $z=d_k$
A_k : residue

$$A_k = (1-d_k z^{-1})H(z)\Big|_{z=d_k}$$

Example: $H(z) = \frac{A_1}{(1-d_1 z^{-1})} + \frac{A_2}{(1-d_2 z^{-1})}$

$A_1 = (1-d_1 z^{-1})H(z)\Big|_{z=d_1} =$

$(1-d_1 z^{-1})\left(\frac{A_1}{1-d_1 z^{-1}} + \frac{A_2}{1-d_2 z^{-1}}\right) = A_1 + 0$

Example: $H(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$ $M=N=2$

$|z|>1$ region of convergence

dividend: $z^{-2}+2z^{-1}+1$ divisor: order 2: $\frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1$

$-(z^{-2}-3z^{-1}+2)$ $\boxed{2}$ quotient

$\underbrace{0+5z^{-1}+3}_{\text{order } 1<2 \text{ STOP}}$

$= 2 + \frac{5z^{-1}+3}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} =$

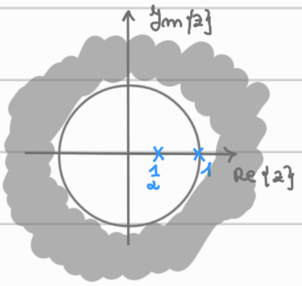
$= 2 + \frac{5z^{-1}+3}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$ poles in $\frac{1}{2}$ and 1
$d_1=\frac{1}{2}$ $d_2=1$

$= 2 + \frac{A_1}{(1-\frac{1}{2}z^{-1})} + \frac{A_2}{(1-z^{-1})} =$

$= 2 - \frac{9}{(1-\frac{1}{2}z^{-1})} + \frac{8}{(1-z^{-1})}$ $|z|>1$

$$A_1 = \left(2 + \frac{5z^{-1}+3}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}\right)\underline{(1-\tfrac{1}{2}z^{-1})}\Big|_{z=\frac{1}{2}} = -9$$

$$A_2 = \left(2 + \frac{5z^{-1}+3}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}\right)\underline{(1-z^{-1})}\Big|_{z=1} = 8$$



$H(z)$: z-transform → z^{-1} transform : $h[m]$

RECALL: ① $x[m] = a^m u[m]$ (exp. series) $\xrightarrow{z}$ $X(z) = \frac{1}{1-az^{-1}}$ z transf. of exp. series — simple pole in z domain ≡ exp. term in discrete time domain

② $x[m]=\delta[m] \xrightarrow{z} X(z)=1$
$x[m]=2\delta[m] \xrightarrow{z} X(z)=2$ } same multiplication by const. in time and z domain
⋮

Example: $H(z) = 2 - \frac{9}{(1-\frac{1}{2}z^{-1})} + \frac{8}{(1-z^{-1})}$

$$h[m] = 2\delta[m] - 9\left(\tfrac{1}{2}\right)^m u[m] + 8\,u[m]$$

$$H(z) = \underbrace{\sum_0^{M-N} {}_r B_r z^{-r}}_{②} + \underbrace{\sum_1^{N} {}_k \frac{A_k}{1-d_k z^{-1}}}_{①} \xrightarrow{z^{-1}} \boxed{h[m] = \underbrace{\sum_0^{M-N} {}_r B_r \delta[m-r]}_{} + \underbrace{\sum_1^{N} {}_k A_k d_k^m u[m]}_{}} \quad |d_k|<1$$

② $\delta[m] \to 1$
$B_r\delta[m] \to B_r$
$B_r\delta[m-r] \to B_r z^{-r}$

① $a^m u[m] \to \frac{1}{1-az^{-1}}$
$|z|>1$

each δ has height B_r
0 … M−N m
M−N+1 (finite length)
[no analogy with analog case] ∄ in analog filters

not limited in time (from −∞ to +∞)
IMPULSE RESPONSE is an exp. which → 0 but with ∞ length

FILTERS
- different classes
- different properties
- different design

IIR: INFINITE IMPULSE RESPONSE ($|d_k|<1$) — ∞ length — MA + AR (it can be present or not) → vector a_k at denominator ⟹ study of stability is necessary $\sum_{k=1}^{N} A_k d_k^m u[m]$

FIR: FINITE IMPULSE RESPONSE (No AR part) — only MA → intrinsically stable (no poles at denom)

ASYMPTOTIC STABILITY: $h[m] \to 0$ for $n \to \infty \iff |d_k|<1$
$\Re\{pole\}<0$ in analog case

SIMPLE STABILITY: $|d_k| \le 1$
if $|d_k|=1$ for $m \to \infty$, $h[m] \to 1$ — not asymptotically at 0 but limited

Specifications of digital filters

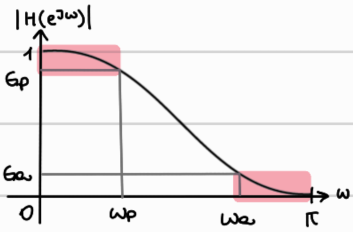
characterization:
DIFFERENCE EQ ⟷ $H(z)$ ⟷ $h[m]$ } dependency on:
- VECTOR OF a_k (AR)
- VECTOR OF b_k (MA)

} knowing them, we know everything

definition based on some specifications

NB. ANALOG CASE: spectrum is not periodic
DISCRETE CASE: spectrum is periodic → pulsation is periodic (up to π)
but spectra are quite similar

LP FILTER
- G_a, G_p
- ω_a, ω_p



PASSBAND by spec. — gain $\in [G_p, 1]$, $\omega \in [0, \omega_p]$ (ideal case)
TRANSITION BAND unknown
ATTENUATION BAND by spec. — gain $\in [0, G_a]$, $\omega \in [\omega_a, \pi]$

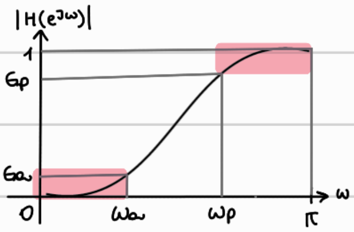
analog case: $\omega_p \to \Omega_p$
$\omega_a \to \Omega_a$ and no limit of attenuation

- GAIN $G_p = \frac{1}{A_p}$ ATTENUATION $\qquad G_a = \frac{1}{A_a}$

- G natural unit
$(G)_{dB} = 20\log_{10}(G)$
$(A)_{dB} = 20\log_{10}(A) = 20\log_{10}\left(\frac{1}{G}\right) = -20\log_{10}(G) = -(G)_{dB}$

$\begin{cases} G = 10^{(G)_{dB}/20} \\ A = 10^{(A)_{dB}/20} \end{cases}$

HP FILTER
- G_a, G_p
- ω_a, ω_p

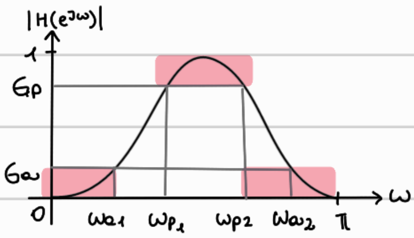


ATTENUATION BAND by spec. — gain $\in [0, G_a]$, $\omega \in [\omega_p, \pi]$
TRANSITION BAND unknown
PASSBAND by spec. — gain $\in [G_p, 1]$, $\omega \in [0, \omega_a]$

Typical values:

AdB	G	
0,5	~0,94	PASSBAND
1	~0,89	
3	~0,71	
10	~0,31	ATTENUATION BAND (ideally at zero)
20	~0,1	
40	~0,01	
60	~0,001	

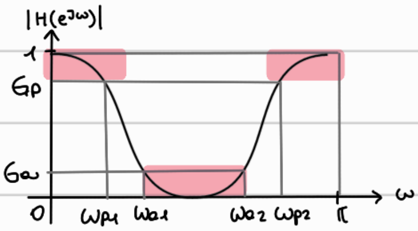
BAND PASS FILTER
- G_a, G_p
- $\omega_{a1} < \omega_{p1} < \omega_{p2} < \omega_{a2}$



ATTENUATION BAND by spec. — gain $\in [0, G_a]$, $\omega \in [0, \omega_{a1}]$
PASSBAND by spec. — gain $\in [G_p, 1]$, $\omega \in [\omega_{p1}, \omega_{p2}]$
ATTENUATION BAND by spec. — gain $\in [0, G_a]$, $\omega \in [\omega_{a2}, \pi]$

BAND STOP FILTER

- G_ω, G_p
- $\omega_{p1} < \omega_{a1} < \omega_{a2} < \omega_{p2}$



NB. Normalized pulsation : $\frac{\omega_c}{\pi}$ or $\frac{\Omega_c}{2}$ $\in [0,1]$

PASSBAND by spec. → gain $\in [G_p, 1]$, $\omega \in [0, \omega_{p1}]$

ATTENUATION by spec. → gain $\in [0, G_a]$, $\omega \in [\omega_{a1}, \omega_{a2}]$

PASSBAND by spec. → gain $\in [G_p, 1]$, $\omega \in [\omega_{p2}, \pi]$

- Relationship between poles and transfer function?

SIMPLE POLE : $z = -d$ multiplicity $= 1$, $|d| < 1$ asymp. stability

DTFT : restriction of z-transform, when $z = e^{j\omega}$ (circle: centered in origin, radius = 1)

TRANSFER FUNCTION: $H(z) = \frac{b_0}{1 + dz^{-1}} = \frac{b_0}{1 + de^{-j\omega}} =$ (EULER'S F.) $= \frac{b_0}{1 + d(\cos\omega - j\sin\omega)} = H(e^{j\omega})$

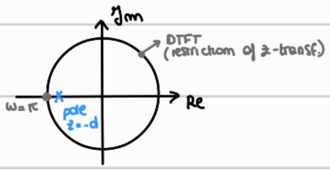
MODULUS at the power of 2 (simpler to plot)

$$|H(e^{j\omega})|^2 = \frac{|b_0|^2}{|\underbrace{1 + d\cos\omega}_{Re} - j\underbrace{d\sin\omega}_{Im}|^2} = \frac{|b_0|^2}{(1 + d\cos\omega)^2 + d^2(\sin\omega)^2} = \frac{|b_0|^2}{1 + d^2\cos^2\omega + 2d\cos\omega + d^2\sin^2\omega} = \boxed{\frac{|b_0|^2}{1 + d^2 + 2d\cos\omega}}$$

$d^2\cos^2\omega + d^2\sin^2\omega = d^2$

⟹ EFFECT OF POLES on DTFT

① $d > 0$ (real and positive) $d \in \mathbb{R}$ for simplicity, $|d| < 1$ for stability

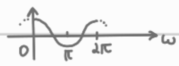


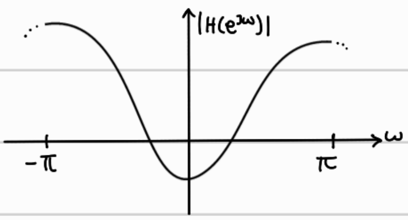
$F = \frac{F_c}{2}$ $\Omega = \frac{\Omega_c}{2}$ $(\omega = \Omega T_c)$

$z \to$ pole, $H(z) \to \infty$

max when denom is min

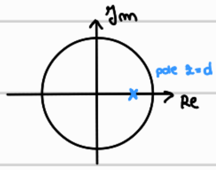
↳ cos is min when $\omega = \pi$ $\omega = \pm\pi$





~ LP FILTER

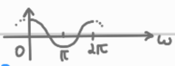
② $d < 0$ (real and negative) $d \in \mathbb{R}$ for simplicity, $|d| < 1$ for stability

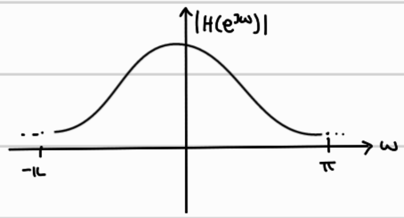


$z \to$ pole, $H(z) \to \infty$

min when denom is max

↳ cos is max hen $\omega = 0$ ~~$\omega = 2\pi$~~ $\omega \in [-\pi, \pi]$





~ HP FILTER

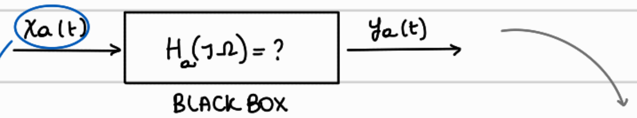
- z-transf HIGH ($\to \infty$) in the pole
- z-transf LOW outside the pole

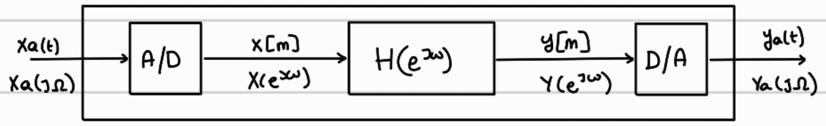
⟶ DTFT high when circle close to the pole

DISTANCE btw pole and circle :
- pole inside the circle
- $d = 0$ ⟶ DTFT CONSTANT (pole is equidistant from all points of circle)

- Relationship between the transfer function related to the analog filter (analog processing) and the transfer function corresponding to the equivalent digital filter (digital processing)

We often have an analog signal that must be processed with analog filters, but we prefer to use a *completely digital pipeline*. We want to obtain filtered signal from a digital filter; we sample and convert an analog signal into a digital one, we design a digital filter, and then we apply *digital to analog conversion* to obtain the analog output.





HYPOTHESIS
- BANDLIMITED analog signal (⇒ NS th)
- sampling freq. f_c according to NS th.
- same f_c for both A/D and D/A conversion

A/D CONVERTER

$x[n] = x_a(nT_c)$ — by sampling, output A/D converter

$$X(e^{j\omega}) = \frac{1}{T_c}\sum_{\ell=-\infty}^{+\infty} X_a\big(j(\Omega-\ell\Omega_c)\big)$$

sampling causes a repetition of spectrum (relationship btw $\mathcal{F}$ transforms)

D/A CONVERTER

TIME DOMAIN:

$$y_a(t) = \sum_{n=-\infty}^{+\infty} y[n]\,\mathrm{sinc}\left((t-nT_c)\frac{\Omega_c}{2}\right)$$

at the end of D/A converter

RECONSTRUCTION FORMULA — use of samples to reconstruct analog signal

FREQ DOMAIN:

$\mathcal{F}\{\text{impulse modulated signal}\} = Y(e^{j\omega})$

$$Y_a(j\Omega) = \underbrace{H_r(j\Omega)}_{\substack{\text{TRANSFER FUNCTION OF FILTER}\\ \text{(ideal LP filter)}}} Y_s(j\Omega) = H_r(j\Omega)\,Y(e^{j\omega})$$

$$H_r(j\Omega) = \begin{cases} T_c & \text{for } |\Omega| \le \frac{\Omega_c}{2} \\ 0 & \text{for } |\Omega| > \frac{\Omega_c}{2} \end{cases}$$

LINK btw A/D and D/A CONVERTER

DISCRETE TIME LTI SYSTEM: $Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$ — DTFT output, DTFT filter, DTFT input

Digital pipeline:

$$\underbrace{Y_a(j\Omega)}_{\text{output of pipeline}} = H_r(j\Omega)\underbrace{Y(e^{j\omega})}_{= H(e^{j\omega})X(e^{j\omega})} =$$

$$= H_r(j\Omega)H(e^{j\omega})\underbrace{X(e^{j\omega})}_{=\frac{1}{T_c}\sum_{\ell=-\infty}^{+\infty} X_a(j(\Omega-\ell\Omega_c))} =$$

$$= \underbrace{H_r(j\Omega)}_{\substack{T_c \text{ for } |\Omega|\le\Omega_c\\ 0 \text{ otherwise}}}H(e^{j\omega})\frac{1}{T_c}\sum_{\ell=-\infty}^{+\infty} X_a(j(\Omega-\ell\Omega_c))$$

↳ band limited spectrum (hp)
↳ $\mathcal{F}\{\cdot\} = 0$ outside cut off pulsation (filter remove replicas)

$$= \begin{cases} \cancel{T_c}\,H(e^{j\omega})\,\frac{1}{\cancel{T_c}}X_a(j\Omega) & \text{for } |\Omega| \geqslant \frac{\Omega_c}{2} \\ 0 & \text{for } |\Omega| < \frac{\Omega_c}{2} \end{cases}$$

$\omega = \Omega T_c$

$$= \begin{cases} H(e^{j\Omega T_c})\,X_a(j\Omega) & \text{for } |\Omega| \geqslant \frac{\Omega_c}{2} \\ 0 & \text{for } |\Omega| < \frac{\Omega_c}{2} \end{cases}$$

where $H(e^{j\omega}) = H(e^{j\Omega T_c})$ — TRANSFER FUNCTION of digital filter

Analog pipeline:

$$Y_a(j\Omega) = H_a(j\Omega)\,X_a(j\Omega)$$

$\mathcal{F}\{y_a(t)\}$ — $\mathcal{F}\{\text{analog filter}\}$ — $\mathcal{F}\{x_a(t)\}$

where $H_a(j\Omega) = \begin{cases} H(e^{j\Omega T_c}) & \text{for } |\Omega| \geqslant \frac{\Omega_c}{2} \\ 0 & \text{for } |\Omega| < \frac{\Omega_c}{2} \end{cases}$

We want to compare the output of the two pipelines (analog and digital): analog and digital filters are equivalent when pulsation is within the band, and zero otherwise. We have an **equivalence in the two implementations** (digital and analog domains).

We acquire a signal in the digital domain; designing and applying a digital pipeline according to the hypothesis, we directly know the *effects of the digital filter to the analog signal*.

DESIGN OF ANALOG FILTERS

Analog techniques are used also for designing digital filters.
We start with the design of LP filters, since the design is mainly for LP digital filters, especially for IIR (which have an equivalent in analog domain). The choice of the type of filter is based on the properties and the differences among filters.

LP Filters

- BUTTERWORTH
- CHEBYCHEV (I and II)
- ELLIPTIC

BUTTERWORTH FILTER

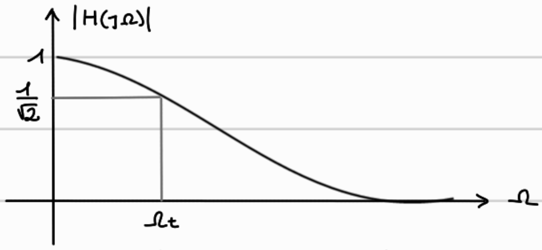
simplest LP filter

Compute PARAMETERS so that TF passes through specifications

$$\boxed{|H(j\Omega)|^2 = \frac{1}{1+\left(\frac{j\Omega}{j\Omega_t}\right)^{2N}}}$$

smooth expression

2 parameters:
- N = # poles (= order of filter)
- Ω_t = cutoff pulsation

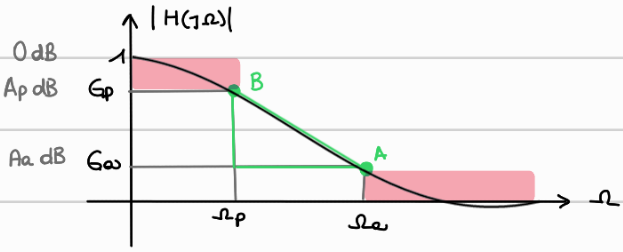


Modulus ↓ when ω ↑

$\Omega \to 0, \ |H(j\Omega)| \to 1$
$\Omega \to \infty, \ H(j\Omega) \to 0$
$\Omega \to \Omega_t, \ H(j\Omega) \to \frac{1}{\sqrt{2}}$ $\quad 20\log_{10}\left(\frac{1}{\sqrt{2}}\right) \simeq -3\text{dB}$

1. Filter is maximally flat in passband (no oscillatory term)
2. Transition band is quite flat (not quick, not so high slope)

$N = ? \quad \Omega_t = ?$



N minimum ≡ min. computational cost : GOAL OF THE DESIGN OF ANY FILTER

even if filter better for N↑

Start the design with the WORST CASE ⟶ 2 points
- $A(\Omega_a, G_a)$ — higher attenuation gain
- $B(\Omega_p, G_p)$ — higher passband gain

$$\begin{cases} G_p^2 = \dfrac{1}{1+\left(\frac{\Omega_p}{\Omega_t}\right)^{2N}} \longrightarrow \left(\dfrac{\Omega_p}{\Omega_t}\right)^{2N} = \dfrac{1}{G_p^2}-1 \\ G_a^2 = \dfrac{1}{1+\left(\frac{\Omega_a}{\Omega_t}\right)^{2N}} \longrightarrow \left(\dfrac{\Omega_a}{\Omega_t}\right)^{2N} = \dfrac{1}{G_a^2}-1 \end{cases}$$

$$\left(\frac{\Omega_p}{\Omega_a}\right)^{2N} = \frac{\frac{1}{G_p^2}-1}{\frac{1}{G_a^2}-1} \longrightarrow 2N\log_{10}\left(\frac{\Omega_p}{\Omega_a}\right) = \log_{10}\left(\frac{\frac{1}{G_p^2}-1}{\frac{1}{G_a^2}-1}\right) \longrightarrow N = \frac{\log_{10}\left(\frac{\frac{1}{G_p^2}-1}{\frac{1}{G_a^2}-1}\right)}{2\log_{10}\left(\frac{\Omega_p}{\Omega_a}\right)}$$

$$\Omega_t = \frac{\Omega_p}{\left(\frac{1}{G_p^2}-1\right)^{\frac{1}{2N}}} \quad \text{or} \quad \Omega_t = \frac{\Omega_a}{\left(\frac{1}{G_a^2}-1\right)^{1/2N}}$$

Example: $\Omega_p = 3000\,2\pi$ $\quad A_p = 1\text{dB} \longrightarrow G_b = 10^{-1/20} \simeq 0{,}89$ $\quad \Rightarrow N = 2{,}53$ PROBLEM not an integer

$\Omega_a = 6000\,2\pi$ $\quad A_a = 10\text{dB} \longrightarrow G_a = 10^{-10/20} \simeq 0{,}32$ $\quad \Rightarrow N = 3$ select worst case

$\Omega_t \simeq 3{,}91\ 10^3\ 2\pi$

- Filter through B, but below A ⟶ higher slope (better in attenuation band)
- Filter through A, but below B

} design of LP filter according to specifications

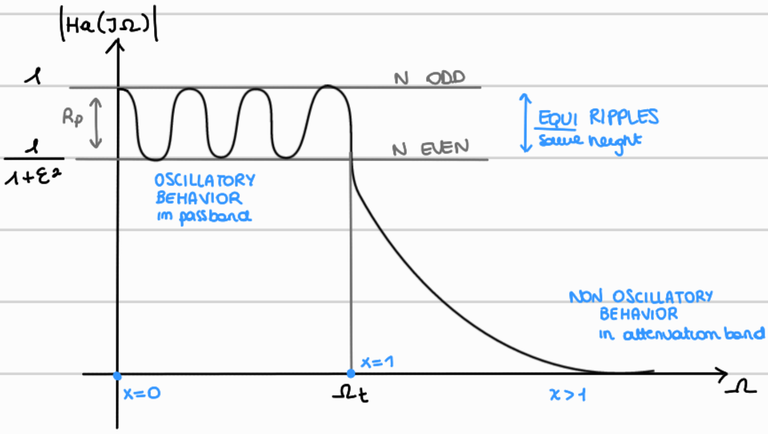
CHEBYCHEV FILTER (type I)

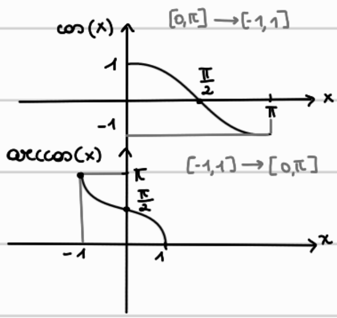
$$\boxed{|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 V_N^2\left(\frac{\Omega}{\Omega_t}\right)}}$$

3 parameters
- N
- Ω_t
- ε^2

$V_N(x) = \cos(N \arccos(x))$ where $x = \frac{\Omega}{\Omega_t}$

$\arccos = \cos^{-1}$ restricted to $[0,\pi]$ ↳ non invertable





$x = \frac{\Omega}{\Omega_t}$
- $x = 1$ for $\Omega = \Omega_t$
- $x < 1$ for $\Omega < \Omega_t$
- $x > 1$ for $\Omega > \Omega_t$

Ω_t : cutoff freq btw oscillatory and non oscillatory term

$\Omega = 0$ $\quad x = 0$ $\quad V_N(0) = \cos(N \underbrace{\arccos(0)}_{\frac{\pi}{2}}) = \cos\left(N\frac{\pi}{2}\right) =$
- N ODD $= 0 \longrightarrow \boxed{|H_a(j\Omega)| = 0}$
- N EVEN $= 1 \longrightarrow \boxed{|H_a(j\Omega)| = \frac{1}{1+\varepsilon^2}}$

$\Omega = \Omega_t$ $\quad x = 1$ $\quad V_N(1) = \cos(N \underbrace{\arccos(1)}_{0}) = \cos(N \cdot 0) = 1$ $\quad \forall N \longrightarrow \boxed{|H_a(j\Omega)| = \frac{1}{1+\varepsilon^2}}$ STARTING POINT of transition band

$0 < \Omega < \Omega_t$ $\quad x < 1$ $\quad V_N(x) = \cos(N \underbrace{\arccos(x)}) \longrightarrow$ RIPPLE

PASSBAND

$\arccos(x) = \frac{\pi}{2} \longrightarrow 0$ for $x \to 0$

$N\arccos(x) = N\frac{\pi}{2} \longrightarrow 0$ $\quad$ smooth continuous function ⊕ cosine = an oscillatory term

$\Omega > \Omega_t$ $\quad x > 1$ $\quad \cos(x) = \frac{e^{jx}+e^{-jx}}{2}$ $\quad x \in \mathbb{R} \longrightarrow \cos(z) = \frac{e^{jz}+e^{-jz}}{2}$ $\quad z \in \mathbb{C}$

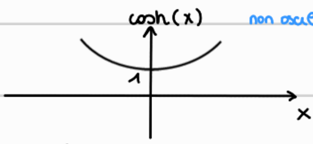
$\cosh(x) = \frac{e^{x}+e^{-x}}{2}$ $\quad x \in \mathbb{R} \longrightarrow \cosh(z) = \frac{e^{z}+e^{-z}}{2}$ $\quad z \in \mathbb{C}$

HYPERBOLIC COSINE

particular case $\quad z = jy$ $\quad y \in \mathbb{R}$ $\quad$ IMAGINARY NUMBER

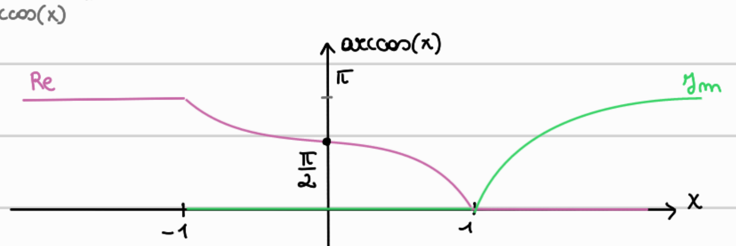
$\cos(jy) = \frac{e^{-y}+e^{y}}{2} = \cosh(y)$

$j(jy) = -y$ (cos of imaginary number) $\quad -j(jy) = y$ (hypercos of imaginary part)



non oscillatory function

arccos(x) can be defined outside [-1,1] considering Im part of the arccos(x)



For $x \in [-1,1]$: arccos(x) has only real part

Considering Im part of arccos(x) where $x \in \mathbb{R}$ → we can extend outside [-1,1]
- $x > 1$ $\quad Re\{\arccos(x)\} = 0$: only Im
- $x < -1$ $\quad Re\{\arccos(x)\} = \pi$: $Re \neq 0$

When $x > 1$, $\underbrace{\arccos(x)}_{\text{pure Im}} = \underbrace{j\,\underbrace{\operatorname{arccosh}(x)}_{\text{Re } (x \in \mathbb{R})}}_{\text{pure Im}}$ $\longrightarrow$ $V_N(x) = \cos(N \arccos(x)) =$

$= \cos(N\, \underbrace{j \underbrace{\operatorname{arccosh}(x)}_{\in \mathbb{R}}}_{\text{pure Im}})$

$\cos(jx) = \cosh(x)$

$= \cosh(N \operatorname{arccosh}(x))$ real, non oscillatory

BUTTERWORTH
- flat ✓
- no quick slope ($\simeq 0$) ✗

CHEBYCHEV I
- oscillatory ✗
- fast entering in transition band ✓ ~ ideal filter

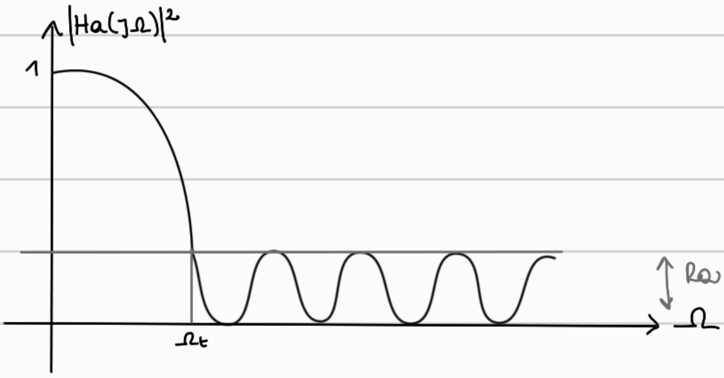
- FIXED WIDTH of transition band $\longrightarrow$ N less $\Rightarrow$ cheapest filter
- FIXED N (# poles) $\longrightarrow$ stricter band with Ch. type I
- $N\uparrow \longrightarrow$ # oscillation $\uparrow$ (but same height)

CHEBYCHEV FILTER (TYPE II)

$$\boxed{|H_a(j\Omega)|^2 = \frac{1}{1 + \left(\varepsilon^2 V_N^2\left(\frac{\Omega_t}{\Omega}\right)\right)^{-1}}}$$

parameters
- N
- Ω_t
- ε^2

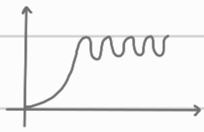
Two modifications wrt type I :



1 Argument is inverted: $x \longrightarrow \frac{1}{x}$ $\left(\frac{\Omega_t}{\Omega}\right)$

↳ behavior in low and high Ω is inverted $\Rightarrow$ HP filter

$\left[\begin{array}{l}\Omega \to 0 \longrightarrow \Omega \to \infty \\ \Omega \to \infty \longrightarrow \Omega = 0\end{array}\right]$



2 Elevation at power of (-1) $\Rightarrow$ LP filter again

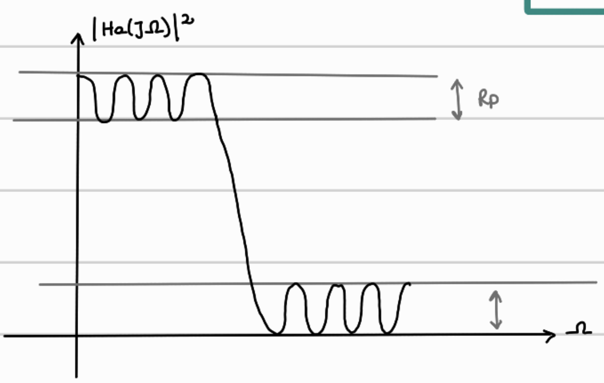
↳ value of magnitude is just inverted

ELLIPTIC FILTER

more complex

$$\boxed{|H_a(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 U_N(\Omega)}}$$

parameter
- N
- ε
- R_a } they can be ≠
- R_p



- FIXED WIDTH of transition band $\longrightarrow$ N less than Cheb. $\Rightarrow$ faster
- FIXED N (# poles) $\longrightarrow$ stricter band than Cheb.
- $N\uparrow \longrightarrow$ # oscillation $\uparrow$ (but same height)

DESIGN OF DIGITAL IIR FILTER through analog filters

SEVERAL TECHNIQUES

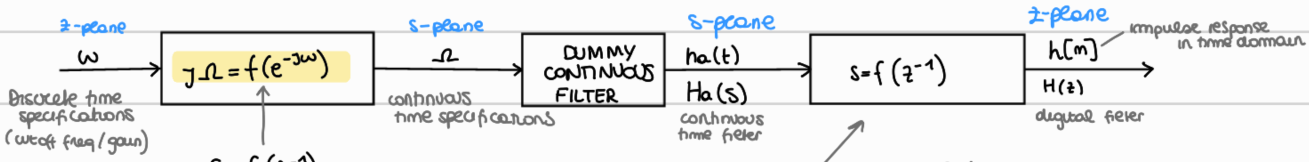
↳ correspondance with analog domain

Most ofthe5techniques for the design of IIR digital filtersuse the design of analog filters. Why?

1. The techniques for analog filters are well known, thus the 1st tentatives were to use them.
2. The densign of analog filters uses very simple and analytical formulas, which are useful to design also digital filters. Could we apply the same processing to digital filters, but without an analog filters? No, because in digital domain tge spectrum is periodic, thus we cannot directly apply the same idea.

Thus, we design a digital filter using a **dummy analog filter** (used but not implemented).

If we want to design a digital filter, we need specifications for the discrete case. We want to use techniques for the design of analog filters to design a digtal filter, thus we need a *trasformation* from digital specifications into continuous specifications. Then, we use known techniques for analog filters, and we obtain the transfer function in the Laplace space and so the impulse response of the filter. Then, we come back to the discrete domain just applying the same transformation.



GENERAL SCHEME to design IIR filter with analog techniques

$S = f(z^{-1})$

① Transformation btw discrete ω and continuous Ω pulsations

Im axis of s-plane

circumference of radius 1 centered in origin in z-plane

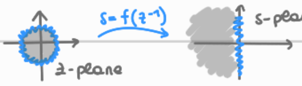
↓ ≡ restriction of transformation btw z- and s-plane

we can use it to convert TF of analog filter to TF of digital filter ③

$s = f(z^{-1})$ must preserve

- FREQUENCY RESPONSE : circle in z-plane is associated to Im axis in s-plane
- STABILITY : filters are stable after transformation

↳ pole inside circle ≡ pole in the negative half of the plane



⇒ the two freq. response of filters can be associated

GOAL of the design of a filter: DEFINE a_k, b_k VECTORS

⟶ we use the DUMMY ANALOG FILTER ②

There are two techniques that use this general sheme:

- Impulse invariance
- Bilinear transformation

Bilinear transformation for LP filter

⟶ ALGEBRIC TRANSFORMATION btw s and z^{-1} (both num and den are linear)

$s = f(z^{-1})$

$$\boxed{s = \frac{2}{T_d}\,\frac{1-z^{-1}}{1+z^{-1}}}$$

$k = \frac{2}{T_d}$ ∀ value

INVERSE TRANSFORM : $z = f(s) = \frac{2+T_d s}{2-T_d s}$

$s = \sigma + j\Omega$

$$= \frac{2+T_d\sigma + jT_d\Omega}{2-T_d\sigma - jT_d\Omega} = \frac{(2+T_d\sigma)+jT_d\Omega}{(2-T_d\sigma)-jT_d\Omega} \longrightarrow |z| = \frac{\sqrt{(2+T_d\sigma)^2+(T_d\Omega)^2}}{\sqrt{(2-T_d\sigma)^2+(T_d\Omega)^2}}$$

- $\sigma > 0 \longrightarrow |z| > 1$
  $Re\{s\} > 0$, $T_d > 0$; $|num| > |den|$
- $\sigma < 0 \longrightarrow |z| < 1$
  $Re\{s\} < 0$ ⟶ STABLE POLE; $|num| < |den|$ pole inside circumference

STABILITY IS PRESERVED

- $\sigma = 0 \longrightarrow |z| = 1$
  $s = j\Omega$ Im axis; radius = 1

PRESERVATION OF FREQ. RESP.

$s = \sigma + j\Omega$ pulsations on Im axis

$$z = \frac{2+T_d j\Omega}{2-T_d j\Omega} = e^{j\omega}$$

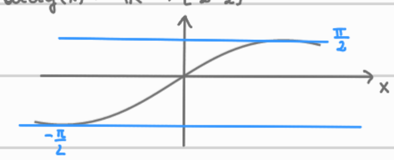
$s = j\Omega$

RELATIONSHIP ?

↓

$\omega = \text{phase}\left(e^{j\omega}\right)$

$\text{arctg}(x) : \mathbb{R} \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



$\alpha = 1$

$\text{arctg}(1) = \frac{\pi}{4} \Big/ -\frac{3\pi}{4}$

⟶ if $Re\{\cdot\} > 0$, ✓

if $Re\{\cdot\} < 0$, add π

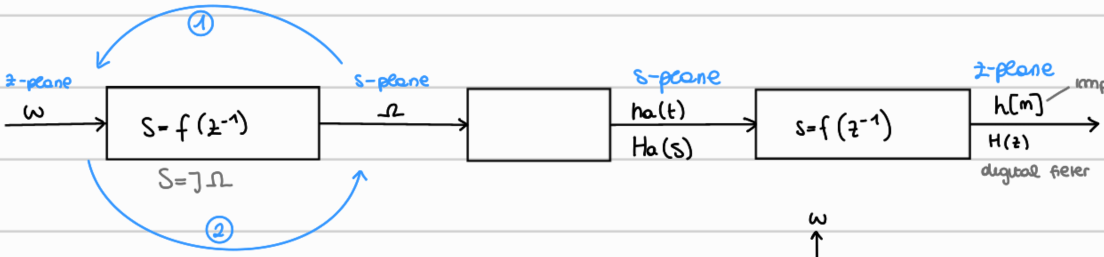
$\arctan(-x) = -\arctan(x)$

$$\omega = \arctan\left(\frac{T_d\Omega}{2}\right) - \arctan\left(-\frac{T_d\Omega}{2}\right) = \arctan\left(\frac{T_d\Omega}{2}\right) + \arctan\left(\frac{T_d\Omega}{2}\right) = 2\arctan\left(\frac{T_d\Omega}{2}\right)$$

BILINEAR TRANSFORMATION

$$\boxed{\begin{array}{ll} (1) & \omega = 2\arctan\left(\frac{T_d\Omega}{2}\right) \\ (2) & \Omega = \frac{2}{T_d}\,tg\left(\frac{\omega}{2}\right) \end{array}}$$

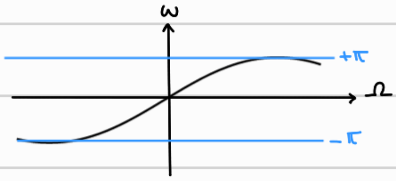
particular case
(restriction on circle / Im axis)



GENERAL SCHEME to design IIR filter with analog techniques

$S = f(z^{-1})$ NON LINEAR

$\omega = 2\arctan\left(\frac{T_d\Omega}{2}\right)$



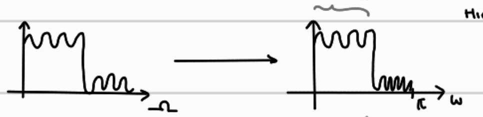
- $\lim\limits_{\Omega\to\infty} \omega = 2\left(\lim\limits_{\Omega\to\infty} \arctan\right) = 2\left(\frac{\pi}{2}\right) = \pi$
- $\lim\limits_{\Omega\to-\infty} \omega = 2\left(\lim\limits_{\Omega\to\infty} \arctan\right) = 2\left(-\frac{\pi}{2}\right) = -\pi$
- $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots \simeq x$ for $x \to 0$

$$\omega = 2\arctan\left(\frac{T_d\Omega}{2}\right) \underset{\Omega\to 0}{\simeq} \cancel{2}\frac{\Omega T_d}{\cancel{2}} = \Omega T_d$$

ALMOST LINEAR for $\Omega \to 0$

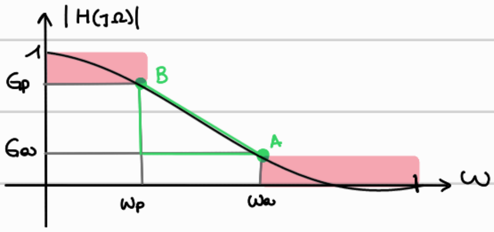
$\simeq$ COMPRESSED btw $-\pi$ and π for $\Omega \to \pm\infty$ $\longrightarrow$

- Design filter $\forall\,\Omega$ since $\omega < \pi$ and spectrum is periodic
  ALIASING for $\omega > \pi$
-



High pulsation only are affected

Example:



$\Omega_p = 0{,}2\pi$
$\Omega_a = 0{,}3\pi$

$A_p = 1\,dB \longrightarrow G_p = -1\,dB \simeq 0{,}89$
$A_a = 10\,dB \longrightarrow A_a = -10\,dB \simeq 0{,}18$
$T_d = 1$

} same for analog or digital filters

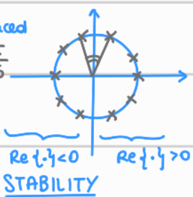
(1) Conversion of specifications in continuous time $\omega \to \Omega$ $\quad \Omega = \frac{2}{T_d}\,tg\left(\frac{\omega}{2}\right)$ $\quad \Omega_p = 2\,tg(0{,}1\pi)$, $\Omega_a = 2\,tg(0{,}15\pi)$ $\quad |H_a(J\Omega_p)| \geq 0{,}89$, $|H_a(J\Omega_a)| \leq 0{,}18$

(2) Design of Butterworth analog filter $\quad |H_a(J\Omega)|^2 = \dfrac{1}{1 + \left(\frac{J\Omega}{J\Omega_t}\right)^{2N}}$

$N = 5{,}305 \longrightarrow N = 6$
$\Omega_t = 0{,}766$

poles are 2N and they are on a circle centered in origin and of radius Ω_t

equally spaced $\frac{2\pi}{12} = \frac{\pi}{6}$



$2N = 12$

$$H_a(s) = \frac{K}{\prod_{0}^{N-1}{}_k (s - p_k)}$$

$K \longrightarrow$ unknown constant $(s=0)$

6 chosen poles

$H_a(0) = 1$ (LP filter) $= \dfrac{K}{\prod_k(-p_k)}$

$$= \frac{\prod_{0}^{N-1}{}_k (-p_k)}{\prod_{0}^{N-1}{}_k (s-p_k)}$$

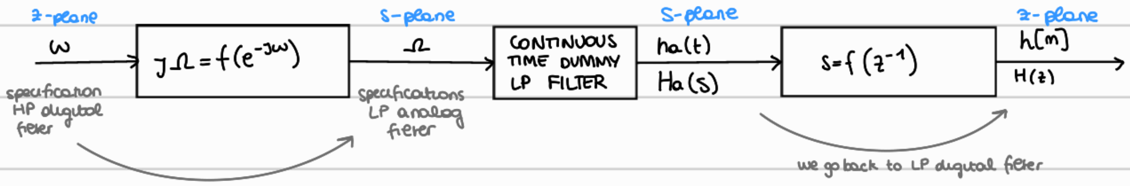
$\longrightarrow K = \prod_{0}^{N-1}{}_k (-p_k)$

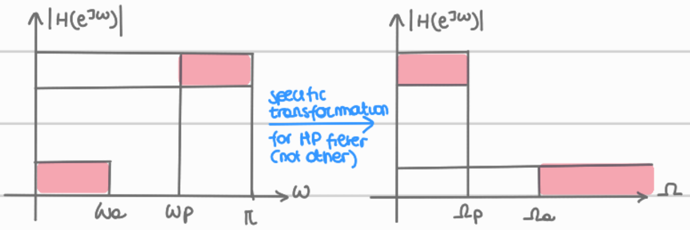
(3) $H(z) = H_a(s)\Big|_{s = \frac{2}{T_d}\frac{1-z^{-1}}{1+z^{-1}}} = \dfrac{\sum_{0}^{M}{}_k\, b_k z^{-k}}{\sum_{0}^{N}{}_k\, a_k z^{-k}}$

DESIGN OF NON-LP FILTER

#1: Bilinear transformation

LIMITED TO HP FILTER





BILINEAR TRANSFORMATION:

$$\boxed{\begin{aligned} s &= \frac{2}{T_d}\,\frac{1+z^{-1}}{1-z^{-1}} \\ z &= \frac{2+T_d s}{-2+T_d s} \end{aligned}}$$

freq response and stability are preserved

Im axis: $s = j\Omega \longrightarrow z = \frac{2+j\Omega T_d}{-2+j\Omega T_d} = e^{j\omega}$

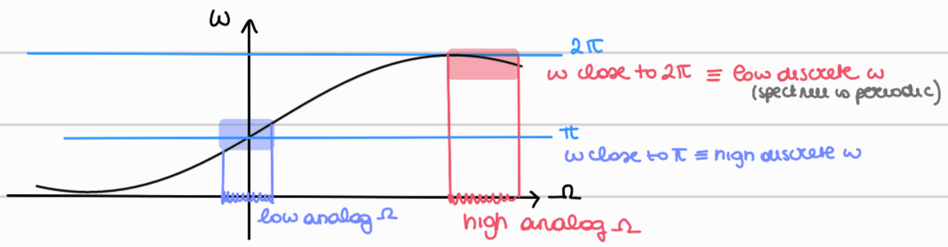
$\omega = \text{arctg}\left(\frac{\Omega T_d}{2}\right) - \text{arctg}\left(\underbrace{-\frac{\Omega T_d}{2}}_{Re\{\}<0}\right)$

$= 2\,\text{arctg}\left(\frac{\Omega T_d}{2}\right) \pm \pi$

$$\boxed{\begin{aligned} \omega &= 2\,\text{arctg}\left(\frac{\Omega T_d}{2}\right) + \pi \\ \Omega &= -\frac{2}{T_d}\,\text{cotg}\left(\frac{\omega}{2}\right) \end{aligned}}$$

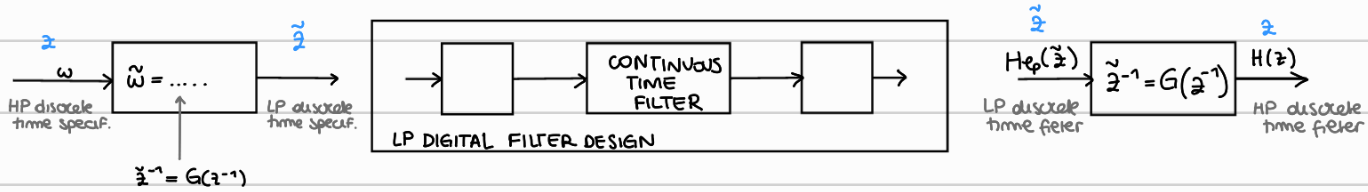
$tg(-x) = -tg(x)$

$\text{cotg}(x) = -tg\left(x + \frac{\pi}{2}\right)$



#2

NOT LIMITED TO HP FILTERS need of transformation btw non-LP and LP digital filter



1. $G(z^{-1})$ rational function of z^{-1} — transformation btw z and $\tilde{z}$ must preserve the form of polynomial ratio for HP digital filter
2. Preserve stability: $|\tilde{z}| < 1 \iff |z| < 1$ — both stable filter (inner part of circle)
3. Preserve Fourier analysis (freq. response): unit circles must be associated — $\tilde{z} = e^{j\tilde{\omega}}$ on unit circle in $\tilde{z}$ plane, $z = e^{j\omega}$ on unit circle in z plane

$\tilde{z}^{-1} = e^{-j\tilde{\omega}} = G(z^{-1}) = G(e^{-j\omega})$

$= \underbrace{|G(e^{-j\omega})|}_{\substack{\text{we need } = 1,\ \forall\omega \\ \text{as } |e^{-j\omega}| = 1,\ \forall\tilde{\omega}}}\, e^{j\,\underbrace{\arg(G(e^{-j\omega}))}_{\substack{-\tilde{\omega} = \arg(G(e^{-j\omega})) \\ \tilde{\omega} = -\arg(G(e^{-j\omega}))}}}$

$\Longrightarrow \begin{cases} |G(e^{-j\omega})| = 1 \quad \forall\omega & \text{SAME MODULUS} \\ \tilde{\omega} = -\text{arctg}(G(e^{-j\omega})) & \text{SAME PHASE} \end{cases}$

$$\boxed{\tilde{z}^{-1} = G(z^{-1}) = \pm \prod_{1}^{N} {}_k\, \frac{z^{-1} - \alpha_k}{1 - \alpha_k z^{-1}} \qquad |\alpha_k| < 1 \text{ for stability}}$$

⟶ changing N and α_k, we obtain the NON LP FILTER

Example: HP FILTER $\quad \underset{\text{LP variable}}{\tilde{z}^{-1}} = \frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$ (HP variable: z^{-1}) where $\cos\alpha = \frac{\cos\left(\frac{\tilde{\omega}_p + \omega_p}{2}\right)}{\cos\left(\frac{\tilde{\omega}_p - \omega_p}{2}\right)}$

$0 \le \tilde{\omega}_p \le \pi$ LP
$0 \le \omega_p \le \pi$ HP } specif. for the type of filter

$\tilde{\omega}_p = \pi - \omega_p$ (we start from digital HP to digital LP filter choosing $\tilde{\omega}_p = \pi - \omega_p$)

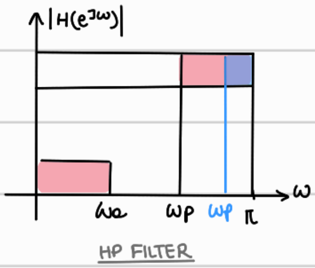
$$\cos\left(\frac{\tilde\omega_p+\omega_p}{2}\right) \overset{!}{=} \cos\left(\frac{\pi}{2}\right)=0 \longrightarrow \cos(\alpha)=0 \longrightarrow \alpha=0$$

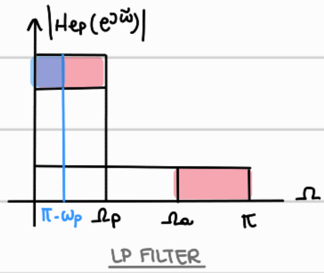
$\tilde\omega_p=\pi-\omega_p$

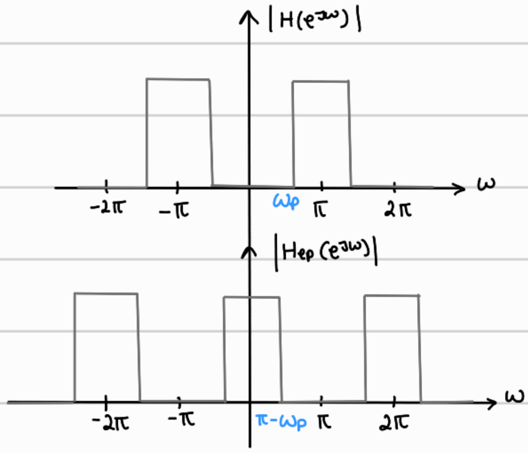
Thus, we obtain $\tilde z^{-1}=z^{-1}$ relationship btw $\tilde z$ and z

$$\tilde\omega = -\arg\big(\underbrace{G(e^{-j\omega})}_{z^{-1}}\big) = \arg\left(-e^{-j\omega}\right) = -\arg\left(e^{j\pm\pi}e^{-j\omega}\right) = \pm\pi+\omega$$

HP FILTER shifted by π







IIR FILTERS
- more selective with low order $\Rightarrow$ faster
- specification only on magnitude $\Rightarrow$ phase distorsion

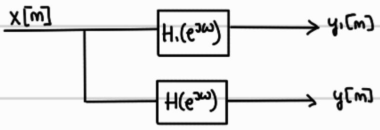
opposite for FIR FILTER

IDEAL LP FILTER: $H_i(e^{j\omega}) = \begin{cases} 1 & |\omega|\le\omega_t \\ 0 & |\omega|>\omega_t \end{cases}$ $\omega\in[-\pi,\pi]$

phase $=0$
gain $=1$ in passband
NO DISTORSION

REAL FILTER: $H(e^{j\omega}) = H_i(e^{j\omega})\,e^{-j\omega k}$

PHASE TERM magnitude $=1$: LINEAR TERM
phase $=-\omega k$



relationship btw real and ideal output ?

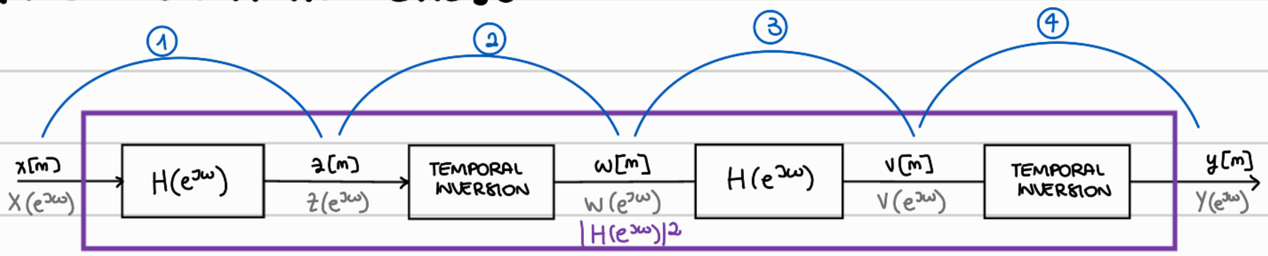
$$Y(e^{j\omega}) = H(e^{j\omega})\,X(e^{j\omega}) = H_i(e^{j\omega})\,e^{-j\omega k}\,X(e^{j\omega}) = Y_i(e^{j\omega})\,e^{-j\omega k} \longrightarrow \underbrace{y[m]}_{\text{real output}} = \underbrace{y_i[m-k]}_{\text{ideal output shifted by k samples}}$$

LINEAR PHASE $\neq$ null phase
(FIR FILTER)

Good output since rigid shift

DISTORSION if non linear phase (IIR FILTER)

IIR Filter with null phase



$z[m]=x[m]*h[m]$ $\quad$ $w[m]=z[-m]$ $\quad$ $v[m]=w[m]*h[m]$ $\quad$ $y[m]=v[-m]$

1 $\quad Z(e^{j\omega}) = H(e^{j\omega})\,X(e^{j\omega})$

2 $\quad z[-m] \Rightarrow Z(e^{-j\omega})$ $\quad W(e^{j\omega}) = Z(e^{-j\omega}) = Z^*(e^{j\omega}) = H^*(e^{j\omega})X^*(e^{j\omega})$

real signal

③ $V(e^{j\omega}) = H(e^{j\omega})\,W(e^{j\omega}) =$

$= \underbrace{H(e^{j\omega})H^*(e^{j\omega})}\,X^*(e^{j\omega}) =$

$= |H(e^{j\omega})|^2\, X^*(e^{j\omega})$

④ $Y(e^{j\omega}) = V(e^{-j\omega}) =$

real signal

$= V^*(e^{j\omega}) =$

$= |H(e^{j\omega})|^2\, X(e^{j\omega})$

$$\boxed{Y(e^{j\omega}) = |H(e^{j\omega})|^2\, X(e^{j\omega})}$$

DESIGN OF FIR FILTERS

for all types of filters

- more expensive → ↑ coefficients wrt IIR with specifications (to reach same performance)
- INTRINSICALLY STABLE No AR part → no poles
- LINEAR PHASE by design → NO PHASE DISTORSION

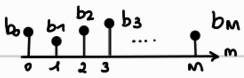
I/O RELATIONSHIP: $y[m] = \sum_{0}^{M}{}_k\, b_k\, x[n-k]$ — b_k: weights

- weighted sum of current and past inputs
- causal

$x[m] = \delta[m]$ — particular input to define impulse response

$h[m] = \sum_{0}^{M}{}_k\, b_k\, \delta[m-k] =$

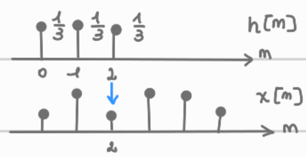
$= b_0\,\delta[m] + b_1\,\delta[m-1] + \ldots + b_M\,\delta[m-M]$ ($\equiv$ coefficients)



FINITE LENGTH $= M+1$

CAUSAL FIR FILTER

Example: $M=2$

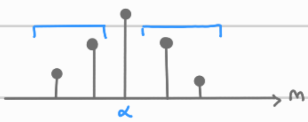


moving average of current and past 2 inputs

THEOREM If coefficients b_k (impulse response) are SYMMETRIC: $b_k = b_{M-k}$ (real value)

$\Longrightarrow$ FIR filter has LINEAR PHASE

sufficient condition



SEMPLICITY M EVEN $\longrightarrow$ $\alpha = \frac{M}{2}$ (integer)

$M+1$ ODD

CENTRAL COEFF + 2 SIDES of equal length

$H(e^{j\omega}) = \sum_{0}^{M}{}_k\, b_k\, e^{-j\omega k}$ — $h[k]$ = coefficients

split 2 in 3 parts

$= \underbrace{\sum_{0}^{\alpha-1}{}_k\, h[k]\,e^{-j\omega k}}_{\text{left side}} + \underbrace{h\left[\tfrac{M}{2}\right] e^{-j\omega\frac{M}{2}}}_{\text{central sample}} + \underbrace{\sum_{\alpha+1}^{M}{}_k\, h[k]\,e^{-j\omega k}}_{\text{right side}}$

→ hp of symmetric coeff. $h[k] = h[M-k]$

$\sum_{\alpha+1}^{M}{}_k\, h[M-k]\,e^{-j\omega k} \overset{CV}{=} \sum_{0}^{\alpha-1}{}_\ell\, h[\ell]\,e^{-j\omega(M-\ell)}$ — $\ell = M-k$, $k = M-\ell$

k	ℓ
$\alpha+1$	$M-(\alpha+1)$
M	0

$M-(\alpha+1) = M-\alpha-1 = M-\frac{M}{2}-1 = \frac{M-2}{2} = \frac{M}{2}-1 = \alpha-1$

CV: $\ell=k$ $= \sum_{0}^{\alpha-1}{}_k\, h[k]\,e^{-j\omega(M-k)}$

$= \sum_{0}^{\alpha-1}{}_k\, h[k]\,e^{-j\omega k} + \sum_{0}^{\alpha-1}{}_k\, h[k]\,e^{-j\omega(M-k)} + h\left[\tfrac{M}{2}\right]e^{-j\omega\frac{M}{2}} =$

collect $e^{-j\frac{M}{2}}$

$= e^{-j\omega\frac{M}{2}}\left[\sum_{0}^{\alpha-1}{}_k\, h[k]\underbrace{\left(e^{j\omega(\frac{M}{2}-k)} + e^{-j\omega(\frac{M}{2}-k)}\right)}_{2\cos(\omega(\frac{M}{2}-k))} + h\left[\tfrac{M}{2}\right]\right] =$

$= \underbrace{e^{-j\omega\frac{M}{2}}}_{\text{phase term}}\left[\underbrace{2\cos\left(\omega\left(\tfrac{M}{2}-k\right)\right) + h\left[\tfrac{M}{2}\right]}_{\text{real term}}\right] =$

$= e^{-j\omega\frac{M}{2}}\, A_H(\omega)$ DELAY OF $\frac{M}{2}$ SAMPLES where M real

LINEAR PHASE

If M ODD $\longrightarrow$ $\alpha = \frac{M}{2}$ not integer (not good)

$M+1$ EVEN

↳ no central term

Window Method

LP filter but for any types of filters
SEMPLICITY

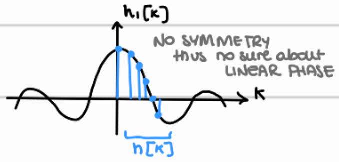
IDEAL LP FILTER: impulse response with ∞ length → WINDOWING of ∞ length ideal impulse response → SELECT window of M+1 SAMPLES → finite length sequence

$$h_i[k] = \begin{cases} \frac{\sin[\omega_t m]}{\pi k} & \text{for } k \neq 0 \\ \frac{\omega_t}{\pi} & \text{for } k=0 \end{cases}$$

$$w[k] = \begin{cases} 1 & \text{for } 0 \le k \le M \\ 0 & \text{otherwise} \end{cases}$$ M+1 samples

RECT. WINDOW

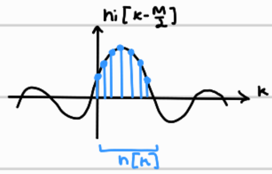
$h[k] = h_i[k]\, w[k]$ no good windowing



⟹ SHIFT OF $\frac{M}{2}$ of IDEAL RESPONSE ⊕ WINDOWING OF 1st (M+1) SAMPLES

$h[k] = h_i\left[k-\frac{M}{2}\right] w[k]$

FINITE and SIMMETRIC IMPULSE RESPONSE



IMPULSE RESPONSE OF:
- CAUSAL FIR FILTER: $h[k]=0$ for $k<0$
- LINEAR PHASE
- output in time will be shifted of $\frac{M}{2}$ samples (already proved)

What happens in freq domain? comparison of TRANSFER FUNCTIONS of ideal and real cases

CONVOLUTION of what?

$h[k] = h_i\left[k-\frac{M}{2}\right] w[k]$ product in time

btw DTFT

$H_i(e^{j\omega})\, e^{-j\omega\frac{M}{2}}$ $\begin{cases} 1 & \text{for } |\omega| \le \omega_t \\ 0 & \text{otherwise} \end{cases}$ for $\omega \in [-\pi, \pi]$ (period)

$$W(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} \frac{\sin\left(\frac{\omega}{2}N\right)}{\sin\left(\frac{\omega}{2}\right)} = e^{-j\omega\frac{M}{2}} \underbrace{\frac{\sin\left(\frac{\omega}{2}(M+1)\right)}{\sin\left(\frac{\omega}{2}\right)}}_{\in \mathbb{R}}$$

$N-1=M$

$$= \underbrace{e^{-j\omega\frac{M}{2}}}_{\text{phase term}} \underbrace{A_w(e^{j\omega})}_{\text{magnitude}}$$

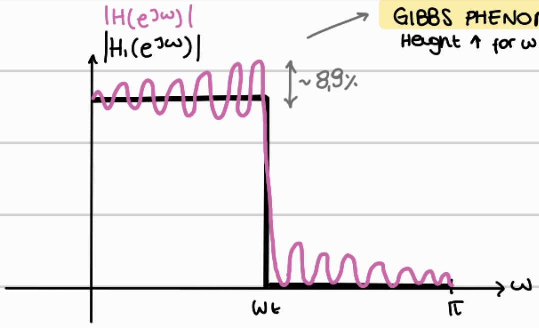
independency on θ

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{H_i(e^{j\theta})\, \cancel{e^{-j\theta\frac{M}{2}}}}_{\text{evaluated in } \theta \text{ (dummy variable)}} \underbrace{e^{-j(\omega-\cancel{\theta})\frac{M}{2}} A_w(e^{j(\omega-\theta)})}_{\text{evaluated in } \omega-\theta} d\theta =$$

$H_i = 1$ for $|\omega| \le \omega_t$
$H_i = 0$ outside

$$= \frac{1}{2\pi} e^{-j\omega\frac{M}{2}} \int_{-\omega_t}^{\omega_t} \underbrace{A_w\left(e^{j(\omega-\theta)}\right)}_{\in \mathbb{R}} d\theta$$

again: split btw real term and phase term



GIBBS PHENOMENON
Height ↑ for $\omega \to \omega_t$

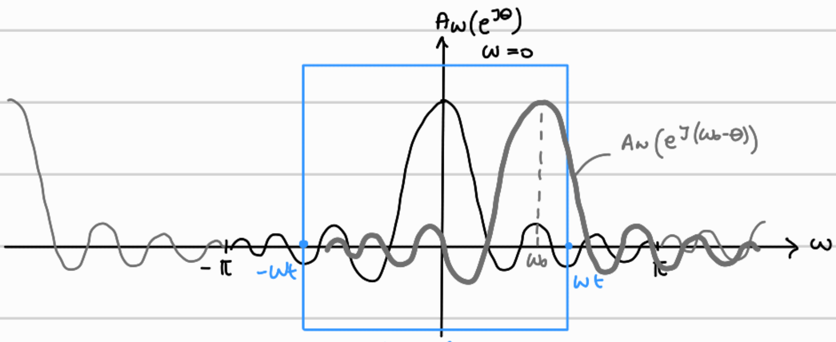
- no equiripples
- if M↑ real → ideal filter but oscillatory component close to transition is always present and equal to 8,9% ca.

effect of troncation decrease

$$\int_{-\omega_t}^{\omega_t} A_w\left(e^{j(\omega-\theta)}\right) d\theta$$

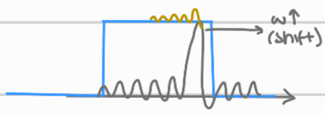
where $A_w = \dfrac{\sin\left(\omega\frac{M+1}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$

- EVEN SYMMETRY: $A_w(e^{-j\theta}) = A_w(e^{j\theta})$
- centered in zero, with MAIN and SIDE LOBES
- PERIODIC of 2π



window always the same

- **HAMMING**: low lateral lobes → reduced overshoot (instead of rect) → low transition band's width
- M↑, main lobe ↓ ⟹ width transition ↓ but overshoot is the same; more delay in time



some side lobes go inside and other go outside

When $\omega \simeq \omega_t$, main lobe goes outside the integration while several side lobes go inside
⟹ result of integral = oscillatory component

- **KAISER WINDOW**: family of window α: parameter to pass from hard to smoother window