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STAT 5810 Homework #4:

I worked with:	Jake, Mina	on	#1a just on the limits of integration & how to get to u & v graphically.
	Logan	on	#1a and 1b, discussed 3+4 briefly,
	Ari	on	#1a, 3a,b,d, 4, some of 5
I got help from	Andres	on	#5 (how solution space ≠ Range), 6
	Dr Moon	on	understanding solution space and some clarifying questions with Linear Mappings.

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① a) Compute the following integrals

$$\iint_S (x+y)e^{x-y}\,dx\,dy$$

Where S the region inside the triangle with vertices $(0,0)$, $(-1,1)$, and $(1,1)$. First, find the integral directly using the variables x, y. Then compute it by performing the following substitution: $x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$.

b) Given a vector $\vec{x} = (x_1, x_2)$ find the following integral:

$$\iint_S \vec{x}\,(x_1^2 + x_1 x_2^2)\,dx_1\,dx_2$$

where S is the region inside the rectangle with vertices $(0,0)$; $(0,1)$; $(1,0)$; $(1,1)$.

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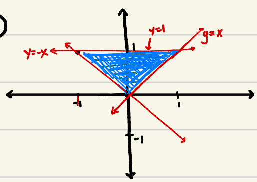
a)



$$\int_0^1 \int_{-y}^{y} (x+y)e^{x-y}\,dx\,dy$$

$$\Rightarrow \int_0^1\int_{-y}^{y} xe^{x-y}\,dx\,dy \quad + \int_0^1\int_{-y}^{y} ye^{x-y}\,dx\,dy$$

$u = x$ $\quad dv = e^{x-y}\,dx$

$du = dx$ $\quad v = e^{x-y}$

$$+ \int_0^1 y \int_{-y}^{y} e^{x-y}\,dx\,dy$$

$$\int_0^1 \left[xe^{x-y} - \int_{-y}^{y} e^{x-y}\,dx \right] dy$$

$$+ \int_0^1 y \int_{-y}^{y} e^{-y}e^{x}\,dx\,dy$$

$$\int_0^1 \left[xe^{x-y} - e^{x-y} \Big|_{-y}^{y} \right] dy$$

$$+ \int_0^1 y\cdot e^{-y}(e^{y} - e^{-y})\,dy \Rightarrow ye^{-y}(e^{y}-e^{-y}) : y - ye^{-2y}$$

$$\int_0^1 \overset{y\cdot e^0}{ye^{y-y}} - \overset{e^0}{e^{y-y}} - (-ye^{-y-y} - e^{-y-y})\,dy \quad + \int_0^1 y - ye^{-2y}\,dy \Rightarrow \ldots + \int_0^1 y\,dy - \int_0^1 e^{-2y}\,dy$$

$$\int_0^1 y\,dy - \int_0^1 1\,dy - \int_0^1 ye^{-2y} - e^{-2y}\,dy \quad + \quad \frac{1}{2} - \frac{e^2+3}{4e^2} \quad - (\cancel{0} + \cancel{\tfrac{1}{2}e^{-2(0)}})$$

$u = y$ $\quad dv = e^{-2y}$

$du = dy$ $\quad v = -\frac{1}{2}e^{-2y}$

$$\frac{1}{2}e^{-2y}\cdot y + \frac{1}{2}\int_0^1 ye^{-2y}\,dy \quad - \frac{1}{2}\left(1 - \frac{1}{e^2}\right)$$

$$\frac{-e^2+5}{2} - 4$$

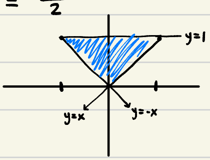
$$\frac{1}{2} - 1 \rightarrow \cancel{\tfrac{1}{2}} - \frac{-3e^2+5}{4\cancel{e}^2} + \cancel{\tfrac{1}{2}} - \frac{\cancel{e}^2+3}{4e^2} = \boxed{\frac{1-e^2}{2}}$$

Next Page for U-V substitution.

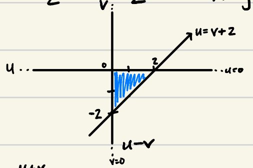
$$X = \frac{u+v}{2} \qquad x+y = \frac{u+v}{2} + \frac{u-v}{2} \Rightarrow x+y = u$$

$$y = \frac{u-v}{2} \qquad x-y = \frac{u-v}{2} - \frac{(u-v)}{2} \Rightarrow x-y = v$$

$$\iint (u)\cdot e^{v} \cdot \text{Jacobian}$$



$x - y$



Jacobian: $\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac12 & \frac12 \\ \frac12 & -\frac12 \end{bmatrix}$

$\det(\text{Jacobian}) = \frac12\left(-\frac12\right) - \frac12\left(\frac12\right) = -\frac24 = -\frac12$

$\left|-\frac12\right| = \frac12$

$y = x \qquad \frac{u-v}{2} = \frac{u+v}{2} \to u-v = u+v \Rightarrow 2v=0 \Rightarrow \boxed{v=0}$

$y = -x \qquad \frac{u-v}{2} = \frac{-(u+v)}{2} \to u-v = -u-v \to u = -u \to 2u = 0 \to \boxed{u=0}$

$y = 1 \qquad \frac{u-v}{2} = 1 \to u-v = 2 \to \boxed{u = v+2} \quad v = u-2$

$$\int_0^2\int_0^{v+2} u\cdot e^{v}\cdot \frac12\,dudv = \int_{-2}^{0}\int_0^{v+2} \frac12 u e^{v}\,du\,dv \Rightarrow \int_0^2 e^{v}\int_0^{v+2} \frac12 u\,dudv \Rightarrow \frac12\left(\frac12 u^2\right) = \frac14 u^2\Big|_0^{v+2} = \frac14\left((v+2)^2 - 0^2\right)$$

$$\Rightarrow \frac14\int_{-2}^{0} e^{v}(v+2)^2\,dv \qquad \begin{matrix} x = v+2 & w = x^2 & dz = e^{v} \\ dx = dv & dw = 2x\,dx & z = e^{v}\end{matrix}$$

$$\Rightarrow \frac14\left(x^2e^{v} - \int_{-2}^{0} e^{v}2x\,dx\right) \qquad \begin{matrix} w_1 = 2x & dz_1 = e^{v} \\ dw_1 = 2 & z_1 = e^{v}\end{matrix}$$

$$\Rightarrow \frac14\left(x^2e^{v} - 2xe^{v} - \int_{-2}^{0} e^{v}\cdot 2\,dv\right)$$

$$\Rightarrow \frac14\left((v+2)^2e^{v} - 2(v+2)e^{v} - 2e^{v}\right)\Big|_{-2}^{0}$$

$$\Rightarrow \frac14\cdot 4e^{0} - \frac14\cdot 4e^{0} - \frac14\cdot 2e^{0} - \left(\frac14\cdot\overset{0}{(-2+2)^2}\cdot e^{-2} - \frac14\cdot 2(-2+2)e^{-2} - \frac14\cdot 2e^{-2}\right)$$

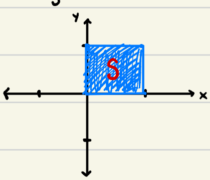
$$\Rightarrow \cancel{1} - \cancel{1} - \frac12 - \cancel{0} + \cancel{0} - \frac12 e^{-2}$$

$$\Rightarrow \frac{1-e^{-2}}{2}$$

b) $\iint_S \vec{x}\left(x_1^2 + x_1x_2^2\right)dx_1dx_2$

S is square inside $(0,0), (0,1), (1,0), (1,1)$

$\vec{x} = (x_1, x_2)$



Solution will be of form

$$\left(\iint x_1(x_1^2 + x_1x_2^2)dx_1dx_2,\ \iint x_2(x_1^2+x_1x_2^2)\,dx_1dx_2\right)$$

Next page for work for 1b.

$$\iint_S \vec{x}\,(x_1^2 + x_1 x_2^2)\,dx_1\,dx_2 \longrightarrow \int_0^1\int_0^1 x_1(x_1^2 + x_1x_2^2)\,dx_1\,dx_2 \quad , \quad \int_0^1\int_0^1 x_2(x_1^2 + x_1x_2^2)\,dx_1\,dx_2$$

$$\int_0^1\int_0^1 x_1^3 + x_1^2x_2^2\,dx_1\,dx_2 \quad , \quad \int_0^1\int_0^1 x_2x_1^2 + x_1x_2^3\,dx_1\,dx_2$$

$$\int_0^1 \frac{1}{4}x_1^4 + \frac{1}{3}x_1^3\cdot x_2^2\Big|_0^1 dx_2 \quad , \quad \int_0^1 \frac{1}{3}x^3x_2 + \frac{1}{2}x_1^2x_2^3\Big|_0^1 dx_2 \qquad \underbrace{\frac{1}{3}(1)^3\cdot x_2}_{\frac{1}{3}} + \underbrace{\frac{1}{2}(1)^2\cdot x_2^2}_{\frac{1}{2}} - \underbrace{\frac{1}{3}(0)^3\cdot x_2}_{0} - \underbrace{\frac{1}{2}(0)^2\cdot x_2^2}_{0}$$

$$\int_0^1 \frac{1}{4}(1)^4 + \frac{1}{3}(1)^3\cdot x_2^2 - \cancel{\frac{1}{4}(0)^4 + \frac{1}{3}(0)^3\cdot x_2^2}\,dx_2 \quad , \quad \int_0^1 \frac{1}{3}x_2 + \frac{1}{2}x_2^3\,dx_2$$

$$\int_0^1 \frac{1}{4} + \frac{1}{3}x_2^2\,dx_2 \quad , \quad \frac{1}{3}\int_0^1 x_2\,dx_2 + \frac{1}{2}\int_0^1 x_2^3\,dx_2$$

$$\frac{1}{4}x_2 + \frac{1}{9}x_2^3\Big|_0^1 \quad , \quad \frac{1}{3}\cdot\frac{1}{2}x_2^2\Big|_0^1 + \frac{1}{2}\cdot\frac{1}{4}x_2^4\Big|_0^1 \rightarrow \frac{1}{6} + \frac{1}{8} - 0$$

$$\frac{1}{4}(1) + \frac{1}{9}(1)^3 - 0 + 0 \quad , \quad \frac{1}{6} = \frac{4}{24} \qquad \frac{1}{8} = \frac{3}{24} \rightarrow \frac{4}{24} + \frac{3}{24} = \underline{\frac{7}{24}}$$

Answer $= \left(\frac{13}{36}, \frac{7}{24}\right)$

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② <u>Solve 2.3 from the book:</u>

Consider the set G of 3x3 matrices defined as follows:

$$G = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times3} \;\middle|\; x, y, z \in \mathbb{R} \right\}$$

We define $\cdot$ as the standard matrix multiplication. Is $(G, \cdot)$ a group? If yes, is it Alebian? Justify your answer.

Check 1) closure under $\cdot$ $\forall x, y \in G$: $x \cdot y = G$ 2) Associativity: $\forall x, y, z \in G$: $(x\cdot y)\cdot z = x\cdot(y\cdot z)$ 3) Neutral: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in G$

4) Inverse element 5) Communtivity (Alebian)

① For G, As a matrix $\in \mathbb{R}^{3\times3}$ this matrix G is closed under multiplication as it is element wise with scalar multiplication, or Matrix multiplication,

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}, \quad i = 1, \ldots, m;\; j = 1, \ldots, k.$$

② $(A\cdot B)\cdot C = A\cdot(B\cdot C)$ follows from the above definition of matrix multip

③ The Neutral element for $\mathbb{R}^{3\times3}$ is

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and when multiplied by G $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$, so the Identity $\in \mathbb{R}^{3\times3}$ is the Neutral Element.

④ Inverse. $\left[\begin{array}{ccc|ccc} 1 & x & z & 1 & 0 & 0 \\ 0 & 1 & y & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow$ Reduce to RREF $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -x & -z+xy \\ 0 & 1 & 0 & 0 & 1 & -y \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

$$\therefore\; G^{-1} = \begin{bmatrix} 1 & -x & -z+xy \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, G is a Group. To see if G is Alebian G has to be Communitive, or $G\cdot A = A\cdot G$

⑤ $A = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$ $\quad G = \begin{bmatrix} 1 & c & d \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$ $\quad G\cdot A = \begin{bmatrix} 1 & x+c & by+d+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix}$

$A\cdot G = \begin{bmatrix} 1 & x+c & xb+d+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix}$ $\qquad A\cdot G \neq G\cdot A$

So $(G, \cdot)$ is not <u>Alebian, but is a group</u>

Next page for 3.

③ Solve exercise 2.9 in the book. Provide a justification for your answer.

Which of the following sets are subspaces of $\mathbb{R}^3$?

a) $A = \{(\lambda, \lambda+\mu^3, \lambda-\mu^3) \mid \lambda, \mu \in \mathbb{R}\}$

b) $B = \{(\lambda^2, -\lambda^2, 0) \mid \lambda \in \mathbb{R}\}$

c) Let $\gamma \in \mathbb{R}$. $C = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_1 - 2\xi_2 + 3\xi_3 = \gamma\}$

d) $D = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_2 \in \mathbb{Z}\}$

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a) 1) does A contain Neutral element?

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in A$, $\lambda = 0$ $\mu = 0$, $0 \in \mathbb{R}$, this is true.

2) Closed under scalar multiplication?

Let $\gamma \in \mathbb{R}$ and we have some vector in A as defined below:

$$\gamma \begin{bmatrix} \lambda \\ \lambda+\mu^3 \\ \lambda-\mu^3 \end{bmatrix} = \begin{bmatrix} \gamma\lambda \\ \gamma\lambda + \gamma\mu^3 \\ \gamma\lambda - \gamma\mu^3 \end{bmatrix}$$

$$\begin{bmatrix} \gamma\lambda \\ \gamma\lambda + \gamma\mu^3 \\ \gamma\lambda - \gamma\mu^3 \end{bmatrix} \in A$$ with $\lambda' = \gamma\lambda$ and $\mu' = \sqrt[3]{\gamma} \cdot \mu$

Thus A is closed under scalar multiplication.

3) Is A closed under vector addition?

$$\vec{a} = \begin{bmatrix} \lambda \\ \lambda+\mu^3 \\ \lambda-\mu^3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} \psi \\ \psi+\eta^3 \\ \psi-\eta^3 \end{bmatrix} \quad \vec{a}+\vec{b} = \begin{bmatrix} \lambda+\psi \\ \lambda+\mu^3+\psi+\eta^3 \\ \lambda-\mu^3+\psi-\eta^3 \end{bmatrix}$$

Let $\lambda' = \lambda + \psi$ $\mu' = \sqrt[3]{\mu^3 + \eta^3}$ (only the Real part). Thus, A is closed under vector addition. Therefore, A is a subspace for $\mathbb{R}^3$.

Next page for b.

b) $B = \{(\lambda^2, -\lambda^2, 0) \mid \lambda \in \mathbb{R}\}$

1) Neutral element $\in B$ when $\lambda = 0$.

2) scalar multiplication

2 cases $+\psi$, $-\psi$

$$\psi \begin{bmatrix} \lambda^2 \\ -\lambda^2 \\ 0 \end{bmatrix} = \begin{bmatrix} \psi\lambda^2 \\ -\psi\lambda^2 \\ 0 \end{bmatrix} \in B \qquad -\psi \begin{bmatrix} \lambda^2 \\ -\lambda^2 \\ 0 \end{bmatrix} = \begin{bmatrix} -\psi\lambda^2 \\ \psi\lambda^2 \\ 0 \end{bmatrix}$$

$\lambda' = \sqrt{\psi}$

in this case λ' cannot be in a negative form in the $\mathbb{R}$ that yields something in the proposed subspace

Thus, B is not a subspace.

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c) Let $r \in \mathbb{R}$. $C = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_1 - 2\xi_2 + 3\xi_3 = r\}$

Let $x_1 = \xi_1$, $\xi_2 = x_2$, $\xi_3 = x_3 \Rightarrow x_1 - 2x_2 + 3x_3 = r$

1	-2	3	
-1	-2	+3	=0
(·-1)	·1	·1	

$\gamma = [-1, 1, 1]^T$

1) The neutral element vector $\in C$ when $\gamma = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ $r \in C$

There are no other values of γ that yield the $\vec{0}$.

2) Scalar multiplication:

for some $\psi \in \mathbb{R}$

$$\psi \cdot \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} \psi\xi_1 \\ \psi\xi_2 \\ \psi\xi_3 \end{bmatrix},$$ this is true for any value of $\psi \in \mathbb{R}$ for the vector to be in C.

Thus, there is closure under scalar multiplication.

3) vector addition:

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} + \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} \xi_1 + \lambda_1 \\ \xi_1 + \lambda_2 \\ \xi_3 + \lambda_3 \end{bmatrix},$$ let $\xi + \lambda = \Delta$, then $$\begin{bmatrix} \xi_1 + \lambda_1 \\ \xi_2 + \lambda_2 \\ \xi_3 + \lambda_3 \end{bmatrix} = \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix}$$ and $\Delta \in C$.

Thus, C is a subspace.

Next page for d.

d) $\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_2 \in \mathbb{Z}\}$

1) Neutral element $0 \in \mathbb{R}$ + $0 \in \mathbb{Z}$.

2) Scalar multiplication

$$\psi \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} \psi \xi_1 \\ \psi \xi_2 \\ \psi \xi_3 \end{bmatrix}$$

$$\psi = \frac{a}{b} \text{ or } \psi = \pi \text{ or } \psi = \sqrt{a} \not\subseteq \mathbb{Z}$$

Thus D is not closed under scalar multiplication since $\xi_2 \in \mathbb{Z}$.

Therefore, D is not a subspace of $\mathbb{R}^3$.

2.10:

4) Are the following sets of vectors linearly independant?

a) $x_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $x_3 = \begin{bmatrix} 3 \\ -3 \\ 8 \end{bmatrix}$

$\text{Ref}\left(\begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & -3 \\ 3 & -2 & 8 \end{bmatrix}\right)$ I used software to calculate, and got : $\begin{bmatrix} 3 & -2 & 8 \\ 0 & -\frac{7}{3} & \frac{7}{3} \\ 0 & 0 & 0 \end{bmatrix}$

Due to Rank being 2 and the dimensions is 3, this is not linearly independant.

b) $x_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ $x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

$$\text{Ref}\left(\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_1, x_2, x_3 are linearly independant but it is not a complete basis set for $\mathbb{R}^5$

Next page for 5

⑤ Consider two subspaces U_1 and U_2, where U_1 is the solution space for the homogenous equation system $A_1 x = 0$ and U_2 is the solution space of the homogeneous system $A_2 x = 0$ with

$$A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 1 & -1 & 2 \end{bmatrix}$$

a) Determine the dimension of U_1, U_2.

RRef forms found using MatLab and verified on a calculator.

$$\text{RRef}(A_1) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which means the system to solve for the solution space U_1:

$$x_1 + x_3 = 0$$
$$x_2 + x_3 = 0$$

$$x_1 + x_3 = x_2 + x_3$$
$$-x_3 \qquad -x_3$$
$$x_1 = x_2$$

$$x_1 + x_3 = 0$$
$$x_1 = -x_3$$

The $\underline{\underline{\dim(U_1) = 1}}$.

The vector $\in U_1$ is $\begin{bmatrix} \alpha \\ \alpha \\ -\alpha \end{bmatrix}$, $\alpha \in \mathbb{R}$

$$\text{RRef}(A_2) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which means the system to solve for the solution space U_1:

$$x_1 + x_3 = 0$$
$$x_2 + x_3 = 0$$

$$x_1 + x_3 = x_2 + x_3$$
$$-x_3 \qquad -x_3$$
$$x_1 = x_2$$

$$x_1 + x_3 = 0$$
$$x_1 = -x_3$$

The $\underline{\underline{\dim(U_2) = 1}}$.

The vector $\in U_1$ is $\begin{bmatrix} \alpha \\ \alpha \\ -\alpha \end{bmatrix}$ $\alpha \in \mathbb{R}$

b) Determine the Bases of U_1 + U_2.

The basis of $U_1 = \begin{bmatrix} \alpha \\ \alpha \\ -\alpha \end{bmatrix}$ = The basis of U_2, $\alpha \in \mathbb{R}$.

c) Determine a basis for $U_1 \cap U_2$.

The basis of $U_1 \cap U_2 = \begin{bmatrix} \alpha \\ \alpha \\ -\alpha \end{bmatrix}$.

Next page for 6.

⑥ 2.16: Are the following mappings linear? Justify your answer.

a) Let $a,b \in \mathbb{R}$.

$$\Phi : L^1([a,b]) \longrightarrow \mathbb{R}$$
$$f \mapsto \Phi(f) = \int_a^b f(x)\,dx$$

where $L^1([a,b])$ denotes the set of integreble functions on $[a,b]$.

b)
$$\phi : C^1 \to C^0 \quad , \quad f \mapsto \phi(f) = f' ,$$

where for $k \geq 1$, C^k denotes the set of k times continuously differentiable functions, and C^0 denotes the set of continuous functions.

c)
$$\phi : \mathbb{R} \to \mathbb{R} \quad , \quad x \mapsto \phi(x) = \cos(x)$$

d)
$$\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \quad , \quad \vec{x} \mapsto \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} \vec{x}$$

e) Let θ be in $[0, 2\pi]$ and

$$\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\vec{x} \longmapsto \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \vec{x}$$

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a) If the mapping proposed is linear, the following relations are true:

$$\phi(x+y) = \phi(x) + \phi(y) \qquad \text{and} \qquad \phi(\lambda x) = \lambda \phi(x)$$

ϕ here denotes $\int_a^b f(x)\,dx$

so

$$\phi(x+y) = \phi(x) + \phi(y) \checkmark$$

$\int_a^b (f+g)\,dx = \int_a^b f\,dx + \int_a^b g\,dx$ holds since summing 2 functions doesn't change if I integrate 1 at a time and then add them together or if I integrate the sum, the integrated value will be the same.

$$\phi(\lambda x) = \lambda \phi(x)$$

$\int_a^b \lambda \cdot f\,dx$ by properties of integration I can pull out the scalar λ to be applied after the integration,

$= \lambda \cdot \int_a^b f\,dx$

Thus, $\phi : L^1[a,b] \longrightarrow \mathbb{R}$, $f \mapsto \phi(f) = \int_a^b f(x)\,dx$ is a Linear Mapping.

b) $\phi(f) = f' = \frac{df}{dx}$

$$\phi(f+g) = \frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx} \checkmark$$

$$\phi(\lambda f) = \frac{d(\lambda f)}{dx} = \frac{\lambda df}{dx} = \lambda \cdot \frac{df}{dx} \checkmark$$

Thus, $\phi(f) = f'$ is a linear Mapping.

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c) $\phi : \mathbb{R} \to \mathbb{R}$, $x \mapsto \phi(x) = \cos(x)$

$\phi(x+y) = \cos(x+y)$ But $\cos(x+y) \neq \cos(x) + \cos(y)$ because the inner value is being added then the cosine is taken. This is not the same as taking the cosine of 2 values and then adding those values.

$\phi(\lambda x) = \cos(\lambda x) \neq \lambda \cos(x)$

Thus, $\phi(x)$ is not a linear mapping.

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d) $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $\vec{x} \mapsto \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} \vec{x}$

$$\phi(\vec{x}) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \phi(\vec{x}) = \begin{cases} x_1 + 2x_2 + 3x_3 \\ x_1 + 4x_2 + 3x_3 \end{cases}$$

$$\phi(x+y) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} (\vec{x} + \vec{y}) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} \vec{y} \checkmark$$

$$\phi(\lambda x) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} (\lambda \vec{x}) = \lambda \cdot \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} \vec{x} \checkmark$$

This is a linear mapping.

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e) $\theta \in [0, 2\pi]$, $\phi(\vec{x}) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \vec{x}$

$$\phi(\vec{x}+\vec{y}) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} (\vec{x} + \vec{y})$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \vec{x} + \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \vec{y} \checkmark$$

$$\phi(\lambda \vec{x}) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} (\lambda \vec{x})$$

$$= \lambda \cdot \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \vec{x} \checkmark$$

Thus, $\phi(\vec{x})$ is a Linear Mapping.

Next page for 7.

7 **2.17**: Consider the linear mapping

$$\phi: \mathbb{R}^3 \rightarrow \mathbb{R}^4$$

$$\phi\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 - 3x_2 \\ 2x_1 + 3x_2 + x_3 \end{bmatrix}$$

A • Find the transformation matrix A_ϕ.

B • Determine the Rank of A_ϕ.

C • Compute the kernal and image of ϕ. What are $\dim(\ker(\phi))$ and $\dim(\text{Im}(\phi))$?

A:
$$\begin{bmatrix} 3x_1 + 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 - 3x_2 \\ 2x_1 + 3x_2 + x_3 \end{bmatrix} \longrightarrow$$

A_ϕ is the system of coefficients in the equation $A\vec{x} = 0$.

so

$$A_\phi = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

B: To determine the $\text{Rk}(A_\phi)$ I put A_ϕ into Reduced Row echelon form.

$$\text{RRef}(A_\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so $\text{Rk}(A_\phi) = 3$

C: Using $\text{RRef}(A_\phi)$ it is seen that $x_3 = 0$, $x_2 = 0$, $x_1 = 0$

so the kernal or solution space contains 1 vector, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Thus $\dim(\ker(\phi)) = 1$

For the image I looked at A:

$$\begin{bmatrix} 3x_1 + 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 - 3x_2 \\ 2x_1 + 3x_3 + x_3 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

so $\text{Im}(A_\phi) = \text{span}\left(\begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right)$

Thus $\dim(\text{Im}(\phi)) = 3$.

$\text{Rk}(A) = \dim(\text{Im}(\phi))$

$3 = 3$ ✓

Fin.