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STAT5810 Homework 5:

I went to Andres office hours and got help on 3, 3.5 b) and some thoughts on how to do 1 d).

I happened to be in there with Teresa, and with some of her questions, adjusted some of my answers.

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① Provide a proof for the following statements:

a) Any Norm is a convex function.

b) The orthogonal complement of a subspace is also a subspace.

c) The non-zero vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$, are linearly independant if $\langle \vec{v}_i, \vec{v}_j \rangle = 0$ for $i \neq j$

Hint: Assume they are linearly dependent and reach a contradiction.

d) Show that the norm induced by an inner product satisfies the properties of a norm.

Hint: You may need to use the Cauchy-Schwarz inequality when proving the triangle inequality.

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a) A function is called a Norm if it is non-negative everywhere, positive outside the origin,

is homogeneous: $\|tx\| = |t| \cdot \|x\|$, and satisfies the triangle inequality: $\|x+y\| \leq \|x\| + \|y\|$.

So if we let $\|x\|$ be a real-valued function $\in \mathbb{R}^n$ that is positively homogeneous,

$$\|tx\| = |t| \cdot \|x\| \quad \forall x \in \mathbb{R}^n,\ t \geq 0.$$

implies $\|x\|$ to be convex if and only if it is sub-additive

$$\|x+y\| \leq \|x+y\| \quad \forall x, y \in \mathbb{R}^n$$

Then it follows that

$$\|\lambda x + (1-\lambda)y\| \leq \|\lambda x\| + \|(1-\lambda)y\| \quad \text{(triangle inequality)}$$

$$= |\lambda| \cdot \|x\| + |1-\lambda| \cdot \|y\| \quad \text{(homogenarity)}$$

$$= |\lambda| \cdot \|x\| + |1-\lambda| \cdot \|y\| \quad \forall x, y \in \mathbb{R}^n,\ 0 \leq \lambda \leq 1.$$

Thus, the Norm is convex.

Next page for 1b.

b) If we have $W \subseteq V$, then the orthogonal complement is a subspace, call the orthogonal compliment $W^{\perp}$

Let $\vec{u}_1$ and $\vec{u}_2$ be vectors $\in W^{\perp}$, where $W^{\perp}$ is the orthogonal compliment of W, and W and V are vector spaces.

Then if $\vec{w} \in W$, and $c \in \mathbb{R}$

$$(\vec{u}_1 + \vec{u}_2) \cdot \vec{w} = \vec{u}_1 \cdot \vec{w} + \vec{u}_2 \cdot \vec{w} = \vec{0}$$

$$(c \cdot \vec{u}_1) \cdot w = c(\vec{u}_1 \cdot w) = c(0) = \vec{0}$$

The intersection of W and $W^{\perp}$ is zero. If $\vec{v}$ is in the intersection, then $\vec{v} \in W$ first then $\vec{v} \in W^{\perp}$. Thus,

$$\vec{v} \cdot \vec{v} = 0 \Rightarrow \vec{v} = \vec{0}.$$

Therefore, since $W^{\perp}$ is closed under vector multiplication with W and results in $\vec{0}$, and the intersection point of $(W \cap V) \cap W^{\perp}$ is $\vec{0}$. This shows

$W^{\perp}$ is also a subspace.

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c) The non-zero vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ are linearly independant if $\langle \vec{v}_i, \vec{v}_j \rangle = 0$ if $i \neq j$.

In this instance the inner product is not necissarily the dot product. Where $\langle \vec{v}_i, \vec{v}_j \rangle = 0$ makes this is is an orthogonal set. This means $\{\vec{v}_i\}$ are mutually orthogonal and non-zero means the following:

$$\langle \vec{v}_i, \vec{v}_j \rangle = \vec{v}_i^{T} \vec{v}_j = \vec{0} \quad \forall\, i \neq j, \text{ and } \langle \vec{v}_i, \vec{v}_i \rangle \neq 0$$

Suppose we have the relationship

$$a_1 \vec{v}_1 + a_2 \vec{v}_2 + \ldots + a_k \vec{v}_k = 0.$$

Then, taking the inner product of both sides with $\vec{v}$,

$$\langle \vec{v}_i, a_1 \vec{v}_1 \rangle + \ldots + \langle \vec{v}_i, a_i \vec{v}_i \rangle + \ldots + \langle \vec{v}_n, a_n \vec{v}_n \rangle = \langle \vec{v}_i, 0 \rangle = 0.$$

Using the mutual orthogonality, we see that $0 = \langle \vec{v}_i, a_i \vec{v}_i \rangle = a_i \langle \vec{v}_i, \vec{v}_i \rangle$. Due to $\langle \vec{v}_i, \vec{v}_i \rangle \neq 0$ it is seen that $a_i = 0$, where i is arbitrary, $a_i = 0 \ \forall\, i = 0, 1 \ldots$ Thus $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is a linearly independent set.

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d) There are 3 conditions needed to show the norm induced by an inner product satisfies the properties of a norm: 1) it must be absolutely homogeneous, 2) it must satisfy the triangle inequality, 3) it is positive definite.

1) Show its absolutely homogeneous.

Let $\lambda \in \mathbb{R}$. Then

$$\|\lambda x\| = \sqrt{\langle \lambda x, \lambda x \rangle} = |\lambda| \sqrt{\langle x, x \rangle} = |\lambda| \cdot \|x\|$$

Since $\|\lambda x\| = |\lambda| \cdot \|x\|$ this proves it is absolutely homogeneous.

Next page for 1d.2

2) Triangle Inequality

The Cauchy - Schwarz inequality states

$$|\langle x,y\rangle| \leq \|x\|\cdot\|y\| \qquad (a)$$

where $\|x\| = \sqrt{\langle x,x\rangle}$ which causes (a) to become

$$|\langle x,y\rangle| = \sqrt{\langle x,x\rangle}\cdot\sqrt{\langle y,y\rangle}$$

$$\Rightarrow \|x+y\|^2 = \|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2$$

$$\Rightarrow \|x+y\|^2 \leq \|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2$$

$$\Rightarrow \|x+y\|^2 \leq \|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2$$

$$\Rightarrow \|x+y\|^2 = (\|x\|+\|y\|)^2$$

$$\Rightarrow \sqrt{\|x+y\|^2} \leq \sqrt{(\|x\|+\|y\|)^2}$$

$$\Rightarrow \|x+y\| \leq \|x\| + \|y\| \qquad (b)$$

where (b) is the desired result.

3) Show it is Positive Definite. From the definition of inner product, the inner product is always >0, so $\|x\| \geq 0$ is true, but we need to show when $\|x\| = 0$.

Let $\|x\| = 0$. We want to show $\|x\|=0$. Then

$$\|0\| = \sqrt{\langle 0,0\rangle} \geq 0.$$

Thus, we have shown that the norm induced by an inner product satisfies the properties of a norm.

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② Compute the inner product of the following continuous functions defined as $\langle f(x), g(x)\rangle = \int_a^b f(x)\cdot g(x)\,dx$.

a) $f(x) = x^2+2x$, $g(x) = x+1$, $a=0$ and $b=1$

b) $f(x) = x$, $g(x) = 3x^2-1$, $a=-1$ and $b=1$

c) $f(x) = x$, $h(x) = 1$, $a=-1$ and $b=1$

d) What can you say about f, g, and h in parts b) and c), i.e. do they constitute a basis for the space of second order polynomials? Are they orthogonal vectors? If you changed a and b would that change the relationships between f, g, and h?

Next page for 2a

a) $\langle f(x), g(x)\rangle = \int_0^1 (x^2+2x)\cdot(x+1)\,dx = \int_0^1 x^3+3x^2+2x \; dx \Rightarrow \frac{1}{4}x^4 + x^3 + x^2 \Big|_0^1$

$= \left(\frac{1}{4}+1+1\right) - 0 = \frac{1}{4}+\frac{4}{4}+\frac{4}{4} = \underline{\underline{\frac{9}{4}}}$

b) $\langle f(x), g(x)\rangle = \int_{-1}^1 (x)\cdot(3x^2-1)\,dx = \int_{-1}^1 3x^3 - x \; dx = \frac{3}{4}x^4 - \frac{1}{2}x^2 \Big|_{-1}^1$

$= \left(\frac{3}{4}-\frac{1}{2}\right) - \left(\frac{3}{4}-\frac{1}{2}\right) = \cancel{\frac{3}{4}} - \cancel{\frac{1}{2}} - \cancel{\frac{3}{4}} + \cancel{\frac{1}{2}} = \underline{\underline{0}}$

c) $\langle f(x), h(x)\rangle = \int_{-1}^1 (x)\cdot(1)\,dx = \int_{-1}^1 x \; dx = \frac{1}{2}x^2 \Big|_{-1}^1 = \frac{1}{2} - \left(\frac{1}{2}\right) = \underline{\underline{0}}$

d) The vectors f, g, and h are linearly independent. There is no way to get each of these by a combination of the other two.

$f = x$, $g = 3x^2-1$, $h = 1$.

Consider $S = [1, x, x^2]^T$ be the standard basis of second order polynomial. Where the number of linear independent vectors is the same dimension as S so $\vec{f}, \vec{g}, \vec{h}$ $(x, 3x^2-1, 1)$ form a basis. These vectors w.r.t. S are

$$S_h = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad S_f = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad S_g = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}.$$

To check if orthogonal, $\langle f, g\rangle = 0$ $f \perp g$, $\langle g, h\rangle = 0$ $g \perp h$, $\langle f, h\rangle = 0$ $f \perp h$

in b it is shown $\langle f, g\rangle = 0$ thus $f \perp g$.

in c it is shown $\langle f, h\rangle = 0$ thus $f \perp h$.

so for all to be orthogonal it is needed to show $\langle g, h\rangle = 0$.

$$\langle g(x), h(x)\rangle = \int_{-1}^1 (3x^2-1)\cdot(1)\,dx = \int_{-1}^1 3x^2 - 1 \; dx = x^3 - x \Big|_{-1}^1 = (1-1)-(-1+1) = \cancel{1} - \cancel{1} + \cancel{1} - \cancel{1} = \underline{\underline{0}}.$$

Thus, they are orthogonal.

If we change the values of a and b even to just $a = 0$, $b = 1$, we do not have orthogonal vectors with the given f, g, and h. In part a, it is seen $\langle f, g\rangle \neq 0$,

$\langle f, h\rangle = \int_0^1 1 \cdot x \, dx = \frac{1}{2}x^2 \Big|_0^1 = \frac{1}{2} - 0 = \underline{\frac{1}{2}} \neq 0$, so h is not orthogonal to f.

$\langle g, h\rangle = \int_0^1 1 \cdot 3x^2 - 1 \, dx = x^3 - x \Big|_0^1 = (1-1) - 0 = 0$ so g is still orthogonal to h.

If we change a + b, if the value of $a = -b$ then f, g, h are orthogonal and form a complete basis. If $a \neq -b$, then there is no guarentee that f, g, h are orthogonal, even thougn they are linearly independent vectors in the space.

Next page for 3.

③ Consider an inner product given by $\langle\vec{x},\vec{y}\rangle_W = \vec{x}^T W\vec{y}$ where W is a symmetric PD matrix. The orthogonal projection matrix P_A onto the column space of a matrix A with respect to this inner product is given by

$$P_A = A(A^TWA)^{-1}A^TW.$$

a) Show that P_A is an idempotent matrix.

b) The matrix $I-P_A$ is the projection matrix for the orthogonal complement of the column space of A. Show that $I-P_A$ is orthogonal to P_A in the sense that $P_A^TW(I-P_A)=0$ (a matrix of 0's in all it's entries.)

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a) For P_A to be idempotent, $P_A^2 = P_A$.

$$P_A^2 = \left(A(A^TWA)^{-1}A^TW\right)\cdot\left(A(A^TWA)^{-1}A^TW\right)$$

$$= A(A^TWA)^{-1}\cdot A^TWA(A^TWA)^{-1}\cdot A^TW$$

$A^TWA\cdot(A^TWA)^{-1} = I$, identity matrix, so the above becomes

$$= A(A^TWA)^{-1}\cdot I\cdot A^TW$$

$$= A(A^TWA)^{-1}A^TW$$

$$= P_A$$

Thus, P_A is an idempotent matrix.

b) Show $P_A^TW(I-P_A)=0$

$$P_A^TW(I-P_A) = \left(A(A^TWA)^{-1}A^TW\right)^TW\left(I-A(A^TWA)^{-1}A^TW\right)$$

using $(AB)^T = A^TB^T$, this becomes

$$= \left(W^TA((A^TWA)^{-1})^TA^TW\right)\cdot\left(I-A(A^TWA)^{-1}A^TW\right)$$

$$= W^TA((A^TWA)^{-1})^TA^TW - W^TA((A^TWA)^{-1})^T\cdot A^TWA(A^TWA)^{-1}\cdot A^TW$$

$$= W^TA((A^TWA)^{-1})^TA^TW - W^TA((A^TWA)^{-1})^T\cdot \quad I \quad \cdot A^TW$$

$$= W^TA((A^TWA)^{-1})^TA^TW - W^TA((A^TWA)^{-1})^TA^TW$$

$$= 0$$

This shows $I-P_A$ is orthogonal to P_A.

Next page for 4.

(4) Solve 3.1, 3.2, 3.5 from the book.

$\boxed{3.1}$: Show that $\langle \cdot , \cdot \rangle$ defined for all $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$ by

$$\langle \vec{x}, \vec{y} \rangle := x_1 y_1 - (x_1 y_2 - x_2 y_1) + 2 x_2 y_2$$

is an inner product

A symmetric Positive Definite bilinear mapping is an inner product. This means it has to have Linearity, Symmetry, and be PD.

a) Linearity means $\langle \lambda \vec{x} + \psi \vec{y}, \vec{z} \rangle = \lambda \langle \vec{x}, \vec{z} \rangle + \psi \langle \vec{y}, \vec{z} \rangle$

$$\langle \lambda \vec{x} + \psi \vec{y}, \vec{z} \rangle = (\lambda x_1 + \psi y_1) z_1 - ((\lambda x_1 + \psi y_1) z_2 + (\lambda x_2 + \psi y_2) z_1) + 2((\lambda x_2 + \psi y_2) z_2)$$

$$= \lambda x_1 z_1 + \psi y_1 z_1 - \lambda x_1 z_2 - \psi y_1 z_2 - \lambda x_2 z_1 - \psi y_2 z_1 + 2\lambda x_2 z_2 + 2\psi y_2 z_2$$

$$= \lambda (x_1 z_1 - (x_1 z_2 + x_2 z_1) + 2 x_2 z_2) + \psi (y_1 z_1 - (y_1 z_2 + y_2 z_1) + 2 y_2 z_2)$$

$$= \lambda \cdot \langle x, z \rangle + \psi \cdot \langle y, z \rangle$$

b) Symmetry $\langle x, y \rangle = \langle y, x \rangle$

$$\langle y, x \rangle = y_1 x_1 - (y_1 x_2 + y_2 x_1) + 2 y_2 x_2$$

where $y_i x_i$ has the communitive property so $y_i x_i = x_i y_i$

$$y_1 x_1 - (y_1 x_2 + y_2 x_1) + 2 y_2 x_2 = x_1 y_1 - (x_2 y_1 + x_1 y_2) + 2 x_2 y_2$$

$$= x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2 x_2 y_2$$

$$= \langle x, y \rangle$$

$$\langle y, x \rangle = \langle x, y \rangle$$

c) Positive Definite means $\langle x, x \rangle > 0$, $\langle 0, 0 \rangle = 0$

$$\langle x, x \rangle = x_1 x_1 - (x_1 x_2 + x_2 x_1) + 2(x_2 x_2)$$

$$= x_1^2 - (x_1 x_2 + x_2 x_1) + 2 x_2^2$$

$$= x_1^2 - 2 x_1 x_2 + 2 x_2^2$$

$$= (x_1 - x_2)^2 + x_2^2 \geq 0$$

So $\forall x_2 \neq 0$, $\langle x, x \rangle > 0$, if the inner product equals 0, it implies $\vec{x} = \vec{0}$. The only time $\langle x, x \rangle = 0$ is when $x_1 = 0$ and $x_2 = 0$ so $\vec{x} = 0$.

Thus, this is an inner product.

Next Page for 3.2

$\boxed{3.2}$: Consider $\mathbb{R}^2$ with $\langle\cdot,\cdot\rangle$ defined for all $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^2$ as

$$\langle x, y\rangle = \vec{x}^T \underbrace{\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}}_{:=A} \vec{y}$$

Is $\langle\cdot,\cdot\rangle$ an inner product?

For this to be an inner product, it needs to be shown that its symmetric Positive Definite and has linearity.

a) Symmetry $\langle x,y\rangle = \langle y,x\rangle$

$$\text{Consider } \langle x,y\rangle = [x_1, x_2]\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= [2x_1 + x_2, \ 2x_2]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= 2x_1y_1 + x_2y_1 + 2x_2y_2$$

$$\text{and } \langle y, x\rangle = [y_1, y_2]\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [2y_1 + y_2, \ y_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 2y_1x_1 + y_2x_1 + 2y_2x_2$$

Notice $2x_1y_1 + x_2y_1 + 2x_2y_2 \neq 2y_1x_1 + y_2x_1 + 2y_2x_2$

$$x_2y_1 \neq y_2x_1$$

Therefore, this is not symmetric, so $\langle\cdot,\cdot\rangle$ is not an inner product.

$\boxed{3.5}$: Consider the Euclidean vector space $\mathbb{R}^5$ with the dot product. A subspace $U \subseteq \mathbb{R}^5$ and $\vec{x} \in \mathbb{R}^5$ are given by

$$U = \text{span}\left(\begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 5 \\ 0 \\ 7 \end{bmatrix}\right), \quad \vec{x} = \begin{bmatrix} -1 \\ -9 \\ -1 \\ 4 \\ 1 \end{bmatrix}.$$

a) Determine the orthogonal projection $\pi_U(\vec{x})$ of $\vec{x}$ onto U.

b) Determine the distance $d(\vec{x}, U)$.

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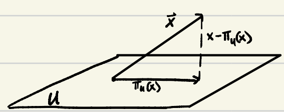
a) We need to find the coordinates λ in terms of x, to find the projection point to $\pi_U(x)$.



First its needed to see if the generating set U is a basis.

so

$$U = \begin{bmatrix} 0 & 1 & -3 & -1 \\ -1 & -3 & 4 & -3 \\ 2 & 1 & 1 & 5 \\ 0 & -1 & 2 & 0 \\ 2 & 2 & 1 & 7 \end{bmatrix}, \quad \text{rref}(U) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

It is seen the first 3 columns are linearly independent and form a basis for U.

$$B_u = \begin{bmatrix} 0 & 1 & -3 \\ -1 & -3 & 4 \\ 2 & 1 & 1 \\ 0 & -1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$ With the equations in the book, it is needed to show $B_u^T B_u \lambda = B_u^T x$, to solve for the values of λ. First, compute the known parts.

$$B_u^T B_u = \begin{bmatrix} 0 & -1 & 2 & 0 & 2 \\ 1 & -3 & 1 & -1 & 2 \\ -3 & 4 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -3 \\ -1 & -3 & 4 \\ 2 & 1 & 1 \\ 0 & -1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 9 & 0 \\ 9 & 16 & -14 \\ 0 & -14 & 31 \end{bmatrix}$$

$$B_u^T \vec{x} = \begin{bmatrix} 0 & 1 & -3 \\ -1 & -3 & 4 \\ 2 & 1 & 1 \\ 0 & -1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -9 \\ -1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 23 \\ -25 \end{bmatrix}$$

This makes the equation to solve

$$\begin{bmatrix} 9 & 9 & 0 \\ 9 & 16 & -14 \\ 0 & -14 & 31 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 23 \\ -25 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 9 & 9 & 0 & 9 \\ 0 & 7 & -14 & 14 \\ 0 & 0 & 3 & 3 \end{array}\right] \Rightarrow \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} (9-0-36)/9 \\ (14-(-14))/7 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

Then the projection $\pi_u(x) = B\lambda$

$$B\lambda = \begin{bmatrix} 0 & 1 & -3 \\ -1 & -3 & 4 \\ 2 & 1 & 1 \\ 0 & -1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -1 \\ -2 \\ 3 \end{bmatrix}$$

Thus the orthoganal projection $\pi_u(x) = \begin{bmatrix} 1 \\ -5 \\ -1 \\ -2 \\ 3 \end{bmatrix}$

b) The distance between $\vec{x}$ and $\pi_u(\vec{x})$ is represented by the formula $\|\vec{x} - \pi_u(\vec{x})\|$.

$$\vec{x} - \pi_u(\vec{x}) = \begin{bmatrix} -1 \\ -9 \\ -1 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -5 \\ -1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 0 \\ 6 \\ -2 \end{bmatrix}$$

Now $$\|\vec{x} - \pi_u(\vec{x})\| = \Bigg\| \underbrace{\begin{bmatrix} -2 \\ -4 \\ 0 \\ 6 \\ -2 \end{bmatrix}}_{a} \Bigg\| = \sqrt{\sum_{i=1}^{5} a_i^2} = \sqrt{60}$$

Thus, $d(\vec{x}, U) = \sqrt{60}$.

FIN.