Math 4610 - Tasksheet 2 #4

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(4) Analysis for centerd difference approximation with Taylor Series Expansions. Show that the approximation is 2nd Order.

The centered difference approximation has this form for a second order derivative

$$f''(a) \approx \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

in order to verify this is 2nd order each component must be written out in a Taylor Series expansion.

for

$$f(a+h) = f(a) + hf'(a) + \tfrac{1}{2}h^2 f''(a) + \tfrac{1}{6}h^3 f'''(a) + \tfrac{1}{24}h^4 f^{(4)}(a) + \dots$$

$$f(a) = f(a)$$

$$f(a-h) = f(a) - hf'(a) + \tfrac{1}{2}h^2 f''(a) + \tfrac{1}{6}h^3 f'''(a) + \tfrac{1}{24}h^4 f^{(4)}(a) + \dots$$

With Taylor Series expansions an error term ξ can be defined as the remaining f(a) terms in the expansion after the order desired. This error term is defined as the analysis is done.

Now Consider,

$$f''(a) = \frac{1}{h^2}\left(f(a+h) - 2f(a) + f(a-h)\right)$$

when put into the Taylor Series expansion

$$f(a+h) = \cancel{f(a)} + \cancel{hf'(a)} + \tfrac{1}{2}h^2 f''(a) + \cancel{\tfrac{1}{6}h^3 f'''(a)} + \tfrac{1}{24}h^4 f^4(a) + \dots$$

$$-2f(a) = \cancel{-2f(a)}$$

$$f(a-h) = \cancel{f(a)} - \cancel{hf'(a)} + \tfrac{1}{2}h^2 f''(a) - \cancel{\tfrac{1}{6}h^3 f'''(a)} + \tfrac{1}{24}h^4 f^4(a) + \dots$$

when canceling out the terms in the expansions for this formula we are left with:

$$f''(a) = \frac{1}{h^2}\left(h^2 f''(a) + \tfrac{1}{12}h^4 f^4(a) + \dots\right)$$

$$\Rightarrow f''(a) + \underbrace{\tfrac{1}{12}h^2 f^4(a) + \dots}_{\text{error term.}}$$

The error part of this equation is

$$\tfrac{1}{24}h^4 f^4(a) + \tfrac{1}{720}h^6 f^6(a) + \dots$$

which for f(a+h) can be represented

$$\tfrac{1}{24}h^4 f^4(\xi_1)$$

and for f(a-h)

$$\tfrac{1}{24}h^4 f^4(\xi_2)$$

and the error for the entire approximation is

$$\xi_1 + \xi_2 = \xi_3$$

so

$$\tfrac{1}{12}h^2 f^4(\xi_3)$$

Thus the central difference approximation is shown to be 2nd Order Accurate:

$$\underline{\underline{f''(a) \approx f''(a) + \tfrac{1}{12}h^2 f^4(\xi_3)}}$$

Though as with any approximation, there is a certain error associated with each entry into the approximation.