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Exam 1

Due Friday, March 27.

Write a brief paragraph on each of the questions provided below.

Questions:

1: Define the order of accuracy of a finite difference approximation of the derivative.

Response:

$$u(x+h) \approx u(x) + h u'(x) + \frac{h^2}{2} u''(x) + \frac{h^3}{6} u'''(x) + \dots$$
$$u(x) \approx u(x)$$
$$\frac{1}{h}\left(\cancel{u(x)} + u'(x) + \frac{h}{2} u''(x) + \frac{h^2}{6} u'''(x) + \dots\right) - \cancel{u(x)} \cdot \frac{1}{h}$$
$$\boxed{\text{It is } \frac{1}{h} u'(x) + \frac{h^2}{2} u''(x) + \dots + \frac{1}{k!} \cdot h^k u^{(k)}(x)}$$

The order of accuracy is defined above, using mathematical notation. The order of accuracy helps in allowing one to know how far the difference approximation needs to be in order for it to be accurate enough, even though there will be some truncation errors in the approximation method.

2: Show that the one sided finite difference method,

f(x) ≈ $\frac{f(x+h)+f(x)}{h}$

is first order accurate using Taylor series expansion.

Response:

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + \frac{h^4}{24} f''''(x) + \dots$$
$$f(x) = f(x)$$
$$\longrightarrow \frac{1}{h}\left(\cancel{f(x)} + f'(x) + \frac{h}{2} f''(x) + \frac{h^2}{6} f'''(x) + \dots\right) - \cancel{\frac{1}{h} f(x)}$$
$$\Rightarrow \frac{1}{h}\left(f(x+h) - f(x)\right) - f'(x) \Rightarrow \frac{1}{2} h y''(\xi)$$
$$\Rightarrow \boxed{D_+(f(x)) \approx \frac{1}{h}\left(f(x+h) - f(x)\right)}$$

3: Define local truncation error and describe the difference between the local truncation error and roundoff error/machine precision.

Response:

Local Truncation Error is defined by $\tau_j = O(h^{(k)})$ as $h \to 0$. The local error is the error when the approximation U_j replaces the real solution $u(x_j)$. Global Error occurs across the whole system, along with the machine precision error. This error could be due to a roundoff error, where the machine being used might not handle the numbers as precisely as are needed. The error could be from a user not using a data type large enough to get close enough to the actual answer, and this might need to be changed to get closer to the desired answer.

4: Derive the finite difference method for the approximation of f //(x), with the highest order of accuracy given the form

f//(x) ≈ a-1 f(x − h) + a0 f(x) + a1 f(x + h)

Response:

For this problem, the Central Difference approximation is the best.

$$a_{-1}\left(f(x-h)\right) + a_0\left(f(x)\right) + a_1\left(f(x+h)\right)$$

$\rightarrow$ f(x) terms

$$f(x) = a_{-1} + a_0 + a_1 = 0$$
$$f'(x) = a_1 h - a_{-1} h = 1$$
$$f''(x) = \tfrac{1}{2}h^2 a_1 + \tfrac{1}{2}h^2 a_{-1} = 0$$
$$f'''(x) = \ldots$$
$$u'(x) = \frac{1}{2h}u(x+h) - \frac{1}{2h}u(x-h)$$

5: For the following two-point boundary value problem

u// + σ u = f

with u(a) = ua and u(b) = ub define a second order accurate approximation for the problem.

Response:

For the problem defined below

$$\begin{cases} u'' + \sigma u = f \\ u(a) = u_a \\ u(b) - b \end{cases}$$

$$\sigma = c_i \qquad a_i = 1$$

$$\Rightarrow a_i u'' + c_i u = f$$

$$= \underbrace{a_i\left(\frac{u_{i-1} - 2u_i + u_{i+1}}{h} + c_i u_i = f_i.\right.}$$

$$\rightarrow \underline{\underline{k(x_{i+1}) - \tfrac{1}{2} h k'(x_{i+1}) + O(h^2)}}$$

The answer is the last line, defined by $k(x_{i+1}) - \frac{1}{2} h k'(x_{i+1}) + O(h^2)$.

6: Define the terms consistent finite difference method and stable finite difference method. How are these related to convergence of the approximations for a two-point boundary value problem?

Response:

Consistency is defined as $\|\tau^h\| \to 0$ as $h \to 0$, $\|\tau^h\| = O(h^P)$ where $\rho \to 0$, then the system is consistent. Stability is defined as $\left\|\left(A^h\right)^1\right\| \leq C$ for all *h* small such that $\|E^h\| \leq C \cdot \|\tau^h\|$ where *E* is the error, such that error → 0 at least as fast as $\|\tau^h\| \to 0$. If this is true then the system is stable. These are related to prove convergence. If there is consistency and stability a system will converge. If there is one but not the other the system will doesn't converge.

7: Describe the steps needed to show that a finite difference method converges in the 2-norm. What is needed to prove stability of the finite difference method in the 2-norm?

Response:

From my understanding, there are 3 steps used to prove convergence in the 2-Norm. First, the system/method should be proved to be continuous. Second, the system/method needs to be proven that it is stable. Lastly, it needs to be proved consistent. If all these are true, then the system or method converges.

8: What issues arise in finite difference methods in one dimension when Neumann boundary conditions are prescribed?

Response:

The biggest issue with using Neumann Boundary Conditions in one dimension is that they require a derivative for the conditions. In two dimensions the boundary conditions are the measure of flux at the boundary. However, in one dimension, there is no way to measure the flux as there is no way to have a valid derivative of a one dimensional function. There is also no way to know that the system will converge for sure.

9: Define and compare Dirichlet and Neumman boundary conditions in terms of a simple two-point boundary value problem.

Response:

Dirichlet boundary conditions are used to give the actual boundary of a given space that the system or function occupies. These are like a wall in a swimming pool, where the water cannot move past the walls. These are given of the form u(a) = a. Neumann boundary conditions measure the flux at the boundary. Or in the pool, it is the measure of the water's direction as it hits the walls of the pool. These conditions are given in the form of u'(b) = b. Neumann conditions are the rate of change at the boundary points specified, where Dirichlet are the boundary being specified.

10: In terms of linear algebra compare the finite difference schemes for the elliptic differential equations in one-dimension, two-dimensions, and three-dimensions? Use the form of the matrix in your discussions.

Response:

For some reason each time I read this question, I understand it differently, and I don't think any way I think to answer the prompt is the correct answer or a valid acceptable answer to this prompt. Part of me wants to talk about the ways we solved the elliptic problems in class and on the homework assignments, but the thing that I am getting really confused with is the discussion of one dimension, two dimensions, and three dimensions for the elliptic problem. If the dimension corresponds to order, the classic form of the elliptic problem we have used in class is

$$\nabla^2 u = f$$

Where the boundary conditions must be specified. In this type of problem for a 1-D case, $\nabla u = f$, where in general it would have the form Lu = f. Where

$$L = \sum_{j,k=1}^{N} A_{jk} \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j+1}^{N} B_j \frac{\partial}{\partial x_j} + C$$

In the 1D problem, A is 0, but B = some constant a. I believe. In the 2-D problem, the matrix A is written as

$$A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}.$$

In 3-D, since the matrix is symmetrical for an N x N system,

$$A = \begin{bmatrix} a & \frac{d}{2} & 0 \\ \frac{d}{2} & b & 0 \\ 0 & \frac{d}{2} & c \end{bmatrix}.$$

This is just based on my understanding. I was referencing page312 from the textbook in trying to understand this question better.

11: Compare three methods for ordering mesh points in a two-dimensional finite difference method for the elliptic problem. State the pros and cons of each of these orderings.

Response:

The three methods that are mentioned to lexicographically order mesh points are the 5-point stencil, the 9-point stencil, and the diagonal or 3-point stencil. The 3-point stencil was the least helpful of the three, and there was no in-depth discussion on the benefits of this one. The 5-point stencil is good to use, in that it has a smaller error and it can be used on graphs that aren't very smooth. The downside to the 5-point stencil is that it requires multiple passes to be accurate. The 9-point stencil is good in that it only requires one pass and it is very useful on smoother graphs. However, the 9-point stencil may have a bigger error.

12: In the solution of linear systems of equations, give a definition of the term, direct method. Give at least two examples of direct methods for the solution of a linear system of equations.

Response:

A Direct Method is a way to solve for a set of unknowns in an equation in a brute force method. This is a way to just solve the system, to get the set of answers quickly. Two examples of these are Gaussian-Elimination and back-substitution, and LU Factorization.

13: Define the term iterative method for the approximate solution of a linear system of differential equations. Give at least two examples of these types of methods.

Response:

Iterative methods are a way to approximate the set of unknown's values with in initial guess, that is refined as it iterates through the solving method over and over. Two examples are the Jacobi iteration method, and Gauss-Seidel's iteration method.

14: Compare and contrast the 5-point stencil and 9-point stencil in the approximate solution of two-point boundary value problems.

Response:

The 5-point stencil is very beneficial with functions that aren't very smooth. For the boundary specifically multiple passes will be used to find all the specifics within the boundaries specified. The error is smaller due to the multiple passes over the function. However, the 9-point stencil only needs one pass over the function to be complete. The 9-point stencil in comparison has a larger error and is very effective on smoother graphs of functions. Both find an approximate solution, but depending on what is desired for the function, one might be more effective to use than the other.

15: Define the term diagonally dominant in terms of linear systems of equations. What can be said about linear solution methods like Gauss Elimination with back-substitution when the coefficient matrix is diagonally dominant?

Response:

A diagonally dominant matrix a square matrix is said to be diagonally dominant if, for every row of the matrix, the magnitude of the diagonal entry in a row is larger than or equal to the sum of the magnitudes of all the other (non-diagonal) entries in that row. The simplest sparse diagonally dominant matrix is the Identity matrix with 1's along the diagonal. In class we have used sparse matrices that are diagonally dominant such as tri-diagonal, or penta-diagonal where there are 0's on the top corner and the bottom corner with numbers on main diagonals, with a possibility for groups of other upper and lower diagonal groupings. With linear solution methods, the process of solving becomes easier with diagonally dominant matrices, especially sparse ones.

16: Define the 2-condition number of a square matrix. How can we use the condition number of a matrix to determine how accurate an approximate solution of a linear system is?

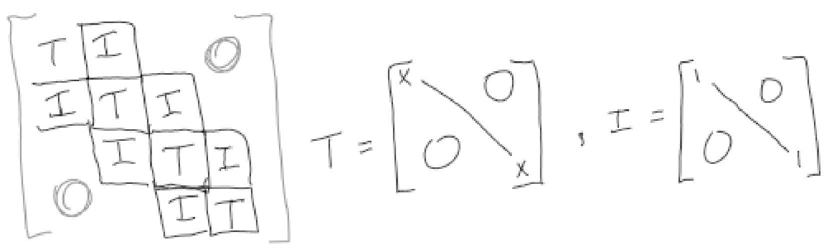
Response:

The 2-condiont number of a square matrix is defined as $k(A) = \|A\| \cdot \|A^{-1}\|$, only if the square matrix is non-singular. It is the ration of the largest to the smallest eigenvalues. The condition number plays a role in the convergence rate of many iterative methods for solving systems of equations.

17: Define the term Toeplitz matrix. Give some examples that arise in the approximate solution of differential equations.

Response:

The Toeplitz matrix is a diagonally dominant matrix that has a general form below. There may be other diagonals around the main diagonal in the T matrix, but the general form has one on each diagonal. The Toeplitz are used in solving higher order systems of Ordinary and Partial Differential Equations, and mesh points.



18: Give examples of when we might consider the use of LU-factorization instead of Gaussian Elimination with Backsubstitution.

Response:

If the system is needing to be solved multiple times, LU-Factorization is more efficient, since it takes less computation time. LU-factorization also factors before solving the system, so if there is a need to use the lower or upper pieces later, it is advantageous to use LU. Gaussian-Elimination does everything in one quick go.

19: Define the term vector-norm and give at least three useful vector norms. Make sure you list the properties that must be satisfied to be a norm.

Response:

A vector-norm is a mapping of all vectors in $\mathbb{R}^S$ to non-negative real numbers where $\|x\| \geq 0$, if *a* is singular then $\|ax\| = |a| \cdot \|x\|$, and $\|x + y\| \leq \|x\| \cdot \|y\|$ is true. Three useful vector norms are:

1. The 2-Norm, $\|x\|_2$

2. The 1-Norm, $\|x\|_1$

3. The infinity Norm, $\|x\|_\infty$

20: Define the term matrix-norm and give at least three examples of matrix norms and the properties that must be satisfied.

Response:

The matrix norm is defined in mathematical terms $A \epsilon \mathbb{R}^{SxS} \to \|A\|$ such that $\|Ax\| \leq C\|x\|$ holds each vector $x \in \mathbb{R}^s$ to compute $\|A\|$ direct from matrix. The three examples of matrix norms and the properties are:

1. $\|A\|_1 = \max_{1\leq j\leq s} \sum_{i=1}^{s} |a_{i,j}|$ The maximum column sum.

2. $\|A\|_\infty = \max_{1\leq i\leq s} \sum_{j=1}^{s} |a_{i,j}|$ The maximum row sum.

3. $\|A\|_1 = \sqrt{\rho(A^T A)}$, $\rho(B)$ is the spectral radius of A. If A is normal and symmetric, then $\|A\|_2 = \rho(A)$.

21: What is a consistent matrix-norm? Give an example of a consistent matrix norm. Give an example of where a consistency norm is needed in our analysis.

Response:

A matrix norm $\|\cdot\|$ on $K^{m\times n}$ is called *consistent* with a vector norm $\|\cdot\|_a$ on K^n and a vector norm $\|\cdot\|_b$ on K^m if: $\|Ax\|_b \leq \|A\| \cdot \|x\|_a$ for all $A \in K^{m\times n}, x \in K^n$. All induced norms are consistent by definition. A consistency norm is needed in our analysis as it allows a lot of operations to be valid. It also allows the max row and max column operations to be true.

22: What is an induced matrix norm? Use the 2-matrix norm to illustrate the parts of your discussion?

Response:

This describes how the matrix stretches unit vectors with respect to the given norm. The following illustrates the induced matrix norm. All induced Norms are consistent.

$$\|A\| = \sup_{x\neq 1} \frac{||Ax||}{||x||} = \max_{\|x\| = 1} ||Ax||.$$

23: Discuss how one can perform a computational convergence analysis of a sequence of approximations to determine the rate of convergence of a particular finite difference method.

Response:

One can perform a convergence analysis on the series of approximations found in a computational analysis. In a method to test the convergence, one needs the old x values, and the new-found x values, along with the size of the dimension of x. With a defined epsilon, there needs to be a comparison with the absolute value of the difference between the two x values. Then if this value is greater than zero, it is the max value. After this, the max value is compared to epsilon. If the max value is greater than epsilon, the method or system converges. If not, it does not converge. There is code that illustrates this below.

```
static int test_convergence(double []x, double []oldx, int n)
{
    double maxvalue, tempvalue;
    int i;
    double epsilon = 0.000000001;

    maxvalue = 0.0;
    for(i=0; i < n; i++)
    {
        tempvalue = Math.abs(x[i] - oldx[i]);
        if (tempvalue > maxvalue)
            maxvalue = tempvalue;

    } // for
    if (maxvalue > epsilon)
        return 0;
    else
        return 1;

}
```