POSSIBLE GEOMETRIES OF THE SHAPE OF THE UNIVERSE

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Contents

1. Introduction

This paper discusses how the composition of the matter and energy in the Universe determines its geometry. In particular, I will derive the Friedman Equation, present its implications for the shape of the Universe, and describe and prove the characteristics of the possible geometries.

Cosmology is the study of the Universe. The Universe is made up of matter and radiation. The gases, liquids, solids, and plasmas that form the planets, stars, nebulae, and galaxies are all examples of baryonic matter. This is matter whose mass is mainly comprised of baryons, which are protons, neutrons, and other more obscure particles[1]. Although the luminous objects in the Universe are all composed of baryonic matter, astronomical observations strongly imply the existence of a non-luminous dark matter. This dark matter has only been detected through its gravitational influence on the luminous matter[1]. Although the nature of dark matter is still unknown, it comprises about five sixths of all the matter in the Universe, with the baryonic matter making up the remaining one sixth[1]. The radiation in the Universe, on the other hand, is much more understood by scientists. We can observe electromagnetic radiation in the form of radio waves, microwaves, infrared radiation, visible light, ultraviolet radiation, X-rays, and γ-rays. Although visible light constitutes a good portion of this, the majority of the radiation in the Universe is in the form of

Date: June 2, 2023.

microwaves. Called the cosmic microwave background (CMB), this dominant source of radiation is believed to be a product of the Big Bang[1].

Another feature of the Universe is that it is expanding. Observations of distant galaxies show that the light received from these galaxies is redshifted. This means that the galaxies are receding. But the redshfits cannot be attributed to the normal Doppler effect, which applies to moving bodies[1]. Instead, the space between us and the galaxies is actually expanding. The rate of cosmic expansion at the current time is measure by the Hubble constant, H_0. The Hubble constant is related to the redshift, z, and distance, d, of a galaxy by $z = \frac{H_0}{c}d$, where c is the speed of light.[1]. Further observations have found that the cosmic expansion rate has been increasing for several billion years, meaning the Universe is expanding at an accelerating rate[1].

One of the main goals of modern cosmolgy is to formulate mathematical models which predict the behavior of the Universe. These so-called *cosmological models* are groups of equations that relate observable quantities and take in parameters which are determined by observation [1]. The models are based on Einstein's general relativity. Einstein showed that space and time are not separate entities; together they form a four-dimensional space-time. He also showed that the density of matter and radiation contained in the space-time affects its geometric properties[1]. In the following sections I will describe the cosmological model dictated by the Friedman Equation and the possible geometries which could arise from the distribution of mass and radiation in the Universe.

2. Comoving Coordinates

The expansion of space can be described mathematically by comoving coordinates. Comoving coordinates are based on a coordinate plane which expands or contracts with space[1]. Thus, points in an expanding space have unchanging coordinates when using comoving coordinates. Comoving coordinates are related to proper coordinates, which are coordinates on a static plane, by the equation $\bar{r} = a(t)\bar{x}$, where $\bar{r}$ are the proper coordinates, $\bar{x}$ are the comoving coordinates, and $a(t)$ is a parameter called the scale factor. The scale factor describes the expansion of the space as a function of time[1]. At the present time, $a(t)$ is taken to be 1.

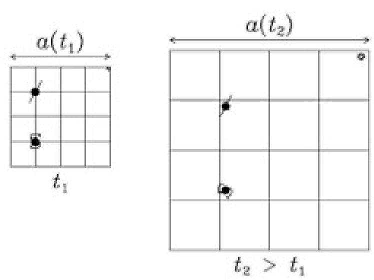

Figure 1. A an expanding coordinate plane which describes the expansion of space with comoving coordinates and the scale factor, $a(t)$ [2].

3. Derivation of the Friedman Equation

We will use the Friedman equations, along with other equation from astronomy, to deduce the Friedman Equation in its well-known form. The two Friedman equations are

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3} \tag{1}$$

$$(\dot{a})^2 = \frac{8\pi G}{3}\rho a^2 - kc^2 + \frac{1}{3}\Lambda c^2 a^2, \tag{2}$$

where a is the scale factor, G is the gravitational constant, ρ is density of the Universe, p is the pressure in the Universe, Λ is the cosmological constant from Einstein's field equations, c is the speed of light, and k is the curvature of space at the present time [3]. $k = 0$ implies a flat geometry with zero curvature, $k = +1$ implies a spherical geometry with positive curvature, and $k = -1$ implies a hyperbolic geometry with negative curvature. I will give further discussion on the curvature of space in later sections.

We divide Equation 2 by a^2 and use the equation $H(t) = \frac{1}{a}\frac{da}{dt}$. $H(t)$ is called the Hubble parameter and is usually denoted by H. Thus, we have

$$H^2 = \frac{8\pi G}{3}\rho - kc^2\frac{1}{a^2} + \frac{1}{3}c^2\Lambda. \tag{3}$$

Now, we will express the Friedman equation in terms of density parameters. The current density parameters of the mass, radiation, curvature, and dark energy are represented by Ω_m, Ω_r, Ω_k, and Ω_Λ, respectively. Each represents the ratio of the observed density of each to the critical density. The critical density is the density of the Universe required for the geometry of the Universe to be flat. In this case, $k = 0$, so the Friedman equation implies that the current critical density is $\rho_{\text{crit}} = \frac{3H_0^2}{8\pi G}$ [1]. We also know that $\rho_\lambda = \frac{\Lambda c^2}{8\pi G}$, so the current density parameter of dark energy is $\Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2}$ [1]. If we plug these values into the equation and divide by H_0^2 we get

$$\frac{H^2}{H_0^2} = \frac{\rho}{\rho_{\text{crit}}} - \frac{kc^2}{H_0^2}\frac{1}{a^2} + \Omega_\Lambda. \tag{4}$$

$\frac{\rho}{\rho_{\text{crit}}}$ is determined by the density of both matter (baryonic and dark) and radiation. The density of matter is proportional to a^{-3} and the density of radiation is proportional to a^{-4} [1]. Thus, $\rho = \rho_m a^{-3} + \rho_r a^{-4}$. Thus, $\frac{\rho}{\rho_{\text{crit}}} = \frac{\rho_m a^{-3} + \rho_r a^{-4}}{\rho_{\text{crit}}}$. By the definition of the density parameter, $\Omega_m = \frac{\rho_m}{\rho_{\text{crit}}}$ and $\Omega_r = \frac{\rho_r}{\rho_{\text{crit}}}$. Thus,

$$\frac{H^2}{H_0^2} = \Omega_r a^{-4} + \Omega_m a^{-3} - \frac{kc^2}{H_0^2}a^{-2} + \Omega_\Lambda. \tag{5}$$

Finally, we we replace $\frac{kc^2}{H_0^2}$ with Ω_k, a constant related to spatial curvature, and take each density parameter to be at the current time. Hence, we get the Friedman equation

$$\frac{H^2}{H_0^2} = \Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} - \Omega_{k,0}a^{-2} + \Omega_{\Lambda,0}. \tag{6}$$

4. Implications of the Friedman Equation

Now, we will show how the density parameters affect the curvature of the universe. Consider Equation 4. We want to generalize this to describe the curvature at any point in time, not just at the current time. So, we replace H_0 with $H(t)$, which we will continue to denote as H. We also know that $\frac{\rho}{\rho_{\text{crit}}} = \Omega_m + \Omega_r$. So, we have

$$\frac{H^2}{H^2} = \Omega_m + \Omega_r - \frac{kc^2}{H^2} + \Omega_\Lambda. \tag{7}$$

Solving for $\frac{kc^2}{H^2}$ gives

$$\frac{kc^2}{H^2} = \Omega_m + \Omega_r + \Omega_\Lambda - 1. \tag{8}$$

The sign of $\frac{kc^2}{H^2}$ is determined by k since c and H are always positive. Thus, if $\Omega_m + \Omega_r + \Omega_\Lambda = 1$, then $k = 0$; if $\Omega_m + \Omega_r + \Omega_\Lambda < 1$, then $k = -1$; and if $\Omega_m + \Omega_r + \Omega_\Lambda > 1$, then $k = +1$. The most recent estimates for the density parameters indicate that $\Omega_m \approx 0.3$ and $\Omega_\Lambda \approx 0.7$. Currently the Universe is matter-dominated.

This means there is much more matter than radiation in the Universe at the current time, so $\Omega_r \approx 0$. (This was not the case in the early Universe, which was radiation dominated). Thus, observations strongly favor a universe in which $\Omega = \Omega_m + \Omega_r + \Omega_\Lambda = 1$ and $k = 0$, which implies a universe with a flat geometry.

If Ω were slightly larger or smaller, this would imply a universe with a spherical or hyperbolic geometry, respectively. In the following sections I will describe the properties of a flat geometry, and also what the geometry of space would be like if it were spherical or hyperbolic.

5. Definitions

Definition 1. [4] We say two lines, l_1, l_2, *intersect* if $l_1 \cap l_2 \neq$. We say l_1, l_2 are *parallel* if either they don't intersect or $l_1 = l_2$.

Definition 2. The *spherical distance* between x and y is defined to be the real number $d_S(x, y) = \theta(x, y)$ [5].

Definition 3. [4] An *isometry* is a function f with the property $|f(P_1)f(P_2)| = |P_1P_2|$ for points P_1, P_2.

Definition 4. [4] *Möbius transformations* are the extension of projective transformations $x \mapsto \frac{ax+b}{cx+d}$ of $\mathbb{R}P^1$ to the upper half plane $z \mapsto \frac{az+b}{cz+d}$ and $z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}$.

6. Flat Geometry

The parallel postulate for Euclidean geometry says that, "Through a point outside a given infinite straight line there is one and only one infinite straight line parallel to the given line" [5].
The circumference of a circle with radius r is $2\pi r$. The interior angles of a triangle sum to π.

7. Spherical Geometry

7.1. Defining Spherical Geometry.

In $\mathbb{R}^3$ spherical geometry, the unit sphere is S^2 is defined by

$$S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$$

The points are the ordinary points on the sphere in $\mathbb{R}^3$, satisfying the equation $x^2 + y^2 + z^2 = 1$. The lines are the great circles on the sphere. Great circles are the intersection of any plane that passes through the origin in $\mathbb{R}^3$ with the sphere [4].

In spherical geometry, Euclid's "parallel postulate" takes the form, "Through a point outside a given line there is no line parallel to the given line" [5].

7.2. Spherical Circumference.

One characteristic of spherical geometry is the circumference of a circle is less than or equal to $2\pi r$, where r is the radius of the circle.

Consider a circle of radius r on S^2, as seen in Figure 2. The radius of the circle as calculated in Euclidean geometry is $\sin(r)$. Thus, the circumference of the circle is $2\pi \sin(r)$. Since $sin(r) \leq r$, the circumference of a circle in spherical geometry will be less that or equal to $2\pi r$. In the case where $r = \frac{\pi}{2}$, the circle on the sphere is a great circle and the circumference of the circle is equal to π. □

7.3. Spherical Trigonometry.

Let points $x, y, z \in S^2$ be spherically non-collinear. Let $C(x, y)$ be the unique great circle such that $x, y \in C(x, y)$. Let $H(x, y, z)$ be the closed hemisphere such that $C(x, y)$ is the boundary and $z \in H(x, y, z)$. The spherical triangle with vertices x, y, z is defined as

$$T(x, y, z) = H(x, y, z) \cap H(y, z, x) \cap H(z, x, y).$$

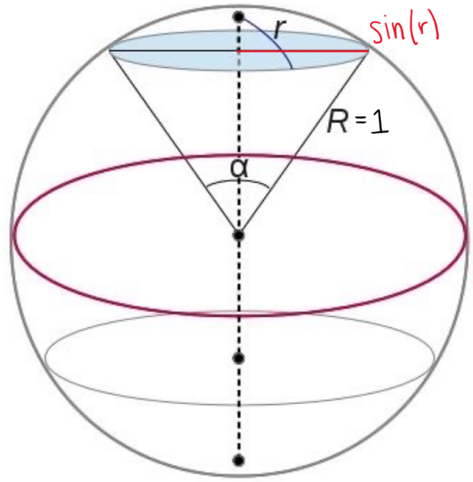

FIGURE 2. A circle in spherical geometry [6]

Let $[x,y]$ be the minor arc joining x and y. The sides of $T(x,y,z)$ are $[x,y],[y,z],[z,x]$. Let the side lengths be denoted by $a=\theta(y,z), b=\theta(z,x), c=\theta(x,y)$. Let

$$f:[0,a]\to S^2, g:[0,b]\to S^2, h:[0,c]\to S^2$$

be the geodesic arcs from y to z, z to x, x to y, respectively [5].

Let the angles of $T(x,y,s)$ be α,β,γ. Let α be the angle between $[z,x]$ and $[x,y]$, defined by the angle between $-g'(b)$ and $h'(0)$. Let β be the angle between $[x,y]$ and $[y,z]$, defined by the angle between $-h'(c)$ and $f'(0)$. Let γ be the angle between $[y,z]$ and $[z,x]$, defined by the angle between $-f'(a)$ and $g'(0)$ [5].

Theorem 5. *If α,β,γ are the angles of a spherical triangle, then $\alpha+\beta+\gamma>\pi$.*

Proof. Let α,β,γ be the angles of a spherical triangle $T(x,y,z)$. Then [5]

$$\begin{aligned}&((x\times y)\times(z\times y))\cdot(z\times x)\\ &=[(x\cdot(z\times y))y-(y\cdot(z\times y))x]\cdot(z\times x)\\ &=(x\cdot(z\times y))(y\cdot(z\times x))\\ &=-(y\cdot(z\times x))^2<0\end{aligned}$$

Theorem 6. *If w,x,y,z are vectors in $\mathbb{R}^3$, then* [5]

$$(x\times y)\cdot z=\begin{vmatrix}x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\\ z_1 & z_2 & z_3\end{vmatrix}$$

By Theorem 6, $x\times y, z\times y, z\times x$ are linearly independent. So, their associated unit vectors are non-collinear on S^2 [5].

Lemma 7. *If $x,y,z\in S^2$ and $\theta(x,y)+\theta(y,z)=\theta(x,z)$, then x,y,z are spherically collinear* [5].

Proof. We know $(x\times y)\cdot(y\times z)=|x\times y||y\times z|$. Hence, $x\times y$ and $y\times z$ are linearly independent. Thus, $(x\times y)\times(y\times z)=0$. We can re-write this as $(x\cdot(y\times z))y$. By Theorem 6, x,y,z are linearly independent. Hence, x,y,z are on a two-dimensional vector subspace of $\mathbb{R}^3$ and are spherically collinear [5]. ☐

By Lemma 7,

$$\theta(x \times y, z \times x) < \theta(x \times y, z \times y) + \theta(z \times y, z \times x).$$

Lemma 8. *If α, β, γ are the angles of a spherical triangle $T(x, y, z)$, then* [5]

(1) $\theta(z \times x, x \times y) = \pi - \alpha$
(2) $\theta(x \times y, y \times z) = \pi - \beta$
(3) $\theta(y \times z, z \times x) = \pi - \gamma$

Proof. The proof is evident from Figure 3 [5]. □

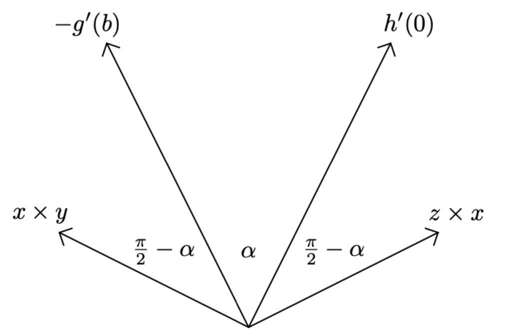

FIGURE 3. Four vectors on the tangent plane T_x with $\alpha < \frac{\pi}{2}$ [5].

By Lemma 8, $\pi - \alpha < \beta + \gamma$. Hence,

$$\alpha + \beta + \gamma > \pi,$$

as desired. □

8. Hyperbolic Geometry

8.1. **Defining Spherical Geometry.** One representation of the hyperbolic plane is the upper half-plane. The half-plane geometry is denoted by $\mathbb{H}$. The metric on $\mathbb{H}$ is $\mathscr{H} = \frac{du^2+dv^2}{v^2}$. In the upper half-plane model, the points are the complex numbers above the real axis in the complex pane. The lines are the regular vertical lines contained in the upper-half, and semi-circles whose centers lie on the real axis [4].

Another model of the hyperbolic plane is the Poincaré disk model or the unit disk, denoted $\mathbb{P}$. It's metric is $\mathscr{P} = \frac{4(du^2+dv^2)}{1-u^2-v^2)^2}$. In the Poincaré disk model, the points are the points inside the unit disk. The lines are the arcs in the unit disk which are orthogonal to the boundary, and also the diameters of the disk [7].

In hyperbolic geometry, Euclid's "parallel postulate" takes the form, "Through a point outside a given line there are infinitely many lines parallel to the given line" [5].

8.2. **Hyperbolic Circumference.** One characteristic of hyperbolic geometry is the the circumference of a circle is greater than or equal to $2\pi r$, where r is the radius of the circle.
Claim. In either $\mathbb{H}$ or $\mathbb{P}$, a circle with radius r has circumference $2\pi \sinh(r)$

Proof.

Corollary 9. *Möbius transformations map circles of $\mathbb{C}^*$ onto circles of $\mathbb{C}^*$* [8].

Corollary 10. *The geometries $\mathbb{H}$ and $\mathbb{P}$ are isometric* [8].

Proof. For the Möbius transformation

$$T = \begin{pmatrix} i & -1 \\ -1 & i \end{pmatrix}_\mu, \quad T^{-1} = \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}_\mu$$

we have $T(-i) = 0, T(1) = 1, T(-1) = -1, T(0) = i$. It follows from Corollary 9 that T transforms the unit circle onto the u-axis and the unit disk onto the upper half-plane. Moreover, since

$$T(u+iv) = \frac{2u + i(1-u^2-v^2)}{u^2+(v-1)^2},$$

it follows that T is an isometry from $\mathbb{P}$ to $\mathbb{H}$ [8]. □

By Corollary 10, it suffices to prove the Theorem for $\mathbb{P}$. We assume the circle is centered at the origin. Let the Euclidean radius of the circle be R. Then $r = \ln \frac{1+R}{1-R}$. Hence, $R = \tanh\left(\frac{r}{2}\right)$. We parametrize the circle as $u = R\cos t, v = R\sin t$. Then, its perimeter relative to $\mathscr{P}$ is [8]

$$\int_{\text{circle}} \frac{2\sqrt{du^2+dv^2}}{1-u^2-v^2} = \int_0^{2\pi} \frac{2Rdt}{1-R^2} = \frac{4\pi R}{1-R^2} = \frac{4\pi \tanh\left(\frac{r}{2}\right)}{1-\tanh^2\left(\frac{r}{2}\right)} = 4\pi \frac{\frac{\sinh\left(\frac{r}{2}\right)}{\cosh\left(\frac{r}{2}\right)}}{\frac{1}{\cosh^2\left(\frac{r}{2}\right)}}$$
$$= 2\pi\sinh(r).$$

Since $\sinh(r) > r$, the circumference of a circle with radius r is greater than $2\pi r$. □

8.3. **Hyperbolic Trigonometry.** First, we will define Lorentzian n-space. *Lorentzian n-space* is the inner product space consisting of $\mathbb{R}^n$, combined with the *Lorentzian inner product* $\langle x, y\rangle = x_1y_1 + \ldots + x_{n-1}y_{n-1} - x_ny_n$ [5]. The *Lorentzian norm* of $x \in \mathbb{R}^n$ is the complex number $||x|| = (x \circ x)^{\frac{1}{2}}$. A vector is said to be *space-like* if $||x|| > 0$. A vector is said to be *time-like* if $||x||$ is imaginary [5].

Now, we will define a triangle in hyperbolic space. Let x, y, z be three hyperbollically noncollinear points of H^2. Let $L(x, y)$ be the unique line in H^2 such that x, y are on $L(x, y)$. Let $H(x, y, z)$ be the closed half-plane such that $L(x, y)$ is the boundary and $z \in H(x, y, z)$. The hyperbolic triangle with vertices x, y, z is defined as

$$T(x, y, z) = H(x, y, z) \cap H(y, z, x) \cap H(z, x, y).$$

[5] Let $[x, y]$ be the segment of $L(x, y)$ joining x and y . The sides of $T(x, y, z)$ are $[x, y], [y, z], [z, x]$. Let the side lengths be denoted by $a = \eta(y, z), b = \eta(z, x), c = \eta(x, y)$. Let

$$f : [0, a] \to H^2, g : [0, b] \to H^2, h : [0, c] \to H^2$$

be the geodesic arcs from y to z, z to x, x to y, respectively [5].

Let the angles of $T(x, y, s)$ be α, β, γ. Let α be the angle between $[z, x]$ and $[x, y]$, defined by the Lorentzian angle between $-g'(b)$ and $h'(0)$. Let β be the angle between $[x, y]$ and $[y, z]$, defined by the Lorentzian angle between $-h'(c)$ and $f'(0)$. Let γ be the angle between $[y, z]$ and $[z, x]$, defined by the Lorentzian angle between $-f'(a)$ and $g'(0)$ [5].

Theorem 11. *If α, β, γ are the angles of a hyperbolic triangle, then $\alpha + \beta + \gamma < \pi$* [5].

Proof. Let α, β, γ be the angles of a hyperbolic triangle $T(x, y, z)$. The vectors $x \otimes y, z \otimes y, z \otimes x$ are linearly independent [5]. Let

$$u = \frac{x \otimes y}{||x \otimes y||}, \quad v = \frac{z \otimes y}{||z \otimes y||}, \quad w = \frac{z \otimes x}{||z \otimes x||}.$$

Consider $(x \otimes y) \otimes (z \otimes y)$ and $(z \otimes y) \otimes (z \otimes x)$. We can re-write these as

$$(x \otimes y) \otimes (z \otimes y) = ((x \otimes y) \circ z)y$$

and

$$(z \otimes y) \otimes (z \otimes x) = ((x \otimes y) \circ z)z.$$

[5] So, $u \otimes v$ and $v \otimes w$ are time-like vectors [5].

Theorem 12. *If w, x, y, z are vectors in $\mathbb{R}^3$, then*

$$(x \otimes y) \circ (z \otimes w) = \begin{vmatrix} x \circ w & x \circ z \\ y \circ w & y \circ z \end{vmatrix}$$

[5]

Lemma 13. *Let x, y be space-like vectors in $\mathbb{R}^3$. If $x \otimes y$ is time-like, then $|||x \otimes y||| = ||x|| \ \ ||y|| \sin \eta(x, y)$.* [5]

Proof. Since $x \otimes y$ is time-like, the vector subspace of $\mathbb{R}^3$ spanned by x and y is space-like [5]. By Theorem 12,

$$\begin{aligned} ||x \otimes y||^2 &= (x \circ y)^2 - ||x||^2 ||y||^2 \\ &= ||x||^2 ||y||^2 \cos^2 \eta(x, y) - ||x||^2 ||y^2|| \\ &= -||x||^2 ||y||^2 \sin^2 \eta(x, y). \end{aligned}$$

[5] □

By Lemma 13 and Theorem 12,

$$\begin{aligned} &\cos(\eta(u, v) + \eta(v, w)) \\ &= \cos \eta(u, v) \cos \eta(v, w) - \sin \eta(u, v) \sin \eta(v, w) \\ &= (u \circ v)(v \circ w) + ||u \otimes v|| \ \ ||v \otimes w|| \\ &> (u \circ v)(v \circ w) + ((u \otimes v) \circ (v \otimes w)) \\ &= (u \circ v)(v \circ w) + ((u \circ w)(v \circ v) - (v \circ w)(u \circ v)) \\ &= u \circ w \\ &= \cos \eta(u, w) \end{aligned}$$

Thus, either

$$\eta(u, w) > \eta(u, v) + \eta(v, w)$$

or

$$2\pi - \eta(u, w) < \eta(u, v) + \eta(v, w).$$

[5]

Lemma 14. *If α, β, γ are the angles of the hyperbolic triangle $T(x, y, z)$, then*

(1) $\eta(z \otimes x, x \otimes y) = \pi - \alpha$
(2) $\eta(x \otimes y, y \otimes z) = \pi - \beta$
(3) $\eta(y \otimes z, z \otimes x) = \pi = \gamma$

[5]

Proof. We assume $x = e_1$, without loss of generality. The proof is evident from Figure 4. □

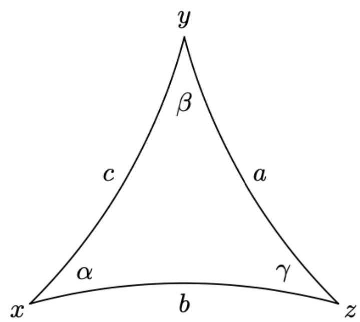

FIGURE 4. A hyperbolic triangle $T(x, y, z)$ [5].

By Lemma 14, $\eta(u, w) = \pi - \alpha, \eta(u, v) = \beta, \eta(v, w) = \gamma$. Therefore, either $\pi > \alpha + \beta + \gamma$ or $\pi + \alpha < \beta + \gamma$ [5]. We assume $\alpha > \beta$ and $\alpha > \gamma$, without loss of generality. But then $\pi + \alpha < \beta + \gamma < \pi + \alpha$, which is a contradiction. So,

$$\alpha + \beta + \gamma < \pi,$$

as desired [5].

□

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