Assignment 1

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1 Use the logical identities given in the text/class to verify the following, showing each step. For each step, state the name of the identity you are using.

1.1 (a)

$(\neg(p \vee \neg q) \rightarrow (p \wedge q))$
$\equiv (\neg p \wedge q) \rightarrow p \wedge q$ * By de Morgans Law
$\equiv \neg(\neg p \wedge q) \vee p \wedge q$ * By definition/property of twin tables
$\equiv (p \vee q) \vee p \wedge q$ * By de Morgans Law
$\equiv (p \vee \neg q)$ * By absorption law
$\equiv \neg q \vee p$ * By association law
$\equiv q \rightarrow p$ * By logical equivalence property

Therefore, $(\neg(p \vee \neg q) \rightarrow (p \wedge q)) \equiv (q \rightarrow p)$

1.2 (b)

$(p \longleftrightarrow q) \wedge \neg(r \rightarrow (p \wedge q)) \wedge p$
$\equiv [(p \rightarrow q) \wedge (q \rightarrow p) \wedge \neg(r \rightarrow (p \wedge q))] \wedge p$ * By the definition of biconditional statements.
$\equiv [(\neg p \vee q) \wedge (\neg q \vee p) \wedge \neg(\neg r \vee (p \wedge q))] \wedge p$ * By logical equivalences
$\equiv [(\neg p \vee q) \wedge (\neg q \vee p) \wedge r \wedge \neg(p \wedge q)] \wedge p$ * By de Morgans Law
$\equiv [(\neg p \vee q) \wedge (\neg q \vee p) \wedge r \wedge (\neg p \vee q)] \wedge p$ * By de Morgans Law
$\equiv [(\neg p \wedge p) \wedge (\neg q \vee q) \wedge (r \vee \neg p \vee q)] \wedge p$ * By the association law
$\equiv [F \wedge T \wedge (r \vee \neg p \vee q)] \wedge p$ * By the Negation law
$\equiv [F \wedge (r \vee \neg p \vee q)] \wedge p$ * By Domination laws
$\equiv [F] \wedge p$ * By the domination law ($F\wedge$ anything is always False.)
$\equiv F$ * By the domination law

Therefore, it is a contradiction.

2 Restate the following statements from English into predicate logic using the propositional functions F(X) for "X is a finite set" and S(X, Y) for X is a subset of Y , where the domain of the variables is all sets.

2.1 (a)"Not all sets are finite"

$\forall X F(X)$
$\neg(\forall X F(X))$
$\exists X \neg F(X)$

2.2 (b)"Every set is a subset of itself"

$\forall X(S(X, X))$

2.3 (c) "Some set is a subset of all sets"

$\exists X \forall Y S(X, Y)$

2.4 (d) "Finite sets have only finite subsets"

$\forall X[F(X) \rightarrow \forall Y(S(Y, X) \rightarrow F(Y))]$

2.5 (e) "There is a set whose only subset is itself"

$\exists X \forall Y(S(Y, X) \rightarrow (Y = X))$

3 Let P(n, a, b, c) denote the propositional function "$a^n + b^n = c^n$"

3.1 (a) Determine the truth values, if possible, of each of the following, using real numbers as the domain for all variables. Justify your answers.

i. P(2, 3, 4, 5): $3^2 + 4^2 = 5^2$
$25 = 25$, therefore True.

ii. P(2, a, 0, a): $a^2 + 0 = a^2$
$a^2 = a^2$, therefore True

iii. $\forall a$P(2, a, 0, a): $a^2 + 0^2 = a^2$
$a^2 = a^2$, therefore True.

iv. $\exists a \exists b \exists c$P(2, a, b, c): There indeed exists an a, b, c where n = 2, such as a = 3, b = 4, c = 5.
$3^2 + 4^2 = 5^2$
$25 = 25$, therefore True

v. $\forall a \exists b \exists c$P(2, a, b, c): This means that for every real number of a, there exists a number b and c for which the function has a certain truth value.
Here is an example that proves that this statement is True:
n =2. a is a. b = 1, $c = \sqrt{(a^2 + 1)}$.
Therefore, $a^2 + 1 = (\sqrt{a^2 + 1})^2$
$a^2 + 1 = a^2 + 1$, so $(a^2 + b^2 = c^n)$ holds. Indeed, for all values of a, there exists a number b and c that follow the condition.

vi. $\exists b \exists c \forall a$P(2, a, b, c): This has a False value as there does not exist a b and a c for any value of a

vii. $\forall b \forall c \exists a$P(2, a, b, c): This is False, for every value of b and c there exists an a cannot be true as $a = \sqrt{c^2 - b^2}$ and $c < 2$, therefore it cannot be for every value of c or b.

viii. $\forall a \forall b \exists c$P(2, a, b, c): This is True as this translates to for any value of a and any value of b there exists a c to make the function true. This works since we can transform the expression into: $c = \sqrt{a^2 + b^2}$, where there has to exist a c for every a and every b.

3.2 (b) Symbolize the following statement, using positive integers as your domain, P(n, a, b, c) as above, and G(x, y) for $x > y$: "The equation $x^n + y^n = z^n$ has no solution in the positive integers if $n > 2$."

$\forall n \forall a \forall b \forall c (G(n,2) \rightarrow \neg P(n,a,b,c))$

We symbolized using the universal quantifiers to indicate that the statement applies to all positive integers. G(n,2) makes sure that $n > 2$. We say "not" P to indicate that $a^n + b^n = c^n$ does not hold.

3.3 (c) Give the symbolic negation of (b), simplifying as much as possible.

Expression and its Negation in Symbolic Form

This is the expression found in (b):

$$\forall n \forall a \forall b \forall c \big(G(n,2) \rightarrow \neg P(n,a,b,c)\big)$$

Negation of the Expression

Step 1: Expanding the Implication

$$G(n,2) \rightarrow \neg P(n,a,b,c) \equiv \neg G(n,2) \lor \neg P(n,a,b,c)$$

Thus, the original statement is rewritten as:

$$\forall n \forall a \forall b \forall c \big(\neg G(n,2) \lor \neg P(n,a,b,c)\big)$$

Step 2: Negating the Entire Expression

$$\neg \forall n \forall a \forall b \forall c \big(\neg G(n,2) \lor \neg P(n,a,b,c)\big)$$

Step 3: Applying De Morgan's Laws

$$\exists n \exists a \exists b \exists c \neg (\neg G(n,2) \lor \neg P(n,a,b,c))$$

Using De Morgan's law again:

$$\neg(\neg G(n,2) \lor \neg P(n,a,b,c)) = G(n,2) \land P(n,a,b,c)$$

Final negated expression:

$$\exists n \exists a \exists b \exists c \big(G(n,2) \land P(n,a,b,c)\big)$$

3.4 (d) Restate (c) in plain, simple English.

The negation of the statement in b translates to "There exists positive integers n, a, b, c such that $n > 2$ and $a^n + b^n = c^n$."

4 Prove that the following inference rule is valid by (i) using other inference rules and identities, (ii) proving that the corresponding conditional statement is a tautology:

i) Argument and Proof using Inference Rules

1.	$p \rightarrow (q \land r)$	Premise
2.	$\neg q$	Premise
3.	$q \land r$	Assumption
4.	q	Conjunction Elimination on (3)
5.	$\bot$	Contradiction from (2) and (4)
6.	$\neg(q \land r)$	Negation Introduction on (3) and (5)
7.	$\neg p$	Modus Tollens on (1) and (6)

Thus, we have proven $\neg p$.

ii) Proof that the Conditional Statement is a Tautology Using Rules of Inference

We want to prove that the statement:

$$\big((p \to (q \land r)) \land \neg q\big) \to \neg p$$

is a tautology by using rules of inference.

1.	$p \to (q \land r)$	Premise
2.	$\neg q$	Premise
3.	$(p \to (q \land r)) \land \neg q$	Assumption
4.	p	Assumption
5.	$q \land r$	Modus Ponens on (1) and (4)
6.	q	Conjunction Elimination on (5)
7.	$\bot$	Contradiction from (2) and (6)
8.	$\neg p$	Negation Introduction on (4) and (7)
9.	$\big((p \to (q \land r)) \land \neg q\big) \to \neg p$	Implication Introduction on (3) to (8)

Since the derivation holds under any possible truth values, the statement is a tautology.

5 Determine if the following argument is valid: I save money when I don't go shopping or I go to work. I don't go to work, or I buy a coffee. Buying a tea is necessary to go shopping. I don't go shopping and I save money only when I don't buy a coffee. I didn't buy a tea. Therefore, I didn't go to work.

Let:

- S = "I go shopping"
- W = "I go to work"
- M = "I save money"
- C = "I buy a coffee"
- T = "I buy a tea"

1. "I save money when I don't go shopping or I go to work."
$$(\neg S \lor W) \to M$$
2. "I don't go to work, or I buy a coffee."
$$\neg W \lor C$$
3. "Buying a tea is necessary to go shopping."
$$S \to T$$
4. "I don't go shopping and I save money only when I don't buy a coffee."
$$(\neg S \land M) \to \neg C$$
5. "I didn't buy a tea."
$$\neg T$$
6. Conclusion: "Therefore, I didn't go to work."
$$\neg W$$

Checking validity

Step 1: Applying Modus Tollens to Premise (3)

$$\begin{aligned}
&S \to T \quad \text{(Premise 3)}\\
&\neg T \quad \text{(Premise 5)}\\
&\therefore \neg S \quad \text{(Modus Tollens)}
\end{aligned}$$

Step 2: Substituting $\neg S$ into Premise (4)

$$\begin{aligned}
&(\neg S \wedge M) \to \neg C \quad \text{(Premise 4)}\\
&\neg S \text{ is true, so for the antecedent to hold, } M \text{ must also be true.}\\
&\therefore M \text{ is true.}\\
&\therefore \neg C \quad \text{(from Premise 4)}
\end{aligned}$$

Step 3: Substituting $\neg C$ into Premise (2)

$$\begin{aligned}
&\neg W \vee C \quad \text{(Premise 2)}\\
&\neg C \text{ is true. The only way for } \neg W \vee C \text{ to be true is if } \neg W \text{ is true.}\\
&\therefore \neg W
\end{aligned}$$

Therefore,

Since we have logically gotten the answer $\neg W$ from the premises, the argument is valid.

6 Prove or disprove each of the following statements.

6.1 (a)

Let a and b be integers. We want to prove or disprove the statement:

If $a^3(b^2-1)$ is even, then a is even or b is odd.

We want to use the contrapositive:

If a is odd and b is even, then $a^3(b^2-1)$ is odd.

Assume a is odd and b is even. Then,

$$\begin{aligned}
a &= 2m+1 \quad \text{for some integer } m,\\
b &= 2n \quad \text{for some integer } n.
\end{aligned}$$

Compute a^3 and b^2-1

$$\begin{aligned}
a^3 &= (2m+1)^3 = 8m^3 + 12m^2 + 6m + 1 \quad \text{(which is odd),}\\
b^2 - 1 &= (2n)^2 - 1 = 4n^2 - 1 = (2n-1)(2n+1) \quad \text{(which is odd).}
\end{aligned}$$

Compute $a^3(b^2-1)$

$$a^3(b^2-1) = (\text{odd integer}) \times (\text{odd integer}) = \text{odd integer}.$$

Since $a^3(b^2-1)$ is odd, the contrapositive statement is true. Thus, the original statement is also true.

6.2 (b)

Let a and b be integers. We are given that $a+b$ is odd. We want to determine whether a^2+b^2 is necessarily odd.

An integer sum is odd if and only if one of the numbers is even and the other is odd. That is, we assume without loss of generality that:

- a is odd ($a = 2m+1$ for some integer m), and
- b is even ($b = 2n$ for some integer n).

Now, we compute the squares:

$$\begin{aligned} a^2 &= (2m+1)^2 = 4m^2+4m+1 = 2(2m^2+2m)+1 \quad \text{(odd)} \\ b^2 &= (2n)^2 = 4n^2 = 2(2n^2) \quad \text{(even)} \end{aligned}$$

Thus, their sum is:

$$a^2+b^2 = \text{odd} + \text{even} = \text{odd}.$$

Since the same reasoning applies if a is even and b is odd, we conclude that a^2+b^2 is always odd when $a+b$ is odd.

Conclusion: The statement is true: If $a+b$ is odd, then a^2+b^2 is odd.

6.3 (c)

Let a and b be integers. We are given that ab is even. We want to determine whether a^2+b^2 is necessarily even.

To disprove the statement, we provide a counterexample. Consider:

- $a = 2$ (even),
- $b = 1$ (odd).

Now, we compute the squares:

$$\begin{aligned} a^2 &= 2^2 = 4 \quad \text{(even)} \\ b^2 &= 1^2 = 1 \quad \text{(odd)} \end{aligned}$$

Thus, their sum is:

$$a^2+b^2 = 4+1 = 5, \quad \text{(odd)}.$$

Since we found a case where ab is even but a^2+b^2 is odd, the statement is false in general.

Conclusion: The given statement is false. If ab is even, it does not necessarily imply that a^2+b^2 is even.

6.4 (d)

We want to determine the validity of the statement:

$$\sqrt{x+y} = \sqrt{x} + \sqrt{y}$$

for real numbers x and y, and whether it holds if and only if $x=0$ or $y=0$.

Step 1: Proving from left to right.

For the equality to hold, squaring both sides must yield the same result:

$$(\sqrt{x}+\sqrt{y})^2 = x+y.$$

$$x + 2\sqrt{xy} + y = x + y.$$

$$2\sqrt{xy} = 0.$$

$$\sqrt{xy} = 0.$$

Since the square root function is only zero when its argument is zero, we must have:

$$xy = 0.$$

This implies that either $x = 0$ or $y = 0$.

Step 2: From right to left

Assume $x = 0$

$$\sqrt{0 + y} = \sqrt{y}$$

Then, Assume $y = 0$

$$\sqrt{x + 0} = \sqrt{x}$$

Conclusion: Since we have proven both directions, the given equality holds if and only if $x = 0$ or $y = 0$ and the statement is True.

6.5 (e)

If the average of four distinct integers is 10, then one of the integers is greater than 11.

Let the four distinct integers be a, b, c, d. The given condition states that their average is 10:

$$\frac{a + b + c + d}{4} = 10.$$

Multiplying both sides by 4 gives:

$$a + b + c + d = 40.$$

Construct a Counterexample

Consider four distinct integers that sum to 40:

$$7, 8, 10, 15.$$

The sum is:

$$7 + 8 + 10 + 15 = 40.$$

Since 15 is greater than 11, this does not disprove the statement.

$$9, 9, 10, 12.$$

This set is not valid since 9 appears twice. If we try to have distinct numbers, lower or equal 11:

$$8, 9, 10, 11.$$

This gives a total of 38, therefore the statement is not respected. However, 38 is lower than 40 and if we use other distinct numbers we will have an even lower sum as we have used to highest distinct numbers in this example.

Conclusion

At least one number must be greater than 11 to ensure the sum reaches 40 while keeping all values distinct. Thus, the original statement is True.

7 Prove that $\log_2 9$ is irrational. Use this to give a constructive proof that there exist irrational numbers x and y such that x^y is rational.

Suppose, for contradiction, that $\log_2 9$ is rational. Then we can write:

$$\log_2 9 = \frac{p}{q},$$

where p and q are integers with $q > 0$.

Rewriting the equation in exponential form,

$$2^{p/q} = 9.$$

Raising both sides to the power of q gives:

$$2^p = 9^q.$$

Since 2^p is a power of 2 and $9^q = (3^2)^q = 3^{2q}$ is a power of 3, we obtain:

$$2^p = 3^{2q}.$$

This is a contradiction because the left-hand side is a power of 2 while the right-hand side is a power of 3, and no power of 2 can equal a power of 3 unless both sides are 1, which is not possible here. Thus, our assumption was false, and $\log_2 9$ is irrational.

Proving x^y is Rational

We now construct irrational numbers x and y such that x^y is rational. Consider:

$$x = \log_2 9, \quad y = \log_2 4.$$

Since we have already shown that $\log_2 9$ is irrational, and it is well known that $\log_2 4 = 2$, we compute:

$$x^y = (\log_2 9)^2.$$

Rewriting using exponent properties,

$$x^y = (\log_2 9)^2 = \log_2 9^2 = \log_2 81.$$

Since $81 = 2^6$, we have:

$$\log_2 81 = 6,$$

which is a rational number. Thus, we have found irrational numbers x and y such that x^y is rational.

Conclusion: We have proven that $\log_2 9$ is irrational and used it to construct irrational numbers x and y such that x^y is rational.