GAME THEORETICAL INFERENCE OF HUMAN BEHAVIOUR IN SOCIAL NETWORKS

[N. Pagan & F. Dörfler, *"Game theoretical inference of human behaviour in social networks"*. Nature Communications (forthcoming).]

Torino, 28.11.2019

NICOLÒ PAGAN
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AUTOMATIC CONTROL LABORATORY

ETH zürich

OBSERVATIONS

Actors decide with whom they want to interact.

01

OBSERVATIONS

- Actors decide with whom they want to interact.
- Network positions provide benefits to the actors.


Forbes

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Using Social Networks To Advance Your Career

Adi Gaskell Contributor

Shutterstock

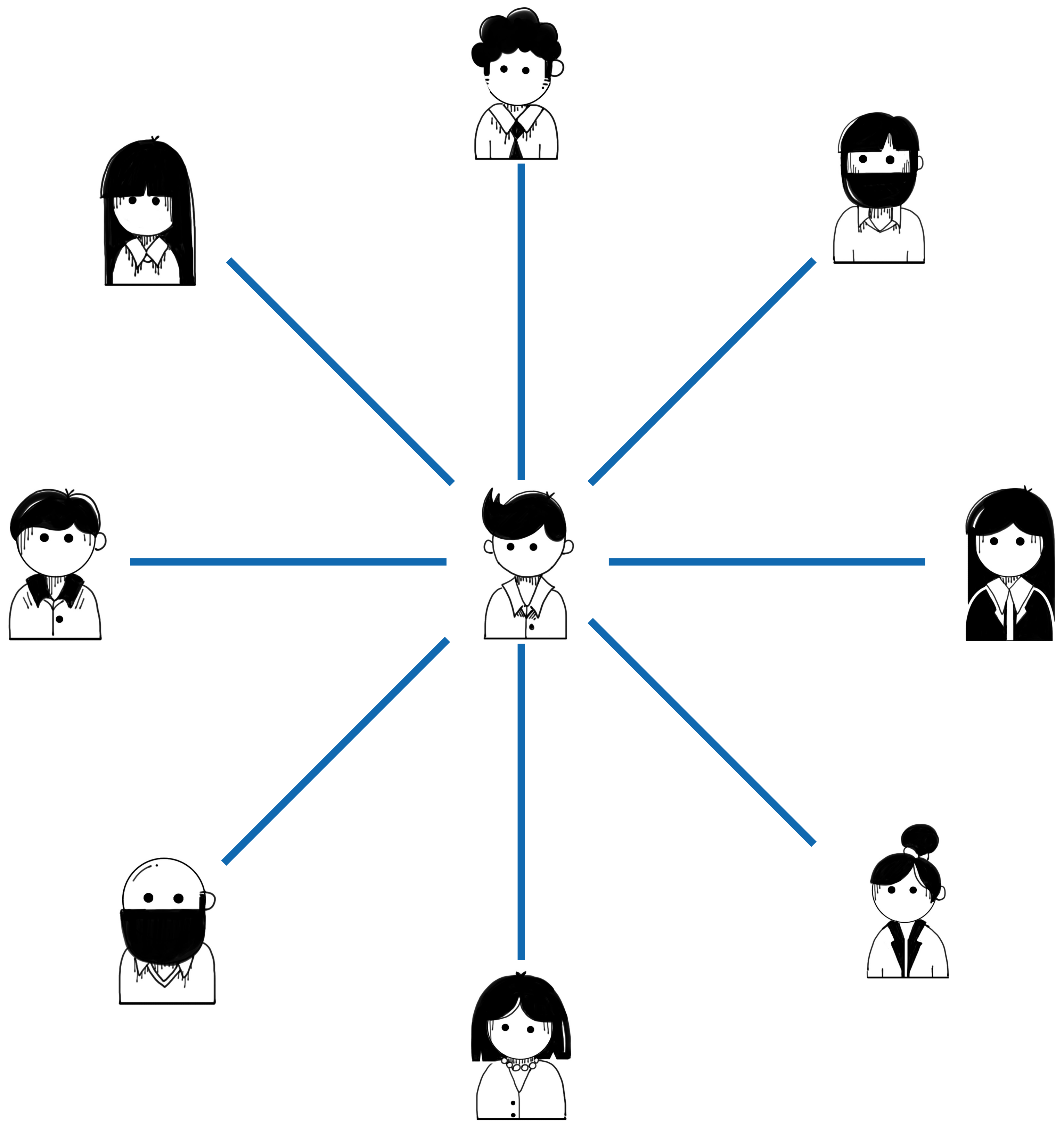
"It's not what you know, it's who you know" is one of those phrases

01

SOCIAL NETWORK POSITIONS' BENEFITS

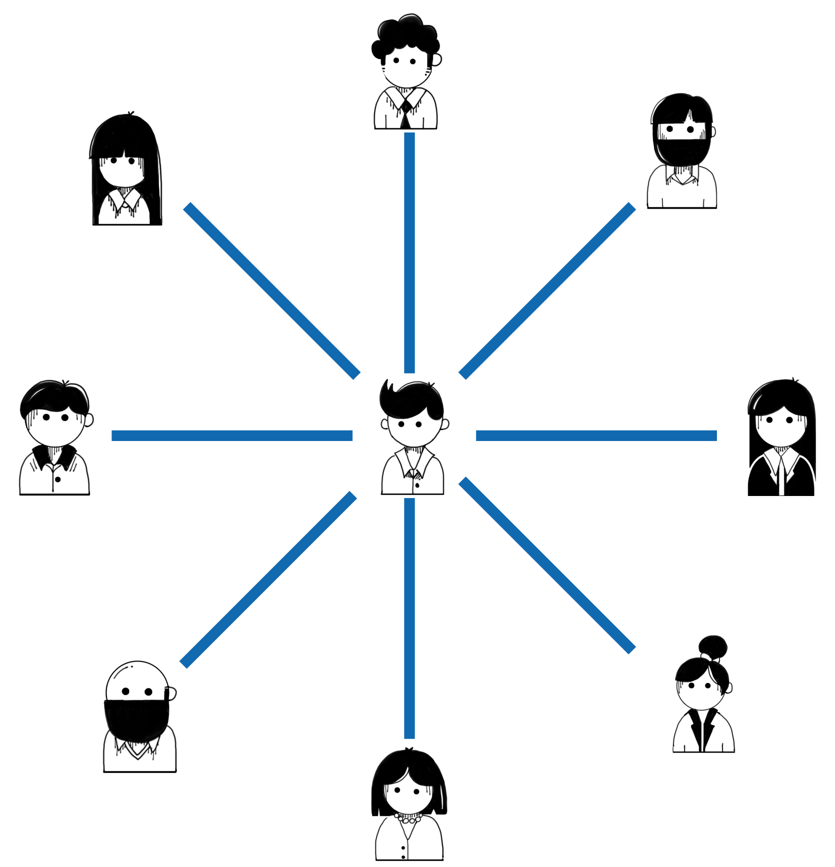
Social Influence

The more people we are connected to, the more we can influence them.



02

SOCIAL NETWORK POSITIONS' BENEFITS



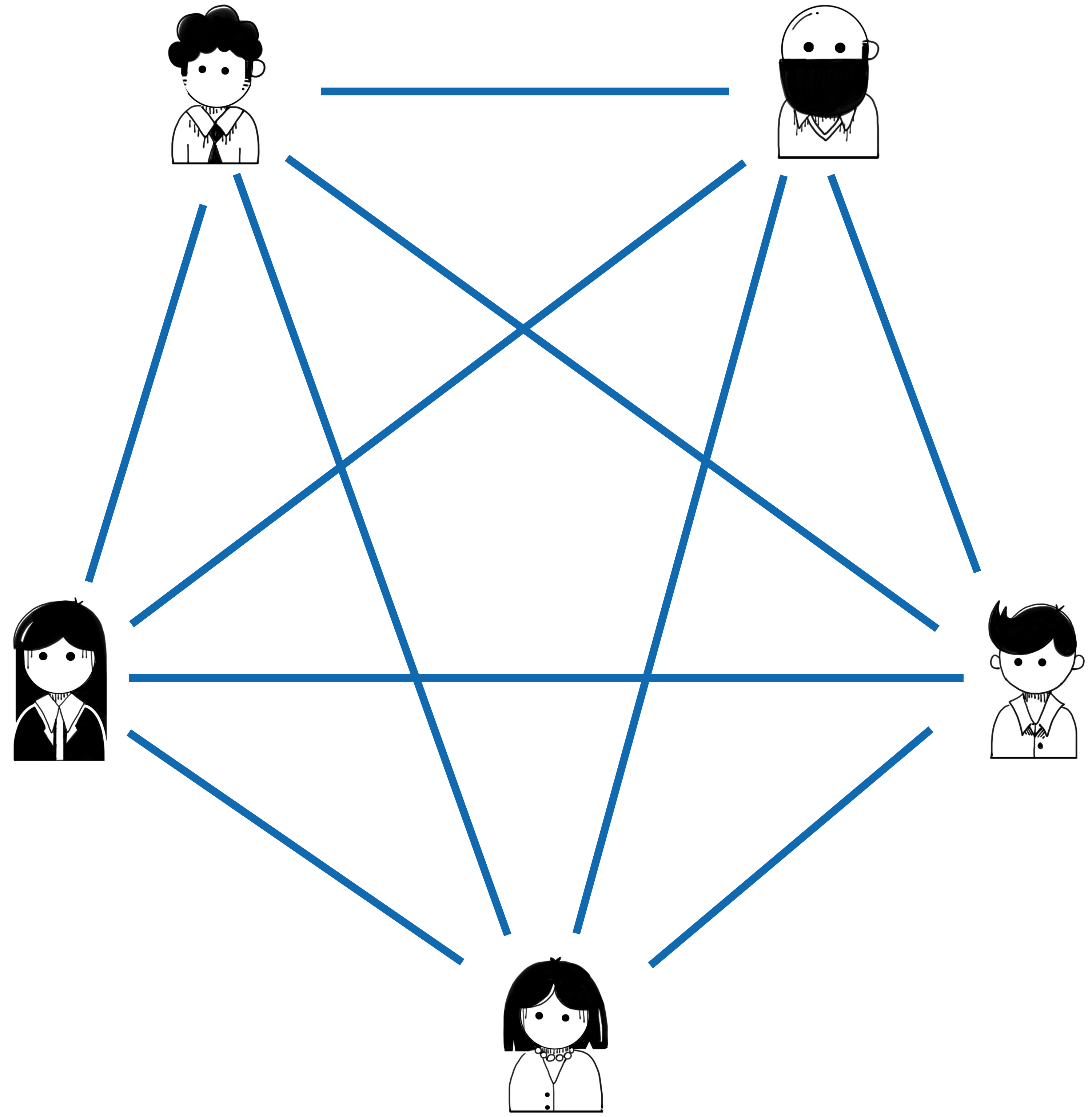
Social Influence

The more people we are connected to, the more we can influence them.

Social Support

The more our friends' friends are our friends, the safer we feel.

[Coleman, J. Foundations of Social Theory, 1990]



02

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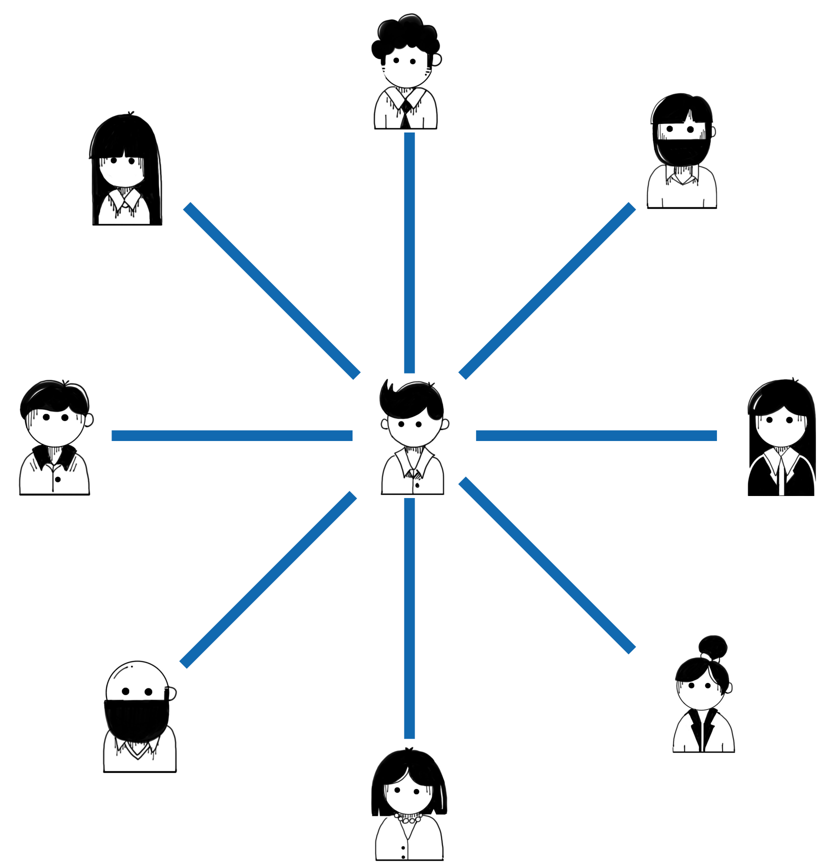

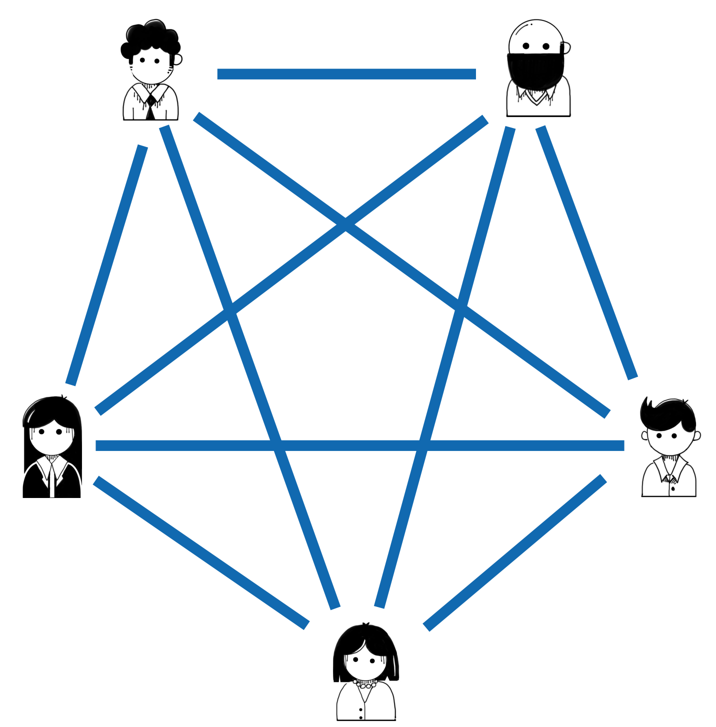

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Brokerage

The more we are on the path between people, the more we can control.

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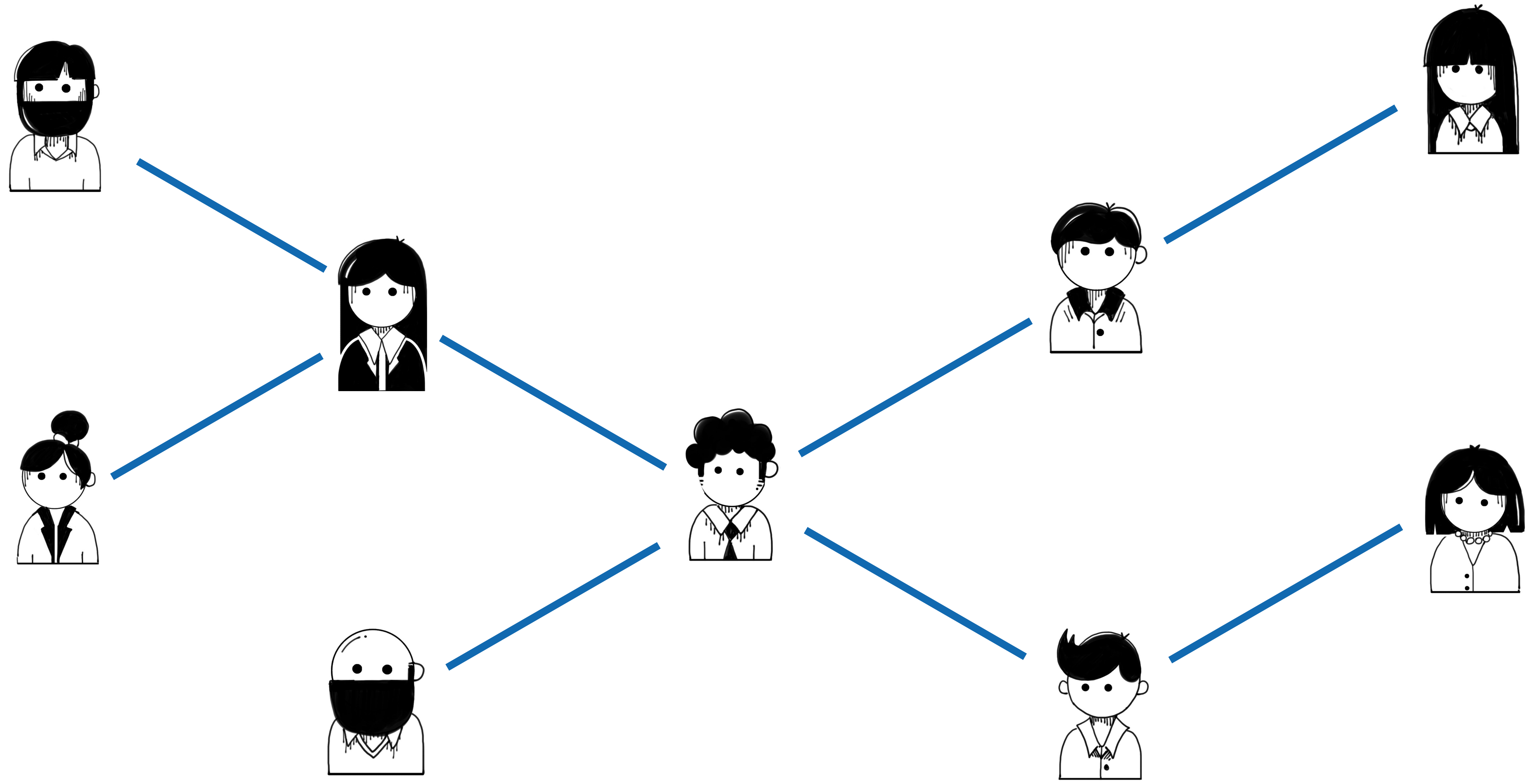

02

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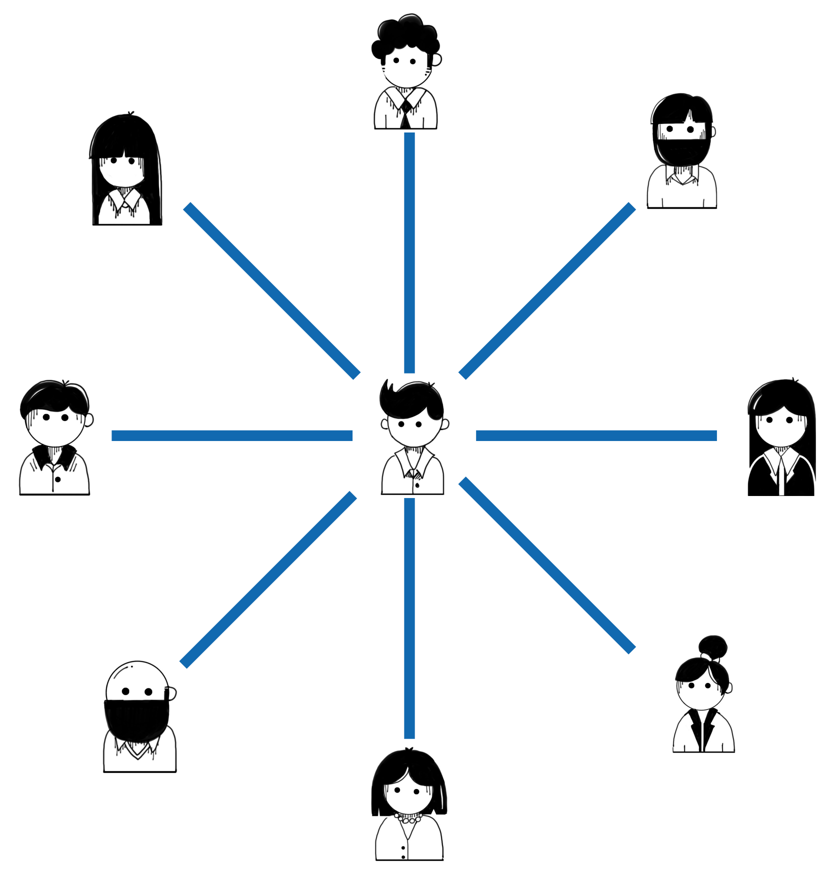
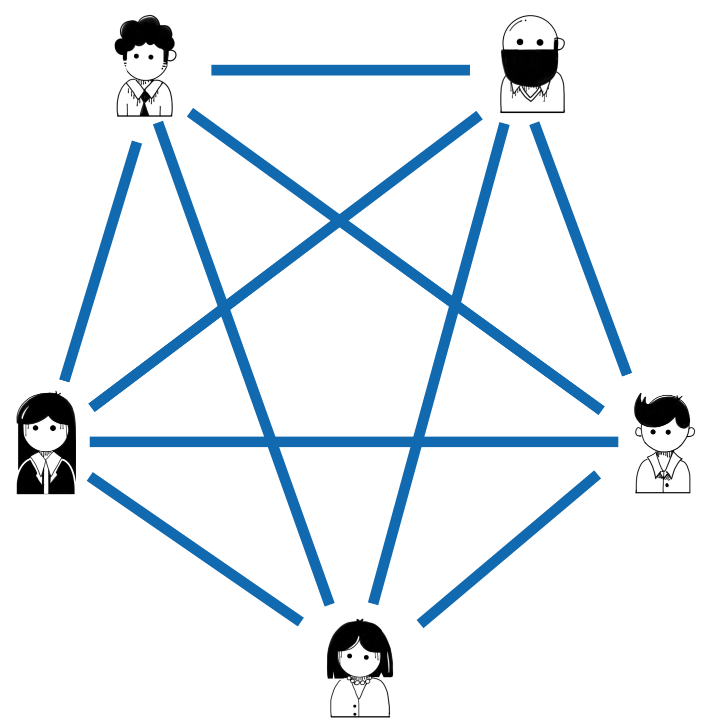
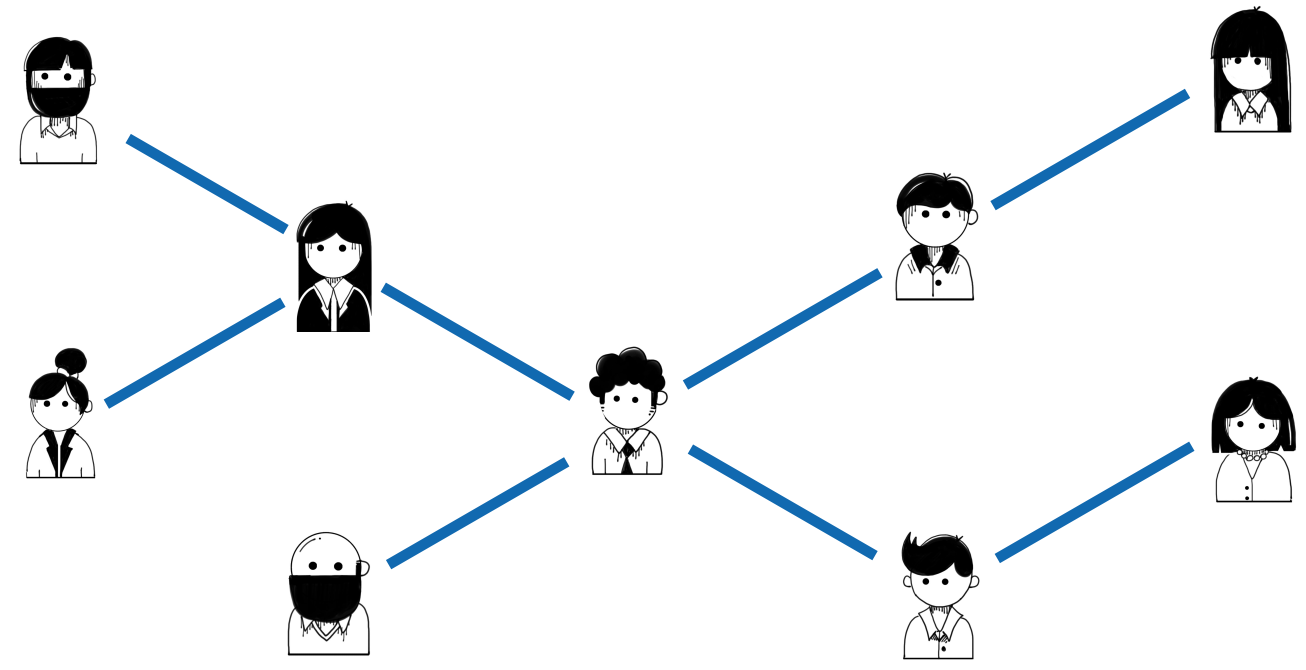


Degree Centrality



Clustering coefficient



Betwenness Centrality

02

N. Pagan & F. Dörfler, *“Game theoretical inference of human behaviour in social networks”*

SOCIAL NETWORK FORMATION MODEL

Directed weighted network $\mathscr{G}$ with $\mathscr{N} = \{1,\ldots,N\}$ agents.

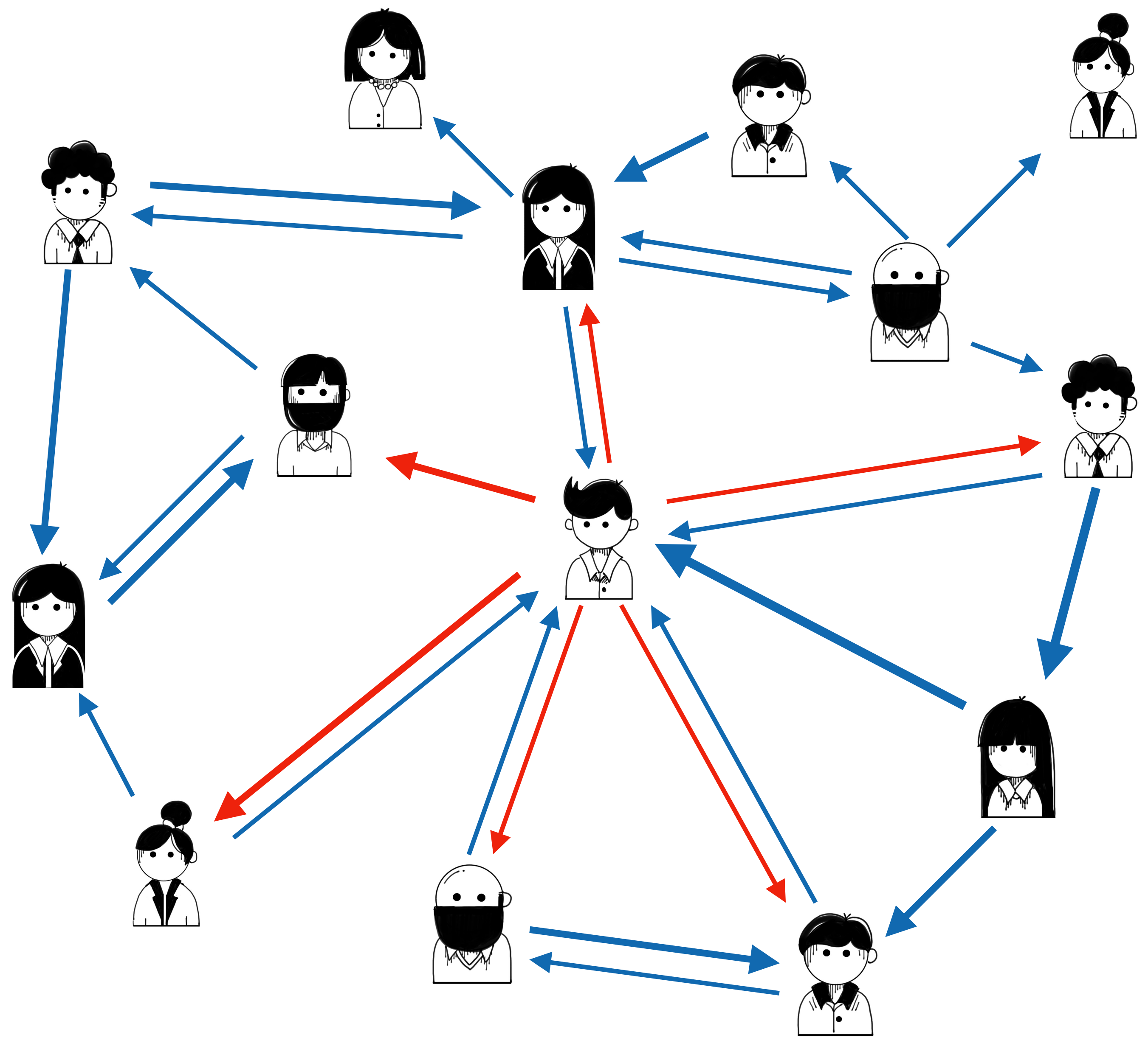
$a_{ij} \in [0,1]$ quantifies the importance of the friendship among i and j from i's point of view.

A typical action of agent i is:

$$a_i = \left[a_{i1}, \ldots, a_{i,i-1}, a_{i,i+1}, \ldots, a_{iN}\right] \in \mathscr{A} = [0,1]^{N-1},$$

Every agent i is endowed with a payoff function V_i and is looking for

$$a_i^\star \in \arg\max_{a_i \in \mathscr{A}} V_i(a_i, \mathbf{a}_{-i})$$



03

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N. Pagan & F. Dörfler, *"Game theoretical inference of human behaviour in social networks"*

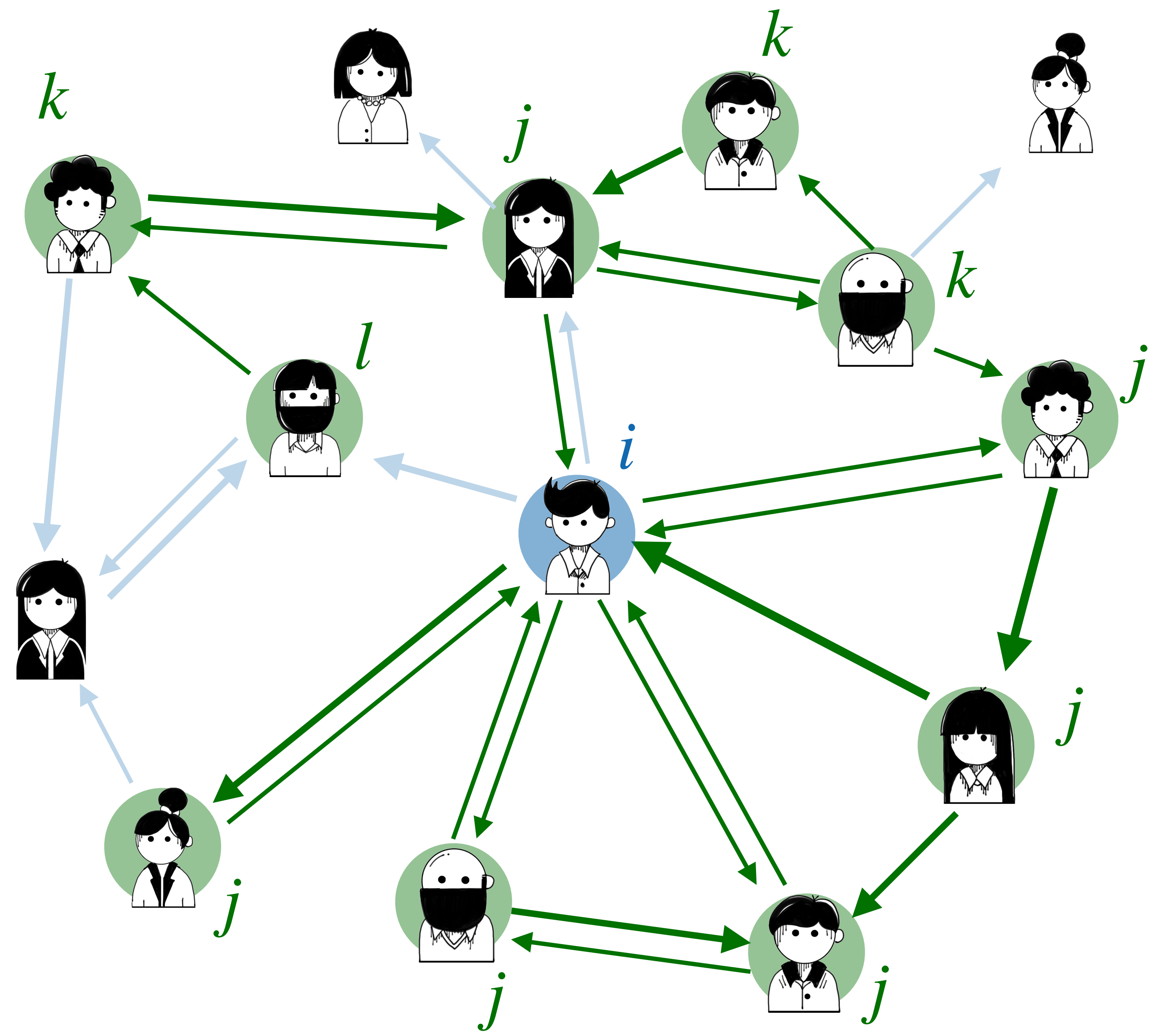
PAYOFF FUNCTION

$$V_i(a_i, \mathbf{a}_{-i}) = \boxed{t_i(a_i, \mathbf{a}_{-i}, \delta_i)}$$

Social Influence on friends

$$t_i(a_i, \mathbf{a}_{-i}, \delta_i) = \sum_{j \neq i} a_{ji} \underbrace{+\delta_i \sum_{k \neq j} \sum_{j \neq i} a_{kj} a_{ji}}_{\text{paths of length 2}} \underbrace{+ \delta_i^2 \sum_{l \neq k} \sum_{k \neq j} \sum_{j \neq i} a_{lk} a_{kj} a_{ji}}_{\text{paths of length 3}}$$

[Jackson, M. O. & Wolinsky, A. A strategic model of social and economic networks. J. Econom. Theory 71, 44–74 (1996).]



04

PAYOFF FUNCTION

$$V_i(a_i, \mathbf{a}_{-i}) = t_i(a_i, \mathbf{a}_{-i}, \delta_i) + u_i(a_i, \mathbf{a}_{-i})$$

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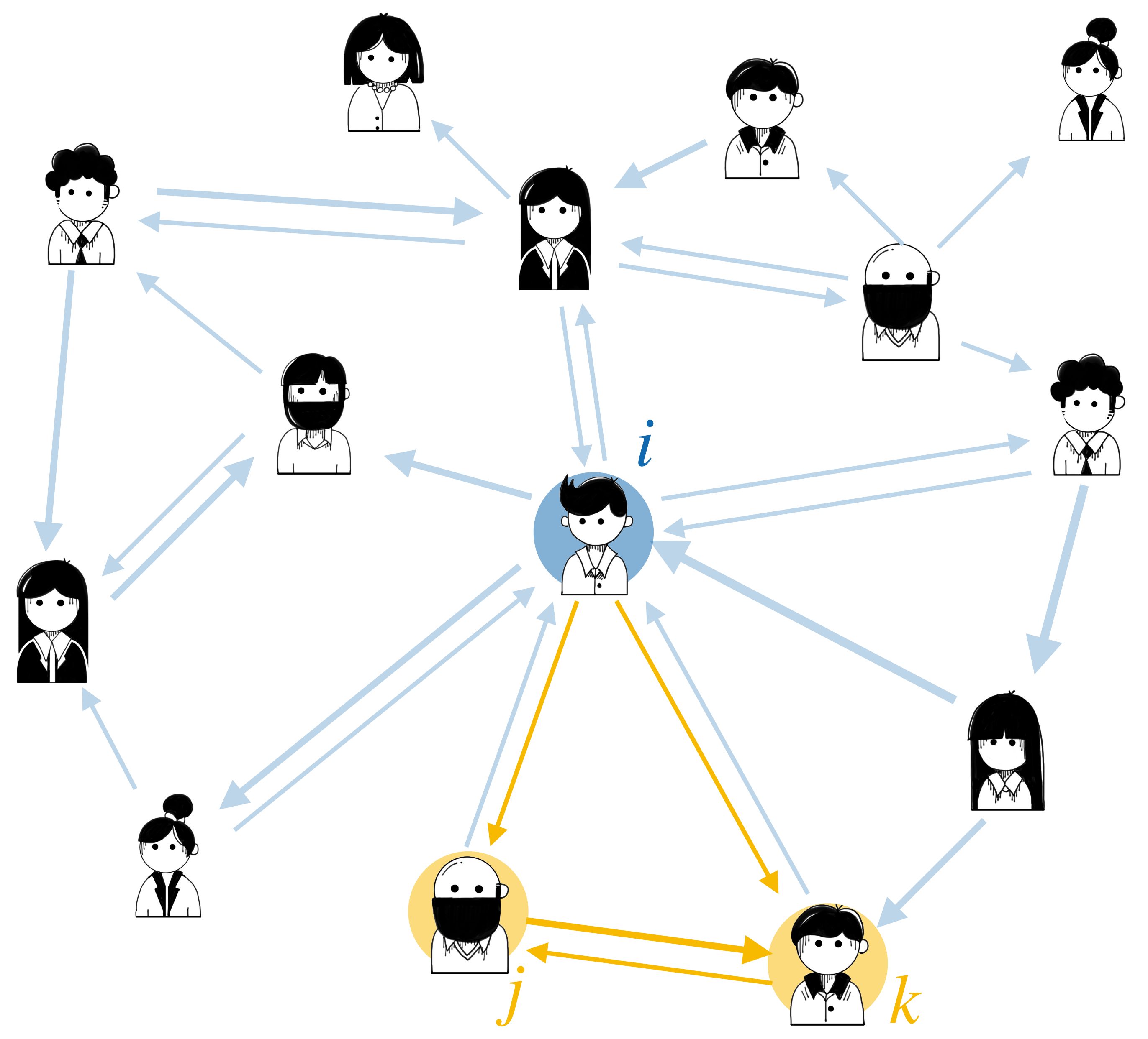
[Jackson, M. O. & Wolinsky, A. A strategic model of social and economic networks. J. Econom. Theory 71, 44–74 (1996).]

Clustering Coefficient

$$u_i(a_i, \mathbf{a}_{-i}) = \sum_{j \neq i} a_{ij} \left(\sum_{k \neq i,j} a_{ik} a_{kj} \right)$$

[Burger, M. J. & Buskens, V. Social context and network formation: an experimental study. Social Networks 31, 63–75 (2009)].



04

PAYOFF FUNCTION

$$V_i(a_i, \mathbf{a}_{-i}) = t_i(a_i, \mathbf{a}_{-i}, \delta_i) + u_i(a_i, \mathbf{a}_{-i}) - \boxed{c_i(a_i)}$$

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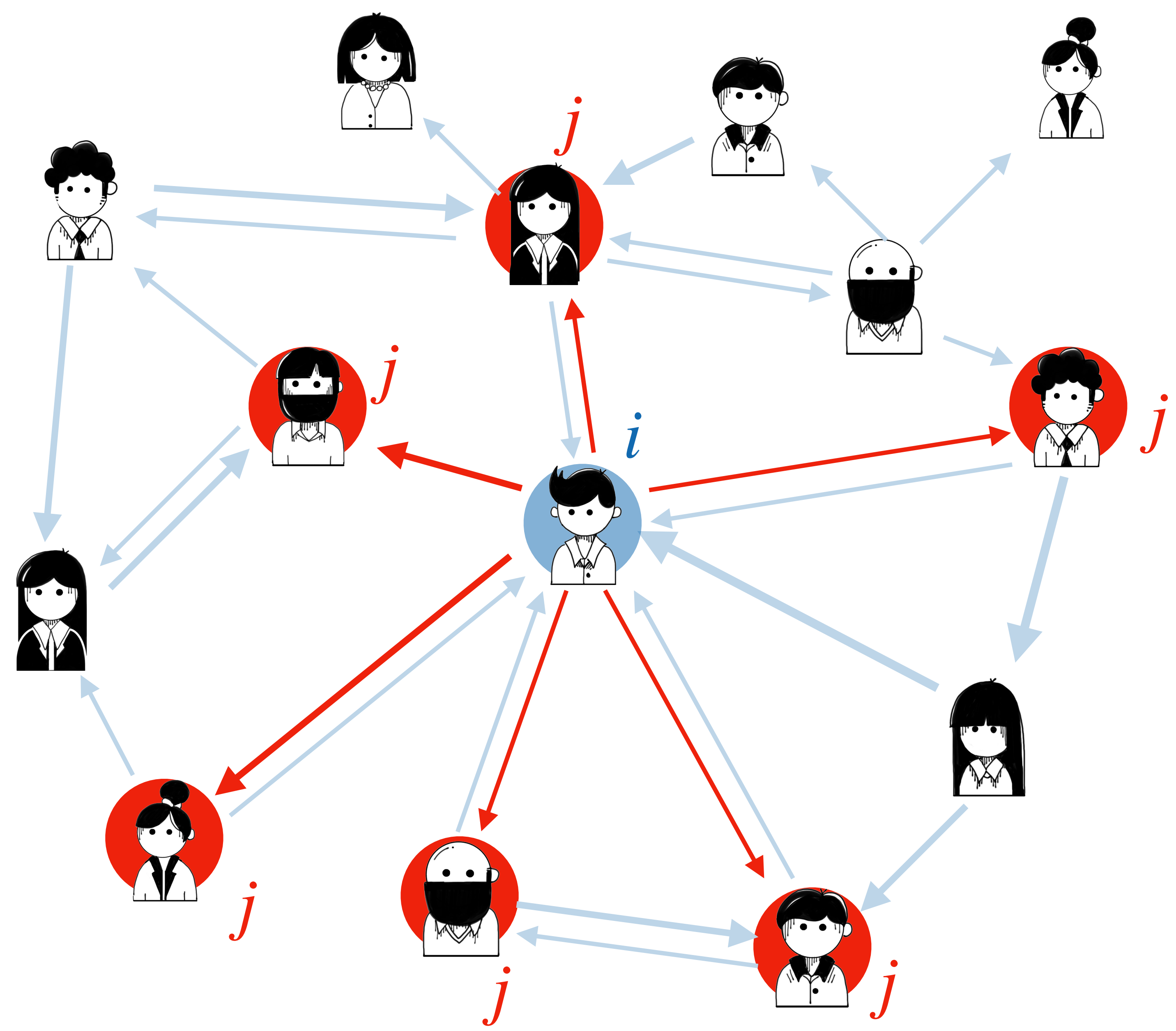
Cost

$$c_i(a_i) = \sum_{j \neq i} a_{ij}$$



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PAYOFF FUNCTION

$$V_i(a_i, \mathbf{a}_{-i} \,|\, \theta_i) = \alpha_i \, t_i(a_i, \mathbf{a}_{-i}, \delta_i) + \beta_i \, u_i(a_i, \mathbf{a}_{-i}) - \gamma_i \, c_i(a_i)$$

$$\theta_i = \{\alpha_i, \beta_i, \gamma_i, \delta_i\}$$

$$\alpha_i \geq 0, \beta_i \in \mathbb{R}, \gamma_i > 0, \delta_i \in [0,1]$$

Social Influence on friends

$$t_i(a_i, \mathbf{a}_{-i}, \delta_i) = \sum_{j \neq i} a_{ji} \underbrace{+\delta_i \sum_{k \neq j} \sum_{j \neq i} a_{kj} a_{ji}}_{\text{paths of length 2}} \underbrace{+ \delta_i^2 \sum_{l \neq k} \sum_{k \neq j} \sum_{j \neq i} a_{lk} a_{kj} a_{ji}}_{\text{paths of length 3}}$$

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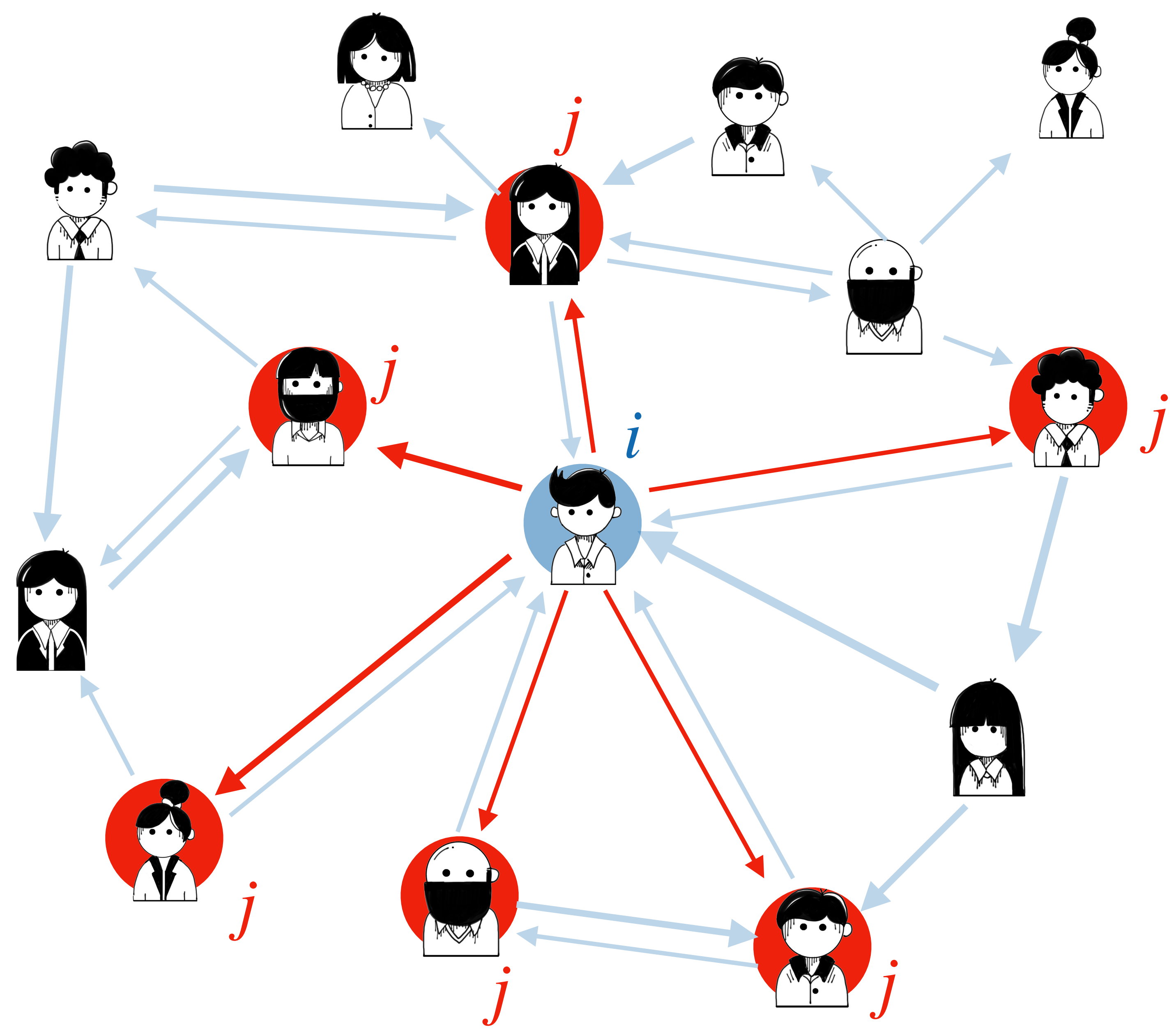


04

$$
\begin{tikzcd}
\mathcal{N} \arrow[dr] & \\
V_i \arrow[r] & \text{Game } (\mathcal{N}, V_i, \mathcal{A}) \\
\mathcal{A} \arrow[ur] &
\end{tikzcd}
$$

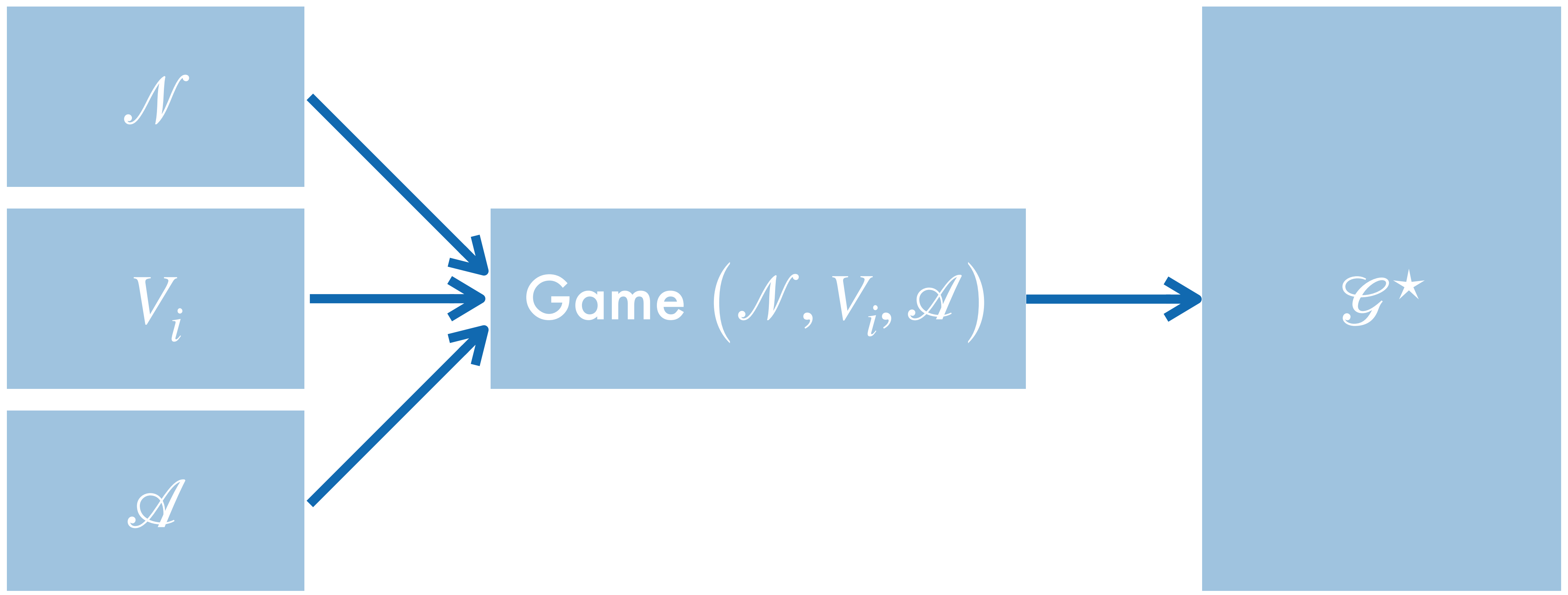
05

NASH EQUILIBRIUM

Definition.

The network $\mathscr{G}^\star$ is a Nash Equilibrium if for all agents i:

$$V_i\left(a_i, \mathbf{a}_{-\mathbf{i}}^\star \,|\, \theta_i\right) \leq V_i\left(a_i^\star, \mathbf{a}_{-\mathbf{i}}^\star \,|\, \theta_i\right),\ \forall a_i \in \mathscr{A} \quad \Longrightarrow \quad \forall i,\ a_i^\star \in \arg\max_{a_i \in \mathscr{A}} V_i\left(a_i, \mathbf{a}^\star_{-i} \,|\, \theta_i\right)$$



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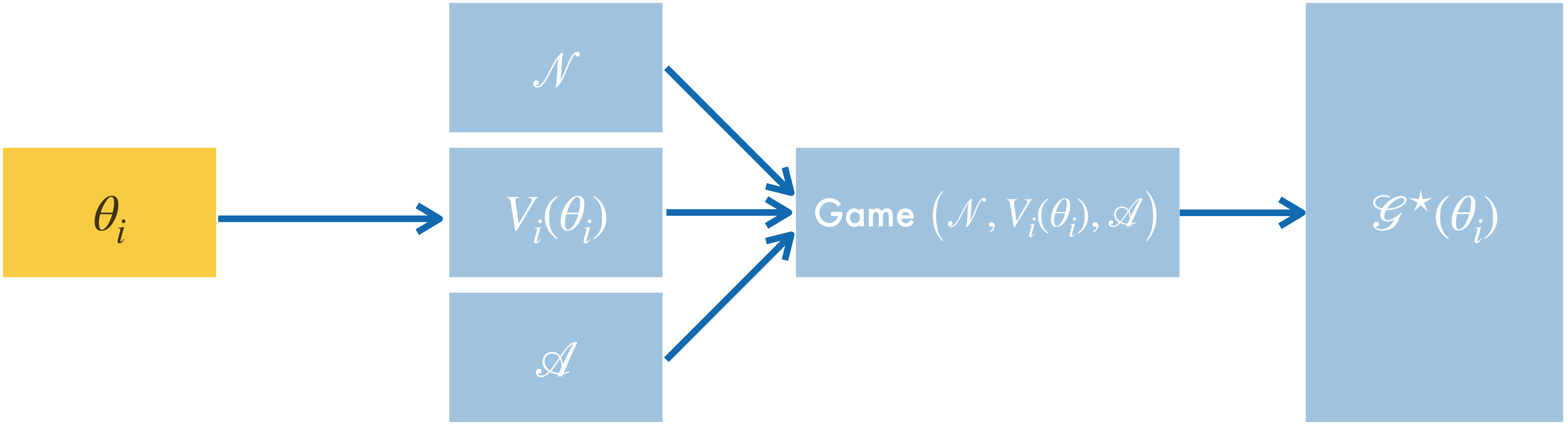
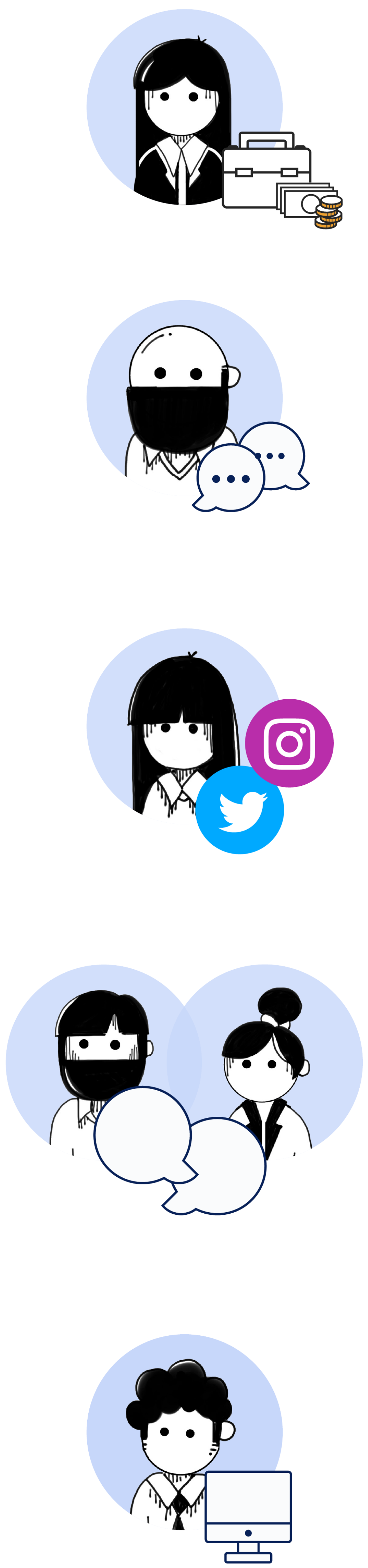


Question: Given θ_i, which $\mathscr{G}^\star$ is in equilibrium?

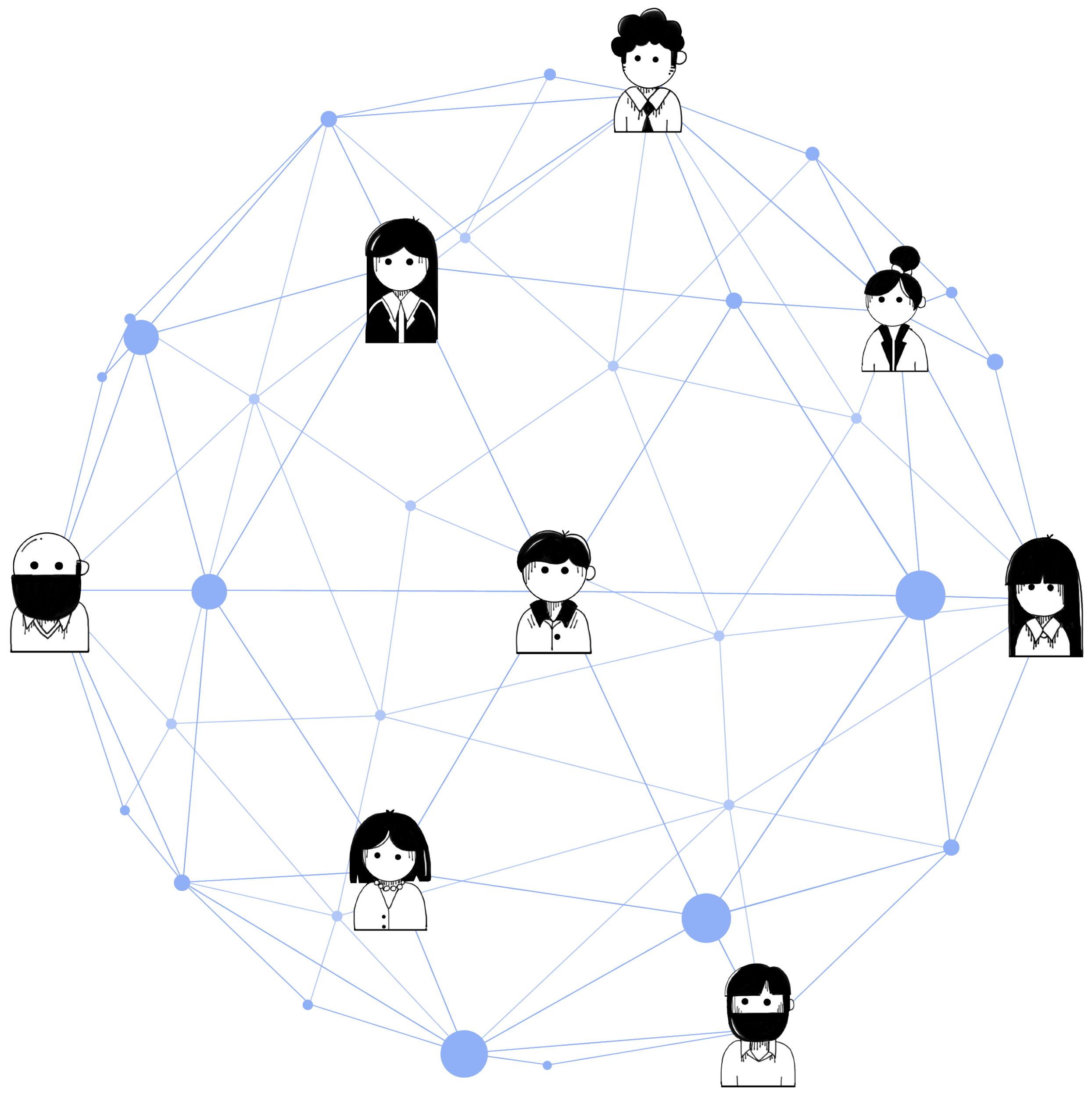
INDIVIDUAL BEHAVIOUR θ_i



DETERMINE

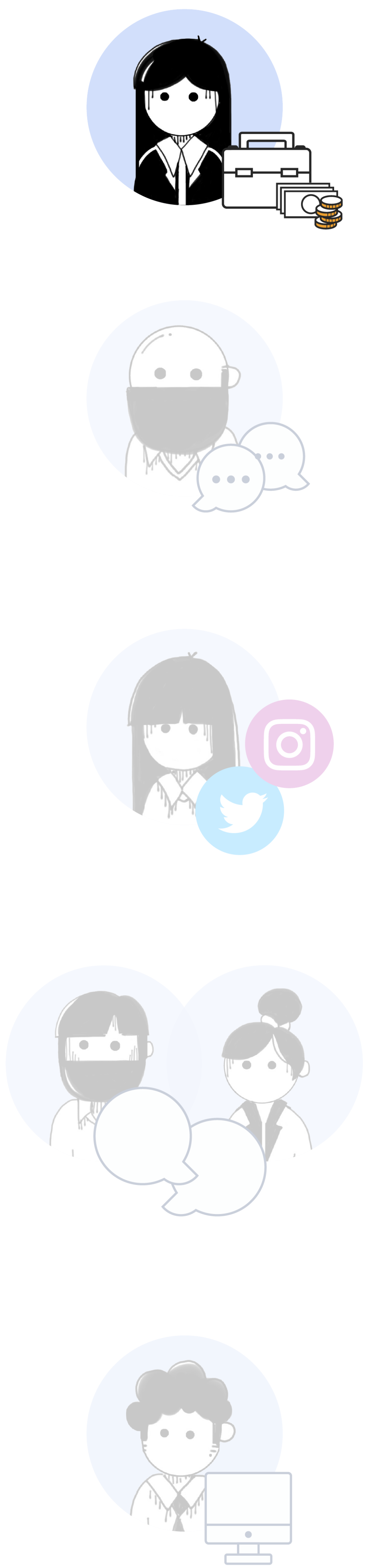
$$\forall i,\ a_i^\star \in \arg\max_{a_i \in \mathscr{A}} V_i\left(a_i, \mathbf{a}_{-i}^\star \,|\, \theta_i\right)$$

STRATEGIC NETWORK FORMATION MODEL

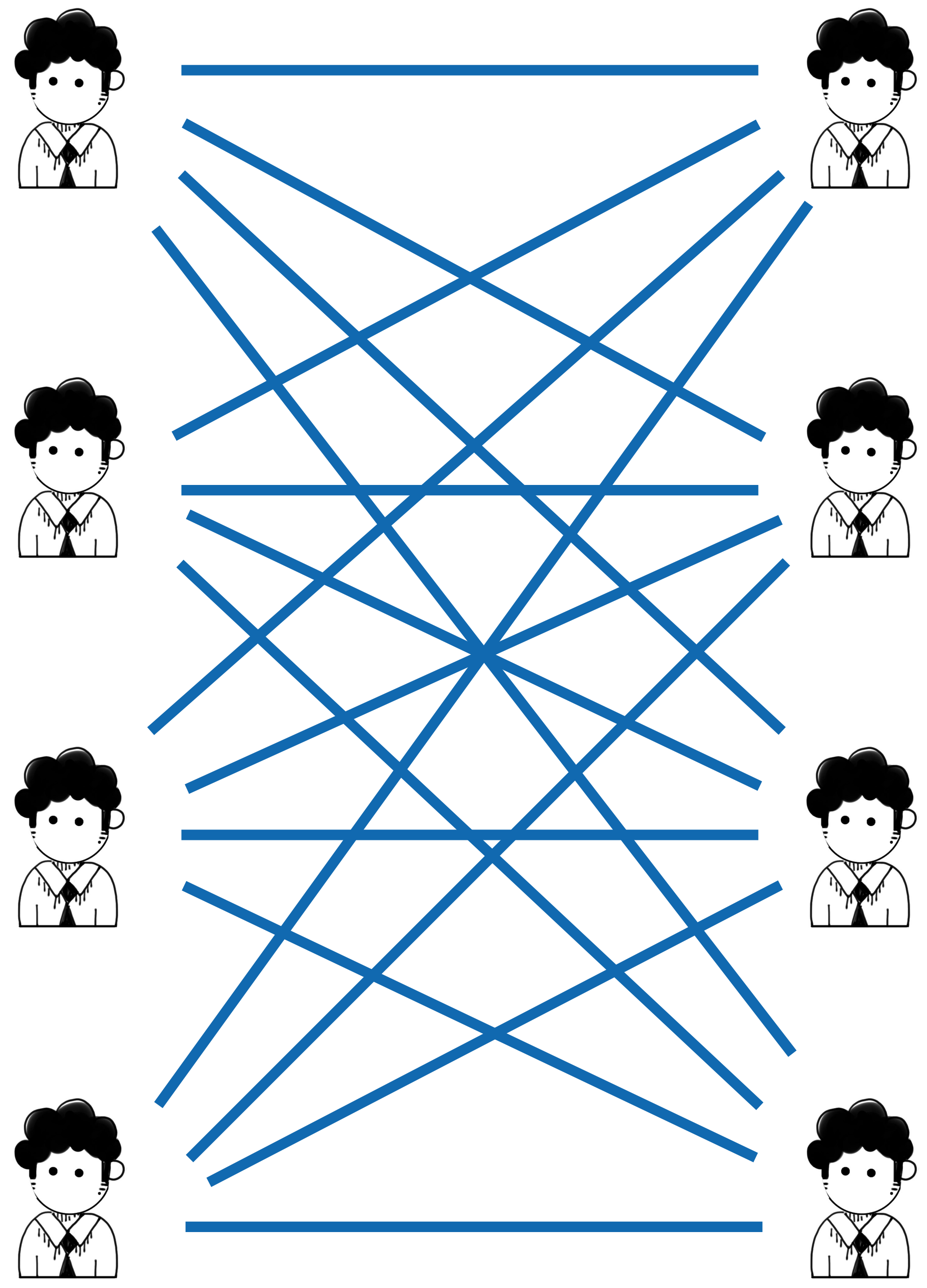


SOCIAL NETWORK STRUCTURE $\mathscr{G}^\star\left(\theta_i\right)$

STRATEGIC NETWORK
FORMATION MODEL

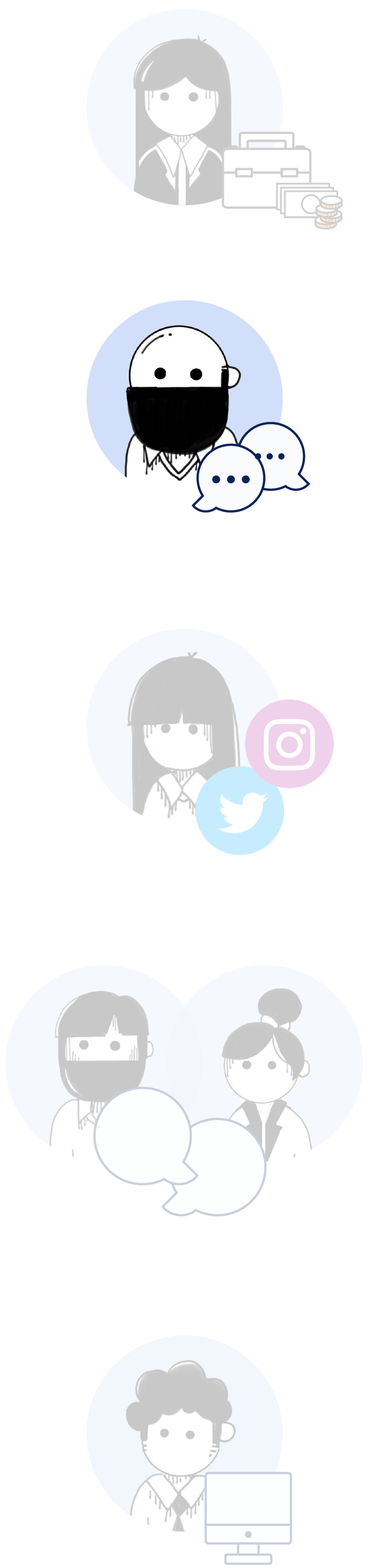


Assumption: Homogeneous agents



[Buechel, B. & Buskens, V. The dynamics of closeness and betweenness. J.Math. Sociol. 37(), 159–191 (2013)]

STRATEGIC NETWORK FORMATION MODEL



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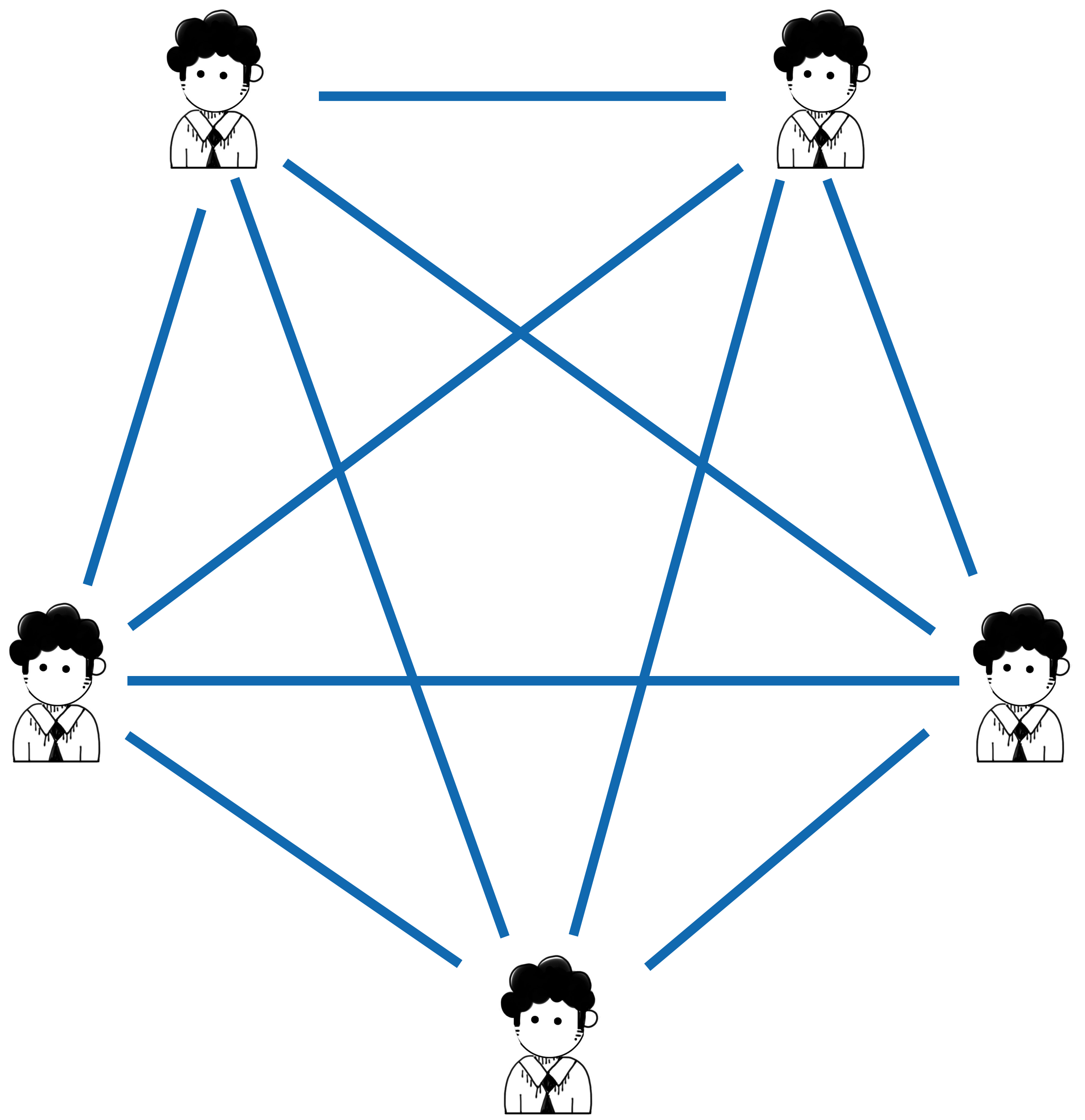

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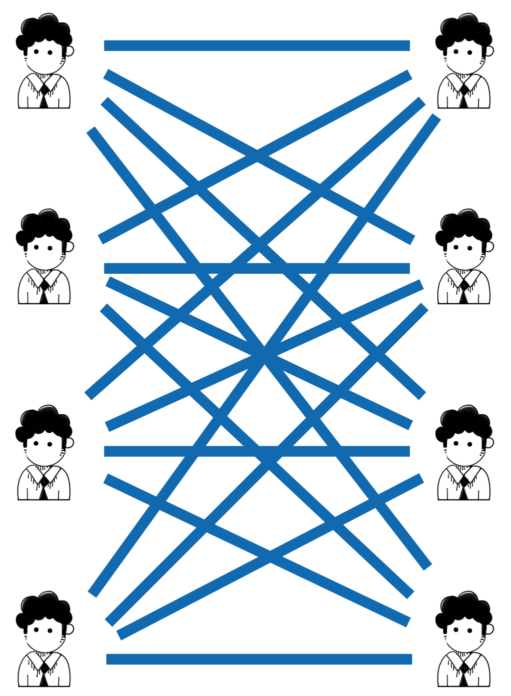

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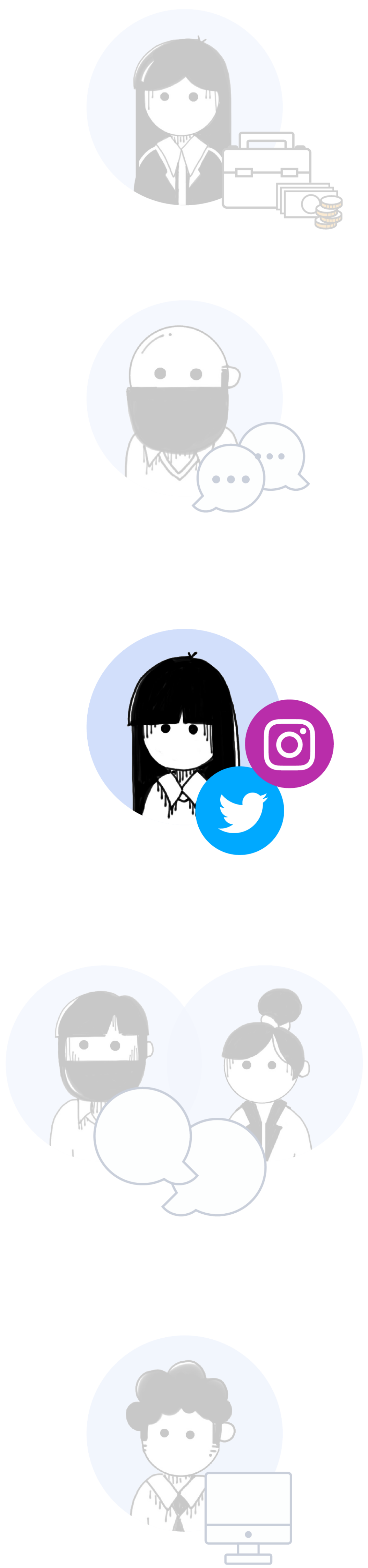

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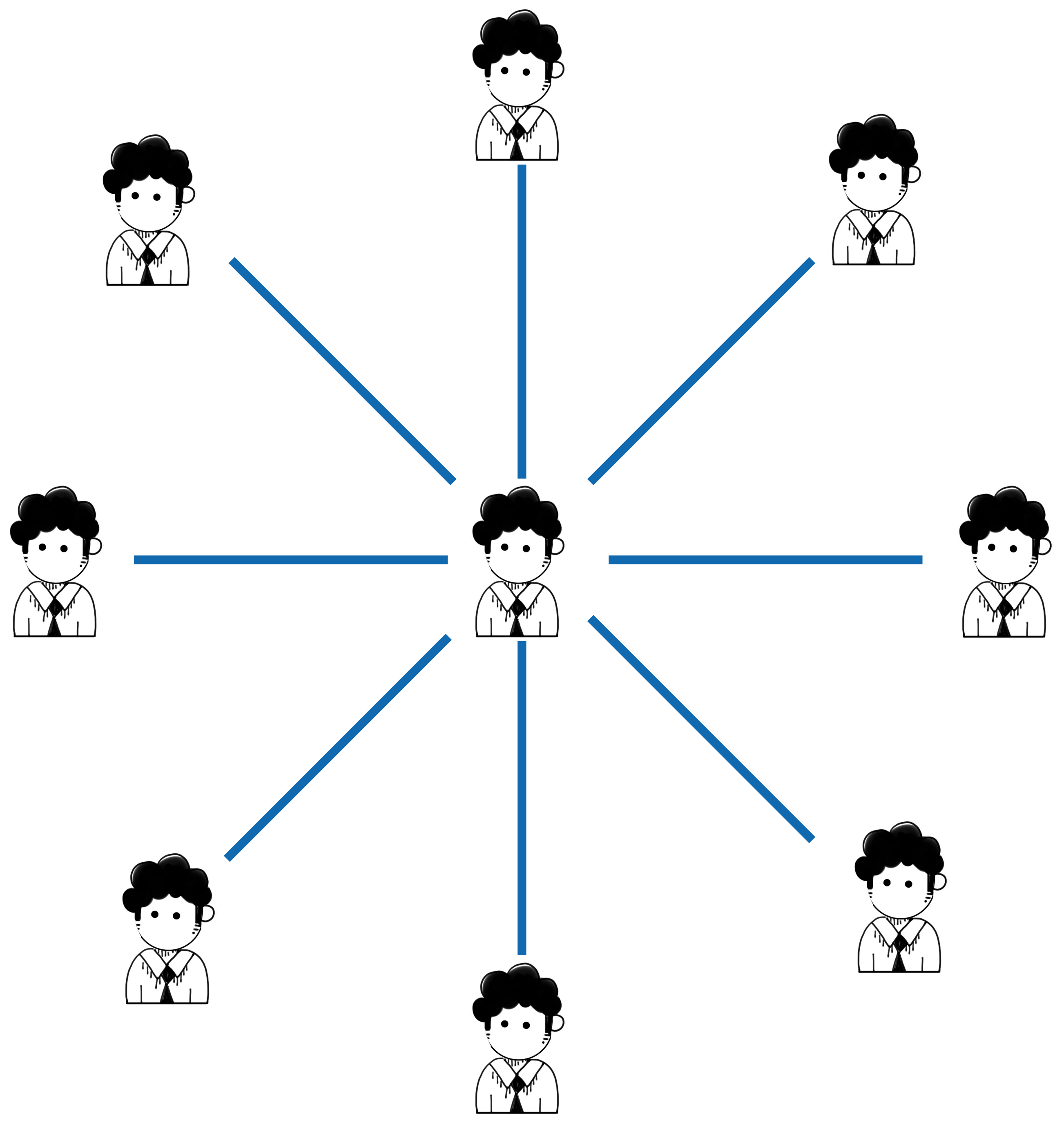
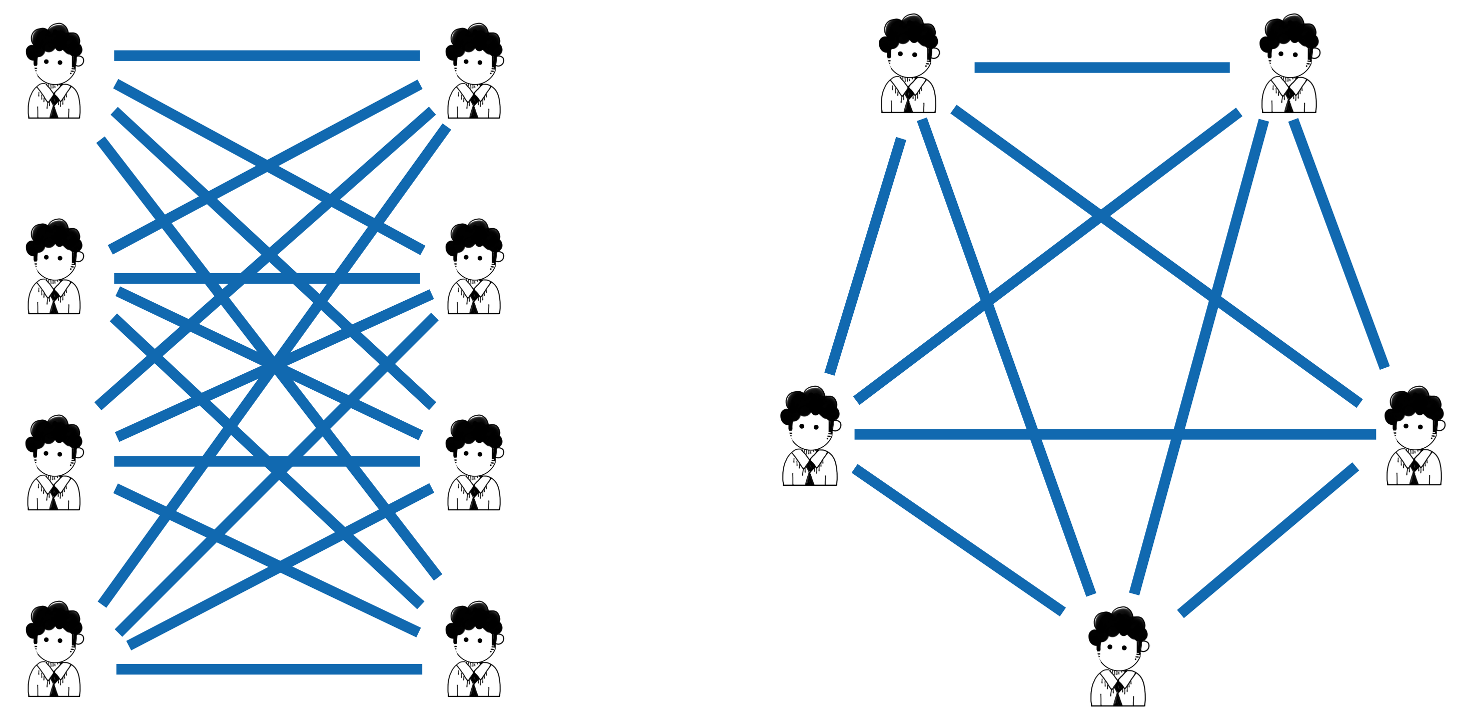


[Buechel, B. In Networks, Topology and Dynamics. Springer Lecture Notes in Economic and Mathematical Systems Vol. 613, 95–109 (Springer, 2008)]



HOMOGENEOUS AGENTS

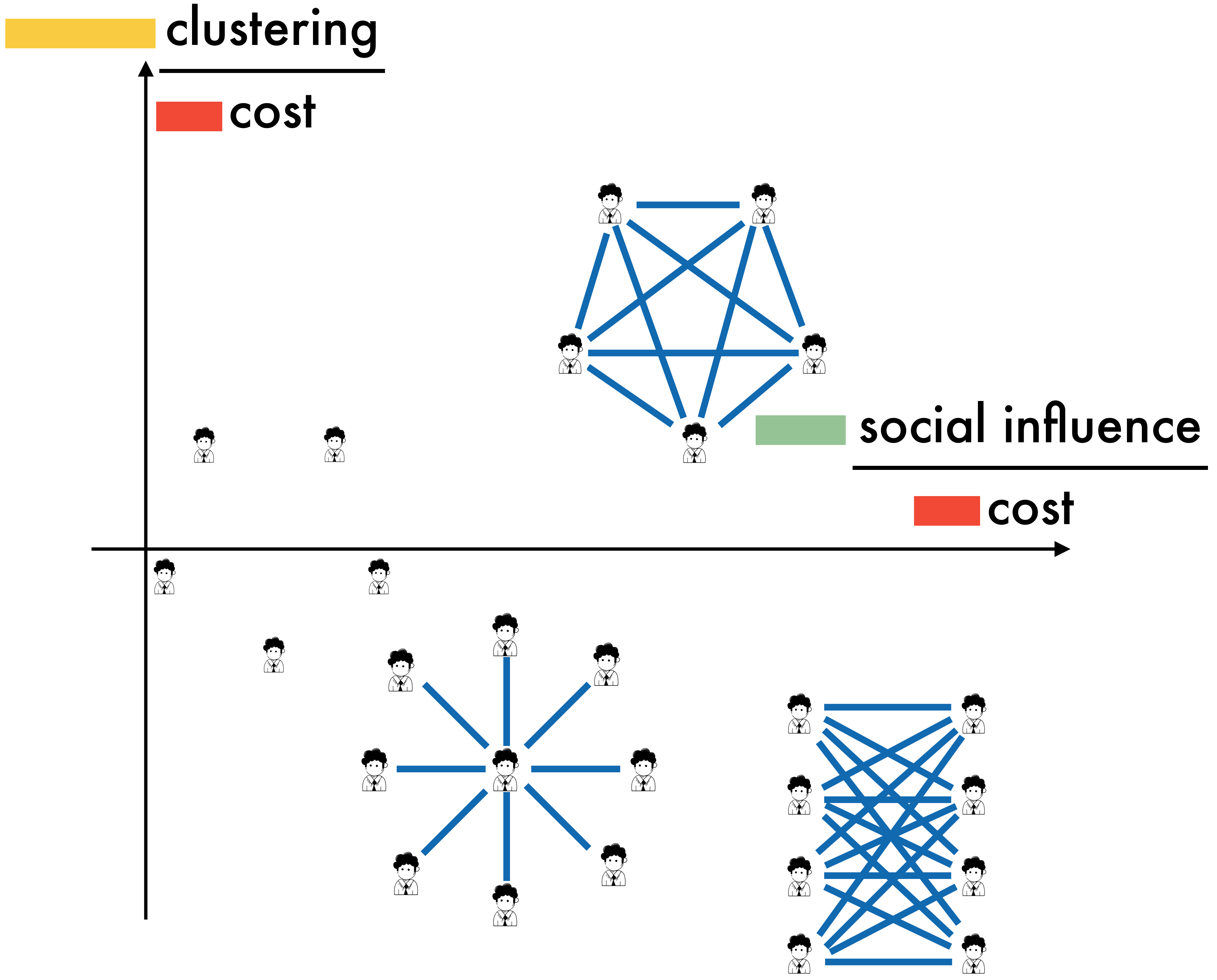
$$V_i(a_i, \mathbf{a}_{-i} \,|\, \theta_i) = \frac{\alpha_i}{\gamma_i}\, t_i(a_i, \mathbf{a}_{-i}, \delta_i) + \frac{\beta_i}{\gamma_i}\, u_i(a_i, \mathbf{a}_{-i}) - c_i(a_i)$$

$$\theta_i = \{\alpha_i, \beta_i, \gamma_i, \delta_i\}$$

$$\alpha_i \geq 0, \beta_i \in \mathbb{R}, \gamma_i > 0, \delta_i \in [0,1]$$

Assumption.

Individual preferences $\theta_i = \theta$, for all agents i.



06

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$$V_i(a_i, \mathbf{a}_{-i} \,|\, \theta_i) = \frac{\alpha_i}{\gamma_i} t_i(a_i, \mathbf{a}_{-i}, \delta_i) + \frac{\beta_i}{\gamma_i} u_i(a_i, \mathbf{a}_{-i}) - c_i(a_i)$$

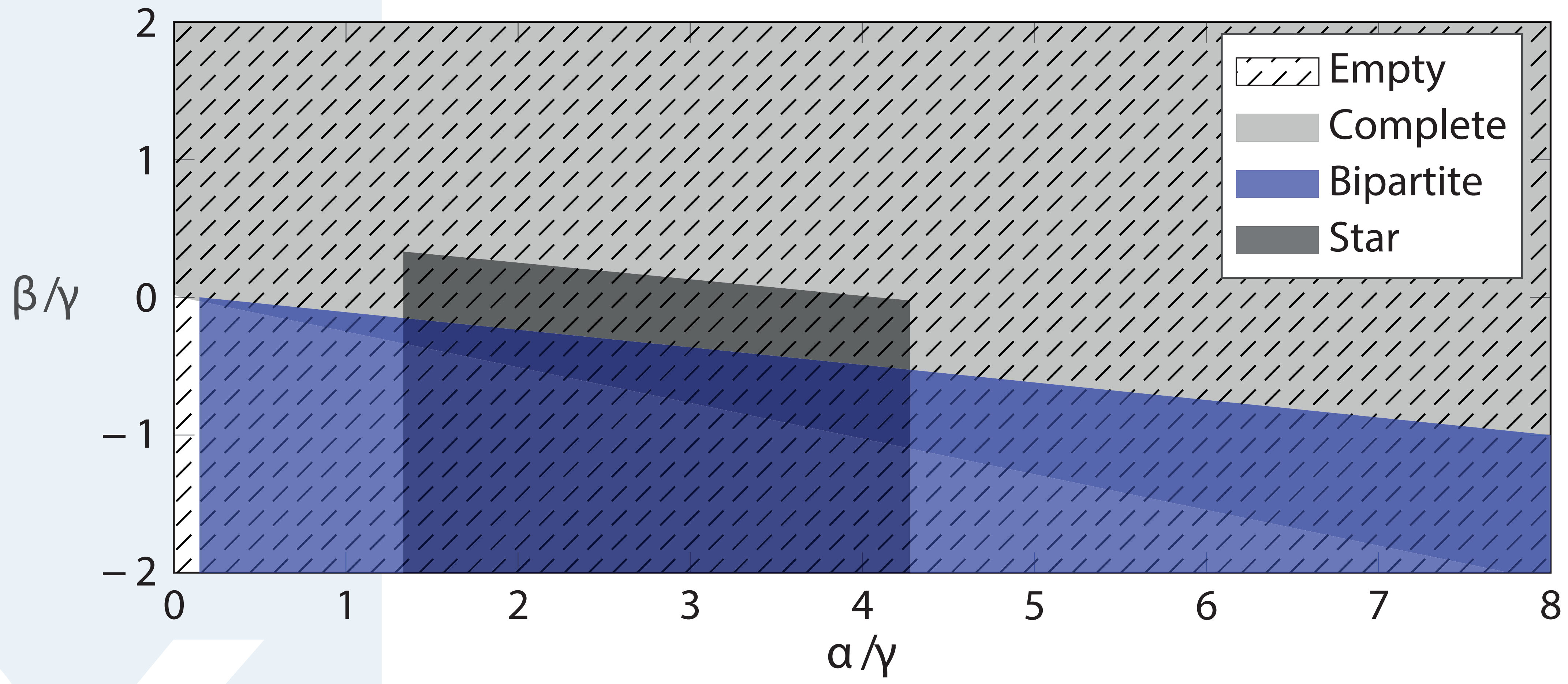
$$\theta_i = \{\alpha_i, \beta_i, \gamma_i, \delta_i\}$$

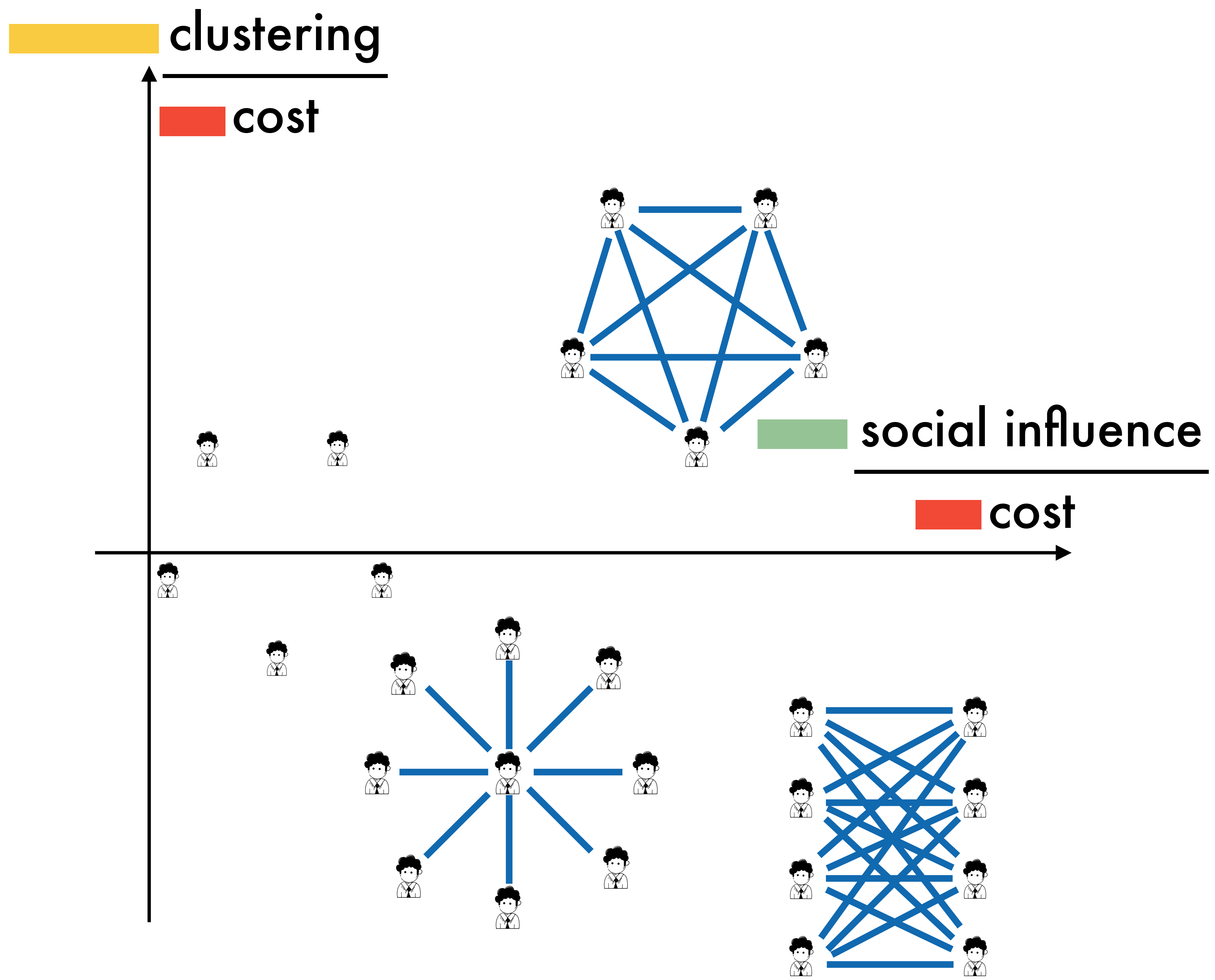
$$\alpha_i \geq 0, \beta_i \in \mathbb{R}, \gamma_i > 0, \delta_i \in [0,1]$$

Assumption.

Individual preferences $\theta_i = \theta$, for all agents i.

Analytical parametric **necessary** conditions can be derived through Variational Inequality.
Sufficiency can also be established.





06

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N. Pagan & F. Dörfler, *"Game theoretical inference of human behaviour in social networks"*

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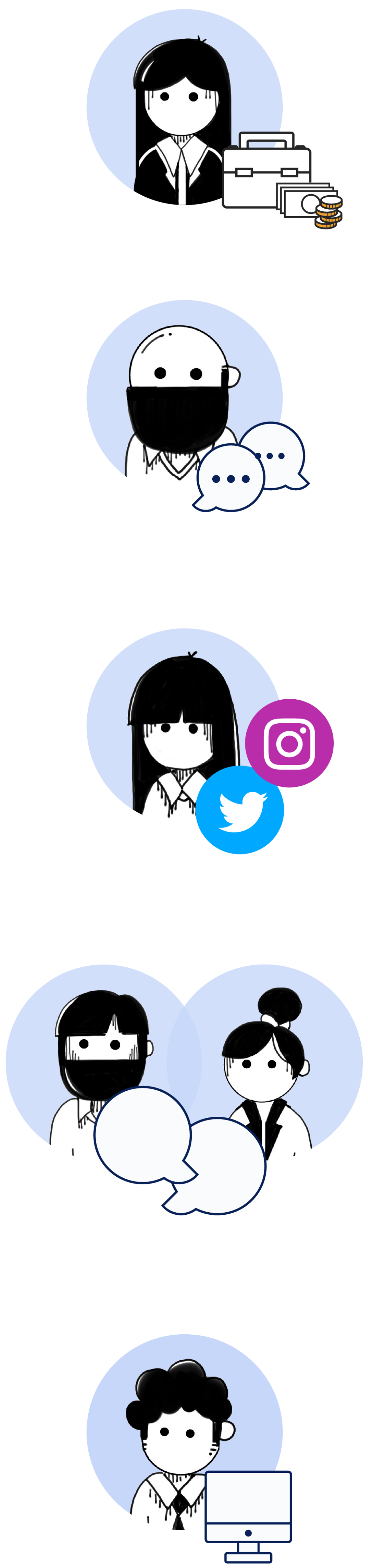
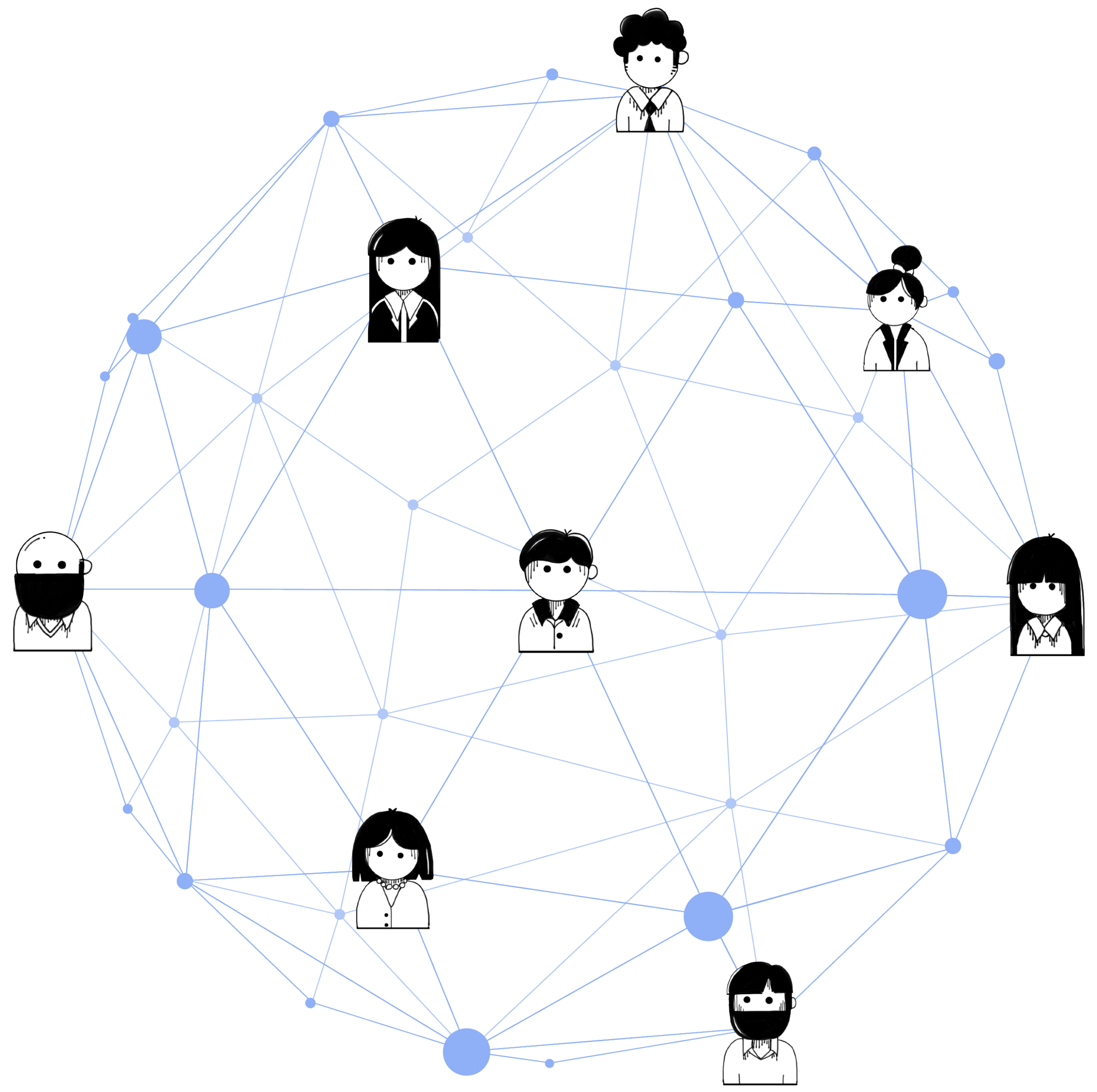
INDIVIDUAL BEHAVIOUR θ_i



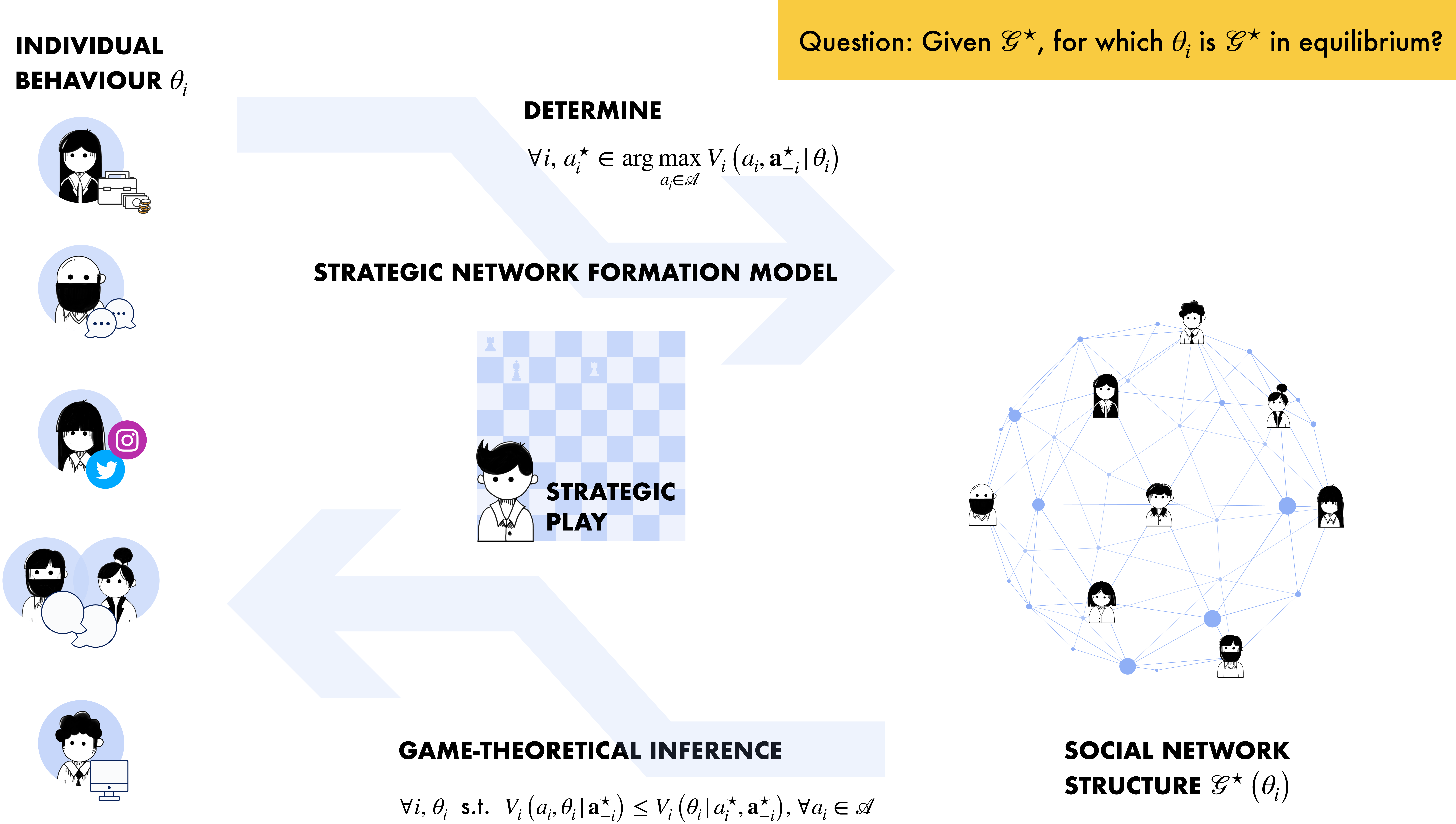
DETERMINE

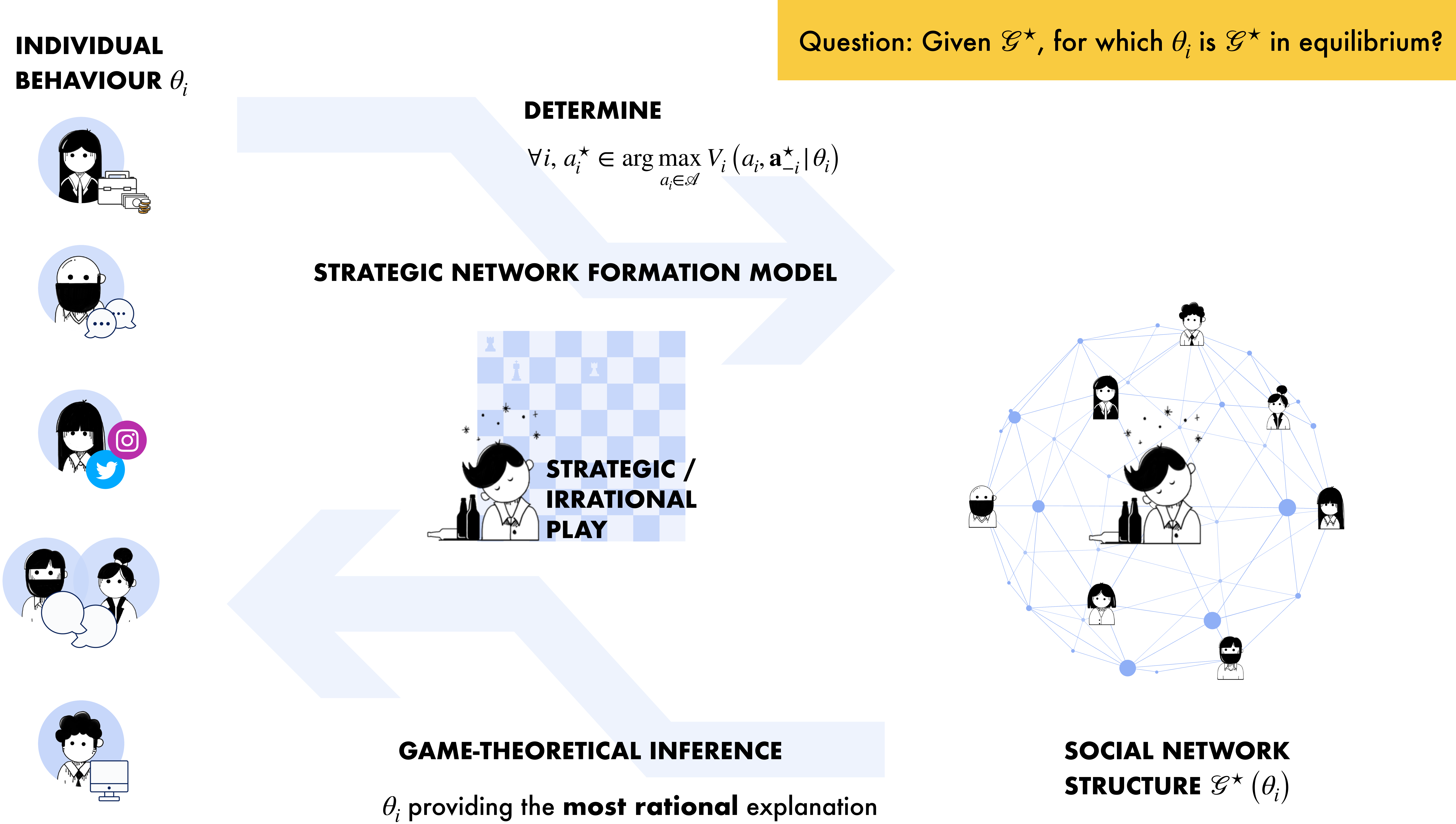
$$\forall i,\ a_i^\star \in \arg\max_{a_i \in \mathscr{A}} V_i\left(a_i, \mathbf{a}_{-i}^\star \,|\, \theta_i\right)$$

STRATEGIC NETWORK FORMATION MODEL



SOCIAL NETWORK STRUCTURE $\mathscr{G}^\star\left(\theta_i\right)$


Question: Given $\mathscr{G}^\star$, for which θ_i is $\mathscr{G}^\star$ in equilibrium?
INDIVIDUAL BEHAVIOUR θ_i
DETERMINE
$\forall i,\ a_i^\star \in \arg\max_{a_i \in \mathscr{A}} V_i\left(a_i, \mathbf{a}_{-i}^\star \,|\, \theta_i\right)$
STRATEGIC NETWORK FORMATION MODEL
STRATEGIC PLAY
GAME-THEORETICAL INFERENCE
$\forall i, \theta_i$ s.t. $V_i\left(a_i, \theta_i \,|\, \mathbf{a}_{-i}^\star\right) \leq V_i\left(\theta_i \,|\, a_i^\star, \mathbf{a}_{-i}^\star\right), \forall a_i \in \mathscr{A}$
SOCIAL NETWORK STRUCTURE $\mathscr{G}^\star\left(\theta_i\right)$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INDIVIDUAL BEHAVIOUR θ_i
DETERMINE
$\forall i,\ a_i^\star \in \arg\max_{a_i \in \mathscr{A}} V_i\left(a_i, \mathbf{a}_{-i}^\star \mid \theta_i\right)$
STRATEGIC NETWORK FORMATION MODEL
STRATEGIC / IRRATIONAL PLAY
GAME-THEORETICAL INFERENCE
θ_i providing the **most rational** explanation
SOCIAL NETWORK STRUCTURE $\mathscr{G}^\star\left(\theta_i\right)$

INVERSE OPTIMIZATION PROBLEM

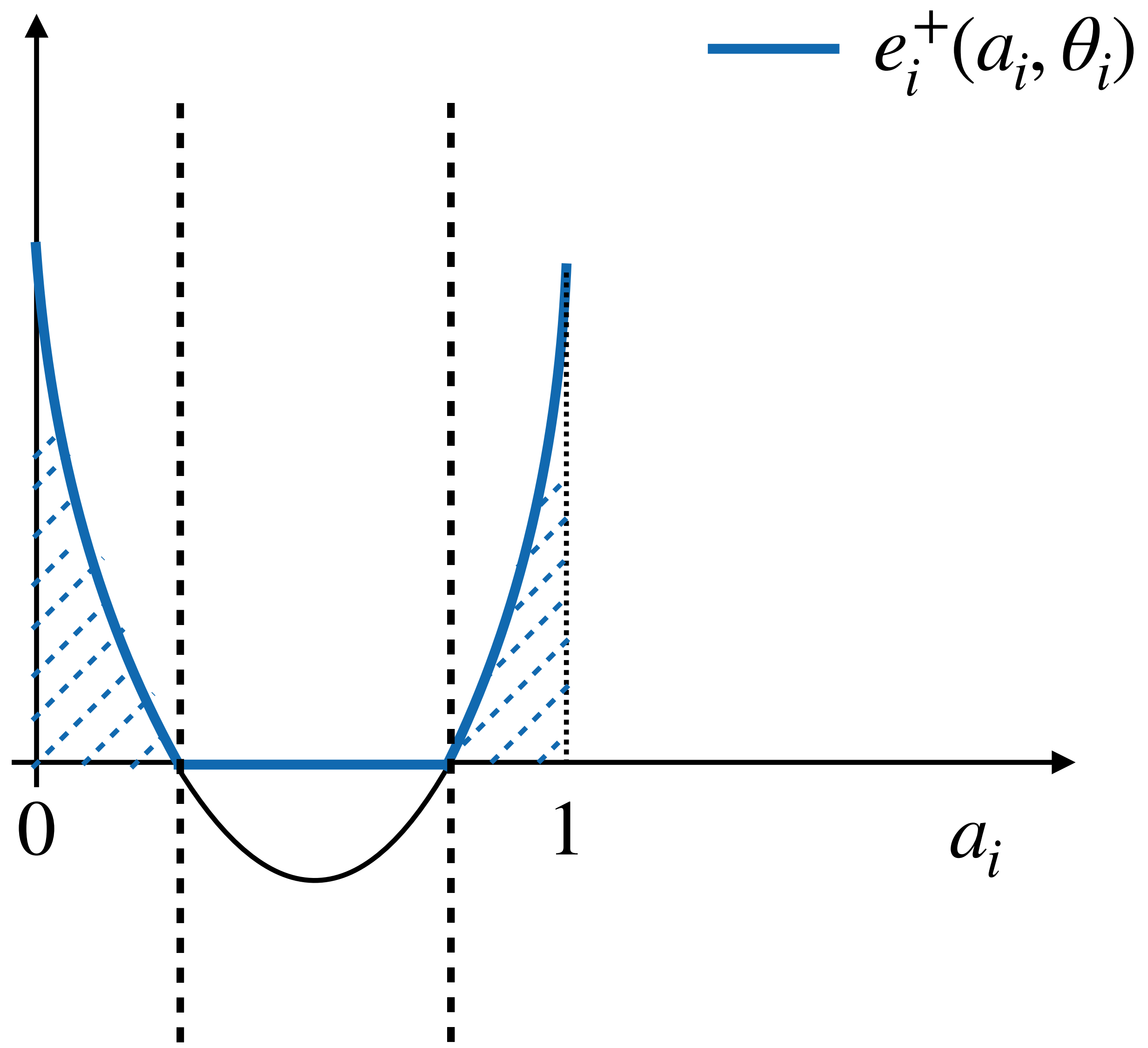
Error function.

$$e_i(a_i, \theta_i) := V_i\left(a_i, \theta_i \,|\, \mathbf{a}^\star_{-i}\right) - V_i\left(\theta_i \,|\, a_i^\star, \mathbf{a}^\star_{-i}\right)$$

$e_i^+(a_i, \theta_i) := \max\left\{0, e_i(a_i, \theta_i)\right\} > 0$ corresponds to a violation of the Nash equilibrium condition

Distance function.

$$d_i(\theta_i) := \left(\int_{\mathscr{A}} e_i^+(a_i, \theta_i)^2 da_i\right)^{1/2} = \|e_i^+(a_i, \theta_i)\|_{L_2(\mathscr{A})}$$



$e_i(a_i, \theta_i) < 0$

No violations: can be neglected

08

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INVERSE OPTIMIZATION PROBLEM

Problem [Minimum NE-Distance Problem].

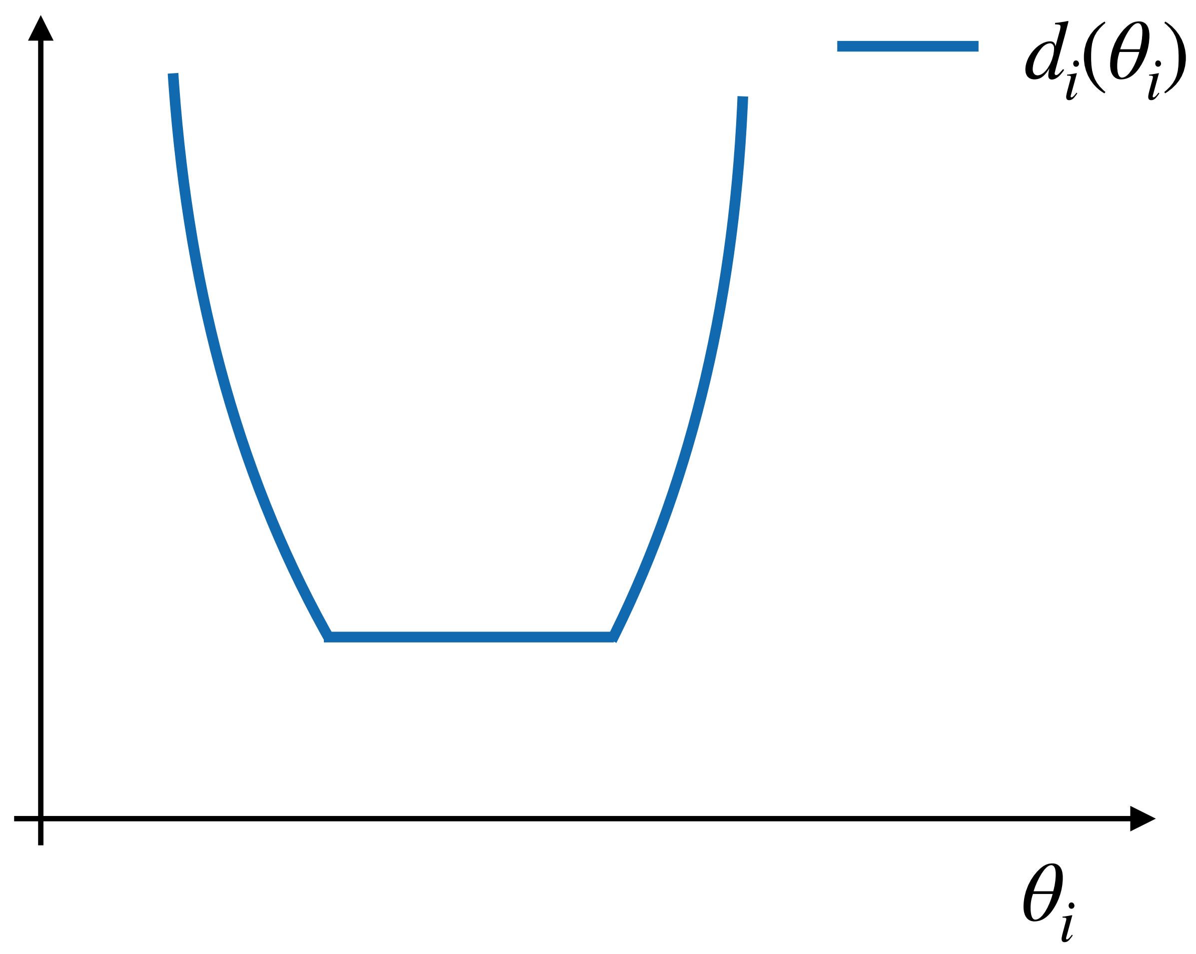
Given a network $\mathscr{G}^\star$ of N agents, for all agents i find the vectors of preferences $\theta_i^\star$ such that

$$\theta_i^\star \in \arg\min_{\theta_i \in \Theta} \; d_i^2(\theta_i)$$

Theorem [Convexity of the objective function].

Let

$$d_i^2(\theta_i) := \int_{\mathscr{A}} \left(\max\left\{0,\, e_i(a_i, \theta_i)\right\} \right)^2 dx = \| e_i^+(a_i, \theta_i) \|^2_{L_2(\mathscr{A})}$$



Then $d_i^2(\theta_i)$ is continuously differentiable, and its gradient reads as

$$\nabla_\theta d_i^2(\theta) = \int_{\mathscr{A}} 2\, \nabla_{\theta_i}\big(e_i(a_i, \theta_i)\big) \max\left\{0,\, e_i(a_i, \theta_i)\right\} da_i\,.$$

Moreover, d_i^2 is convex.

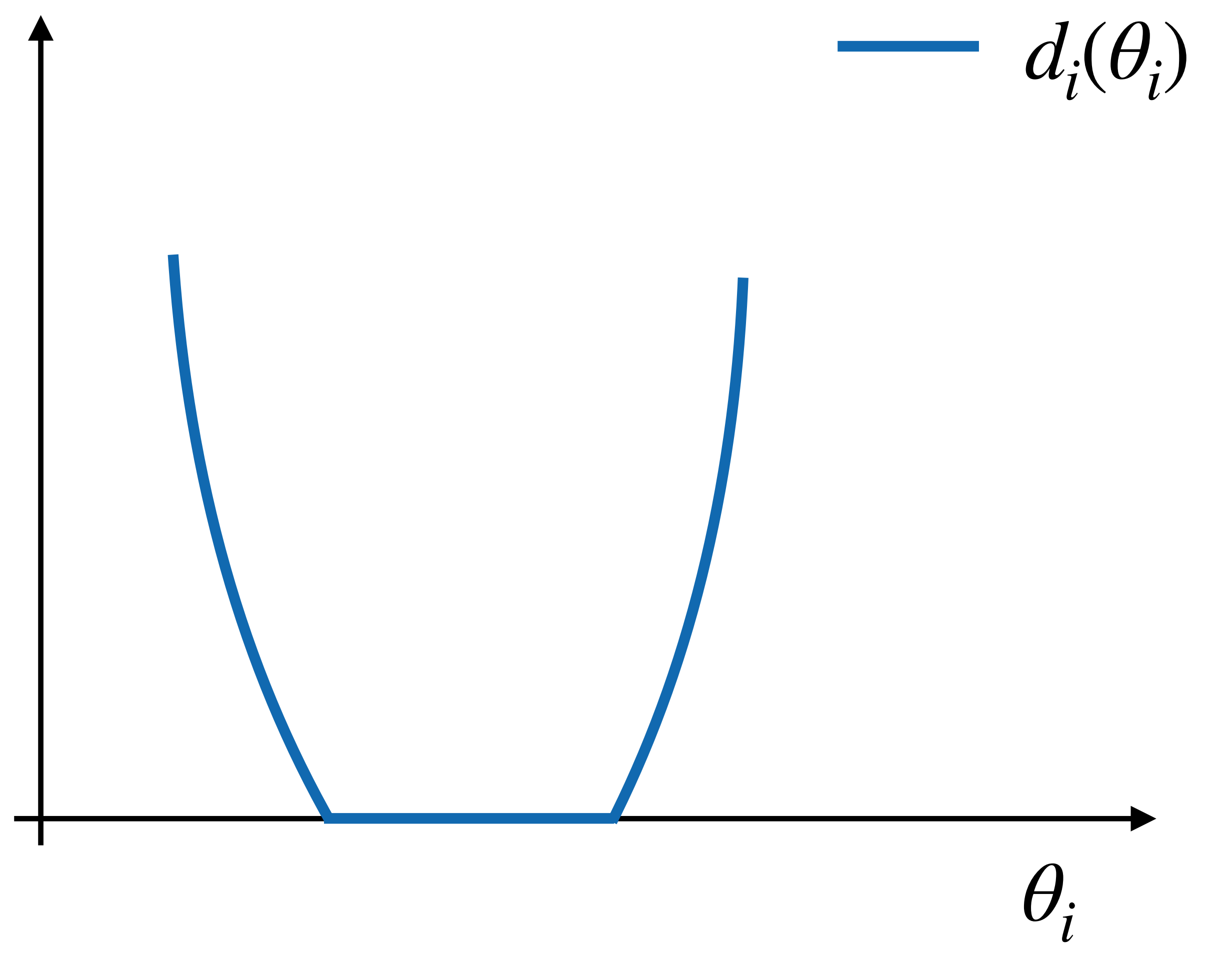
09

INVERSE OPTIMIZATION PROBLEM - SOLUTION (I)

$$d_i(\theta_i) := \left(\int_{\mathcal{A}} e_i^+(a_i,\theta_i)^2 da_i\right)^{1/2} = \|e_i^+(a_i,\theta_i)\|_{L_2(\mathcal{A})}$$

Theorem.

Assume $\min_{\theta_i \in \Theta} d_i^2(\theta_i) = 0$. Then, the set of solutions $\Theta_{i,0}$ is a polyhedron.



INVERSE OPTIMIZATION PROBLEM - SOLUTION (I)

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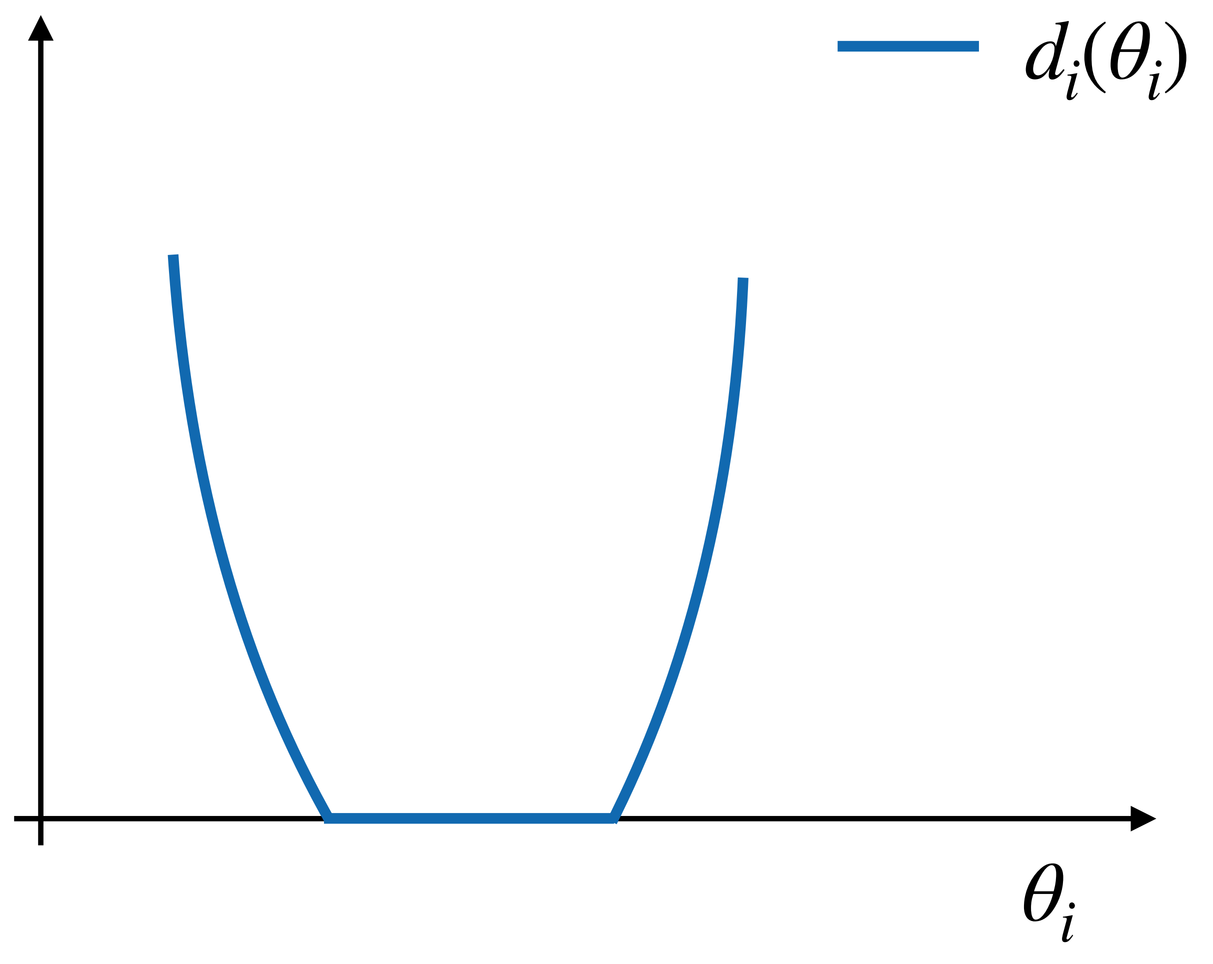

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Observe that $e_i(a_i, \theta_i) = x_i(a_i)\theta_i - y_i(a_i)$ is a combination of linear and mixed quadratic terms in the components of a_i, but there are not no pure quadratic terms of the form a_{ij}^2.

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$$Hess\left(e_i(\theta_i)\right) = \begin{bmatrix} 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot \\ & & \ddots & & \\ \cdot & \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

INVERSE OPTIMIZATION PROBLEM - SOLUTION (I)

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Then $Trace\left(Hess\, e_i(a_i)\right) = 0$, i.e., $\sum_{j=1}^{N-1} \lambda_j = 0$.

Thus, any critical point in the compact set $\mathscr{A}$ is a saddle point, and cannot be a strict maximum.

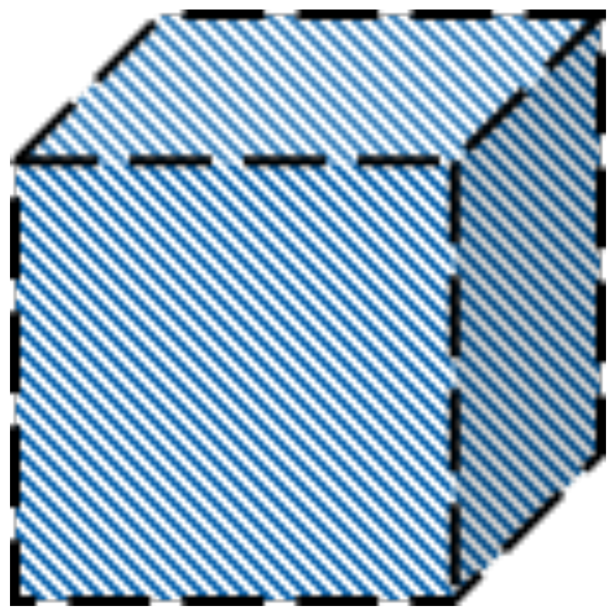
INVERSE OPTIMIZATION PROBLEM - SOLUTION (I)

Theorem.

Assume $\min_{\theta_i \in \Theta} d_i^2(\theta_i) = 0$. Then, the set of solutions $\Theta_{i,0}$ is a polyhedron.

Proof (sketch).

Then the maximum must be attained on the boundary, which is composed by $2(N-1)$ sides, each side being the compact set $[0,1]^{N-2}$.



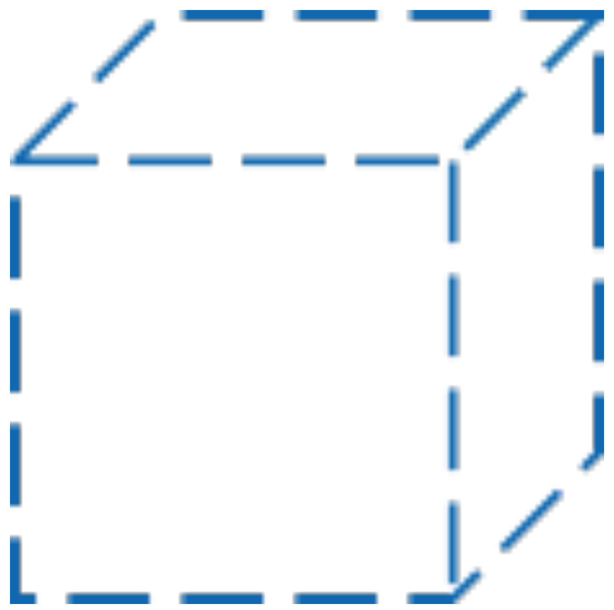
INVERSE OPTIMIZATION PROBLEM - SOLUTION (I)

Theorem.

Assume $\min_{\theta_i \in \Theta} d_i^2(\theta_i) = 0$. Then, the set of solutions $\Theta_{i,0}$ is a polyhedron.

Proof (sketch).

On the side, the error function has the same property.
By induction...



INVERSE OPTIMIZATION PROBLEM - SOLUTION (I)

Theorem.

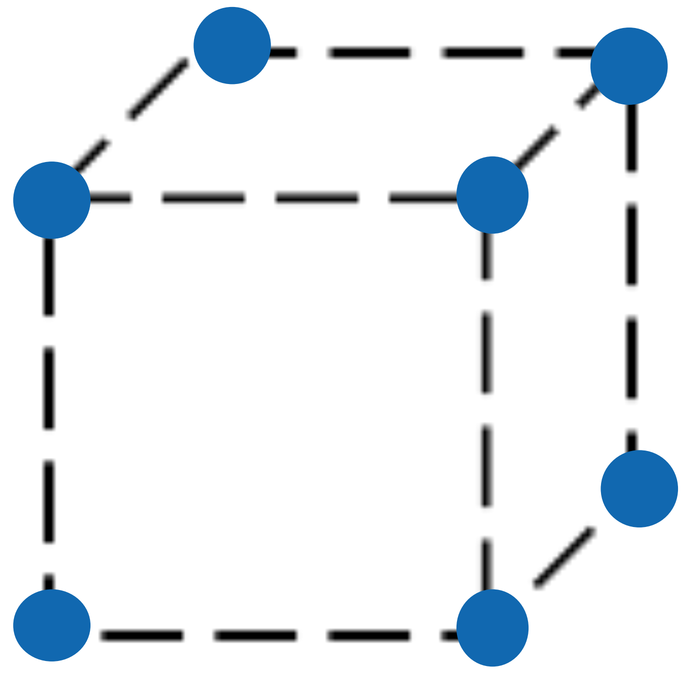
Assume $\min_{\theta_i \in \Theta} d_i^2(\theta_i) = 0$. Then, the set of solutions $\Theta_{i,0}$ is a polyhedron.

Proof (sketch).

Finally...

$\Theta_{i,0} := \left\{ \theta_i \in \Theta, \text{ s.t. } \forall a_i \in \mathscr{A}_{\{0,\,1\}},\, e_i(a_i, \theta_i) \leq 0 \right\},$

where $\mathscr{A}_{\{0,\,1\}} = \{0,\,1\}^{N-1}$.



$\mathscr{A}_{\{0,\,1\}} = \{0,\,1\}^{N-1}$

INVERSE OPTIMIZATION PROBLEM - SOLUTION (I)

$e_i(a_i, \theta_i) = x_i(a_i)\theta_i - y_i(a_i)$

Theorem.

Assume $\min_{\theta_i \in \Theta} d_i^2(\theta_i) = 0$. Then, the set of solutions $\Theta_{i,0}$ is a polyhedron.

Proof (sketch).

$\Theta_{i,0} = \left\{ \theta_i \in \Theta, \text{ s.t. } \forall a_i \in \mathscr{A}_{\{0,\,1\}},\, x_i(a_i)\theta_i - y_i(a_i) \leq 0 \right\}$, where $\mathscr{A}_{\{0,\,1\}} = \{0,\,1\}^{N-1}$.

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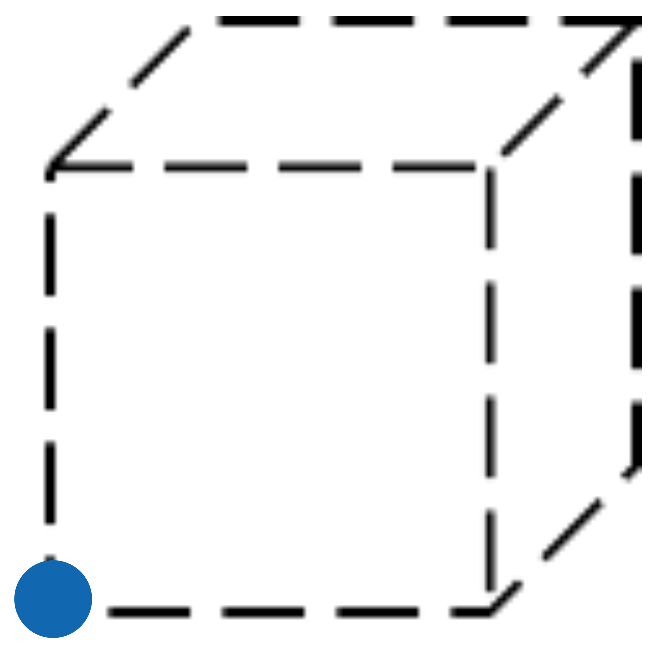

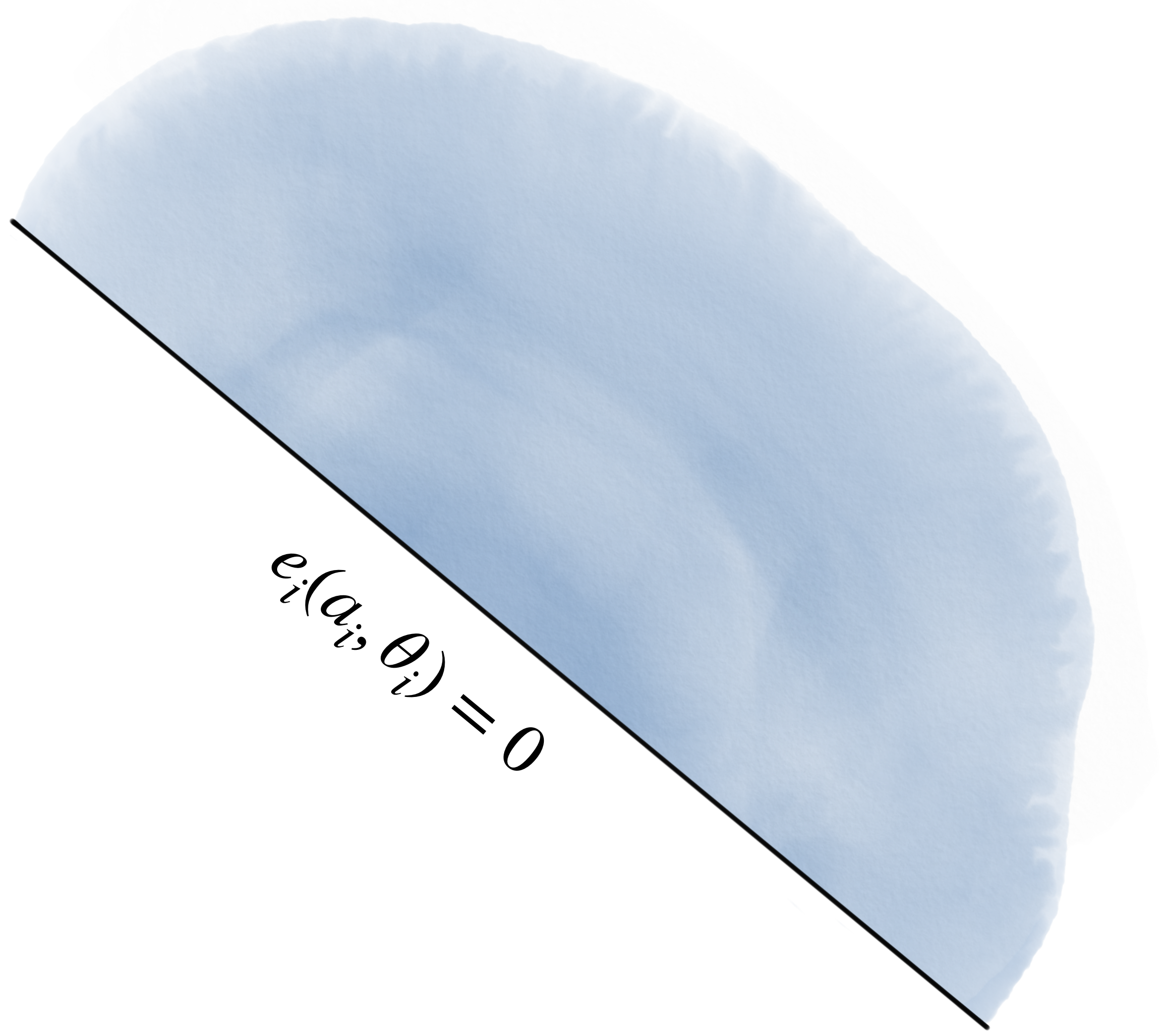

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INVERSE OPTIMIZATION PROBLEM - SOLUTION (I)

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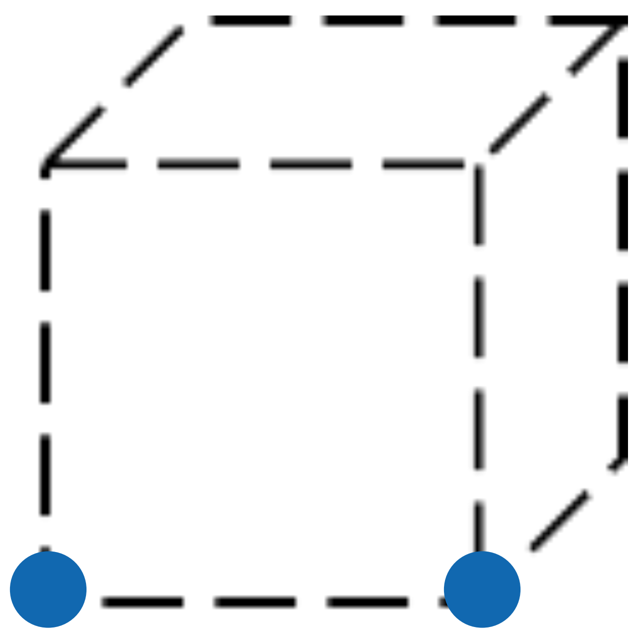

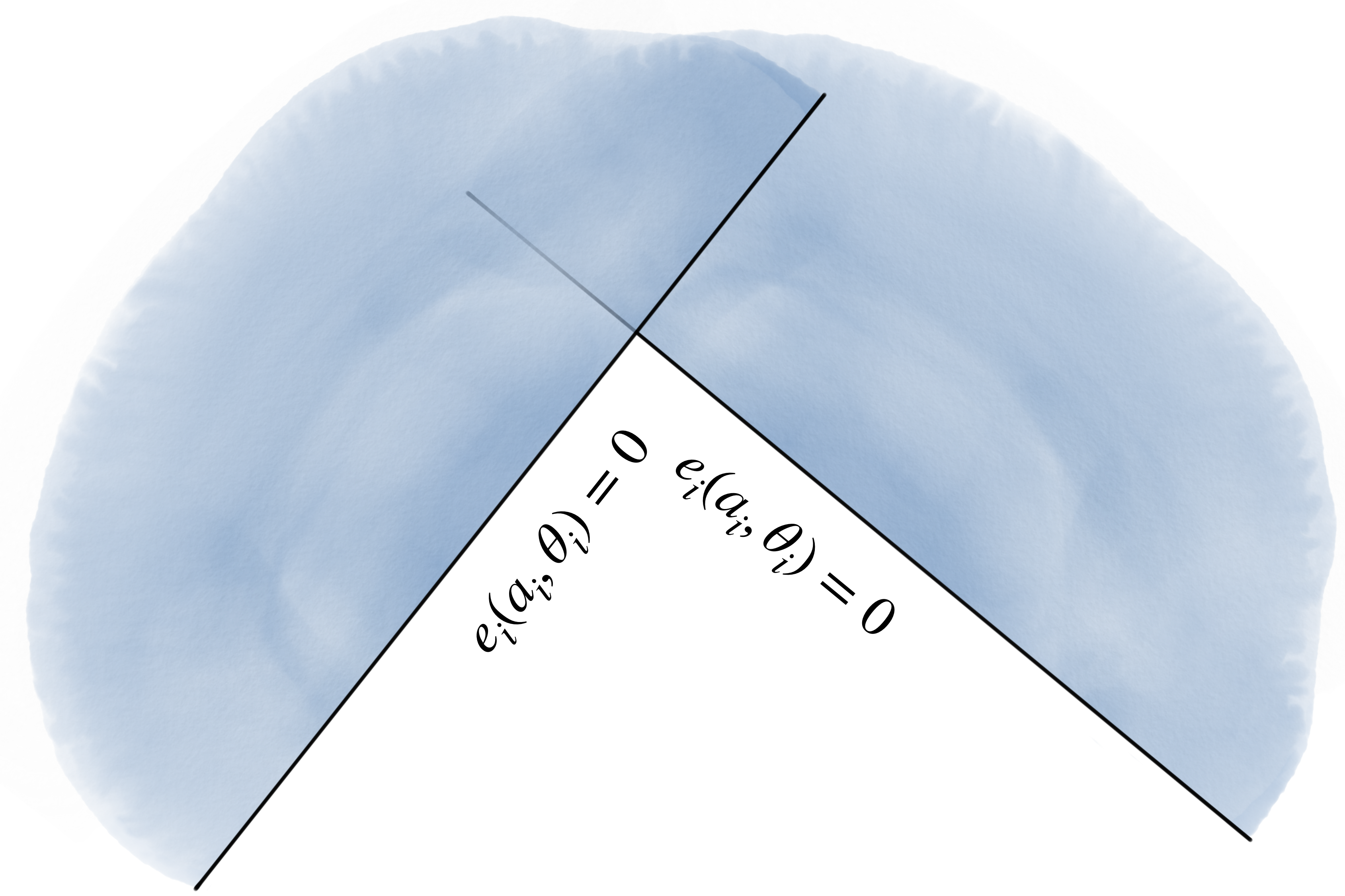

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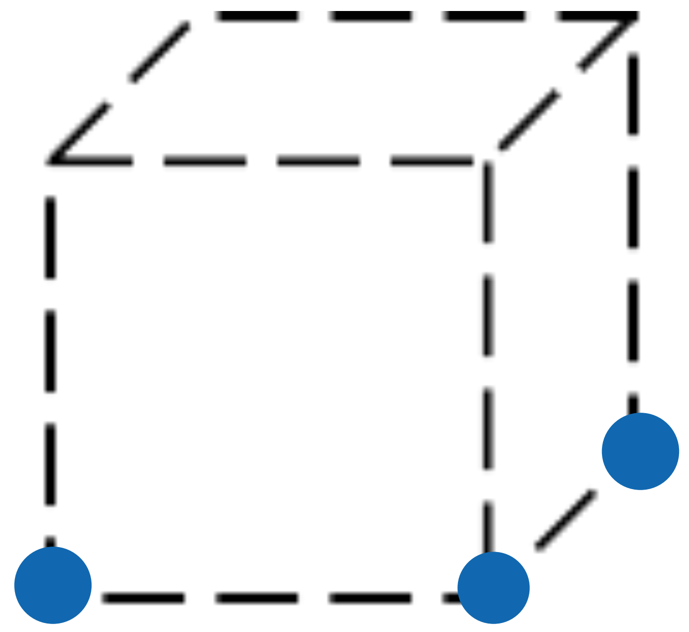

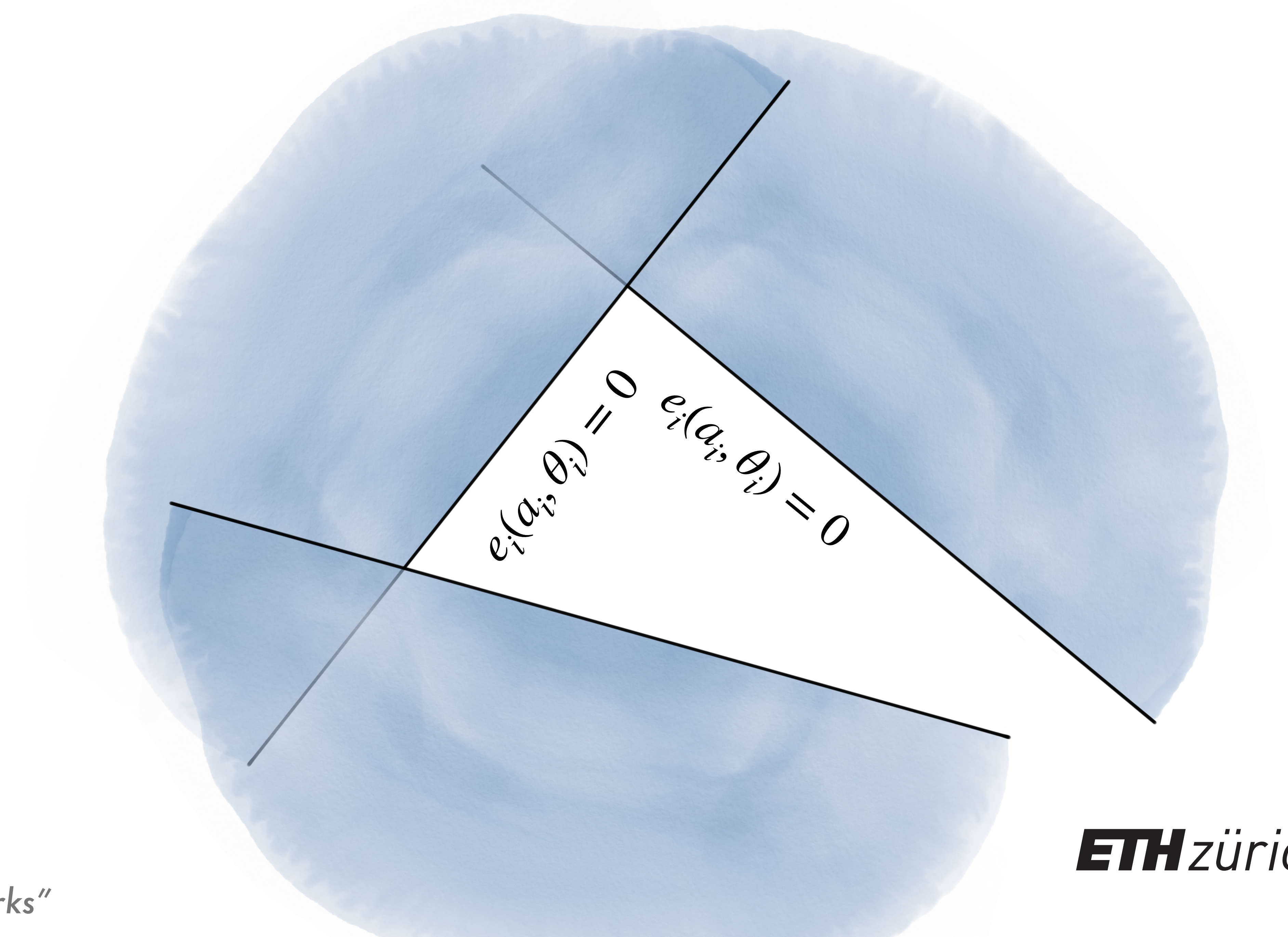

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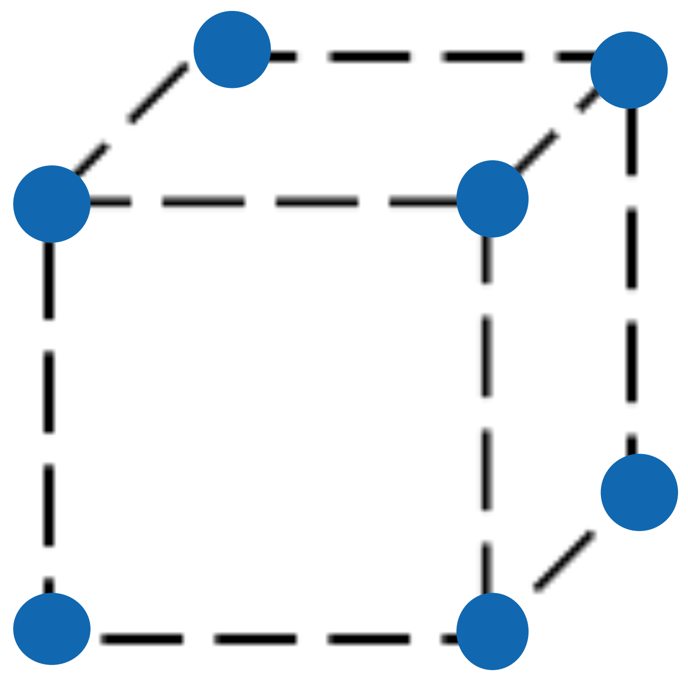

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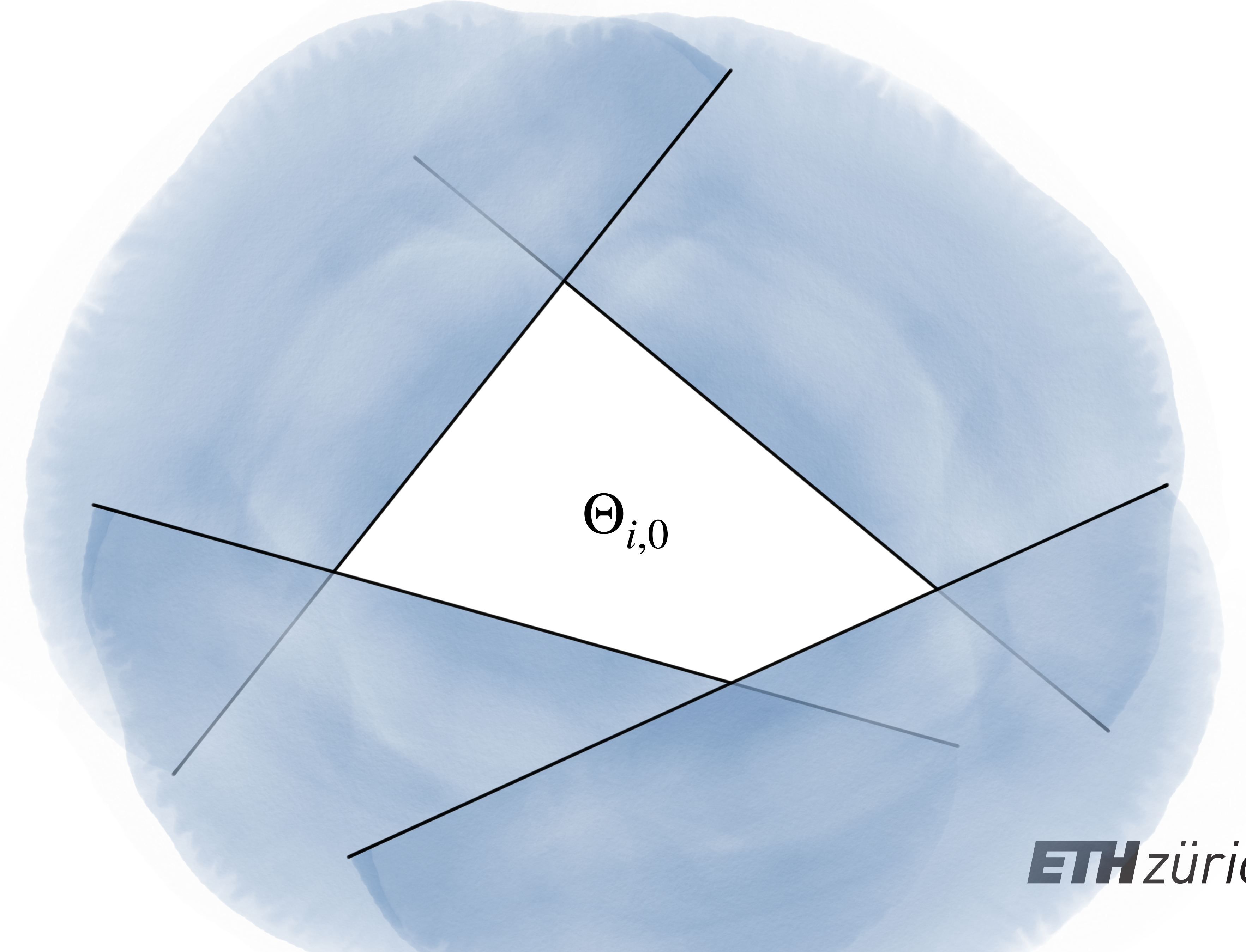

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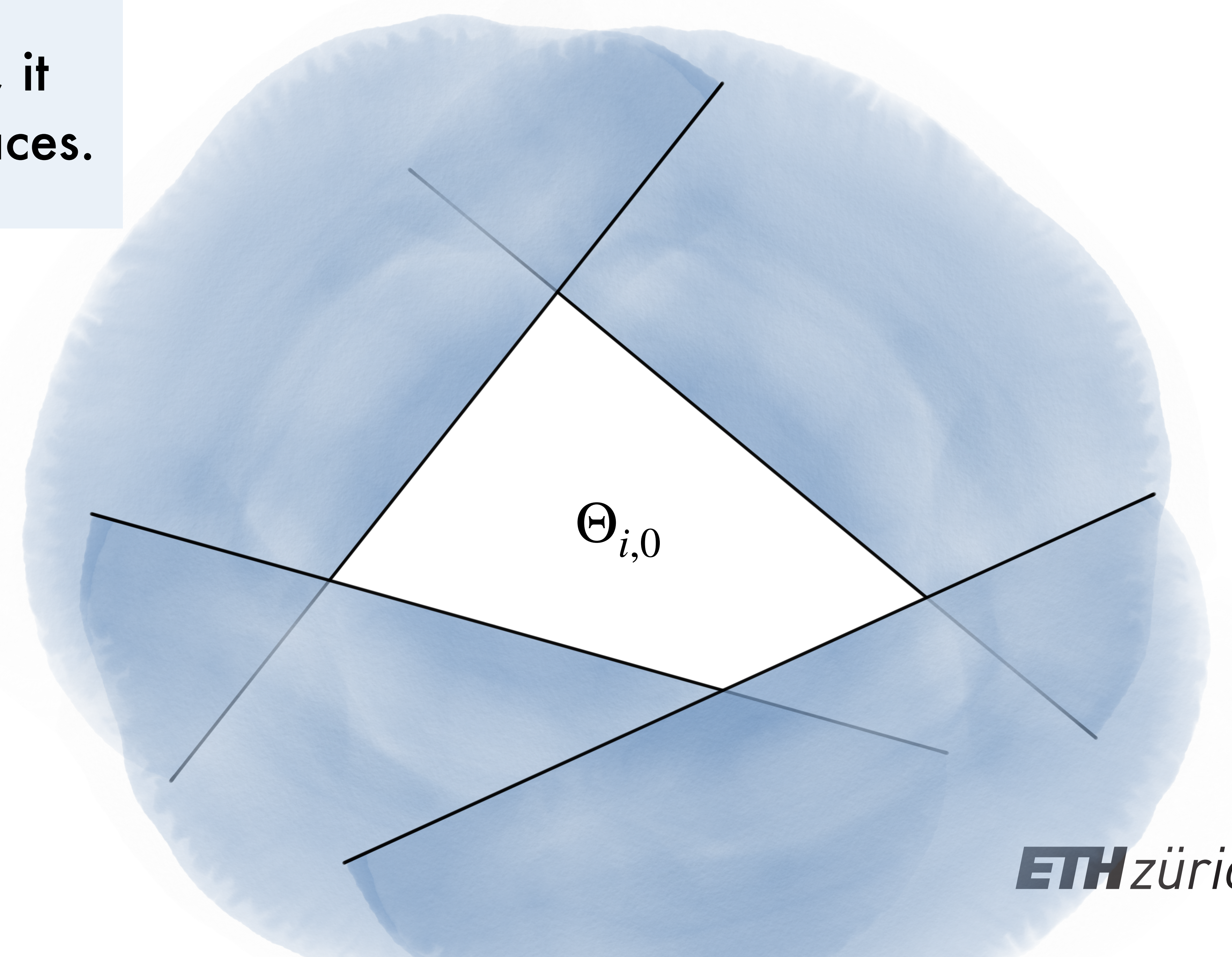
Thus, if a set of exact solutions to the NE conditions exists, it can be found as the intersection of **finitely** many half-spaces.



INVERSE OPTIMIZATION PROBLEM - SOLUTION (II)

If $\Theta_{i,0} = \varnothing$*?*

First-order optimality condition

$$0 = \nabla_{\theta_i}\left(d_i^2(\theta_i)\right) = 2\int_{\mathscr{A}} \nabla_{\theta_i}\left(e_i(a_i, \theta_i)\right) \max\left\{0,\, e_i(a_i, \theta_i)\right\} da_i\,.$$

max operator within $(N-1)$ - dimensional integral

INVERSE OPTIMIZATION PROBLEM - SOLUTION (II)

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max operator within $(N-1)$ - dimensional integral

Removing the max operator, integral over $\mathcal{A}_+(\theta_i) = \left\{a_i \in \mathcal{A} \,|\, e_i(a_i,\theta_i) > 0\right\}$,

$$0 = \nabla_{\theta_i}\left(d_i^2(\theta_i)\right) = 2\int_{\mathcal{A}_+(\theta_i)} \nabla_{\theta_i}\left(e_i(a_i,\theta_i)\right) e_i(a_i,\theta_i) da_i,$$

cannot be solved in closed form for θ_i (in the general case).

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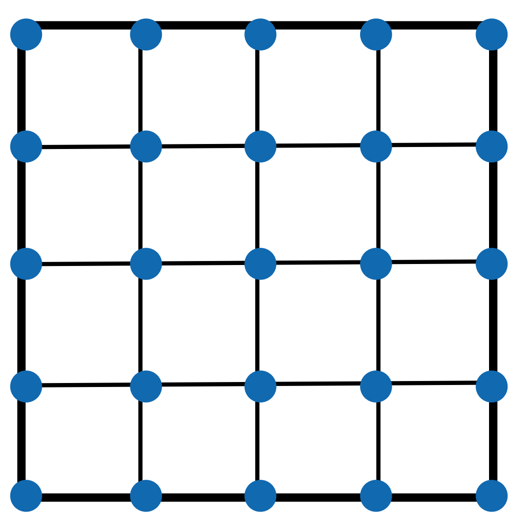
INVERSE OPTIMIZATION PROBLEM - SOLUTION (II)

Search for an approximate solution. Consider a finite set of possible actions (samples)

$$\left\{a_i^j\right\}_{j=1}^{n_i} \subset \mathscr{A}$$



Let $e_i^j(\theta_i) = e_i(a_i^j, \theta_i)$ and $e_i^{j,+}(\theta_i) = e_i^+(a_i^j, \theta_i)$ be the corresponding error and positive error at the samples.

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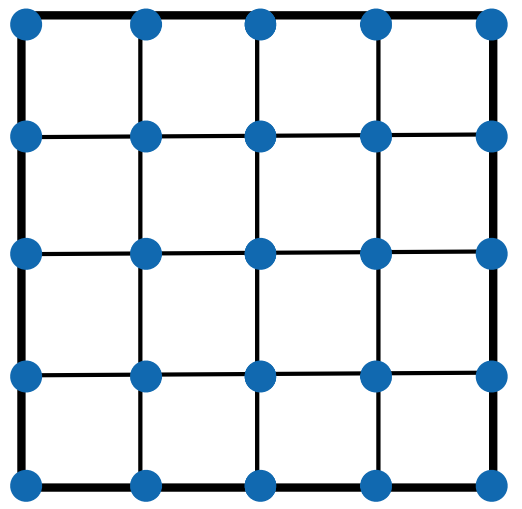

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Approximate the distance function as

$$\tilde{d}_i(\theta_i) := \left(\sum_{j=1}^{n_i} \left(e_i^{j,+}(\theta_i) \right)^2 \right)^{1/2} = \|\mathbf{e}_i^+\|_2$$

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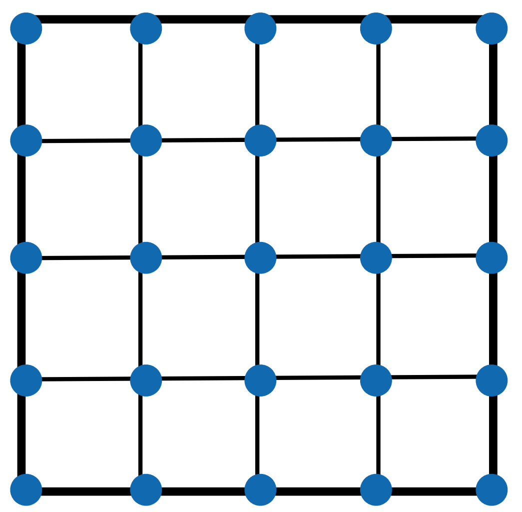

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Problem [Discrete Minimum NE-Distance Problem].

Given a network $\mathscr{G}^\star$ of N agents, for all agents i find the vectors of preferences $\hat{\theta}_i$ such that

$$\hat{\theta}_i \in \arg\min_{\theta_i \in \Theta} \tilde{d}_i^2(\theta_i)$$

Same property of the original problem (Convexity)

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N. Pagan & F. Dörfler, *"Game theoretical inference of human behaviour in social networks"*

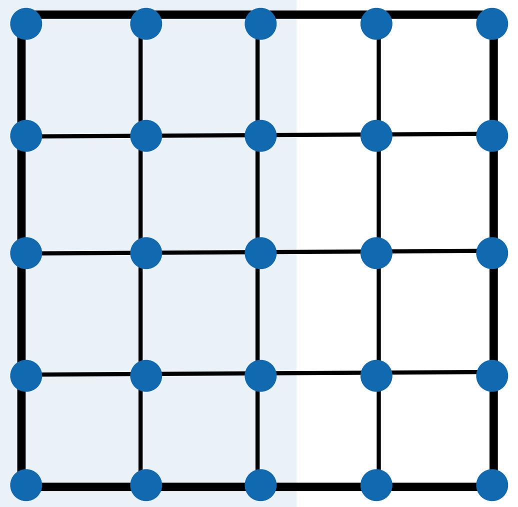
$e_i(a_i, \theta_i) = x_i(a_i)\theta_i - y_i(a_i)$

INVERSE OPTIMIZATION PROBLEM - SOLUTION (II)

The first-order optimality condition $\nabla_{\theta_i}\left(\tilde{d}_i^2(\theta_i)\right) = 2\sum_{j=1}^{n_i} \nabla_{\theta_i} e_i^j(\theta_i) \max\left\{0,\, e_i^j(\theta_i)\right\} = 0$ can be written as:

$$\mathbf{X}_i^T\left(\max\left\{\mathbf{0}_{n_i\times 1}, \mathbf{X}_i\theta_i - \mathbf{y}_i\right\}\right) = 0.$$

where



$\mathscr{A}$

samples

$$\underbrace{\begin{bmatrix} x_{i,1}(a_i^1) & x_{i,2}(a_i^1) & x_{i,3}(a_i^1) \\ x_{i,1}(a_i^2) & x_{i,2}(a_i^2) & x_{i,3}(a_i^2) \\ \vdots & \vdots & \vdots \\ x_{i,1}(a_i^{n_i}) & x_{i,2}(a_i^{n_i}) & x_{i,3}(a_i^{n_i}) \end{bmatrix}}_{\mathbf{X}_i} \begin{bmatrix} \theta_{i,1} \\ \theta_{i,2} \\ \theta_{i,3} \end{bmatrix} - \underbrace{\begin{bmatrix} y_i(a_i^1) \\ y_i(a_i^2) \\ \vdots \\ y_i(a_i^{n_i}) \end{bmatrix}}_{\mathbf{y}_i} = \underbrace{\begin{bmatrix} e_i^1 \\ e_i^2 \\ \vdots \\ e_i^{n_i} \end{bmatrix}}_{\mathbf{e}_i}$$

INVERSE OPTIMIZATION PROBLEM - SOLUTION (II)

The first-order optimality condition $\nabla_{\theta_i}\left(\tilde{d}_i^{\,2}(\theta_i)\right) = 2\sum_{j=1}^{n_i} \nabla_{\theta_i} e_i^j(\theta_i) e_i^{j,+}(\theta_i) = 0$ can be written as:

$$\mathbf{X}_i^T \left(\max\left\{ \mathbf{0}_{n_i \times 1}, \mathbf{X}_i\theta_i - \mathbf{y}_i \right\} \right) = 0.$$

In the particular case where $\mathbf{X}_i\theta_i - \mathbf{y}_i = \mathbf{e}_i \geq \mathbf{0}_{n_i \times 1}$ component wise, i.e., when $\mathbf{e}_i = \mathbf{e}_i^+$,

$$\hat{\theta}_i = \left(\mathbf{X}_i^T \mathbf{X}_i\right)^{-1} \mathbf{X}_i^T \mathbf{y}_i$$

INVERSE OPTIMIZATION PROBLEM - SOLUTION (II)

The first-order optimality condition $\nabla_{\theta_i}\left(\tilde{d}_i^2(\theta_i)\right) = 2\sum_{j=1}^{n_i} \nabla_{\theta_i} e_i^j(\theta_i) e_i^{j,+}(\theta_i) = 0$ can be written as:

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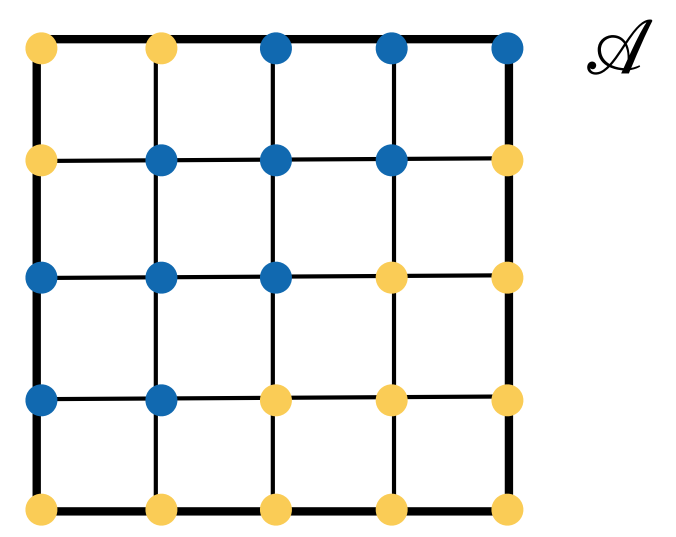
In the general case, there will be a subset of samples $J_i(\theta_i) \subset \left\{1,\ldots,n_1\right\}$ such that

$$x_i^j \theta_i - y_i^j = e_i^j = e_i^{j,+} \geq 0 \quad \forall j \in J_i(\theta_i)$$

while the remaining samples satisfy the NE condition. Define $\mathbf{X}_i(\theta_i)$ from $\mathbf{X}_i$, keeping only $j \in J_i(\theta_i)$

Violate NE conditions

$$\underbrace{\begin{bmatrix} x_{i,1}(a_i^1) & x_{i,2}(a_i^1) & x_{i,3}(a_i^1) \\ x_{i,1}(a_i^2) & x_{i,2}(a_i^2) & x_{i,3}(a_i^2) \\ \vdots & \vdots & \vdots \\ x_{i,1}(a_i^{n_i}) & x_{i,2}(a_i^{n_i}) & x_{i,3}(a_i^{n_i}) \end{bmatrix}}_{\mathbf{X}_i} \qquad e_i^j < 0$$



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INVERSE OPTIMIZATION PROBLEM - SOLUTION (II)

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In the general case, there will be a subset of samples $J_i(\theta_i) \subset \left\{1, \ldots, n_1\right\}$ such that

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$$\hat{\theta}_i = \left(\mathbf{X}_i^T(\hat{\theta}_i) \mathbf{X}_i(\hat{\theta}_i) \right)^{-1} \mathbf{X}_i^T(\hat{\theta}_i) \mathbf{y}_i(\hat{\theta}_i)$$

INVERSE OPTIMIZATION PROBLEM - SOLUTION (II)

Problem [Discrete Minimum NE-Distance Problem].

Given a network $\mathscr{G}^\star$ of N agents, for all agents i find the vectors of preferences $\hat{\theta}_i$ such that

$$\hat{\theta}_i \in \arg\min_{\theta_i \in \Theta} \tilde{d}_i^2(\theta_i)$$

$$\hat{\theta}_i = \left(\mathbf{X}_i^T(\hat{\theta}_i)\mathbf{X}_i(\hat{\theta}_i)\right)^{-1} \mathbf{X}_i^T(\hat{\theta}_i)\mathbf{y}_i(\hat{\theta}_i)$$

It still cannot be solved explicitly. However, the (projected) gradient method can be used to reach a solution.

$$\nabla_{\theta_i}\left(\tilde{d}_i^{\,2}(\theta_i)\right) = 2\sum_{j=1}^{n_i} \nabla_{\theta_i} e_i^j(\theta_i) e_i^{j,+}(\theta_i)$$

CONFIDENCE INTERVALS

Question: How "good" is the approximate solution?

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$$\hat{\theta}_i \in \arg\min_{\theta_i \in \Theta} \|\mathbf{e}_i^+(\theta_i)\|^2$$

where $e_i^+ = \max\left\{0, \mathbf{X_i}\theta - y\right\}$ is the vector of positive errors. Then

$$\hat{\theta}_i = \left(\mathbf{X}_i^T(\hat{\theta}_i)\mathbf{X}_i(\hat{\theta}_i)\right)^{-1}\mathbf{X}_i^T(\hat{\theta}_i)\mathbf{y}_i(\hat{\theta}_i)$$

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Problem [Ordinary Least Square (OLS) Regression].

Given a set of n samples $\{x_i\}_{i=1}^n$ and n observations $\{y_i\}_{i=1}^n$, where each scalar y_i is the response to the row vector x_i of values of p predictors (regressors) x_{ij} for $j = 1,\ldots,p$, find $\hat{\theta} \in \mathbb{R}^p$ such that

$$\hat{\theta} = \arg\min_{\theta} \|e(\theta)\|_2^2,$$

where $e = \mathbf{X}\theta - y$ is the vector of residuals. Then

$$\hat{\theta} = \left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{y}$$

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Since the residuals are all non negative, we assume $\varepsilon_i \sim Exp\left(\lambda_i \mathbf{I}_n\right)$ and i.i.d..
Bias and confidence interval can still be derived.

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If $\varepsilon \sim \mathcal{N}\left(0, \sigma^2 \mathbf{I}_n\right)$ and i.i.d, then the estimator $\hat{\theta}$ is unbiased, normally distributed, and its covariance matrix reads as

$$\sigma^2 \mathbf{X}^T\mathbf{X}$$

AUSTRALIAN BANK

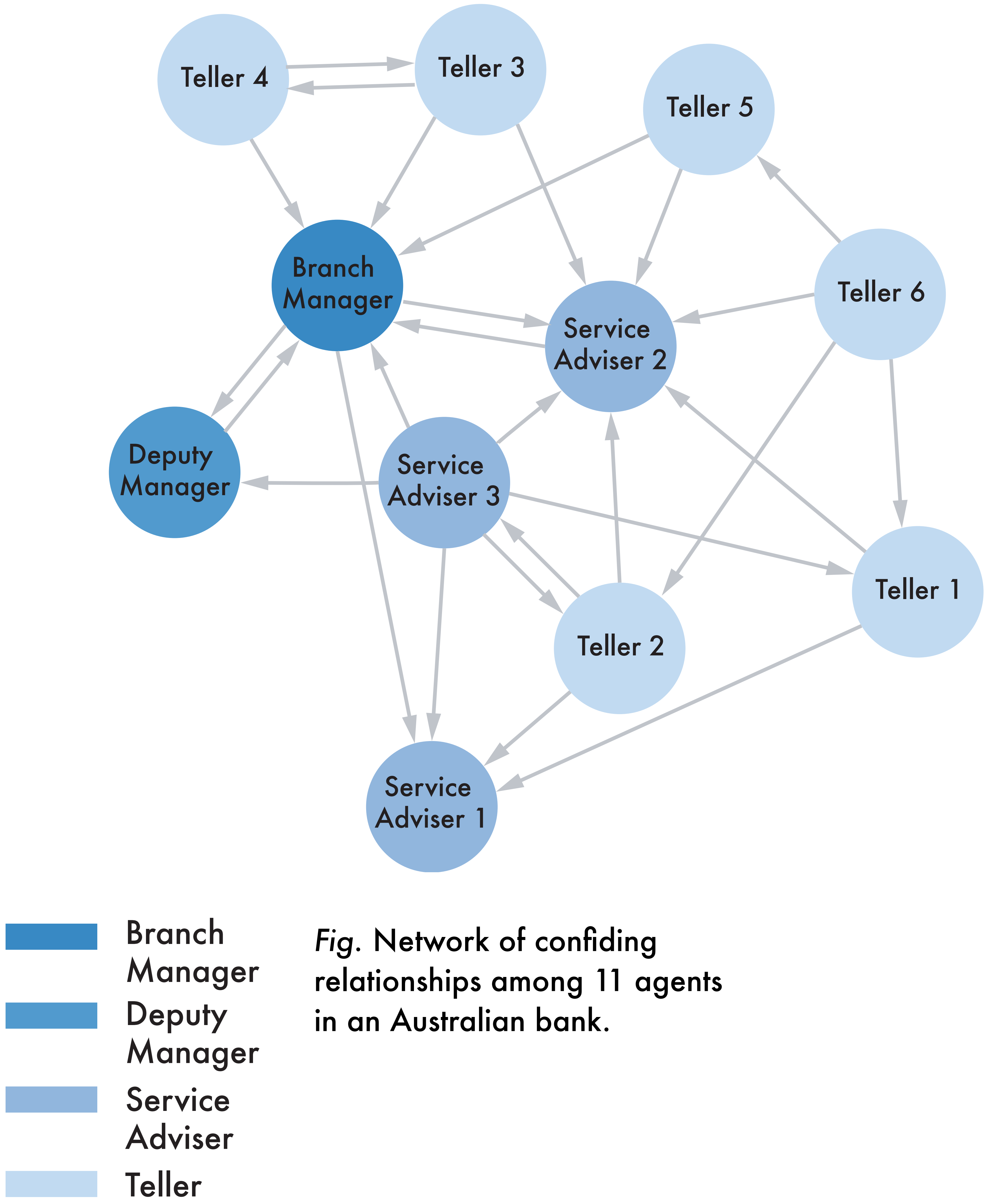
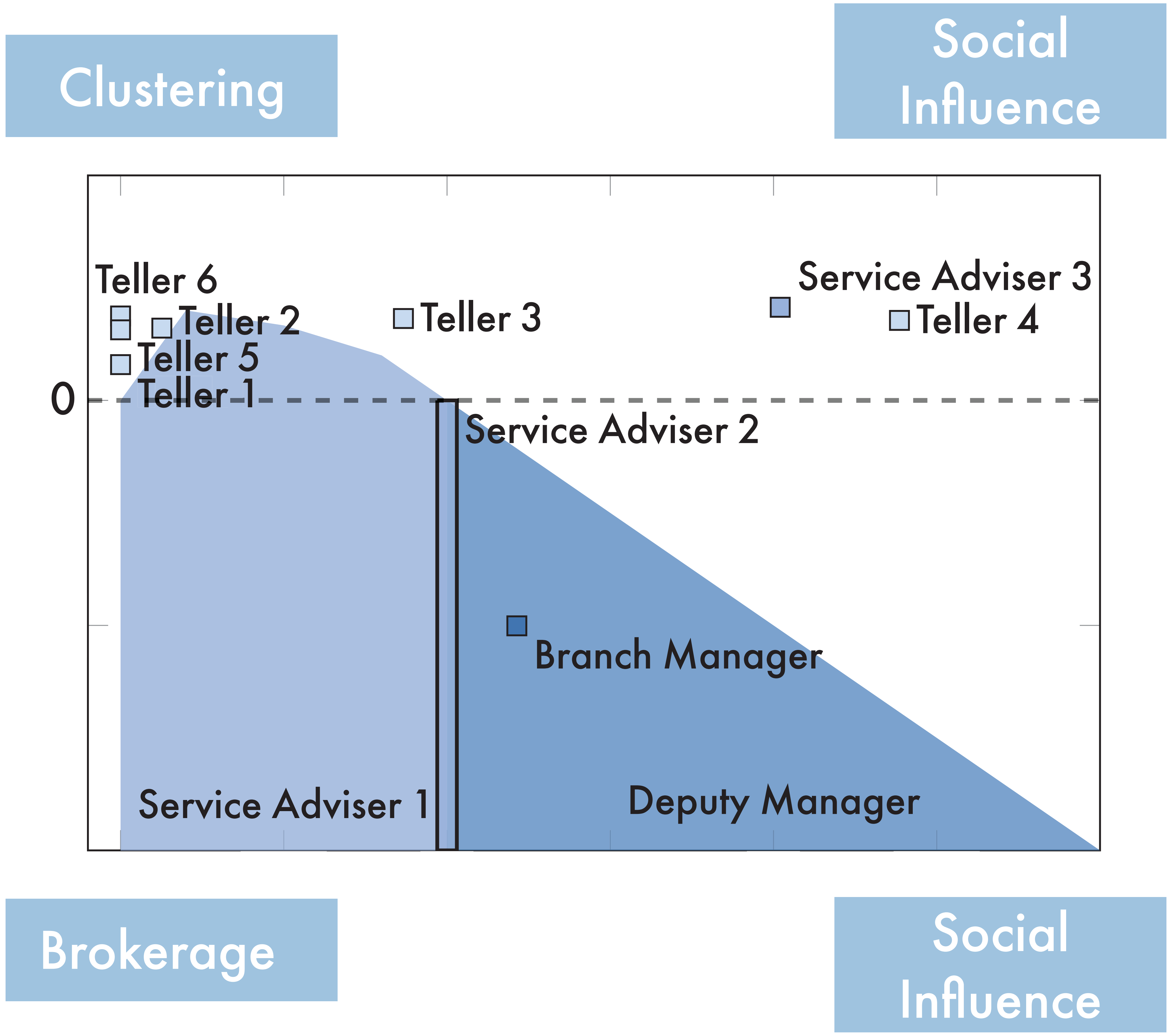


Fig. Network of confiding relationships among 11 agents in an Australian bank.



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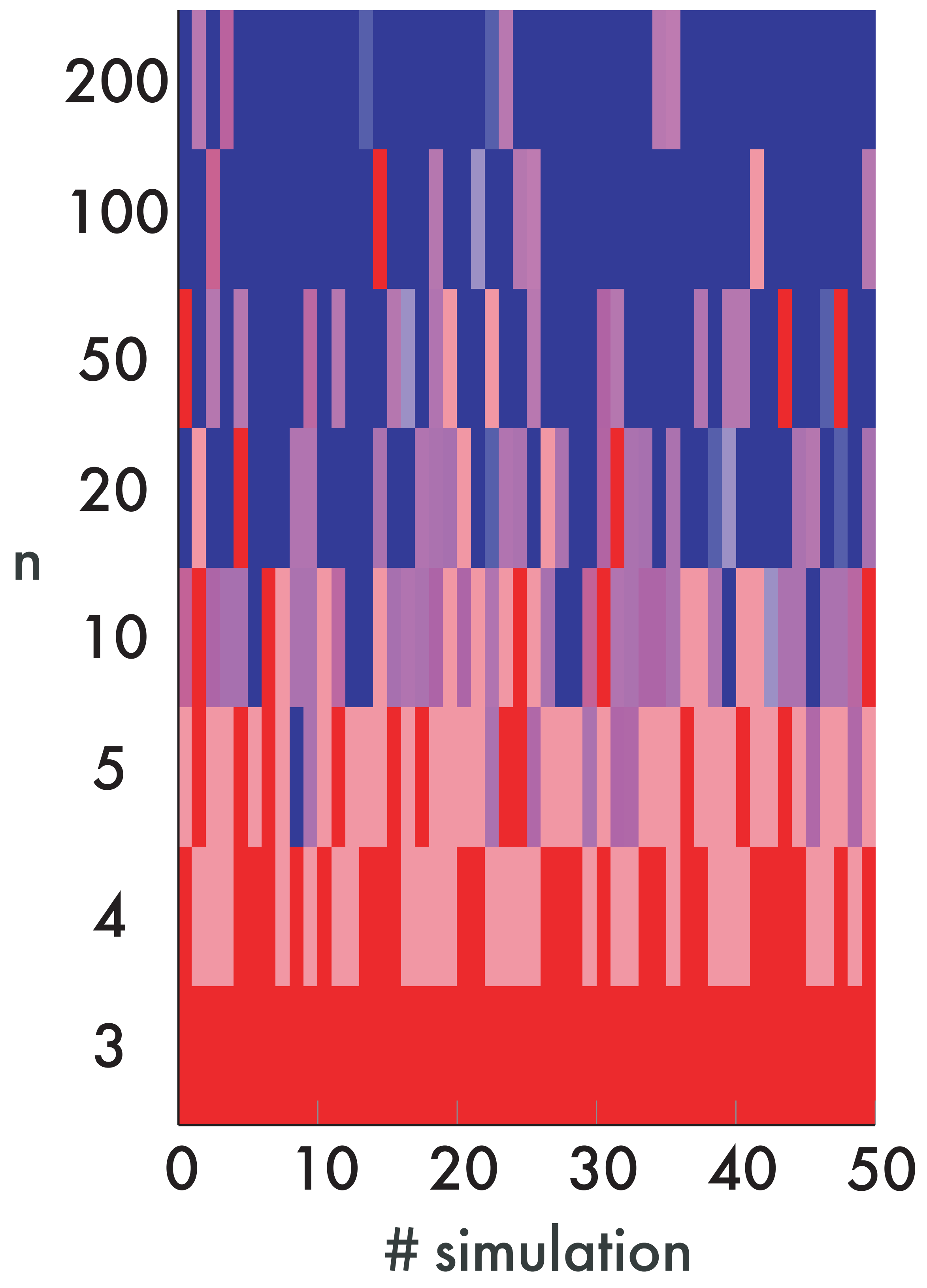
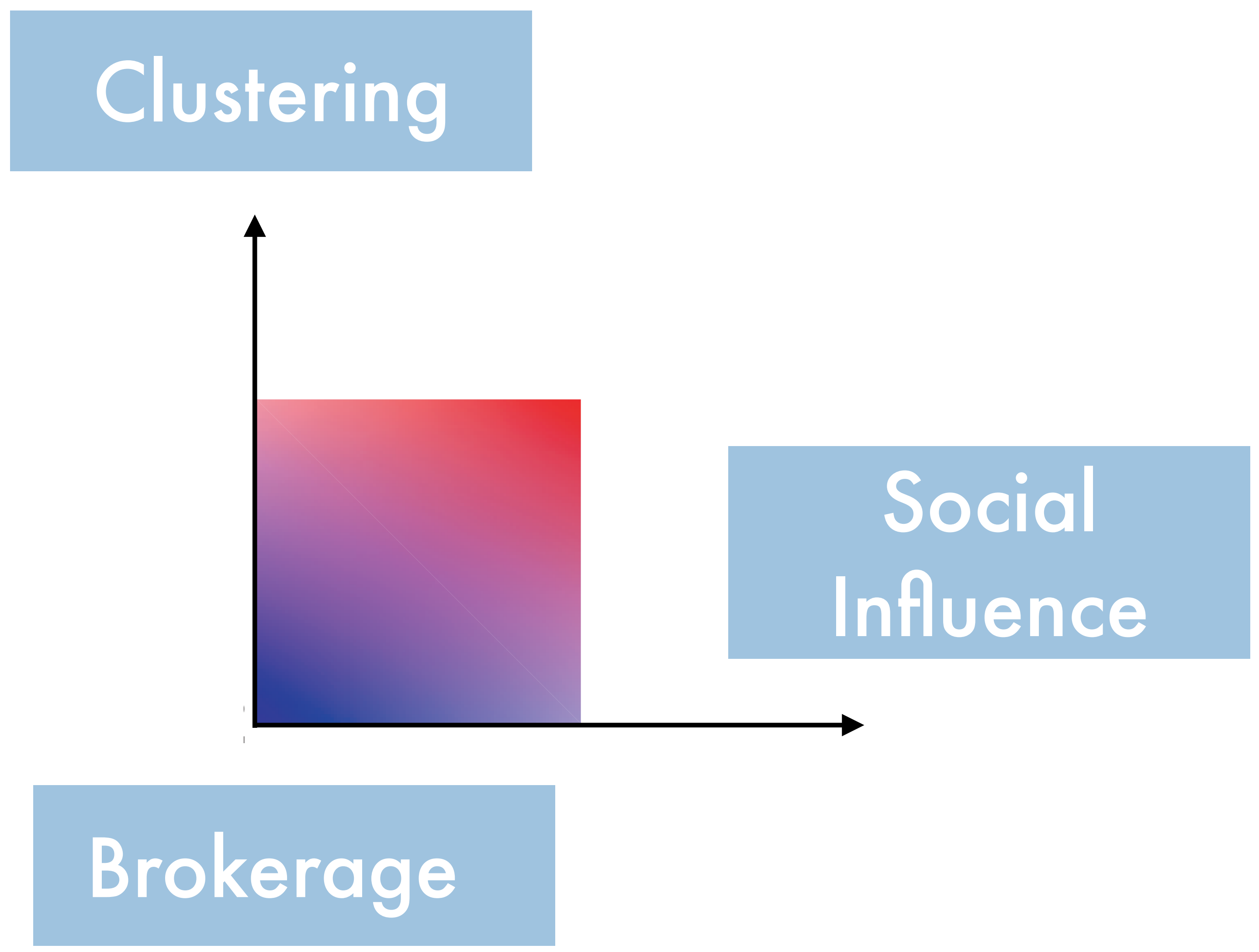
PREFERENTIAL ATTACHMENT MODEL

Nodes are introduced sequentially.
Each newborn **receives 2 incoming links** from existing nodes (randomly selected, proportionally to the outdegree), and **creates 2 outgoing ties** to existing nodes (randomly selected, proportionally to the indegree).





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N. Pagan & F. Dörfler, *"Game theoretical inference of human behaviour in social networks"*. Nature Communications (forthcoming).

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