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Solutions Series 2

Exercise 1

The mathematical formulation of the problem can be found at the end of this document.

a) See the Excel file.

b) For every vehicle type we obtain an individual independent transportation problem of the mentioned form. Hence, no additional model needs to be formulated.

c) The explicit formulation for the specified instance with 3 branches is given by:

$$\begin{aligned}
\min\ & 20x_{12} + 35x_{13} + 20x_{21} + 19x_{23} + 35x_{31} + 24x_{32} \\
& x_{11} + x_{12} + x_{13} \le 7 \\
& x_{21} + x_{22} + x_{23} \le 5 \\
& x_{31} + x_{32} + x_{33} \le 9 \\
& x_{11} + x_{21} + x_{31} \ge 4 \\
& x_{12} + x_{22} + x_{32} \ge 11 \\
& x_{13} + x_{23} + x_{33} \ge 6 \\
& x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33} \ge 0
\end{aligned}$$

Here, the integrality constraints for all variables can be skipped because of the *natural integrality property* (see the explanation at the end of this file).

In the case without "empty transfers" (variables x_{ii} are skipped, see the explanation at the end of this document) the model reads as follows:

$$\begin{aligned}
\min\ & 20x_{12} + 35x_{13} + 20x_{21} + 19x_{23} + 35x_{31} + 24x_{32} \\
& x_{12} + x_{13} \le 3 \\
& x_{21} + x_{23} \le 0 \\
& x_{31} + x_{32} \le 3 \\
& x_{21} + x_{31} \ge 0 \\
& x_{12} + x_{32} \ge 6 \\
& x_{13} + x_{23} \ge 0 \\
& x_{12}, x_{13}, x_{21}, x_{23}, x_{31}, x_{32} \ge 0
\end{aligned}$$

Exercise 2

a) See the problem description at the end of this document.

b) The explicit formulation for the specified instance is given by:

$$\max\ 122.81(x_1+x_2+x_3+x_4)+175.04(y_{11}+y_{21}+y_{31}+y_{41})+152.68(y_{12}+y_{22}+y_{32}+y_{42}) \quad \text{(A.10)}$$

$$107y_{11}+93y_{21}+87y_{31}+108y_{41} \geq 100(y_{11}+y_{21}+y_{31}+y_{41}) \quad \text{(A.11)}$$

$$107y_{12}+93y_{22}+87y_{32}+108y_{42} \geq 91(y_{12}+y_{22}+y_{32}+y_{42})$$

$$5y_{11}+8y_{21}+4y_{31}+21y_{41} \leq 7(y_{11}+y_{21}+y_{31}+y_{41}) \quad \text{(A.12)}$$

$$5y_{12}+8y_{22}+4y_{32}+21y_{42} \leq 7(y_{12}+y_{22}+y_{32}+y_{42})$$

$$x_1+y_{11}+y_{12} \leq 3814 \quad \text{(A.13)}$$

$$x_2+y_{21}+y_{22} \leq 2666$$

$$x_3+y_{31}+y_{32} \leq 4016$$

$$x_4+y_{41}+y_{42} \leq 1300$$

$$x_1,x_2,x_3,x_4 \geq 0 \quad \text{(A.14)}$$

$$y_{11},y_{12},y_{21},y_{22},y_{31},y_{32},y_{41},y_{42} \geq 0 \quad \text{(A.15)}$$

The conditions (A.11) and (A.12) can be re-written as follows (see the end of this file):

$$(107-100)y_{11}+(93-100)y_{21}+(87-100)y_{31}+(108-100)y_{41} \geq 0 \quad \text{(A.11')}$$

$$(107-91)y_{12}+(93-91)y_{22}+(87-91)y_{32}+(108-91)y_{42} \geq 0$$

$$(5-7)y_{11}+(8-7)y_{21}+(4-7)y_{31}+(21-7)y_{41} \leq 0 \quad \text{(A.12')}$$

$$(5-7)y_{12}+(8-7)y_{22}+(4-7)y_{32}+(21-7)y_{42} \leq 0$$

c) See the Excel file.

Exercise 3

a), b) Notice first that the set S describes the set of all vectors consisting of five components where three of them are ones and two of them are zeros. The cardinality of S is given as

$$|S| = \binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5\cdot 4}{1\cdot 2} = 10,$$

i.e. the solution space consists of exactly 10 solutions. This can easily be verified, since the number of solutions corresponds to the number of possibilities to choose 3 components out of the 5 vector components where to assign the value 1.

The neighborhood $N(\boldsymbol{x})$ is defined as the union of three sets where the last set just contains the vector $\boldsymbol{x}$ itself:

$$N(\boldsymbol{x}) = \{\boldsymbol{x}' \in S : x'_i = x_{i \bmod 5+1} \text{ for } i=1,\ldots,5\} \cup \{\boldsymbol{x}' \in S : x'_i = x_{(i+3) \bmod 5+1} \text{ for } i=1,\ldots,5\} \cup \{\boldsymbol{x}\}$$

In order to understand the definition of the other two sets, it can be helpful to insert different values for index i and observe the resulting pattern. The first set is defined as:

$$\{\boldsymbol{x}' \in S : x'_i = x_{i \bmod 5+1} \text{ for } i=1,\ldots,5\}$$

Inserting the possible values for index i, we obtain:

$$x'_1 = x_{1 \bmod 5+1} = x_2$$
$$x'_2 = x_{2 \bmod 5+1} = x_3$$
$$x'_3 = x_{3 \bmod 5+1} = x_4$$
$$x'_4 = x_{4 \bmod 5+1} = x_5$$
$$x'_5 = x_{5 \bmod 5+1} = x_1$$

Thus, we observe that the first set contains exactly one vector $\boldsymbol{x}'$ obtained by applying a *cyclic left-shift* to vector $\boldsymbol{x}$. Analogously, we can see that the second set

$$\{\boldsymbol{x}' \in S : x'_i = x_{(i+3) \bmod 5+1} \text{ for } i = 1,\ldots,5\}$$

comprises exactly one vector $\boldsymbol{x}'$ obtained by applying a *cyclic right-shift* to vector $\boldsymbol{x}$.

Consequently, the neighborhood $N(\boldsymbol{x})$ contains exactly three vectors for every $\boldsymbol{x}$:

$$N(\boldsymbol{x}) = \{\boldsymbol{x}^{left}, \boldsymbol{x}^{right}, \boldsymbol{x}\}$$

where $\boldsymbol{x}^{left}$ is obtained from $\boldsymbol{x}$ by applying a cyclic left-shift, e.g.

$$\boldsymbol{x} = (0,1,1,0,1)^T \quad \rightarrow \quad \boldsymbol{x}^{left} = (1,1,0,1,0)^T$$

and $\boldsymbol{x}^{right}$ is obtained from $\boldsymbol{x}$ by applying a cyclic right-shift, e.g.

$$\boldsymbol{x} = (0,1,1,0,1)^T \quad \rightarrow \quad \boldsymbol{x}^{right} = (1,0,1,1,0)^T$$

The following table shows all vectors together with their both neighbouring solutions and the corresponding function values. The last two columns indicate whether the considered solution is a local/global maximal/minimal solution. (Notice: every global solution is locally optimal.)

Notice further that the upper (lower) five solutions can be achieved by applying cyclical shifts to each other, respectively.

$\boldsymbol{x}$:	$f(\boldsymbol{x})$:	$\boldsymbol{x}^{right}$:	$f(\boldsymbol{x}^{right})$:	$\boldsymbol{x}^{left}$:	$f(\boldsymbol{x}^{left})$:	Max:	Min:
(1,1,1,0,0)	15	(0,1,1,1,0)	11	(1,1,0,0,1)	10	global	-
(0,1,1,1,0)	11	(0,0,1,1,1)	10	(1,1,1,0,0)	15	-	-
(0,0,1,1,1)	10	(1,0,0,1,1)	8	(0,1,1,1,0)	11	-	-
(1,0,0,1,1)	8	(1,1,0,0,1)	10	(0,0,1,1,1)	10	-	local
(1,1,0,0,1)	10	(1,1,1,0,0)	15	(1,0,0,1,1)	8	-	-
(1,1,0,1,0)	9	(0,1,1,0,1)	12	(1,0,1,0,1)	14	-	local
(0,1,1,0,1)	12	(1,0,1,1,0)	13	(1,1,0,1,0)	9	-	-
(1,0,1,1,0)	13	(0,1,0,1,1)	6	(0,1,1,0,1)	12	local	-
(0,1,0,1,1)	6	(1,0,1,0,1)	14	(1,0,1,1,0)	13	-	global
(1,0,1,0,1)	14	(1,1,0,1,0)	9	(0,1,0,1,1)	6	local	

c) For instance, choose $\boldsymbol{x}^* = (1,1,0,1,0)$ which is a local minimum. We have to show that the function value in all solutions belonging to the neighbourhood of $\boldsymbol{x}^*$ cannot be smaller than the function value in $\boldsymbol{x}^*$ itself. This clearly is the case as $12 \geq 9$ and $14 \geq 9$.

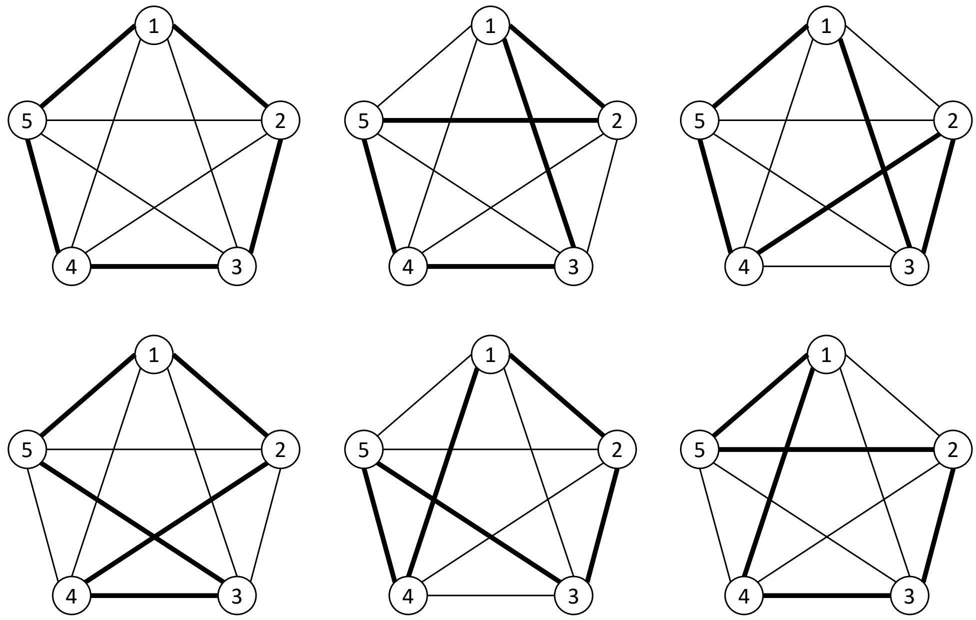
Exercise 4

a) The set V of vertices and the set E of edges of the graph $G=(V,E)$ are given by

$$V=\{1,2,3,4,5\},\quad E=\{(1,2),(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,4),(3,5),(4,5)\}.$$

b) The neighborhood $N(\boldsymbol{x})$ consists of the following elements (here, the incidence vectors have the following component order: $\boldsymbol{x}=(x_{12},x_{13},x_{14},x_{15},x_{23},x_{24},x_{25},x_{34},x_{35},x_{45})^T$):

$$\begin{aligned} N(\boldsymbol{x})=\{&(1,0,0,1,1,0,0,1,0,1),\\ &(1,1,0,0,0,0,1,1,0,1),\\ &(0,1,0,1,1,1,0,0,0,1),\\ &(1,0,0,1,0,1,0,1,1,0),\\ &(1,0,1,0,1,0,0,0,1,1),\\ &(0,0,1,1,1,0,1,1,0,0)\}\end{aligned}$$



c) According to the definition of the Euclidian neighbourhood $N_\varepsilon(\boldsymbol{x})$ we get that the set $N_\varepsilon(\boldsymbol{x})\cap S$ consists of all incidence vectors $\boldsymbol{y}$ of tours in the graph $G=(V,E)$ which have a distance from the present tour $\boldsymbol{x}$ of less than ε, i.e.

$$\|\boldsymbol{y}-\boldsymbol{x}\|=\sqrt{(y-x)^2}=\sqrt{(y_1-x_1)^2+(y_2-x_2)^2+\ldots+(y_n-x_n)^2}<\varepsilon$$

All tours y belonging to the 2-opt-neighborhood $N(\boldsymbol{x})$ differ from $\boldsymbol{x}$ in exactly 4 components; they are constructed by eliminating two edges from $\boldsymbol{x}$ and replacing them by two other edges. Hence, we have $\|\boldsymbol{y}-\boldsymbol{x}\|=\sqrt{4}=2$ for all neighbouring tours; thus $N(\boldsymbol{x})\subseteq N_\varepsilon(\boldsymbol{x})\cap S$ is valid for $\varepsilon=2.1$.

It remains to show that there are no other tours with distance to $\boldsymbol{x}$ of less than 2.1.

(1) One can easily see that there are no tours with a distance to $\boldsymbol{x}$ less than 2. Such tours would be allowed to differ from $\boldsymbol{x}$ in at most 3 components. It is not possible to construct from $\boldsymbol{x}$ a new tour by eliminating just one edge and replacing it by another edge. Hence, at least 2 edges have to be eliminated and replaced by 2 other edges; the resulting difference is at least 2.

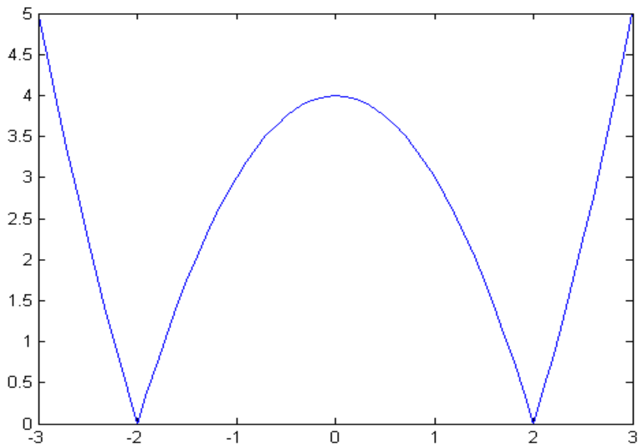
(2) Consider all tours with a distance to $\boldsymbol{x}$ equal to two. (The next biggest possible distance is $\sqrt{5}$ but this is greater than 2.1, hence it is not needed to consider it). Further, one can easily see that all tours constructed by exchange of two edges belong to the 2-opt-neighborhood $N(\boldsymbol{x})$. Namely, when choosing the two edges to be removed one sees that the two edges to be included are determined uniquely, i.e. there is just one possibility how to connect the resulting pieces to produce a tour.

There are $\binom{5}{2}=10$ possibilities how to choose the two edges to be removed from the five edges of the tour $\boldsymbol{x}$. Among them, five possibilities include edges which "don't touch" each other (they don't have a common vertex). These five choices (together with $\boldsymbol{x}$ itself) yield the neighbourhood $N(\boldsymbol{x})$. The other five choices include neighbouring edges. Obviously, one cannot produce a new tour from $\boldsymbol{x}$ by removing two neighbouring edges and replacing them by two other edges.

Thus we have shown that $N(\boldsymbol{x}) = N_\varepsilon(\boldsymbol{x}) \cap S$.

Exercise 5

a) The graph of the function *f* looks as follows:



b) There exist the following local and global maximal solutions:

Solution x^*:	Function value $f(x^*)$:	Local maximum:	Global maximum:
-3	5	yes	yes
0	4	yes	no
3	5	yes	yes

c) There exist the following local and global minimal solutions:

Solution x^*:	Function value $f(x^*)$:	Local minimum:	Global minimum:
-2	0	yes	yes
2	0	yes	yes

d) Consider the solution at $x^* = 0$. We have to show that there exists $\varepsilon > 0$ such that

$$
\begin{aligned}
f(x) \leq f(x^*) \text{ for all } x \in N_\varepsilon(x^*) \cap S \quad &= \{y \in R : |\, y - x^* \,| < \varepsilon\} \cap S \\
&= \{y \in R : x^* - \varepsilon < y < x^* + \varepsilon, -3 \leq y \leq 3\} \\
&= \{y \in R : -\varepsilon < y < \varepsilon, -3 \leq y \leq 3\}
\end{aligned}
$$

Let $\varepsilon = 1$. Hence, we have to show that

$$f(x) = |-x^2 + 4| \leq f(x^*) = 4 \qquad \text{for all } x \text{ with } -1 < x < 1$$

But this is obvious; from $-1 < x < 1$ it follows that $0 \leq x^2 < 1$ and $3 < f(x) = \left|-x^2 + 4\right| \leq 4$.

A.2. Problem 2: Product Mixture in an Oil Refinery

The problem can be formulated as a Linear Program as follows:

Sets:

I	Set of raw fuel types, $I = \{1, \dots, m\}$.
J	Set of jet fuel types, $J = \{1, \dots, n\}$.

Parameters:

a_i^{PN}	Performance number PN of raw fuel i, $i \in I$.
a_i^{RVP}	Vapor pressure RVP of raw fuel i, $i \in I$.
b_i	Production output (in barrel) of raw fuel i, $i \in I$.
c_i	Price (\$ per barrel) for raw fuel i, $i \in I$.
d_j^{PN}	Minimal performance number PN required for jet fuel j, $j \in J$.
d_j^{RVP}	Maximal vapor pressure RVP required for jet fuel j, $j \in J$.
f_j	Price (\$ per barrel) for jet fuel j, $j \in J$.

Variables:

x_i	Amount (in barrels) of raw fuel i sold directly, $i \in I$.
y_{ij}	Amount (in barrels) of raw fuel i used to produce jet fuel j, $i \in I$, $j \in J$.

Objective function and constraints:

$$\max \sum_{i \in I} c_i x_i + \sum_{j \in J} f_j \sum_{i \in I} y_{ij} \tag{A.10}$$

$$\sum_{i \in I} a_i^{PN} y_{ij} \geq d_j^{PN} \sum_{i \in I} y_{ij}, \qquad j \in J \tag{A.11}$$

$$\sum_{i \in I} a_i^{RVP} y_{ij} \leq d_j^{RVP} \sum_{i \in I} y_{ij}, \qquad j \in J \tag{A.12}$$

$$x_i + \sum_{j \in J} y_{ij} \leq b_i, \qquad i \in I \tag{A.13}$$

$$x_i \geq 0, \qquad i \in I \tag{A.14}$$

$$y_{ij} \geq 0, \qquad i \in I, j \in J \tag{A.15}$$

Consider constraints A.11 and A.12. According to the problem definition, we know that the values PN and RVP of a raw fuel mixture are given by the weighted average of the corresponding values of the individual components. For PN, we obtain the following constraint:

$$\frac{\sum_{i \in I} a_i^{PN} y_{ij}}{\sum_{i \in I} y_{ij}} \geq d_j^{PN}, \qquad j \in J$$

Multiplication by $\sum_{i \in I} y_{ij}$ yields the linear constraint A.11. Notice that it can be written equivalently as follows:

$$\sum_{i \in I} (a_i^{PN} - d_j^{PN}) y_{ij} \geq 0, \qquad j \in J$$

Here, the difference $(a_i^{PN} - d_j^{PN})$ specifies the deviation of the PN value of raw fuel i from the minimal PN value required for mixture j. Hence, the constraint states that the weighted total deviation cannot be negative.

A.3. Problem 3: Vehicle Dispatching in a Car Rental Company

The car rental problem is an example of the Classical *Transportation Problem* (or Hitchcock Problem) which can be formulated as follows.

We are given a set of suppliers $I = \{1, 2, \ldots, m\}$ with given production capacities a_i, $i \in I$, and a set of consumers $J = \{1, 2, \ldots, n\}$ with given demands b_j, $j \in J$. Our task is to find a transportation plan x_{ij}, $i \in I$, $j \in J$, resulting in minimal costs. Here x_{ij} denotes the number of production units to be transported from supplier i to consumer j, and c_{ij} denotes the transport costs for one unit from supplier i to consumer j.

The car rental problem is a special case of the transportation problem where the suppliers coincide with the consumers and are represented by the branches, i.e. $I = J = \{1, \ldots, n\}$. The production capacities correspond to current vehicle stocks, i.e. the number of vehicles available in the evening, and the demands correspond to the requested vehicle stocks, i.e. the number of vehicles required on the next morning.

The transportation problem belongs to the class of network flow problems and is one of the first systematically investigated linear programming problems. A fundamental property of network flow problems is the fact that there always exists an *integer* optimal solution (so-called *natural integrality property*). Hence, when using a suitable 'vertex method' (such as the Simplex algorithm) to solve the problem, one can skip the integrality conditions for the variables x_{ij} and solve it as a normal LP.

The car renting problem can be formulated as follows:

Sets:

I	Set of locations (offer), $I = \{1, 2, \ldots, n\}$.
J	Set of locations (demand), $J = \{1, 2, \ldots, n\}$.

Parameters:

a_i	Number of vehicles available at location i, $i \in I$.
b_j	Number of vehicles requested at location j, $j \in J$.
c_{ij}	Transfer distance (in km) from location i to location j, $i \in I, j \in J$.

Variables:

x_{ij} Number of vehicles transferred from location i to location j, $i \in I, j \in J$.

Objective function and constraints:

$$\min \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \tag{A.16}$$

$$\sum_{j \in J} x_{ij} \leq a_i, \qquad i \in I \tag{A.17}$$

$$\sum_{i \in I} x_{ij} \geq b_j, \qquad j \in J \tag{A.18}$$

$$x_{ij} \geq 0, \qquad i \in I, j \in J \tag{A.19}$$

This formulation has a minor flaw: vehicles remaining at location i are modelled as transfers x_{ii} from i to i. This can be avoided by a small change:

$$\min \sum_{i \in I} \sum_{j \in J, j \neq i} c_{ij} x_{ij} \tag{A.20}$$

$$\sum_{j \in J, j \neq i} x_{ij} \leq \max\{a_i - b_i, 0\}, \qquad i \in I \tag{A.21}$$

$$\sum_{i \in I, i \neq j} x_{ij} \geq \max\{b_j - a_j, 0\}, \qquad j \in J \tag{A.22}$$

$$x_{ij} \geq 0, \qquad i \in I, j \in J, i \neq j \tag{A.23}$$

The term $\max\{a_i - b_i, 0\}$ corresponds to the actual offer of vehicles at location i (which is the difference $a_i - b_i$ if $a_i \geq b_i$, otherwise it is 0). Analogously, $\max\{b_j - a_j, 0\}$ corresponds to the actual demand at location j. An optimal solution (with respect to the second formulation) is given by

$$x_{1,3} = 5, x_{5,2} = 1, x_{5,4} = 10, x_{6,2} = 5, x_{7,2} = 3, x_{7,9} = 16, x_{10,3} = 5, x_{10,8} = 3$$

with associated minimal total transfer distance of 3395 km.

In the case of multiple vehicle types, one obtains an individual transportation problem of the above form for every vehicle type.