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Exercise Series 3

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Exercise 1

a) Consider the function $f:\mathbb{R}^2 \to \mathbb{R}$ given by $f(\boldsymbol{x}) = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}$ and let $\boldsymbol{c} = (3,-2)^{\mathsf{T}}$. Draw the level sets corresponding to levels 6 and 12. What is the connection between the level sets of f and the vector $\boldsymbol{c}$?

b) Consider the function $f:\mathbb{R}^3 \to \mathbb{R}$ given by $f(\boldsymbol{x}) = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}$ and let $\boldsymbol{c} = (1,2,1)^{\mathsf{T}}$. Draw the level sets corresponding to levels 2 and 3. What is the connection between the level sets of f and the vector $\boldsymbol{c}$?

Exercise 2

The following LP is given:

$$\begin{aligned} \max\ x_1 + 3x_2 + 2x_3 \\ x_2 + x_3 \le 2 \\ x_1 - 2x_2 \le -2 \\ x_1, x_2, x_3 \ge 0 \end{aligned}$$

Write this LP in the following possible notations:

- In the vector form in „row notation“: $\max\ \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x},\ \boldsymbol{a}^i\boldsymbol{x} \le b_i,\ i = 1,\ldots,m,\ \ \boldsymbol{x} \ge \boldsymbol{0}$
- In the vector form in „column notation“: $\max\ \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x},\ \sum_{j=1}^{n} \boldsymbol{A}_j x_j \le \boldsymbol{b},\ \ \boldsymbol{x} \ge \boldsymbol{0}$
- In „matrix notation“: $\max\ \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x},\ \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b},\ \ \boldsymbol{x} \ge \boldsymbol{0}$

Exercise 3

Consider the following LP in general form:

$$\begin{aligned} \min x_1 - 2x_2 - 3x_3 \\ x_1 + 2x_2 + 4x_3 \ge 12 \\ x_1 - x_2 + x_3 = 2 \\ x_1 + 2x_2 + x_3 \le 14 \\ x_1 \ge 0 \\ x_2\ \text{free} \\ x_3 \le 0 \end{aligned}$$

Transform the LP into the following forms:

- Maximizing problem in canonical form
- Minimizing problem in canonical form
- Minimizing problem in standard form
- Maximizing problem in inequality form.

Exercise 4

Consider the introductory example "Shift planning in a department store" (see the Script, section 1.2.4)

a) Formulate the mentioned shift planning problem as an integer linear optimization problem in the general case where arbitrary number of time periods and shifts are possible (in the example from Script, the considered time period is one hour).

b) Implement the instance from the Script in Excel and determine the optimal solution using Solver.

Hint:
Denote the set of time periods by $I = \{1,\ldots,m\}$ *and the set of shifts by* $J = \{1,\ldots,n\}$ *. Define a 0-1-matrix* $\boldsymbol{A} \in \{0,1\}^{m\times n}$ *, where* $a_{ij} = 1$ *if and only if the time period* i *is included in the shift* j *. The constraints can be then written in the form* $\boldsymbol{Ax} \geq \boldsymbol{b}$ *.*

Exercise 5 (* optional *)

Occasionally, one is supposed to consider maximization problems with an objective function in the form

$$f(\boldsymbol{x}) = \min\{\boldsymbol{a}^i \boldsymbol{x} + b_i : i \in I\} \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{a^i} \in \mathbb{R}^n, b_i \in \mathbb{R}$ for all $i \in I$.

a) Draw the graph of the following function defined on $\mathbb{R}^1$:

$$f(x_1) = \min\left\{\frac{1}{2}x_1 + 1, -\frac{1}{4}x_1 + 4, -x_1 + 10\right\}$$

b) Show that any function of the form (1) is concave on $\mathbb{R}^n$.