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Exercises Series 2

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Exercise 1

Consider the introductory problem “Vehicle dispatching in a car rental company” from the lecture notes.

a) Implement the linear optimization model from the lecture in Excel. Use the distances coming from the attached separate file.

b) Write down the mathematical formulation of an extended optimization problem which can handle dispatching problems with various vehicle types. (You don’t need to implement the extended model in Excel.)

c) Write down the optimization problem (not extended) in the explicit form (i.e. with actual numbers and without sums and vectors) for an instance with 3 car hire locations. The data is given as follows:

	1	*2*	*3*
a_i	7	5	9
b_j	4	11	6

c_{ij}	*1*	*2*	*3*
1	0	20	35
2	20	0	19
3	35	24	0

Exercise 2

Consider the introductory problem “Product mixture in an oil refinery” from the lecture notes.

a) Formulate the problem as a linear optimization problem for the general case with a set $I = \{1, \ldots, m\}$ describing the raw fuels and the set $J = \{1, \ldots, n\}$ describing the jet fuels. Specify in detail the used sets, parameters, variables, further constraints and the objective function.

b) Write down the optimization problem in the explicit form (i.e. with actual numbers and without sums and vectors) for the instance with four raw fuel types and two jet fuel types mentioned in the lecture notes.

c) Implement the optimization model for the instance from the lecture notes in Excel, and determine an optimal solution.

Exercise 3

Consider a base set

$$S = \{\boldsymbol{x} \in \{0,1\}^5 : \sum_{i=1}^{5} x_i = 3\}$$

and a function given by

$$f : S \to \mathbb{R} \text{ with } f(\boldsymbol{x}) = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}\text{, where } \boldsymbol{c}^{\mathsf{T}} = (5,3,7,1,2)\,.$$

Further, we define on S a neighbourhood notion N as follows:

$$\begin{aligned} N(\boldsymbol{x}) = \{\boldsymbol{x}' \in S : x'_i = x_{i \bmod 5+1} \text{ for } i = 1,\dots,5\} &\cup \\ \{\boldsymbol{x}' \in S : x'_i = x_{(i+3) \bmod 5+1} \text{ for } i = 1,\dots,5\} &\cup \{\boldsymbol{x}\} \end{aligned}$$

a) Determine all elements of S and the corresponding function values of f.

b) Determine all local (with regard to N) and all global maximal and minimal solutions of f.

c) Choose any local minimal solution and prove that it is indeed a local minimal solution.

Hint: To understand the terms $x'_i = x_{i \bmod 5+1}$ *and* $x'_i = x_{(i+3) \bmod 5+1}$ *enter some values of* i *into the formulas. The operator "mod" describes the modulo operator, which calculates the remainder when performing integer division. For instance,* $5 \bmod 5 = 0$ *and* $6 \bmod 5 = 1$.

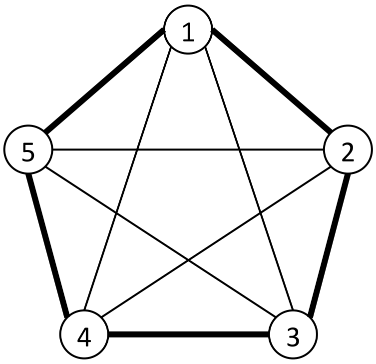
Exercise 4

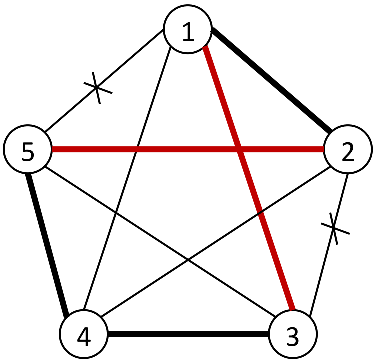
The *Traveling Salesman Problem* (TSP) is a fundamental optimization problem in the graph theory and belongs to the most researched problems of the Combinatorial Optimization. The task (in a symmetric TSP) is to find a tour with shortest length in an undirected graph with given edge lengths. Here, a tour is a round trip where every vertex of the graph is visited exactly once.

There exist numerous (meta-)heuristics for the TSP; many of them are based on the principle of step-wise improvement of the existing solution (*iterative improvement, local search*). Starting at a current solution (i.e. a tour) $\boldsymbol{x}$ we are looking for a better tour in a *neighbourhood* $N(\boldsymbol{x})$ of the tour $\boldsymbol{x}$; if a better tour is found, it will replace the current tour $\boldsymbol{x}$. The method is repeated until no better tours can be found in the neighbourhood. The final tour is then *locally optimal* regarding the used neighbourhood.

One of the most successful neighbourhood notions is the so-called *2-opt-neighbourhood,* which was developed in 1973 by Shen Lin and Brian Kernighan. Here, a neighbouring tour is constructed by eliminating two arbitrary non-adjacent edges from the tour and by connecting the remaining line segments by two new edges, resulting in a new tour (see example below).

As an example, consider the following undirected graph $G = (V, E)$; the edge lengths are not needed here and hence omitted. We are given the marked tour $\boldsymbol{x}$ in the left picture which visits the vertices 1, 2, 3, 4, 5, 1 in this order. The right picture depicts a neighbouring tour from the neighbourhood $N(\boldsymbol{x})$ which can be obtained by eliminating the edges (2, 3) and (1, 5), and simultaneously inserting the edges (1, 3) and (2, 5).





Tours in a graph $G = (V, E)$ can be regarded as subsets of edges and can be described by an *incidence vector* $\boldsymbol{x} \in \{0,1\}^E$. The incidence vector is a 0-1-vector which consists of components x_{vw} for all edges *(v,* w). A component is equal to 1 if the corresponding edge belongs to the tour, otherwise it is 0. The edge order in the incidence vector can be chosen freely but has to be followed consequently. For instance, the current tour in the left picture is given by the incidence vector

$$\boldsymbol{x} = (x_{12}, x_{13}, x_{14}, x_{15}, x_{23}, x_{24}, x_{25}, x_{34}, x_{35}, x_{45})^\mathsf{T} = (1,0,0,1,1,0,0,1,0,1)^\mathsf{T}.$$

a) Write down the set of vertices V and the set of edges E of the graph $G = (V, E)$.

b) Write down all elements (i.e. corresponding incidence vectors) of the neighbourhood $N(\boldsymbol{x})$, where $\boldsymbol{x}$ is the current tour mentioned above.

c) *(* optional *)* Let $S \subseteq \{0,1\}^E$ be the solution space of the TSP, i.e. the set of all incidence vectors of all tours. For the considered example, show that $N(\boldsymbol{x}) = N_\varepsilon(\boldsymbol{x}) \cap S$ for $\varepsilon = 2.1$.

Exercise 5

Consider the base set

$$S = \{x \in \mathbb{R} : -3 \le x \le 3\}$$

and the function *f* given by

$$f: S \to \mathbb{R} \text{ and } f(x) = |-x^2 + 4|$$

a) Plot the graph of the function f .

b) Determine all local (with regard to Euclidean neighbourhoods) and global maximal solutions of f and the corresponding function values.

c) Determine all local (with regard to Euclidean neighbourhoods) and global minimal solutions of f and the corresponding function values.

d) Prove that $x^* = 0$ is a local maximal solution with regard to Euclidean neighbourhoods.