The Bethe-Yang equations generalise the Bethe Ansatz equations to relativistic systems. In particular, they arise as a quantisation condition for the wavefunction of a particle on a torus of radius R.

To derive the Bethe-Yang equations for our graded theory, we assume that on the torus there are L_k, $k = 1 \dots 5$ particles for each species, and that they are sufficiently diluted so that there are regions where each particle can be considered free. We then take one particle of the species k and make it go around the torus; in the transition between free regions, it will pass through an interaction zone with the particle j of the l-th species. The interaction has the effect of multiplying the wavefunction of the system by the S matrix factor $S_{kl}(\theta_{i_k}, \theta_{j_l})$. Then the Bethe-Yang equations are a consequence of the (quasi-)periodic boundary conditions on the torus of radius R and are given by:

$$1 = e^{i\mu_k} e^{iRp_{i_k}} \prod_{l \in \mathbb{Z}_5} \prod_{j_l \neq i_k}^{L_l} S_{kl}(\theta_{i_k}, \theta_{j_l}) \,, \quad i_k = 1 \dots L_k \,,\ k \in \mathbb{Z}_5 \,. \tag{1}$$

From these equations, it is possible to obtain the Thermodynamic Bethe Ansatz in the thermodynamic limit $L_k \to \infty$. Our derivation follows closely (Zamolodchikov paper [**?**]). First, rewrite (1) in logarithmic form:

$$i\mu_i + im\,R \sinh\theta_{i_k} + \sum_{l \in \mathbb{Z}_5} \sum_{j_l \neq i_k}^{L_l} \log S_{kl}(\theta_{i_k}, \theta_{j_l}) = 2\pi n_{i_k} \,, \quad i_k = 1 \dots L_k \,,\ k \in \mathbb{Z}_5 \,, \tag{2}$$

where n_{i_k} is the mode number for the species k. In the thermodynamic limit, we can express the sum over j_l as an integral via the introduction of the particle densities in the rapidity interval $\Delta\theta_{j_l}$, namely ρ_l. We get:

$$i\mu_k + imR \sinh\theta_k + \sum_{l \in \mathbb{Z}_5} \int_{-\infty}^{+\infty} \log S_{kl}(\theta_k, \theta_l)\ \rho_l(\theta_l) d\theta_l = 2\pi n_k \,, \quad k \in \mathbb{Z}_5 \,. \tag{3}$$

We derive this equation by θ_k, introducing the density of levels for the species k, $\rho_{tot,k} = \rho_k + \bar{\rho}_k \equiv \frac{dn_k}{d\theta_k}$, where $\bar{\rho}_k$ is the hole density, obtaining:

$$\frac{mR\cosh\theta_k}{2\pi} + \sum_{l \in \mathbb{Z}_5} \int_{-\infty}^{+\infty} \frac{1}{2\pi i} \frac{d}{d\theta_k} \log S_{kl}(\theta_k, \theta_l)\ \rho_l(\theta_l) d\theta_l = \rho_k + \bar{\rho}_k \,, \quad k \in \mathbb{Z}_5 \,. \tag{4}$$

Now we need to minimise the free energy of the twisted theory:

$$-\sum_{k \in \mathbb{Z}_5} LRf_k = \sum_{k \in \mathbb{Z}_5} (-LE_k + TS_k + i\mu_k) \tag{5}$$

where as usual we have defined:

$$E_k = \int m\cosh(\theta)\rho_k(\theta) d\theta \,, \tag{6}$$

$$S_k = \int d\theta ((\rho_k + \bar{\rho}_k) \log(\rho_k + \bar{\rho}_k) - \rho_k \log\rho_k - \bar{\rho}_k \log\bar{\rho}_k) \,. \tag{7}$$

This minimisation problem must take account the existing constraint between $\delta\rho_k$ and $\delta\bar{\rho}_k$, which is found by varying (4):

$$\delta\rho_k + \delta\bar{\rho}_k = + \sum_{l\in\mathbb{Z}_5} \int_{-\infty}^{+\infty} \frac{1}{2\pi i} \frac{d}{d\theta_k} \log S_{kl}(\theta_k, \theta_l)\ \delta\rho_l(\theta_l) d\theta_l \tag{8}$$