Notes on:
The Mechanics and Thermodynamics of Continua

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1 Motion of a Body

B is a reference body containing points $\mathbf{X}$ which are material points. There is a one-to-one function

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$$

taking reference material points $\mathbf{X}$ to spatial points $\mathbf{x}$ at time t. We require

$$J(\mathbf{X}, t) := \det \nabla \boldsymbol{\chi}_t(\mathbf{X}) > 0$$

where J is the volumetric Jacobian of the mapping $\boldsymbol{\chi}_t$ at $\mathbf{X}$. Region occupied by body B at time t is

$$\mathcal{B}_t = \boldsymbol{\chi}_t(B)$$

is the deformed body at time t.

1.1 Convection of Sets with the Body

A is a material set. Then **deforms to** $\mathcal{A}_t$ at time t. $\mathcal{A}_t$ **convects with the body** if there is a set A of material points such that

$$\mathcal{A}_t = \boldsymbol{\chi}_t(A)$$

for all t. Note that material cannot cross the boundary of a spatial set which convects with the body. Also note that if $\mathbf{X}$ is on ∂B (boundary), then $\boldsymbol{\chi}(\mathbf{X}, t)$ is on $\partial \mathcal{B}_t$ for all time t and conversely.

2 The Deformation Gradient

The **Deformation gradient** of a body is

$$\mathbf{F} = \nabla \boldsymbol{\chi}, \qquad F_{ij} = \frac{\partial \chi_i}{\partial X_j},$$

the Jacobian matrix of $\mathbf{x} = \mathbf{x}(\mathbf{X})$. As above

$$J = \det \mathbf{F} > 0.$$

2.1 Approximation of a Deformation by a Homogeneous Deformation

2.1.1 Homegeneous Deformations

Fix time t so that

$$\boldsymbol{\chi}(\mathbf{X}) \equiv \boldsymbol{\chi}_t(\mathbf{X}).$$

$\boldsymbol{\chi}$ is a **homogeneous deformation** if $\mathbf{F}(\mathbf{X}) \equiv \mathbf{F}(\mathbf{X}, t)$ is independent of $\mathbf{X}$. So

$$\boldsymbol{\chi}(\mathbf{X}) - \boldsymbol{\chi}(\mathbf{Y}) = \mathbf{F}(\mathbf{X} - \mathbf{Y})$$

for **all** material points $\mathbf{X}$ and $\mathbf{Y}$. By the above, $\underline{\mathbf{F}\text{ maps material vectors to spatial vectors.}}$ Then, also, $\mathbf{X} - \mathbf{Y} = \mathbf{F}^{-1}[\boldsymbol{\chi}(\mathbf{X}) - \boldsymbol{\chi}(\mathbf{Y})]$ so that $\mathbf{F}^{-1}$ maps spatial vectors too material vectors. Taking the inner product with a spatial vector $\mathbf{s}$ gives

$$\mathbf{s} \cdot [\boldsymbol{\chi}(\mathbf{X}) - \boldsymbol{\chi}(\mathbf{Y})] = \mathbf{s} \cdot [\mathbf{F}(\mathbf{X} - \mathbf{Y})] = (\mathbf{F}^T \mathbf{s}) \cdot (\mathbf{X} - \mathbf{Y})$$

so that $\mathbf{F}^T$ maps spatial vectors to material vectors.
Summarizing the mapping properties:

1. $\mathbf{F}$ and $\mathbf{F}^{-T}$ map material vectors to spatial vectors
2. $\mathbf{F}^{-1}$ and $\mathbf{F}^T$ map spatial vectors to material vectors

2.1.2 General Deformations

Let χ_t be an arbitrary deformation. Taylor expanding the deformation about material point $\mathbf{X}$ gives

$$\underline{\chi_t(\mathbf{Y}) - \chi(\mathbf{X}) = \mathbf{F}(\mathbf{X}, t)(\mathbf{Y} - \mathbf{X})} + o(|\mathbf{Y} - \mathbf{X}) \qquad \text{as } |\mathbf{Y} - \mathbf{X}| \to 0.$$

Therefore, $\mathbf{F}(\mathbf{X}, t)(\mathbf{Y} - \mathbf{X})$ is an approximation of $\chi_t(\mathbf{Y}) - \chi(\mathbf{X})$. Also, since $\mathbf{X}$ is fixed in the Taylor expansion, $\mathbf{F}(\mathbf{X}, t)$ is constant. Thus the underlined portion is the second definition of a homogeneous deformation. Therefore, *within a neighborhood of a material point $\boldsymbol{X}$ and to within an error of $o(|\boldsymbol{Y} - \boldsymbol{X}|)$, a deformation behaves like a homogeneous deformation.* So with $o(|\mathbf{Y} - \mathbf{X}|)$ small, we have:

1. (M1) $F(\mathbf{X}, t)$ can be thought of as a mapping of an infinitesimal neighborhood of $\mathbf{X}$ in the reference body to an infinitesimal neighborhood of $\mathbf{x} - \chi_t(\mathbf{X})$ in the deformed body.

2. (M2) This gives an asymptotic meaning to the formal relation

$$\mathbf{dx} = \mathbf{F}(\mathbf{X}, t)\mathbf{dX}$$

Now, we have that the mapping properties for a homogeneous deformation hold pointwise for the deformation gradient in an arbitrary deformation. For example, for a given $\mathbf{X}$, the linear transformation $\mathbf{F}(\mathbf{X}, t)$ associates with each material vector $\mathbf{m}$ a spatial vector $\mathbf{s} = \mathbf{F}(\mathbf{X}, t)\mathbf{m}$.

2.2 Convection of Geometric Quantities

2.2.1 Infinitesimal Fibers

Define the temporally constant material vector field $\mathbf{f}_R$ associated with a given spatial vector file $\mathbf{f}$ by

$$\mathbf{f}(\mathbf{x}, t) = \mathbf{F}(\mathbf{X}, t)\mathbf{f}_R(\mathbf{X}) \qquad \mathbf{x} = \chi_t(\mathbf{X}) \tag{6.8}$$

for all $\mathbf{X}$ and t.

Now by the above statements about the local homogeneity of deformation, we can see equation 6.8 becomes

$$\epsilon\mathbf{f}(\mathbf{x}, t) = \mathbf{F}(\mathbf{X}, t)(\epsilon\mathbf{f}_R(\mathbf{X})) \tag{6.9}$$

for $\epsilon > 0$. This can be considered as describing the local deformation when the neighborhood of $\mathbf{X}$ under consideration is magnified by a factor of ϵ^{-1}.

In equation 6.8, $\mathbf{f}_R(\mathbf{X})$ is an **infinitesimal undeformed fiber** and $\mathbf{f}(\mathbf{x}, t) = \mathbf{F}(\mathbf{X}, t)\mathbf{f}_R(\mathbf{X})$ is the corresponding **(infinitesimal) deformed fiber**. We can see the deformed fiber as **embedded** in the deforming body $\mathcal{B}_t$ and we say $\mathbf{f}(\mathbf{x}, t)$ convects with the body.

f convects with the body and **f is convecting** mean that there is a fixed (time independent) material vector field $\mathbf{f}_R(\mathbf{X})$ such that equation 6.8 holds.

2.2.2 Curves

C is a **material curve** with parameterization $\hat{\mathbf{X}}(\lambda)$, $\lambda \in [\lambda_0, \lambda_1]$ which does not intersect itself. The corresponding **spatial curve** is $\mathcal{C}_t = \chi_t(C)$. Note the time-dependent parameterization. Then $\mathcal{C}_t$ is a curve **embedded** in the deforming body.

2.2.3 Tangent Vectors

Given $\mathbf{X}$ on C, the tangent to C at $\mathbf{X}$ is

$$\boldsymbol{\tau}_R(\mathbf{X}) = \frac{d\hat{\mathbf{X}}(\lambda)}{d\lambda} \tag{6.12}$$

Then the corresponding tangent to $\mathcal{C}_t$ at $\mathbf{x}$ is

$$\boldsymbol{\tau}(\mathbf{x}, t) = \frac{\partial \hat{\mathbf{x}}_t(\lambda)}{\partial \lambda} \tag{6.13}$$

which gives

$$\boldsymbol{\tau}(\mathbf{x}, t) = \mathbf{F}(\mathbf{X}, t)\boldsymbol{\tau}_R(\mathbf{X}) \tag{6.15}$$

Theorem 2.1 (Transformation Law for Tangent Vectors) *At each time, the relation (6.15) associates with any vector $\boldsymbol{\tau}_R$ at $\boldsymbol{X}$ a vector $\boldsymbol{\tau}$ at $\boldsymbol{x} = \boldsymbol{\chi}_t(\boldsymbol{X})$ with the following property: if $\boldsymbol{\tau}_R$ is tangent to a material curve at $\boldsymbol{X}$, then $\boldsymbol{\tau}$ is tangent to the corresponding deformed curve through $\boldsymbol{x}$.*

2.2.4 Bases

Fix a material basis

$$\{\mathbf{m}_i(\mathbf{X})\} = \{\mathbf{m}_1(\mathbf{X}), \mathbf{m}_2(\mathbf{X}), \mathbf{m}_3(\mathbf{X})\}.$$

Then the associated spatial basis is

$$\{\mathbf{s}_i(\mathbf{x}, t)\} = \{\mathbf{F}(\mathbf{X}, t)\mathbf{m}_i(\mathbf{X})\} \tag{6.16}$$

at $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$. This spatial basis convects with the body, ie is embedded in the deforming body.

3 Stretch, Strain, and Rotation

3.1 Stretch and Rotation Tensors. Strain

The polar decomposition (rotation $\mathbf{R}$ and positive-definite symmetric tensors $\mathbf{U}$ and $\mathbf{V}$) of the deformation gradient:

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R} \tag{7.1}$$

$\mathbf{U}$ is the **right stretch tensor** and $\mathbf{V}$ is the **right stretch tensor**.
The following is good for theoretical but difficult in application

$$\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}} \qquad \text{and} \qquad \mathbf{V} = \sqrt{\mathbf{F}\mathbf{F}^T} \tag{7.2}$$

Left and Right Cauchy-Green (deformation) tensors C and B:

$$\begin{aligned} \mathbf{C} = \mathbf{U}^2 = \mathbf{F}^T\mathbf{F}, \quad C_{ij} = F_{ki}F_{kj} = \tfrac{\partial \boldsymbol{\chi}_k}{\partial \mathbf{X}_i}\tfrac{\partial \boldsymbol{\chi}_k}{\partial \mathbf{X}_j} \\ \mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^T, \quad B_{ij} = F_{ik}F_{jk} = \tfrac{\partial \boldsymbol{\chi}_i}{\partial \mathbf{X}_k}\tfrac{\partial \boldsymbol{\chi}_j}{\partial \mathbf{X}_k} \end{aligned} \tag{7.3}$$

Then,

$$\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^T \qquad \textit{and} \qquad \mathbf{B} = \mathbf{R}\mathbf{C}\mathbf{R}^T \tag{7.4}$$

and

$$\mathbf{U}, \mathbf{V}, \mathbf{C}, \text{ and } \mathbf{B} \text{ are symmetric and positive-definite.} \tag{7.5}$$

Green - St. Venant strain tensor:

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I}) \tag{7.6,7.7,7.8}$$

As rotation tensors are orthogonal, $\mathbf{E}$ vanishes when $\mathbf{F}$ is a rotation. We now have properties

1. (M3) $\mathbf{U}$, $\mathbf{C}$, and $\mathbf{E}$ map material vectors to material vectors
2. (M4) $\mathbf{V}$ and $\mathbf{B}$ map spatial vectors to spatial vectors
3. (M5) $\mathbf{R}$ maps material vectors to spatial vectors

3.2 Fibers. Properties of the Tensors U and C

3.2.1 Infinitesimal Fibers

Infinitesimal undeformed fibers and $\mathbf{f}_R$ and $\bar{\mathbf{f}}_R$ and corresponding deformed fibers

$$\mathbf{f} = \mathbf{F}\mathbf{f}_R \qquad and \qquad \bar{\mathbf{f}} = \mathbf{F}\bar{\mathbf{f}}_R \tag{7.9}$$

Then

$$\mathbf{f} \cdot \bar{\mathbf{f}} = (\mathbf{R}\mathbf{U}\mathbf{f}_R) \cdot (\mathbf{R}\mathbf{U}\bar{\mathbf{f}}_R) = \mathbf{U}\mathbf{f}_R \cdot \mathbf{U}\bar{\mathbf{f}}_R = \mathbf{f}_R \cdot \mathbf{U}^2\bar{\mathbf{f}}_R = \mathbf{f}_R \cdot \mathbf{C}\bar{\mathbf{f}}_R \tag{7.11}$$

So,

$$|\mathbf{f}| = |\mathbf{U}\mathbf{f}_R| \tag{7.12}$$

So applying the right stretch tensor gives the deformed length of an infinitesimal fiber.

Define: $\theta = \angle(\mathbf{f}_R, \bar{\mathbf{f}_R})$ the angle between fibers.
Then, by 7.1 and 7.12,

$$\frac{\mathbf{f} \cdot \bar{\mathbf{f}}}{|\mathbf{f}||\bar{\mathbf{f}}} = \frac{\mathbf{C}\mathbf{f}_R \cdot \mathbf{U}\bar{\mathbf{f}}_R}{|\mathbf{U}\mathbf{f}_R||\mathbf{U}\bar{\mathbf{f}}_R|}$$

So,

$$\angle(\mathbf{f}, \bar{\mathbf{f}}) = \angle(\mathbf{U}\mathbf{f}_R, \mathbf{U}\bar{\mathbf{f}}_R) \tag{7.13}$$

so applying the right stretch tensor gives the angle between infinitesimal deformed fibers.

3.2.2 Finite Fibers

Consider material and spatial line segments

$$\Delta\mathbf{X} = \mathbf{Y} - \mathbf{X} \qquad and \qquad \Delta\mathbf{x} = \chi(\mathbf{Y}) - \chi(\mathbf{X})$$

with $\Delta\mathbf{X} > 0$.
Then we know

$$\Delta\mathbf{x} = \mathbf{F}(\mathbf{X})\Delta\mathbf{X} + o(|\Delta\mathbf{X}|) \qquad as \qquad |\Delta\mathbf{X}| \to 0. \tag{7.14}$$

Useful to think of $\Delta\mathbf{X}$ as of an undeformed fiber of finite length L and direction $\mathbf{e}$ at $\mathbf{X}$, so

$$\Delta\mathbf{X} = L\mathbf{e}, \qquad |\mathbf{e}| = 1 \tag{7.15}$$

So the corresponding deformed fiber is

$$\Delta\mathbf{x} = L\mathbf{F}(\mathbf{X})\mathbf{e} + o(L) \qquad as \qquad L \to 0$$

Then the following limit gives the **stretch vector**

$$\lim_{L\to 0} \frac{\Delta\mathbf{x}}{L} = \mathbf{F}(\mathbf{X})\mathbf{e} \tag{7.16}$$

This is called the stretch vector because it is the limiting value of the deformed fiber measured per **unit length** of the undeformed fiber in direction $\mathbf{e}$.
Thus the **stretch** is

$$\lambda = \lim_{L\to 0} \frac{|\Delta\mathbf{x}|}{L} = |\mathbf{F}(\mathbf{X})\mathbf{e}| \tag{7.17}$$

From earlier, taking $\mathbf{f}_R = \mathbf{e}$ gives

$$\lambda = |\mathbf{U}(\mathbf{X})\mathbf{e}| \qquad \lambda^2 = \mathbf{e} \cdot \mathbf{C}(\mathbf{X})\mathbf{e} \tag{7.18}$$

Remark 3.1 *The right stretch tensor $\boldsymbol{U}$ determines the stretch λ at $\boldsymbol{X}$ relative to any material direction $\boldsymbol{e}$ by $\lambda = |\boldsymbol{U}(\boldsymbol{X})\boldsymbol{e}|$.*

Now, taking two fibers from $\mathbf{X}$ of the same length L with directions $\mathbf{e}_1$ and $\mathbf{e}_2$. Then

$$\lim_{L\to 0}\left(\frac{(\Delta\mathbf{x})_1}{L}\cdot\frac{(\Delta\mathbf{x})_2}{L}\right) = \lim_{L\to 0}\left(\frac{(\Delta\mathbf{x})_1)}{L}\right)\cdot\lim_{L\to 0}\left(\frac{(\Delta\mathbf{x})_2}{L}\right) = \mathbf{U}(\mathbf{X})\mathbf{e}_1\cdot\mathbf{U}(\mathbf{X})\mathbf{e}_2 = \mathbf{e}_1\cdot\mathbf{C}(\mathbf{X})\mathbf{e}_2 \tag{7.21-7.22}$$

Remark 3.2 *The right Cauchy-Green tensor $\boldsymbol{C}(\boldsymbol{X})$ characterizes inner products of stretch vectors at $\boldsymbol{x}$*

Let θ_L be the angle between the deformed fibers $(\Delta\mathbf{x})_1$ and $(\Delta\mathbf{x})_2$, as before. Then,

$$\theta_L = \angle((\Delta\mathbf{x})_1, (\Delta\mathbf{x})_2) = \cos^{-1}\left(\frac{(\Delta\mathbf{x})_1\cdot(\Delta\mathbf{x})_2}{|(\Delta\mathbf{x})_1||(\Delta\mathbf{x})_2|}\right)$$

After math cheese,

$$\lim_{L\to 0}\theta_L = \angle(\mathbf{U}(\mathbf{X})\mathbf{e}_1, \mathbf{U}(\mathbf{X})\mathbf{e}_2). \tag{7.23}$$

Remark 3.3 *Let $(\Delta\boldsymbol{x})_1$ and $(\Delta\boldsymbol{x})_2$ be the deformed fibers corresponding to fibers at $\boldsymbol{X}$ of finite length L in directions $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. Then, as $L\to 0$, the angle between*

$$\frac{(\Delta\boldsymbol{x}_1)}{L} \text{ and } \frac{(\Delta\boldsymbol{x})_2}{L}$$

tends to the angle between $\boldsymbol{U}(\boldsymbol{X})\boldsymbol{e}_1$ and $\boldsymbol{U}(\boldsymbol{X})\boldsymbol{e}_2$.

3.3 Principle Stretches and Principal Directions

As $\mathbf{U}$ and $\mathbf{V}$ are symmetric and positive-definite, the have spectral representations of the form

$$\mathbf{U} = \lambda_i\mathbf{r}_i\otimes\mathbf{r}_i = \sum_{i=1}^{3}\lambda_i\mathbf{r}_i\otimes\mathbf{r}_i \qquad \mathbf{V} = \lambda_i\mathbf{l}_i\otimes\mathbf{l}_i$$

where

1. $\lambda_i > 0\,\forall i$, the **principal stretches** are eigenvalues of $\mathbf{U}$ and also of $\mathbf{V}$.
2. $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ are the **right principal directions** and eigenvectors of $\mathbf{U}$

$$\mathbf{U}\mathbf{r}_i = \lambda_i\mathbf{r}_i \qquad \text{(no sum on i)}$$

3. $\mathbf{l}_1$, $\mathbf{l}_2$, and $\mathbf{l}_3$ are the **left principal directions** and the eigenvectors of $\mathbf{V}$

$$\mathbf{V}\mathbf{l}_i = \lambda_i\mathbf{l}_i \qquad \text{(no sum on i)}$$

Thus, $\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^T$ gives

$$\sum_{i=1}^{3}\lambda_i\mathbf{R}\mathbf{r}_i\otimes\mathbf{R}\mathbf{r}_i = \sum_{i=1}^{3}\lambda\mathbf{l}_i\otimes\mathbf{l}_i$$

Therefore,

$$\mathbf{l}_i = \mathbf{R}\mathbf{r}_i, \qquad i = 1,2,3 \tag{7.27}$$

We then have the following expressions using the principal stretches and directions:

$$\left.\begin{array}{l} \mathbf{C} = \lambda_i^2\mathbf{r}_i\otimes\mathbf{r}_i \\ \mathbf{B} = \lambda_i^2\mathbf{l}_i\otimes\mathbf{l}_i \\ \mathbf{E} = \frac{1}{2}(\lambda_i^2 - \mathbf{I})\mathbf{r}_i\otimes\mathbf{r}_i \end{array}\right\} \tag{7.28}$$

where each is summed over i.

Also,

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \lambda_i\mathbf{l}_i\otimes\mathbf{r}_i$$

Also the logarithmic strain tensors of Hencky are scary.