1 Motion of a Body

B is a reference body containing points $\mathbf{X}$ which are material points. There is a one-to-one function

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$$

taking reference material points $\mathbf{X}$ to spatial points $\mathbf{x}$ at time t. We require

$$J(\mathbf{X}, t) := \det \nabla \boldsymbol{\chi}_t(\mathbf{X}) > 0$$

where J is the volumetric Jacobian of the mapping $\boldsymbol{\chi}_t$ at $\mathbf{X}$. Region occupied by body B at time t is

$$\mathcal{B}_t = \boldsymbol{\chi}_t(B)$$

is the deformed body at time t.

Convection of Sets with the Body A is a material set. Then **deforms to** $\mathcal{A}_t$ at time t. $\mathcal{A}_t$ **convects with the body** if there is a set A of material points such that

$$\mathcal{A}_t = \boldsymbol{\chi}_T(A)$$

for all t. Note that material cannot cross the boundary of a spatial set which convects with the body. Also note that if $\mathbf{X}$ is on ∂B (boundary), then $\boldsymbol{\chi}(\mathbf{X}, t)$ is on $\partial \mathcal{B}_t$ for all time t and conversely.

2 The Deformation Gradient

The **Deformation gradient** of a body is

$$\mathbf{F} = \nabla \boldsymbol{\chi}, \qquad F_{ij} = \frac{\partial \chi_i}{\partial X_j},$$

the Jacobian matrix of $\mathbf{x} = \mathbf{x}(\mathbf{X})$. As above

$$J = \det \mathbf{F} > 0.$$

Homegeneous Deformations Fix time t so that

$$\boldsymbol{\chi}(\mathbf{X}) \equiv \boldsymbol{\chi}_t(\mathbf{X}).$$

$\boldsymbol{\chi}$ is a **homogeneous deformation** if $\mathbf{F}(\mathbf{X}) \equiv \mathbf{F}(\mathbf{X}, t)$ is independent of $\mathbf{X}$. So

$$\boldsymbol{\chi}(\mathbf{X}) - \boldsymbol{\chi}(\mathbf{Y}) = \mathbf{F}(\mathbf{X} - \mathbf{Y})$$

for **all** material points $\mathbf{X}$ and $\mathbf{Y}$. By the above, $\underline{\mathbf{F}\text{ maps material vectors to spatial vectors.}}$ Then, also, $\mathbf{X} - \mathbf{Y} = \mathbf{F}^{-1}[\boldsymbol{\chi}(\mathbf{X}) - \boldsymbol{\chi}(\mathbf{Y})]$ so that $\mathbf{F}^{-1}$ maps spatial vectors too material vectors. Taking the inner product with a spatial vector $\mathbf{s}$ gives

$$\mathbf{s} \cdot [\boldsymbol{\chi}(\mathbf{X}) - \boldsymbol{\chi}(\mathbf{Y})] = \mathbf{s} \cdot [\mathbf{F}(\mathbf{X} - \mathbf{Y})] = (\mathbf{F}^T \mathbf{s}) \cdot (\mathbf{X} - \mathbf{Y})$$

so that $\mathbf{F}^T$ maps spatial vectors to material vectors.
Summarizing the mapping properties:

1. $\mathbf{F}$ and $\mathbf{F}^{-T}$ map material vectors to spatial vectors

2. $\mathbf{F}^{-1}$ and $\mathbf{F}^{T}$ map spatial vectors to material vectors

General Deformations Let $\boldsymbol{\chi}_t$ be an arbitrary deformation. Taylor expanding the deformation about material point $\mathbf{X}$ gives

$$\underline{\boldsymbol{\chi}_t(\mathbf{Y}) - \boldsymbol{\chi}(\mathbf{X}) = \mathbf{F}(\mathbf{X}, t)(\mathbf{Y} - \mathbf{X})} + o(|\mathbf{Y} - \mathbf{X}) \qquad as|\mathbf{Y} - \mathbf{X}| \to 0$$

. Therefore $\mathbf{F}(\mathbf{X}, t)(\mathbf{Y} - \mathbf{X})$ is an approximation of $\boldsymbol{\chi}_t(\mathbf{Y}) - \boldsymbol{\chi}(\mathbf{X})$. Also, since $\mathbf{X}$ is fixed in the Taylor expansion, $\mathbf{F}(\mathbf{X}, t)$ is constant. Thus the underlined portion is the second definition of a homogeneous deformation. Therefore, *within a neighborhood of a material point $\boldsymbol{X}$ and to within an error of $o(|\,\boldsymbol{Y} - \boldsymbol{X}|)$, a deformation behaves like a homogeneous deformation.*
So with $o(|\mathbf{Y} - \mathbf{X}|)$ small, we have:

1. $F(\mathbf{X}, t)$ can be thought of as a mapping of an infinitesimal neighborhood of $\mathbf{X}$ in the reference body to an infinitesimal neighborhood of $\mathbf{x} - \boldsymbol{\chi}_t(\mathbf{X})$ in the deformed body.

2. This gives an asymptotic meaning to the formal relation

$$\mathbf{dx} = \mathbf{F}(\mathbf{X}, t)\mathbf{dX}$$

Now, we have that the mapping properties for a homogeneous deformation hold pointwise for the deformation gradient in an arbitrary deformation. For example, for a given $\mathbf{X}$, the linear transformation $\mathbf{F}(\mathbf{X}, t)$ associates with each material vector $\mathbf{m}$ a spatial vector $\mathbf{s} = \mathbf{F}(\mathbf{X}, t)\mathbf{m}$.

Convection of Geometric Quantities Define the temporally constant material vector field $\mathbf{f}_R$ associated with a given spatial vector file $\mathbf{f}$ by

$$\mathbf{f}(\mathbf{x}, t) = \mathbf{F}(\mathbf{X}, t)\mathbf{f}_R(\mathbf{X}) \qquad \mathbf{x} = \boldsymbol{\chi}_t(\mathbf{X}) \tag{6.8}$$

for all $\mathbf{X}$ and t.

Now by the above statements about the local homogeneity of deformation, we can see equation 6.8 becomes

$$\epsilon\mathbf{f}(\mathbf{x}, t) = \mathbf{F}(\mathbf{X}, t)(\epsilon\mathbf{f}_R(\mathbf{X})) \tag{6.9}$$

for $\epsilon > 0$. This can be considered as describing the local deformation when the neighborhood of $\mathbf{X}$ under consideration is magnified by a factor of ϵ^{-1}.