Supplementary material for "A Game-Theoretical Control Framework for Transactive Energy Trading in Energy Communities"

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March 2024

Remark 1. *We highlight that matrix $\mathbf{B}$ is both symmetric and orthogonal, i.e., $\mathbf{B} = \mathbf{B}^\top$ and $\mathbf{B}^\top \mathbf{B} = \mathbf{I}_n$ (thus, $\mathbf{B}$ defines an involution: $\mathbf{B}\mathbf{B} = \mathbf{I}_n$). To show the first claim, note that $\mathbf{B}_{ij} = \mathbf{B}_{ji}^\top$, for all $i, j \in \mathcal{A}$. For the second claim, observe that, by construction and symmetry, all rows and columns of $\mathbf{B}$ are linearly independent, and each row and column of $\mathbf{B}$ has exactly one non-zero element, which is equal to 1. Therefore, all the rows and columns of $\mathbf{B}$ are orthonormal vectors.*

Standing Assumption 1. *For all $i \in \mathcal{A}$, the functions $f_i(\cdot,\cdot,\cdot,\cdot)$ and $\mathbf{g}_i(\cdot)$ are continuously differentiable, $f_i(\cdot,\cdot,\mathbf{p}_i,\mathbf{p}_{-i})$ is (jointly) convex for every fixed $(\mathbf{p}_i, \mathbf{p}_{-i})$, and $\nabla_{\hat{\mathbf{x}}_i} f_i(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i, \cdot, \cdot)$ and $\nabla_{\check{\mathbf{x}}_i} f_i(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i, \cdot, \cdot)$ are $\hat{L}_i$-Lipschitz continuous and $\check{L}_i$-Lipschitz continuous for every fixed $(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i)$, respectively. Moreover, the pseudo-gradient*

$$\mathbf{q}(\hat{\mathbf{x}}, \check{\mathbf{x}}, \mathbf{p}) = \begin{bmatrix} \operatorname{col}\left(\nabla_{\hat{\mathbf{x}}_i} f_i(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i, \mathbf{p}_i, \mathbf{p}_{-i})\right)_{i\in\mathcal{A}} \\ \operatorname{col}\left(\nabla_{\check{\mathbf{x}}_i} f_i(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i, \mathbf{p}_i, \mathbf{p}_{-i})\right)_{i\in\mathcal{A}} \\ \operatorname{col}\left(\rho_i\left(\mathbf{p}_i - \mathbf{g}_i(\hat{\mathbf{x}}_{-i})\right)\right)_{i\in\mathcal{A}} \end{bmatrix} \in \mathbb{R}^{3n}$$

is μ-strongly monotone. Finally, Ω is a closed convex set with a non-empty relative interior.

Proof of Proposition 1

Under the considered setup

$$\mathbf{q}(\hat{\mathbf{x}}, \check{\mathbf{x}}, \mathbf{p}) = \begin{bmatrix} \tilde{\nabla}_{\hat{\mathbf{x}}} \boldsymbol{\psi} + \mathbf{B}\mathbf{p} \\ \tilde{\nabla}_{\check{\mathbf{x}}} \boldsymbol{\psi} - \mathbf{p} \\ \rho\mathbf{p} - \rho\mathbf{Q}\mathbf{B}\hat{\mathbf{x}} \end{bmatrix},$$

where we have used $\mathbf{p}_{-i} = \mathbf{B}_i\mathbf{p}$, $\hat{\mathbf{x}}_{-i} = \mathbf{B}_i\hat{\mathbf{x}}$, and we have defined $\mathbf{Q} = \operatorname{diag}(\mathbf{Q}_i)_{i\in\mathcal{A}} \in \mathbb{R}^{n\times n}_{\geq 0}$ and

$$\tilde{\nabla}_{\hat{\mathbf{x}}} \boldsymbol{\psi} := \operatorname{col}\left(\nabla_{\hat{\mathbf{x}}_i} \psi_i(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i)\right)_{i\in\mathcal{A}}$$
$$\tilde{\nabla}_{\check{\mathbf{x}}} \boldsymbol{\psi} := \operatorname{col}\left(\nabla_{\check{\mathbf{x}}_i} \psi_i(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i)\right)_{i\in\mathcal{A}}.$$

Since every $\psi_i(\cdot,\cdot)$ is twice continuously differentiable, it follows that $\mathbf{q}(\cdot,\cdot,\cdot)$ is μ-strongly monotone if it holds that

$$\mathrm{D}\,\mathbf{q}(\hat{\mathbf{x}}, \check{\mathbf{x}}, \mathbf{p}) + \mathrm{D}\,\mathbf{q}(\hat{\mathbf{x}}, \check{\mathbf{x}}, \mathbf{p})^\top - 2\mu\mathbf{I}_{3n} \succeq 0, \tag{1}$$

for all $\hat{\mathbf{x}}, \check{\mathbf{x}}, \mathbf{p} \in \mathbb{R}^n_{\geq 0}$. Here,

$$\mathrm{D}\,\mathbf{q}\left(\hat{\mathbf{x}}, \check{\mathbf{x}}, \mathbf{p}\right) = \begin{bmatrix} \mathrm{D}_{\hat{\mathbf{x}}}\, \tilde{\nabla}_{\hat{\mathbf{x}}} \boldsymbol{\psi} & \mathrm{D}_{\check{\mathbf{x}}}\, \tilde{\nabla}_{\hat{\mathbf{x}}} \boldsymbol{\psi} & \mathbf{B} \\ \mathrm{D}_{\hat{\mathbf{x}}}\, \tilde{\nabla}_{\check{\mathbf{x}}} \boldsymbol{\psi} & \mathrm{D}_{\check{\mathbf{x}}}\, \tilde{\nabla}_{\check{\mathbf{x}}} \boldsymbol{\psi} & -\mathbf{I}_n \\ -\rho \mathbf{Q}\mathbf{B} & \mathbf{0}_{n\times n} & \rho \mathbf{I}_n \end{bmatrix} \in \mathbb{R}^{3n\times 3n}.$$

Denote the top left $2n \times 2n$ block of $\mathrm{D}\,\mathbf{q}\left(\hat{\mathbf{x}}, \check{\mathbf{x}}, \mathbf{p}\right)$ as $\mathbf{J}\left(\hat{\mathbf{x}}, \check{\mathbf{x}}\right)$. It follows that $\mathbf{J}\left(\hat{\mathbf{x}}, \check{\mathbf{x}}\right)$ is the Hessian matrix of the function $\varphi(\hat{\mathbf{x}}, \check{\mathbf{x}}) = \sum_{i\in\mathcal{A}} \psi_i\left(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i\right)$. Therefore, $\mathbf{J}\left(\hat{\mathbf{x}}, \check{\mathbf{x}}\right) = \mathbf{J}\left(\hat{\mathbf{x}}, \check{\mathbf{x}}\right)^\top$ and $\boldsymbol{\zeta}^\top \mathbf{J}\left(\hat{\mathbf{x}}, \check{\mathbf{x}}\right) \boldsymbol{\zeta} \geq \underline{\theta} \boldsymbol{\zeta}^\top \boldsymbol{\zeta}$, for all $\hat{\mathbf{x}}, \check{\mathbf{x}} \in \mathbb{R}^n_{\geq 0}$, and all $\boldsymbol{\zeta} \in \mathbb{R}^{2n}$. Moreover, let

$$\mathbf{M}_1\left(\hat{\mathbf{x}}, \check{\mathbf{x}}\right) = \begin{bmatrix} \mathbf{J}\left(\hat{\mathbf{x}}, \check{\mathbf{x}}\right) - \mu \mathbf{I}_{2n} & \begin{bmatrix} \mathbf{B} \\ -\mathbf{I}_n \end{bmatrix} \\ \begin{bmatrix} \mathbf{B}^\top & -\mathbf{I}_n \end{bmatrix} & (\rho - \mu)\mathbf{I}_n \end{bmatrix}$$

$$\mathbf{M}_2\left(\hat{\mathbf{x}}, \check{\mathbf{x}}\right) = \begin{bmatrix} \mathbf{J}\left(\hat{\mathbf{x}}, \check{\mathbf{x}}\right) - \mu \mathbf{I}_{2n} & \begin{bmatrix} -\rho \mathbf{B}^\top \mathbf{Q}^\top \\ \mathbf{0}_{n\times n} \end{bmatrix} \\ \begin{bmatrix} -\rho \mathbf{Q}\mathbf{B} & \mathbf{0}_{n\times n} \end{bmatrix} & (\rho - \mu)\mathbf{I}_n \end{bmatrix}.$$

As such, the condition in (1) can be equivalently rewritten as $\mathbf{M}_1\left(\hat{\mathbf{x}}, \check{\mathbf{x}}\right) + \mathbf{M}_2\left(\hat{\mathbf{x}}, \check{\mathbf{x}}\right) \succeq 0$. Hence, if $\mathbf{M}_1\left(\hat{\mathbf{x}}, \check{\mathbf{x}}\right)$ and $\mathbf{M}_2\left(\hat{\mathbf{x}}, \check{\mathbf{x}}\right)$ are both shown to be positive semi-definite matrices, for all $\hat{\mathbf{x}}, \check{\mathbf{x}} \in \mathbb{R}^n_{\geq 0}$, then we can conclude that $\mathbf{q}(\cdot, \cdot, \cdot)$ is μ-strongly monotone.

Consider first $\mathbf{M}_1\left(\hat{\mathbf{x}}, \check{\mathbf{x}}\right)$. By the Schur complement, we have that $\mathbf{M}_1\left(\hat{\mathbf{x}}, \check{\mathbf{x}}\right) \succeq 0$ if and only if $\rho > \mu$ and

$$\mathbf{J}\left(\hat{\mathbf{x}}, \check{\mathbf{x}}\right) - \mu \mathbf{I}_{2n} \succeq \frac{1}{\rho - \mu} \begin{bmatrix} \mathbf{B}\mathbf{B}^\top & -\mathbf{B} \\ -\mathbf{B}^\top & \mathbf{I}_n \end{bmatrix}.$$

Note that $\mathbf{J}\left(\hat{\mathbf{x}}, \check{\mathbf{x}}\right) - \mu \mathbf{I}_{2n} \succeq \left(\underline{\theta} - \mu\right) \mathbf{I}_{2n}$, and

$$\frac{1}{\rho - \mu} \begin{bmatrix} \mathbf{B}\mathbf{B}^\top & -\mathbf{B} \\ -\mathbf{B}^\top & \mathbf{I}_n \end{bmatrix} \preceq \frac{2}{\rho - \mu} \mathbf{I}_{2n}.$$

Here, the second claim follows from Remark 1 and the Gershgorin Circle Theorem [1, Fact 4.10.16]. Therefore, if $\rho > \mu$ and $\underline{\theta} - \mu \geq 2/(\rho - \mu)$, then $\mathbf{M}_1\left(\hat{\mathbf{x}}, \check{\mathbf{x}}\right)$ is positive semi-definite, for all $\hat{\mathbf{x}}, \check{\mathbf{x}} \in \mathbb{R}^n_{\geq 0}$.

Similarly, consider $\mathbf{M}_2\left(\hat{\mathbf{x}}, \check{\mathbf{x}}\right)$. By the Schur complement, we have that $\mathbf{M}_2\left(\hat{\mathbf{x}}, \check{\mathbf{x}}\right) \succeq 0$ if and only if $\rho > \mu$ and

$$\mathbf{J}\left(\hat{\mathbf{x}}, \check{\mathbf{x}}\right) - \mu \mathbf{I}_{2n} \succeq \frac{\rho^2}{\rho - \mu} \begin{bmatrix} \mathbf{B}^\top \mathbf{Q}^\top \mathbf{Q}\mathbf{B} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \end{bmatrix}.$$

By Remark 1, it follows that $\mathbf{B}^\top \mathbf{Q}^\top \mathbf{Q}\mathbf{B}$ is similar to $\mathbf{Q}^\top \mathbf{Q}$ and so they have the same eigenvalues. Consequently,

$$\frac{\rho^2}{\rho - \mu} \begin{bmatrix} \mathbf{B}^\top \mathbf{Q}^\top \mathbf{Q}\mathbf{B} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \end{bmatrix} \preceq \frac{\rho^2}{\rho - \mu} \overline{\lambda} \mathbf{I}_{2n}.$$

Thus, if $\rho > \mu$ and $\underline{\theta} - \mu \geq \rho^2 \overline{\lambda}/(\rho - \mu)$, then $\mathbf{M}_2\left(\hat{\mathbf{x}}, \check{\mathbf{x}}\right)$ is positive semi-definite, for all $\hat{\mathbf{x}}, \check{\mathbf{x}} \in \mathbb{R}^n_{\geq 0}$.

Hence, if $\rho > \mu$ and $\underline{\theta} - \mu \geq \max\left\{2,\, \rho^2 \overline{\lambda}\right\} / (\rho - \mu)$, then the conditions for the positive semi-definiteness of $\mathbf{M}_1(\cdot, \cdot)$ and $\mathbf{M}_2(\cdot, \cdot)$ are satisfied, and the proof is completed.

Proof of Corollary 1

Note that under Standing Assumption 1, the augmented pseudo-gradient $\tilde{\mathbf{q}}(\hat{\mathbf{x}}, \check{\mathbf{x}}, \mathbf{y}, \mathbf{p}) = \left[\mathbf{q}(\hat{\mathbf{x}}, \check{\mathbf{x}}, \mathbf{p})^\top, \mathbf{0}_n^\top\right]^\top$ is monotone in all the variables, and μ-strongly monotone in $(\hat{\mathbf{x}}, \check{\mathbf{x}}, \mathbf{p})$. Thus, the result follows from [2, Theorem 4.6].

References

[1] Dennis S. Bernstein. *Matrix Mathematics: Theory, Facts, and Formulas (Second Edition)*. Princeton University Press, 2009. ISBN: 9781400833344. DOI: doi:10.1515/9781400833344.

[2] Eike Börgens and Christian Kanzow. "ADMM-Type Methods for Generalized Nash Equilibrium Problems in Hilbert Spaces". In: *SIAM Journal on Optimization* 31.1 (2021), pp. 377–403. DOI: 10.1137/19M1284336.