Supplementary Material for "Optimal Solar Tracking for Sustainable Crop Cultivation and Energy Generation in Agrivoltaics"

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1 Proof of Proposition 1

The objective in (16) is a piece-wise affine function of $\sin\beta_i$ and $\cos\beta_i$, and, thus, linear in $\mathbf{u}_i$. Apart from the integrality requirements, all the constraints mentioned in ($\mathrm{P}_{\mathbf{u}}$) are linear: (11) is a proper inequality, (14)-(15) can be converted to inequalities, while the remaining ones are linear equalities. However, the condition in the aforementioned logical expression, i.e., the inequality in (6), albeit linear in $\mathbf{u}_i$, is non-trivial. By expanding the product, $g_{ij}(\mathbf{p}_k)$ can be written as a follows

$$g_{ij}(\mathbf{p}_k) = \mathbf{a}_{ijk}^{\top}\mathbf{u}_i + b_{ijk} \tag{1}$$

where, for all $i \in \mathcal{M}$, $j = 1, \dots, 4$, and $k \in \mathcal{P}$, $\mathbf{a}_{ijk}$ is defined as follows

$$\begin{aligned}\mathbf{a}_{ijk} = {} & \frac{d_i w_i}{2}\begin{bmatrix}\frac{\cos(\alpha-\bar{\alpha}_i)}{\tan\eta}\\ 1\end{bmatrix}\\ & + \sigma(j-1) d_i \begin{bmatrix}\frac{\cos\alpha}{\tan\eta}(c_{i,1}-p_{k,1}) + \frac{\sin\alpha}{\tan\eta}(p_{k,2}-c_{i,2})\\ \cos\bar{\alpha}_i(c_{i,1}-p_{k,1}) + \sin\bar{\alpha}_i(p_{k,2}-c_{i,2}) - \frac{\sin(\alpha-\bar{\alpha}_i)}{\tan\eta}c_{i,3}\end{bmatrix}\end{aligned} \tag{2}$$

and b_{ijk} is defined as follows

$$b_{ijk} = w_i\sigma_j\left(\begin{bmatrix}\sin\bar{\alpha}_i\\ \cos\bar{\alpha}_i\end{bmatrix}^{\top}\begin{bmatrix}p_{k,1}-c_{i,1}\\ p_{k,2}-c_{i,2}\end{bmatrix} + \frac{c_{i,3}\cos(\alpha-\bar{\alpha}_i)}{\tan\eta}\right) \tag{3}$$

where σ_j, mapping $j \mapsto \{-1, 0, 1\}$, is the discrete sine wave, defined as $\sigma_j := \sin(\pi j/2)$.

2 Proof of Proposition 2

We start by expanding the left-hand-side of the inequality in Proposition 2 as follows

$$
\begin{aligned}
|J(\tilde{\mathbf{u}}^*) - J(\mathbf{u}^*)|^2 &= \left| \gamma \sum_{i \in \mathcal{M}} (p_i^* - \tilde{p}_i^*) \right|^2 \\
&= |\nabla_{\mathbf{u}} J(\mathbf{u})^\top (\tilde{\mathbf{u}}^* - \mathbf{u}^*)|^2 \\
&\leq \|\nabla_{\mathbf{u}} J(\mathbf{u})\|^2 \|(\tilde{\mathbf{u}}^* - \mathbf{u}^*)\|^2
\end{aligned} \tag{4}
$$

where the objective gradient $\nabla_{\mathbf{u}} J(\mathbf{u}) = \mathrm{col}(\nabla_{\mathbf{u}_i} J(\mathbf{u}))_{i \in \mathcal{M}}$, with $\nabla_{\mathbf{u}_i} J(\mathbf{u})$ being calculated as follows

$$
\nabla_{\mathbf{u}_i} J(\mathbf{u}) = \gamma d_i w_i \begin{bmatrix} r^n \cos\eta \cos(\alpha - \bar{\alpha}_i) \\ r^n \sin\eta + \dfrac{r^d - \rho r^g}{2} \end{bmatrix}, \ \forall i \in \mathcal{M} \tag{5}
$$

and the last line in (4) holds form the Cauchy–Schwarz inequality. Hence, we are now interested in bounding the distance $\|(\mathbf{u}^* - \tilde{\mathbf{u}}^*)\|^2$. For any $i \in \mathcal{M}$, let us consider the most restrictive case., i.e. when constraint (11) corresponds to $\mathbf{e}_2^\top \tilde{\mathbf{u}}(\beta_i) \geq 0$. From the fundamental theorem of linear programming [1, Theorem 3.5], we know that $\tilde{\mathbf{u}}^*$ lies on one of the vertices of the polygonal chain, i.e. $\boldsymbol{\ell}_\nu$, for some $\nu = 1, \ldots, L$. Thus, the furthest away $\mathbf{u}^*$ might lie is at the middle of the arc between $\boldsymbol{\ell}_\nu$ and $\boldsymbol{\ell}_{\nu+1}$ (or equivalently, $\boldsymbol{\ell}_{\nu-1}$). For brevity, let $\delta_\nu := \frac{\pi(2\nu - L - 1)}{2(L-1)}$ be the angle that uniquely defines the position of $\boldsymbol{\ell}_\nu$ on the unitary circle, so that the point in the middle of such arc has coordinates $\left[\sin\left(\frac{\delta_{\nu+1}+\delta_\nu}{2}\right), \cos\left(\frac{\delta_{\nu+1}+\delta_\nu}{2}\right)\right]^\top$, for any $\nu = 1, \ldots, L-1$. Therefore, we have

$$
\begin{aligned}
&\|\tilde{\mathbf{u}}^* - \mathbf{u}^*\|^2 \leq \left\| \begin{bmatrix} \sin\delta_\nu \\ \cos\delta_\nu \end{bmatrix} - \begin{bmatrix} \sin\left(\frac{\delta_{\nu+1}+\delta_\nu}{2}\right) \\ \cos\left(\frac{\delta_{\nu+1}+\delta_\nu}{2}\right) \end{bmatrix} \right\|^2 \\
&= 2\left(1 - \cos\delta_\nu \cos\left(\frac{\delta_{\nu+1}+\delta_\nu}{2}\right) - \sin\delta_\nu \sin\left(\frac{\delta_{\nu+1}+\delta_\nu}{2}\right)\right) \\
&= 2\left(1 - \cos\left(\frac{\delta_{\nu+1}-\delta_\nu}{2}\right)\right) \\
&= 2\left(1 - \cos\left(\frac{\pi}{2(L-1)}\right)\right)
\end{aligned} \tag{6}
$$

where in the last equality we use the fact that, for any angle $\alpha, \beta \in [0, 2\pi]$, $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$. By substituting the last equality in (6) in (4), we obtain the condition in Proposition 2.

3 Proof of Corollary 1

In order for the statement to hold, it is sufficient to show, from (4), that

$$
\begin{aligned}
|J(\hat{\mathbf{u}} - J(\mathbf{u}))|^2 &\leq \gamma \|\nabla J(\mathbf{u})\|^2 \|(\hat{\mathbf{u}}^* - \mathbf{u}^*)\|^2 \\
&\leq \gamma \|\nabla J(\mathbf{u})\|^2 \|(\tilde{\mathbf{u}}^* - \mathbf{u}^*)\|^2
\end{aligned} \tag{7}
$$

which holds iff $\|\hat{\mathbf{u}}^* - \mathbf{u}^*\|^2 \le \|\tilde{\mathbf{u}}^* - \mathbf{u}^*\|^2$. Therefore, by expanding such inequality, we have

$$\begin{aligned}
\|\hat{\mathbf{u}}^* - \mathbf{u}^*\|^2 &\le \|\tilde{\mathbf{u}}^* - \mathbf{u}^*\|^2 \\
\|\hat{\mathbf{u}}^*\|^2 - \|\tilde{\mathbf{u}}^*\|^2 &\le 2(\hat{\mathbf{u}}^* - \tilde{\mathbf{u}}^*)^\top \mathbf{u}^* \\
1 - \|\tilde{\mathbf{u}}^*\|^2 &\le 2\|\hat{\mathbf{u}}^* - \tilde{\mathbf{u}}^*\|\|\mathbf{u}^*\| \cos\theta \\
1 - \|\tilde{\mathbf{u}}^*\|^2 &\le 2(1 - \|\tilde{\mathbf{u}}^*\|) \cos\theta \\
&\le 2(1 - \|\tilde{\mathbf{u}}^*\|^2) \cos\theta \\
\cos\theta &\ge \frac{1}{2} \implies \theta \le \frac{5\pi}{3}
\end{aligned} \tag{8}$$

where we used i) $\|\mathbf{u}^*\| = \|\hat{\mathbf{u}}^*\| = 1$, since $\mathbf{u}^*$ and $\hat{\mathbf{u}}^*$ lie on the unitary circle, ii) $\|\hat{\mathbf{u}}^* - \tilde{\mathbf{u}}^*\| = 1 - \|\tilde{\mathbf{u}}^*\|$, since $\hat{\mathbf{u}}^*$ and $\tilde{\mathbf{u}}^*$ lie on the same ray, and that iii) $\|\mathbf{x}\|^2 \le \|\mathbf{x}\|$ when $\|\mathbf{x}\| \le 1$. In the last inequality, θ represents the angle between vectors $\hat{\mathbf{u}}^* - \tilde{\mathbf{u}}^*$ and $\mathbf{u}^*$. Therefore, $\theta \le \frac{\pi}{2(L-1)}$ always holds, as discussed in Appendix 2. Hence, we have

$$\theta \le \frac{\pi}{2(L-1)} \le \frac{5\pi}{3} \implies L \ge 1 + \frac{3}{10} \ge 3 \tag{9}$$

References

[1] James P Ignizio and Tom M Cavalier. *Linear programming.* Prentice-Hall, Inc., 1994.