ECOLE POLYTECHNIQUE FÉDÉRAL DE LAUSANNE

SEMESTER PROJECT

A solution approach for the Multicommodity Flow Problem within rail freight transportation

Author:
Nicolas PRADIGNAC

Supervisor:
Mr. Stefano BORTOLOMIOL
Mr. Nikola OBRENOVIC

A semester project in fulfillment of the requirements

in the

TRANSP-OR Laboratory

May 5, 2018

ECOLE POLYTECHNIQUE FÉDÉRAL DE LAUSANNE

Abstract

TRANSP-OR Laboratory

A solution approach for the Multicommodity Flow Problem within rail freight transportation

by Nicolas PRADIGNAC

The objective of this project is to study the Muticommodity Flow Problem within the transport industry. In that field, companies aim to minimize the cost of routing the goods given that some resources, such as railways, have limited capacities. The report is divided into two stages. First the two main variants of the MCF : the path-based formulation and the arc-node model are coded and experimented using CPLEX on small datasets. Afterward, another approach called Column-Generation is implemented in Python to solve the MCF for larger instances.

Contents

Chapter 1

Literature Review

The multicommodity flow problem (MCFP) is a well-known network flow problem which has been studied since 1958 by William S. Jewell. As the MCFP comes up in many areas, the problem can be visualized in many ways. For instance, it can be described as follows :

Some producers want to ship their products, through railways, to some consumers. Both producers and consumers are modeled as nodes. Each product, called commodity, is identified by 3 components : its source (the producer of the product), its destination (the consumer of the product) and its quantity. Moreover, each railway, modeled by an arc in the network, is characterized by its cost per quantity unit and its maximal quantity capacity. In some cases, the nodes may be restricted by a maximal capacity as well. Then, the aim is to assign a path to each commodity from its source to its destination, while minimizing the total cost function and without exceeding the maximal capacity constraints.

The MCFP can be expressed as a linear-program and consequently solved in polynomial time using common LP techniques. However, these methods are not very efficient to solve large instances of the problem. That is why, today, heuristic or alternative methods are mainly employed to solve the MCFP.

The MCFP can model a wide variety of real-world problem [1], and a lot of case-specific variants exist. We can split them in three main categories depending on the domain of the decision variables, either they are integers, then the problem is called integer MCFP, approximation algorithm are mainly used in this case. Second, the decision variables take on real values and the problem can be modeled as a LP. The last category is mixed integer MCFP, where some decision variables are integers and others real values. It has been proved that in integer and mixed integer MCFP, the problem becomes NP-hard.

Today most of the methods employed to solve the MCFP use "decomposition" techniques and can be classified in three categories [2] :

1. **Price-directive decomposition** :
   The idea is to replace complicated constraints by adding "prices" variables which penalize bad solutions. This leads to solving easier LP and so gain efficiency. Some widely use price-directive methods are Lagrangian relaxation, Dantzig-Wolfe decomposition or Column-Generation.

2. **Resource-directive decomposition** :
   These methods aim for reducing the MCFP to multiple single commodity flow

problem which can then be solved very efficiently. To do so, a new constraint is introduced which allocate a certain amount of capacity on each arc for each commodity, without exceeding the total original arc capacity. However these methods does not ensure convergence.

3. **Partitioning methods** :
Most of the well-known LP solver methods can be adjusted to the MCFP specifically, for instance the "network simplex" is a speed-up modified version of simplex method.

Besides, it has been shown that the resource-directive algorithms converge rapidly for small problems but are outperformed by the price-directive techniques for large instances.
The above description of the MCFP make us think that this is a mainly transportation problem. However this problem comes up in a lot of other fields ranging from scheduling and telecommunication to image processing.

First, Niluka Rodrigo1 and Lashika Rjapaksha [3], showed that the MCFP can be used to solve the facility location problem (FLP). In this problem the goal is to ship several goods from plants to customers while passing through a distributed center. The goal is thus to determine which distribution center to assign to which plant, while minimizing the transportation cost and satisfying the capacity constraints. Murat Oguz, Tolga Bektas, Julia A. Bennell [4] describe in their paper a restricted version of the FLP based on the interger MCFP. Moreover, they use a Benders decomposition algorithm to optimally solve the problem, and show that the performance of their algorithm has a better running time than the previous best-known algorithm. Unlike the Column-Generation approach, the Benders decomposition method adds new constraints as it converges to the optimal solution.

Besides, Jean-François Cordeau, François Soumis and Jacques Desrosiers [9] propose a method to solve the assignement of locomotives and cars to trains problem. This minimization problem is subjected to compatibility equipment type constraints, making the problem harder. To solve it, they extent their previous approach(2000) based on the MCFP, and use a combination of branch-and-bound, followed by a Bender decomposition to solve a mixed-integer program at each node, in which the subproblems, corresponding to LP-relaxations of the MCFP are optimized by a Dantzig-Wolfe decomposition.

Another major problem in public transport, involving the MCFP, is the line-planning problem. Given a transportation demand, it consists in defining suitable line routes and their frequencies, in a public transport network, made of several modes (bus, tram, subway...). Most of the time, the objective is to minimize the operating costs of lines and the total passenger traveling time. Borndörfer, Ralf, Martin Grötschel, and Marc E. Pfetsch [10] propose a new multicommodity flow model for this problem where the commodities correspond to the demand between two stations. First, the algorithm discussed solve the LP-relaxation of the problem using a Column-Generation approach and then uses a greedy algorithm to construct a good integer solution. Their experiments show that their algorithm has a significant optimization potential and might be efficient to solve large instances.

Moreover, the MCPF comes up in many air transportation systems. Indeed, Dai, Weibin, et al.[11] propose an new heuristic algorithm to assign passenger demands, seen as commodities, to feasible airlines. Most of the time, the cost vector is based on the distance between the nodes. Their algorithm is based on column generation approach, in the sense that, first, they generate an initial set containing short paths. The shortness of a path is first measured in terms of pure distance value, then they adapt a new measure, using the number of nodes that a path contains to define their set of relevant paths. Based on their experiment, it turns out that the node-based distance measure is much more efficient. Even if their path-finding algorithms are time-consuming, a similar approach will be employed in the second part of the report.

As previously said, the MCFP can be found is scheduling problems too. Indeed Arton P. Dornelesa, Olinto C. B. de Araújob and Luciana S. Buriola [5] published a paper where they propose a mixed integer MCF model for the high-school timetabling problem. The goal is to build weekly schedule where a set of classes must be assigned to timeslots. Basically, the arcs represent transition of time periods and each teacher is represented by a commodity. To solve the problem, they propose an algorithm based on Dantzig-Wolfe decomposition which is a price-directive decomposition, and use a column generation approach to solve their Master problem. Moreover, two strategies are proposed to speed-up the algorithm. The analysis shows that the running time of their method is faster than the previous existing algorithm.

Finally, in most of the real-time applications, the commodities are unsplittable. Indeed in the maritime surveillance case, Hela Masri1, Saoussen Krichen, and Adel Guitouni[6] proposed to model the problem as an integer MCFP. Some messages, modeled as commodities, need to be shared among several nodes, surveillance employees, for instance. Then, the goal is to minimize the most congested arc traffic value. An effective heuristic method is proposed, called Hybrid Genetic Algorithm (HGA), and is compared to another method called Ant Colony System (ACS). It turns out that HGA produces slightly better output value than ACS and most importantly, the running time of HGA is extremely smaller than ACS for large instances.

The MCFP can even be found in, a priori, not related field, as image processing. Indeed Horesh Ben Shitrit, Jerome Berclaz, Francois Fleuret, and Pascal Fua [7] formulated a multi-persons tracking problem as a MCFP. In result, they obtained one of the best tracking algorithm so far.

All those examples show that a wide variety of methods exist to solve MCFP. Weibin Dai, Jun Zhang, Xiaoqian Sun [8] discussed the performance of two widely used methods, namely, Column-Generation and Lagrangian relaxation across various implementations and datasets. The results indicate that, in general, column generation approach obtains better output values and running times than Lagrangian relaxation, except in large network with high number of commodities where the running time of Lagrangian relaxation is slightly lower.

Chapter 2

Arc-Node and Path-based models

From all these papers, which present approaches to real-life problems based on the MCFP, we observe that it can be formulated in many ways. In this report we will focus on the two mains formulations used today, to solve the MCFP. Also, as this report is related to the rail freight transportation, the objective is to minimize the total cost to transport all the goods from their source to their recipient. The nodes represent the stations, the arcs correspond to railways and commodities are goods. Throughout this report, each commodity is sent from exactly one node and is shipped to exactly one node. Moreover all the commodities are splittable.

2.1 Arc-Node model

Given a directed graph $G := (N,A)$, the arcs are defined by their start and end nodes (i, j), and their cost per unit c_{ij}. Each node $n \in N$ and arc $(i, j) \in A$ have a maximum capacity u_n and v_{ij} respectively. Moreover, a commodities $k \in K$ is characterized by its start and end nodes, as well as a quantity q_k. Finally, to simplify the expression of the flow conservation constraints, we define a variable δ_n^k which equals 1 if the node n is the start node of the commodity k, equals -1 if n is the end node, otherwise it equals 0. The decision variable x_{ij}^k correspond to the fraction of the total quantity of the commodity k passing through the arc (i, j).
The node-arc formulation is defined as follow :

$$
\begin{aligned}
\text{minimize} \quad & \sum_{k\in K}\sum_{(i,j)\in A} c_{ij}x_{ij}^k q_k && && (1.1)\\
\text{subject to} \quad & \sum_{(i,j)\in A} x_{ij}^k - \sum_{(j,i)\in A} x_{ji}^k = \delta_i^k && \forall i \in N, \forall k \in K && (1.2)\\
& \sum_{k\in K}\sum_{(j,i)\in A} x_{ji}^k q_k \leq u_i && \forall i \in N && (1.3)\\
& \sum_{k\in K} x_{ij}^k q_k \leq v_{ij} && \forall (i,j) \in A && (1.4)\\
& x_{ij}^k \geq 0 && \forall (i,j) \in A, \forall k \in K && (1.5)
\end{aligned}
$$

This formulation aims to minimize the total cost of shipping all the commodities (1.1), while conserving, for each commodity, the entire flow between the start and end nodes (1.2). Moreover, the total quantity of the commodities at each node and arcs is limited (1.3),(1.4), and the commodities are allowed to split during the shipping (1.5).
The arc-node model represents the MCFP in a intuitive and stepwise way. Indeed, it

can be seen as follow, for each arc (i,j), how valuable is it to assign a commodity k to the arc.

2.2 Path-based model

Another widely used MCF formulation is called path-based model. Given a graph $G := (N, A)$ similar to the one defined in the arc-node model, nodes and arcs have a capacity u_n and v_{ij} respectively, the cost per unit of an arc is c_{ij}, and the quantity of the commodity k is q_k. Also, we define the sets $P(k)$ such that it contains all the possible paths from the start and end node of the commodity k. The cost per unit of a path p is denoted c_p, it corresponds to the sum of c_{ij}, for all arcs (i,j) contained in the path p. Unlike the arc-node variant, the decisions variables are x_p^k which is the fraction of the total quantity of the commodity k passing through the path $p \in P(k)$. Besides, to simplify the formulation of the capacity constraints, we define α_{ij}^p to be equals to 1 if the arc (i,j) is a part of the path p. As well as β_n^p to be equal to 1 if the node n is contained in the path p and is not the origin of the path p.
The path-based formulation is described as follow :

$$
\begin{aligned}
\text{minimize} \quad & \sum_{k \in K} \sum_{p \in P(k)} c_p x_p^k q_k && && (2.1)\\
\text{subject to} \quad & \sum_{p \in P(k)} x_p^k = 1 && \forall k \in K && (2.2)\\
& \sum_{k \in K} \sum_{p \in P(k)} x_p^k q_k \beta_i^p \leq u_i && \forall i \in N && (2.3)\\
& \sum_{k \in K} \sum_{p \in P(k)} x_p^k q_k \alpha_{ij}^p \leq v_{ij} && \forall (i,j) \in A && (2.4)\\
& x_p^k \geq 0 && \forall p \in P(k), \forall k \in K && (2.5)
\end{aligned}
$$

Similarly, the objective is the minimize the total cost of transporting all the commodities(2.1), while conserving, for each commodity all the flow from the start node to the end node (2.2). Besides the nodes and arcs have maximal capacities (2.3),(2.4) and the commodities can split during the transportation(2.5).

This formulation allow us to visualize the MCFP in another way, instead of focusing on the arcs, we look at the bigger picture and compute how good it is to assign a certain amount of a commodity to a path.

2.3 Analysis and Experiments

Thus, we defined the MCFP in two different ways. It would be interesting to know in which cases a formulation is more efficient than the other one. Table 1 show the number of variables and constraints in both formulations. Let $P = \bigcup_{k \in K} P(k)$. First, observe that in general, $\|P\|$ is extremely larger than $\|A\|$, so the number of variables in the path-based formulation is highly bigger than in the arc-node model. Besides the number of variable grows exponentially with the number of arcs in the path-based model. However, notice that the number of flow conservation constraints is highly lower in the path-based model. Note that the positivity constraints (1.5),(2.5) are easy to very, and thus not taken into account during the analysis. Besides, the number of nodes and arcs capacities constraints are the same. Thus, as the number of commodities grows, the number of constraints in the arc-node model grows greatly faster than in the path-based model.

	#Variables	#Constraints	
Arc-node formulation	$\|K\|\|A\|$	$\|K\|\|N\|$	(1.2) Flow conservation
		+ $\|N\|$	(1.3) Node capacity
		+ $\|A\|$	(1.4) Arc capacity
		+ $\|K\|\|A\|$	(1.5) Positivity
Path-based formulation	$\|K\|\|P\|$	$\|K\|$	(2.2) Flow conservation
		+ $\|N\|$	(2.3) Node capacity
		+ $\|A\|$	(2.4) Arc capacity
		+ $\|K\|\|P\|$	(2.5) Positivity

TABLE 2.1: Number of variables and constraints in the arc-node and path based formulations

The performance of these two different formulations are tested using data sets of small, medium and large size, while varying the node and arc capacities. All the experiments were run on a PC with a 1,6 GHz Intel Core i5 processor and 8Go of RAM, using CPLEX v12.8.0.0. The tables 2.2, 2.3 and 2.4 present the results obtained on the datasets.

	Constraints values	Objective func.	Running time
	$u_n = \infty, v_{ij} = \infty$	1 623 760	0.63sec
Arc − node	$u_n = \{2800, 5600, 8400\}, v_{ij} = \{560, 980, 1400, 1820\}$	1 628 400	1.01s
	$u_n = \{2600, 5200, 7800\}, v_{ij} = \{520, 910, 1300, 1690\}$	1 657 820	0.91s
Memory peak	$u_n = \{2400, 4800, 7200\}, v_{ij} = \{480, 840, 1200, 1560\}$	1 690 260	0.78s
$\simeq$ 24M	$u_n = \{2200, 4400, 6600\}, v_{ij} = \{440, 770, 1100, 1430\}$	1 724 660	0.81s
	$u_n = \{2100, 4200, 6300\}, v_{ij} = \{420, 735, 1050, 1365\}$	unfeasible	0.58s
	$u_n = \infty, v_{ij} = \infty$	1 623 760	65.86s(30.8 + 1.2 + 8.7)
Path − based	$u_n = \{2800, 5600, 8400\}, v_{ij} = \{560, 980, 1400, 1820\}$	1 628 400	70.98s(30.2 + 1.3 + 11.8)
	$u_n = \{2600, 5200, 7800\}, v_{ij} = \{520, 910, 1300, 1690\}$	1 657 820	66.2s(30.6 + 1.2 + 8.8)
Memory peak	$u_n = \{2400, 4800, 7200\}, v_{ij} = \{480, 840, 1200, 1560\}$	1 690 260	74.9s(32.3 + 1.4 + 9.75)
$\simeq$ 1.5G	$u_n = \{2200, 4400, 6600\}, v_{ij} = \{440, 770, 1100, 1430\}$	1 724 660	69.32s(31.2 + 1.1 + 8.7)
	$u_n = \{2100, 4200, 6300\}, v_{ij} = \{420, 735, 1050, 1365\}$	unfeasible	39.70s(30.1 + 1.68)

TABLE 2.2: Small dataset : 20 nodes, 48 arcs, 202 commodities.
The nodes and arcs constraints are split in 3 and 4 categories respectively. The running time contains 3 main stages : (loading and precompute the data + solving time + postprocessing time)

	Constraints values	Objective function	Running time
Arc − node	$u_n = \{40, 80, 100, 150\}$, $v_{ij} = \infty$	unfeasible	
Memory peak	$u_n = \{45, 80, 150, 200\}$, $v_{ij} = \infty$	43 598 821,5	1h15
$\simeq$	$u_n = \{60, 90, 130, 180\}$, $v_{ij} = \infty$	42 909 708	1h19
	$u_n = \infty$, $v_{ij} = \infty$	42 469 841	1h21
Path − based		Require too much pre-compuation	

TABLE 2.3: Medium dataset : 2172 nodes, 4546 arcs, 242 commodities. The quantity of each commodity is lower than in the small dataset, which explains the low capacity constraints values. For the moment the arcs capacity constraints are not included.

	Constraints values	Objective function	Running time
Arc − node Memory peak $\simeq$	$u_n = \infty$, $v_{ij} = \infty$	536 675 196 (to be confirmed)	12h31 (to be confirmed)
Path − based		Require too much pre-compuation	

TABLE 2.4: Large dataset : 2172 nodes, 4546 arcs, 1173 commodities. For the moment the arcs capacity constraints are not included.

First, notice that CPLEX solve to optimality the problem, which is why the objective function values are the same in both formulation. Moreover, for each dataset, the number of arcs is equals to a bit more than twice the number of nodes. The sparse structure of the network explains why the objective function grow relatively slowly from the unconstraint to unfeasible instance.

For the small dataset the running time of the arc-node formulation is extremely faster than the path-based model. Indeed, the later require a large amount of precomputational time, to generate the set $P(k)$ for each commodity and the number of variables is larger in the path-based formulation. Besides, the path based model requires much more memory than the arc-node formulation, again it can be explained by the large difference of number of variables generated between the two models.

Chapter 3

Column-Generation approach

The experiments performed in last section show that the arc-node formulation is extremely faster, and consumes less memory than the path-based approach. The later tend to be even computationally unfeasible as the size of the network grows. From the running time analysis and the table 2.1, the main reason explaining the huge gap between these models is the fact that the path-based approach contains an enormous number of variable and grows very fast as the size of the network increases, resulting in a large amount of pre-computational time to generate all of them.

However from Table 2.1, we notice that the main benefit of the path-based approach is the low number of constraints. Moreover, most of the time the optimal solution contains only a small amount of variables among all the ones generated, meaning that we do not need all variables to obtain the optimal solution. One way to leverage these observations would be to employ a method which does not require all the variables to find the optimal solution, but only the relevant ones. From the literature review we have seen that a wide variety of approaches exist, with the aim of cutting down the amount of variables considered. One of them is the column generation approach, which is today extensively used. Besides, it is an exact method which is well suited for our datasets given their sparse structures. Finally, the comparative analysis performed in Weibin Dai, Jun Zhang, Xiaoqian Sun paper [8], tend to show that column generation approach outperform Lagrangian decomposition for large instances.

Hence, the column generation approach appears to be a very efficient method in our case, we are going to apply it to our network.

3.1 How does it work ?

First, we call the problem that we desire to solve, the Master problem (MP). In our case, it is the path-based model described from (2.1) to (2.5). As stated earlier, the goal is to narrow down the number of variables generated, and solve the problem to optimality. This problem is called the Restricted Master problem (RMP). The difference between the MP and the RMP is only that the RMP generate a subset of variables instead of all of them.

The optimal solution will be obtained by solving the RMP repeatedly. However, how can we be certain that there is no better solution if we do not even generate all variables ? To resolve this issue, we will make use of the Primal-Dual Optimality conditions.

Theorem 1. *Given a primal linear problem P, let us denote its corresponding dual problem D :*

$x*$ and $y*$ are respectively the optimal solution of the primal and dual problem if and only if the three following conditions hold :

- *Primal feasibility : x* satisfies each constraint of P*
- *Objective function : The value of the primal objective function for x* is the same as the dual objective function value for y*.*
- *Dual feasibility : y* satisfies each constraint of D*

Thus, given the optimal solution to the RMP called $\widetilde{x*}$ and its dual variables $\widetilde{y*}$, we can confirm that $\widetilde{x*}$ satisfies the Primal feasibility condition as the RMP contains the same constraints as the MP. Besides the Objective function condition is also satisfied thanks to the strong duality theorem. The only condition left that we must verify, to ensure that the optimal solution of the RMP is also the optimal solution to the MP, is the dual feasibility. That is, verify that $\widetilde{y*}$ satisfy all the constraints of the dual of the MP.

Let y_k with $k \in K$, $y_{(i)}$ with $i \in N$ and $y_{(i,j)}$ such that $(i,j) \in A$ denote the dual variables corresponding to the k^{th} constraint of (2.2), the i^{th} constraint of (2.3) and to the $(i,j)^{th}$ constraint of (2.4) respectively. The dual problem of the MP is formulated as follows : (TO DO : INCLUDE SMALL FIGURE TO SHOW ALL THE DIFFERENT VARIANTS OF THE MCFP)

$$
\begin{aligned}
\text{maximize} \quad & \sum_{k \in K} y_k + \sum_{i \in N} y_{(i)} u_i + \sum_{(i,j) \in A} y_{(i,j)} v_{ij} && && (3.1)\\
\text{subject to} \quad & y_k + \sum_{i \in N} y_{(i)} q_k \beta_i^p + \sum_{(i,j) \in A} y_{(i,j)} q_k \alpha_{ij}^p \leq c_p^k q_k && \forall k \in K, \forall p \in P(k) && (3.2)\\
& y_k \in \mathbb{R} && \forall k \in K && (3.3)\\
& y_{(i)} \leq 0 && \forall i \in N && (3.4)\\
& y_{(i,j)} \leq 0 && \forall (i,j) \in A && (3.5)
\end{aligned}
$$

From the Theorem 1, the aim is to verify whether $\widetilde{y*}$ satisfies all the (3.2) constraints, this problem is called the Pricing problem. Nonetheless, the amount of (3.2) constraints is equal to the number of variables in the MP, which is extremely large. This huge quantity of variables was the reason why we needed to find a new efficient approach to exploit the path-based model. Thus, the key is to identify an efficient technique to verify all those constraints without explicitly enumerate all of them.

We are going to formulate the (3.2) constraints in another way to tackle the pricing problem from a different point of view :

$$
\begin{aligned}
\forall k \in K, \forall p \in P(k) : \quad & y_k + \sum_{i \in N} y_{(i)} q_k \beta_i^p + \sum_{(i,j) \in A} y_{(i,j)} q_k \alpha_{ij}^p \leq c_p^k q_k \\
\equiv \quad & -y_k - \sum_{i \in N} y_{(i)} q_k \beta_i^p - \sum_{(i,j) \in A} y_{(i,j)} q_k \alpha_{ij}^p + c_p^k q_k \geq 0
\end{aligned}
$$

$$\equiv \quad -y_k - \sum_{(i,j)\in A} y_{(j)} q_k \alpha_{ij}^p - \sum_{(i,j)\in A} y_{(i,j)} q_k \alpha_{ij}^p + q_k \sum_{(i,j)\in A} c_{ij}^k \alpha_{ij}^p \geq 0$$
$$\equiv \quad -y_k + q_k \sum_{(i,j)\in A} \alpha_{ij}^p (c_{ij}^k - y_{(j)} - y_{(i,j)}) \geq 0$$

Consequently, for each commodity $k \in K$ all of the following inequalities need to be satisfied :

$$-y_k + q_k \sum_{(i,j)\in A} \alpha_{ij}^p (c_{ij}^k - y_{(j)} - y_{(i,j)}) \geq 0 \qquad \text{subject to : } p \in P(k)$$

Which is equivalent to the system:

$$\begin{array}{lll} & -y_k + q_k \sum_{(i,j)\in A} \alpha_{ij}^p (c_{ij}^k - y_{(j)} - y_{(i,j)}) \geq 0 & \\ \text{subject to :} & \sum_{(i,j)\in A} \alpha_{ij}^p - \sum_{(j,i)\in A} \alpha_{ji}^p = \gamma_i^k & \forall i \in N \\ & \alpha_{ij}^p \in \{0,1\} & \forall p \in P, \forall (i,j) \in A \end{array}$$

where γ_i^k equals 1 if i is the starting node of the commodity k, equals -1 if i is the destination of k and 0 otherwise. Notice that P denotes all paths in the network.

Finally, we can rewrite the pricing problem as Integer Programs, the following problem will be called sub-pricing problem for commodity k :

$$\begin{array}{llll} \text{minimize} & -y_k + q_k \sum_{(i,j)\in A} \alpha_{ij} (c_{ij}^k - y_{(j)} - y_{(i,j)}) & & (4.1) \\ \text{subject to :} & \sum_{(i,j)\in A} \alpha_{ij} - \sum_{(j,i)\in A} \alpha_{ji} = \gamma_i^k & \forall i \in N & (4.2) \\ & \alpha_{ij} \in \{0,1\} & \forall (i,j) \in A & (4.3) \end{array}$$

Indeed if the objective function value of the optimal solution is negative, then it implies that the path p described by the arcs α_{ij}'s violates a constraint in the dual of the MP. Thus the corresponding variable in the RMP needs to be generated to satisfy the constraint, in order to ensure that the constraint will be satisfied later on. Finally, if for each commodity, the objective function value of the optimal solution is non-negative, it implies that the corresponding RMP solution is also optimal to the MP.

The constraints (4.2) and (4.3) aim to produce a set of arcs such that it contains exactly one arc starting from the origin of the commodity k, and one arc ending at the destination of commodity k, moreover the set of arcs must form a path. Thus, the feasible α_{ij}'s vectors represent all the paths from the origin of commodity k to its destination. On the other hand, the objective function contains one constant y_k, and the other term is a cost vector modified by the dual variables $y_{(i)}$ and $y_{(i,j)}$. Therefore, this integer program is in fact a shortest path problem with penalized arcs costs. Hence, the pricing problem boils down to a set of $\|K\|$ shortest path problems.

Plenty of polynomial times algorithms exist to solve the shortest path problem efficiently. Consequently, we obtained a efficient method to verify if the optimal solution of the RMP is also optimal to the MP.

However, in the first place, we need to find a feasible set of variables to the RMP to guarantee the primal feasibility condition of Theorem 1. As we only request feasibility, a fast heuristic algorithm can be applied.

Finally the column generation approach can be summarized in Figure 1 :

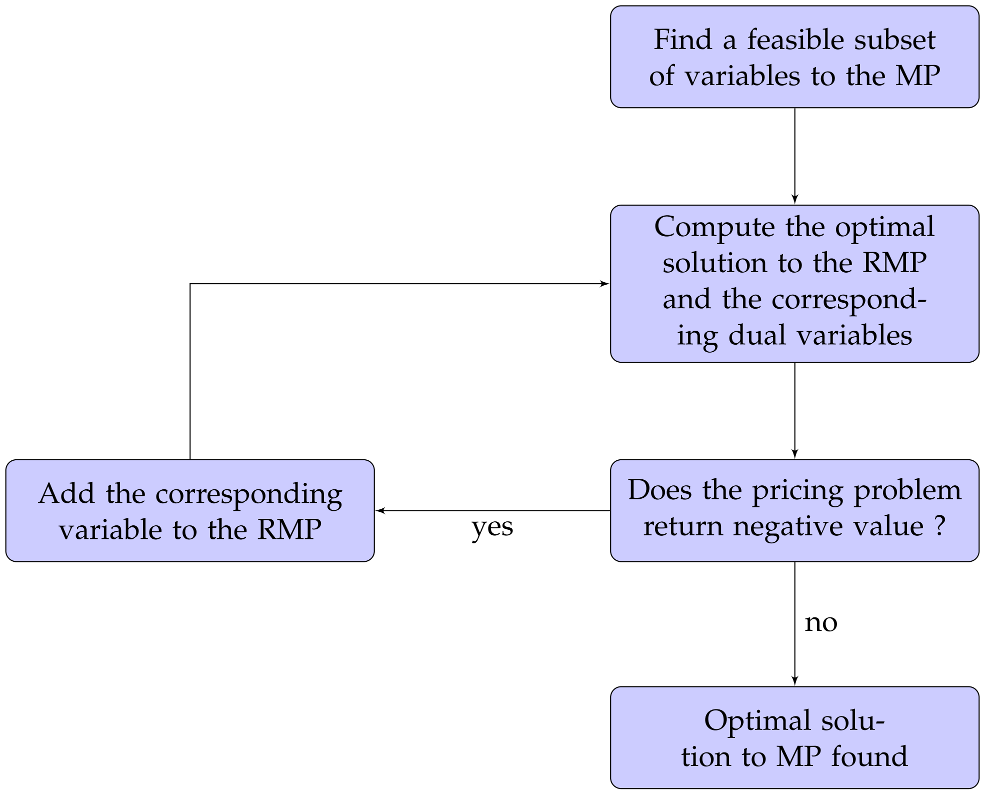


FIGURE 3.1: Sketch of the column generation method

Throughout the experiment part, several strategies will be tested, for instance : Do we obtain a better running time if we add paths to the RMP one by one, or several at one time. Moreover, many different algorithms to find a feasible promising set of variables in the initialization step will be compared. The aim is to find the more efficient column generation approach to solve the MCFP in our network.

3.2 Experiment and Analysis

The following implementations are coded in Python using CPLEX API to solve the different LP's. First the datasets were parsed into Python data structures. Notice that the implementations are described in chronological order and that the results obtained are subject to the code optimization and the data-structures used. Consequently the following analysis and experiments should not be taken for a general truth, but as an indication

3.2.1 1^{st} Implementation :

- **Initial feasible set :** The first task is to find a method to find a promising set of initial variables. We noticed that the network represents a rail freight grid, consequently the cost between two arcs is mainly due to the distance between them. Thus selecting the shortest path for a commodity as its initial variable seems to be an efficient strategy. We choose to run Dijkstra's algorithm to compute them.
  However capacity constraints might prevent us from assigning all commodities to their shortest path. To overcome this difficulty, for each commodity, we select all the paths between the origin i and the destination j with a distance less than up * (shortest path between i and j) where $up \geq 1$. We run Depth First search algorithm with the additional distance stopping condition to find these paths.
  As we want to minimize the number of variables in the RMP, in the first iteration up is fixed to 1, then the RMP is run to verify whether the initial set is feasible. If it is the case, we achieved the initializing phase, otherwise, up is slightly increased until a feasible initial set is found.
  Finally, if up reaches a large value as 4, we assume that the RMP is unfeasible. Notice that, for large value of up all the possible paths will be generated, so if the master problem has a feasible solution, this method will also return one eventually. Algorithm 1 summarizes this approach.

```
up = 1;
initialPaths = ∅ ;
for i ← 1 to nCommodities do
    initialPaths[i] ← Dijkstra(commodity[i].origin, commodity[i].destination);
    shortestPaths[i] ← initialPaths[i];
end
while RMP(initialPaths) is unfeasible do
    for i ← 1 to nCommodities do
        initialPaths[i] ←
          DepthFirstSearch(commodity[i].origin, commodity[i].destination, up ∗
          shortestPaths[i]);
    end
    if up ≥ 4 then
        print('TheMasterproblemisunfeasible');
        break;
    end
    up = up * 1.05
end
```

Algorithm 1: Initialization phase

- **RMP & Pricing Problem :** The RMP and sub-pricing problems are solved through CPLEX API. Regarding the dual feasibility condition, for each commodity the sub-pricing problem is solved and only the solution corresponding to the lowest negative objective value, among all the sub-pricing problems, is added the RMP.

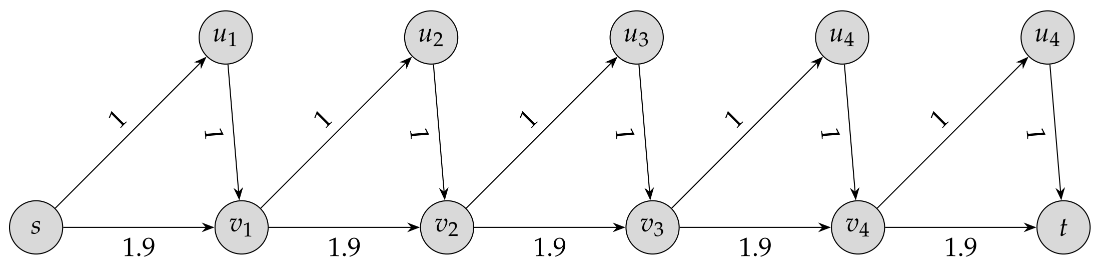
- **Experiment :** The first implementation detailed above is tested for the small dataset.

	Constraints values	Obj. function	Running time
Column Generation 1^{st} Implementation	$u_n = \infty, v_{ij} = \infty$	1 623 760	4.4(0.4+4.0)
	$u_n = \{2800, 5600, 8400\}, v_{ij} = \{560, 980, 1400, 1820\}$	1 628 400	7.5(3.4+4.0)
	$u_n = \{2600, 5200, 7800\}, v_{ij} = \{520, 910, 1300, 1690\}$	1 657 820	8.8(5.0+3.8)
	$u_n = \{2400, 4800, 7200\}, v_{ij} = \{480, 840, 1200, 1560\}$	1 690 260	13.7(9.9+3.8)
	$u_n = \{2200, 4400, 6600\}, v_{ij} = \{440, 770, 1100, 1430\}$	1 724 660	15.5(11.5+3.9)
	$u_n = \{2100, 4200, 6300\}, v_{ij} = \{420, 735, 1050, 1365\}$	unfeasible	33.58

TABLE 3.1: Small dataset : The running time unit is second and the sum (x+y) represents the duration of finding the initial feasible set + solving the MP after having the initial set. (TO DO : Add memory peak)

From Table 1, we observe that, as the column generation is an exact method, the objective function values are the same as the ones we obtained from the arc-node and path-based model in Chapter 2. Then, we remark that the running time grows rather quickly as the constraints gets tighter. The explanation is that, as the constraints gets tighter, the variable up gets bigger so more instances of *DepthFirstSearch* need to be run.
Moreover, the running times are still extremely larger than the ones we obtained using the arc-node formulation, but we can notice an improvement compared to the path-based formulation, indeed, for the unconstraint execution, this implementation takes almost 15 times less time than the path-based model.

Afterwards, the same implementation is tested for the medium dataset. However the execution is stopped after few hours, the Algorithm 1 used to find the initial set takes a huge amount of time and seems to be unsuitable for medium or large instance. Indeed, in order to find the initial set of paths, for each commodity, the algorithm executes *DepthFirstSearch* which generate all possible paths from the origin. But as we have seen, the explicit enumeration of paths is time-consuming, consequently this approach should be avoided. Even the additional distance stopping condition in *DepthFirstSearch* doesn't ensure a relevant running time.
Moreover, this algorithm has other drawbacks, for instance in a network similar to the one below :



The first iteration of the algorithm will only include the straight path from s to t as up equals 1 ($s - v1 - v2 - v3 - v4 - t$). If no feasible solution is found, up will be incremented and a lot of paths will be added to the initial set, such as ($s - u1 - v1 - v2 - v3 - v4 - t$). Among them, a lot might be unnecessary and the algorithm would waste resources to generate them. Thus, the intuitive Algorithm 1 is not effective enough for larger instances, a new approach needs to be found.

3.2.2 2^{nd} Implementation :

- **Initial feasible set :** We noticed that the cost of a path is mainly due to the distance between the nodes. Moreover Dijkstra's algorithm is one of the fastest known single-source shortest-path algorithm for directed graphs. Hence we will design another approach to generate the initial set of paths using only Dijkstra's algorithm.
  First, like in Algorithm 1, we will try to route all the commodities through their respective shortest path. If it results in a feasible solution, the initialization task is achieved, otherwise the algorithm tries to find which node or arc prevent the most from assigning the commodities to their shortest path. Once the node/arc is identified, the shortest paths are recomputed for each commodity, ignoring it. At this stage, the commodities can be routing through two or one path (if ignoring the identified node/arc doesn't change from its original shortest path). Then, we test if the new set of initial variables is feasible, if it is not, once again, we compute the node/arc having its capacity constraint the most violated, and the new shortest paths for each commodity, ignoring the two previously identified nodes/arcs.
  The process is repeated until a feasible solution is found, or a same node/arc is identified twice as the one preventing the most from assigning commodities to their generated paths.

  In order to identify which node/arc has its capacity constraint the most violated, we compute the total flow coming in each node minus its respective capacity. Hence, we obtain the exceeding flow for each node, the one with the highest value is picked.
  However, how to compute the flow coming in a node, if we can route a commodity through multiple different paths ? We consider that the quantity of the commodity is equally split through all its different paths. The algorithm 2 and 3 summarize the method described.

```
initialPaths = ∅ ;
for i ← 1 to nCommodities do
    intialPaths[i] ← Dijkstra(commodity[i].origin, commodity[i].destination);
end
node_arcToIgnore = ∅ ;
while RMP(initialPaths) is unfeasible do
    temp = node_arcPreventingTheMost();
    if temp ∈ node_arcToIgnore then
        print('The Master problem is unfeasible');
        break;
    end
    node_arcToIgnore.add(temp);
    for i ← 1 to nCommodities do
        initialPaths[i].add(Dijkstra(commodity[i].origin, commodity[i].destination,
          node_arcToIgnore));
    end
end
```

Algorithm 2: Initialization phase

```
countNode = ∅ ;
for i ← 1 to nStations do
    countNode[i] ← −capacity[node[i].type];
    for j ← 1 to nCommodities do
        for k ← 1 to length(initialPaths[j]) do
            if i ∈ initialPaths[j][k] then
                countNode[i] += commodity[j].quantity/length(initialPaths[j]);
            end
        end
    end
end
;
countArc = ∅ ;
for i ← 1 to nArc do
    countArc[i] ← −capacity[arc[i].type];
    for j ← 1 to nCommodities do
        for k ← 1 to length(initialPaths[j]) do
            if i ∈ initialPaths[j][k] then
                countArc[i] += commodity[j].quantity/length(initialPaths[j]);
            end
        end
    end
end
return argmax(countNode.union(countArc));
```

Algorithm 3: nodePreventingTheMost()

- **RMP & Pricing Problem :** The RMP and the sub-pricing problems are solved through CPLEX API.
  Regarding the dual feasibility condition, each optimal solution with negative objective function value to the sub-pricing problems are added to the RMP.

Unlike the Algorithm 1, multiple paths are added to the RMP at one time. The main reason is that, given two commodities A and B, adding a new path to the initial set of A won't change the sub-pricing problem of B, if the new path and the paths in the initial set of A don't share any stations with the paths in the initial set of B. Consequently, it means that two sub-pricing problems might be independent from each other.

- **Experiment :** The second implementation detailed above is tested for the medium dataset (For the moment the arcs in the medium dataset do not have maximal capacities).

	Constraints values	Objective function	Running time
Column	$u_n = \{40, 80, 100, 150\}$, $v_{ij} = \infty$	unfeasible	
Generation	$u_n = \{45, 80, 150, 200\}$, $v_{ij} = \infty$	43 598 821,5	1h34(23m+1h11)
2^{nd} Method	$u_n = \{60, 90, 130, 180\}$, $v_{ij} = \infty$	42 909 708	58m(7m+51m)
	$u_n = \infty$, $v_{ij} = \infty$	42 469 841	1h21

TABLE 3.2: Medium dataset. In the running time column, the sum (x+y) represents the duration of finding the initial feasible set + solving the MP after having the initial set.

From the Table 3.2, we can notice that the running times is similar to the ones obtained by the arc-node formulation. However the running time grows quickly as the constraints get tighter and is mostly due to the pricing problem.

At this stage, the implementations were not efficient enough. Thus, several others strategies were tested. I optimized my code and chose more efficient data-structures. I will present the two most efficient implementations I have obtained.

3.2.3 3^{rd} Implementation

- **Initial feasible set :** In this implementation the initial feasible set is computed in a similar way as in the 2^{nd} implementation. The only difference is that, instead of computing and ignoring the node/arc with the most violated capacity, the two most violated nodes and arcs will be ignored. So this strategy is described by the Algorithm 2, however, now the Algorithm 3 returns the two indices with the highest values in $countNode$ and the two indices with the highest values in $countArc$.

- **RMP & Pricing Problem :** The RMP is still solved using CPLEX API.
  In the previous implementation, we noticed that the pricing problem represented most of the running time. To speed-up the implementation, the sub-pricing problems are now solved using Dijkstra's algorithm instead of using CPLEX API. Otherwise, the strategy to add the paths to the RMP is the same as in the 2^{nd} implementation.

- **Experiment :** The third implementation detailed above is tested for the medium dataset and large dataset. Note that the arc capacities are now taken into account. Due to the large size of v_{ij}, only the scaling coefficient is indicated. This coefficient is used to scale down the real-life are capacity to our network size. Besides, in order to represent the real-life network as closely as possible, the node capacities u_n contains three values now, and the capacity values are well-adjusted.

	Constraints values	Objective function	Running time
Column	$u_n = \{\infty, 40, 80\}$, $v_{ij}\_cofficient = 1/80$	unfeasible	53s
Generation	$u_n = \{\infty, 45, 80\}$, $v_{ij}\_cofficient = 1/75$	42 607 124,99	30s(21+9)
3^{nd} Method	$u_n = \{\infty, 50, 80\}$, $v_{ij}\_cofficient = 1/70$	42 526 557,63	25s(16+9)
	$u_n = \infty$, $v_{ij} = \infty$	42 469 841	8s(4+4)

TABLE 3.3: Medium dataset. In the running time column, the sum (x+y) represents the duration of finding the initial feasible set + solving the MP after having the initial set.

	Constraints values	Objective function	Running time
Column	$u_n = \{\infty, 400, 800\}$, $v_{ij}\_cofficient = 1/10$	unfeasible	199s
Generation	$u_n = \{\infty, 500, 1000\}$, $v_{ij}\_cofficient = 1/9$	535 530 970	141s(95+46)
3^{nd} Method	$u_n = \{\infty, 600, 1200\}$, $v_{ij}\_cofficient = 1/9$	534 524 702,56	138s(94+44)
	$u_n = \{\infty, 700, 1400\}$, $v_{ij}\_cofficient = 1/8$	533 882 451	109s(66+43)
	$u_n = \infty$, $v_{ij} = \infty$	533 530 970	23s(12+11)

TABLE 3.4: Large dataset. In the running time column, the sum (x+y) represents the duration of finding the initial feasible set + solving the MP after having the initial set.

On the first hand, the running times are extremely smaller than in the previous implementation. This is due to the fact that the strategies employed to find the initial set and solve the pricing problem are more efficient, along with the fact that the code has been optimized and the data structures used are well suited. Once again, the objective function values grow slowly because of the sparse structure of the graph, and the running time increases moderately as the constraints get tighter.

3.2.4 4^{th} Implementation

- **Initial feasible set :** From now on, the objective is to find another strategy to compute the initial promising set. The main drawback of the Algorithm 2 is that each time we detect that the RMP is unfeasible, then we iterate over all the commodities (l.13), all the nodes (Alg.3, l.2) and all the stations (Alg.3, l.14)
The strategy proposed in this implementation aims to reduce the number of iteration. The first step is similar as before : for each commodity we run Dijkstra's algorithm to find the corresponding shortest path. Next, the commodities are sorted according to the outdegree value of their shortest path. The outdegree of a path corresponds to the sum of the outdegrees of each nodes the path contains, except the destination node. Consequently, the outdegree of a path tend to represent the number of alternative path from two nodes. Once the commodities are sorted, we try to route each commodities through its shortest path. If a node/arc has already reached its maximal capacity, then the shortest path is re-computed ignoring the violated node/arc. In the case where no feasible path exist for a commodity, the latter is put at the top of the

list. The process is repeated until a feasible path is assign to each commodity. The algorithm 4 resumes this strategy.

```
1  initialPaths = ∅ ;
2  for i ← 1 to nCommodities do
3  |  intialPaths[i] ← Dijkstra(commodity[i].origin, commodity[i].destination);
4  end
5  ordered_commoditiy ←= argsort(outdegree(initialPaths));
6
7  for i ∈ ordered_commodity do
8  |  while assigning initialPaths[i] to commodity i violate a capacity constraint do
9  |  |
10 |  |  initialPaths[i] ← Dijkstra(i.origin, i.destination, violated_nodes_arcs);
11 |  |  if initialPaths[i] doesn't exist then
12 |  |  |  ordered_commodity.put_on_top(i);
13 |  |  |  break;
14 |  |  end
15 |  end
16 end
17 if A commodity has not a feasible path then
18 |  Return to l. 7;
19 end
```

Algorithm 4: Initialization phase

The main advantage of this strategy is that, in general, we iterate over all the commodities less time than in Algorithm 2. Besides we do not iterate over the arcs and nodes.

- **RMP & Pricing Problem :** The RMP is still solved using CPLEX API. The pricing problem is solve in the same way as in the 3^{rd} implementation.
- **Experiment :** The forth implementation described above is tested for the medium dataset and large dataset.

	Constraints values	Objective function	Running time
Column Generation 3^{rd} Method	$u_n = \{\infty, 40, 80\}$, $v_{ij}\_cofficient = 1/80$	unfeasible	47s
	$u_n = \{\infty, 45, 80\}$, $v_{ij}\_cofficient = 1/75$	42 607 124,99	34s(11+23)
	$u_n = \{\infty, 50, 80\}$, $v_{ij}\_cofficient = 1/70$	42 526 557,63	31s(9+22)
	$u_n = \infty$, $v_{ij} = \infty$	42 469 841	7s(3+4)

TABLE 3.5: Medium dataset. In the running time column, the sum (x+y) represents the duration of finding the initial feasible set + solving the MP after having the initial set.

	Constraints values	Objective function	Running time
Column Generation 3^{rd} Method	$u_n = \{\infty, 400, 800\}$, $v_{ij}\_cofficient = 1/10$	unfeasible	295s
	$u_n = \{\infty, 500, 1000\}$, $v_{ij}\_cofficient = 1/9$	can't find initial set	270s
	$u_n = \{\infty, 600, 1200\}$, $v_{ij}\_cofficient = 1/9$	534 524 702,56	173s(87+86)
	$u_n = \{\infty, 700, 1400\}$, $v_{ij}\_cofficient = 1/8$	533 882 451	172s(88+84)
	$u_n = \infty$, $v_{ij} = \infty$	533 530 970	19s(7+12)

TABLE 3.6: Large dataset. In the running time column, the sum (x+y) represents the duration of finding the initial feasible set + solving the MP after having the initial set.

First of all, we observe that the total running time are slightly bigger than before, especially in the large dataset. However, the amount of time the algorithm takes to compute the initial feasible set is in general better than in the previous implementation, which is what we expected. The main drawback of this approach is that the initial set contains only one path for each commodity contrary to the 3^{rd} implementation, in which many paths can be assign to a commoditiy in the initial set. Consequently, the number of iteration of the pricing problem is much higher than before, resulting in a higher total running time. This drawback even prevents from finding an initial set when the constraint are too tight.

All the implementations described above highly depends on the code optimization and the data-structures used to store the dataset. Therefore, many improvements could still be made, as well as other techniques to solve the pricing problem and find an initial set could be faster.

Chapter 4

Conclusion

In this report, we have seen that the Multicommodity Flow Problem comes up in a large variety of fields and has been studied since 1958. Therefore, a wide diversity of techniques exist to resolve the problem, especially decomposition techniques.

On the first hand, we implemented and experimented on CPLEX the two mains LP formulations of the MCFP which are the arc-node and path-based model. The results show that the arc-node formulation sorely outperforms the path-based formulation due to number of variables which becomes enormous as the size of the network grows. But the latter have the advantage of containing less constraints. Moreover, most of the time the optimal solution contains a small amount of variables. Based on these observations, the Column Generation approach seemed well suited, consequently we implemented multiples versions of it, in order to find the more efficient one. The experiments tend to show that, in general, the 3^{rd} implementation is the most effective one. Hence, we have showed that the common LP-techniques employed by CPLEX to solve linear problems are completely outperforms by the Column Generation approach.

As previously said, the result obtained are subject to the code optimization. Thus improving some key part of the code could lead to significant running time enhancement. For instance Xu, M. H.[12], et al. presents an improved version of Dijkstra's algorithm for sparse network. Besides, many other heuristic strategies may exist in order to find an initial set more efficiently. As these implementations aim to solve transport related datasets, some commodities might be unsplitable in real life, thus it could be interesting to add a branch-and-cut layer.

Bibliography

[1] C. Barnhart N. Krishnan and P. Vance. *Multicommodity Flow Problems*. In C. A. Floudas and P. M. Pardalos editors Encyclopedia of Optimization pages 2354-2362. Springer US 2009.

[2] J. Lindstrom. *Multicommodity Network Flow - Methods and Application* in Proc. Information Resources Management Association International Conference (IRMA04) Business Process Management Tools and Technologies Track New Orleans Louisiana May 2004 pp. 1-8.

[3] Rodrigo, Niluka, and Lashika Rjapaksha. *Mathematical Model and a Case Study for Multi-Commodity Transportation Problem.* International Journal of Theoretical and Applied Mathematics 4.1 (2018)

[4] Oğuz, Murat, Tolga Bektaş, and Julia A. Bennell. *Multicommodity flows and Benders decomposition for restricted continuous location problems.* European Journal of Operational Research 266.3 (2018): 851-863.

[5] Dorneles, Árton P., Olinto CB de Araújo, and Luciana S. Buriol. *A column generation approach to high school timetabling modeled as a multicommodity flow problem.* European Journal of Operational Research 256.3 (2017): 685-695.

[6] Masri, Hela, Saoussen Krichen, and Adel Guitouni. *A Hybrid Genetic Algorithm for Solving the Unsplittable Multicommodity Flow Problem: The Maritime Surveillance Case.* International Conference on Algorithmic Applications in Management. Springer, Cham, 2014.

[7] Shitrit, Horesh Ben, et al. *Multi-commodity network flow for tracking multiple people.* IEEE transactions on pattern analysis and machine intelligence 36.8 (2014): 1614-1627.

[8] Weibin, D. A. I., Jun Zhang, and S. U. N. Xiaoqian. *On solving multi-commodity flow problems: An experimental evaluation.* Chinese Journal of Aeronautics 30.4 (2017): 1481-1492.

[9] Jean-François Cordeau, François Soumis and Jacques Desrosiers. *Simultaneous assignment of locomotives and cars to passenger trains.* Operations research 49.4 (2001): 531-548.

[10] Borndörfer, Ralf, Martin Grötschel, and Marc E. Pfetsch. *A column-generation approach to line planning in public transport.* Transportation Science 41.1 (2007): 123-132.

[11] Weibin Dai, Jun Zhang, Xiaoqian Sun, and Sebastian Wandelt *Node Dependency in Multi-Commodity Flow Problem with Applications to Transportation Networks.* CICTP 2016. 2016. 1989-2001.

[12] Xu, M. H., et al *An improved Dijkstra's shortest path algorithm for sparse network.* Applied Mathematics and Computation (2007)